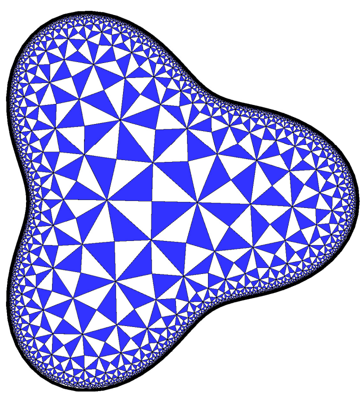
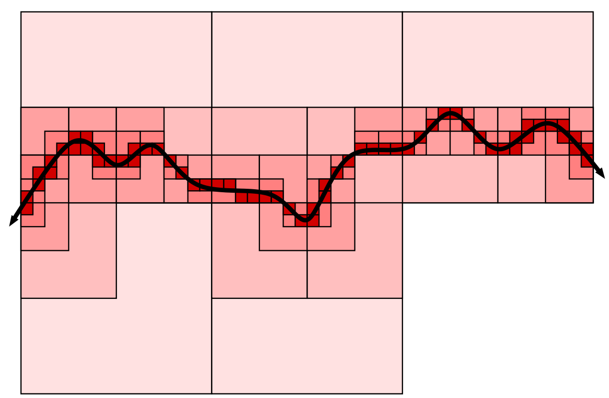
WEIL-PETERSSON CURVES, TRAVELING SALESMAN THEOREMS, AND MINIMAL SURFACES

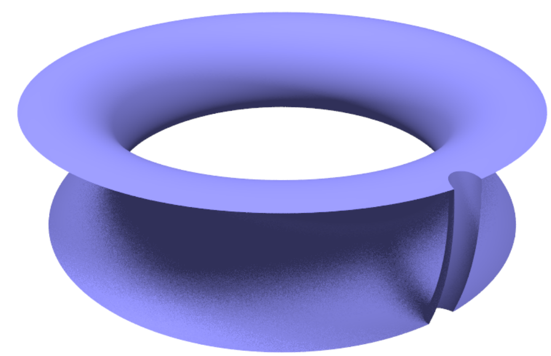
Christopher Bishop, Stony Brook

Friday December 2, 2022
Colloquium, Rutgers University

`www.math.stonybrook.edu/~bishop/lectures`







Goals for today:

(1) Define Weil-Peterson class of curves.

(2) Give some motivation and connections to various areas.

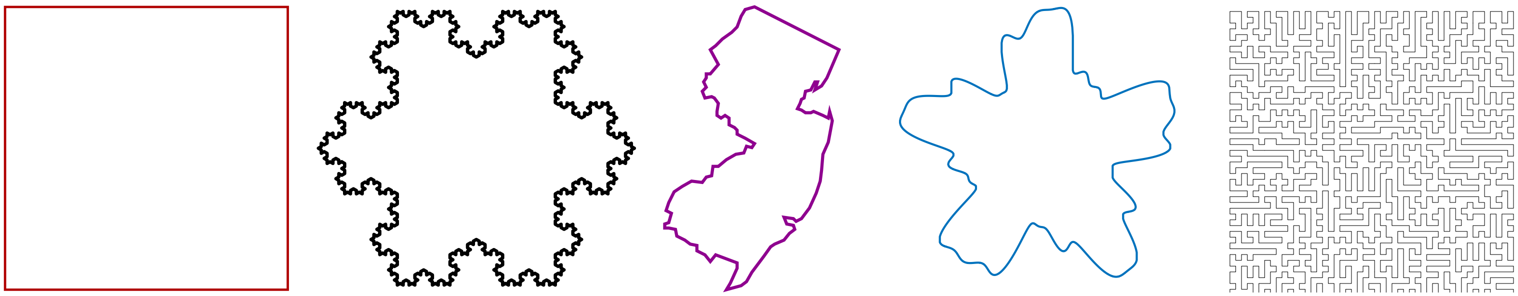
(3) State half a theorem, sketch parts of the proof.

(1) String theory studies spaces of loops.

(2) Computation is easier in Hilbert spaces.

$\Rightarrow$ We want the space of loops to look like a Hilbert space.

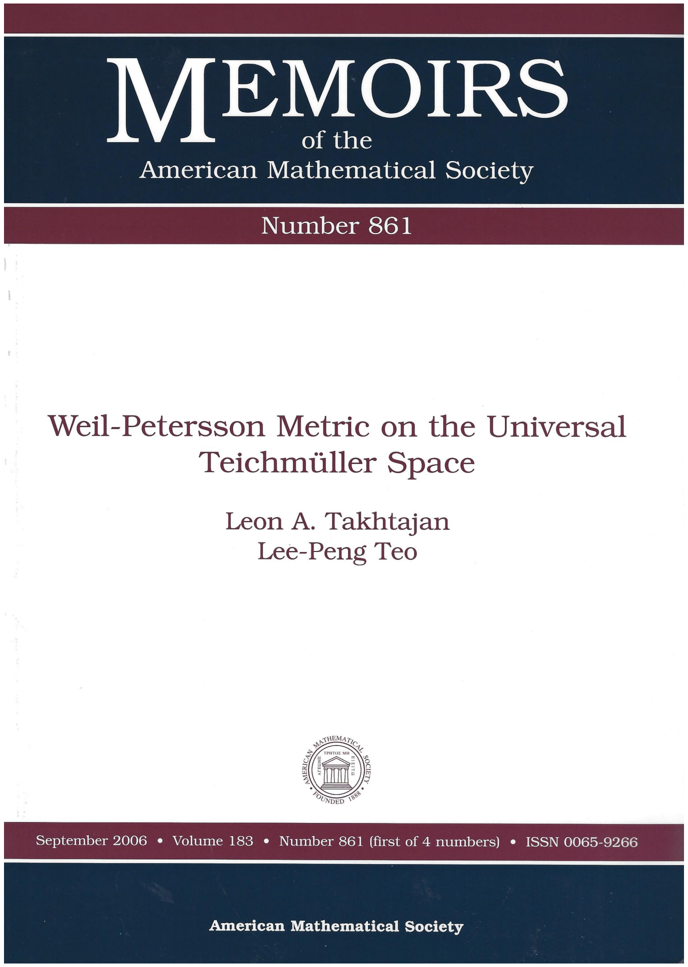
(1) $T(1)$ = universal Teichmüller space = quasicircles



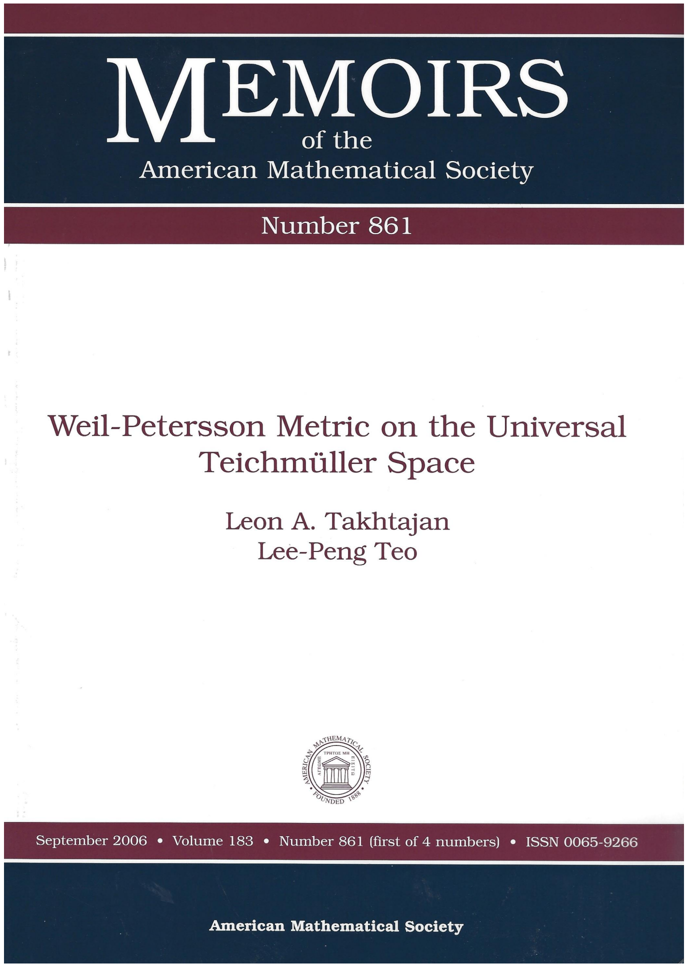
(2) Usual Teichmüller metric based on L^∞ (supremum norm).

$\Rightarrow$ $T(1)$ is Banach manifold, not Hilbert manifold.

$\Rightarrow$ not so good for physics or computations.

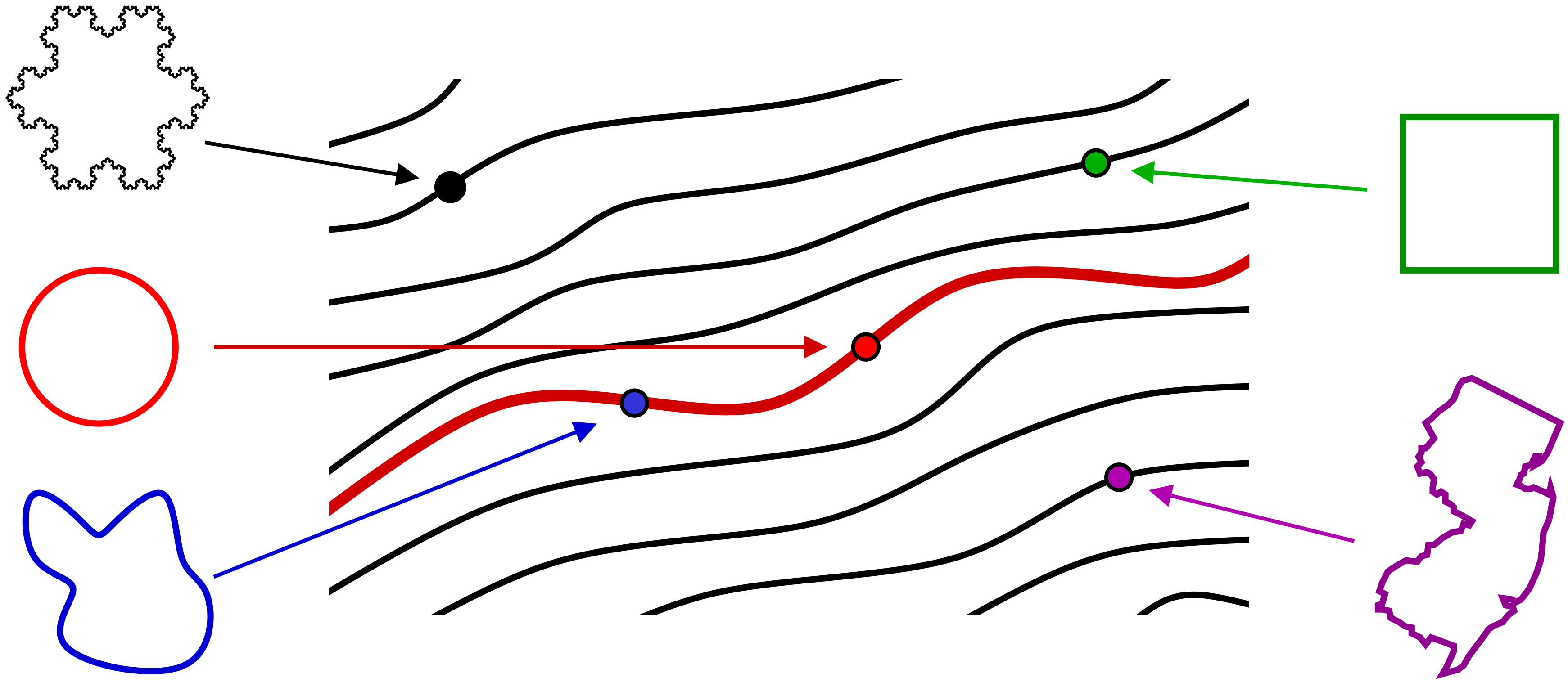


"In this memoir, we prove that the universal Teichmüller space $T(1)$ carries a new structure of a complex Hilbert manifold and show that the connected component of the identity of $T(1)$ — the Hilbert submanifold $T_0(1)$ — is a topological group. ..."



