Random walks in analysis

Christopher J. Bishop
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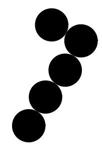
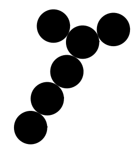
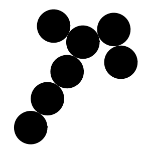
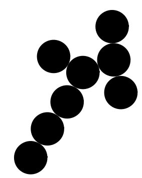
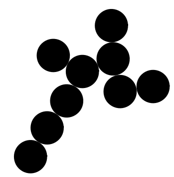
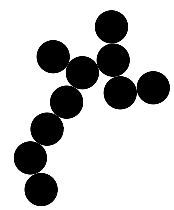


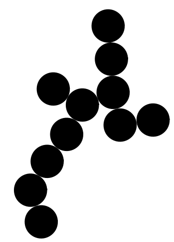


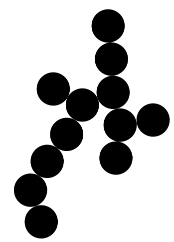


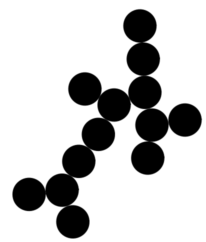


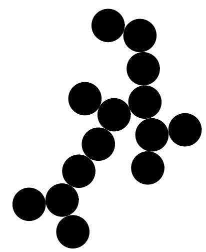


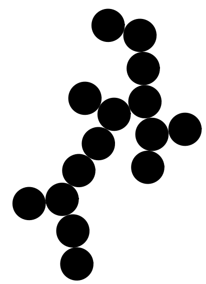


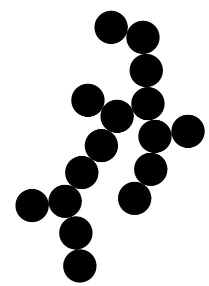


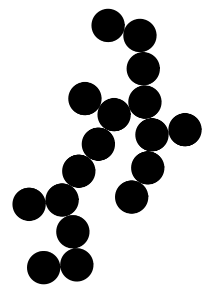


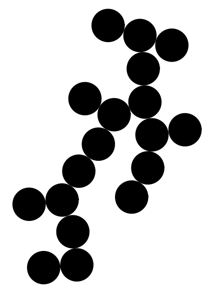


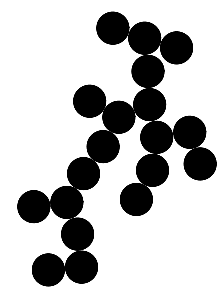


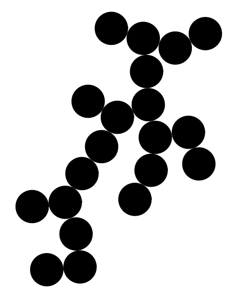


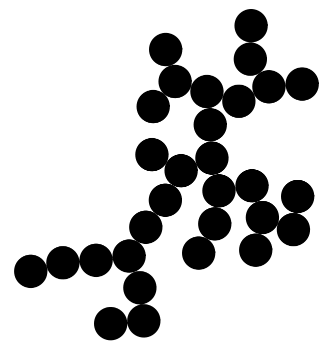


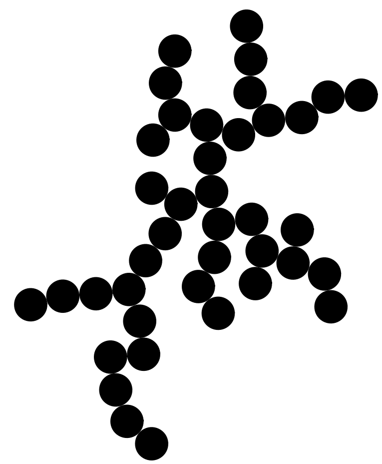


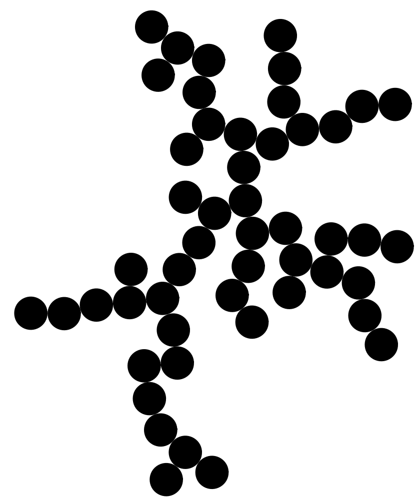


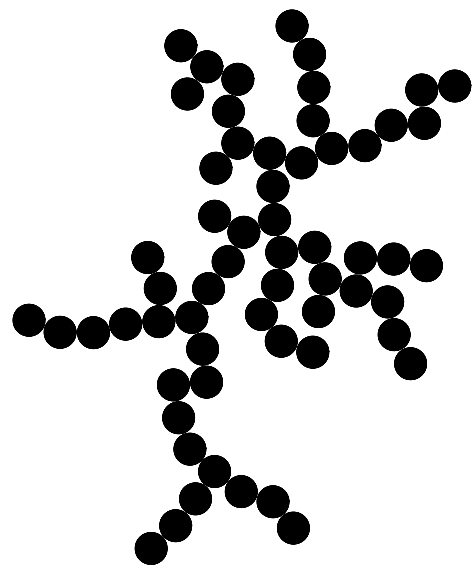


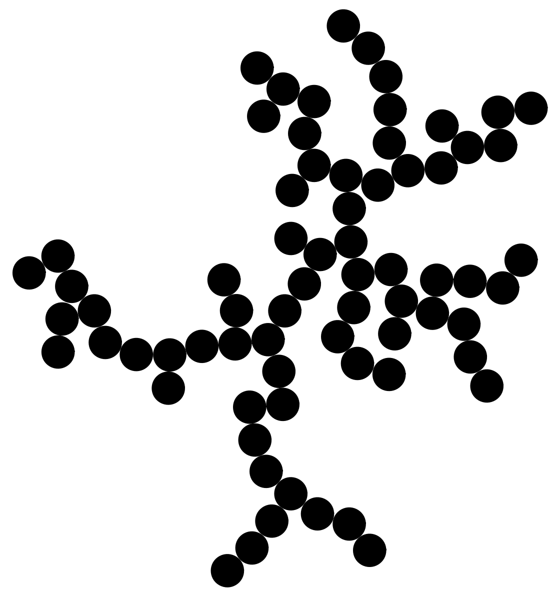


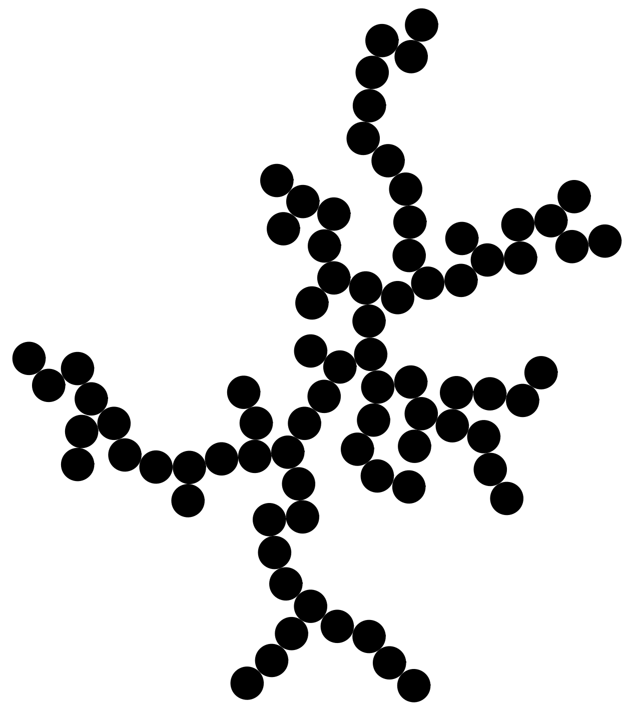


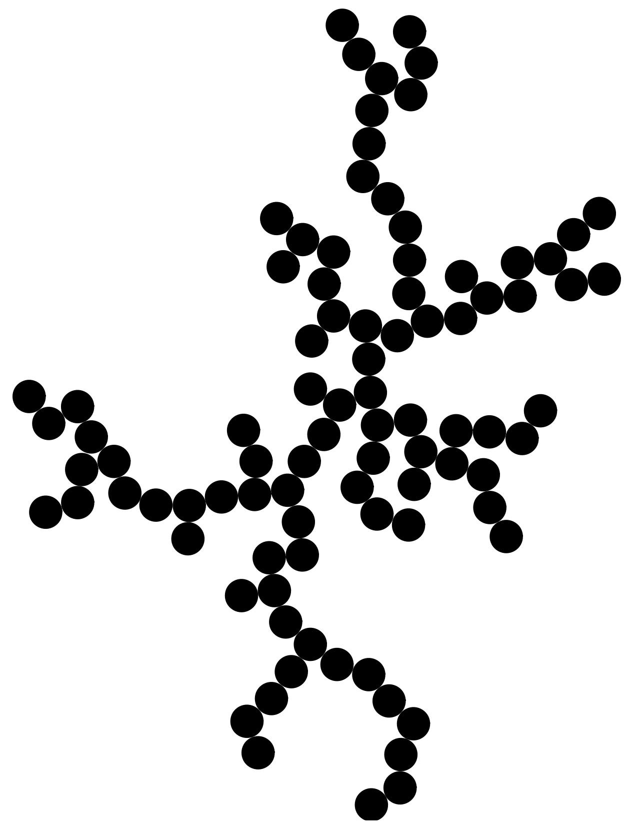


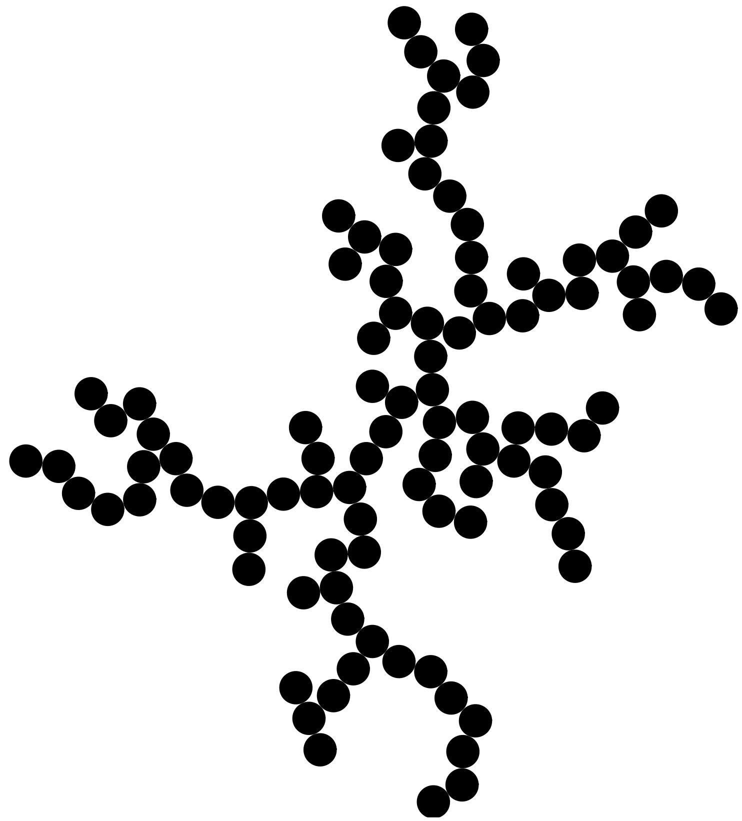




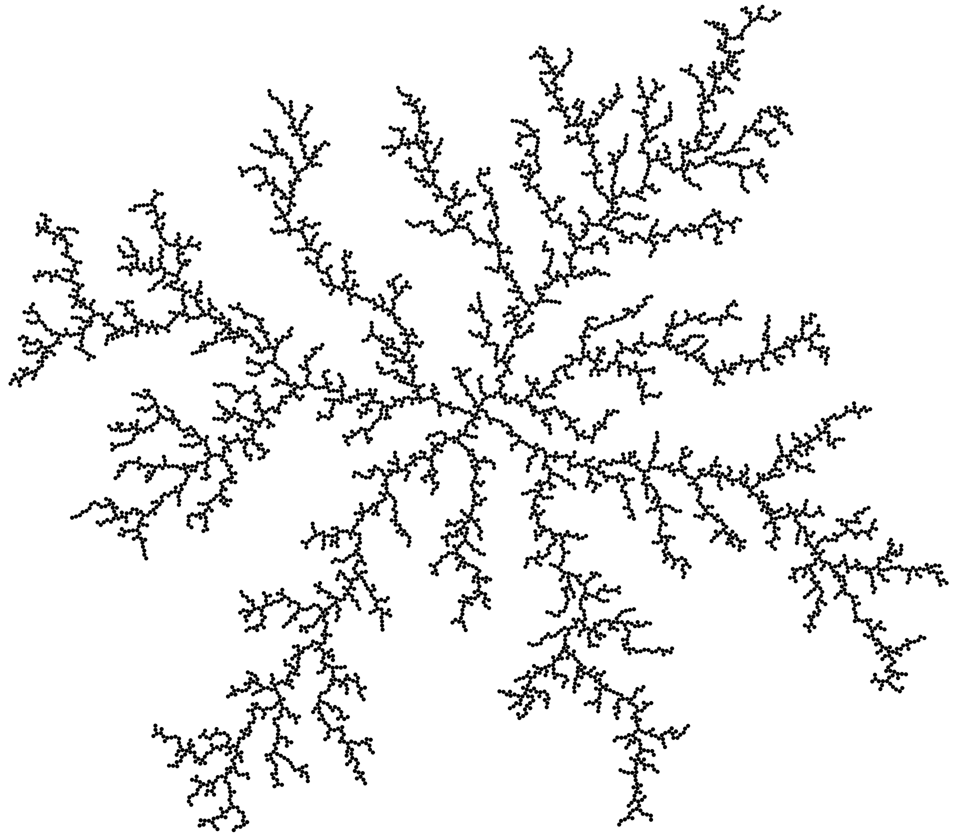








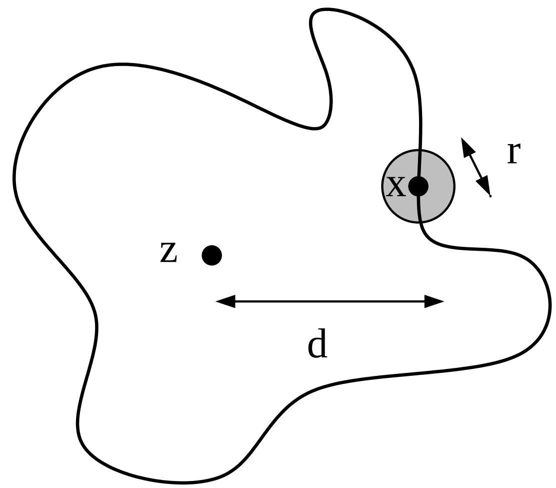
Diffusion Limited Aggregation (DLA)



Harry Kesten: $R \leq CN^{2/3}$. Experiments $\simeq N^{.585}$.

Harmonic Measure: $\omega(z, E, \Omega)$ = Probability that a Brownian motion started at z first hits $\partial\Omega$ in E.

Beurling's estimate: If $D = D(x, r)$ and $d = |z - x|$, then $\omega_z(D) \leq C\sqrt{r/d}$.



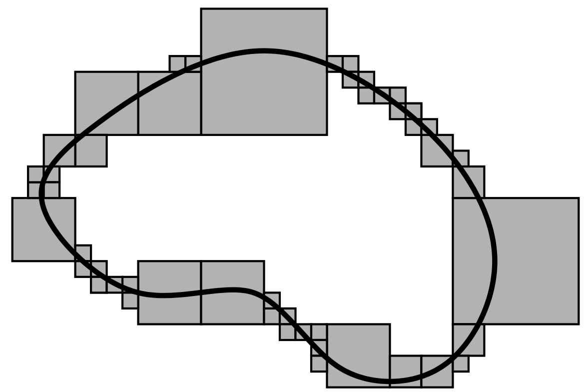
Ω = outside of DLA, $d = N^\alpha$, $r = 1$, ω = harmonic measure of outermost disk. Consider times $N, 2N$.

$$(2^\alpha - 1)N^\alpha = (2N)^\alpha - N^\alpha \approx N\omega \leq Nd^{-1/2} \simeq N^{1-\alpha/2}.$$

$$\Rightarrow \quad \alpha \leq 1 - \alpha/2 \quad \Rightarrow \quad \alpha \leq 2/3.$$

$$\ell(E) = \lim_{\delta\to 0} \min \sum \mathrm{diam}(Q_k) : E \subset \cup Q_k, \mathrm{diam}(Q_k) < \delta.$$

E is **rectifiable** if $E \subset \Gamma$, Γ connected, $\ell(\Gamma) < \infty$.

