True Trees

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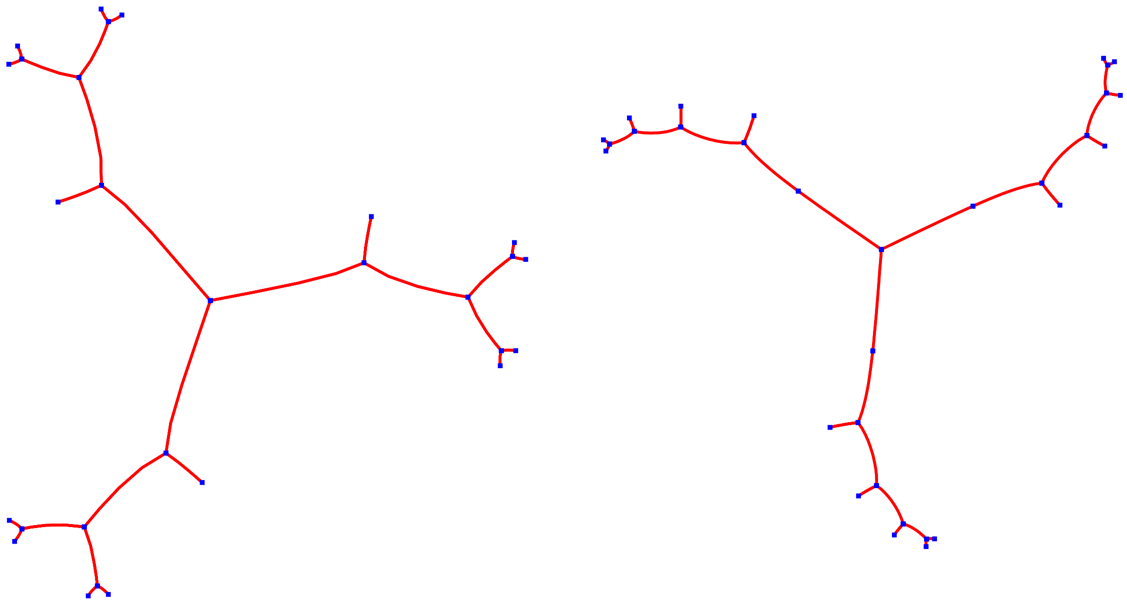
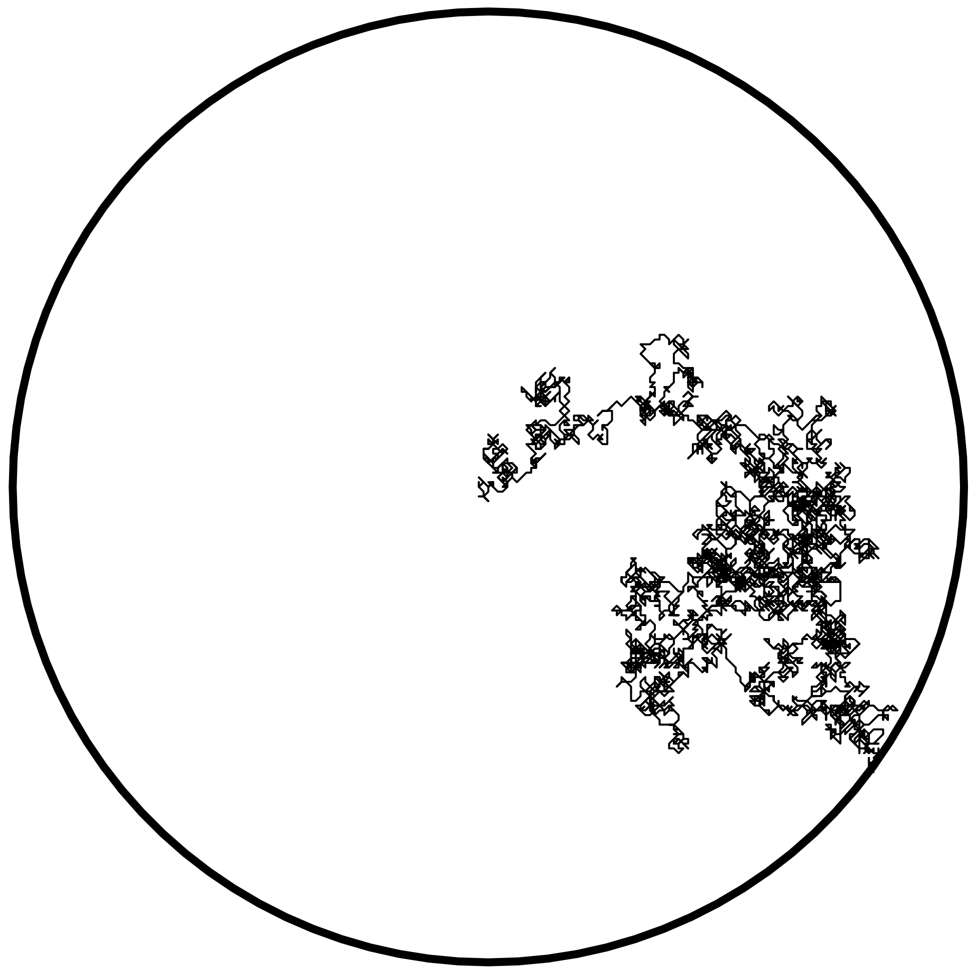
Simons Center for Geometry and Physics December 8, 2015

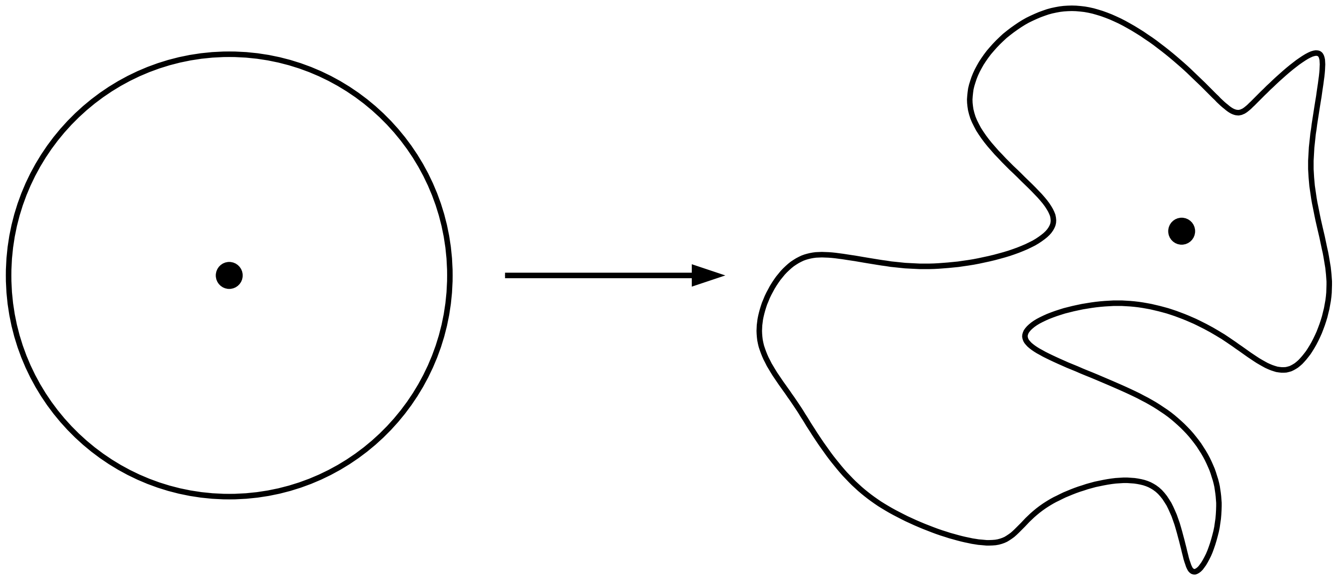


lecture slides available at

`www.math.sunysb.edu/~bishop/lectures`



Harmonic measure = exit distribution of Brownian motion



Harmonic Measure = conformal image of Lebesgue measure.

Depends on base point. Often take 0 or ∞.

By symmetry $\omega_0 = \omega_\infty$ for circle.

If $\omega_0 = \omega_\infty$ must Γ be a circle?

If $\omega_0 = \omega_\infty$ must Γ be a circle? **Yes**.

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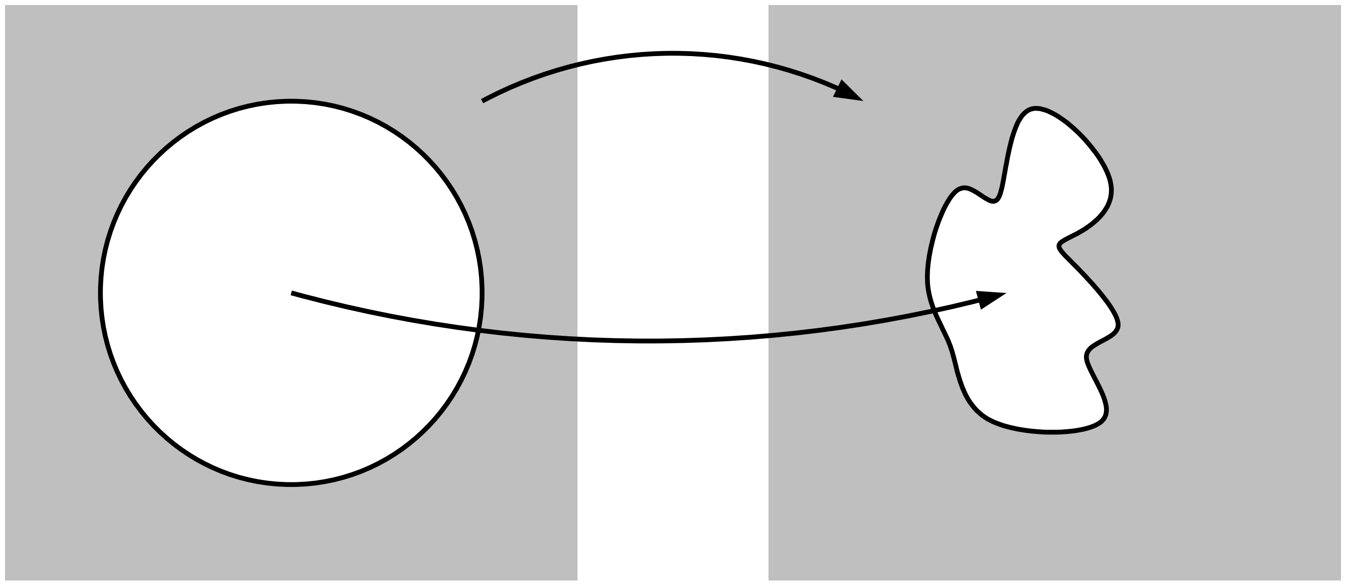

Conformally map inside to inside, outside to outside.

$\omega_0 = \omega_\infty$ means maps agree on boundary.

Get homeomorphism of plane holomorphic off circle.

Is entire by Morera's theorem. Plus 1-1 implies linear.

What happens if only $\omega_0 \sim \omega_\infty$, i.e., for all $E \subset \gamma$:

$$\frac{1}{C} \leq \frac{\omega_o(E)}{\omega_\infty(E)} \leq C$$

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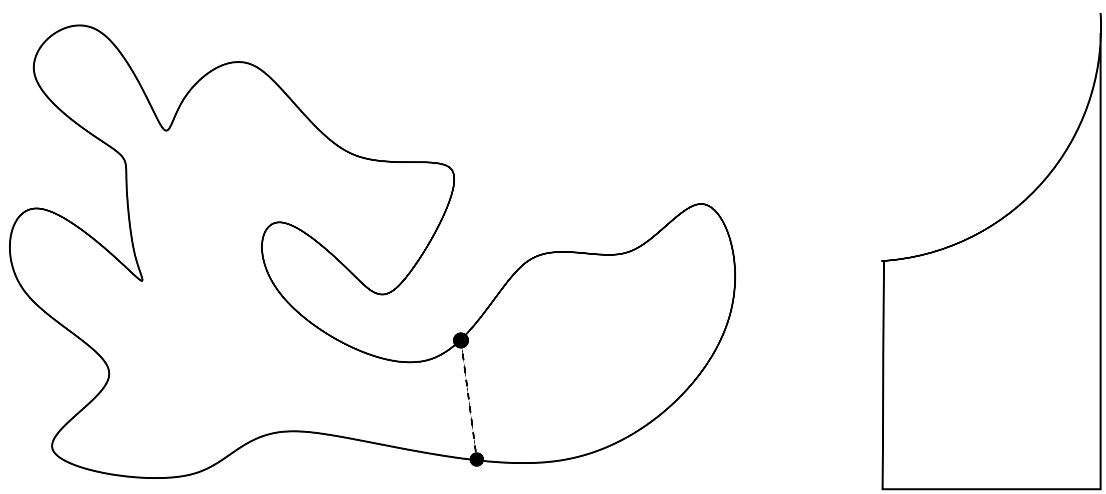
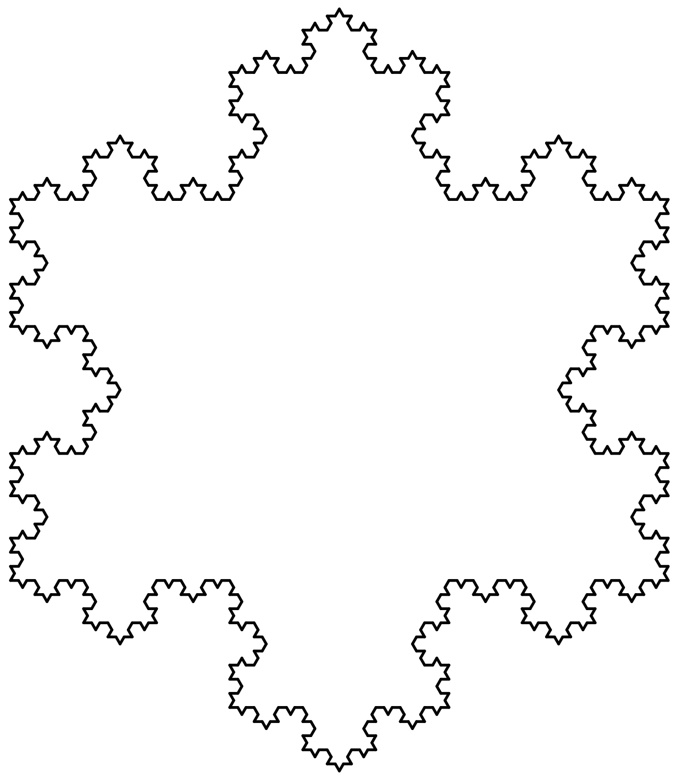
- If $C = 1 + \epsilon$, then γ is rectifiable.
- If $C < \infty$, γ is a quasi-circle.

quasi-circle = quasiconformal image of circle
= smaller arc between $x, y \in \gamma$ has diameter $O(|x - y|)$.



In general, $\omega_0 \perp \omega_\infty$ is possible for quasicircles.



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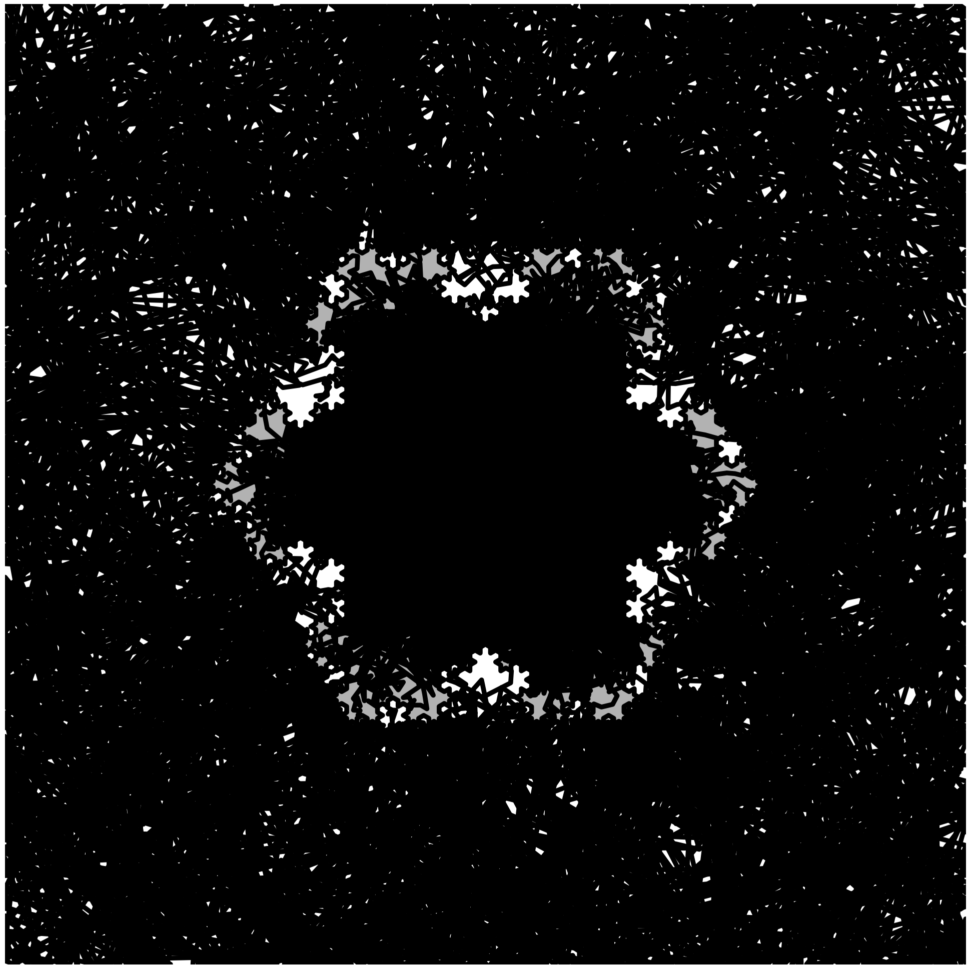
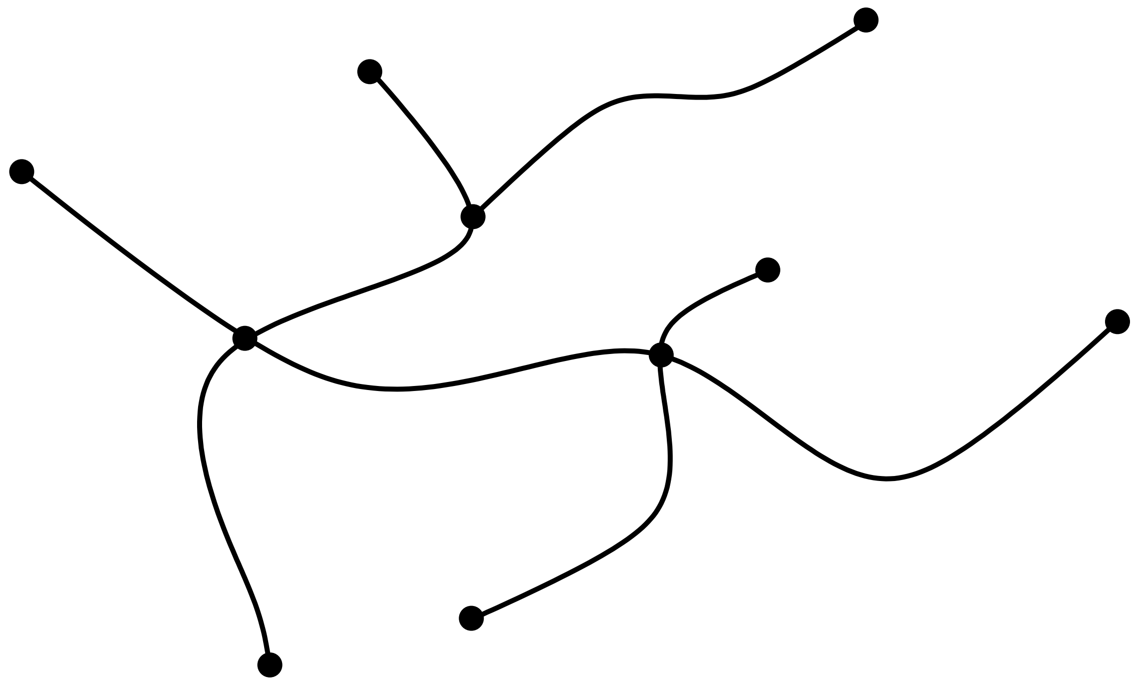
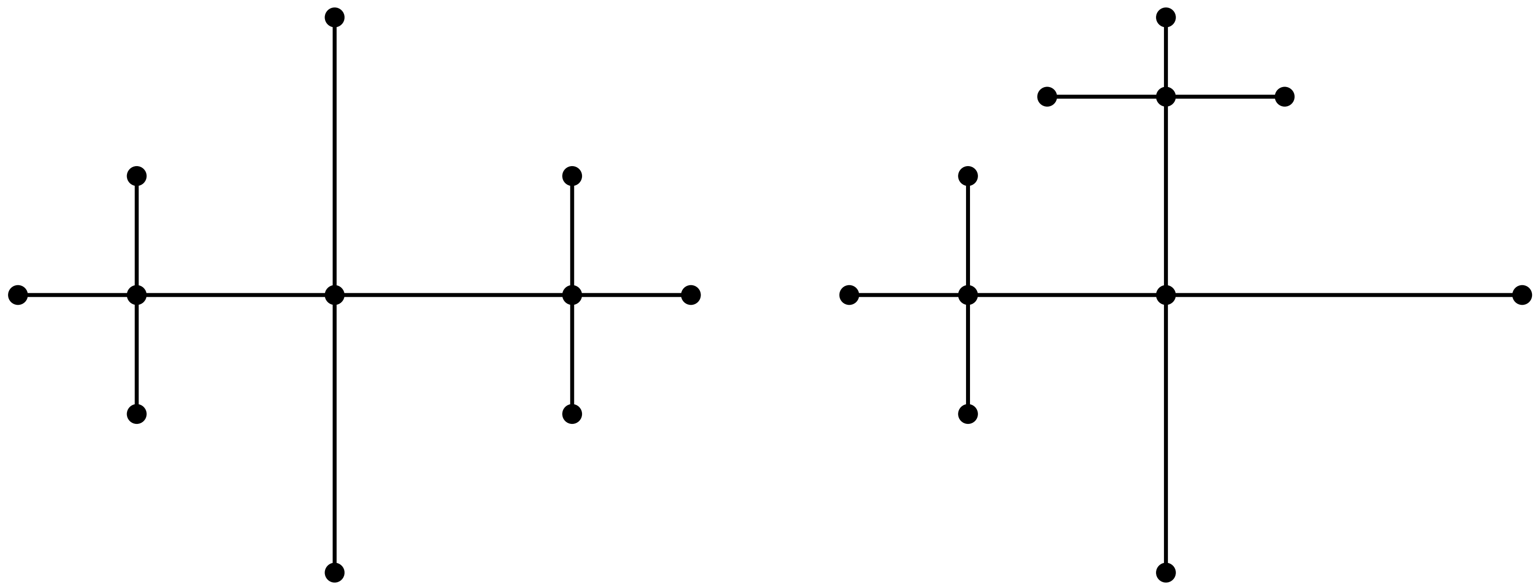


What happens if we replace closed curve by a tree?



Can we make harmonic measure ω_∞ same on "both sides"?



Same tree, different planar trees

A planar tree is **conformally balanced** if

- every edge has equal harmonic measure from ∞
- edge subsets have same measure from both sides

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A line segment is an example.