"Weil-Petersson class boundary parameterizations provide the correct analytic setting for conformal field theory." — Radnell, Schippers and Staubach, 2017

Cartoon of universal Teichmüller space



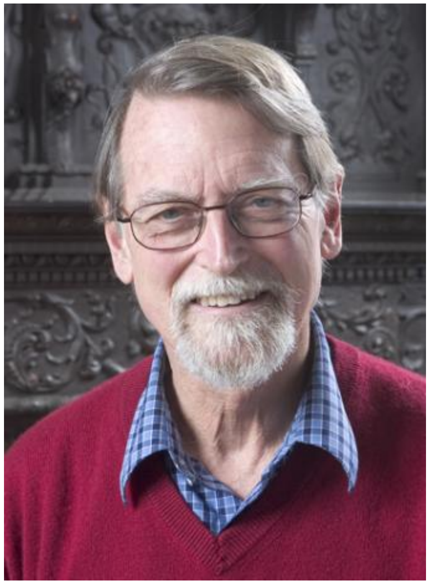
Takhtajan and Teo make $T(1)$ a (disconnected) Hilbert manifold.

$T_0(1)$ = Weil-Petersson class

= connected component containing the circle

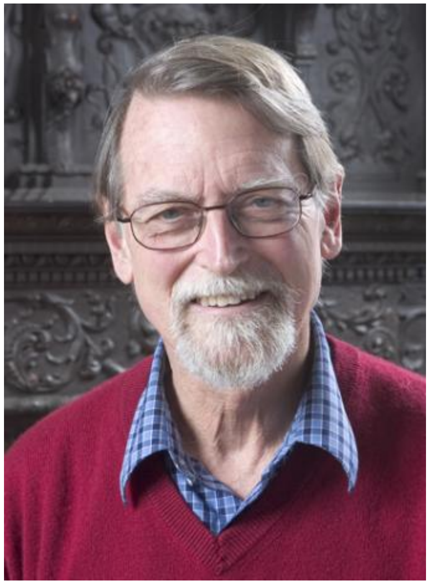
= closure of smooth curves

= ∞-dim Kähler-Einstein manifold.



In Dec 2017 email David Mumford asked me which non-smooth curves are in WP? Motivated by computer vision and pattern recognition.

"Riemannian geometries on spaces of plane curves, Michor and Mumford, *J. Eur. Math. Soc. (JEMS)*, 2006.



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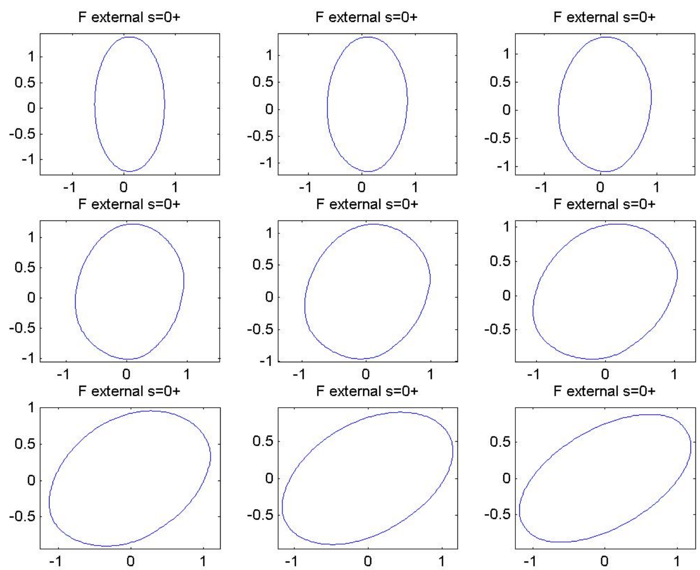
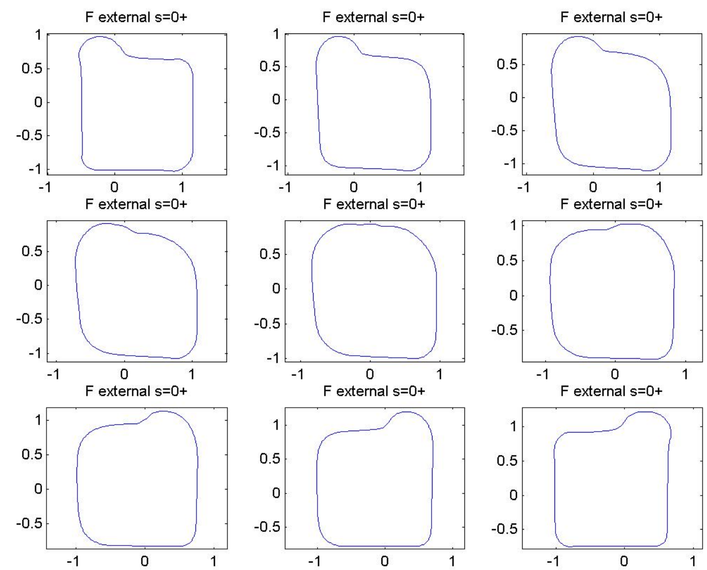
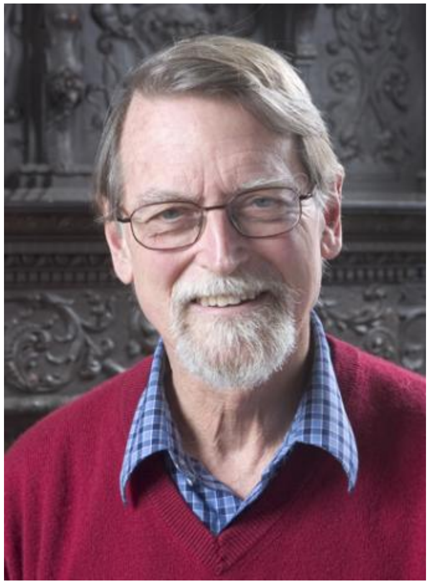




Figures by Eitan Sharon



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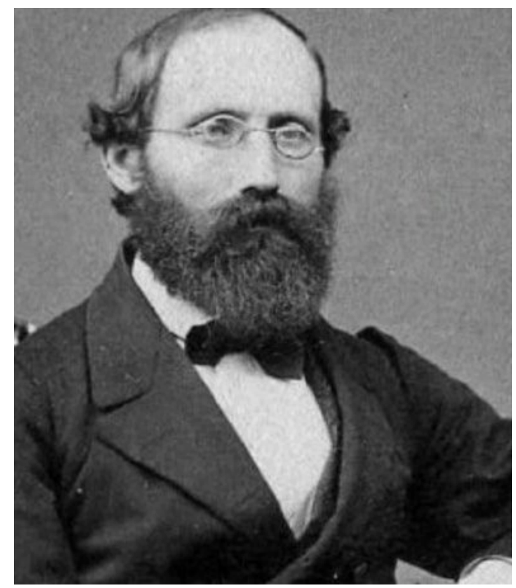
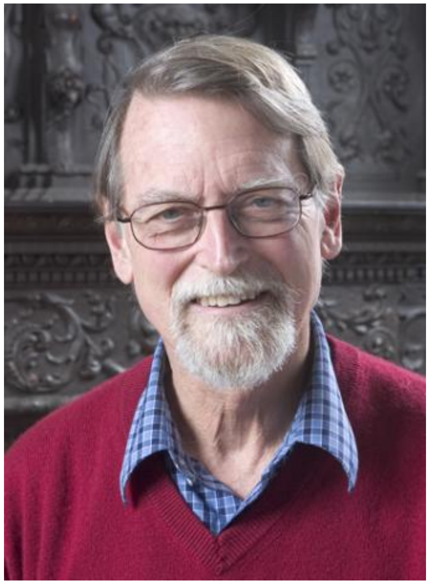
There are, however, manifolds in which the fixing of position requires not a finite number but either an infinite series or a continuous manifold of determinations of quantity. Such manifolds are constituted for example by the possible shapes of a figure in space, ...

Bernhard Riemann, Habilitatsionschrift





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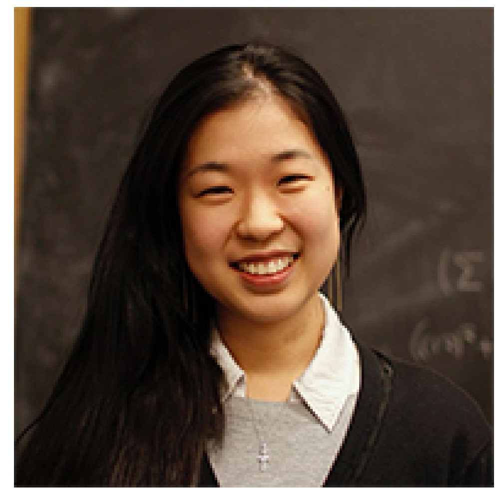
Jan 2019 IPAM workshop: Analysis and Geometry of Random Sets.

Lecture by Yilin Wang:

"Loewner energy via Brownian loop measure and action functional analogs of SLE/GFF couplings"



So the Weil-Petersson class (undefined so far) is linked to:

- String theory
- Kähler-Einstein manifolds
- Teichmüller theory
- Pattern recognition
- Brownian loops, SLE, Gaussian free fields, ...

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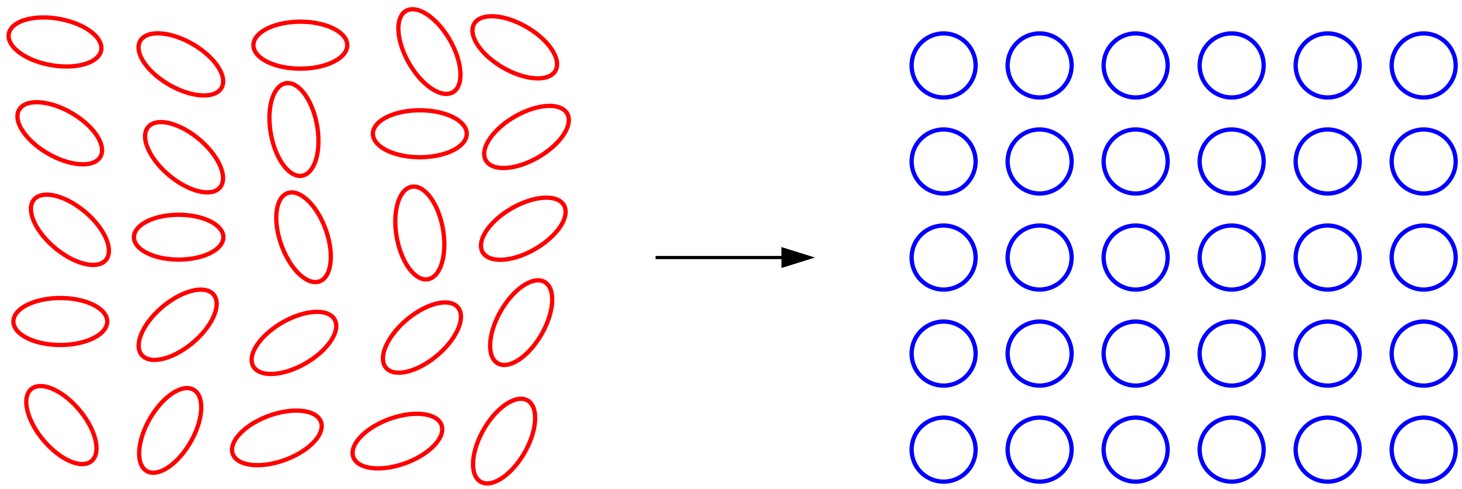
In today's talk I will discuss further connections to:

- Geometric function theory
- Sobolev spaces
- Knot theory
- The traveling salesman theorem
- Convex hulls in hyperbolic space
- Minimal surfaces
- Renormalized area (quantum engtanglement)

We start with a quick review of quasiconformal maps.

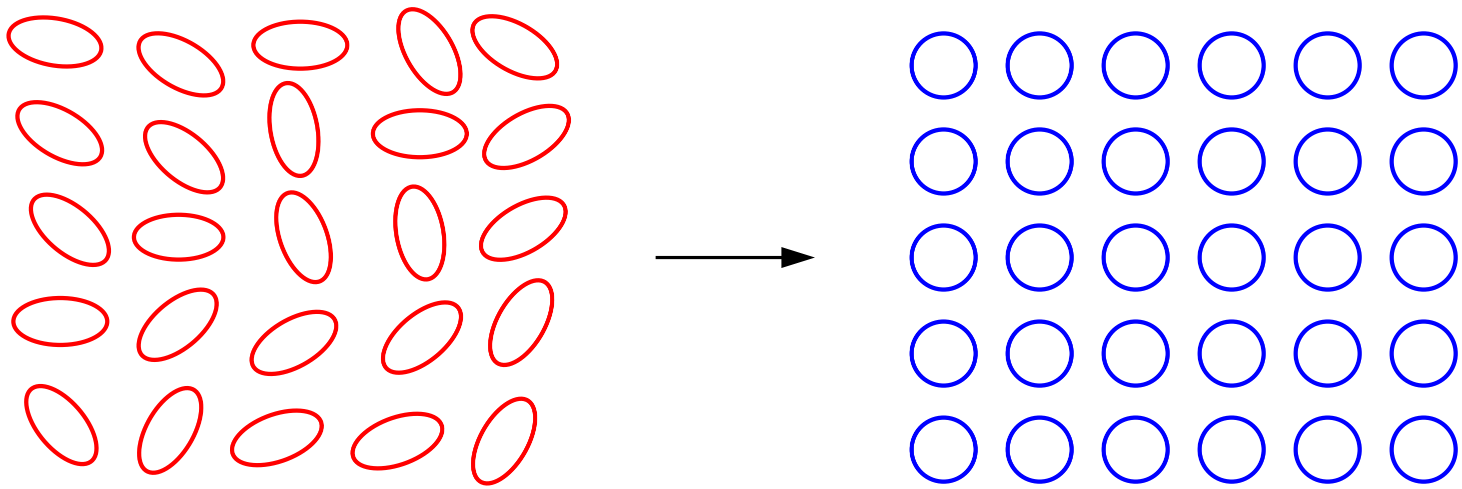
Diffeomorphisms send infinitesimal ellipses to circles.