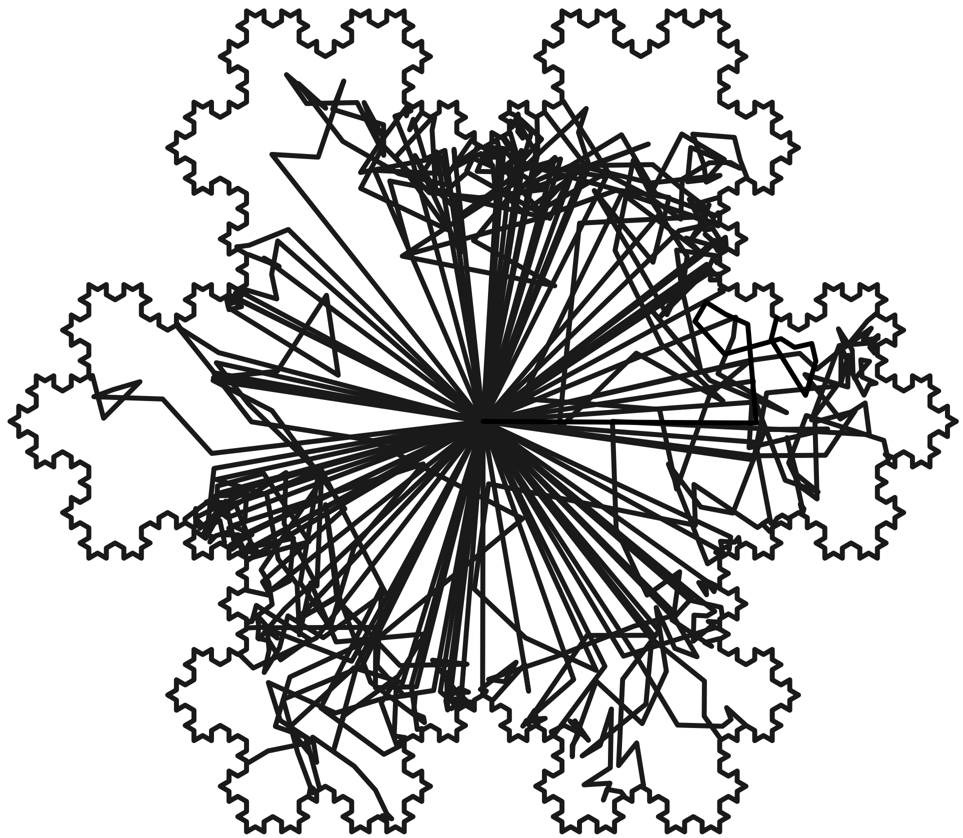


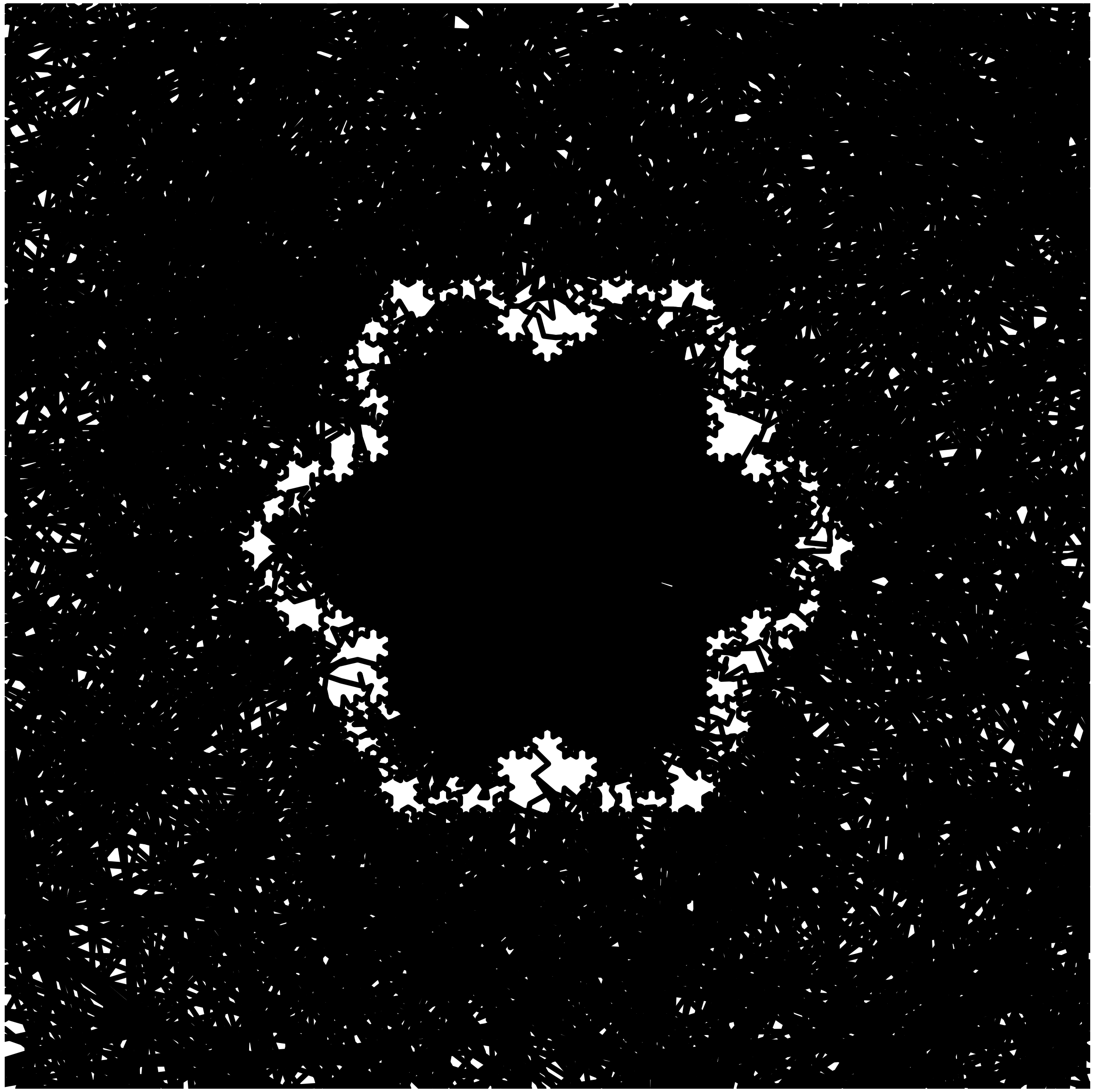
F. and M. Riesz, 1916: If $E \subset \partial\Omega$ and $\partial\Omega$ is rectifiable curve, then $\ell(E) = 0 \Leftrightarrow \omega(E) = 0$.

Think of Q as a device costing $\varphi(Q)$ and detecting when a Brownian path hits it. How expensive to detect exit point a.s.? Riesz says linear cost $= \infty$ for rectifiable domains and $\varphi(t) = t$. What about a fractal domain?

Cost = 0 for fractals! (Lavrentiev '36, Makarov '85).

$$\limsup_{r\to 0} \frac{\omega(D(x,r))}{r} = \infty, \omega \text{a.e.}.$$





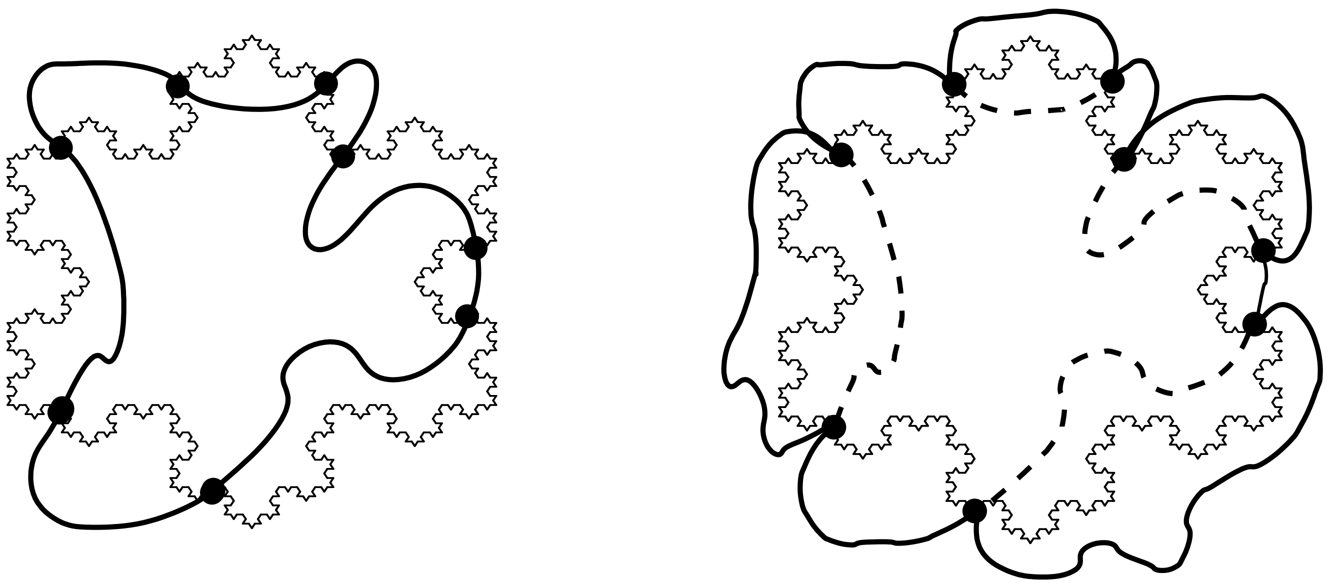
1000 walks per side, 9 of 768 sides hit from both sides.

Theorem: Harmonic measure for two sides of a closed curve are mutually singular on the non-tangent points.

Brownian motions on different sides of a fractal curve hit disjoint sets.

Detectors to find Brownian exit points on a fractal are cheap to buy but expensive to install.

Detectors for snowflake may be cheap, but installing them is expensive. They can't lie along a rectifiable path. Suppose Γ is rectifiable, $E = \Gamma \cap \partial\Omega$, $\ell(E) = 0$.



"Flip" components of $\Gamma \cap \Omega$ to outside to get rectifiable Ω' containing Ω and apply Riesz theorem:

$$\omega(E, \Omega) \leq \omega(E, \Omega') = 0.$$

Generalized Riesz Thm (Øksendal conjecture)

Bishop-Jones: If $E \subset \partial\Omega$ and E is rectifiable, then $\ell(E) = 0 \Rightarrow \omega(E) = 0$.

In a fractal domain, Brownian motion a.s. exits on a non-rectifiable set of zero length.

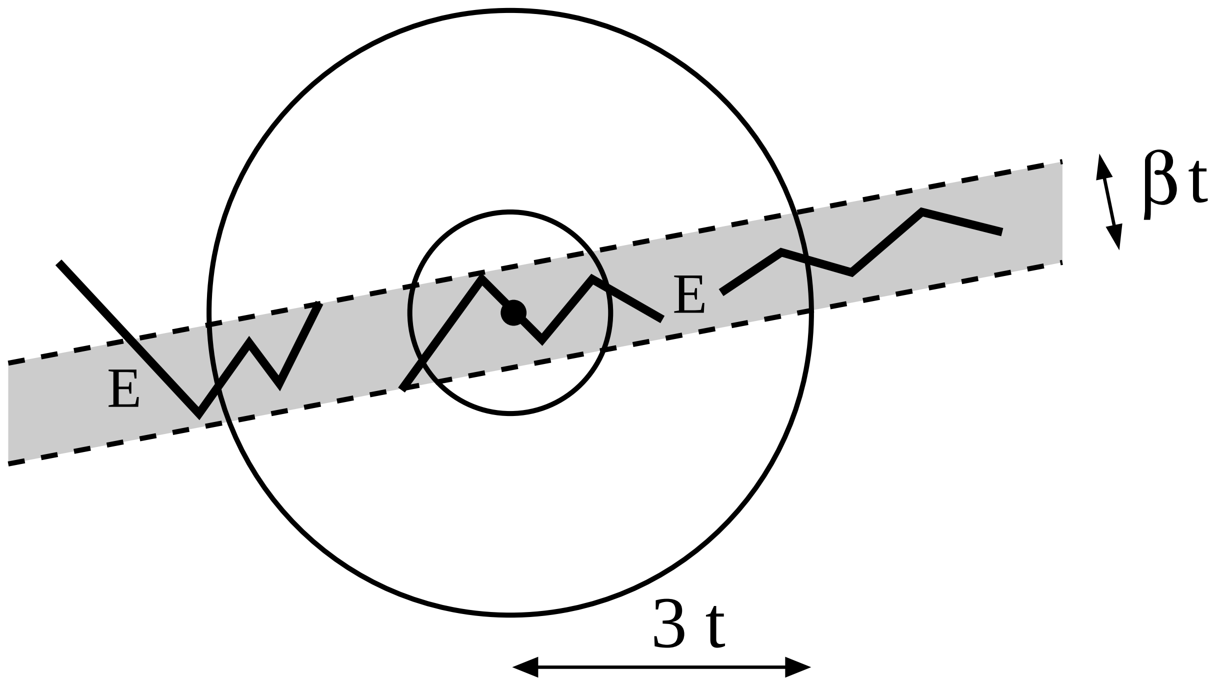
Proof uses:

- Covering map $\mathbb{D} \to \mathbb{R}^2 \setminus E$.
- L^2 theory for Schwarzian derivatives
- The traveling salesman theorem

Peter Jones β's:

$$\beta_E(x,t) = \inf_{\mathcal{L}} \max_{z \in E \cap D(x,3t)} \frac{\operatorname{dist}(z,L)}{t},$$

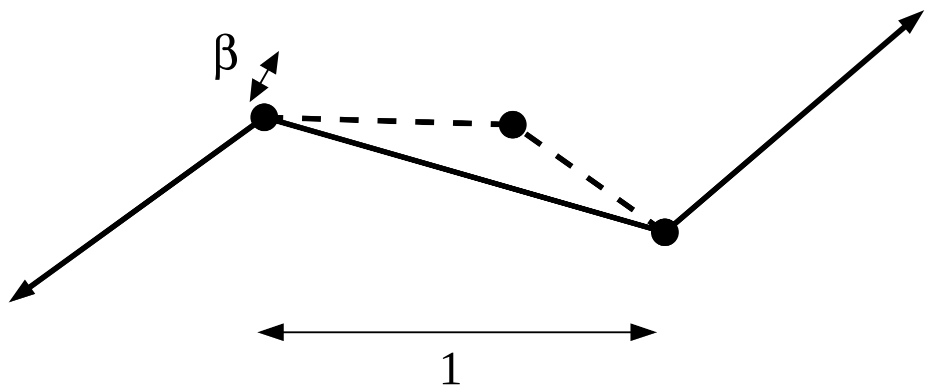
where $\mathcal{L}$ = lines hitting $D(x,t)$.

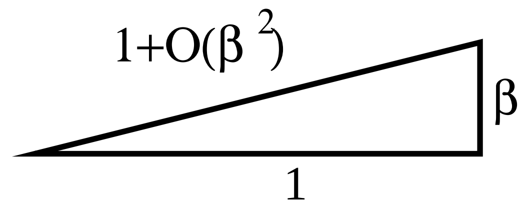


Jones' traveling salesman theorem:

E is rectifiable iff

$$\operatorname{diam}(E) + \int_0^\infty \int_{\mathbb{R}^2} \beta_E(x,t)^2 \quad \frac{dxdt}{t^2} < \infty.$$

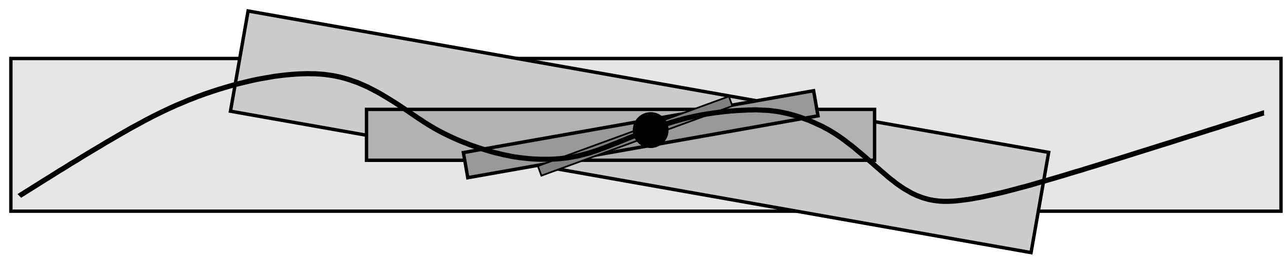




Build polygonal approximation. At scale r replace length r by length $r + O(\beta^2 r)$. Additional length is $O(\beta^2 r)$.