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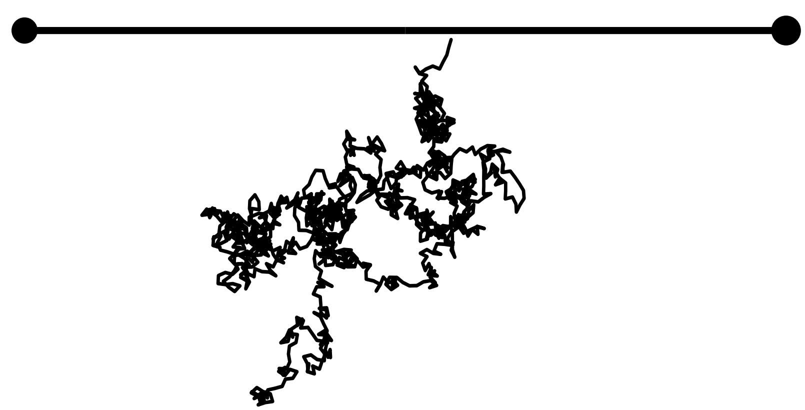
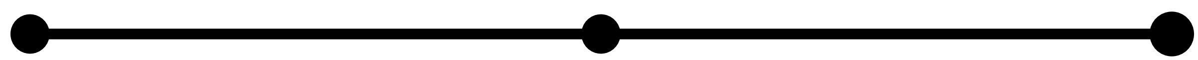
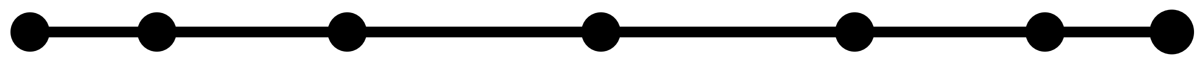
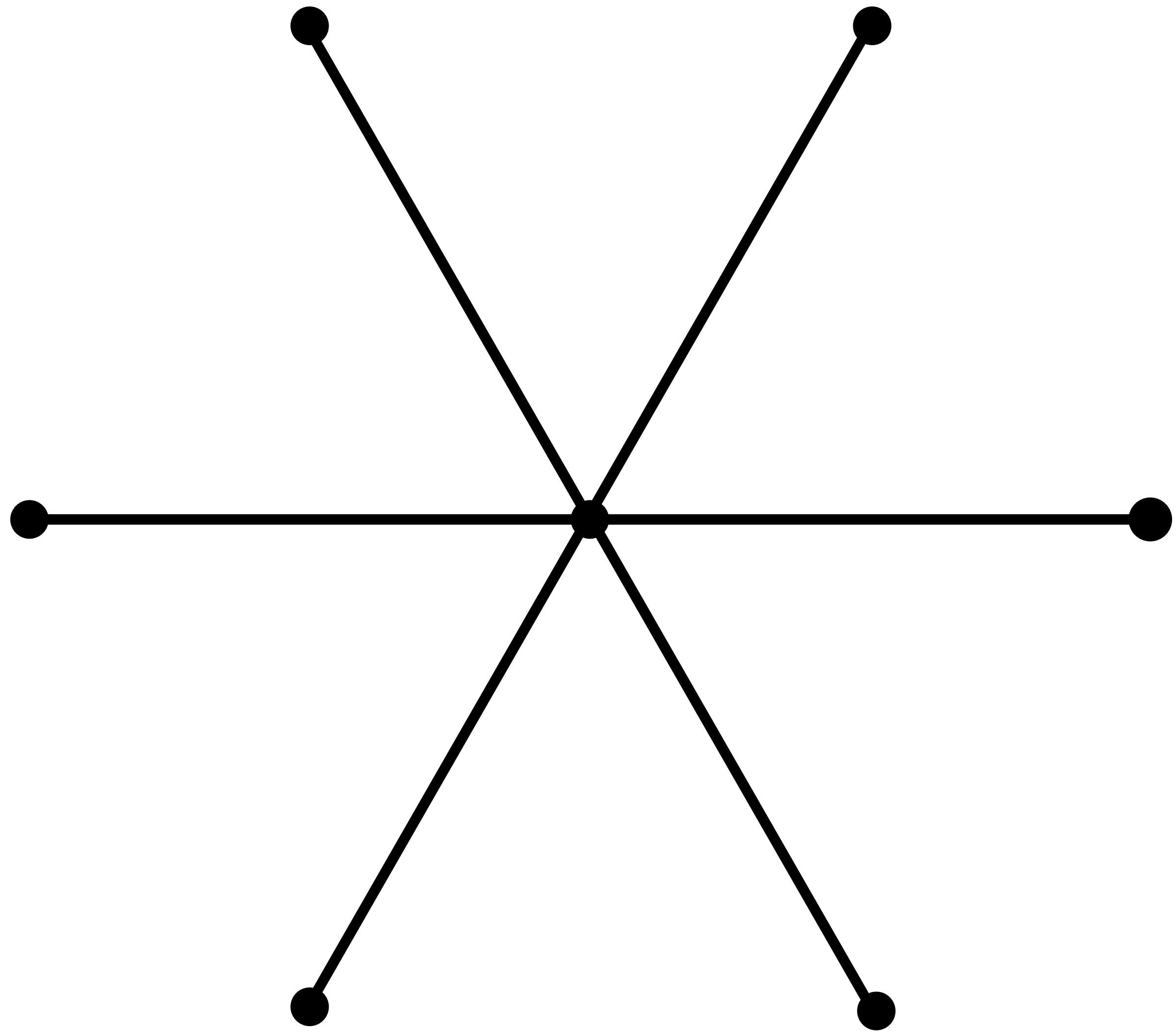
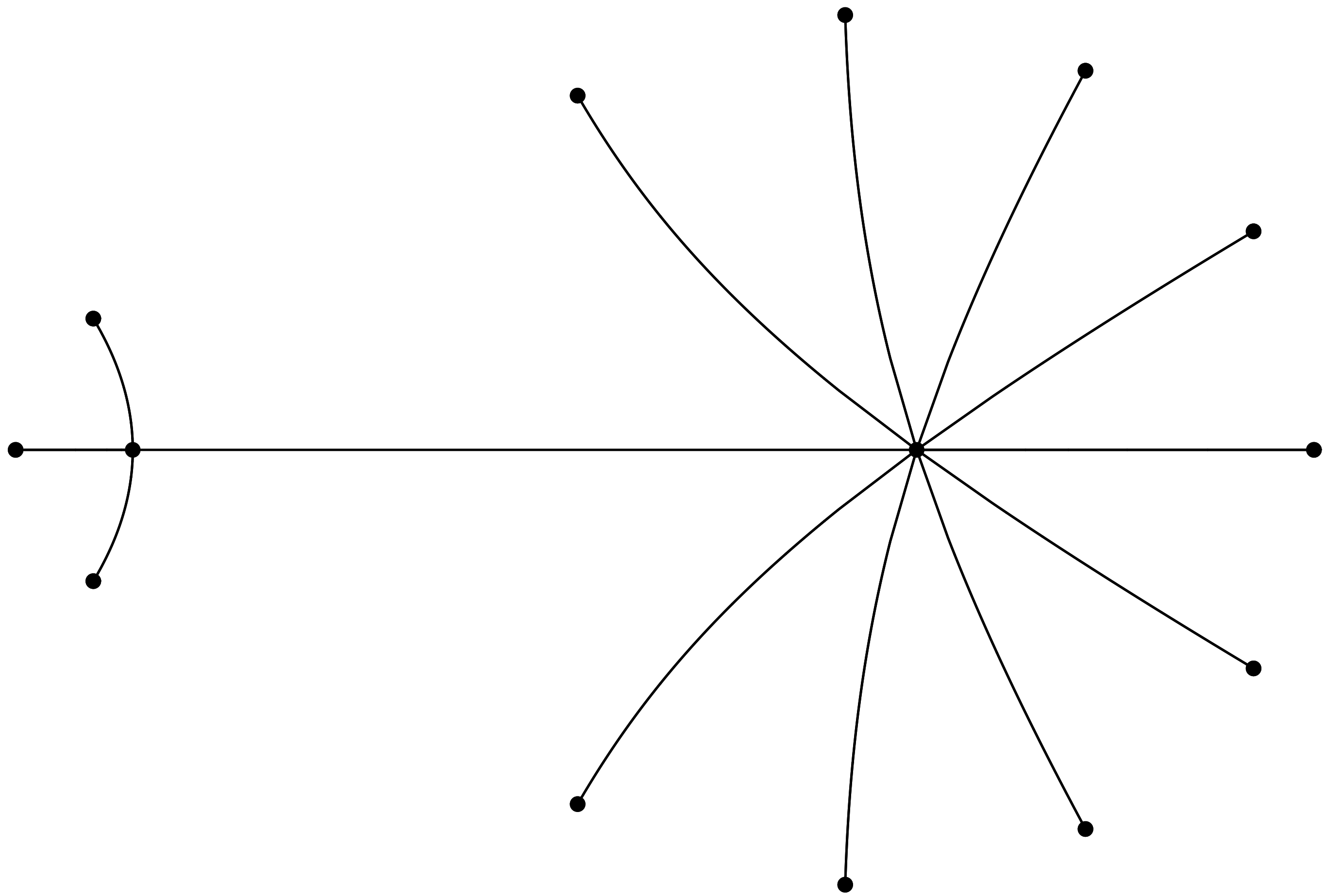
A line segment is an example. Are there others?

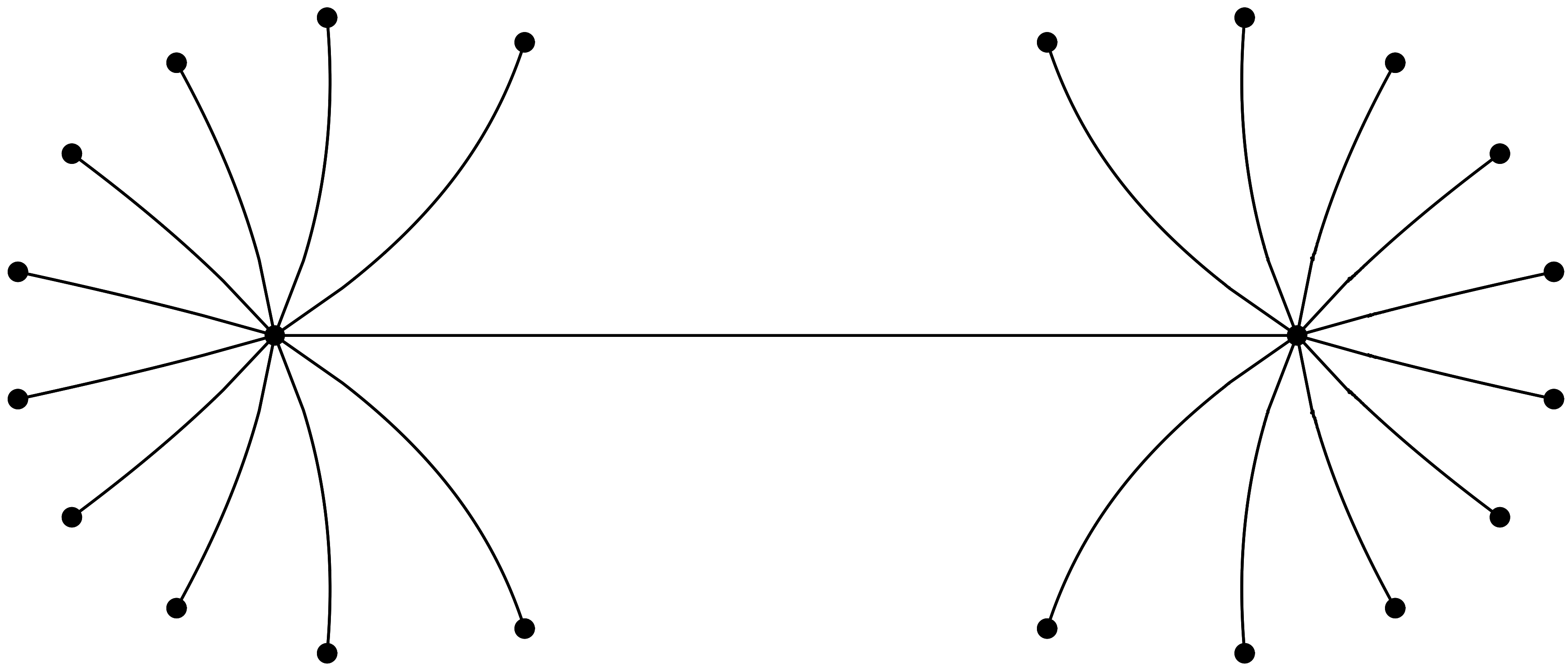












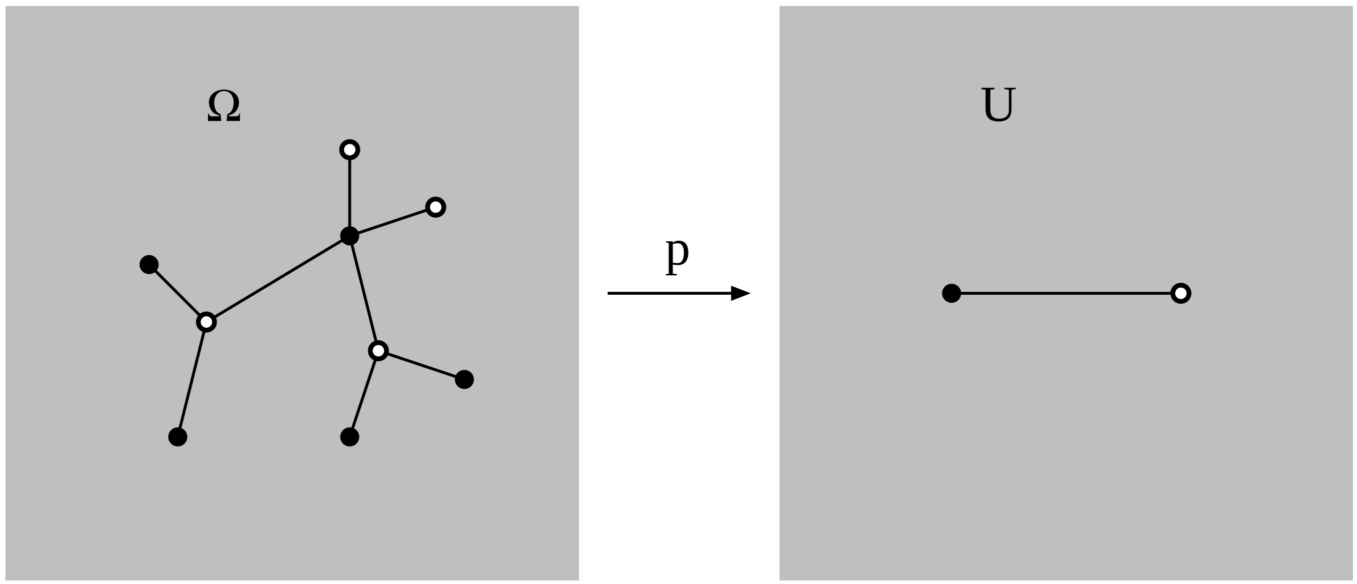
$p =$ polynomial

$\mathrm{CV}(p) = \{p(z) : p'(z) = 0\} =$ critical values

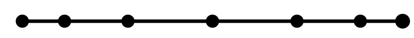
If $\mathrm{CV}(p) = \pm 1$, p is called **generalized Chebyshev** or **Shabat**.

Balanced trees $\leftrightarrow$ Shabat polynomials

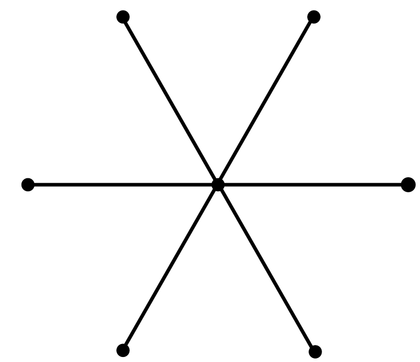
Fact: T is balanced iff $T = p^{-1}([-1,1])$, $p =$ Shabat.



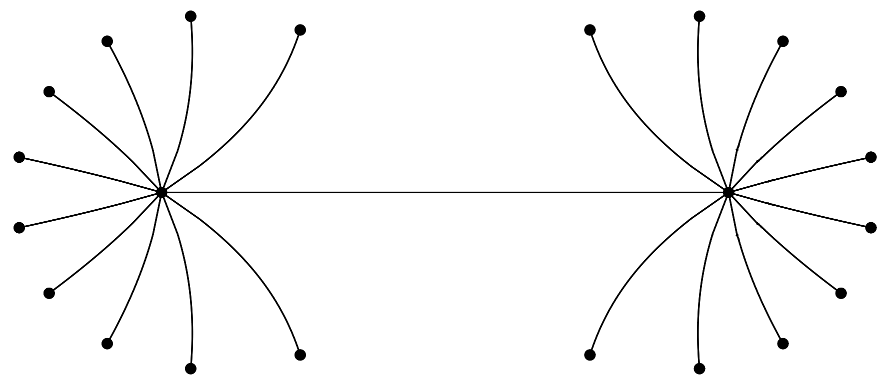
$\Omega = \mathbb{C} \setminus T$ $\qquad$ $U = \mathbb{C} \setminus [-1, 1]$



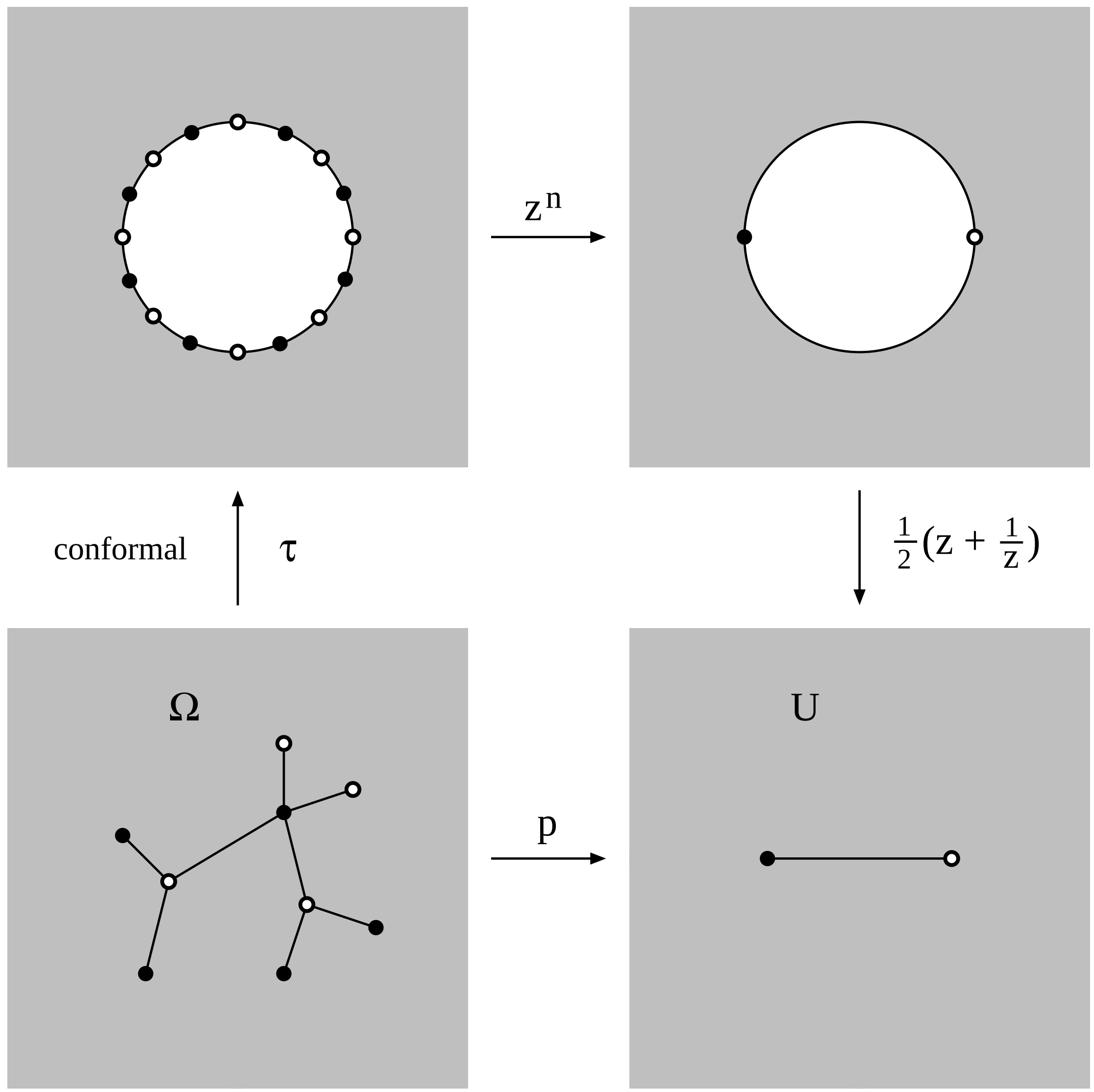
p(z) = 1st type Chebyshev



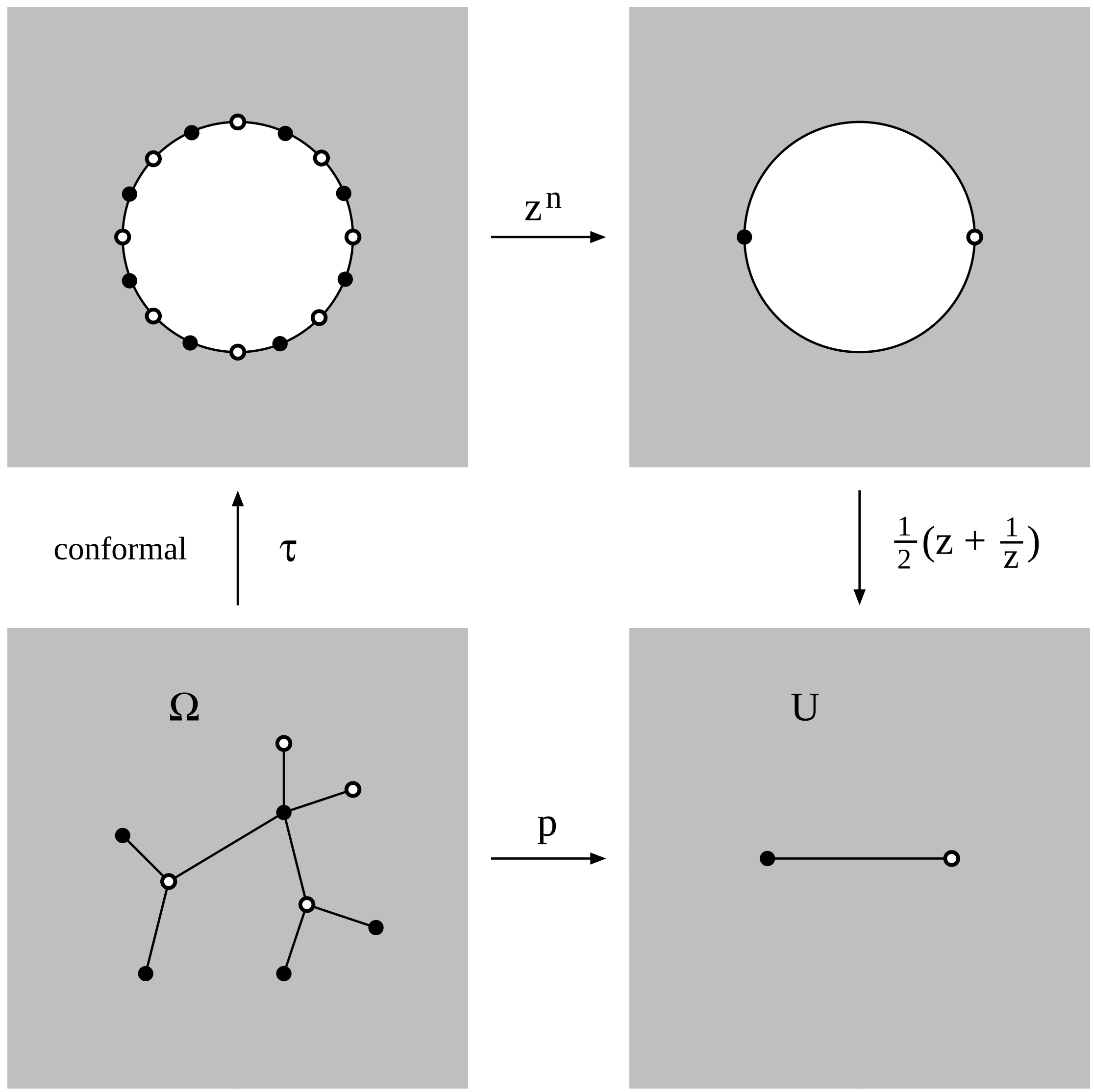
$p(z) = 2z^n - 1$



$p'(z) = c(z+1)^a(z-1)^b$



$$\tau = \text{conformal } \Omega \to \mathbb{D}^* = \{|z| > 1\}.$$



p is entire and n-to-1. $\Rightarrow p =$ polynomial

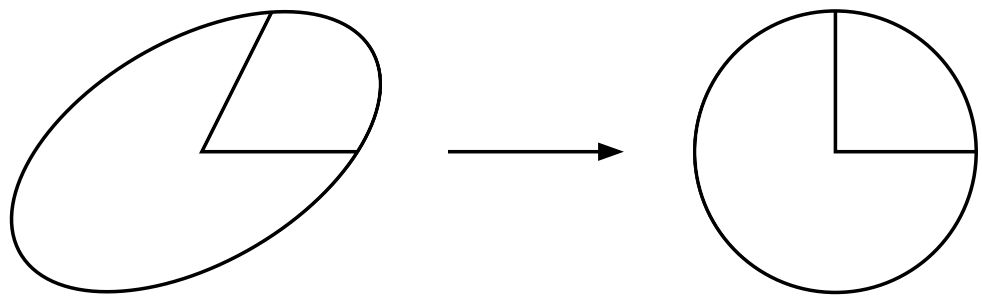
What if the tree is not balanced?

What if the tree is not balanced?

Replace conformal map by quasiconformal map.