Eccentricity = ratio of major to minor axis of ellipse.

K-quasiconformal = ellipses have eccentricity $\leq K$ almost everywhere

Diffeomorphisms send infinitesimal ellipses to circles.



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Ellipses determined by dilatation $\mu = f_{\overline{z}}/f_z$ with $f_{\overline{z}}, f_z = \frac{1}{2}(f_x \pm i f_y)$.

$$|\mu| = \frac{K-1}{K+1} < 1, \qquad \arg(\mu) \text{ gives major axis.}$$

f is QC $\Leftrightarrow \|\mu\|_\infty < 1$. $\qquad f$ is conformal = 1-1 holomorphic $\Leftrightarrow \mu \equiv 0$.

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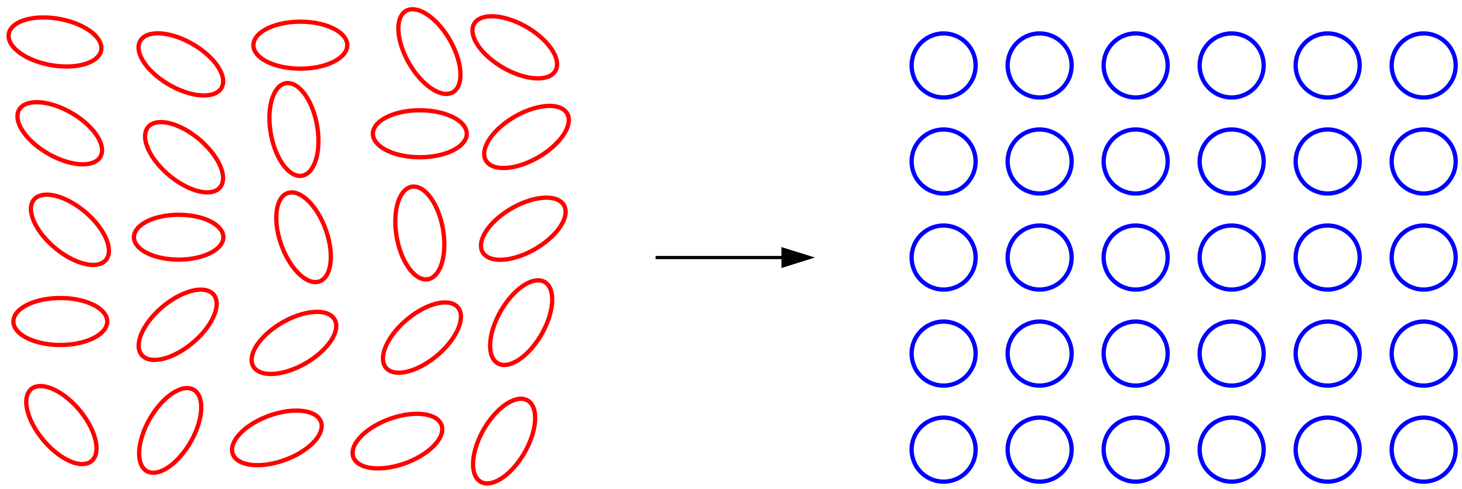

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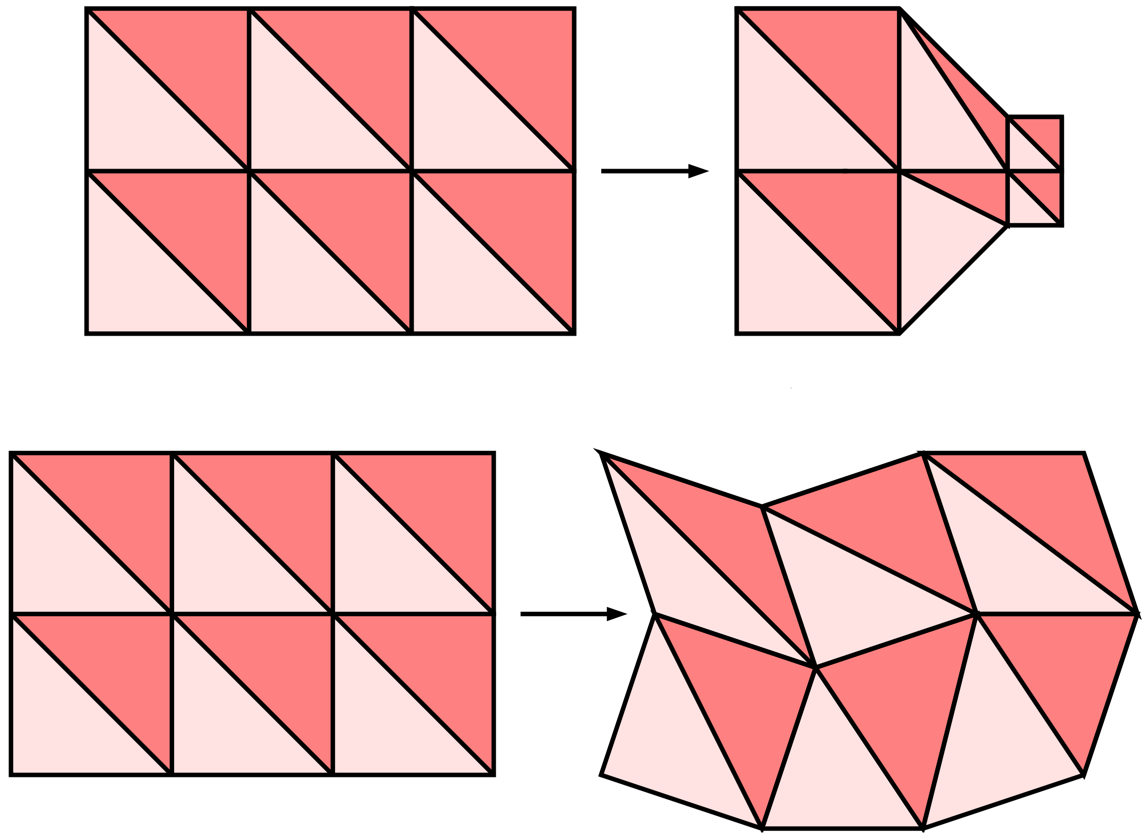
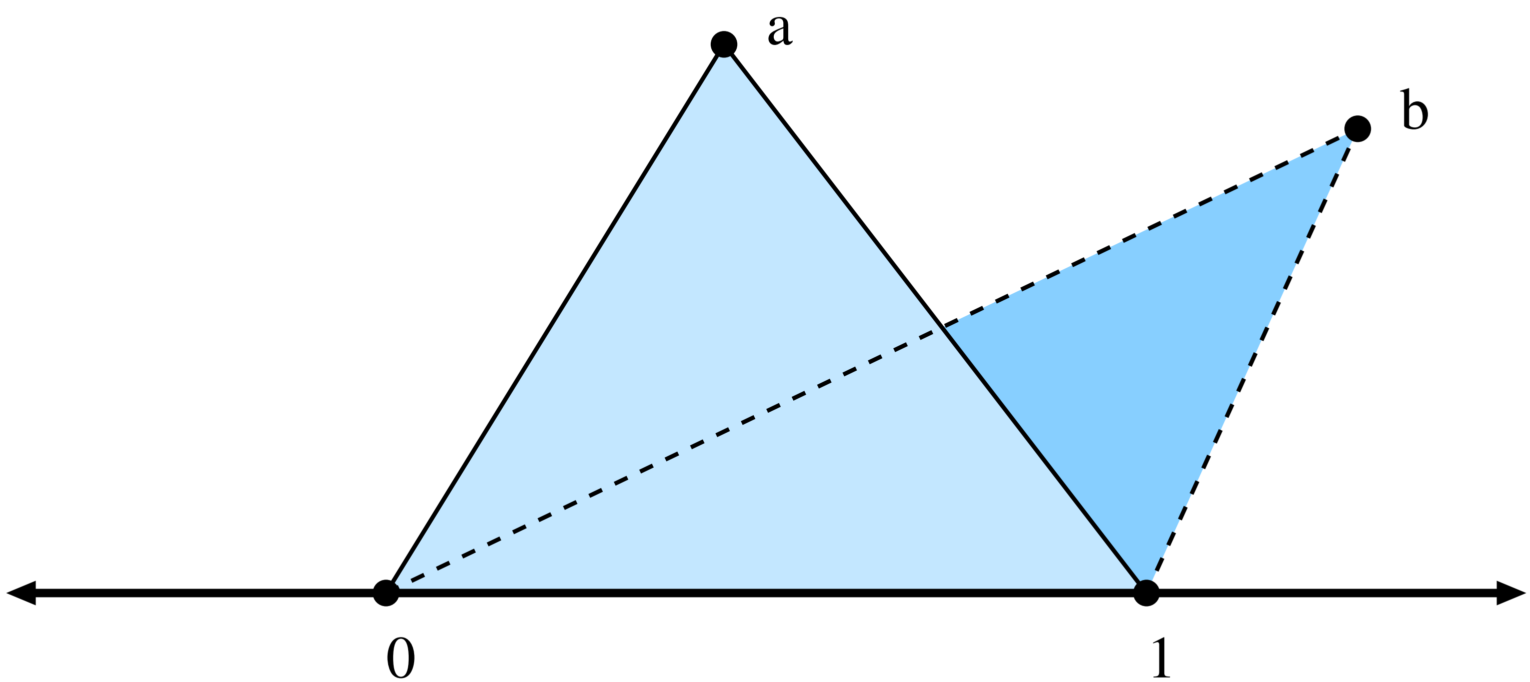
Special case of QC are biLipschitz maps

$$\frac{1}{C} \leq \frac{|f(x) - f(y)|}{|x - y|} \leq C.$$



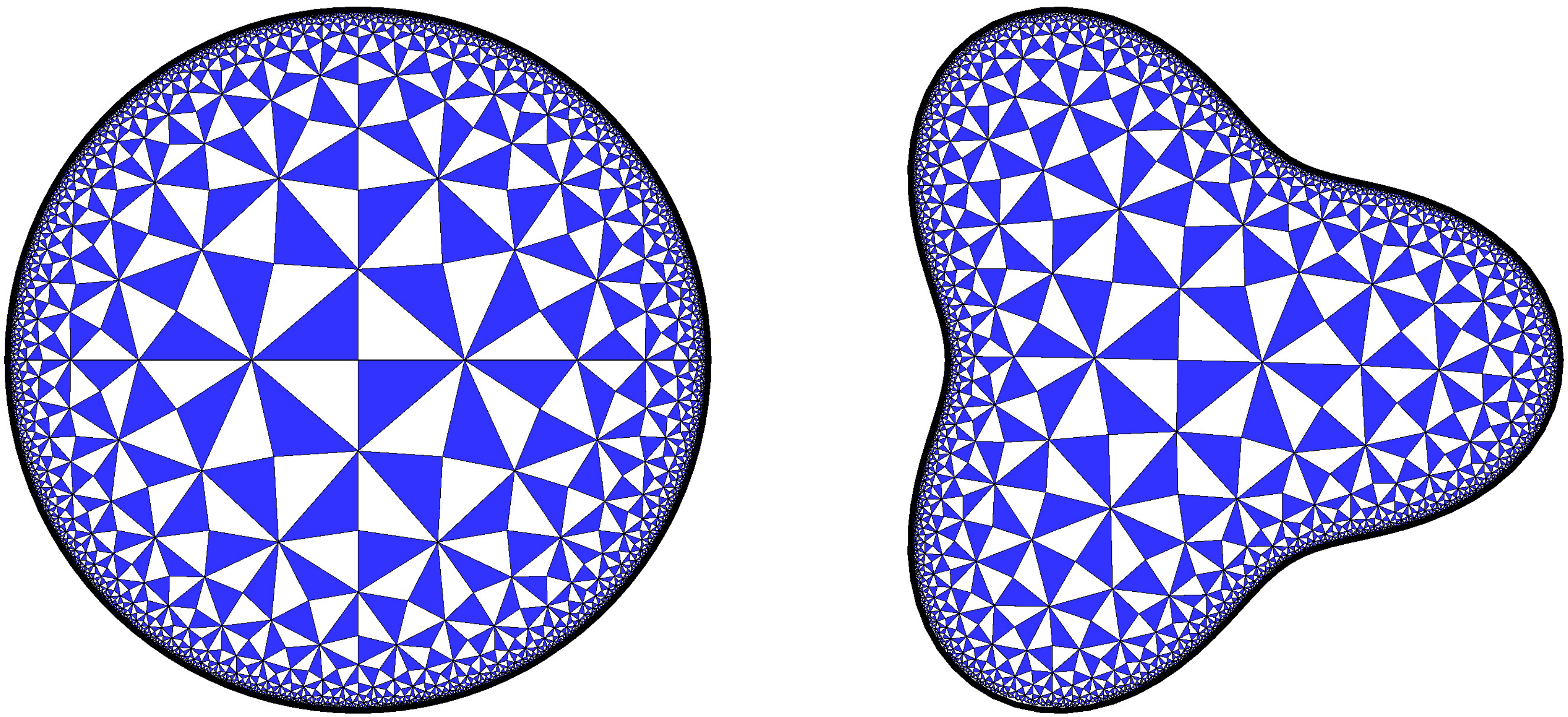
QC maps preserve “shape” up to bounded factor. Scales may change.