Thm 1: If $\int_0^1 \beta_E(x,t)^2 \frac{dt}{t} \le M$ for all $x \in E$, then E lies on a curve of length $\le Ce^M \mathrm{diam}(E)$.

Thm 2: x is tangent of γ a.e. iff $\int \beta_\gamma(x,t)^2 \frac{dt}{t} < \infty$.

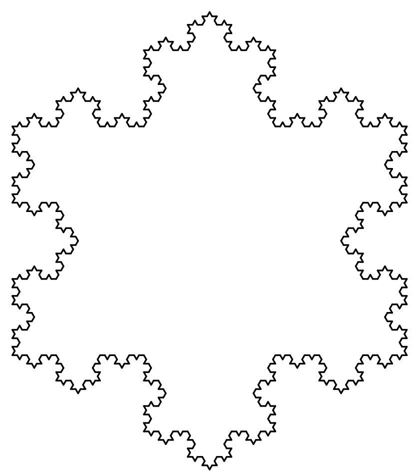


Thm 3: If E is connected and $\beta_E(x,t) > \epsilon$ for all $x \in E$ and $0 < t < \mathrm{diam}(E)$ then $\dim(E) > 1 + C\epsilon^2 > 1$.

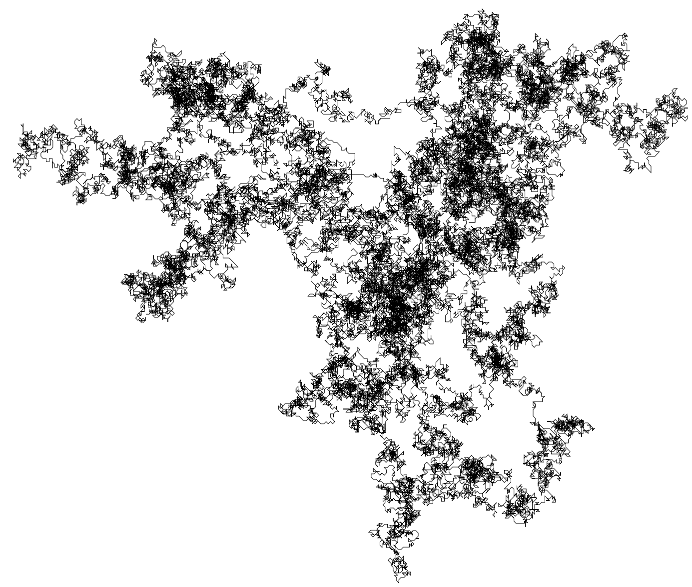
φ-cost of $E = \min \sum \varphi(Q_k)$, $E \subset \cup Q_k$.

$\varphi(Q) = \mathrm{diam}(Q)$ is length, $\varphi(t) = t^2$ is area.

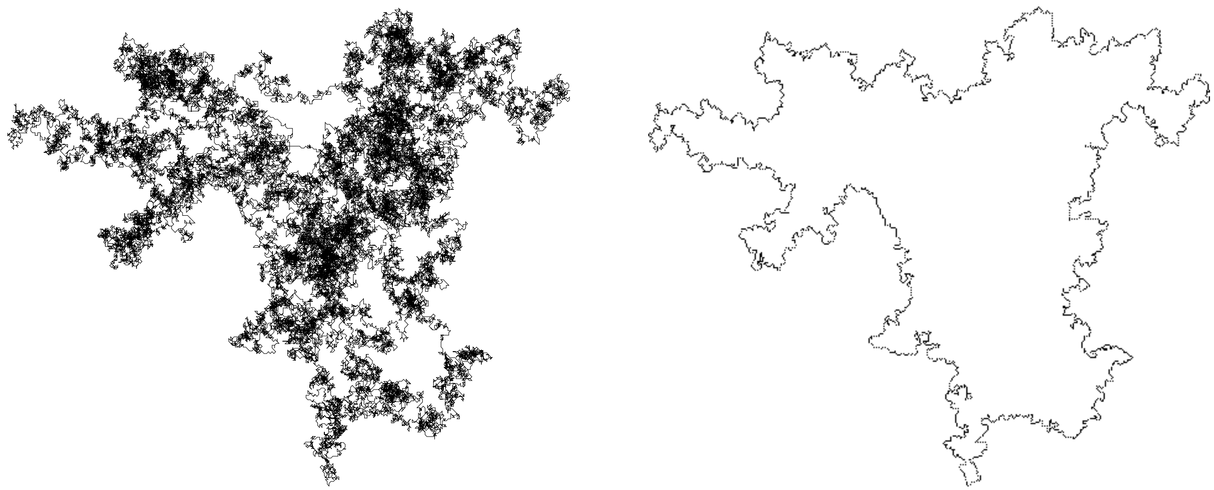
The t^α-cost of covering E infinitely often is 0 or ∞. Sharp α gives **Hausdorff dimension**.



Von Koch Snowflake
HD$= \log 4/\log 3$



Brownian path, HD=2
$t^2 \log \frac{1}{t} \log\log \frac{1}{t}$



Frontier of Brownian motion

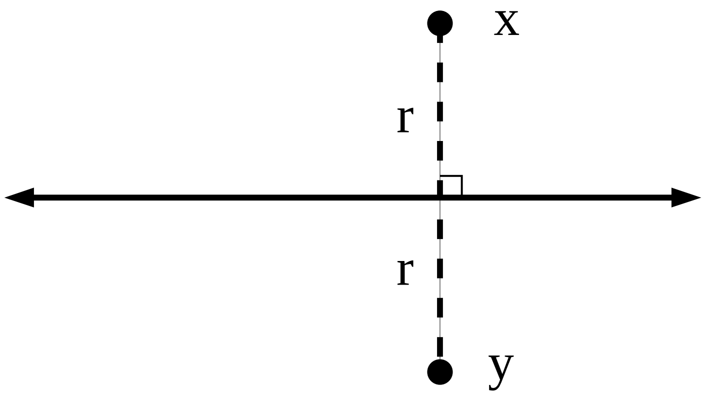
Mandelbrot conjectured it has dimension 4/3 based on physical arguments (Flory '49, Gennes '91).

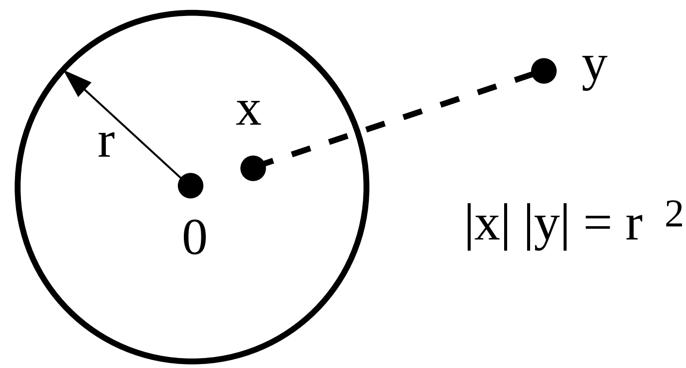
Bishop-Pemantle-Peres: $\dim(\text{Frontier}) > 1 + \epsilon$.

dim = 4/3 proven by Lawler-Schramm-Werner. Also related to work of Aizenmann.

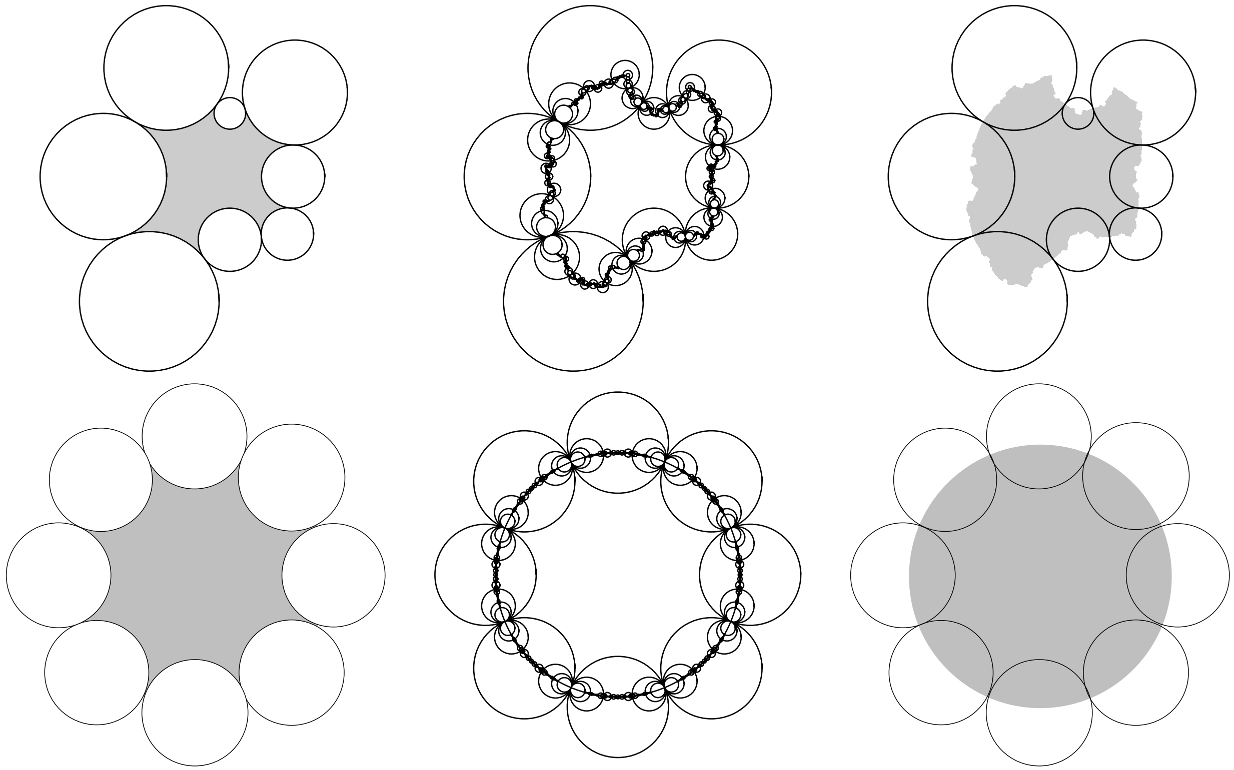
Möbius transformations are maps $z \rightarrow \frac{az+b}{cz+d}$.

This group is generated by reflections in lines and circles.





Mostow rigidity, Bowen Dichotomy

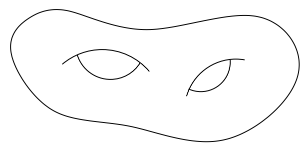


If $\partial\Omega$ is connected and Ω is preserved by a discrete Möbius group, prove $\Omega =$ disk or $\dim(\partial\Omega) > 1$.

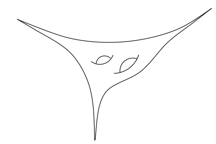
Mostow, Agard, Bowen, Sullivan, Bishop-Jones, Astala-Zinsmeister, Bishop

Theorem: Ω = disk or $\dim(\partial\Omega) > 1$.

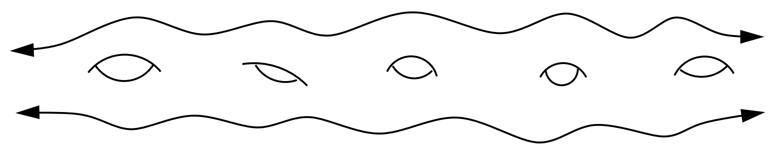
- True if $R = \Omega/G$ is compact (Rufus Bowen)
- True if R is finite area (Dennis Sullivan)
- False if Brownian motion is transient on R (A-Z).
- True iff Brownian motion is recurrent on R.



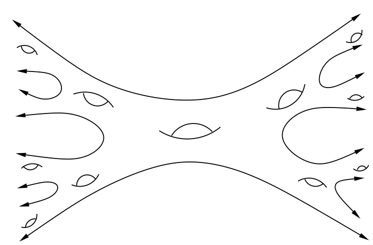
Compact



Finite area



Brownian motion recurrent (divergence type)

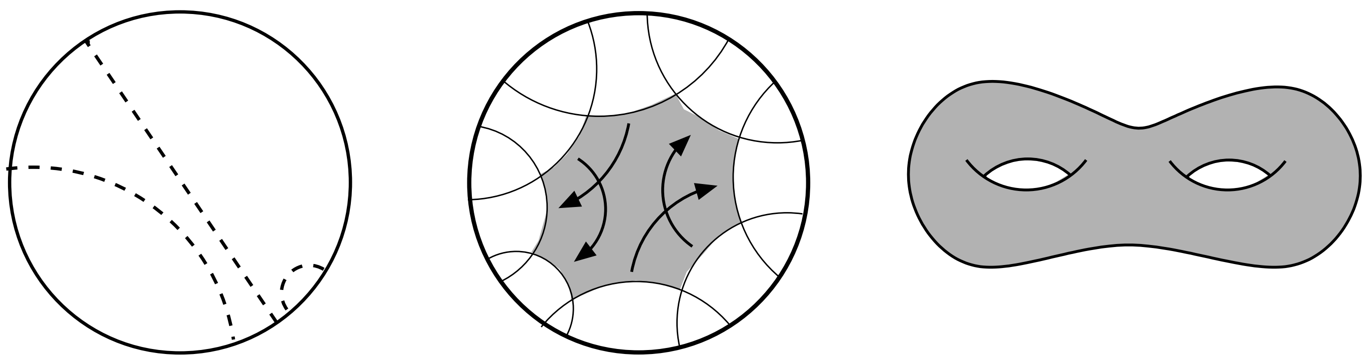


Brownian motion transient (convergence type)

Hyperbolic metric on disk given by

$$d\rho = \frac{ds}{1-|z|^2} \simeq \frac{ds}{\mathrm{dist}(z, \partial D)}.$$

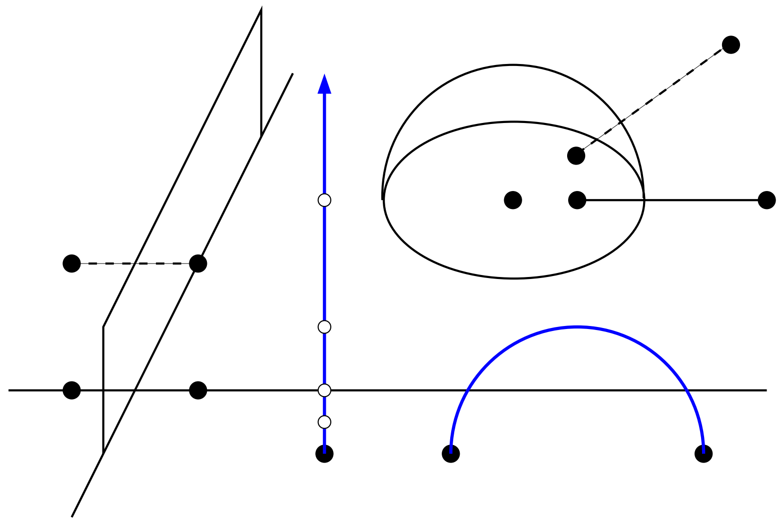
- Isometries are the Möbius transformations.
- Geodesics are circles perpendicular to boundary.
- Volume grows exponentially



Hyperbolic metric on other planar domains defined using conformal map from disk. Still have

$$d\rho \simeq \frac{ds}{\mathrm{dist}(z, \partial D)}.$$

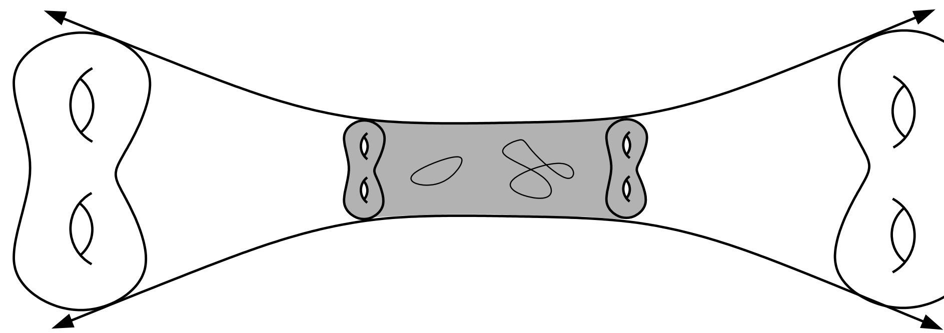
Reflections in $\mathbb{R}^2$ extend to upper 3-space, $\mathbb{R}^3_+$.