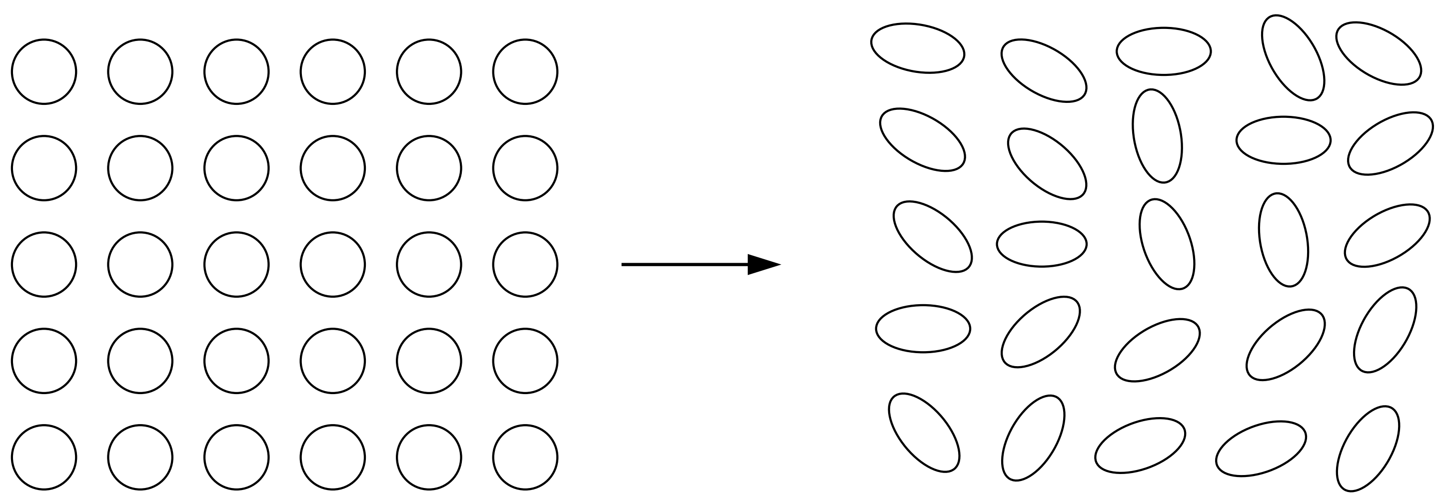
dilatation $= \mu_f = f_{\overline{z}}/f_z =$ measure of non-conformality

$K = (\mu + 1)/(\mu - 1) =$ ratio of singular values of tangent map



quasiconformal = homeomorphism with bounded angle distortion.

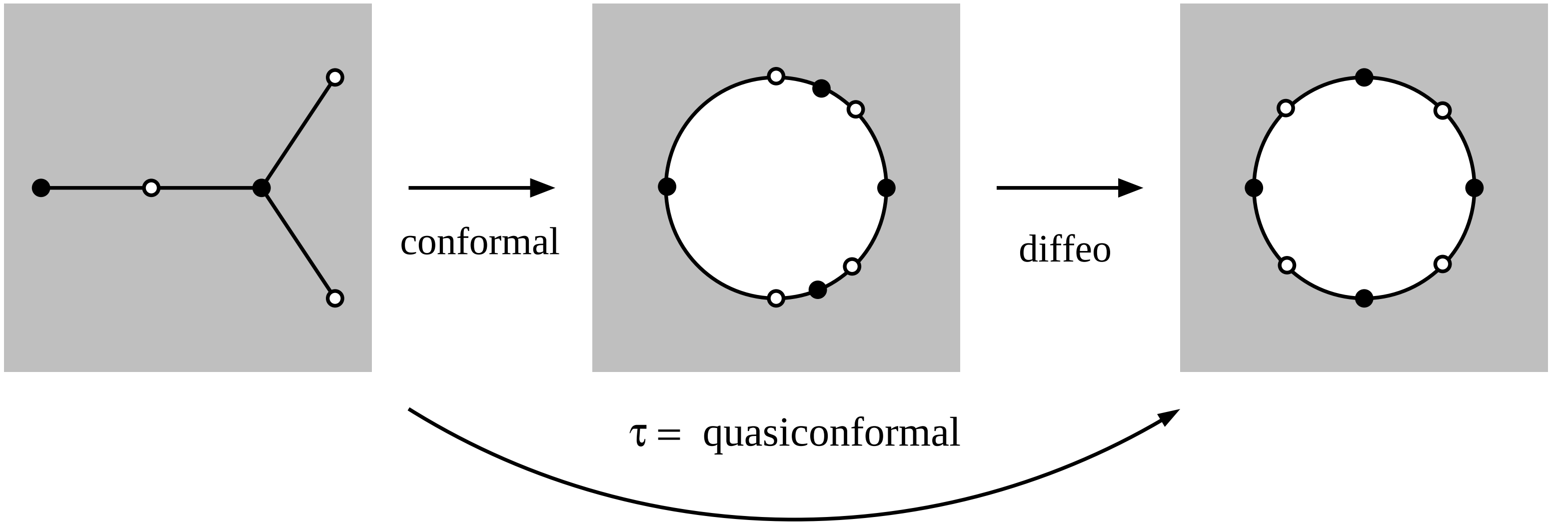
= homeomorphism with $\sup|\mu_f| \leq k < 1$.



quasiregular = function with $\sup |\mu_f| \leq k < 1$.

Theorem: If f is quasi-regular and finite-to-one then $f = g \circ \phi$ where g is polynomial and ϕ is a QC homeomorphism.

This is corollary of **Measurable Riemann Mapping Theorem**.

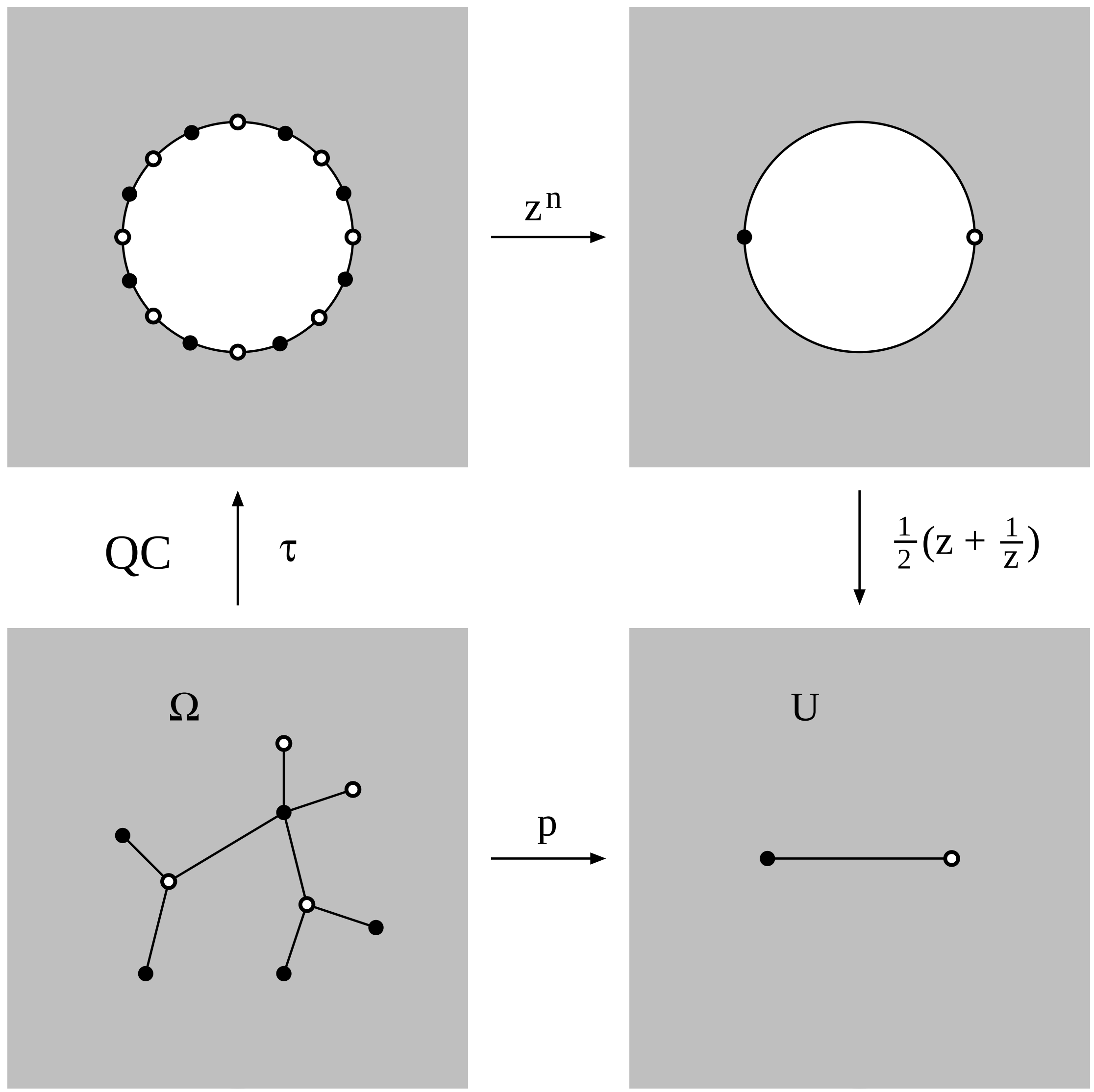


Map $\Omega \to \{|z| > 1\}$ conformally, then "even out" by diffeomorphism.

If tree is "nice" (smooth edges, equal angles at each vertex), then

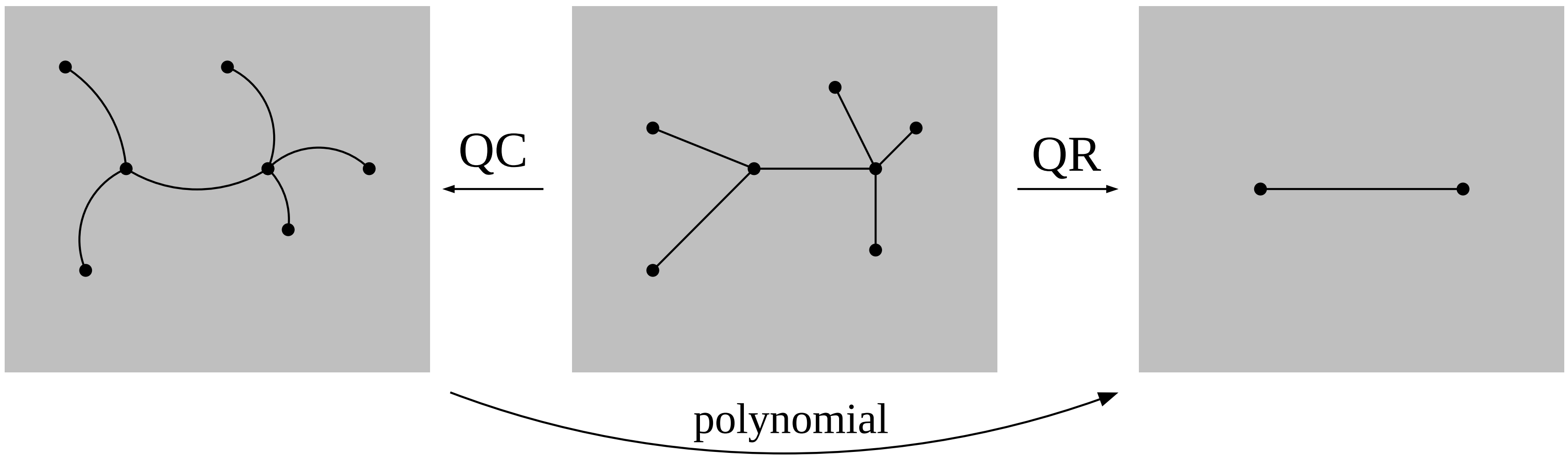
- composition is QC
- any subset of an edge has two images of equal length

QC constant depends on tree. Is finite since tree is finite.



τ is quasiconformal on Ω. p is quasi-regular on plane.

Factor the QR map into a QC map and polynomial:



Cor: Every planar tree has a true form.

In Grothendieck's theory of *dessins d'enfants*, a finite graph of a topological surface determines a conformal structure and a Belyi function (a meromorphic function to sphere branched over 3 values).

Shabat polynomial is special case of Belyi function.
(branch points $= -1, 1, \infty$).

Polynomial has coefficients in a number field (finite extension of $\mathbb{Q}$). Universal Galois group acts on these polynomials, hence has an action on finite trees. Determining orbits is major open problem.

"conformally balanced tree" = "true form of a tree".

Is the polynomial computable from the tree?

Kochetkov, *Planar trees with nine edges: a catalogue*, 2007. "The complete study of trees with 10 edges is a difficult work, and probably no one will do it in the foreseeable future".

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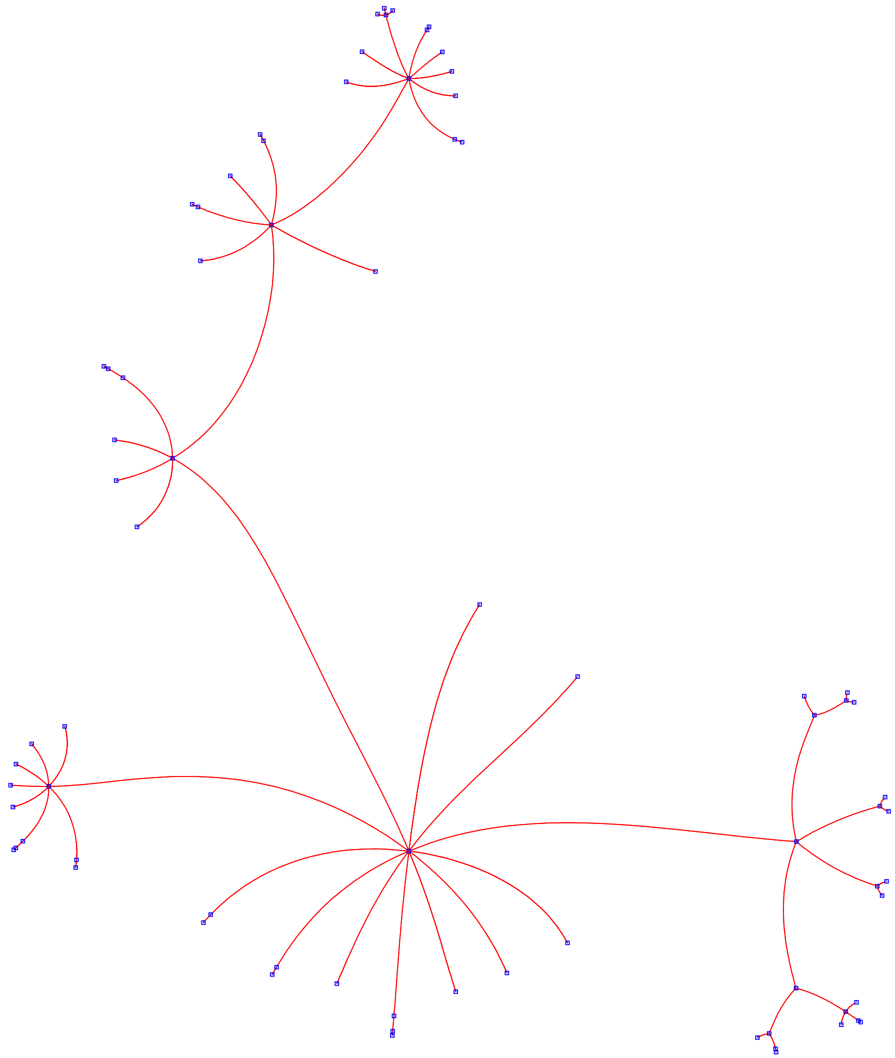
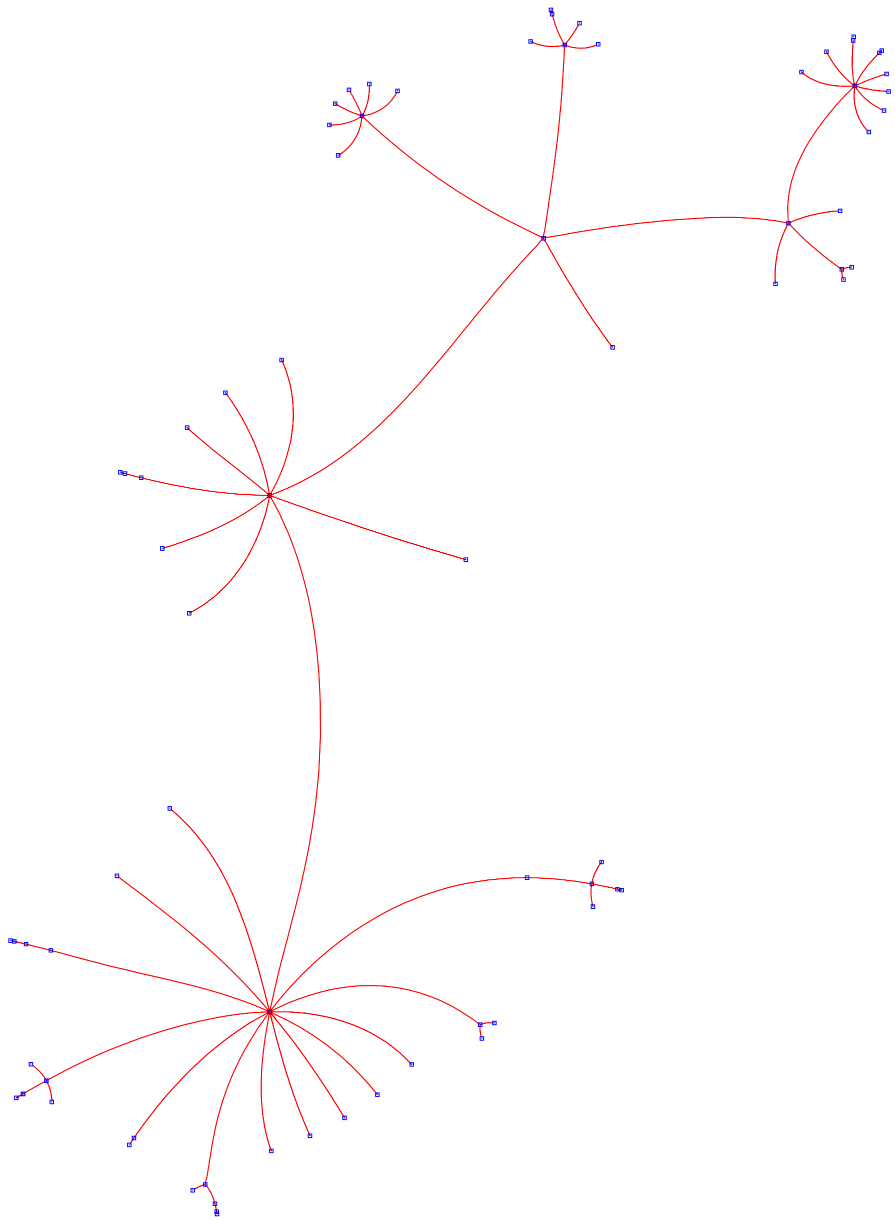
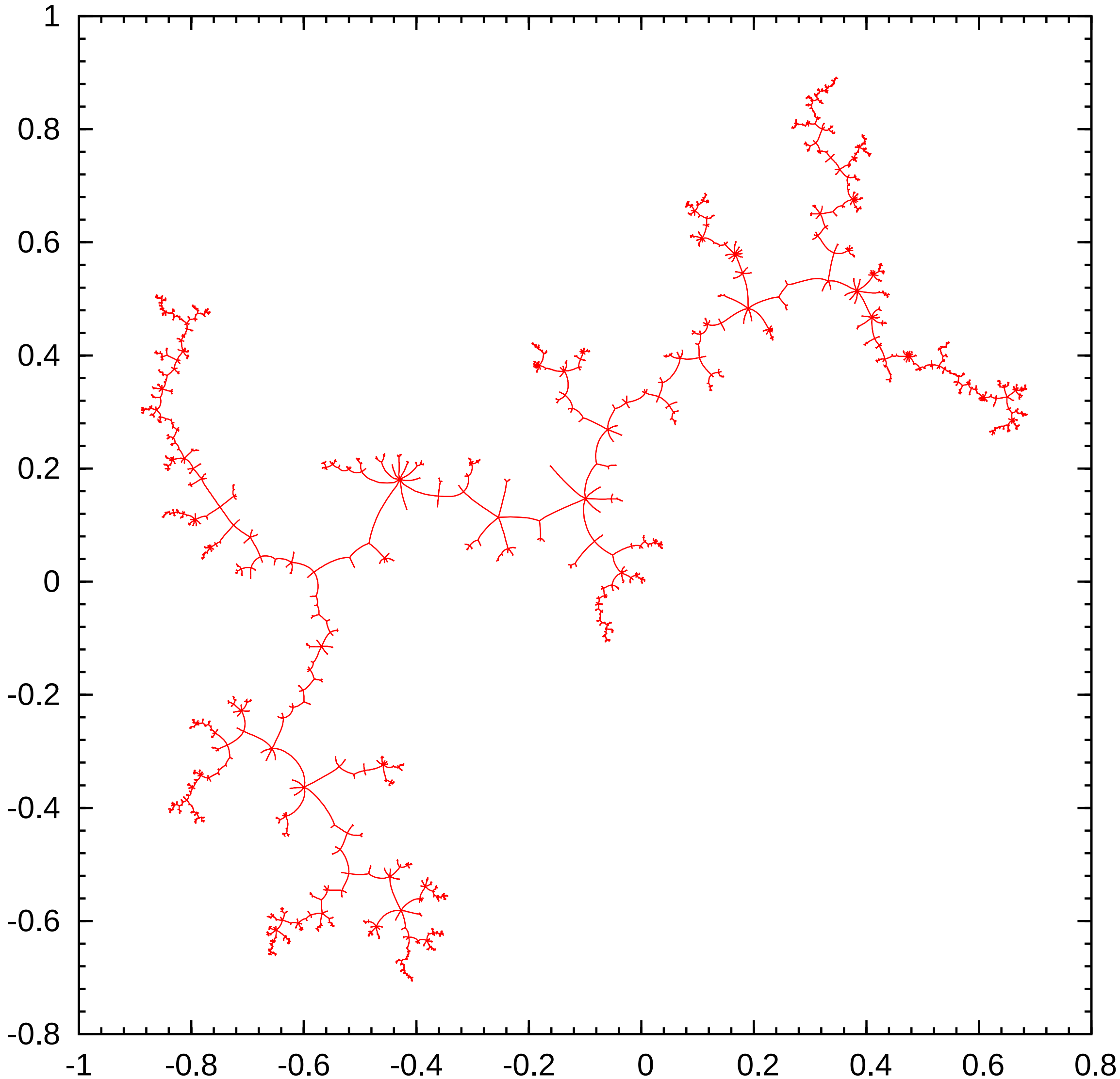
Marshall and Rohde have approximated true trees with thousands of edges. They have catalogued all 95,640 true trees with 14 edges to 25 digits of accuracy. Can obtain 1000's of digits.