BiLipschitz maps preserve both shape and scale up to bounded factor.



Affine map between triangles $\{0, 1, a\}$ and $\{0, 1, b\}$ has constant dilatation

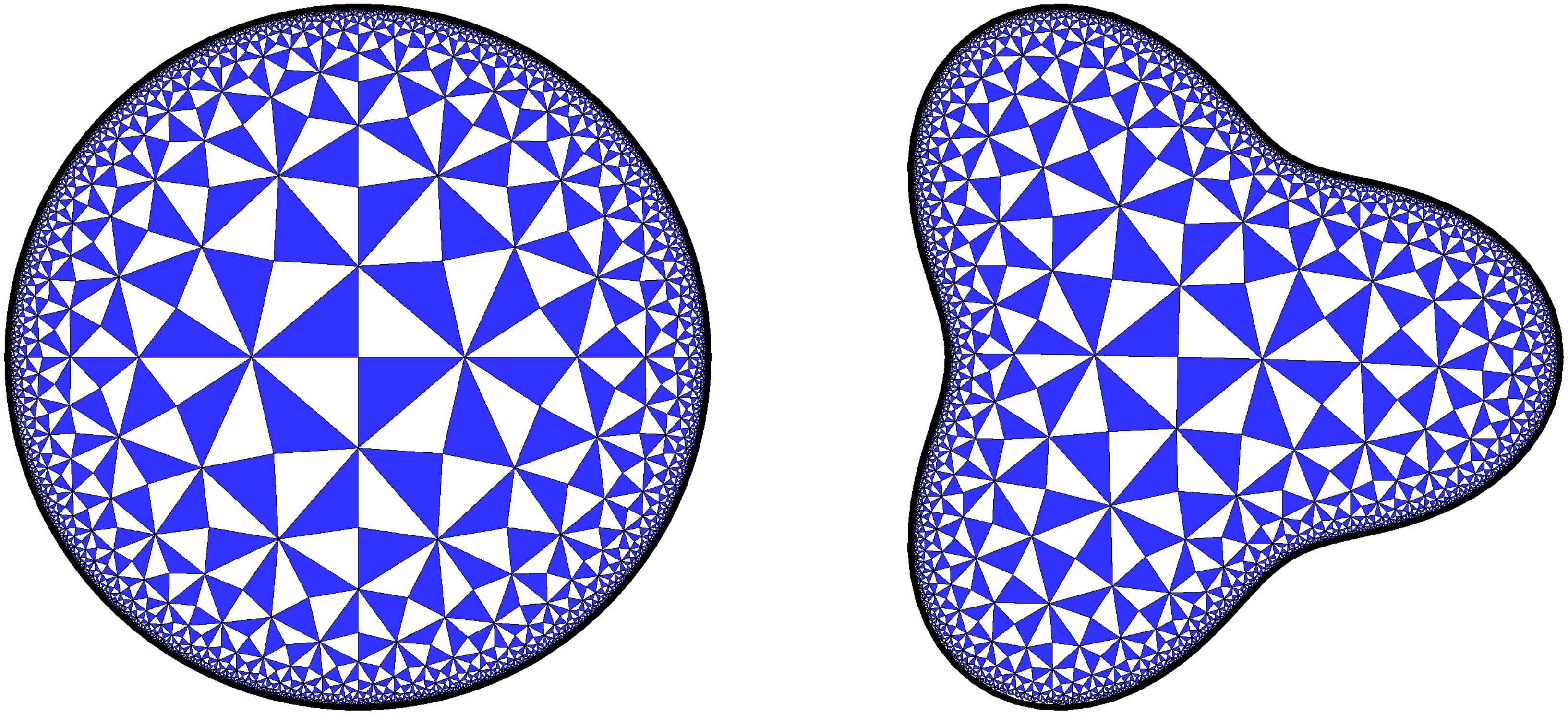
$$\mu = \frac{b - a}{b - \overline{a}}$$



Riemann Mapping Thm: any Jordan domain is conformal image of $\mathbb{D}$.

Liouville's Theorem $\Rightarrow$ any conformal map $\mathbb{C} \to \mathbb{C}$ is linear.

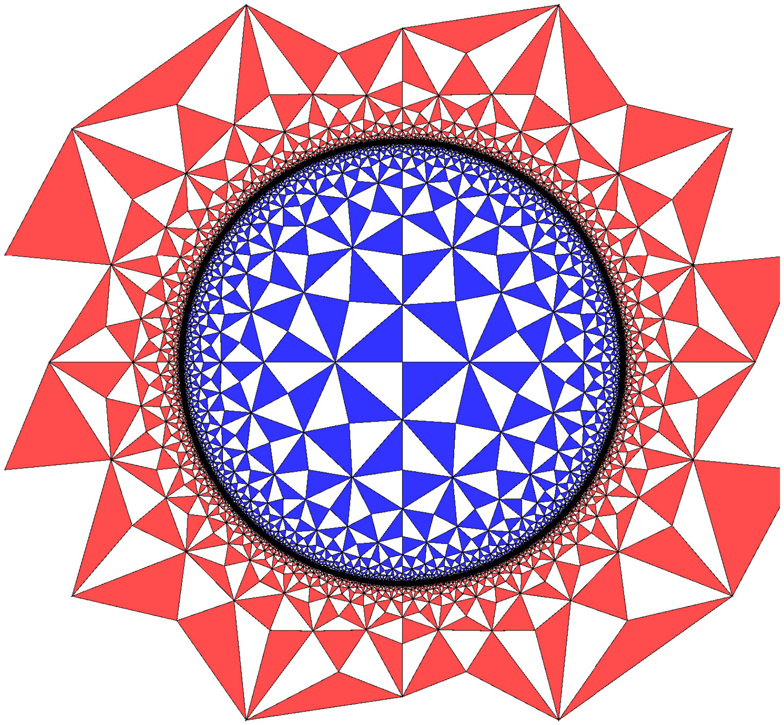
$\Rightarrow$ map above can't be extended to be conformal in whole plane.

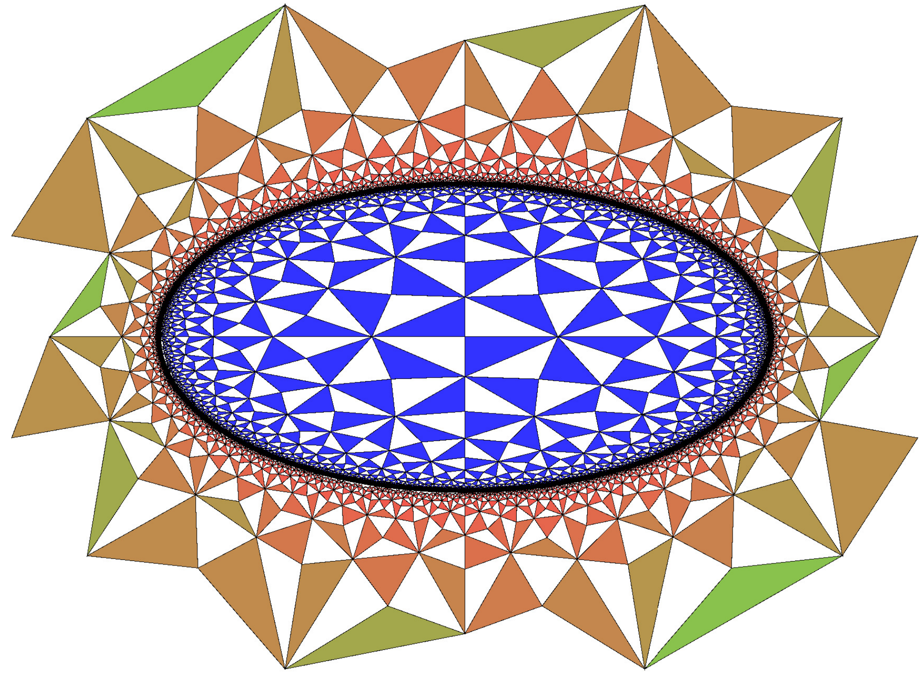


Distortion of triangles indicated by change in color.

Conformal $\Rightarrow$ no visible distortion.

But if we extend map to outside, some distortion must occur.

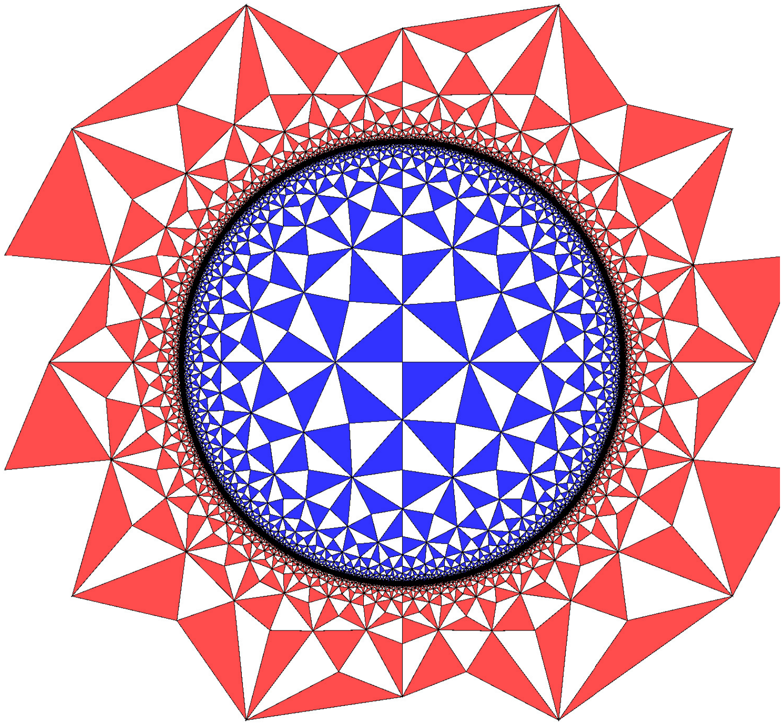


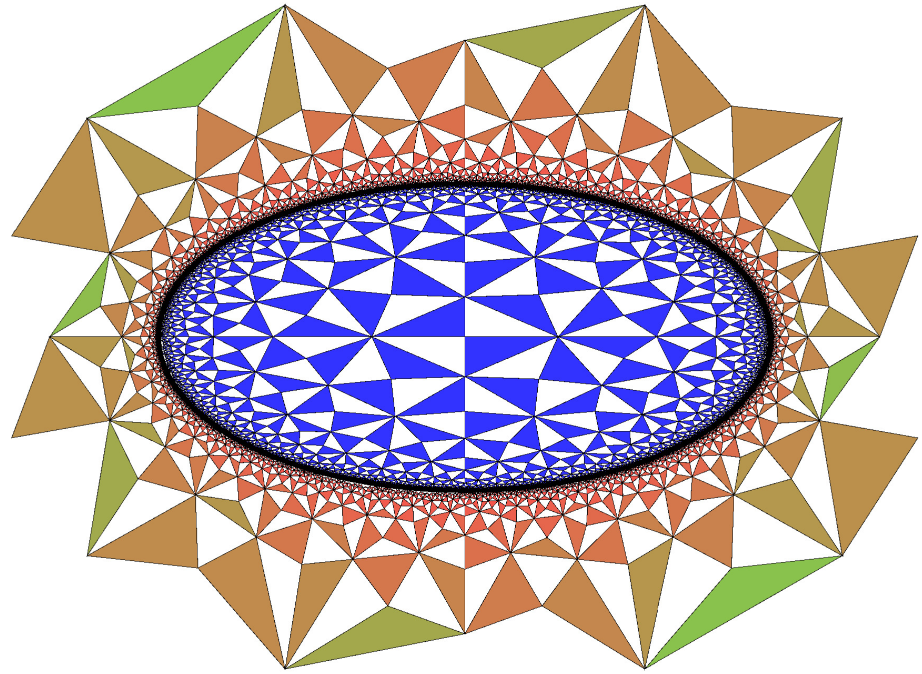


Conformal inside circle, quasiconformal on plane.