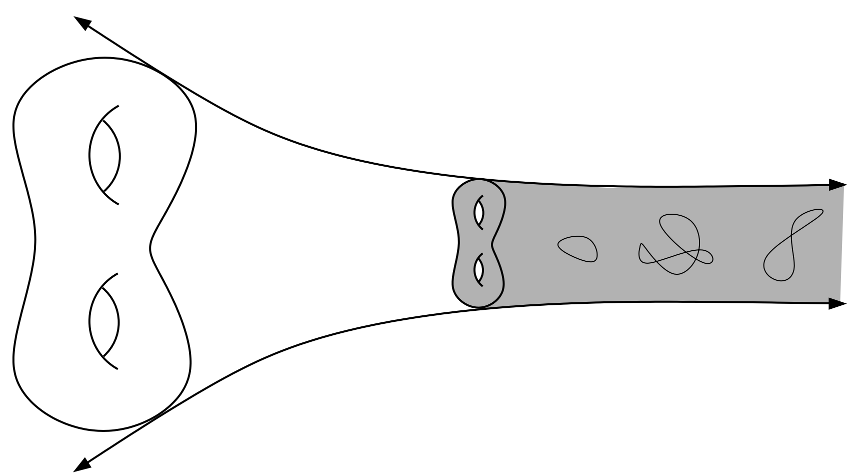
- Hyperbolic metric $d\rho = \frac{ds}{\mathrm{dist}(z,\mathbb{R}^2)}$.
- Möbius transformations = Isometries.
- Geodesics are circles perpendicular to boundary.

Hyperbolic 3-manifold is $M = \mathbb{R}^3_+/G$, G a discrete group of isometries.

Convex hull of closed geodesics is the **convex core**.



Geometrically Finite, vol(core) $< \infty$

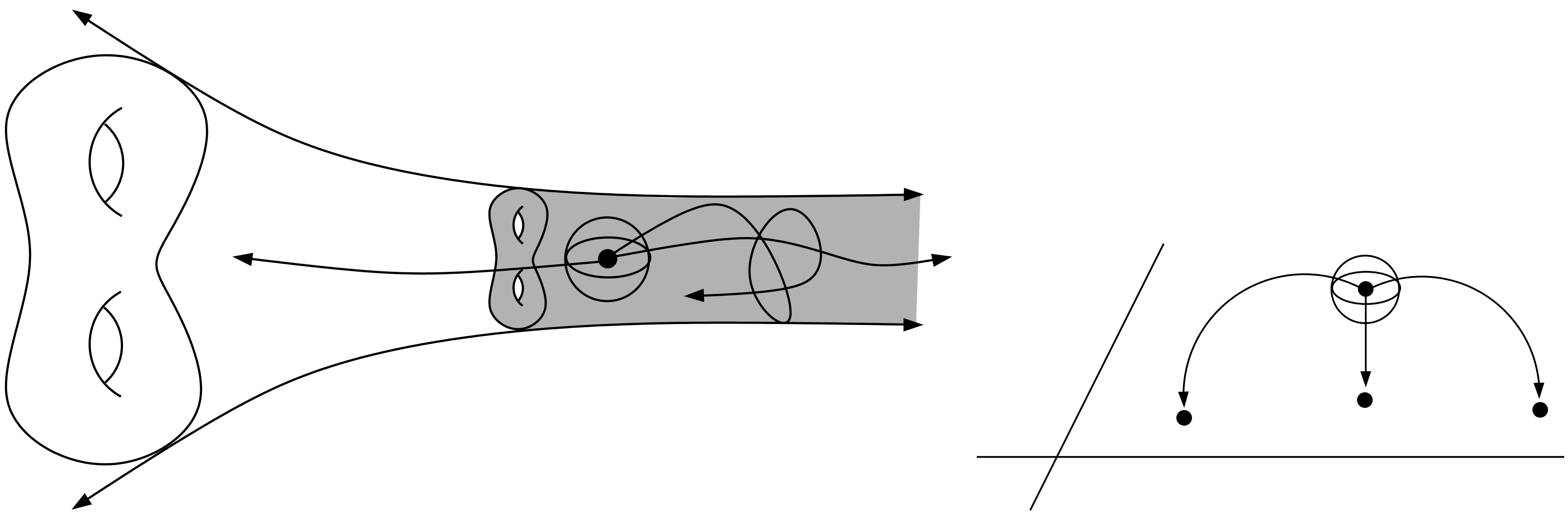


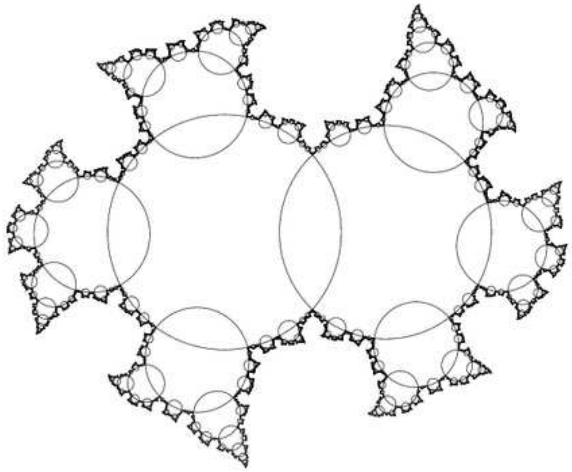
Geometrically Infinite, vol(core) $= \infty$

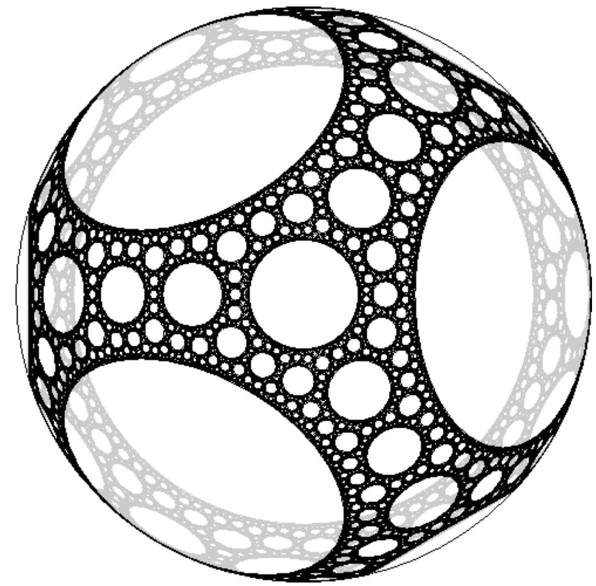
The limit set, Λ

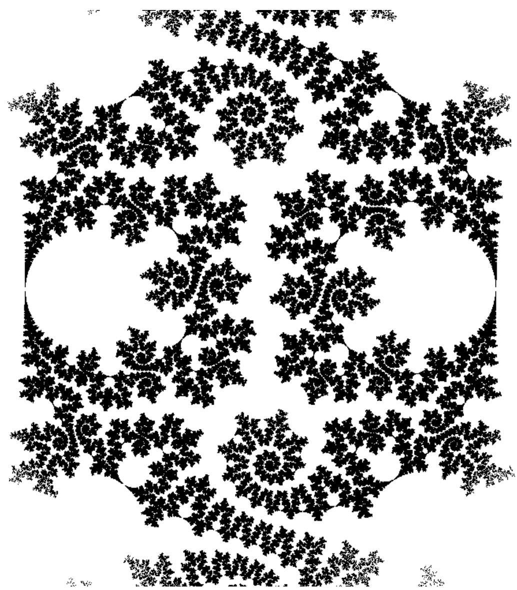
Fix base point in M. Choose direction. Follow geodesic.

- stays in bounded region forever, $\Lambda_b \subset \Lambda$
- stays in convex core forever, Λ (= limit set)
- eventually leaves convex core, $\Omega = \mathbb{R}^2 \setminus \Lambda$.









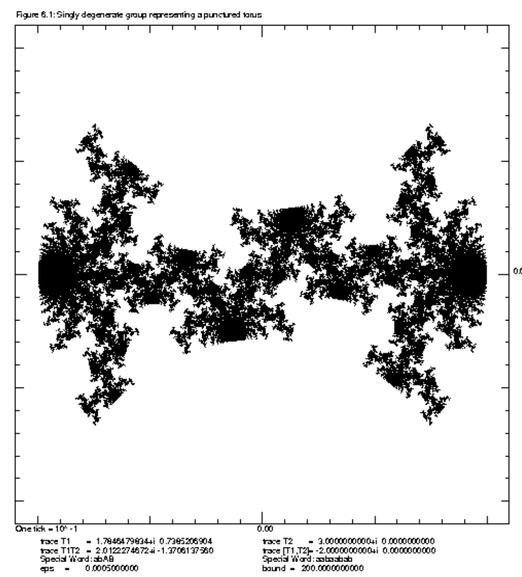


Figure 6.1: Singly degenerate group representing a punctured torus
One tick = 10^ -1
0.00
trace T1 = 1.7846479834+i 0.7385206904
trace T1T2 = 2.0122274672+i -1.3706137560
Special Word : abAB
eps = 0.0005000000
trace T2 = 3.0000000000+i 0.0000000000
trace [T1,T2]= -2.0000000000+i 0.0000000000
bound = 200.0000000000

Suppose G is finitely generated.

Sullivan: If G is geo. finite then $\dim(\Lambda) < 2$.

Bishop-Jones: If G is geo. infinite then $\dim(\Lambda) = 2$.

Cor: If $G_n \to G$ then $\limsup \dim(\Lambda_n) \geq \dim(\Lambda)$.

Cor: Λ is either a Cantor set, a circle or $\dim(\Lambda) > 1$.

Kholodenko, Arkady L.
Boundary conformal field theories, limit sets of Kleinian groups and holography.
Journal of Geometry and Physics.
35 (2000), no. 2-3, 193–238.

The concept of field theories being represented by observations at the boundary of space-time is evident both in S-matrix theory and in the definition of conserved quantities in gravity. The AdS/CFT correspondence, which provides an interpretation of bulk field theories in anti-de Sitter space-times in terms of boundary conformal field theories . . .

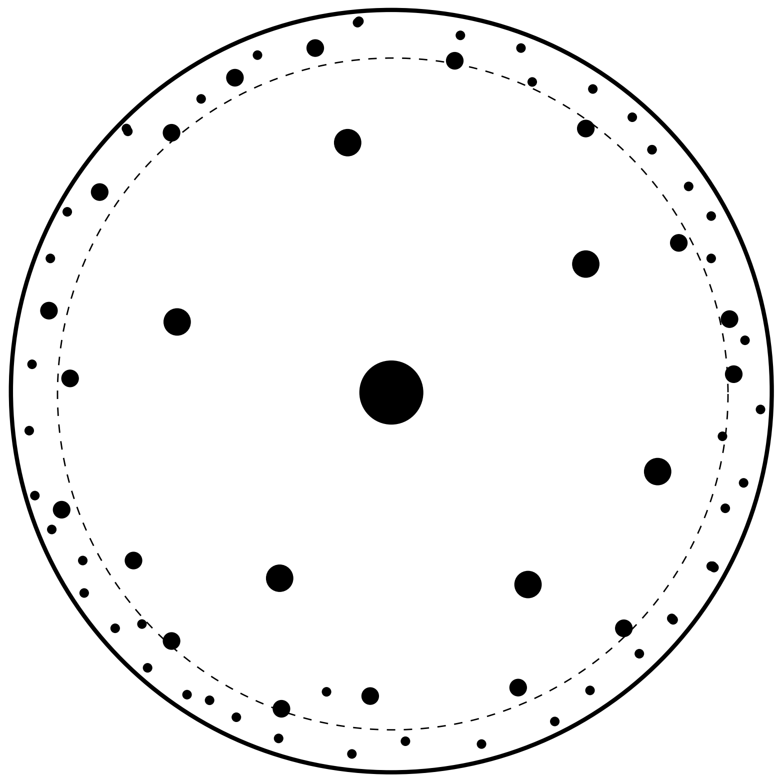
⋮

The Hausdorff dimension of the limit set, in turn, is known to be equal to the value of the exponent separating convergence and divergence of the Poincare series of the Kleinian group Γ.

⋮

Given $M = \mathbb{R}^3_+/G$ define **critical exponent**

$$\delta = \limsup \frac{1}{n} \log \#\{g \in G : \rho(x_0, g(x_0)) < n\}$$



Theorem: For any non-trivial hyperbolic manifold

$$\delta = \dim(\Lambda_b) \leq \dim(\Lambda).$$

Equality for finitely generated groups.

Due to Sullivan in geo. finite case. Later generalized to variable curvature, symmetric spaces, Gromov hyperbolic groups, quasiconformal groups ....

Used in proof of 'Geometrically infinite' $\Rightarrow \dim(\Lambda) = 2$.

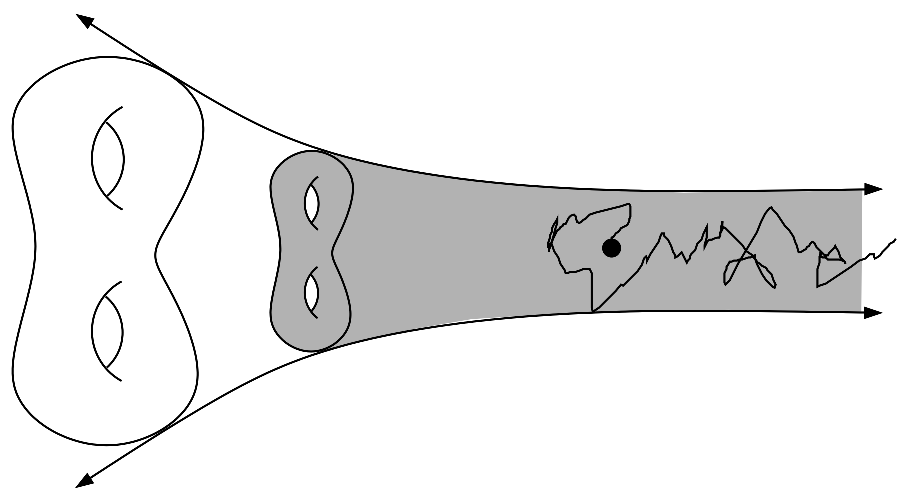
If $\delta = 2$, done. Assume $\delta < 2$. Show $\text{area}(\Lambda) > 0$.

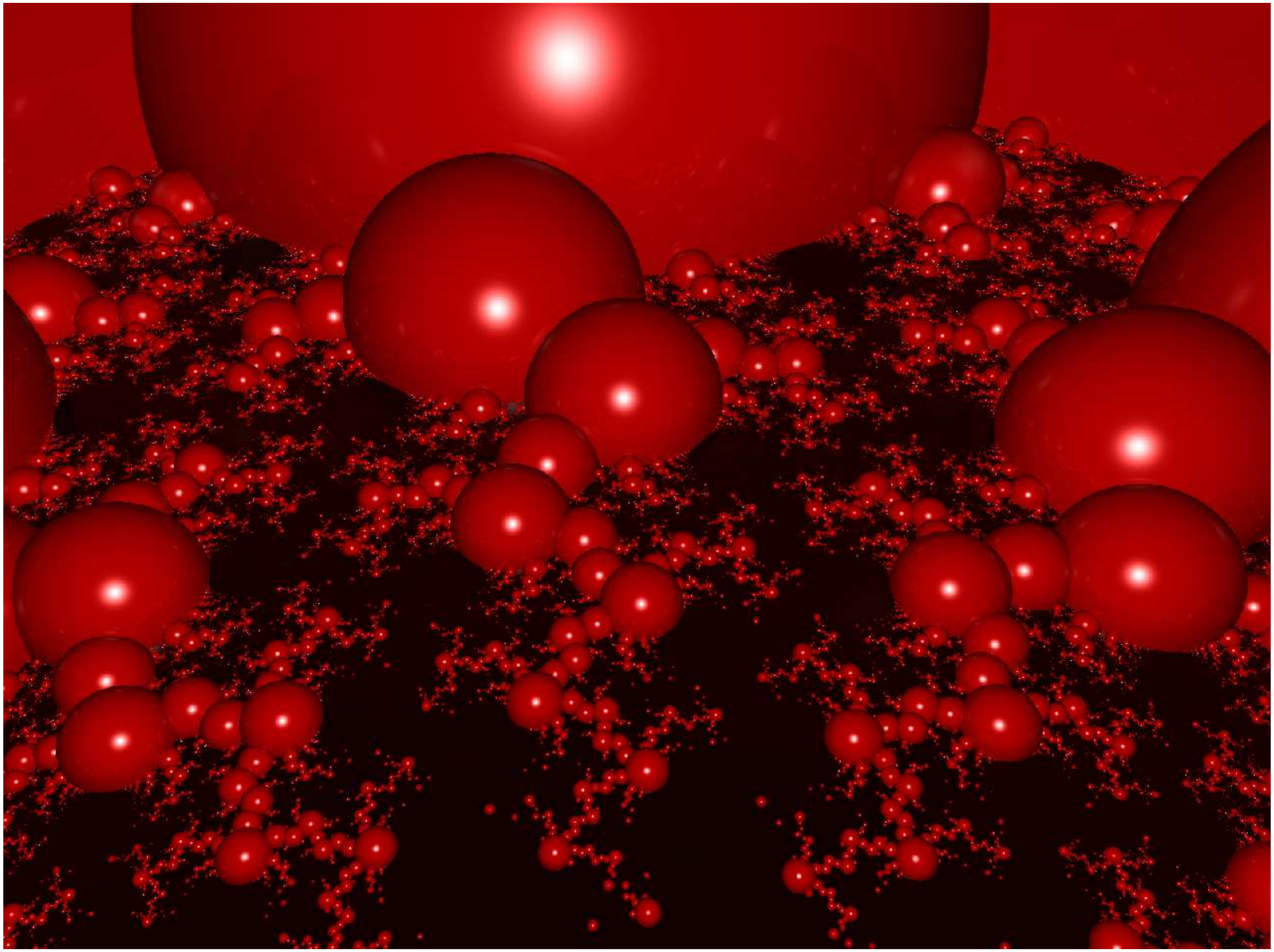
$$k = \text{heat kernel} = \text{Prob}(B(t) = y : B(0) = x).$$

$$k(x, y, t) \le C_{x,y}\ \rho \exp(-c\rho^2/t), \quad t \le \rho$$

$$k(x, y, t) \le C_{x,y}\ \exp(-t\,\delta(2-\delta)), \quad t > \rho.$$

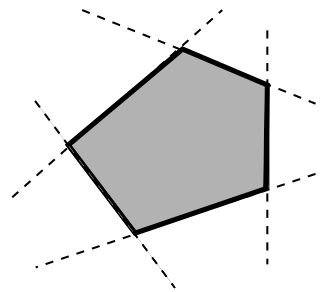
where $\rho = \rho(x, y)$. If $\text{dist}(x, \partial C(M)) \gg 1$ then Brownian path started at x doesn't hit $\partial C(M)$. $\Rightarrow$ area$(\Lambda) > 0 \Rightarrow \dim(\Lambda) = 2$.