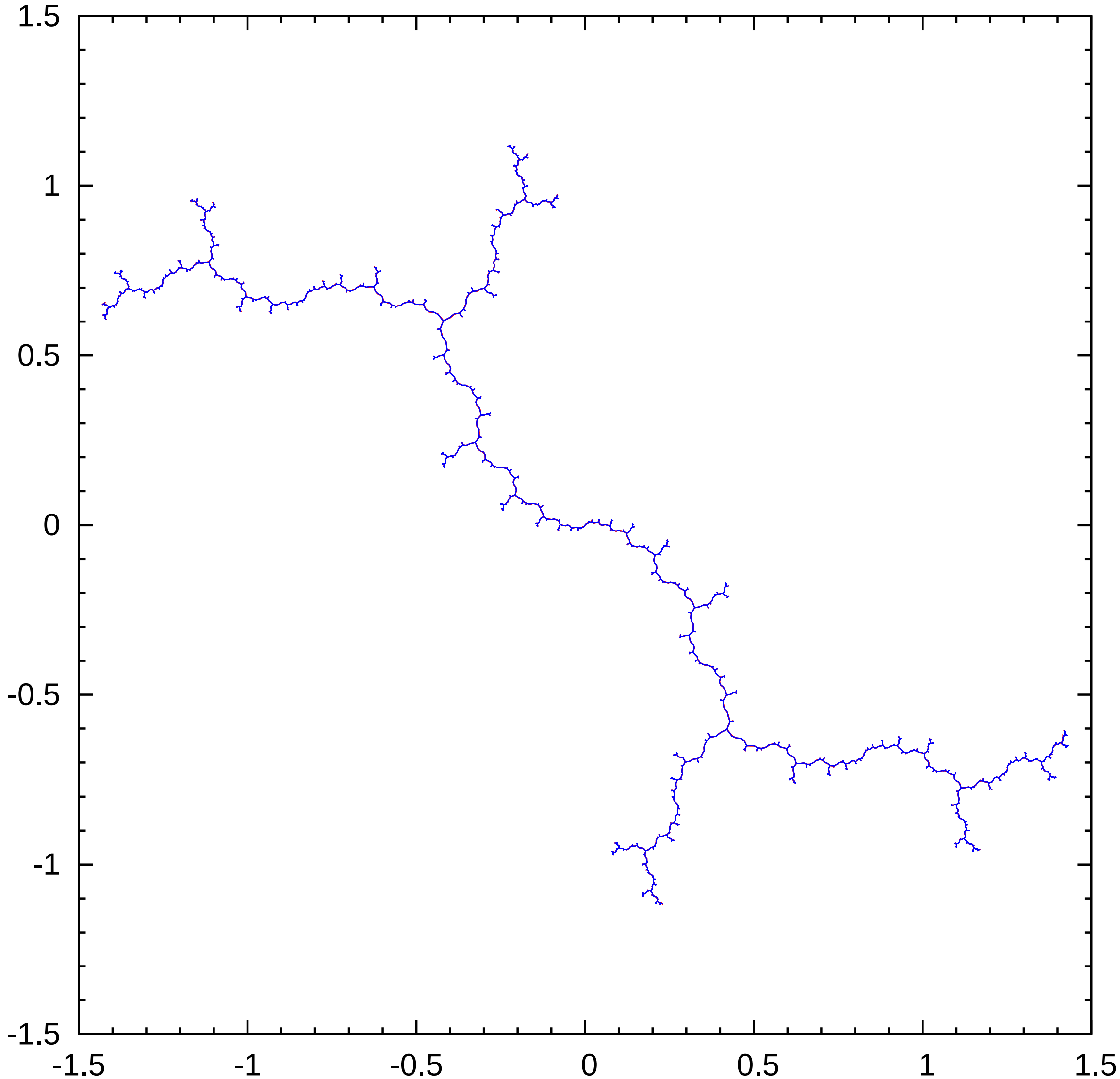
Can we use high precision approximations and lattice reduction (e.g., PSLQ) to find exact algebraic coefficients?







Random true tree with 10,000 edges



Tree with dynamical combinatorics

Every planar tree has a true form.

In other words, all possible **combinatorics** occur.

What about all possible **shapes**?

Every planar tree has a true form.

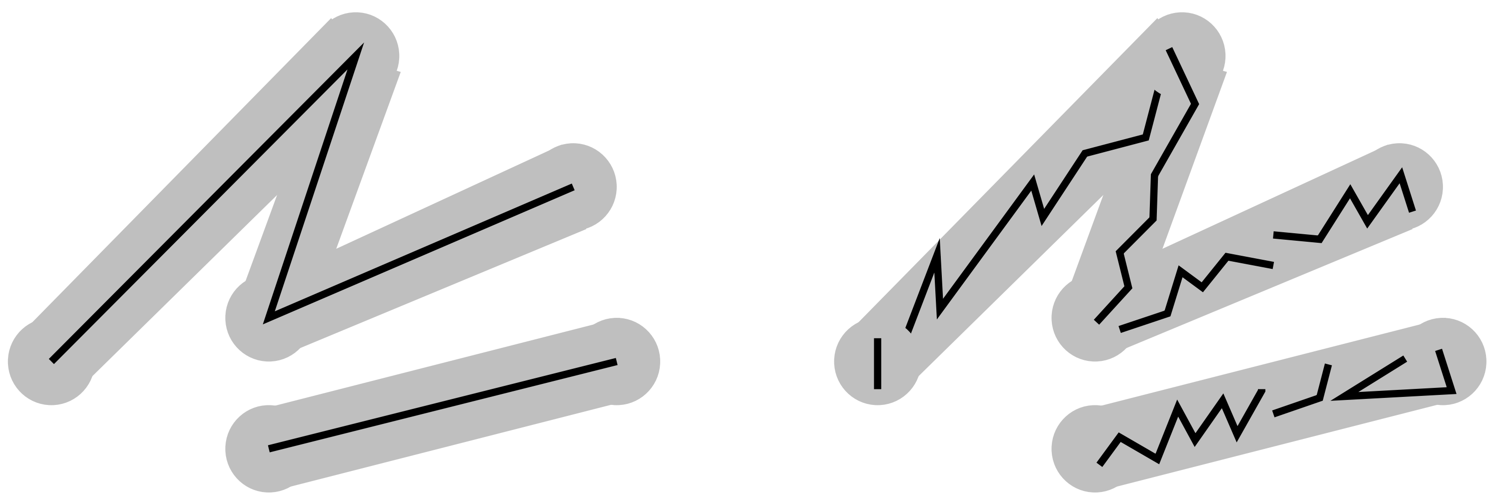
In other words, all possible **combinatorics** occur.

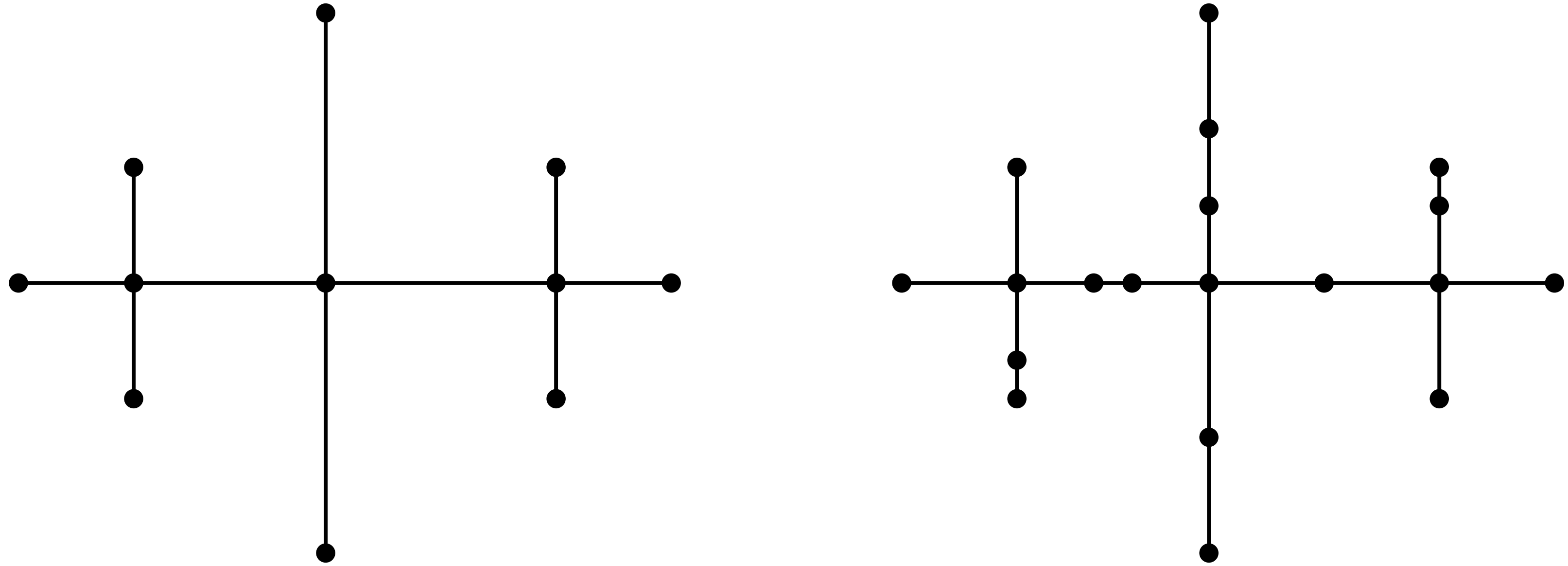
What about all possible **shapes**?

Hausdorff metric: if E is compact,

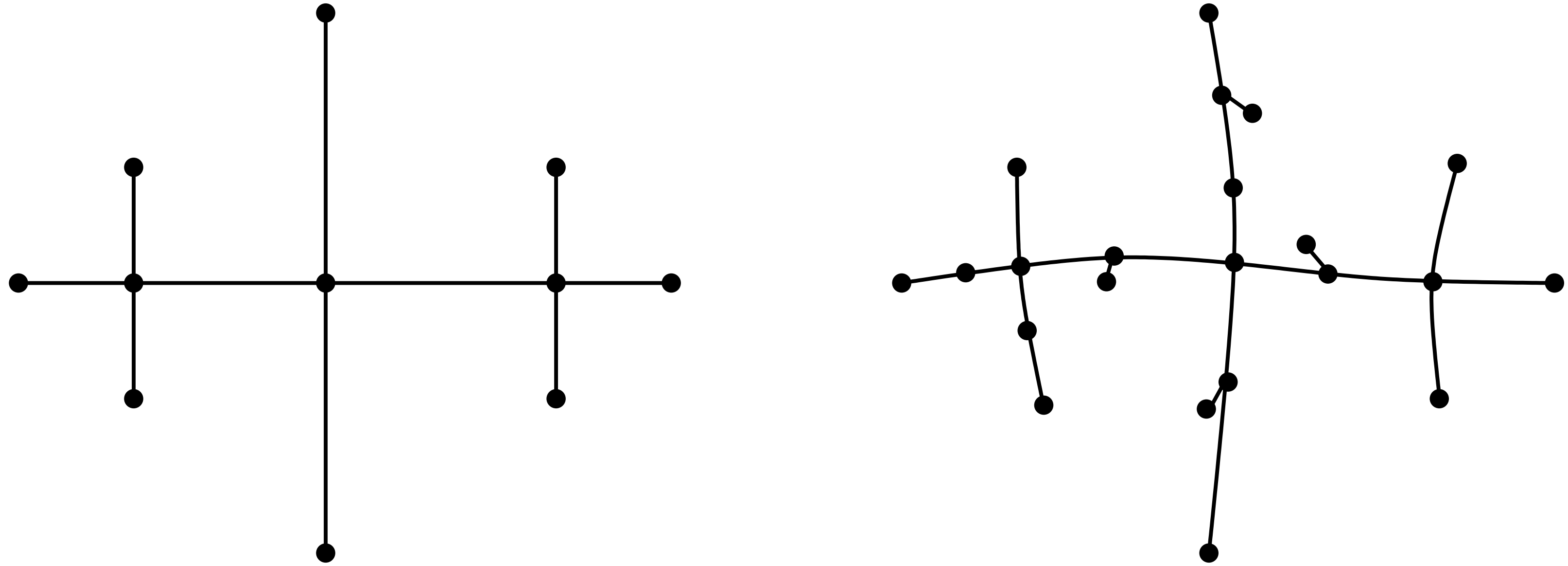
$$E_\epsilon = \{z : \mathrm{dist}(z, E) < \epsilon\}.$$

$$\mathrm{dist}(E, F) = \inf\{\epsilon : E \subset F_\epsilon, F \subset E_\epsilon\}.$$





Different combinatorics, same shape



Different trees, similar shapes

Close in Hausdorff metric

Theorem: Every continuum is a limit of true trees.

True trees are dense, Inventiones Mat., 197(2014), 433-452.

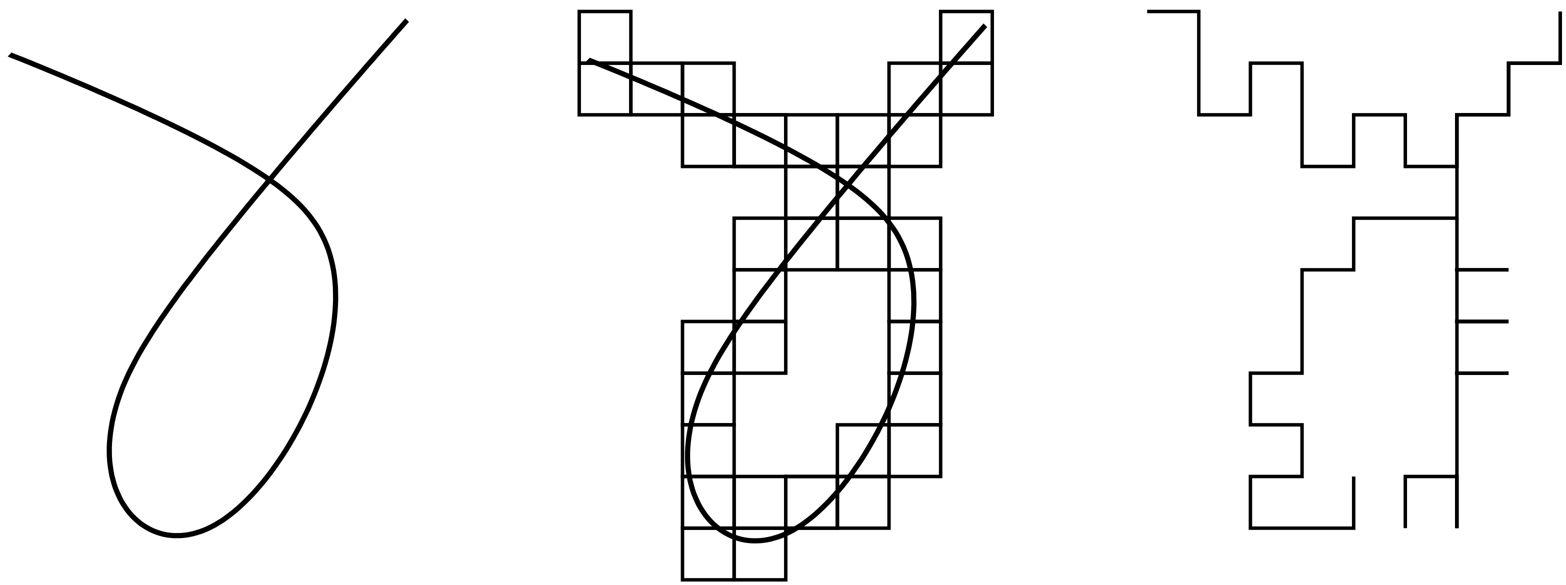
Limit in Hausdorff metric.

continuum = compact, connected set

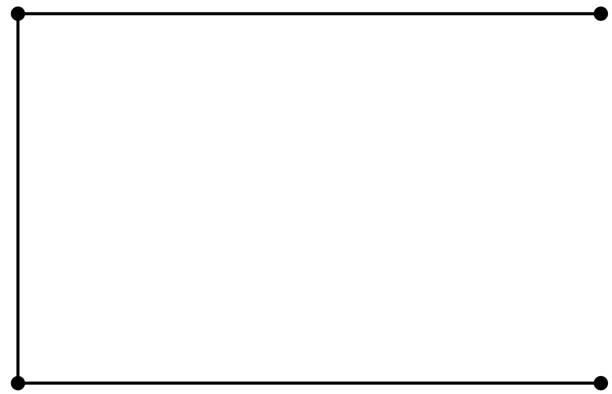
Theorem: Every continuum is a limit of true trees.

Answers question of Alex Eremenko.

Enough to approximate certain finite trees by true trees.

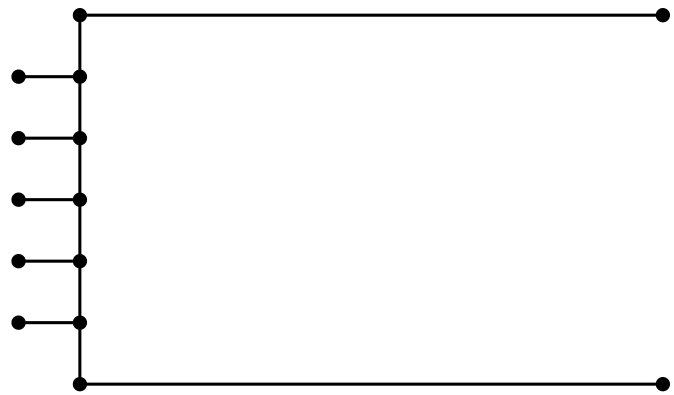


Idea: reduce harmonic measure ratio by adding edges.



Vertical side has larger harmonic measure from left.

Idea: reduce harmonic measure ratio by adding edges.

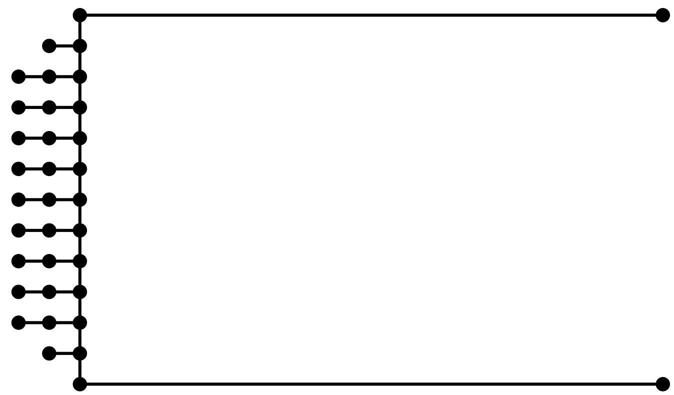


“Left” harmonic measure is reduced.

Roughly 3-to-1 reduction.

New edges are uniformly close to balanced.

Idea: reduce harmonic measure ratio by adding edges.

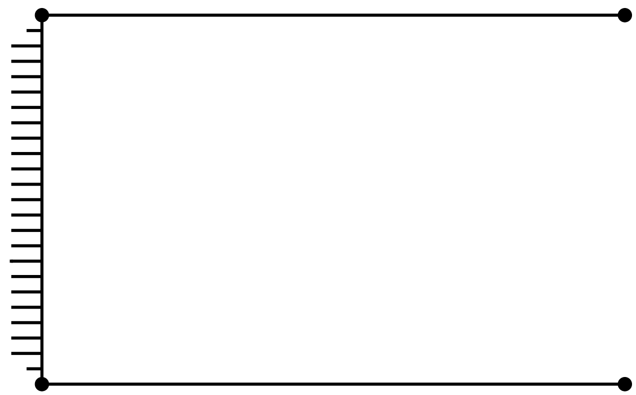


Longer spikes mean more reduction.

Need extra vertices to make edges have equal measure.

Can achieve balance within factor of M (universal).

Idea: reduce harmonic measure ratio by adding edges.



Longer spikes mean more reduction.

Need extra vertices to make edges have equal measure.

Can achieve balance within factor of M (universal).

New edges can be taken in small neighborhood of original tree.

Steps of Proof:

- Given tree T, build M-balanced T' by adding branches.
- Can choose T' in small neighborhood of T.
- T' has QC image $T'' = \phi(T')$ that is a true tree (MRMT).
- We can make ϕ conformal except on area $\to 0$.
- $\sup |\mu_\phi| \leq k(M) < 1$ (independent of T; this is the hard step).
- Hence ϕ is near identity ($\Rightarrow T''$ close to T').
- Hence T'' is close to T.

Theorem: True trees are dense.

Corollary of density of true trees: Every continuum is a limit of Julia sets of polynomials with two post-critical points.

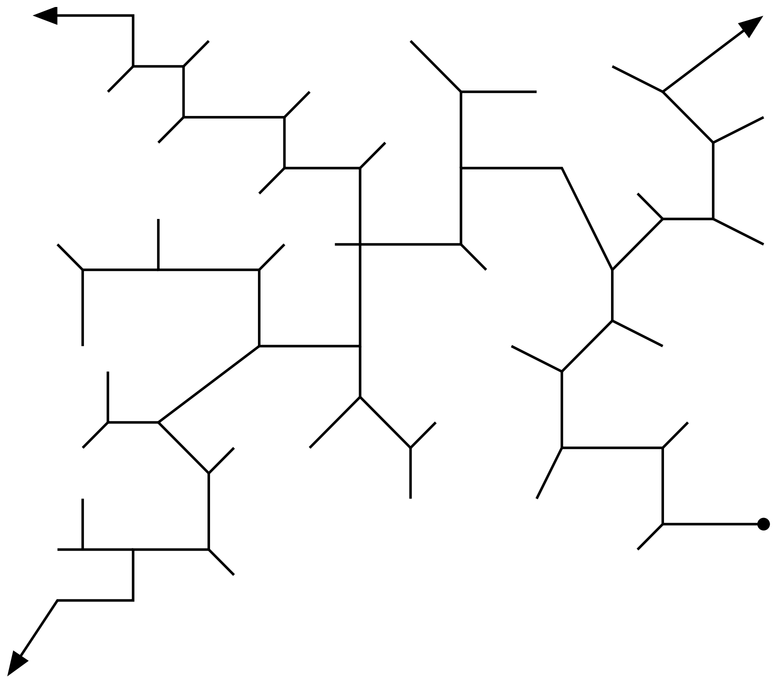
Dynamical dessins are dense, with Kevin Pilgrim, to appear in Revista Mat. Iberoamericana.

PCF Julia sets are dendrites = connected, don't separate plane

PFC = post-critically finite

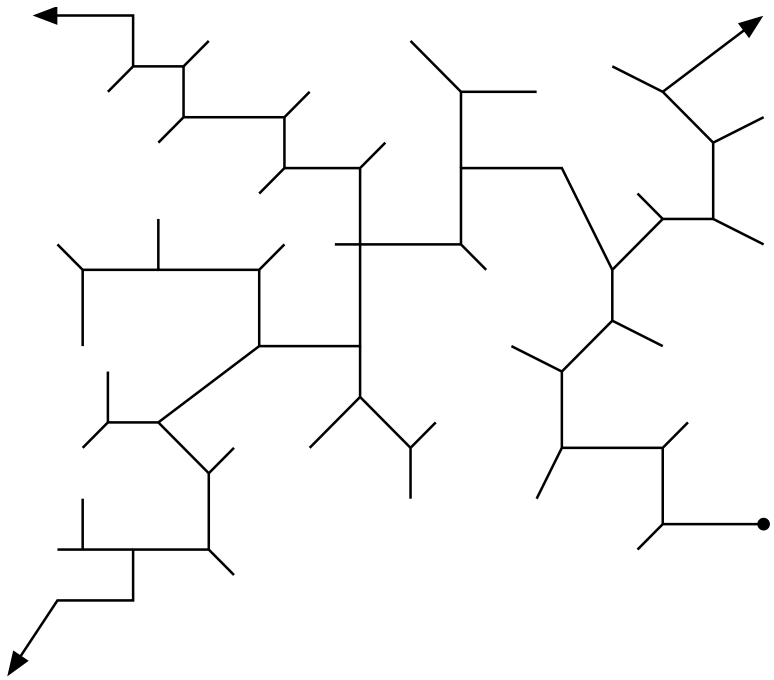
Easy exercise: Any compact set can be approximated by polynomial Julia sets.

What about infinite trees?



Is there a theory of *dessins d'adolescents* that relates infinite trees to entire functions?

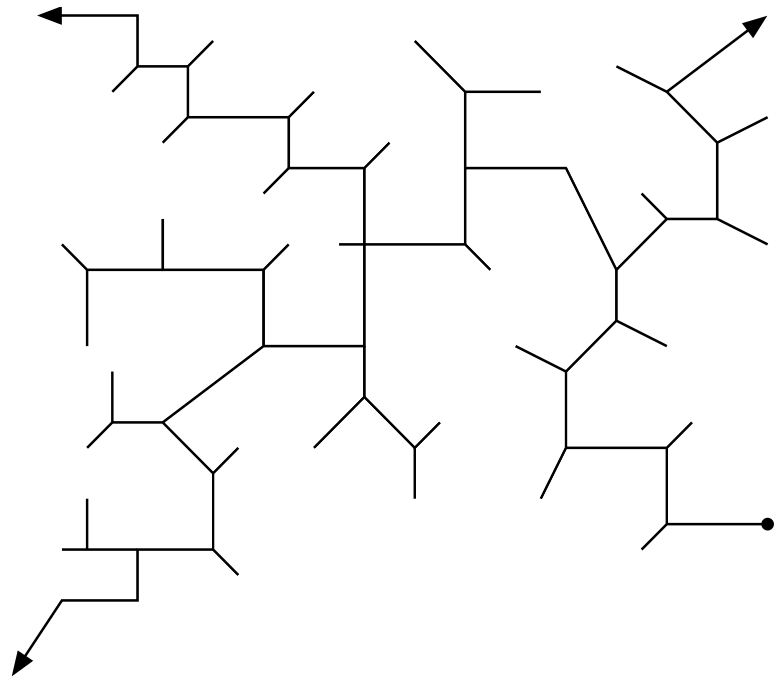
What about infinite trees?



What does "balanced" mean now?

Harmonic measure from ∞ doesn't make sense.