Quasicircle = quasiconformal image of circle.

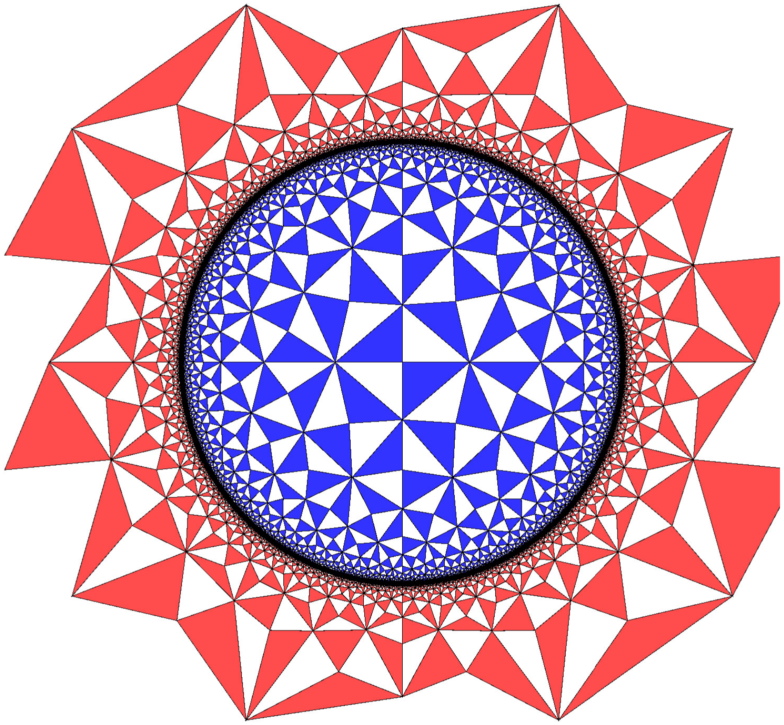
Color distortion = angle distortion

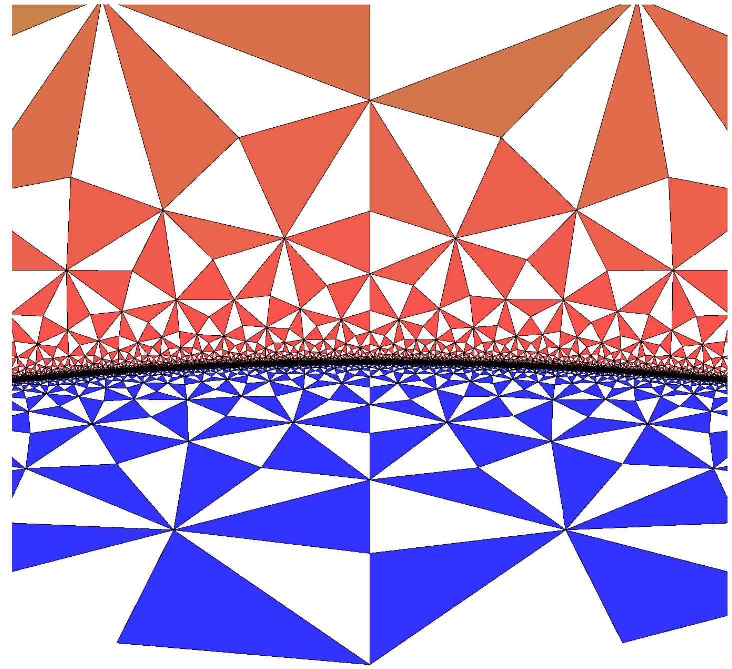




Teichmüller metric = maximum dilatation

Weil-Petersson metric = $\sum(\text{dilatation})^2$

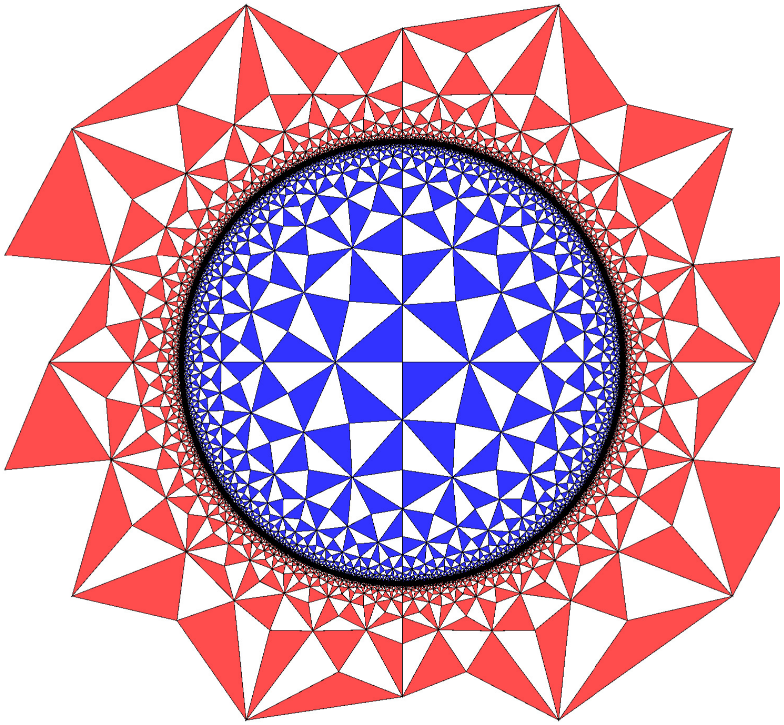
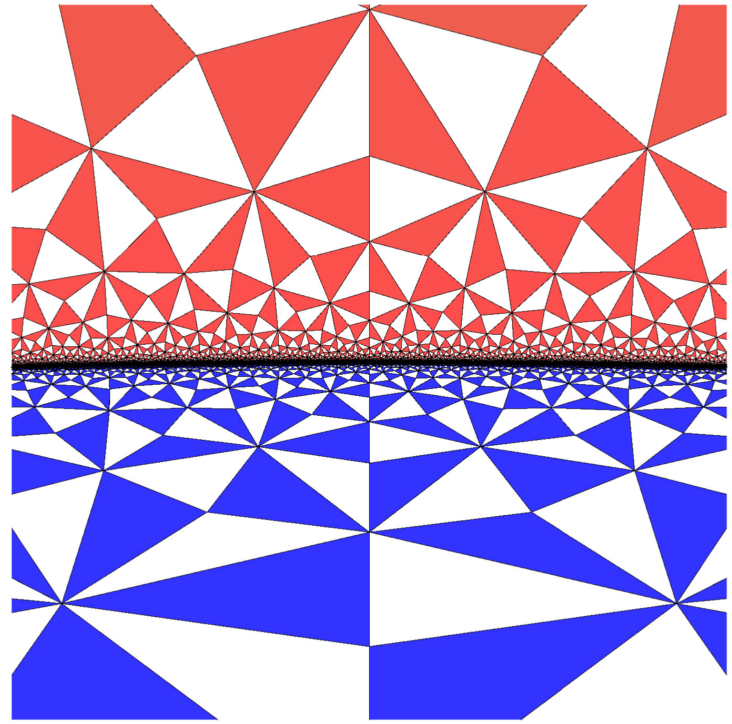




Teichmüller metric = maximum dilatation

Weil-Petersson metric $= \sum(\text{dilatation})^2$

For smooth domains Weil-Petersson sum converges

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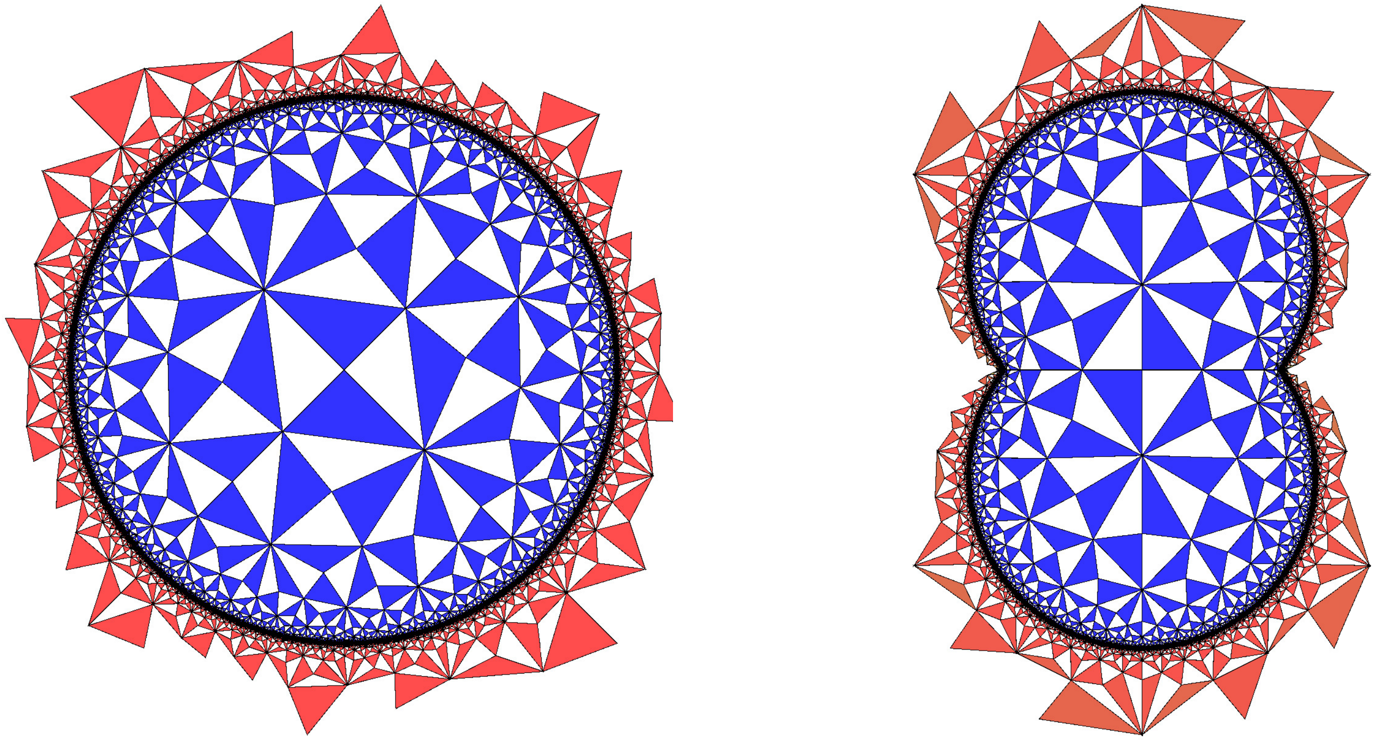
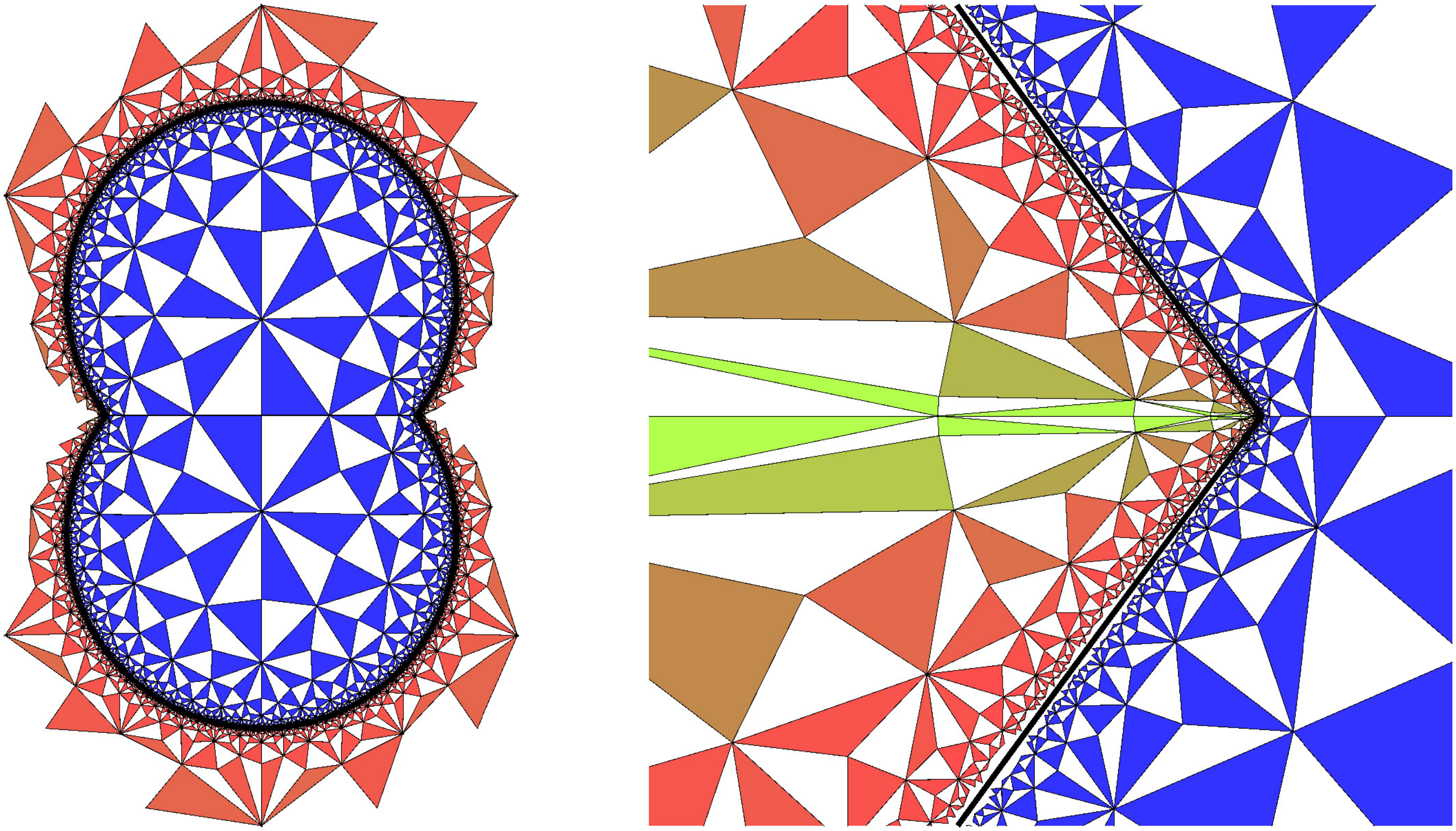