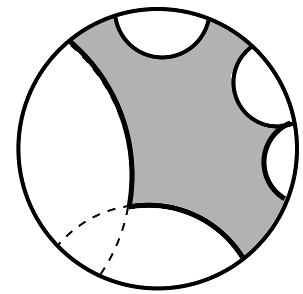




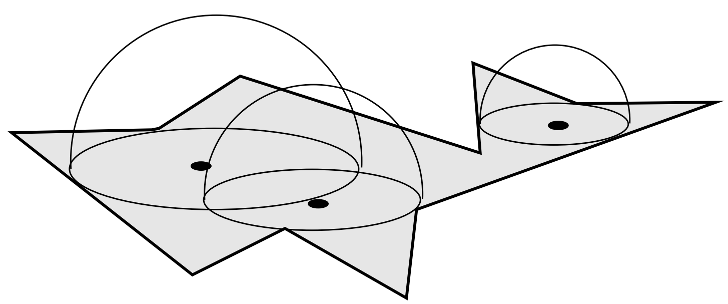
Inside convex core of a geometrically infinite group.

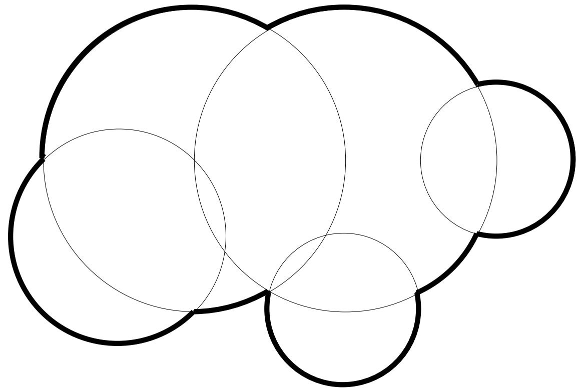
In Euclidean space, a convex set is intersection of half-planes. Same for hyperbolic space, but half-planes are bounded by circles or hemispheres.

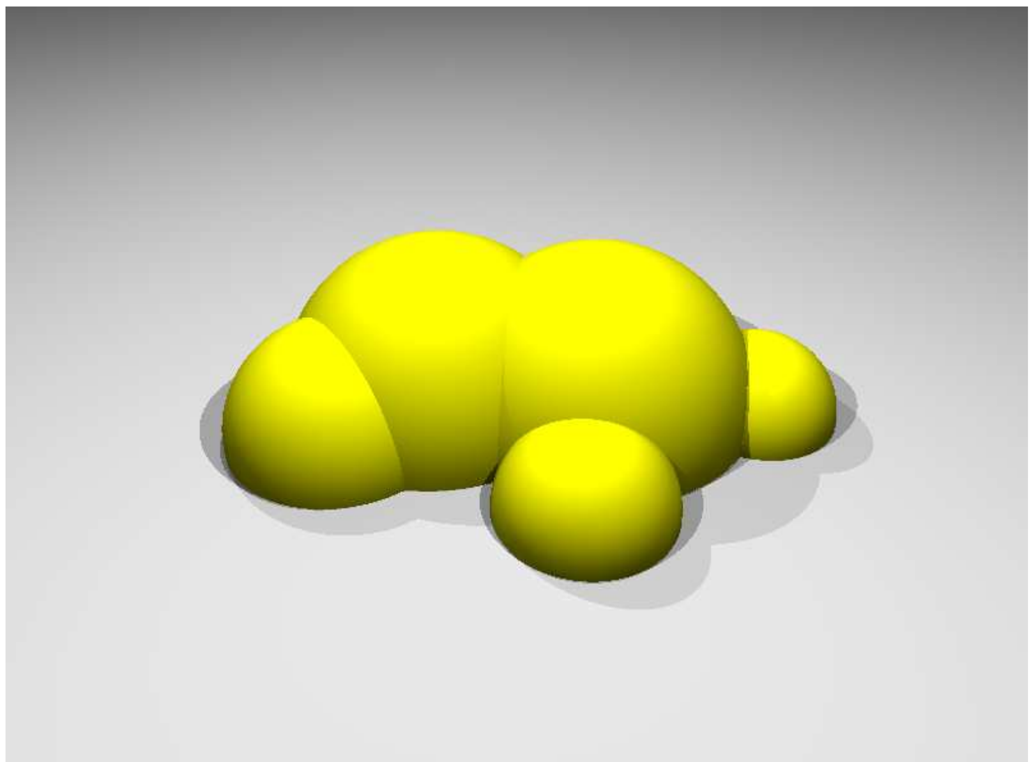


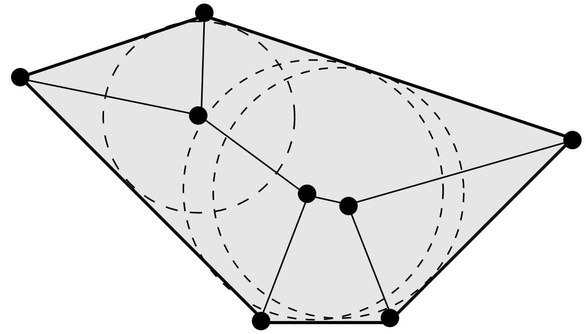


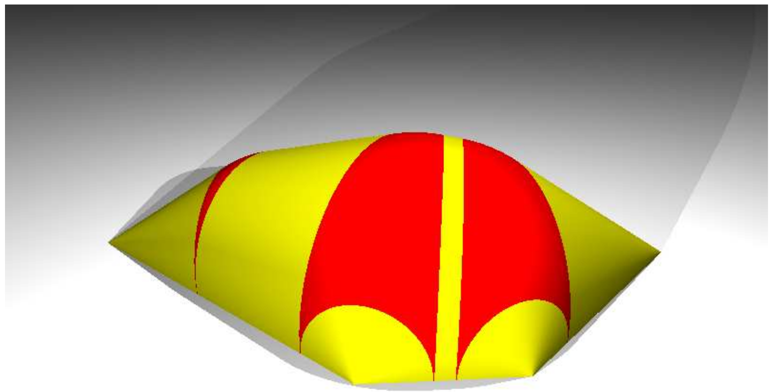
Take union of all hemispheres whose bases lie inside Ω. Upper envelope is the "dome" of Ω. Is boundary of hyperbolic convex hull of Ω^c.





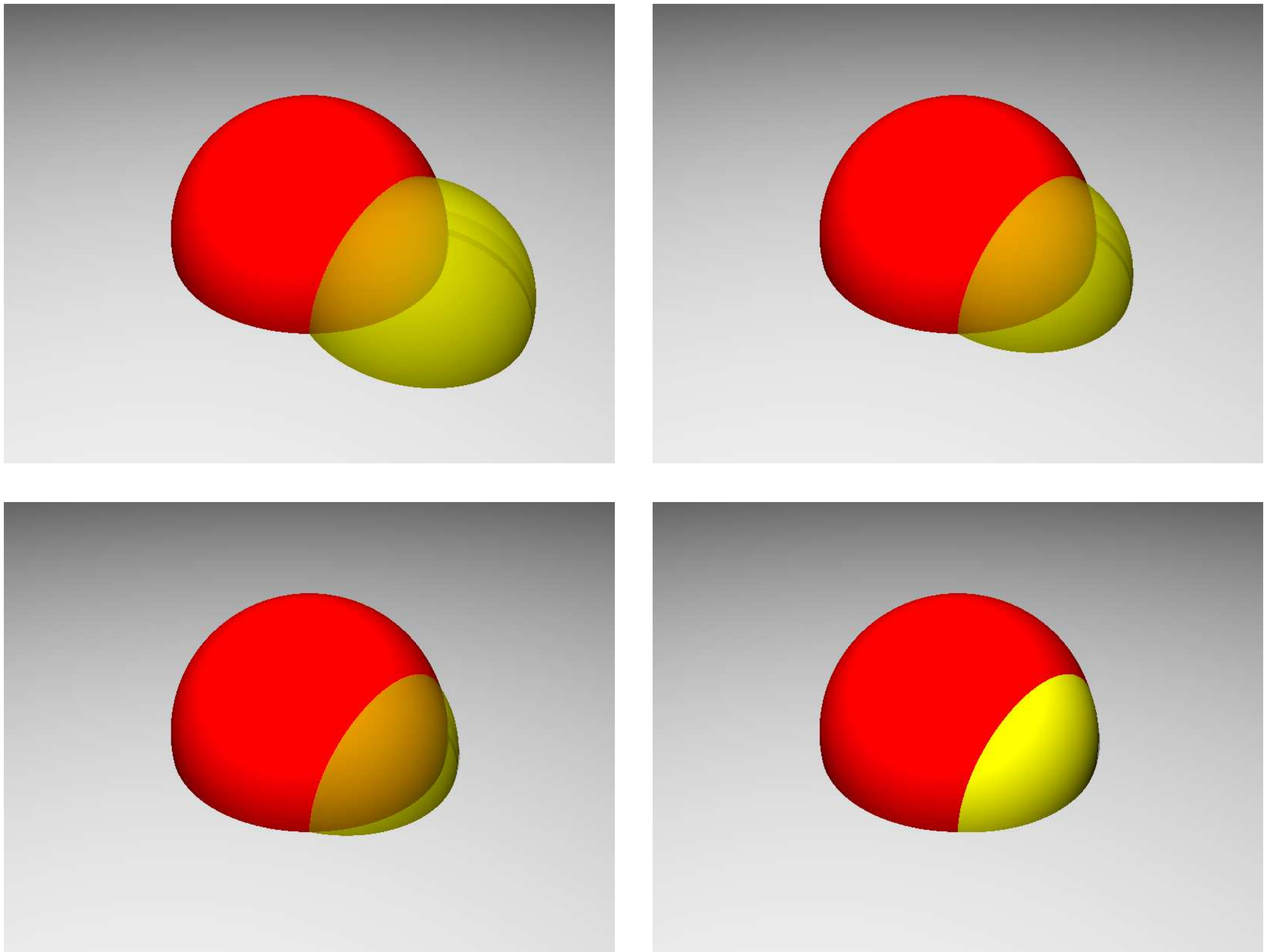


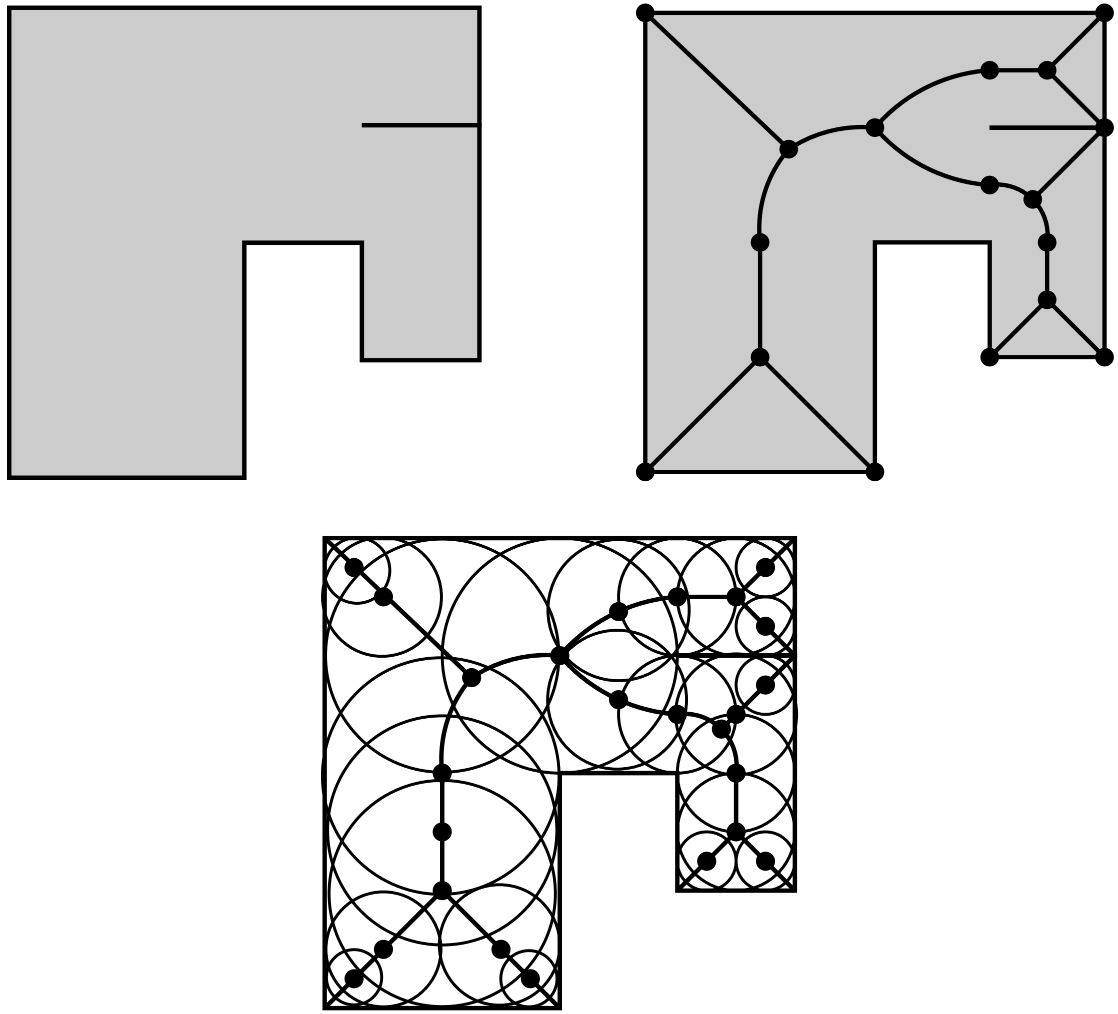




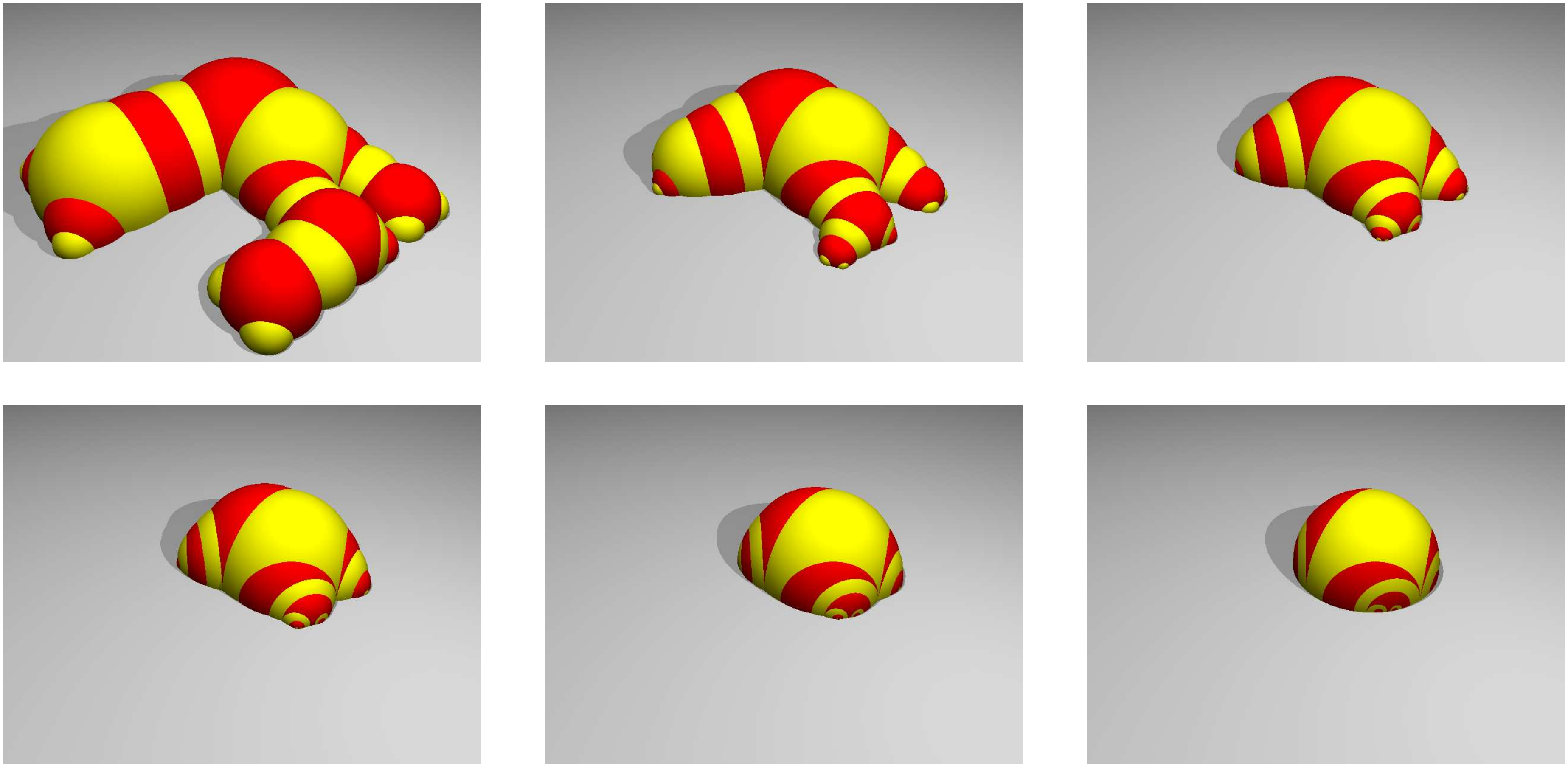
Medial Axis = centers of internal disks hitting boundary in at least two points.