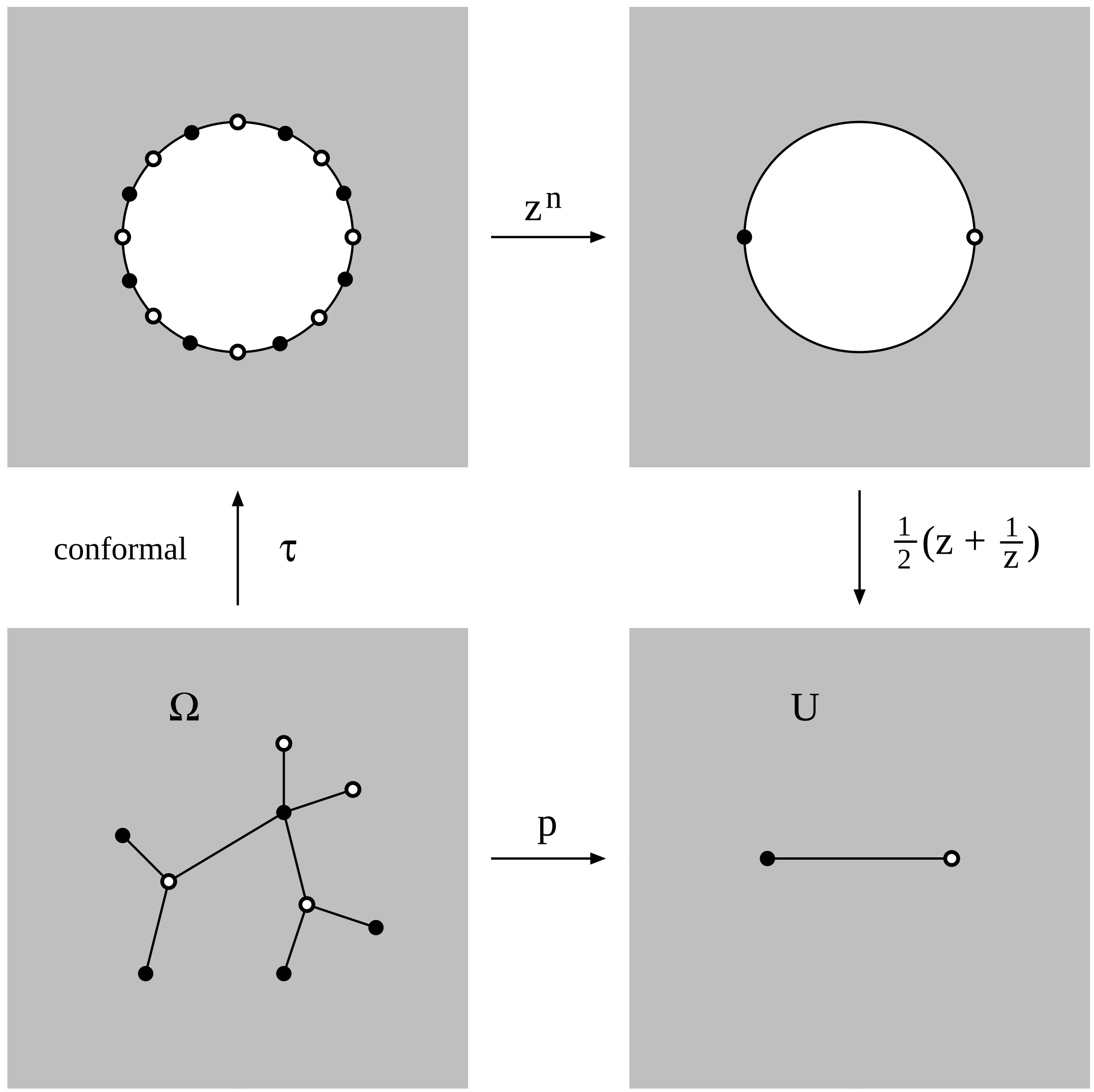
What about infinite trees?



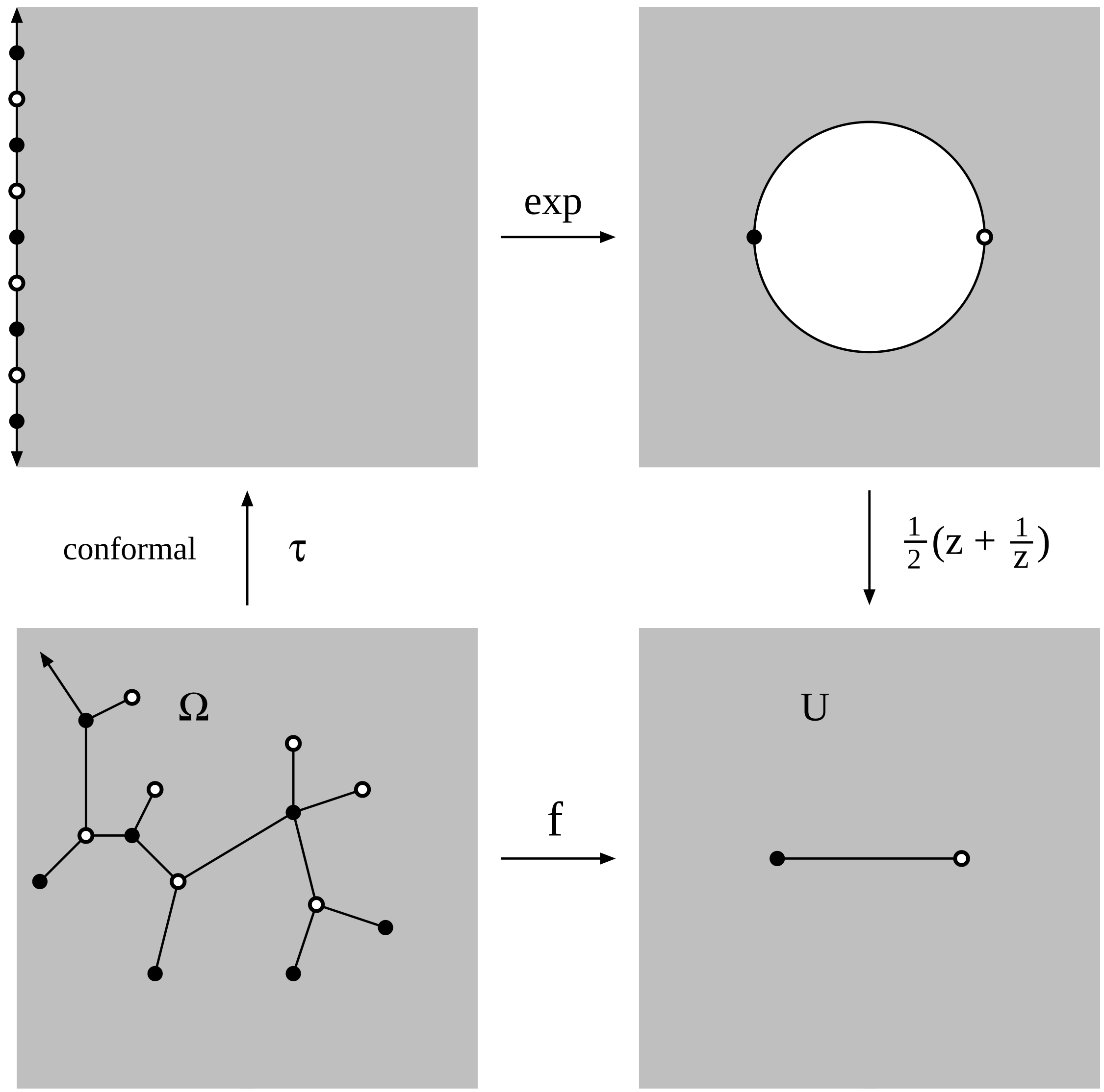
Main difference:

$\mathbb{C}\setminus$ finite tree = one annulus

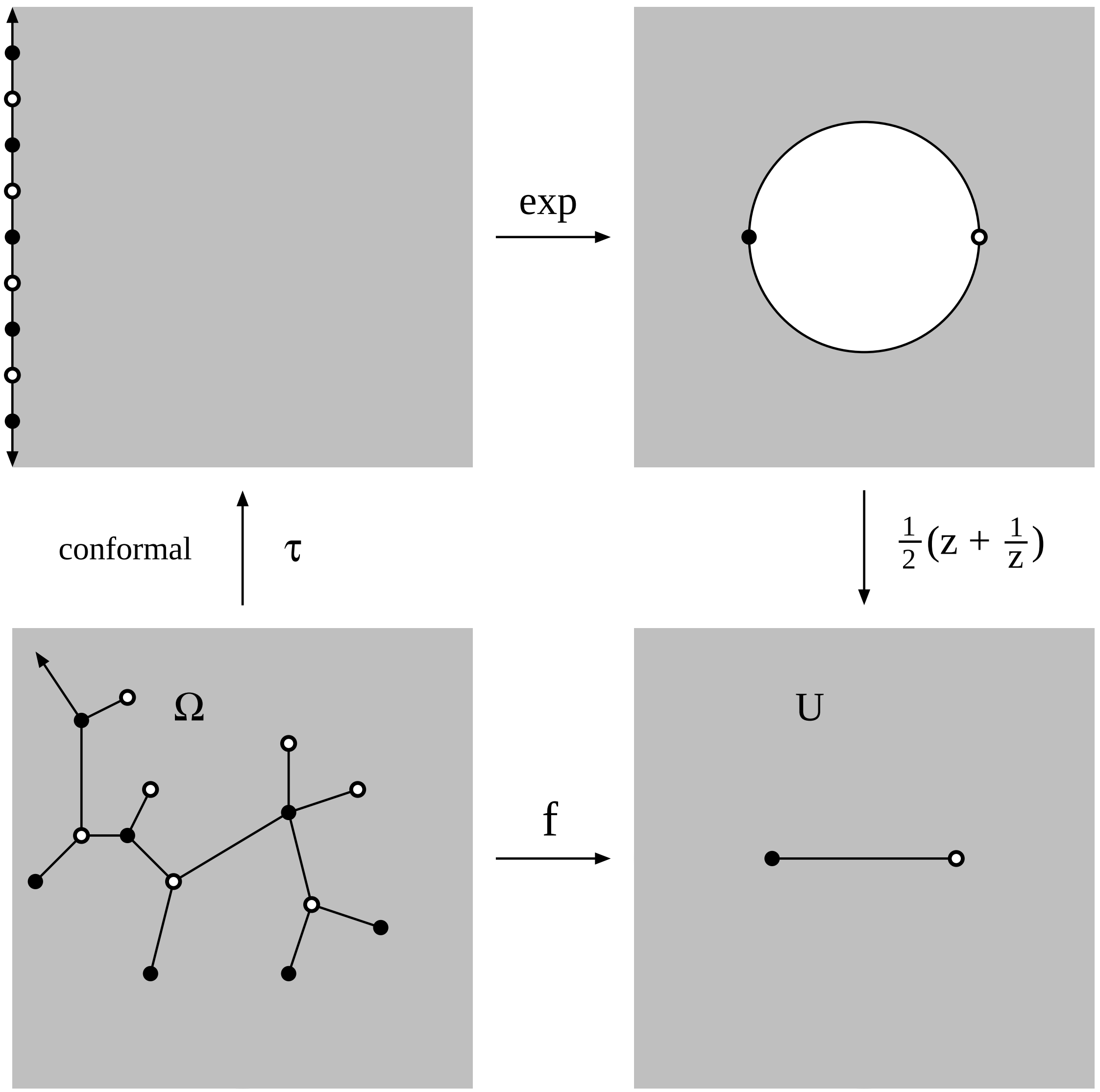
$\mathbb{C}\setminus$ infinite tree = many simply connected components



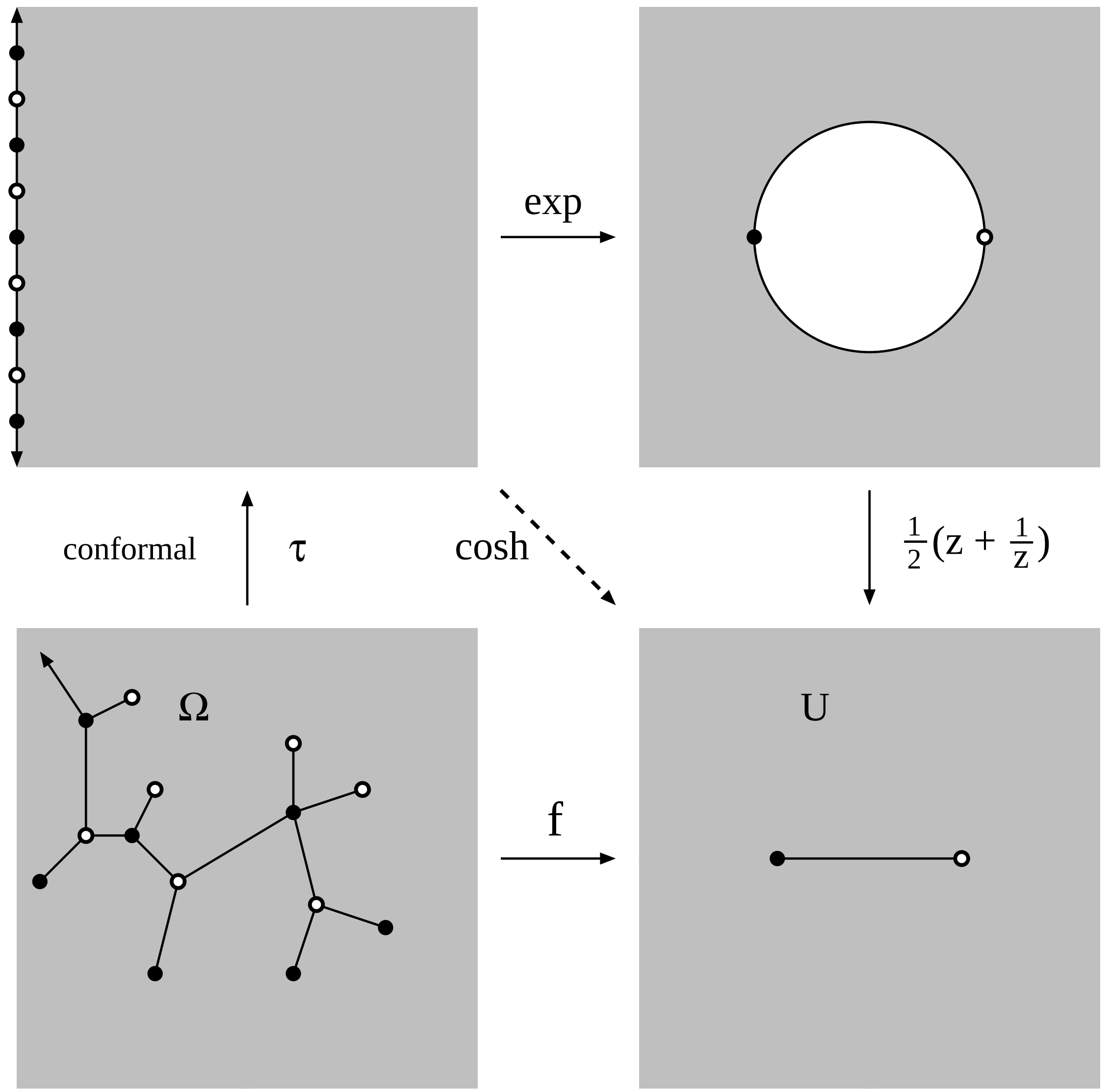
Recall finite case. Infinite case is very similar.



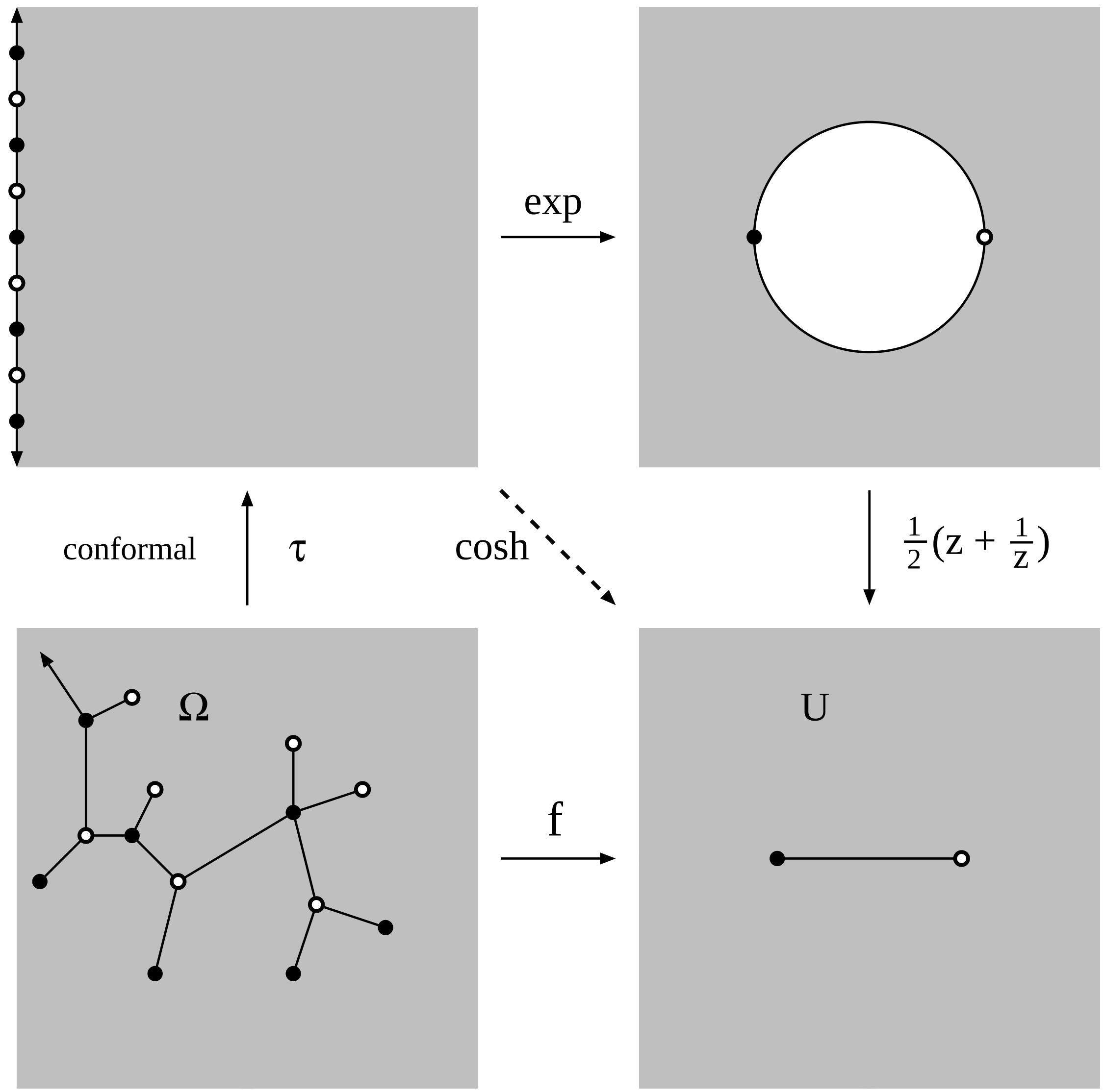
τ maps components of $\mathbb{C} \setminus T$ to right half-plane.



Pullback length to tree. Every side gets τ-length π.



Balanced tree $\Leftrightarrow f = \cosh \circ \tau$ is entire, $\mathrm{CV(f)} = \pm 1$.



Is every "nice" infinite tree QC-balanced?

Problem: Given an infinite tree T, build a quasi-regular g with $\mathrm{CV}(g) = \pm 1$ and $T \approx g^{-1}([-1,1])$.

We make two assumptions about T.

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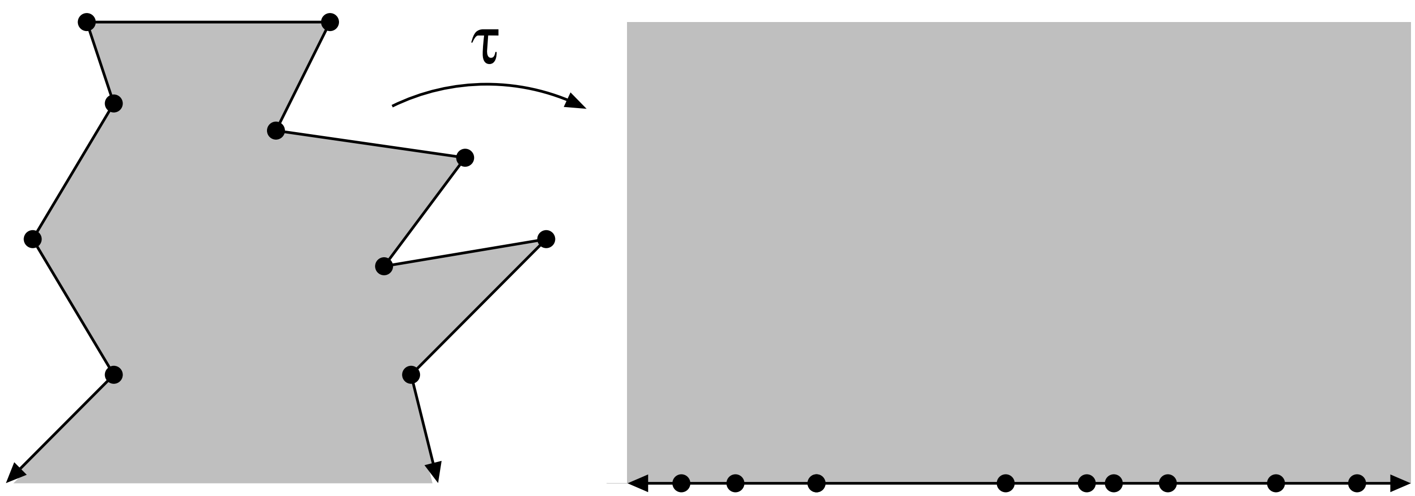
We make two assumptions about T.

1. Adjacent sides have comparable τ-length (local)

2. τ-lengths have positive lower bound (global)



1. Adjacent sides have comparable τ-length.

This follows if T has **bounded geometry**:

- edges are uniformly C^2
- angles are bounded away from 0
- adjacent edges have comparable lengths
- non-adjacent edges satisfy $\mathrm{diam}(e) \leq C \mathrm{dist}(e, f)$.

This is usually **easy** to check.

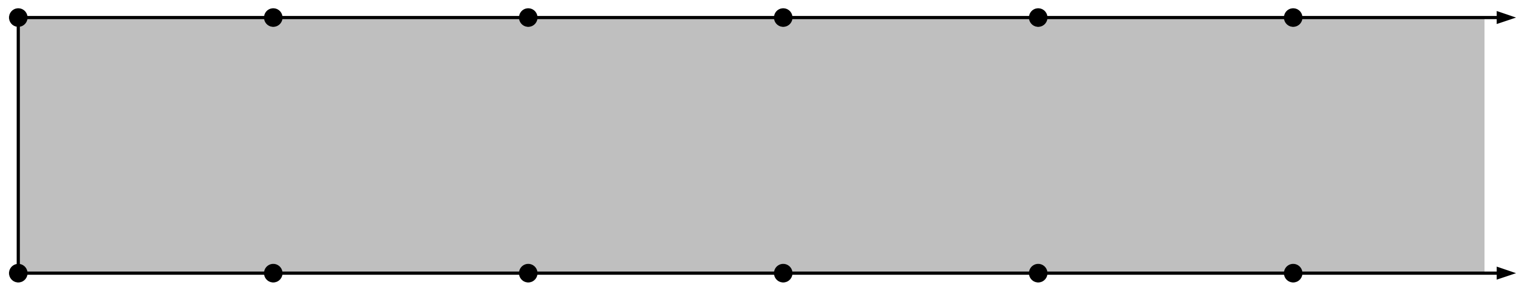
2. τ-lengths have positive lower bound.

Verifying this is usually straightforward.

e.g., read Garnett and Marshall *Harmonic Measure*.

Use standard estimates for hyperbolic metric, extremal length, harmonic measure ...

2. τ-lengths of sides have positive lower bound.