Corners cause distortion on all scales.

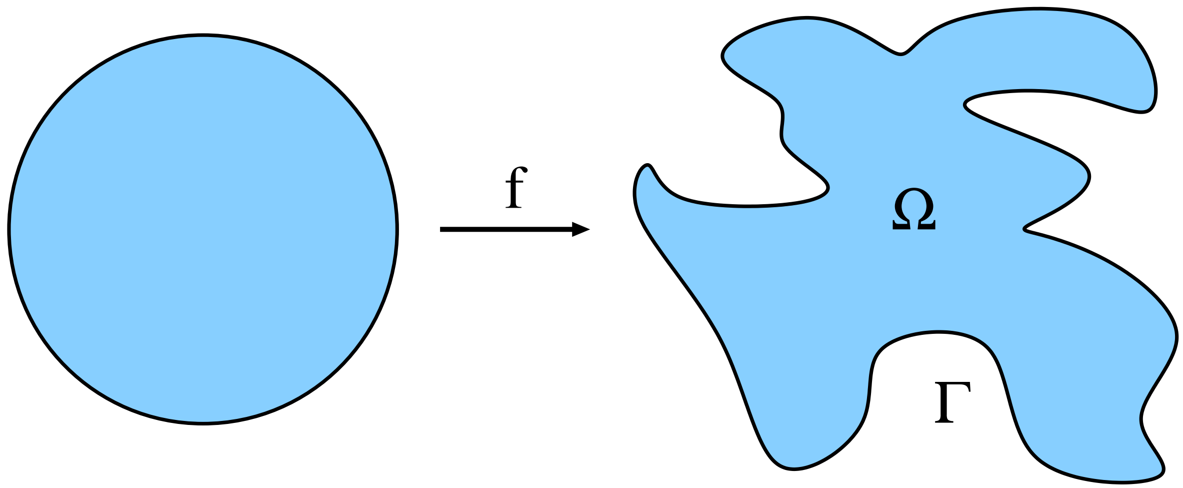
(Plot of Nehari function, suggested by Martin Chuaqui Farrú)



Infinitely many triangles with large distortion

$$\text{WP sum} = \sum(\text{distortion})^2 = \infty$$

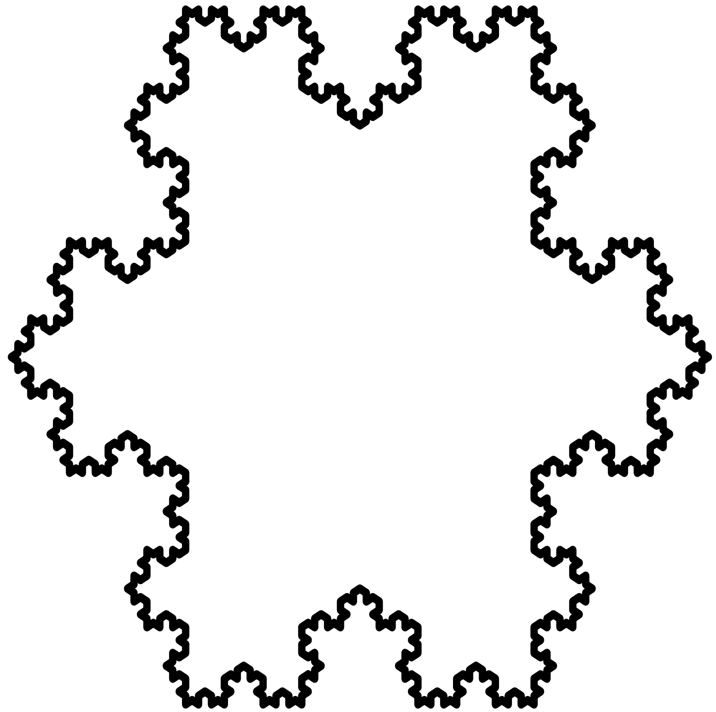
A **quasicircle** is the image of circle under a quasiconformal map of $\mathbb{R}^2$.



$T(1)$ = Universal Teichmüller space = quasicircles modulo similarities.

All smooth closed curves are quasicircles.

But some fractals are also quasicircles.



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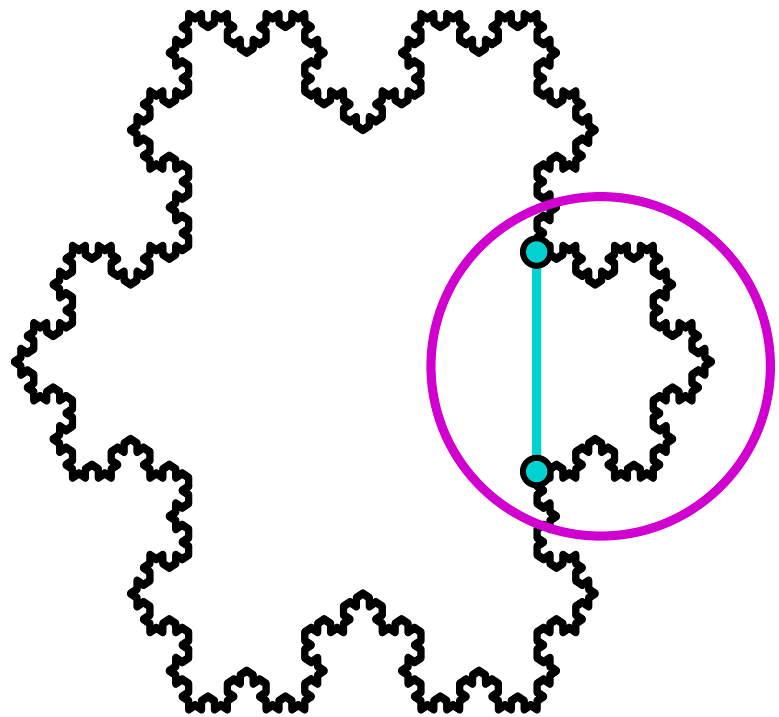
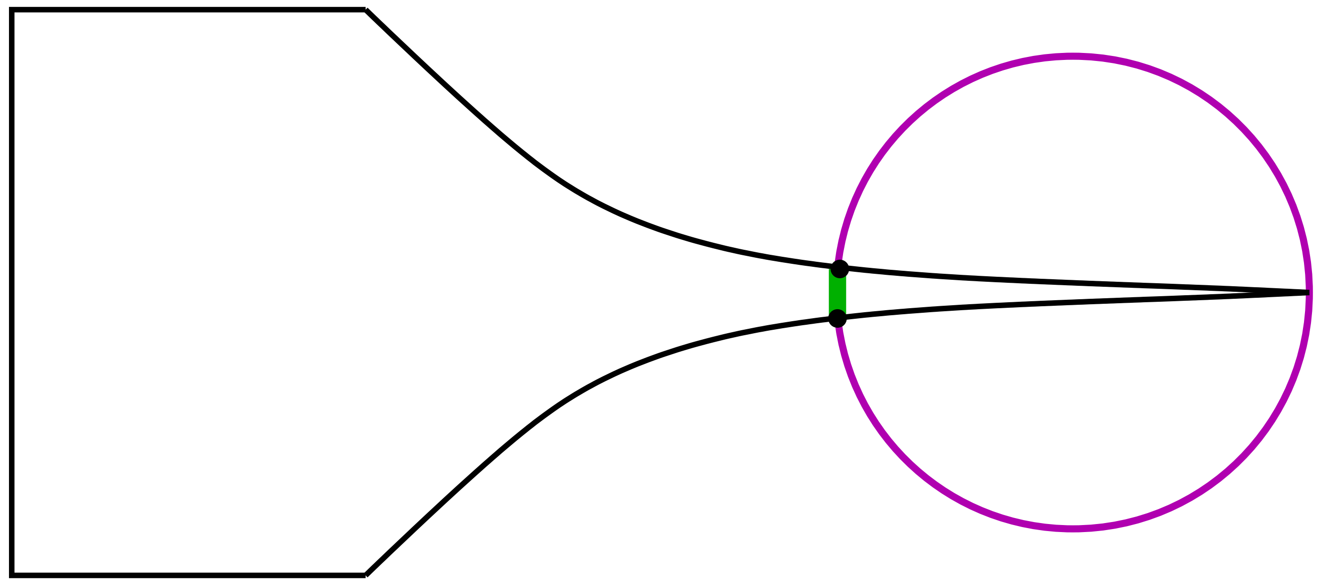


Γ is a quasicircle iff $\quad \mathrm{diam}(\gamma) = O(\mathrm{crd}(\gamma)) \quad$ for all $\gamma \subset \Gamma$.

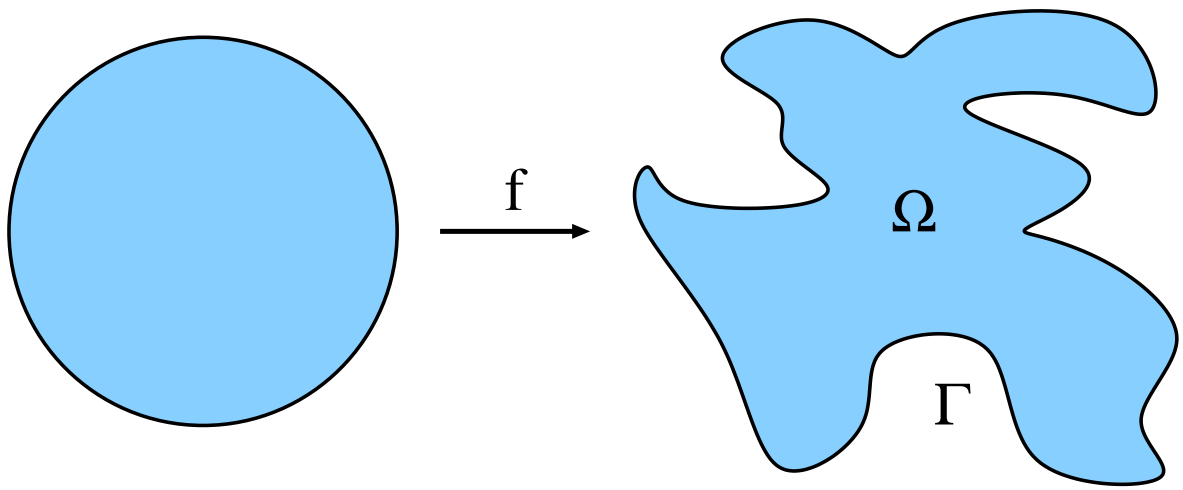
$\mathrm{crd}(\gamma) = |z - w|$, z, w, endpoints of γ.