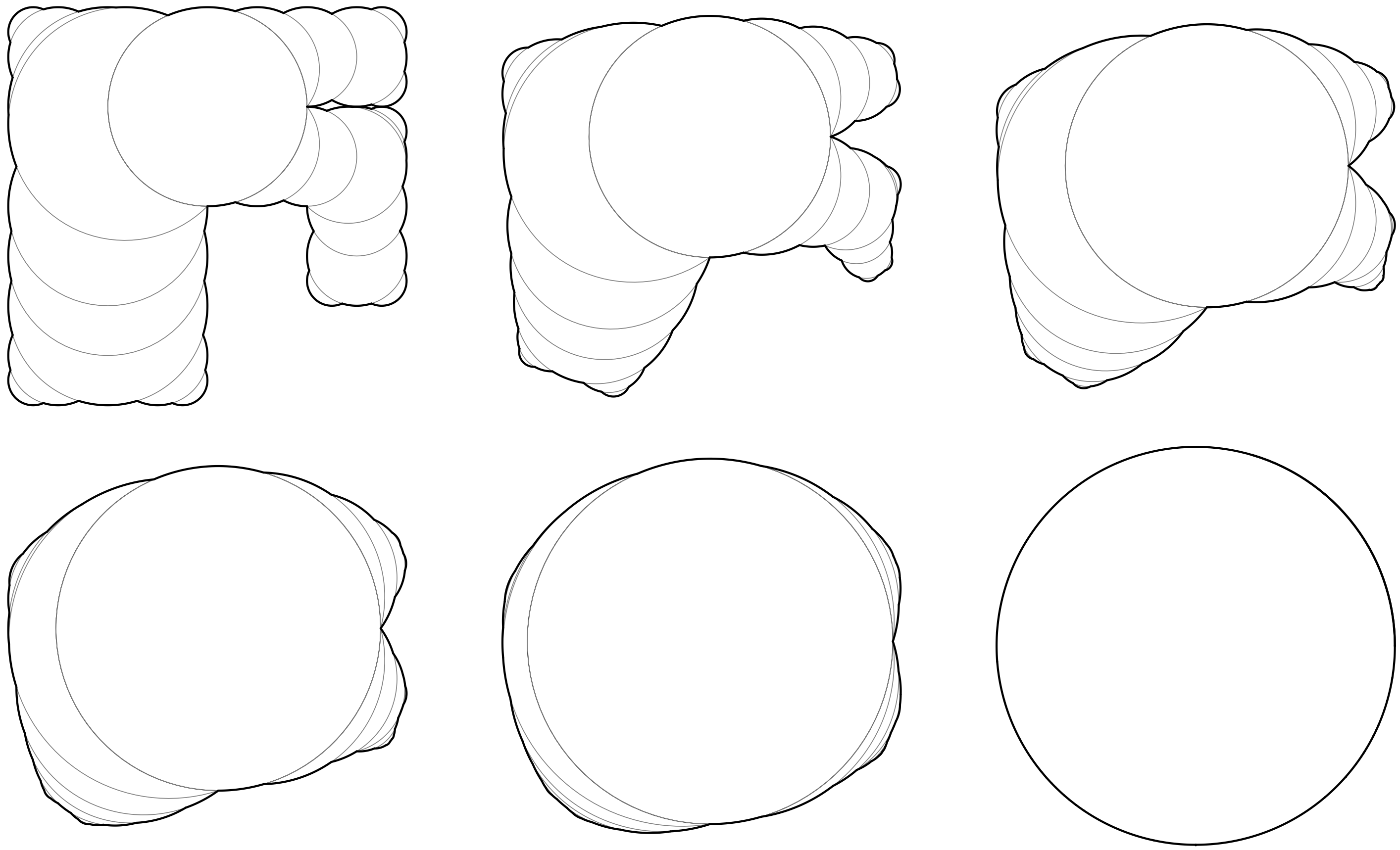
It is easy to map any dome conformally to a disk.





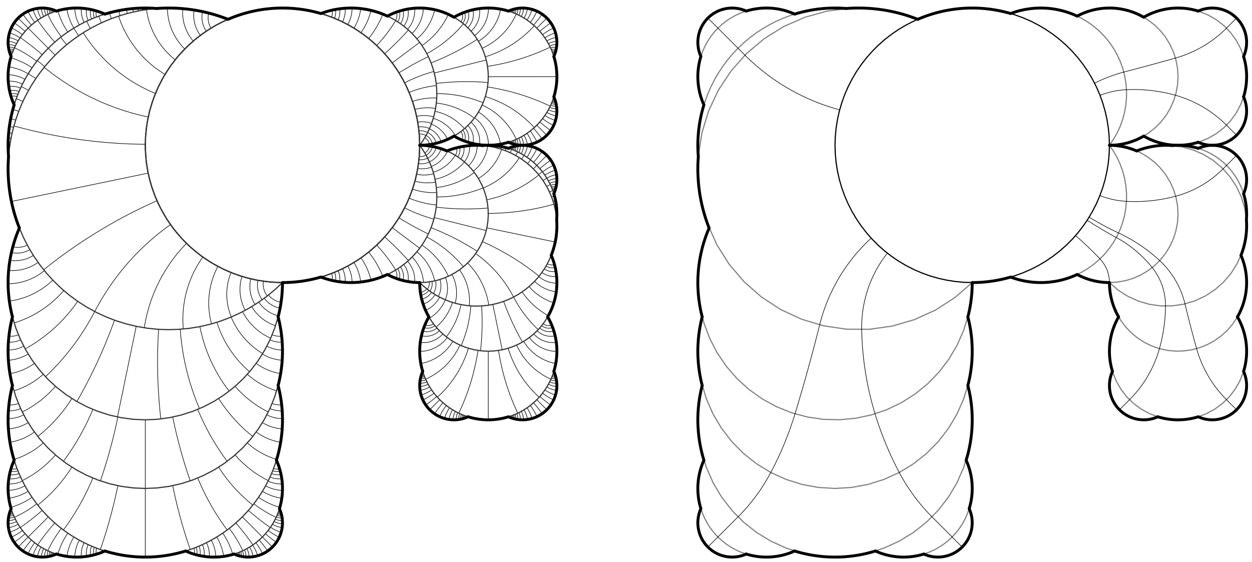
A polygon, medial axis, approximation by disks.





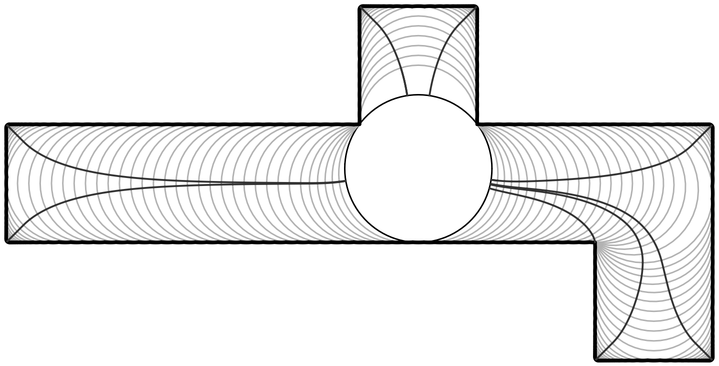
Angle scaling family

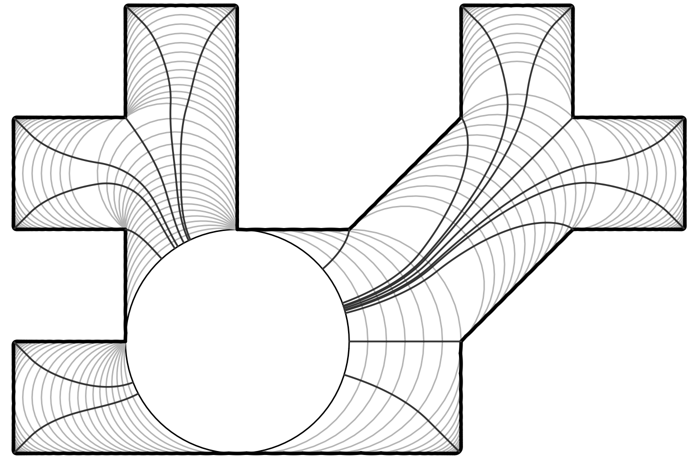
Instead of collapsing all crescents at once, we may do one at a time, from leaves towards root.

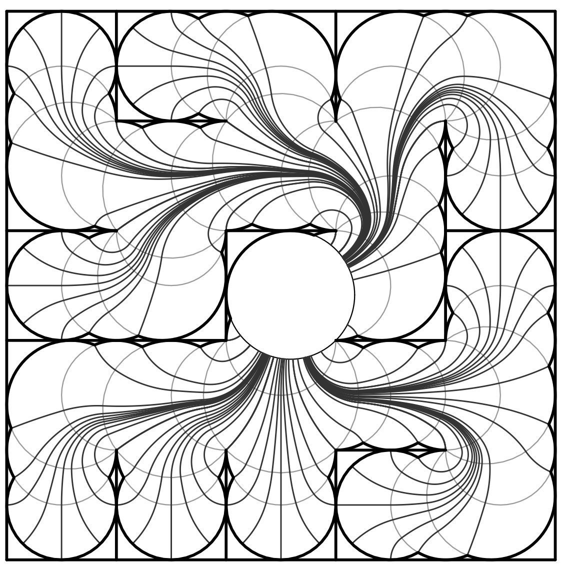


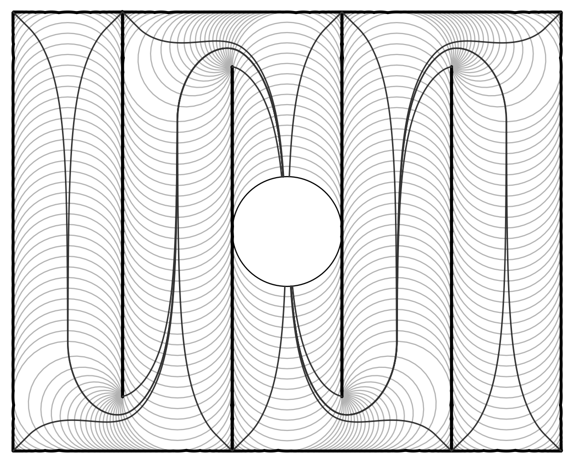
Defines a flow from boundary to disk along foliation of crescents by orthogonal arcs.

Taking limits, this flow exists for any domain.





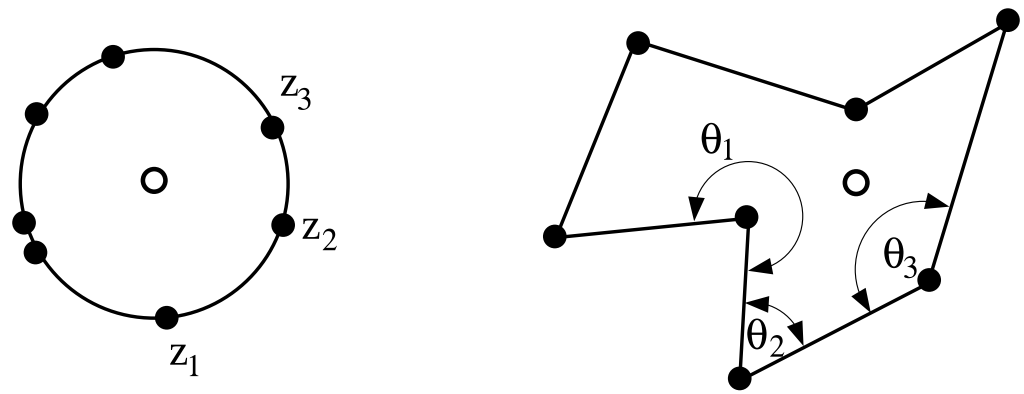




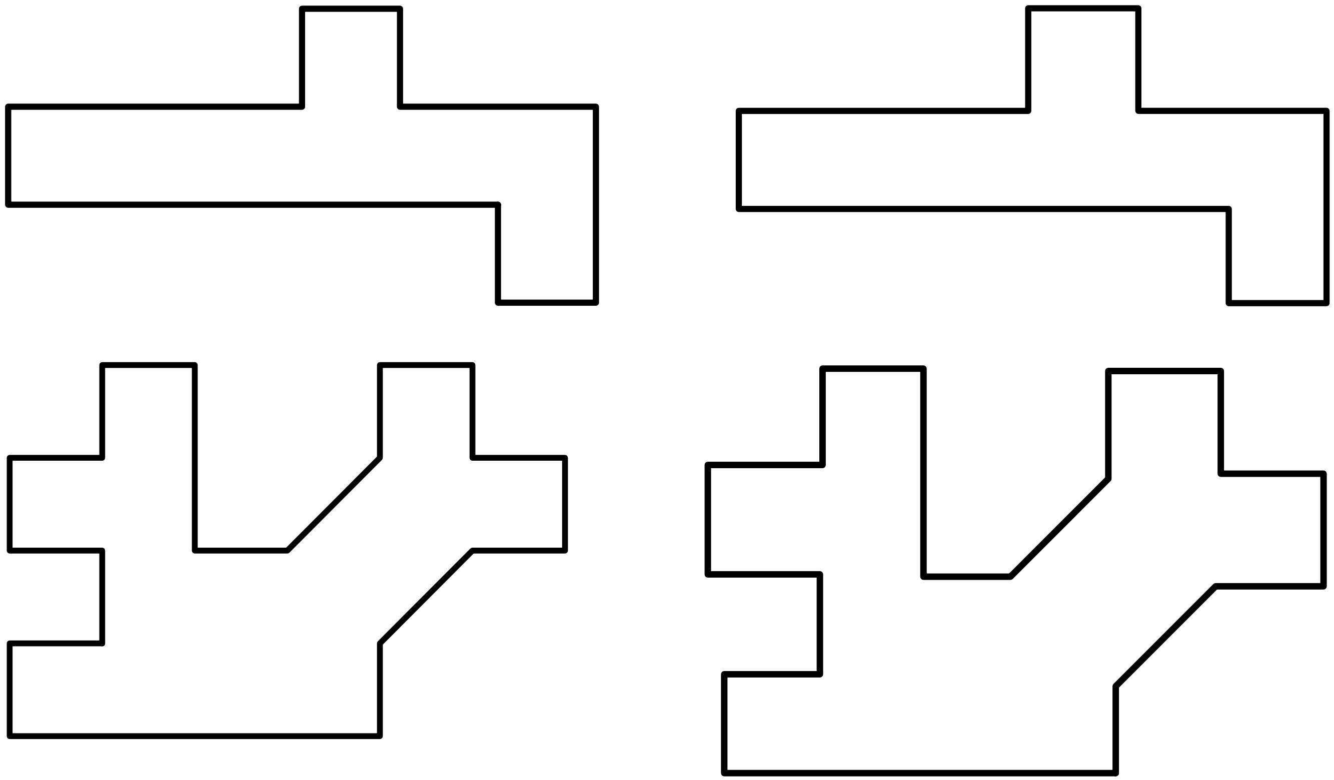
Schwartz-Christoffel, 1867: conformal map to polygon with interior angles $\{\theta_k\}$

$$f'(w) = \prod_{k=1}^{n} (1 - \frac{w}{z_k})^{\frac{\theta_k}{\pi} - 1},$$

where $\{z_k\} \subset \partial\mathbb{D}$ map to vertices.