“inside” the τ-lengths grow exponentially. (good)

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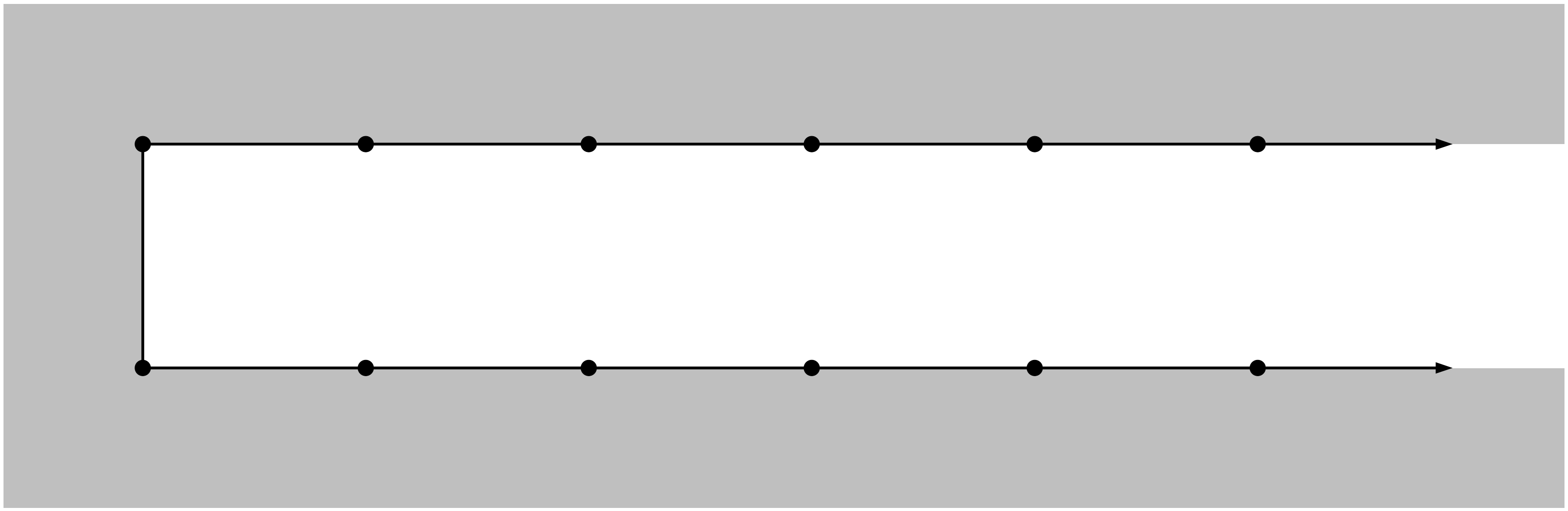
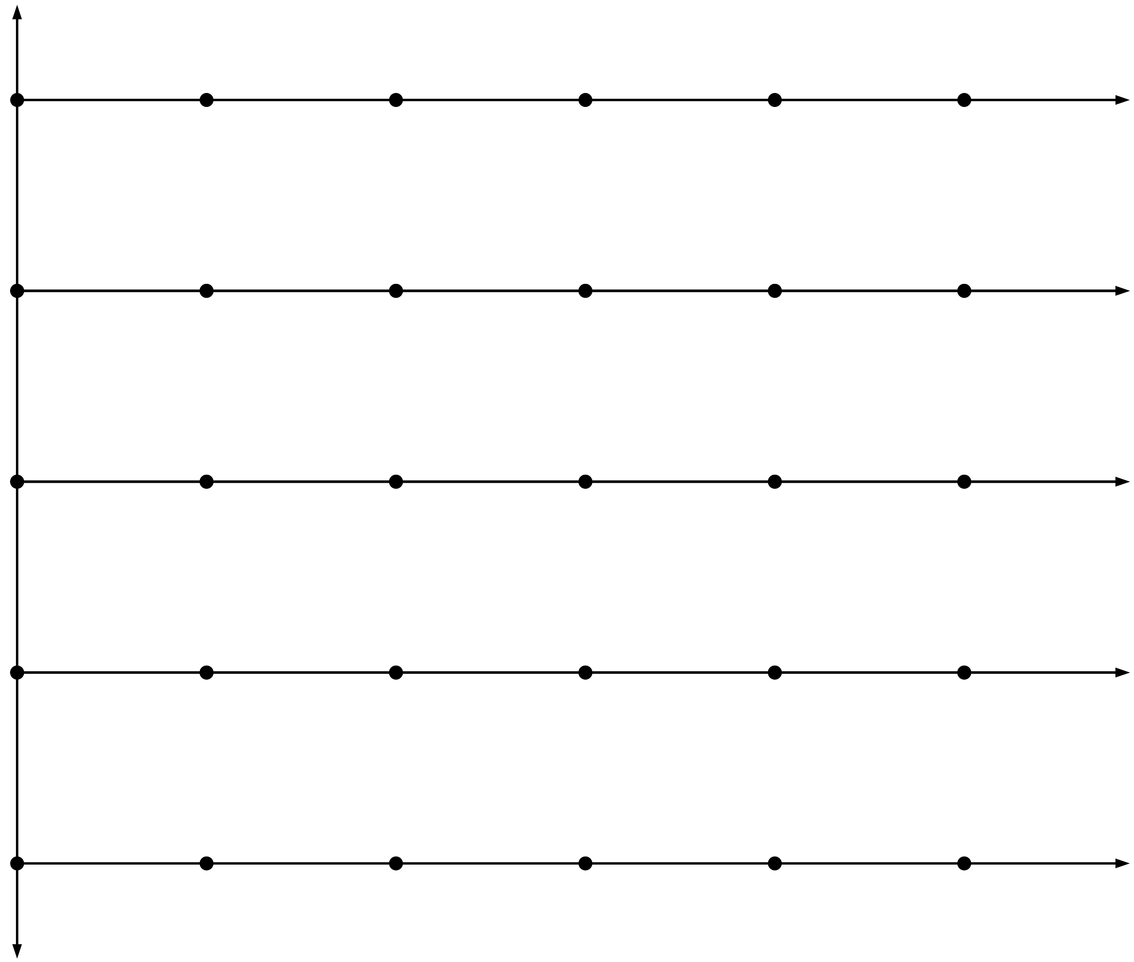


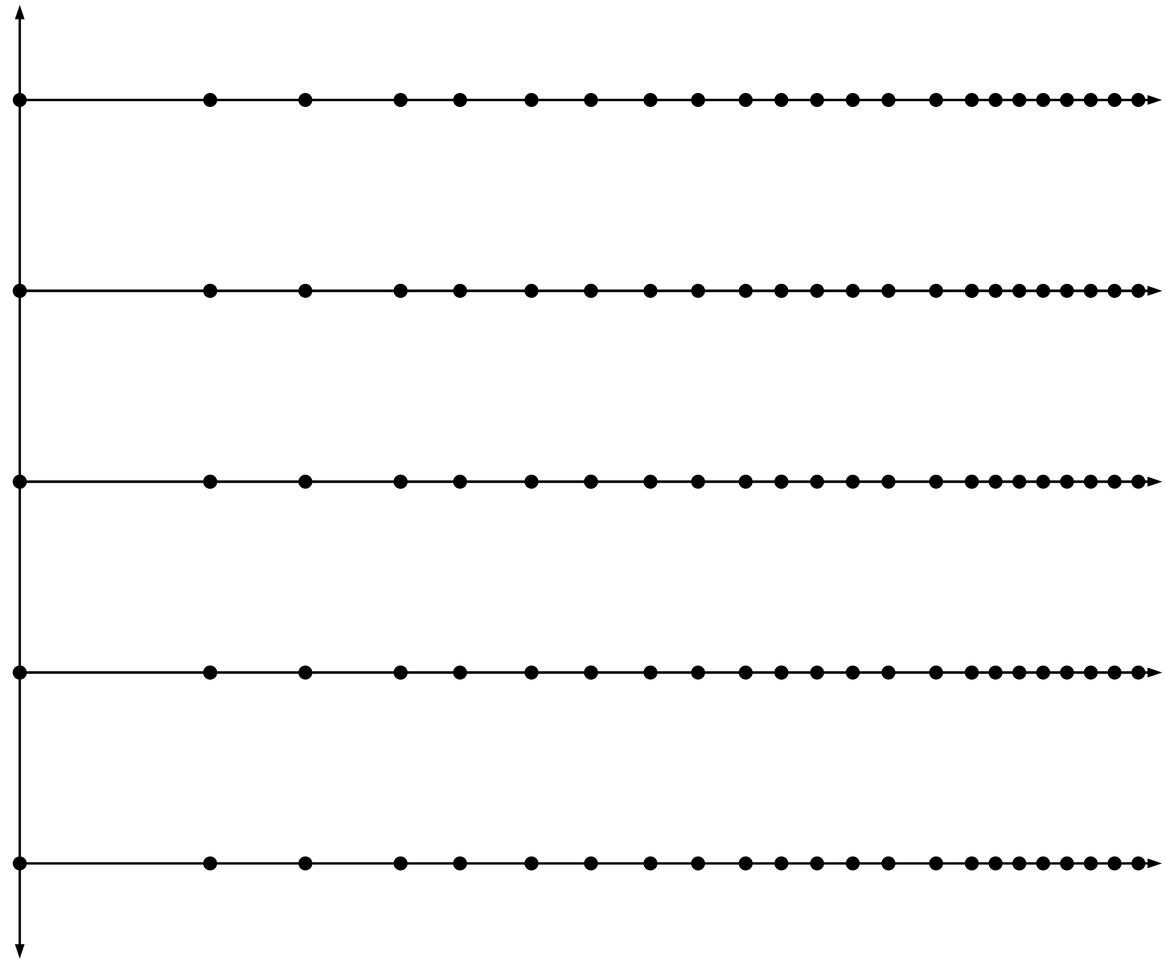
"inside" the τ-lengths grow exponentially. (good)

"outside" the τ-length decrease like $n^{-1/2}$. (bad)

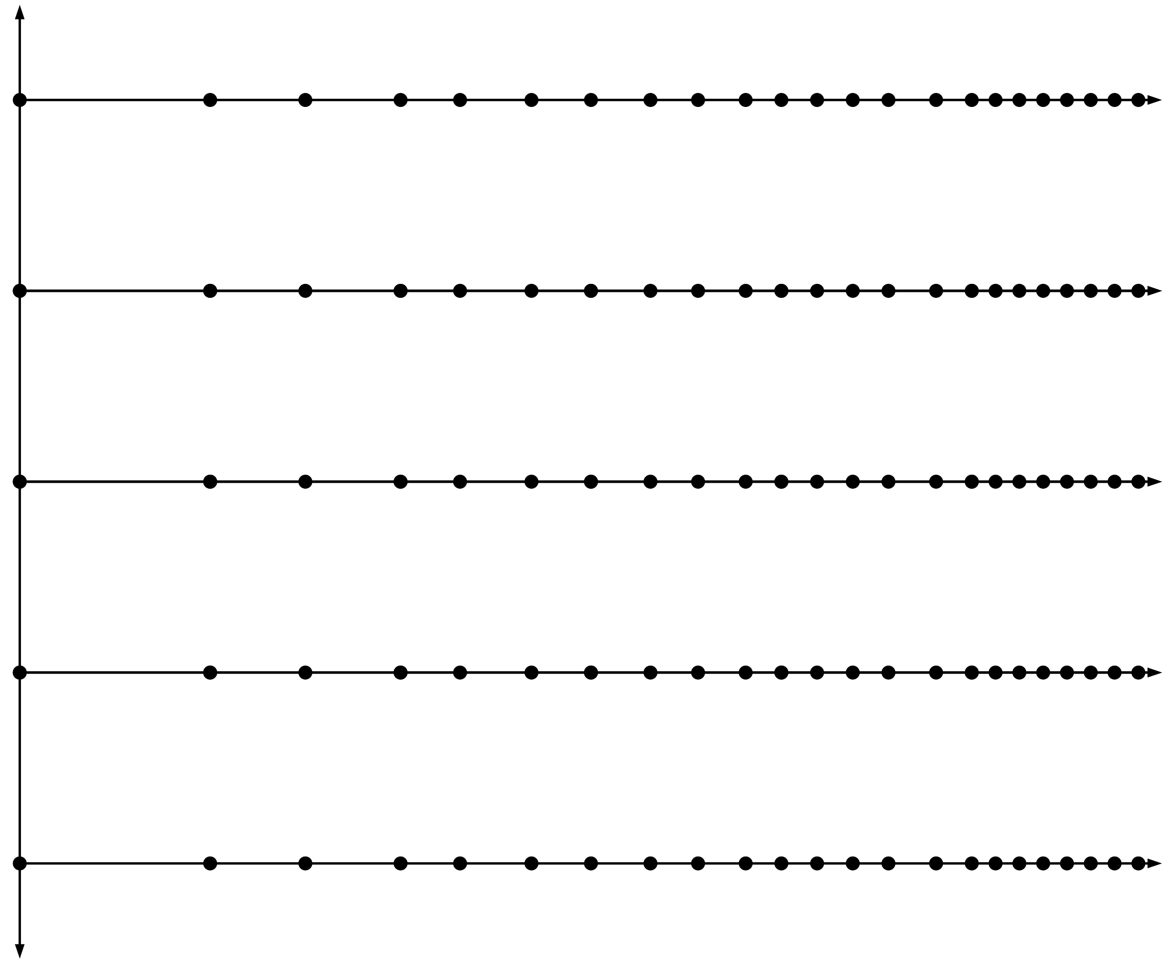
τ-lengths are **not** bounded below.



Bounded geometry and τ-bounded.



Also bounded geometry and τ-bounded.



This tree is "better" than last one. Why?

If e is an edge of T and $r > 0$ let

$$e(r) = \{z : \mathrm{dist}(z, e) \leq r \cdot \mathrm{diam}(e)\}$$

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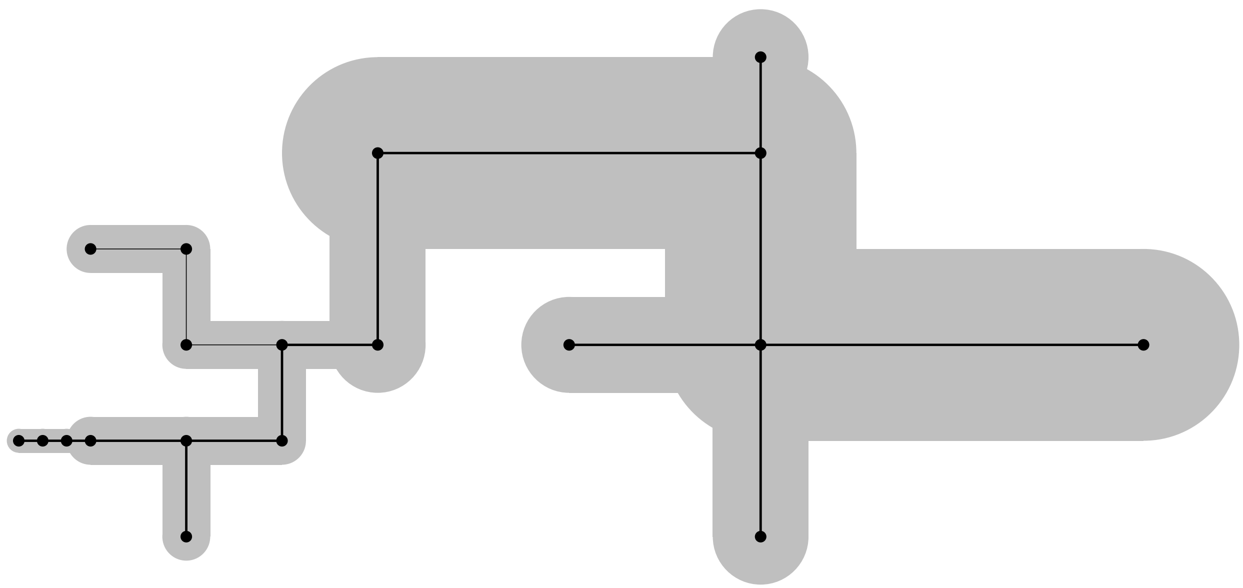
Define neighborhood of T: $T(r) = \cup\{e(r) : e \in T\}$.



$T(r)$ for infinite tree replaces Hausdorff metric in finite case.

If e is an edge of T and $r > 0$ let

$$e(r) = \{z : \mathrm{dist}(z, e) \leq r \cdot \mathrm{diam}(e)\}$$

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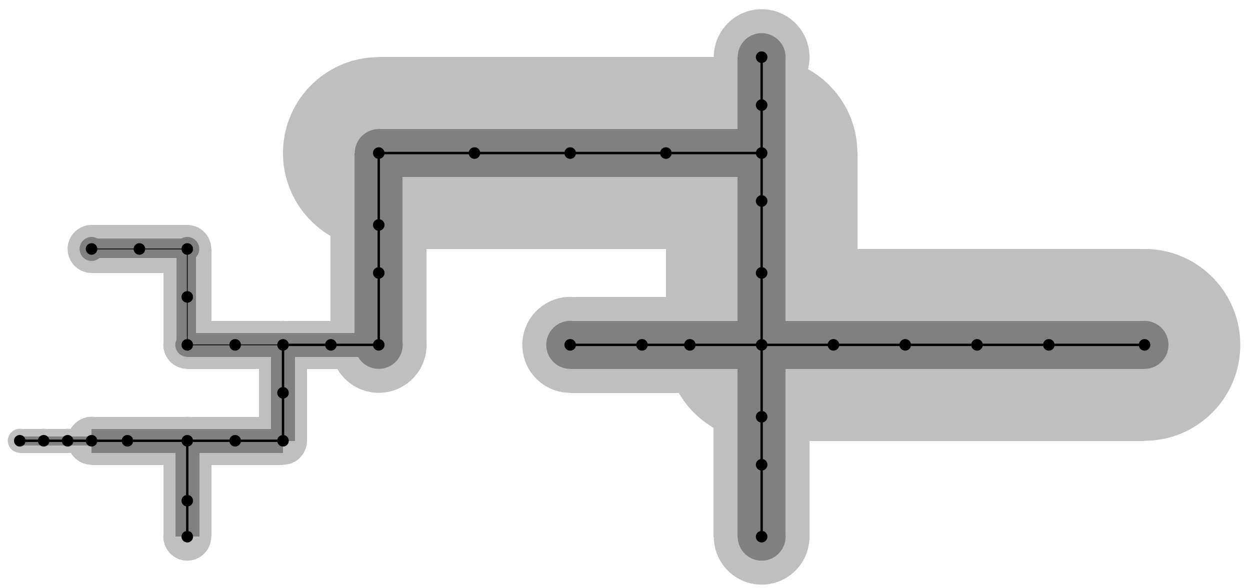
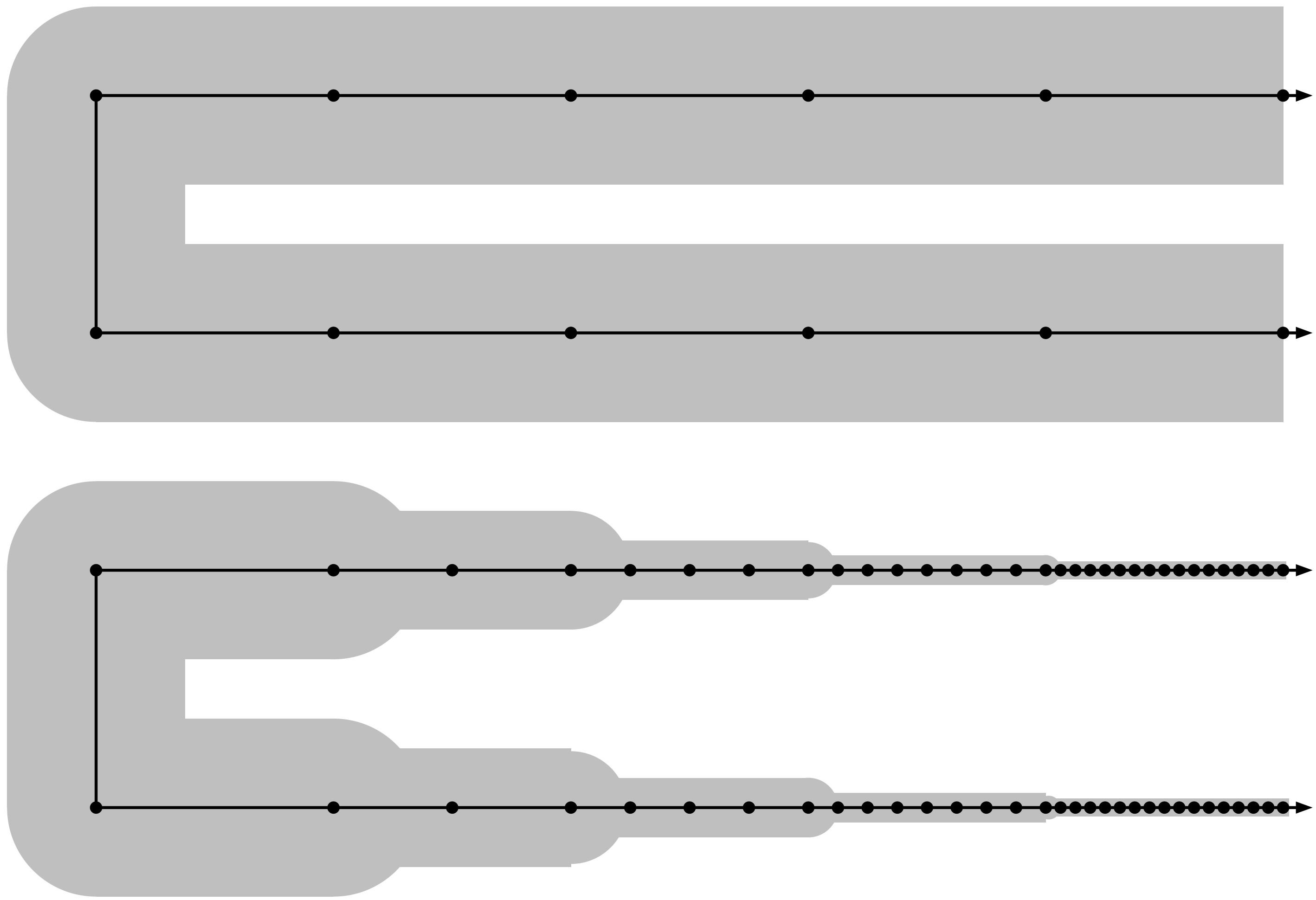
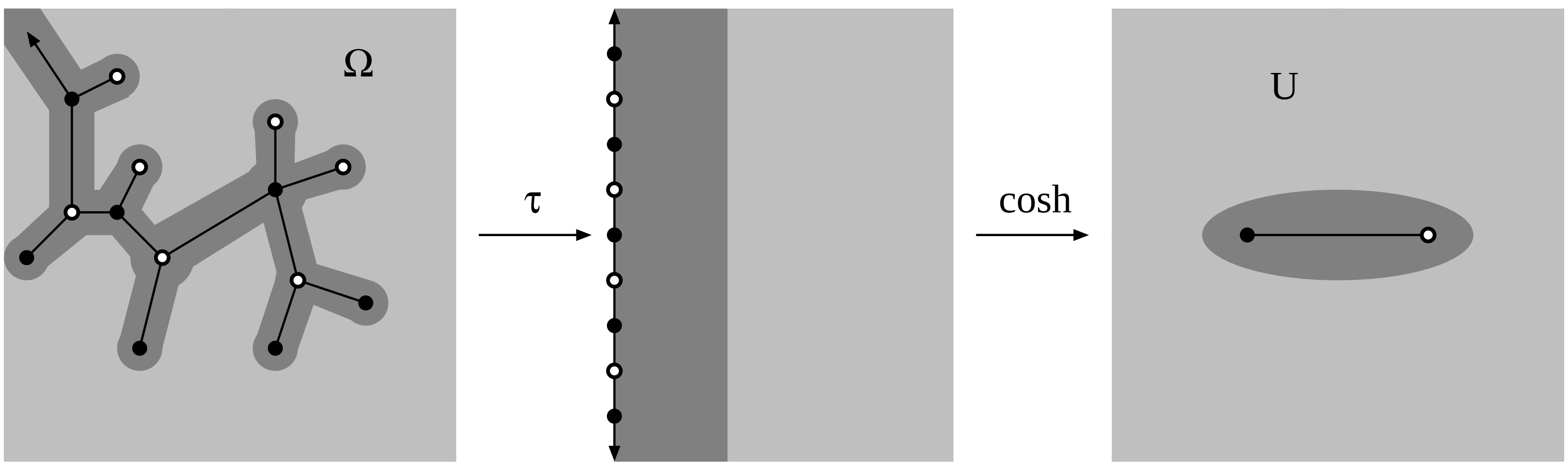


Adding vertices reduces size of $T(r)$.

We use QC maps with dilatations supported in $T(r)$.
If $T(r)$ is small, we get better control.