Not a quasicircle: small segment cuts off large arc.

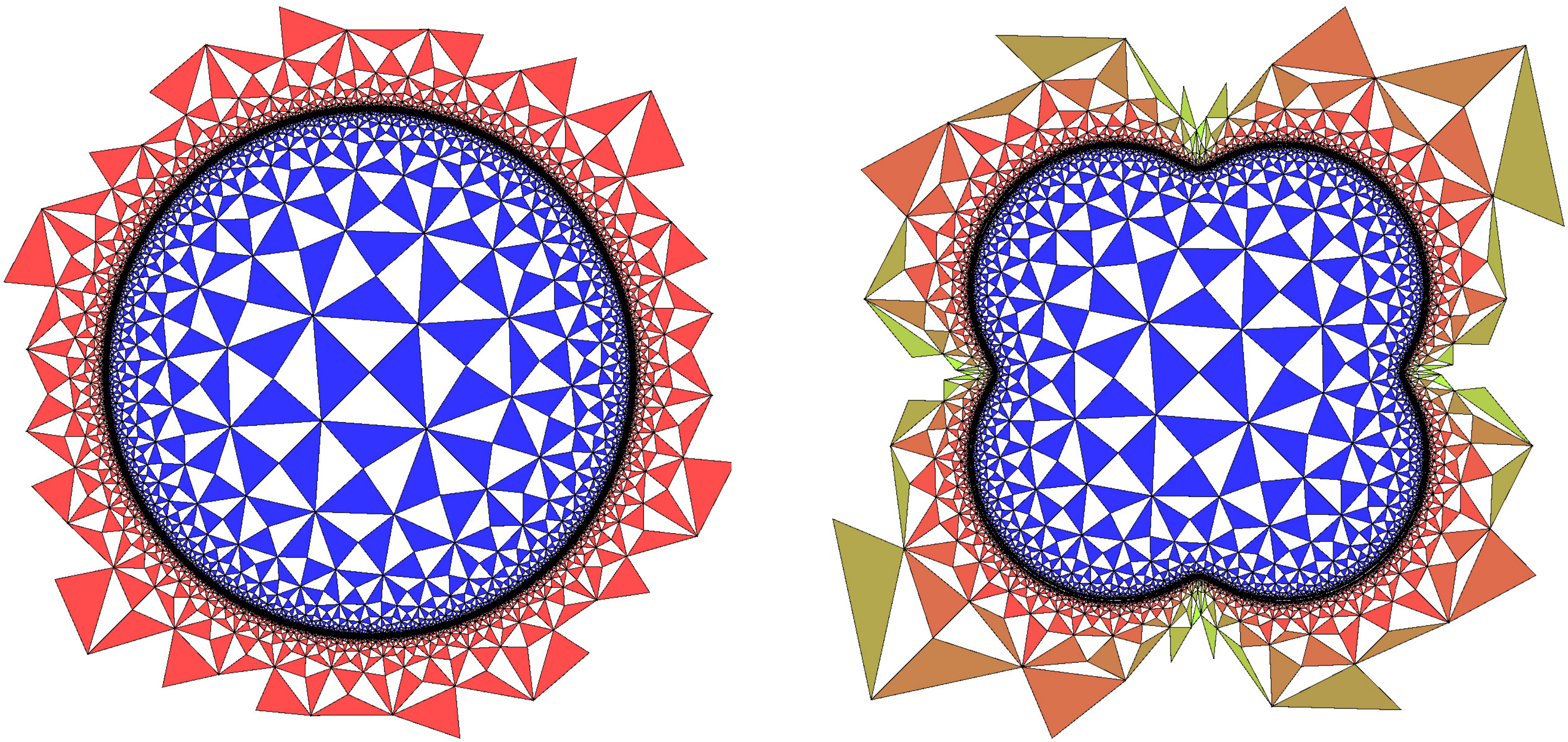
Defn: $\Gamma = f(\mathbb{T})$ is **Weil-Petersson** if $\mu \in L^2(dA_\rho)$.

Here $dA_\rho = \frac{dxdy}{(1-|z|^2)^2}$ = hyperbolic area on $\mathbb{C} \setminus \mathbb{T}$.



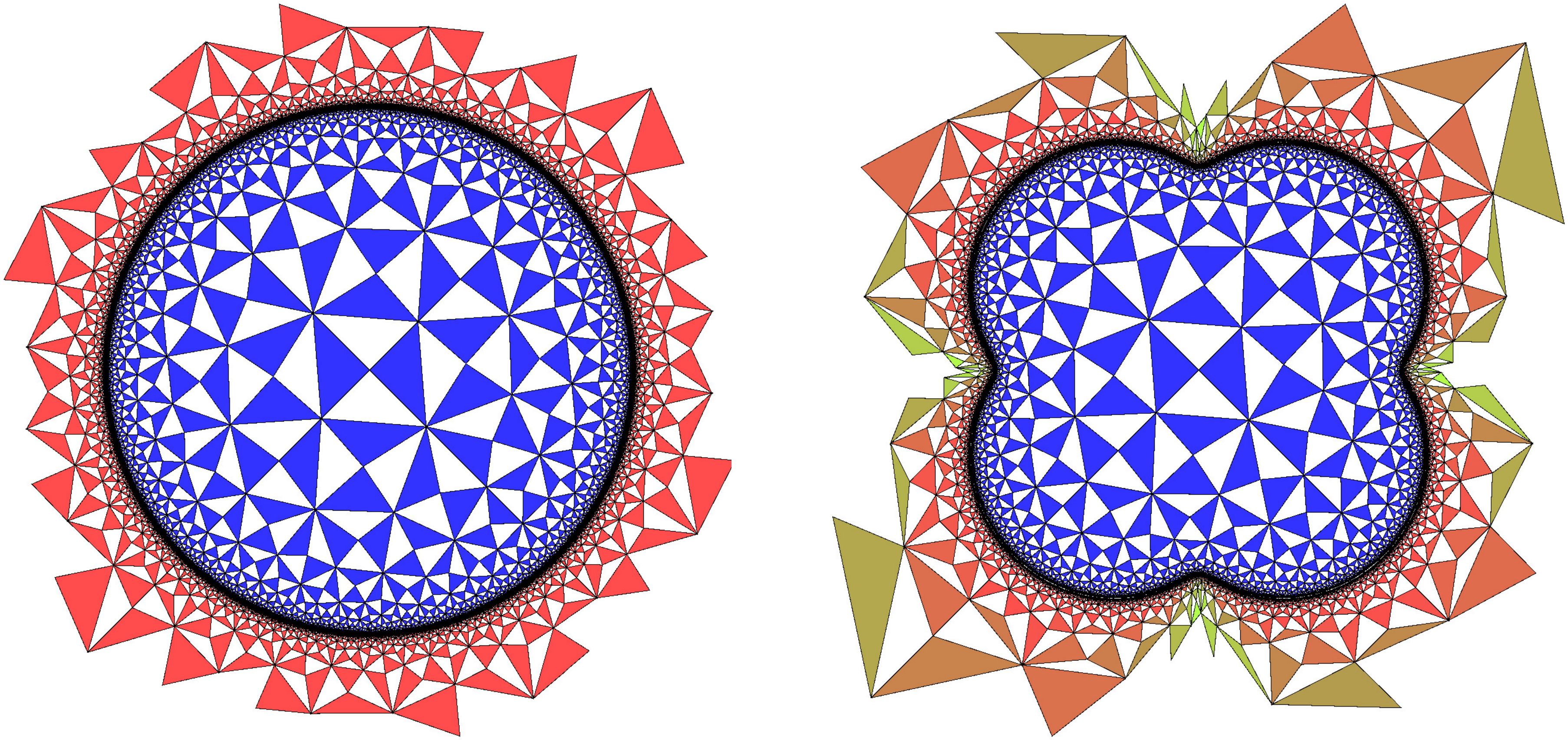
Informally: WP is to L^2, as QC is to L^∞.

The Weil-Petersson class is Möbius invariant.



Triangles are (approximately) unit hyperbolic size.

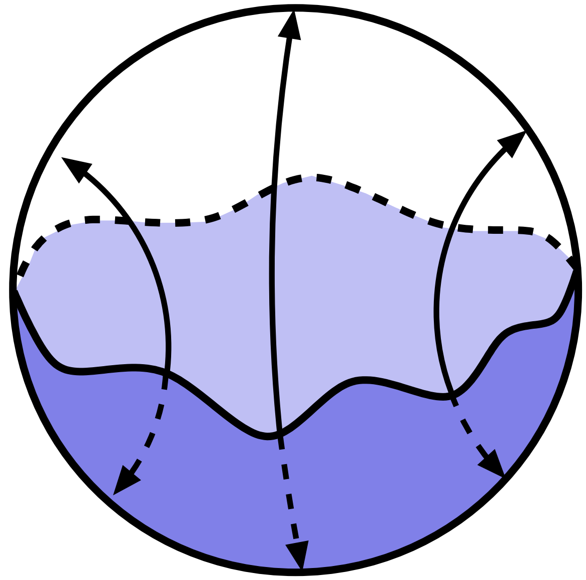
$$\text{Weil-Petersson} = \sum (\text{distortions})^2 < \infty.$$



Circle reflection: $R : z \to 1/\overline{z}$. Triangles reflect across circle.

$f \circ R \circ f^{-1}$ is QC reflection across Γ with L^2 dilatation.

Theorem: $\Gamma = f(\mathbb{T})$ is **Weil-Petersson** if Γ is pointwise fixed for biLipschitz involution of S^2 with $\mu \in L^2$ for hyperbolic area on $\mathbb{S}^2 \setminus \Gamma$.



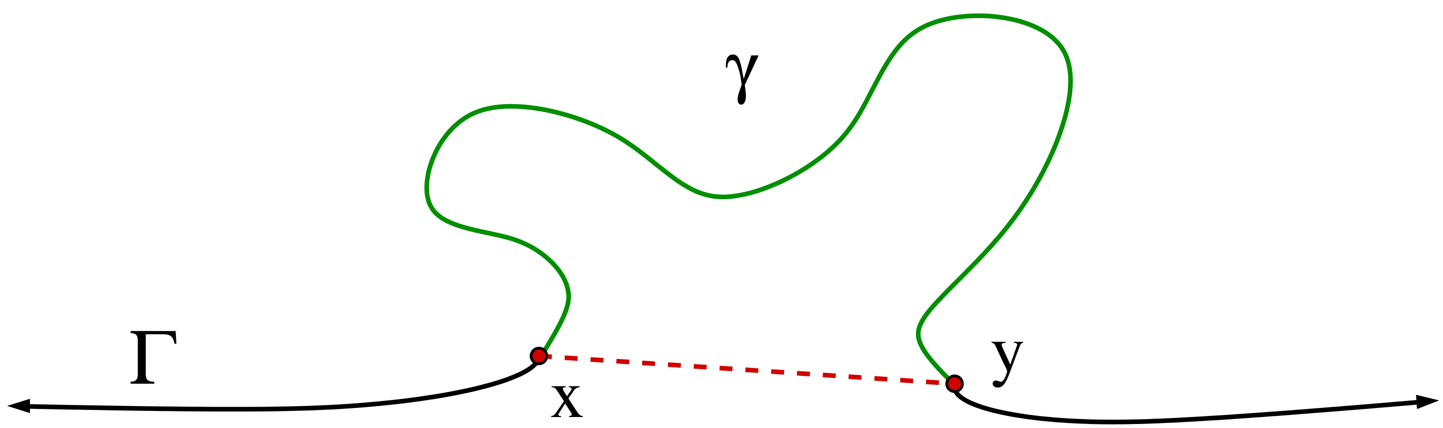
We will use this later. Generalizes to dimensions $d \geq 4$.

Quasicircles **can** be fractals, but Weil-Petersson curves **never are**.

WP-curves are rectifiable (= finite length), in fact, are chord-arc.

Γ is a **chord-arc** iff $\quad \ell(\gamma) = O(|x-y|) \quad$ for all $\gamma \subset \Gamma$.