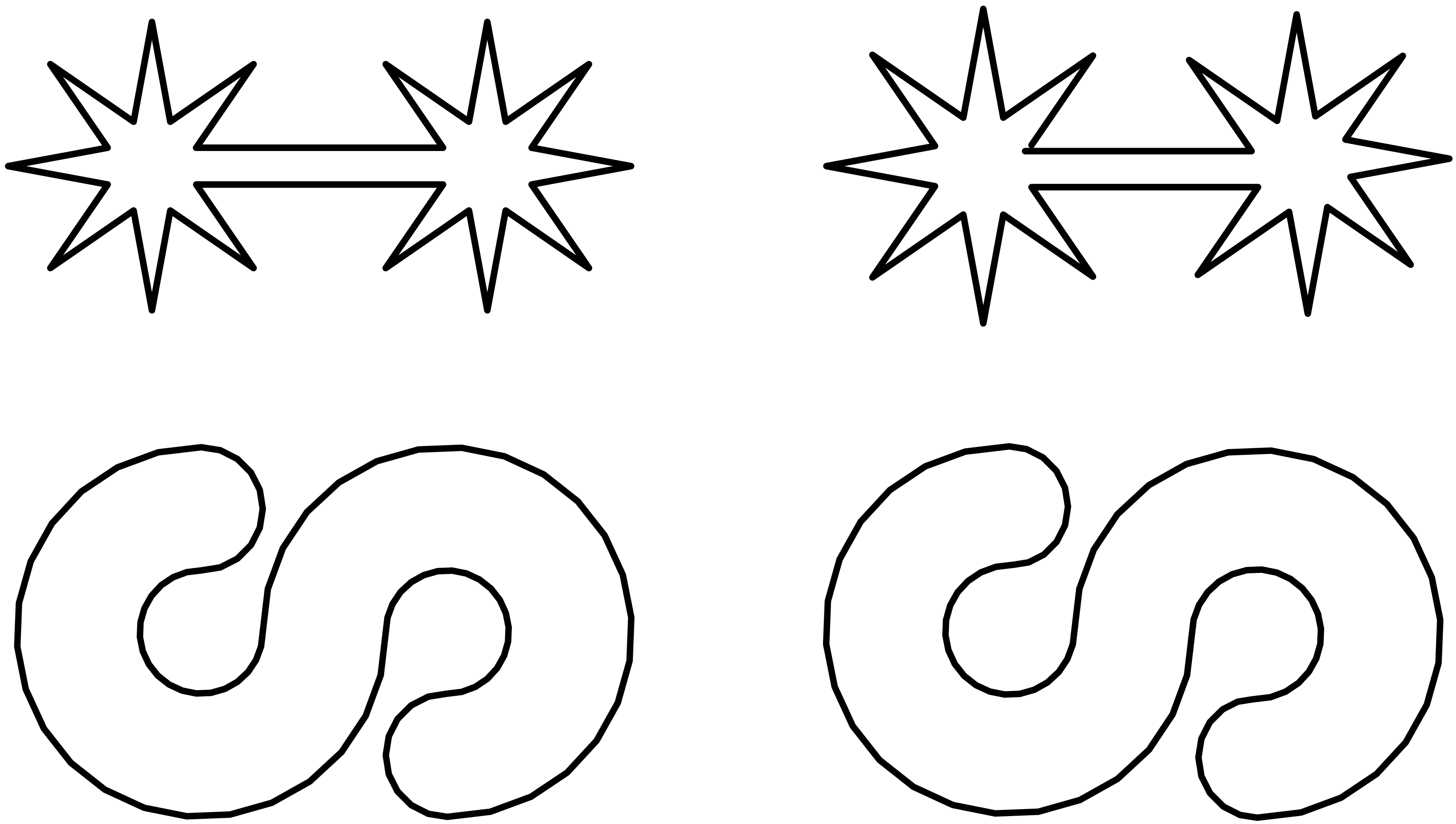


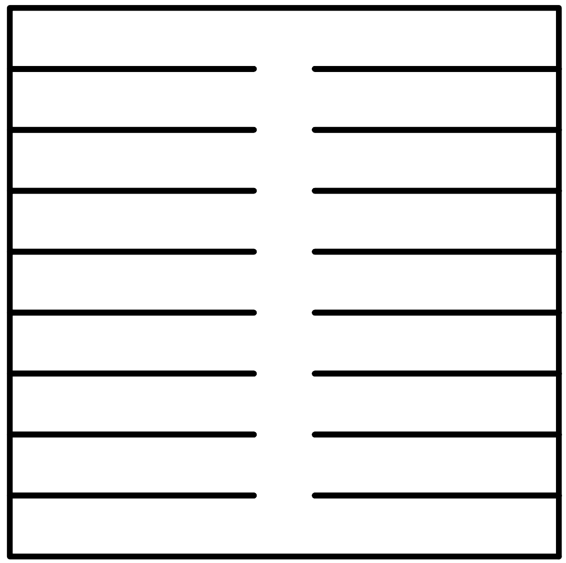
Circular? How do we know $\{z_k\}$ without f? Use the medial axis flow to guess parameters.

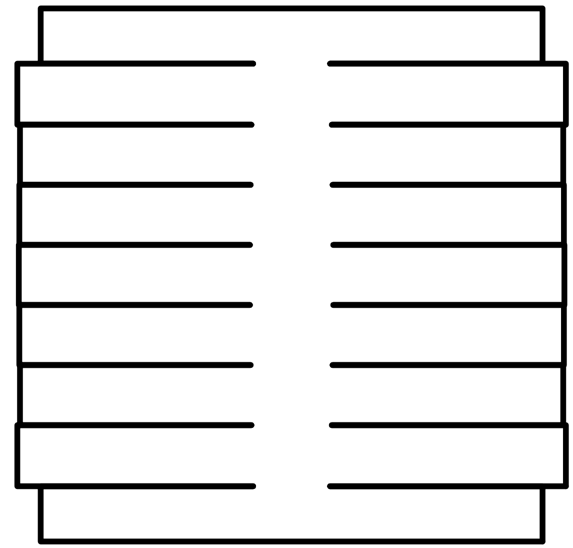


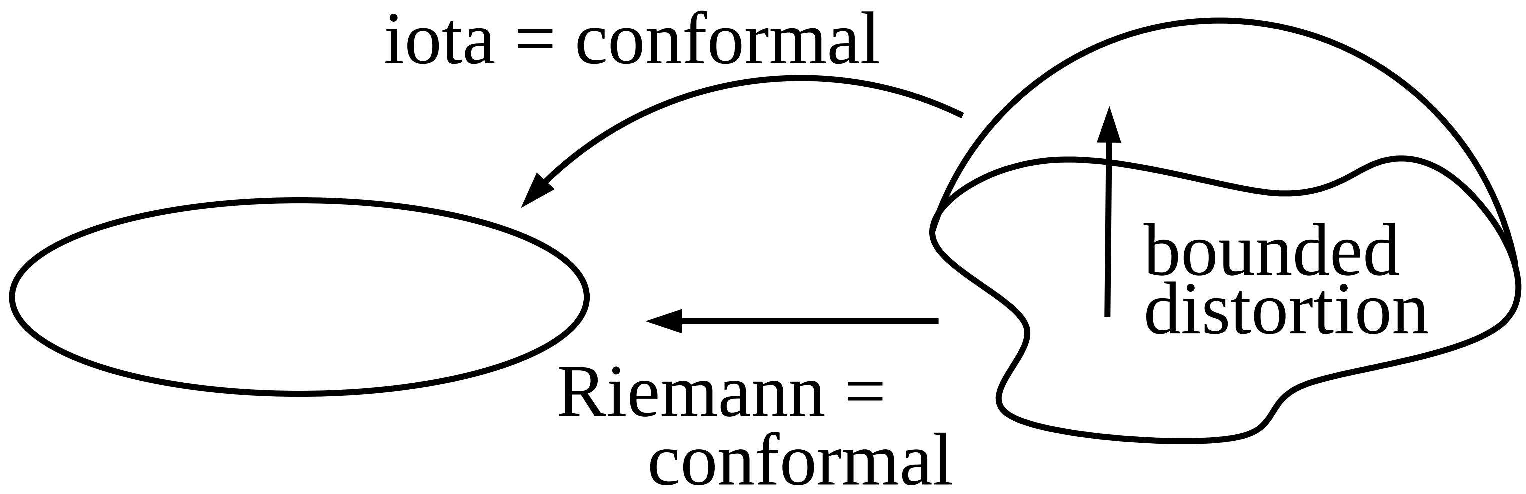
Target Polygons Guessed SC images

MA flow gives "formula" for SC-parameters $\{z_k\}$ which is correct with uniformly bounded error.

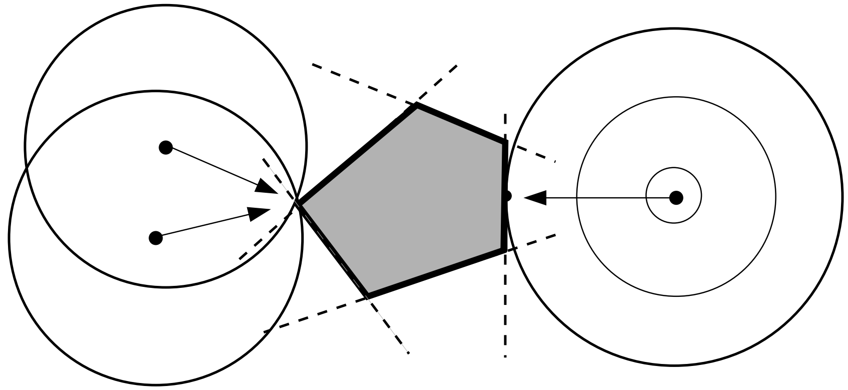






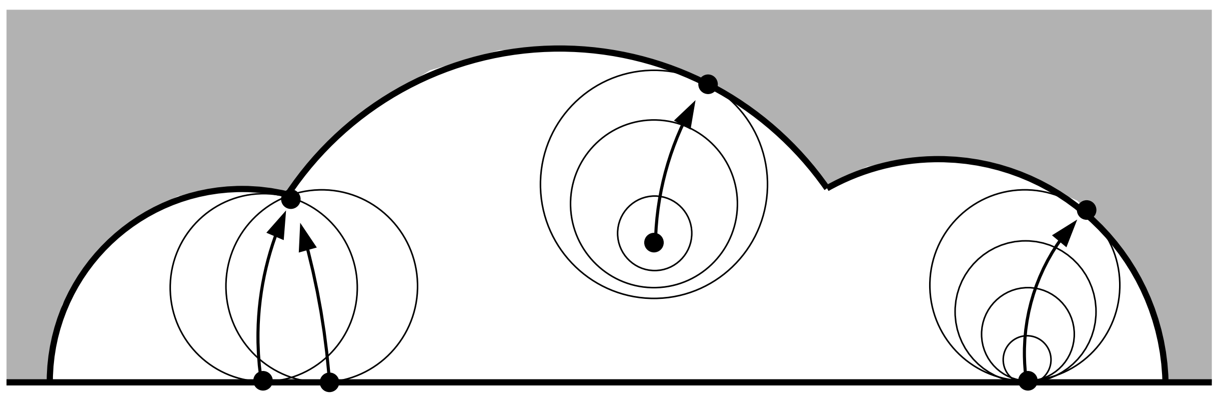

iota = conformal
bounded
distortion
Riemann =
conformal

Nearest point map in $\mathbb{R}^n$ is Lipschitz.



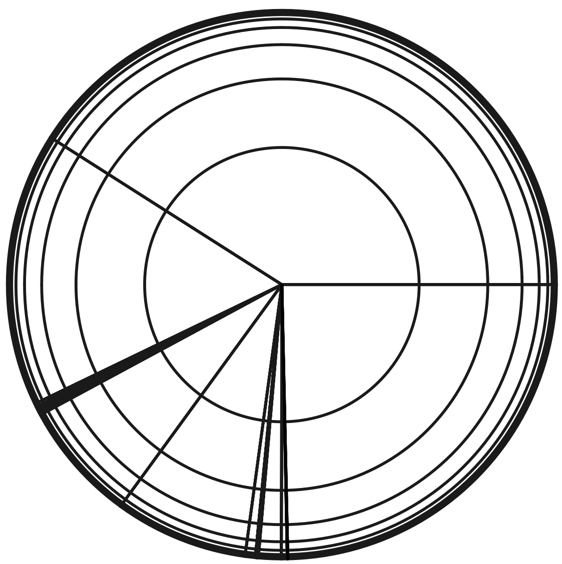
NPM in hyperbolic space extends to boundary and is a quasi-isometry (Sullivan, Epstein-Marden, Bishop)

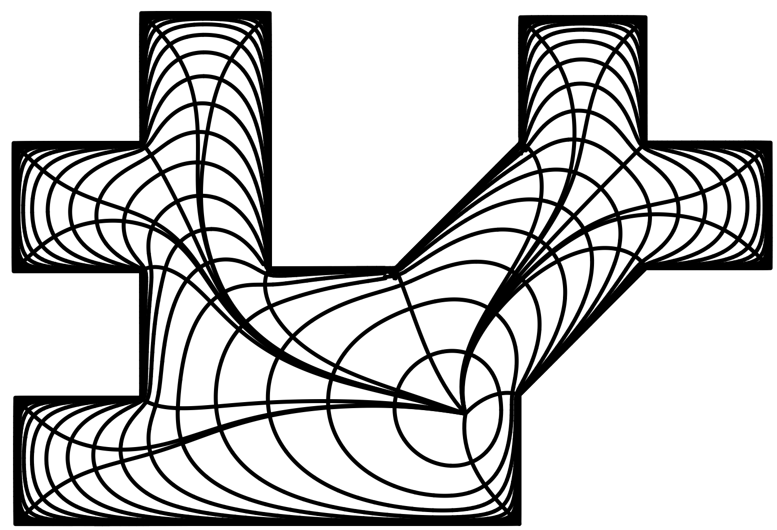
$$\rho(x,y)/A - B \leq \rho(R(x),R(y)) \leq A\rho(x,y).$$

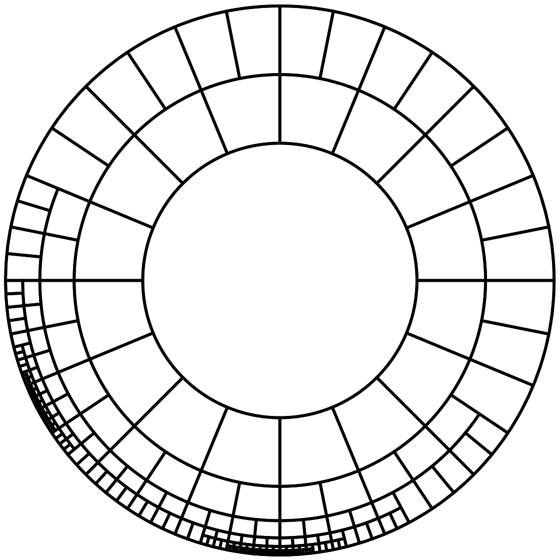


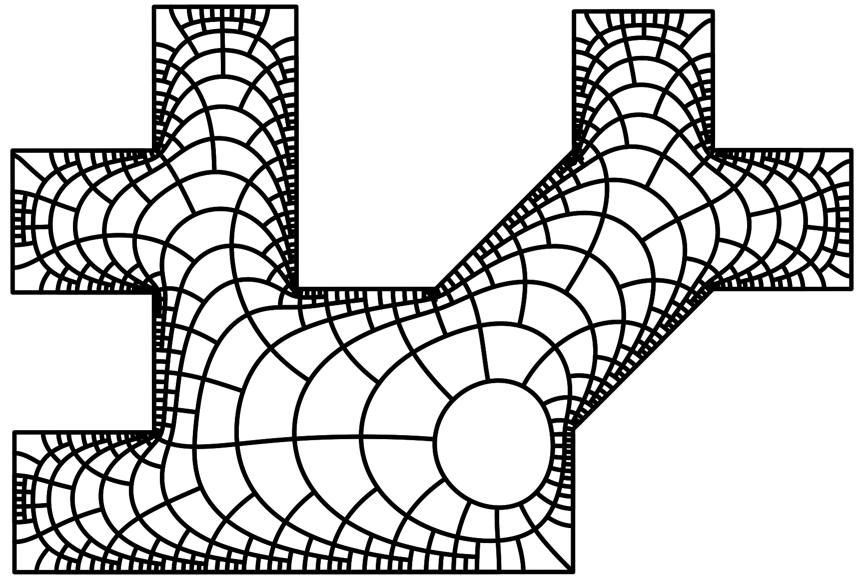
Fast Mapping Theorem: Given an n-gon we can compute an ϵ-map $f : \mathbb{D} \to \Omega$ in time $O(n \log \frac{1}{\epsilon} \log\log \frac{1}{\epsilon})$.