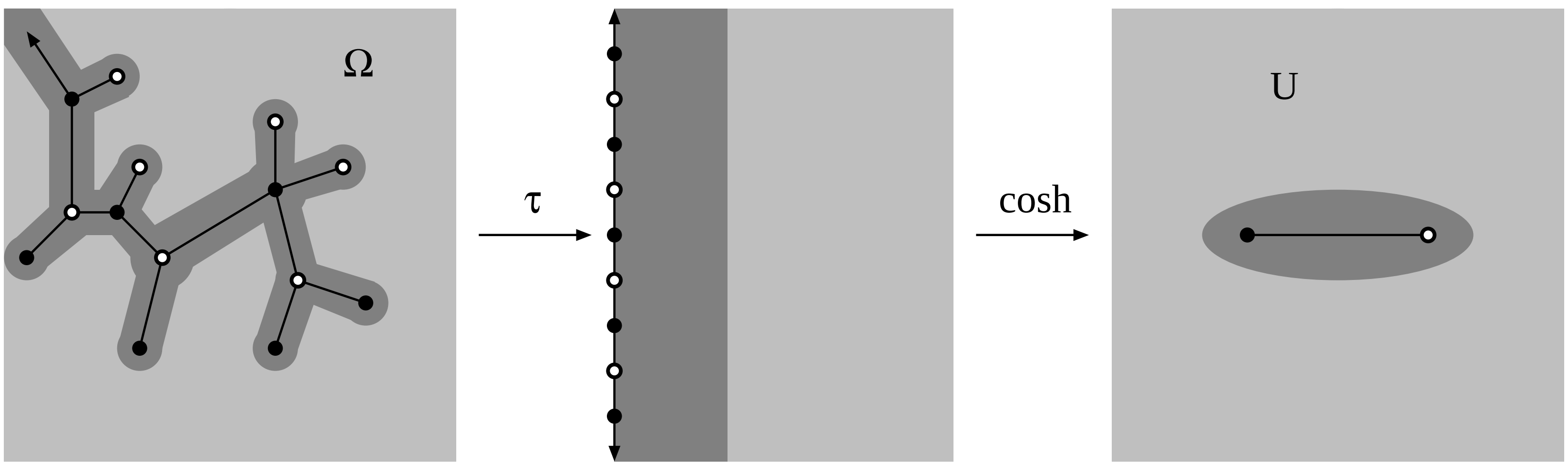


QC Folding Thm: Suppose T has bounded geometry and all τ-lengths $\geq \pi$. Then there is a $r > 0$ and a quasi-regular g such that $g = \cosh \circ \tau$ off $T(r)$ and $\mathrm{CV}(g) = \pm 1$.



Constructing entire functions by quasiconformal folding,
Acta. Math. 214:1(2015) 1-60

QC Folding Thm: Suppose T has bounded geometry and all τ-lengths $\geq \pi$. Then there is a $r > 0$ and a quasi-regular g such that $g = \cosh \circ \tau$ off $T(r)$ and $\mathrm{CV}(g) = \pm 1$.



- $\exists$ QC-tree T' s.t. $T \subset T' \subset T(r)$.

QC Folding Thm: Suppose T has bounded geometry and all τ-lengths $\geq \pi$. Then there is a $r > 0$ and a quasi-regular g such that $g = \cosh \circ \tau$ off $T(r)$ and $\mathrm{CV}(g) = \pm 1$.

- $|\mu_\phi| \leq K$ and $\mathrm{supp}(\mu_\phi) \subset T(r)$.

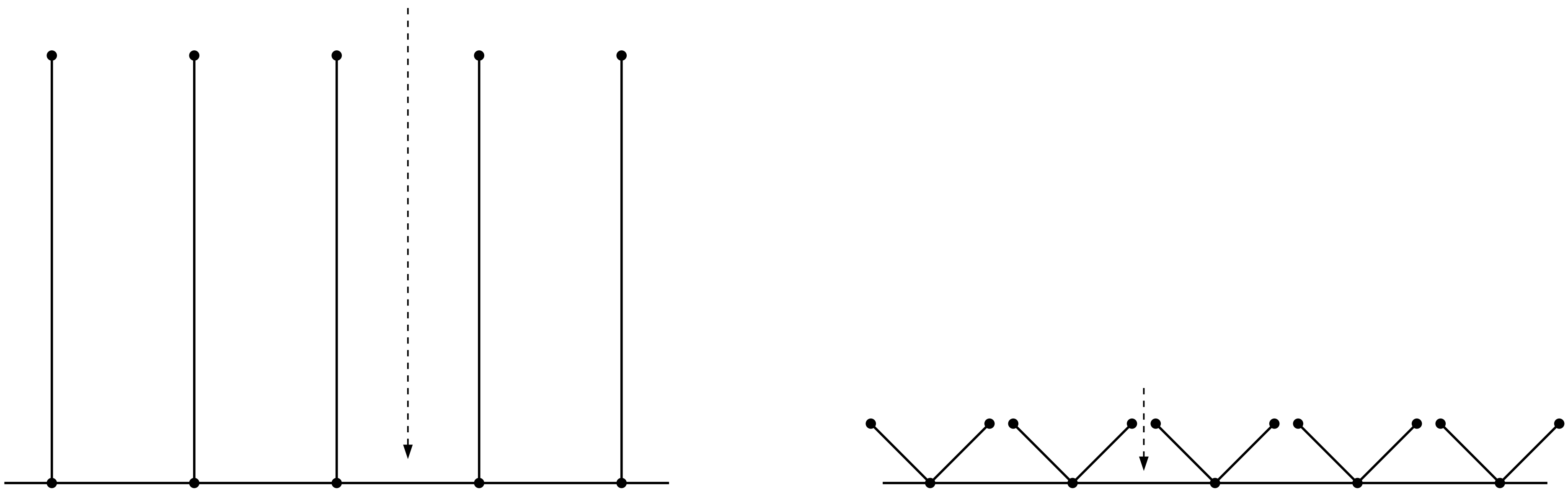
- K and r depend only on bounded geometry constants.

- Can shrink $T(r)$ by subdividing T and rescaling τ.

Cor: There is an entire function f with $\mathrm{CV}(f) = \pm 1$ so that $f^{-1}([-1, 1])$ approximates T.

Finite vs infinite case:

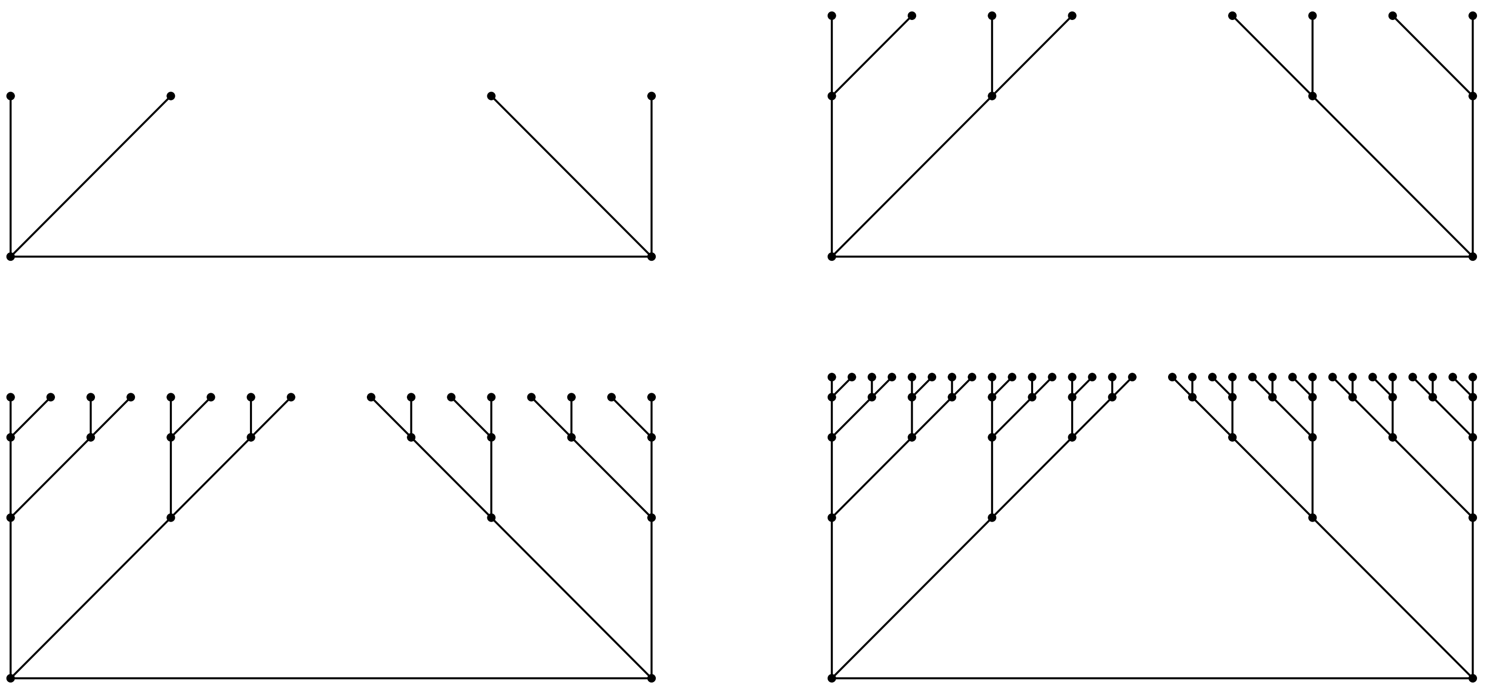
• Instead of adding spikes to edges of trees, we add small finite trees. Original edge is only reached through small gap between trees.



Finite vs infinite case:

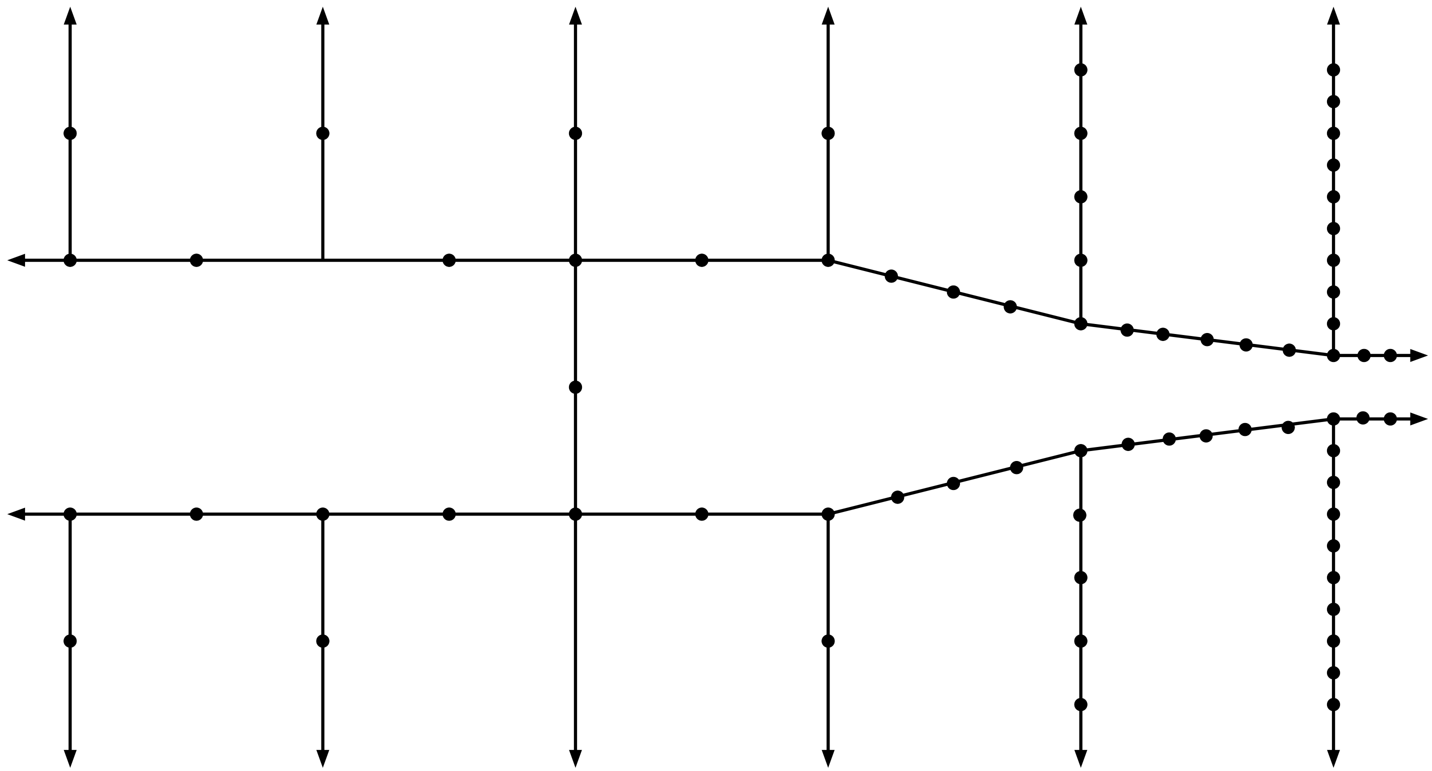
• Instead of adding spikes to edges of trees, we add small finite trees. Original edge is only reached through small gap between trees.

• Add more tree to make all sides are M-balanced. The more complicated combinatorics allow us to handle arbitrarily bad imbalances with uniform QC bounds.



Rest of talk will describe applications of QC folding.

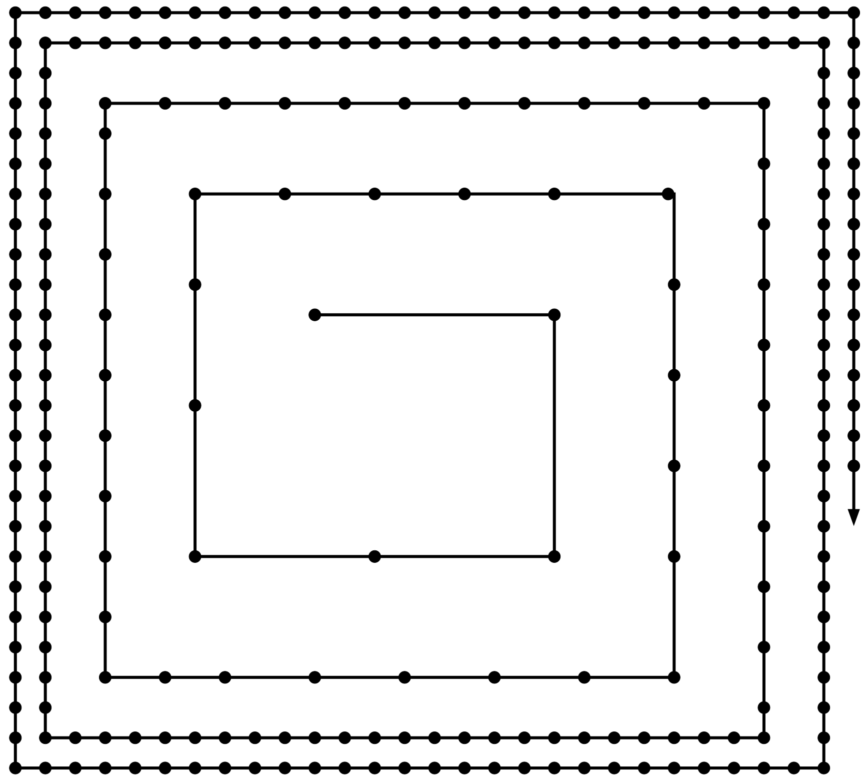
Rapid increase



$\exists f \in \mathcal{S}_2$ so $f(x) \nearrow \infty$ as fast as we wish.

First such example due to Sergei Merenkov.

Fast spirals



$\exists f \in \mathcal{S}_2$ so $\{|f| > 1\}$ spirals as fast as we wish.

Adam Epstein's order conjecture

Order of growth:

$$\rho(f) = \limsup_{|z|\to\infty} \frac{\log\log|f(z)|}{\log|z|}.$$

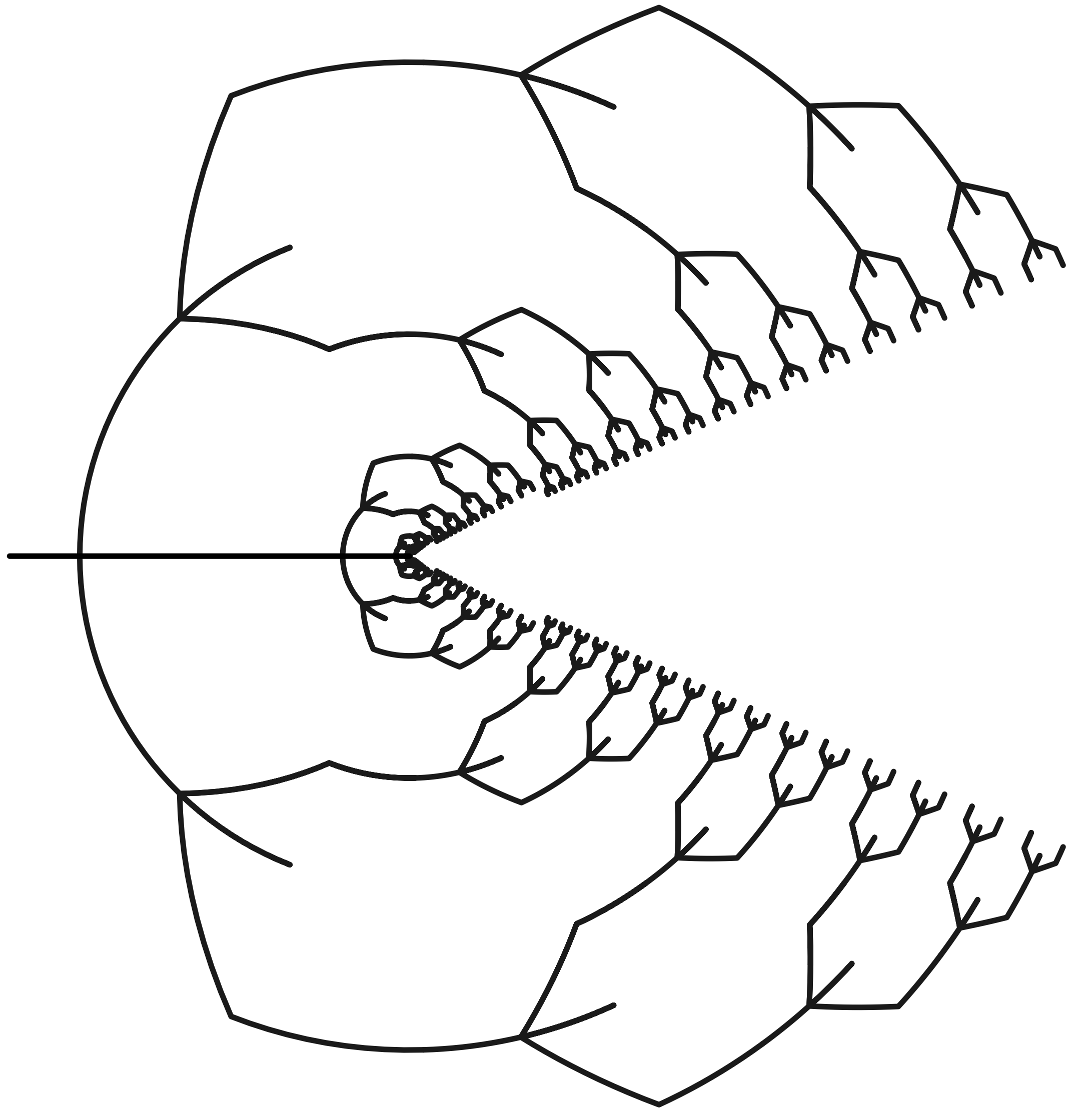
Analogous to degree of polynomial.

f, g QC-equivalent if $\exists$ QC ϕ, ψ s.t. $f \circ \phi = \psi \circ g$.

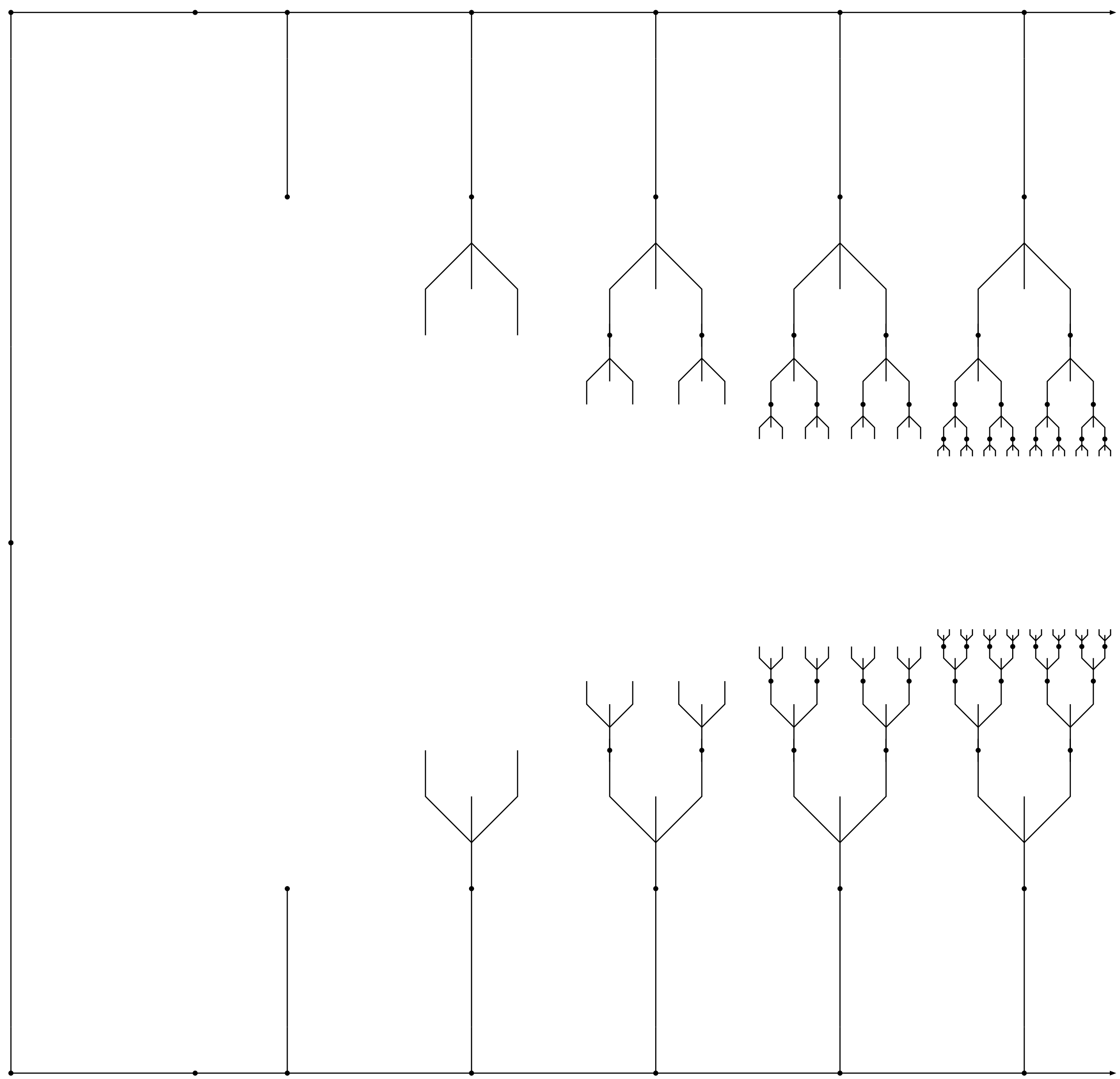
Question: $f, g \in \mathcal{S}$ QC-equivalent $\Rightarrow \rho(f) = \rho(g)$?

False in $\mathcal{B}$ (Epstein-Rempe)

No



Same domain in logarithmic coordinates

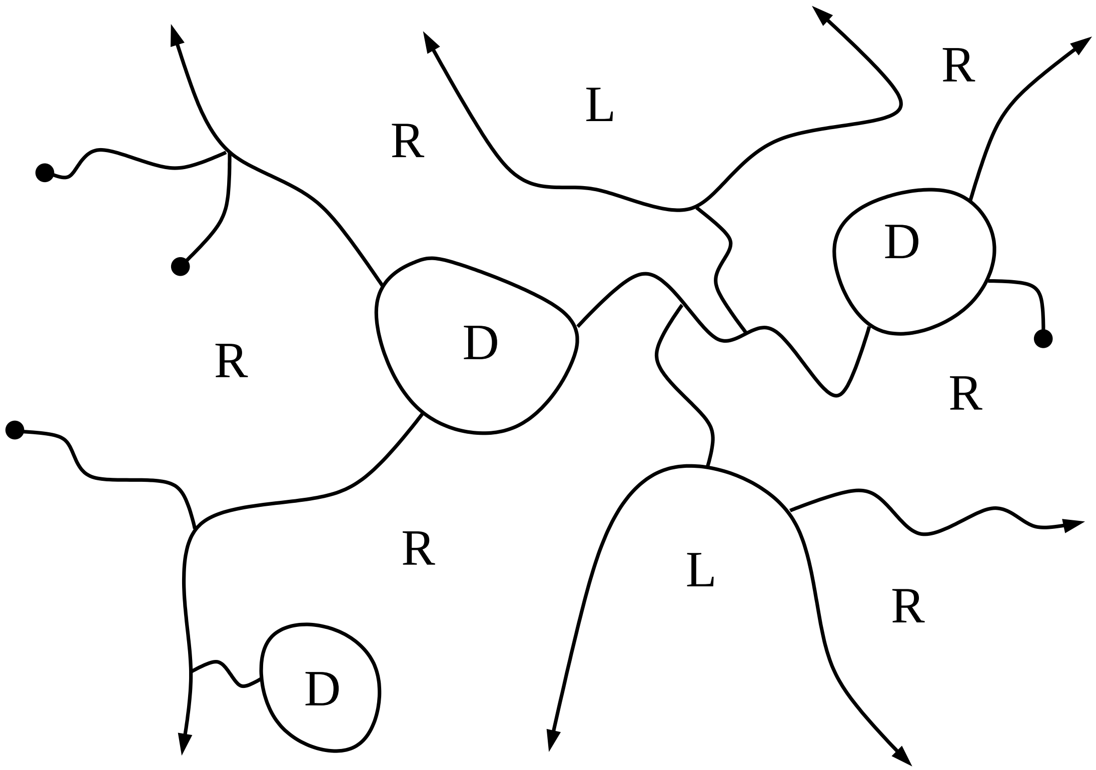


So far, QC-folding always gives critical values ± 1.