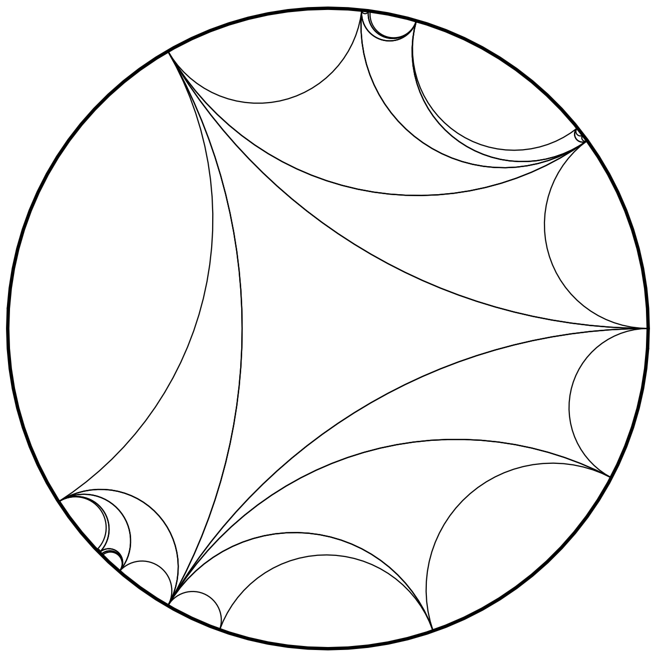
distorts angles by ϵ (same as $(1+\epsilon)$-quasiconformal)

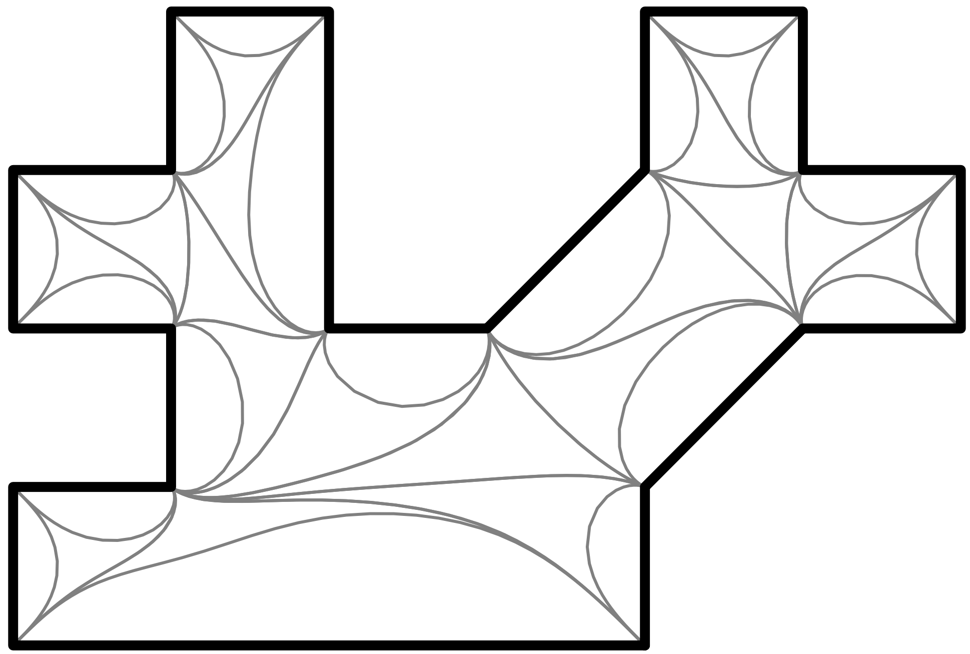


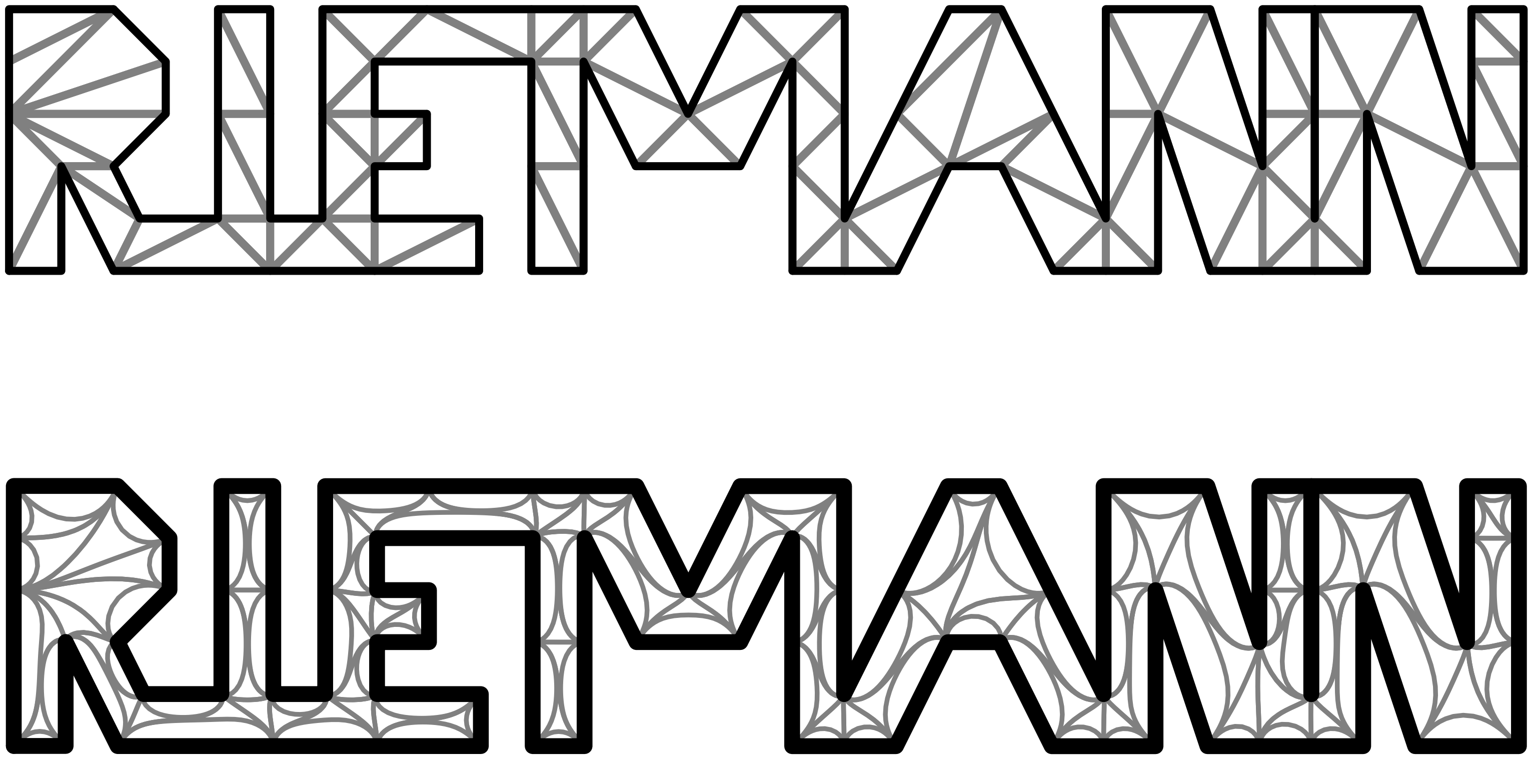






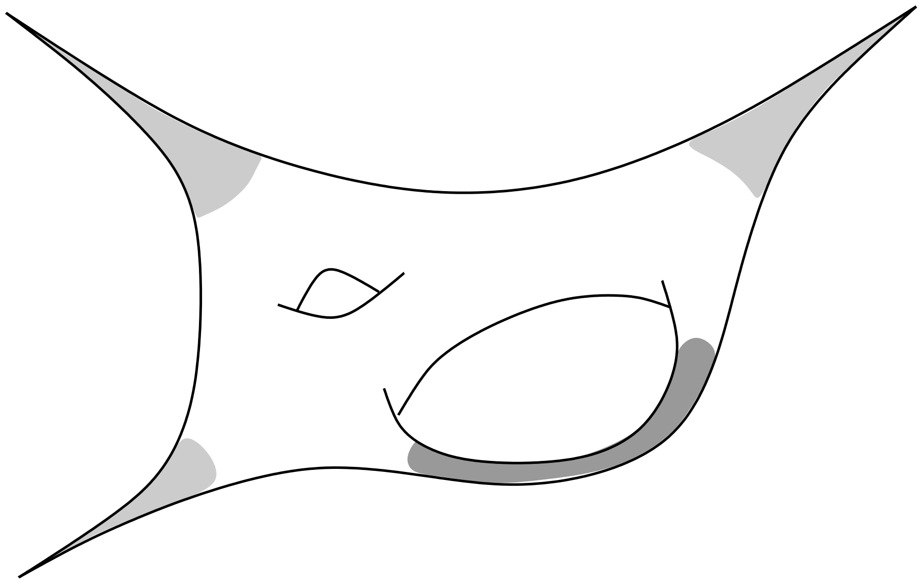


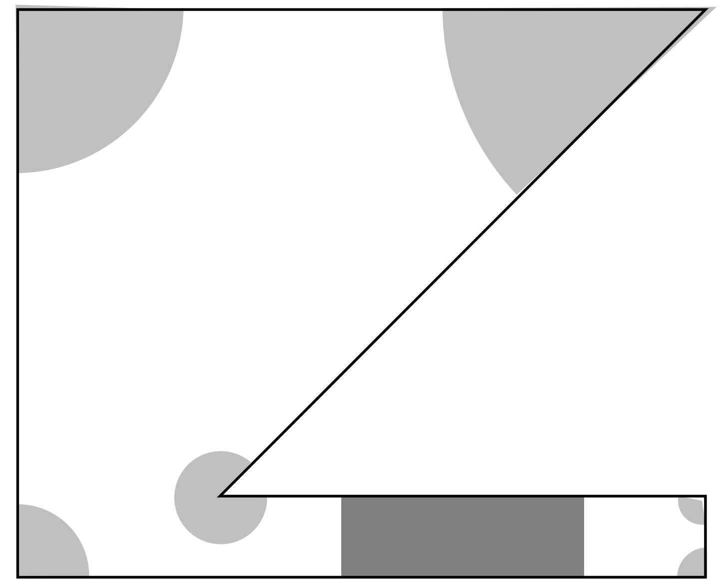


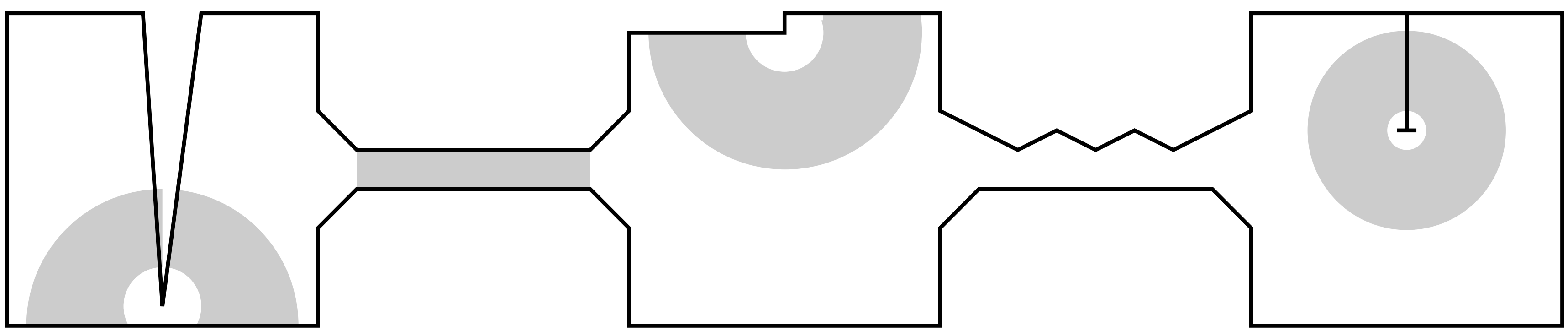

RIEMANN
RIEMANN

Ideas in proof of FMT:

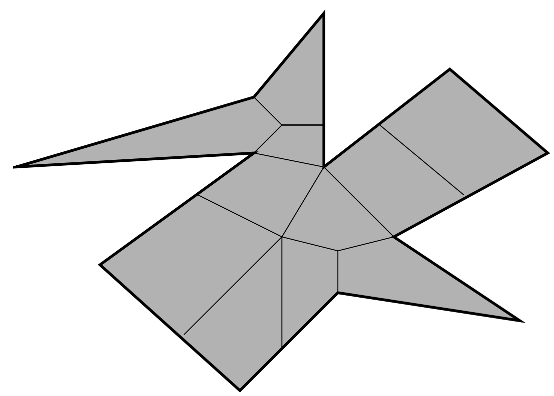
- Thick and thin parts of polygons
- $O(n)$ domain decomposition
- Newton's type iteration via fast multipole
- Angle scaling







Bern and Eppstein (1997): Any n-gon has a quadrilateral mesh with angles $\leq 120°$. At most $O(n)$ points are added. Runs in $O(n \log n)$.

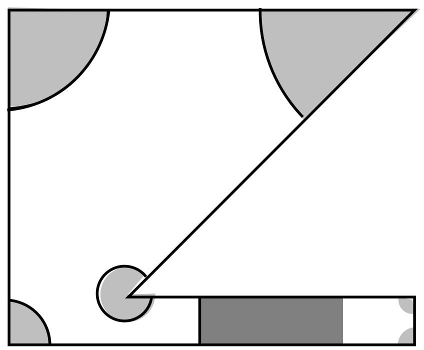


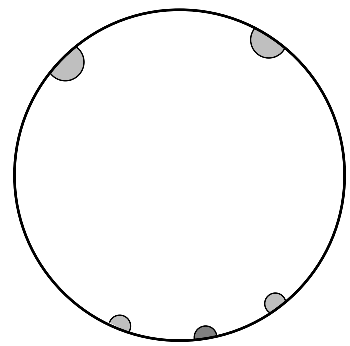
B., 2008: Any n-gon has a $O(n)$ quadrilateral mesh so that all new angles are between $60°$ and $120°$ and which can be computed in $O(n)$.

Angle bounds are best possible.

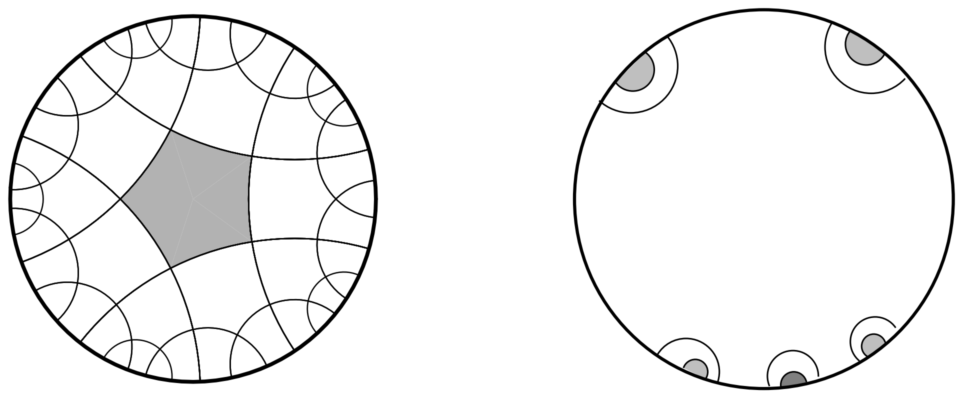
Idea of optimal meshing:

Divide polygon into thick and thin parts. Thick parts look piecewise smooth with 90° angles. Map to disk minus hyperbolic half-planes.





Disk has tesselation by hyperbolic right pentagons. Finite approximation divides disk into pentagons, triangles and quadrilaterals.



Each piece can be meshed consistently.