All critical points have uniformly bounded degree.

A simple modification gives:

- high degree critical points
- critical values other than ± 1
- finite asymptotic values.

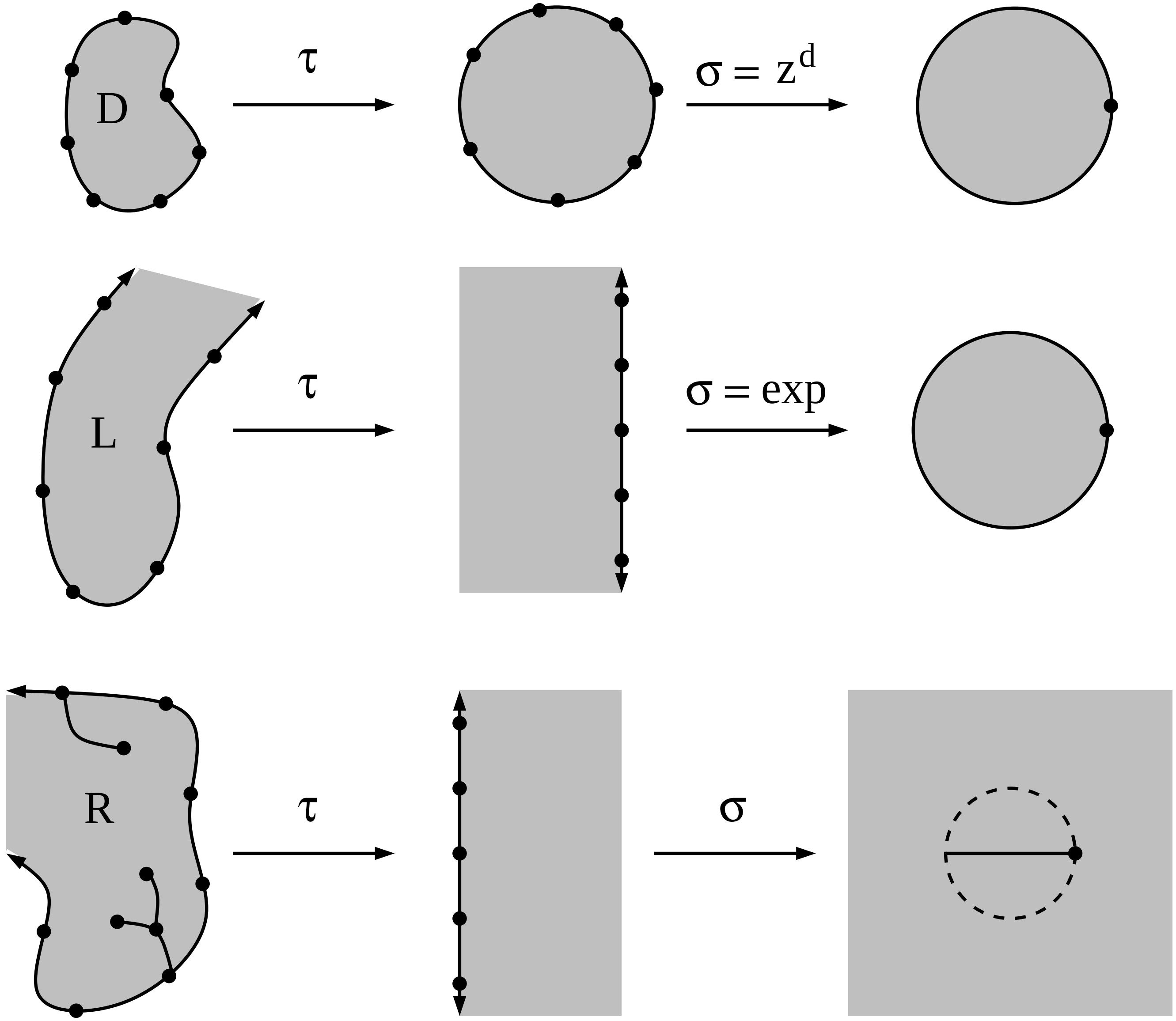


Replace tree by graph. Graph faces labeled D,L,R.

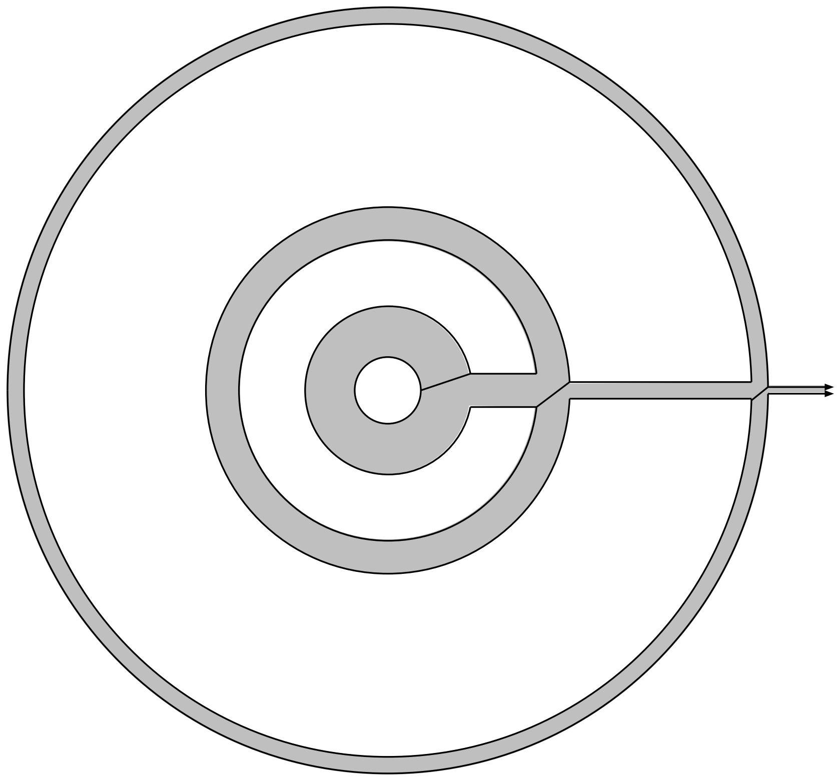
D = bounded Jordan domains (high degree critical points)

L = unbounded Jordan domains (asymptotic values)

D's and L's only touch R's.

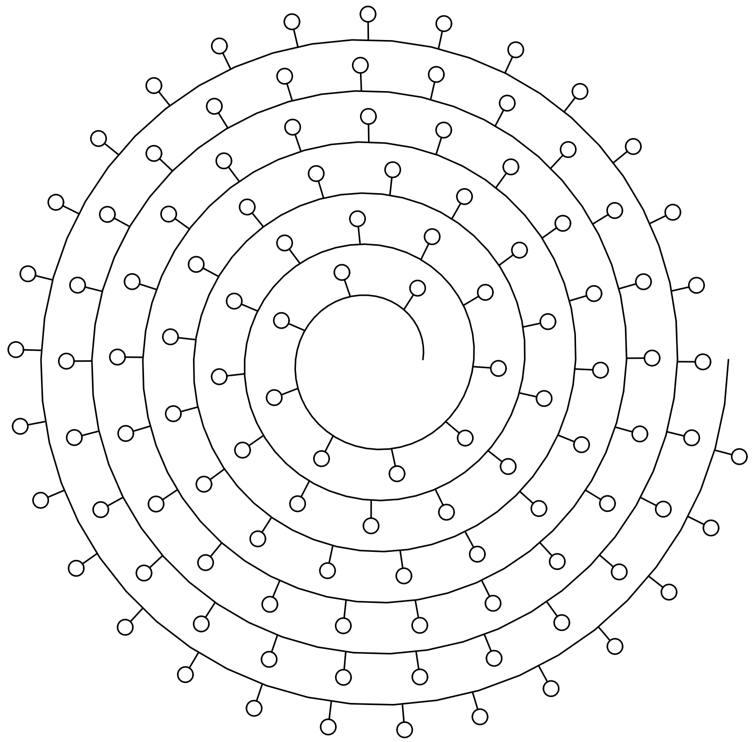

D
τ
σ = z^d
L
τ
σ = exp
R
τ
σ

The area conjecture



One R-component, many D-components

$\exists f \in \mathcal{S}$ s.t. area$(\{z : |f(z)| > \epsilon\}) < \infty$ for all $\epsilon > 0$.



Let $m(r) = \min_{|z|=r} |f(z)|, \quad M(r) = \max_{|z|=r} |f(z)|.$

$$\exists f \in \mathcal{S} \text{ s.t. } \limsup_{r\to\infty} \frac{\log m(r)}{\log M(r)} = -\infty.$$

Wiman conjectured ≥ -1. First counterexample due to Hayman.

Bounded type wandering domain:

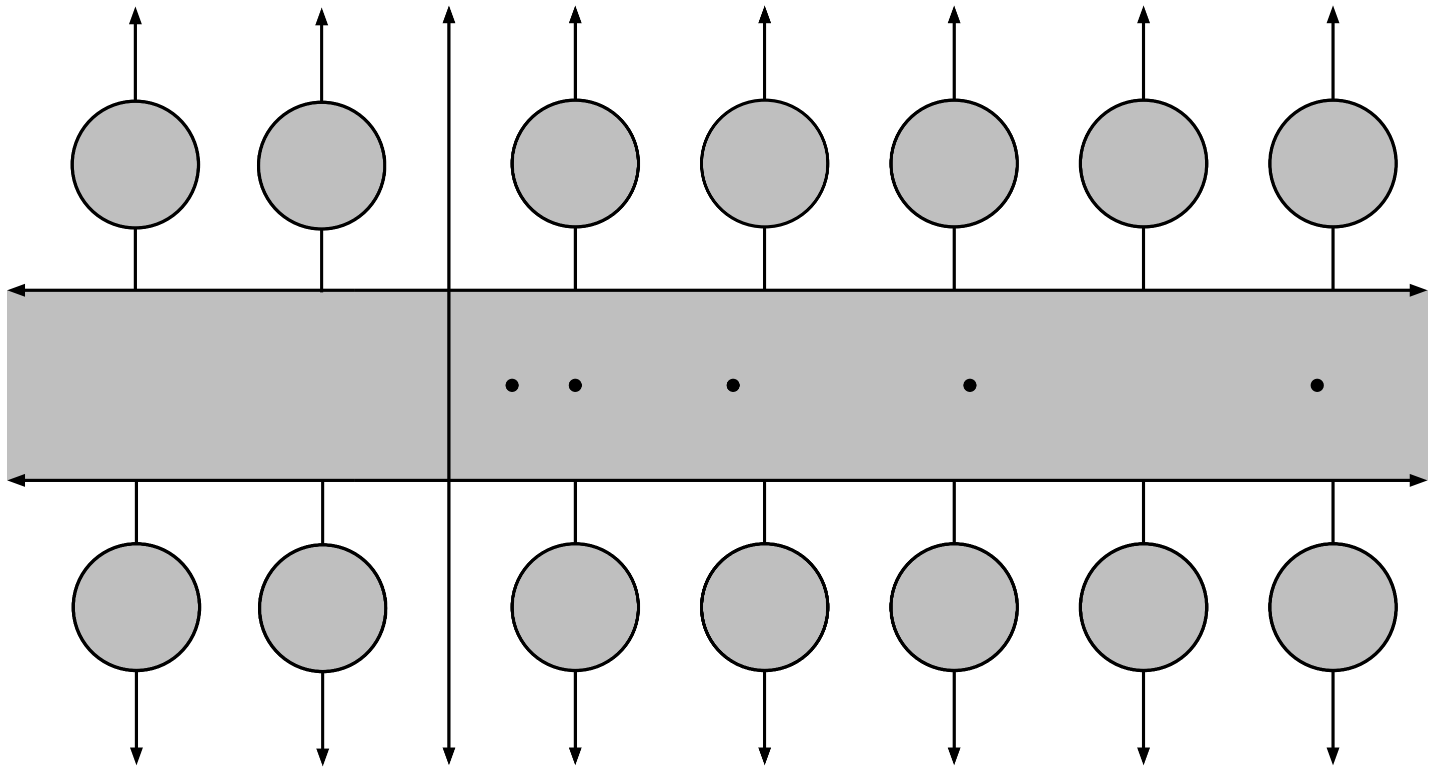
Thm: $\exists\ f \in \mathcal{B}$ that has a wandering domain.

wandering domain = non pre-periodic Fatou component

$\mathcal{B}$ = Eremenko-Lyubich class = bounded singular set

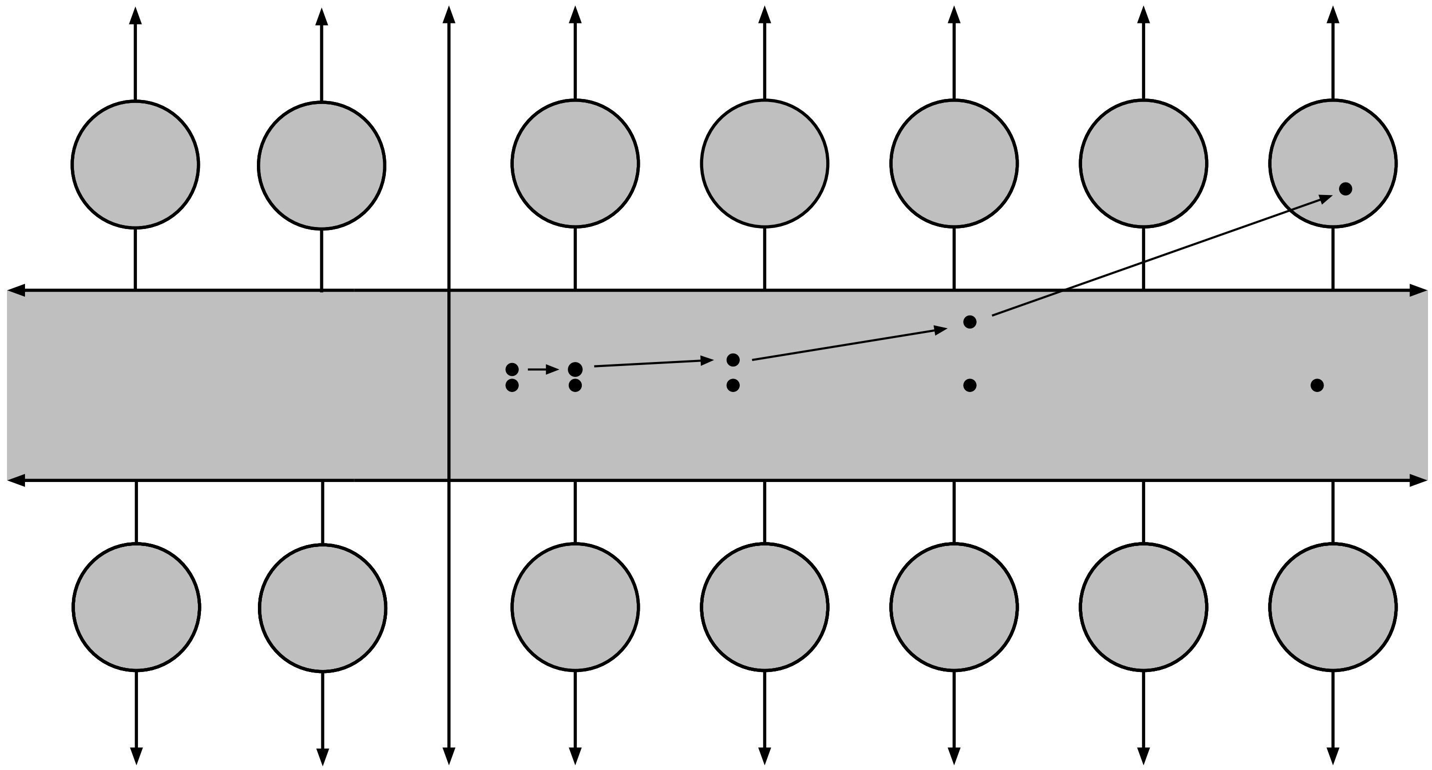
Sullivan's non-wandering theorem extended to $\mathcal{S}$ (finite singular set) by Eremenko-Lyubich, Goldberg-Keen.

First wandering domains for entire functions due to Baker.



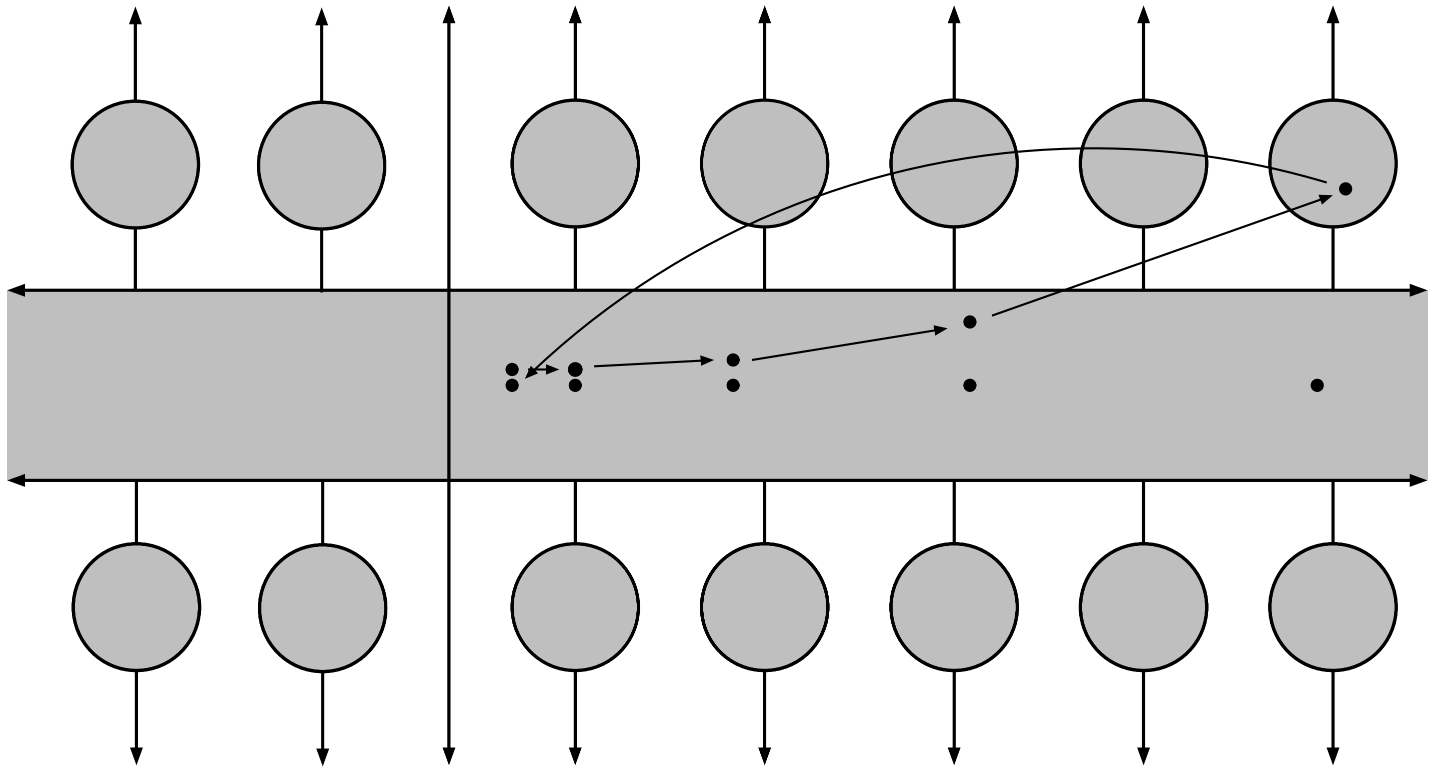
Uses only D and R components. Symmetric.

Dots are orbit of 1/2.



Orbit just above 1/2 diverges from real line.

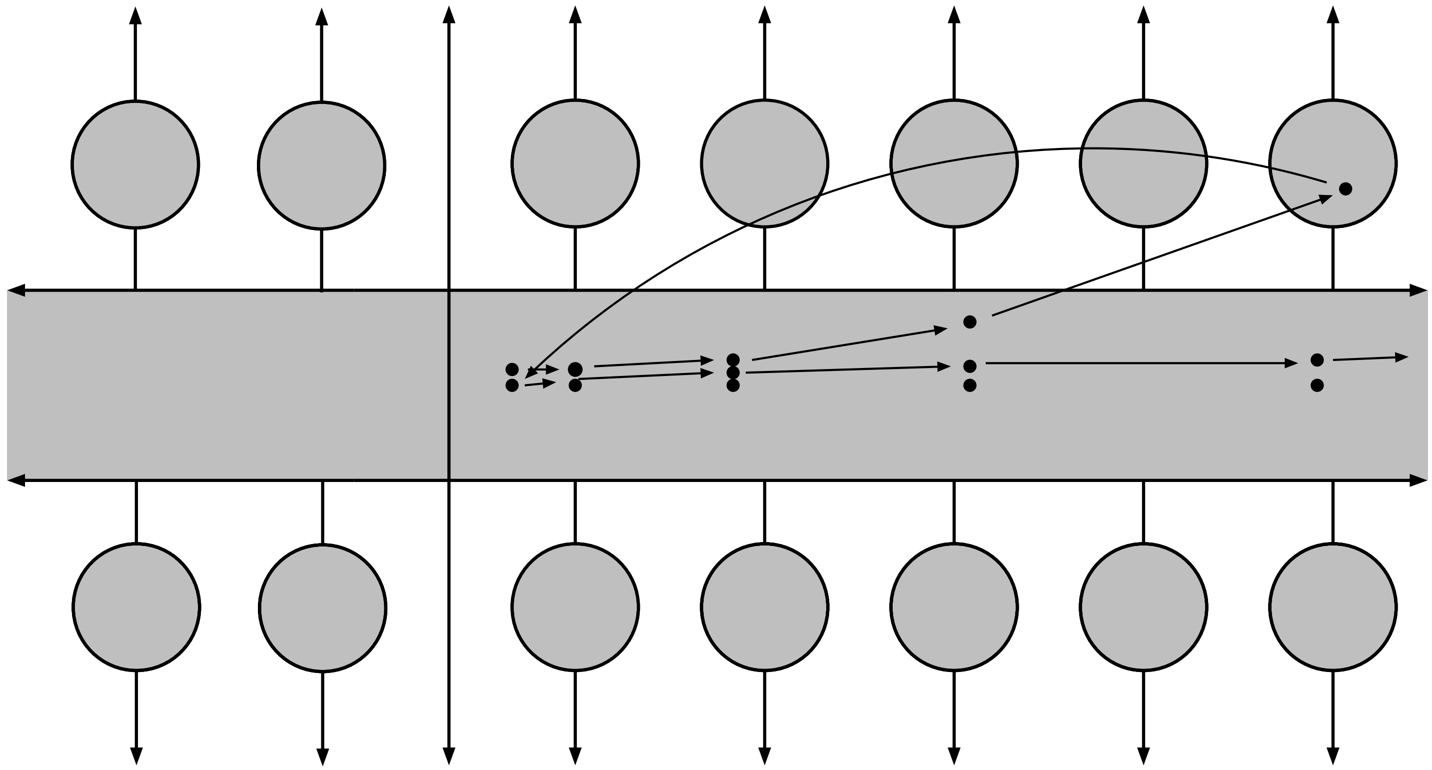
Can choose to land near center of a D-component.



D-component contains high degree critical point.

Critical value is just above 1/2.

But closer to 1/2 than previous starting point.



New orbit follows 1/2 longer before returning.

Orbit is unbounded, but not escaping.

Iterated disk has $\mathrm{diam}(D_n) \to 0$, so is in Fatou set.

Lemma: D_n's are in different Fatou components:

Proof: If not, there are $n < m$ so D_n, D_m are in same Fatou component. Then kth iterates D_{n+k}, D_{k+m} alway always land in same component.

Iterate until D_{k+m} returns near 1/2; D_{k+n} is far away. Contradicts Schwarz lemma (hyperbolic distances decrease under iteration).

Thanks for listening. Questions?