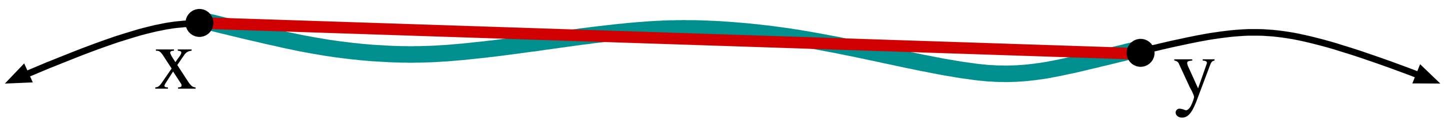
$x, y =$ endpoints of γ.



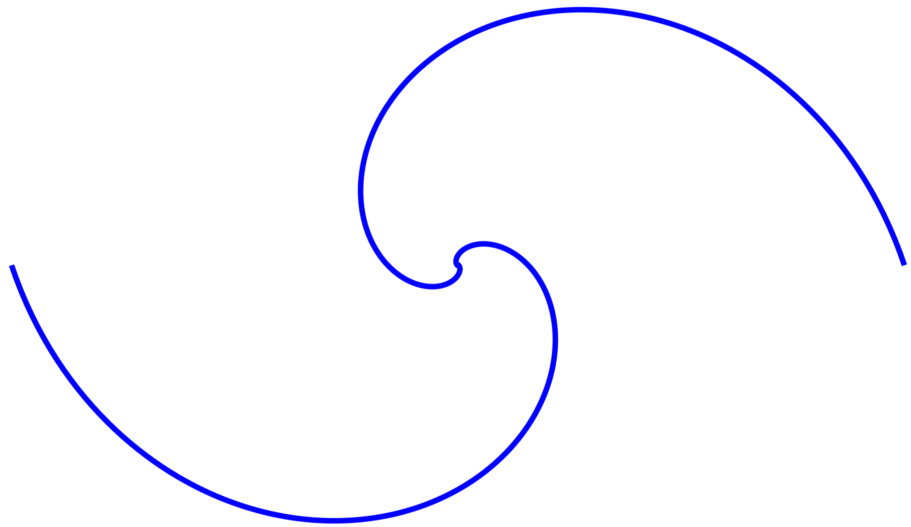
Even stronger: Weil-Petersson $\Rightarrow$ Asymptotically smooth

Asymptotically smooth means that $\gamma \subset \Gamma$, $\ell(\gamma) \to 0$ implies

$$\frac{\ell(\gamma)}{|x-y|} \to 1, \quad \text{or equivalently,} \quad \frac{\ell(\gamma) - |x-y|}{|x-y|} \to 0.$$

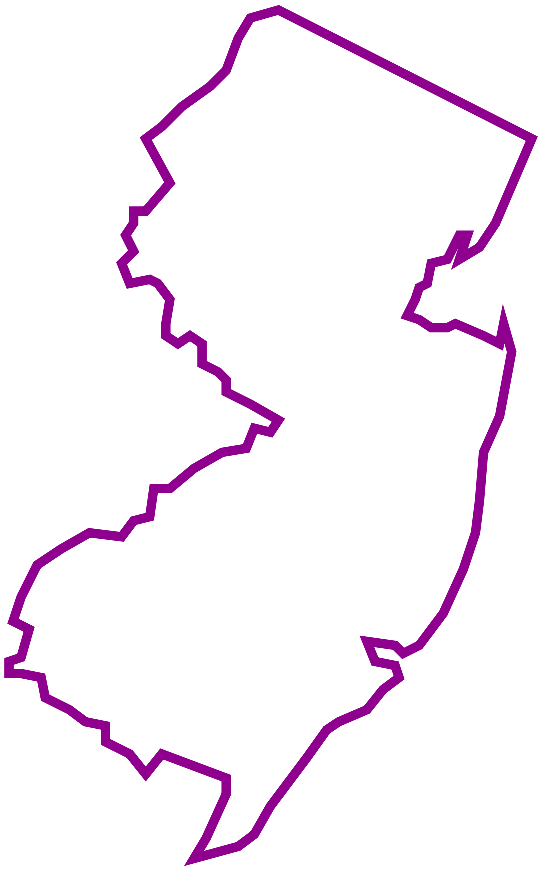


Weil-Petersson curves are almost C^1 (but not quite).

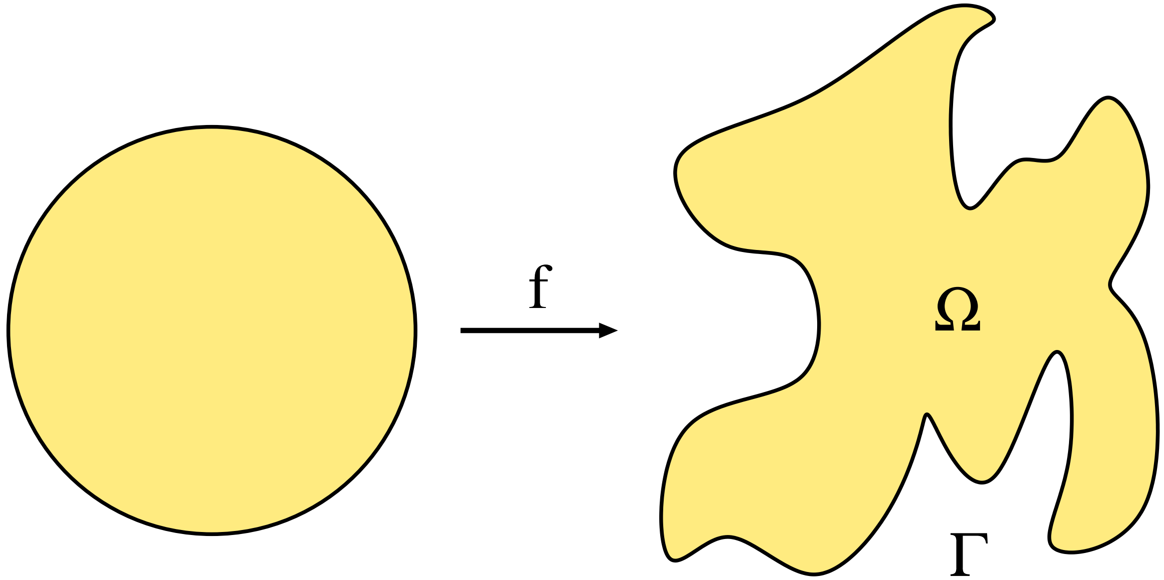


Weil-Petersson curves need not be C^1.

$z(t) = \exp(-t + i \log t)$, infinite spiral.

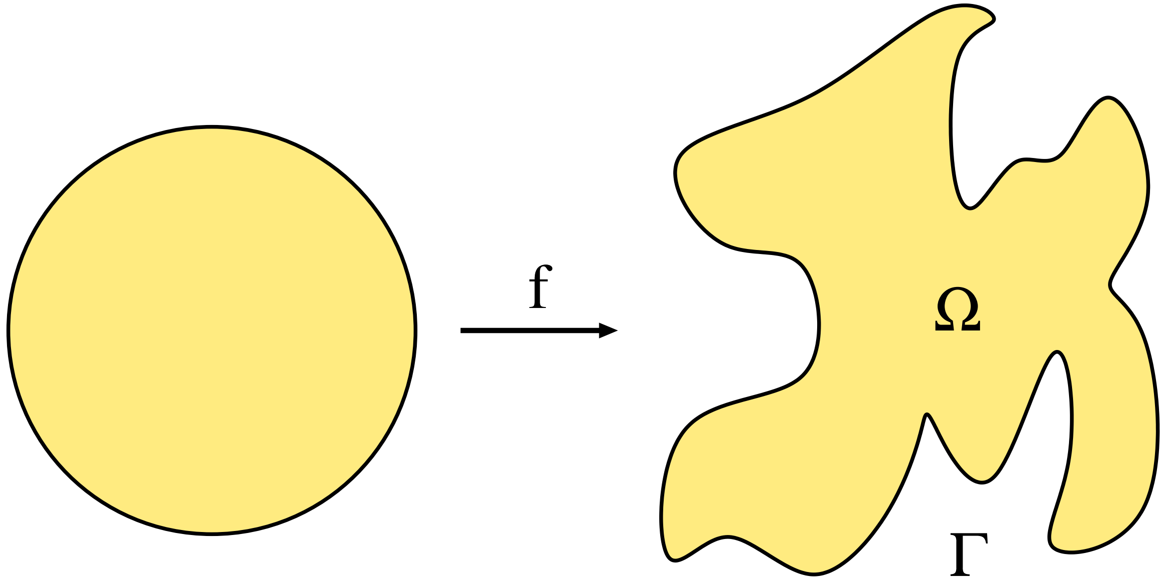


Not Weil-Petersson



For a conformal map $f : \mathbb{D} \to \Omega$, f' is never zero.

$\Rightarrow \log f'$ is well defined and holomorphic.



Suppose X is a space of holomorphic functions on $\mathbb{D}$,

e.g., L^p, VMO, BMO, Hardy spaces, Bergman, Bloch, Sobolev, ...

Problem: Characterize $\Gamma = f(\mathbb{T})$ so that $\log f' \in X$.

This has been solved for some X's.

Kari Astala and Michel Zinsmeister invented "BMO-Teichmüller theory" where $\log f' \in$ BMO (1990's).

(BMO = Bounded Mean Oscillation, close cousin of $L^\infty \subset \mathrm{BMO} \subset L^p$, $p < \infty$, "John-Nirenberg space")

Peter Jones and I characterized curves with $\log f' \in$ BMO.

(roughly speaking, Γ has "good" approximations by chord-arc curves)

In their memoir, Takhtajan and Teo prove:

Theorem: Γ is Weil-Petersson iff $u = \log f' \in W^{1,2}(\mathbb{D})$.

$W^{1,2}(\mathbb{D}) = \{u : |\nabla u| \in L^2(dxdy)\}$ = one derivative in L^2

Γ is WP iff $\int_{\mathbb{D}} |(\log f')'|^2 dxdy < \infty$.

In their memoir, Takhtajan and Teo prove:

Theorem: Γ is Weil-Petersson iff $u = \log f' \in W^{1,2}(\mathbb{D})$.

I learned this in the IPAM lecture of Yilin Wang. She proved $\log f' \in W^{1,2}$ iff Γ has finite **Loewner energy** (defined by her and Steffen Rohde).

Her work connects WP to large deviations of Schramm-Loewner evolutions (SLE(κ)) and the Brownian loop soup of Lawler and Werner.

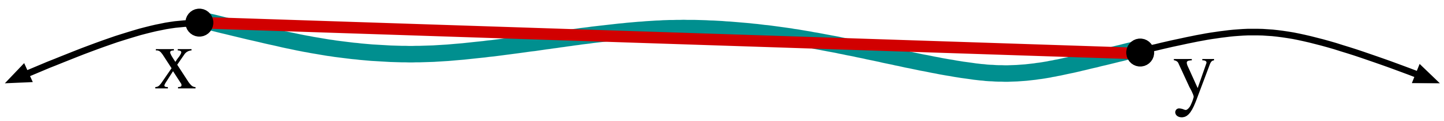
Brownian motion = random curves

SLE = random Jordan curves (no self-intersections)

At same IPAM conference, Atul Shekhar referred me to paper of Gallardo-Gutiérrez *et al.*, discussing a conjecture of Peter Jones, now a

Theorem: Γ is Weil-Petersson iff

$$\int_\Gamma \int_\Gamma \frac{\ell(x,y) - |x-y|}{|x-y|^3} ds dt < \infty$$



Strong version of "asymptotically smooth"

$\ell(x, y)$ = arclength distance between x, y along curve

For a chord-arc curve, $\ell(x,y) \simeq |x-y|$, so

$$\begin{aligned}
\frac{\ell(x,y)-|x-y|}{|x-y|^3} &\simeq \frac{(\ell(x,y)-|x-y|)(\ell(x,y)+|x-y|)}{|x-y|^2\ \ell(x,y)^2} \\
&= \frac{\ell(x,y)^2-|x-y|^2}{|x-y|^2\ \ell(x,y)^2} \\
&= \frac{1}{|x-y|^2} - \frac{1}{\ell(x,y)^2}
\end{aligned}$$

The last term has been considered in knot theory.

The **Möbius energy** of a curve $\Gamma \in \mathbb{R}^n$ is

$$\text{Möb}(\Gamma) = \int_\Gamma \int_\Gamma \left(\frac{1}{|x-y|^2} - \frac{1}{\ell(x,y)^2} \right) ds dt.$$

Theorem: Γ is WP iff $\text{Möb}(\Gamma) < \infty$.

The **Möbius energy** of a curve $\Gamma \in \mathbb{R}^n$ is

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Möbius energy is Hadamard renormalization of divergent integral

$$\int_\Gamma \int_\Gamma \frac{dsdt}{|x-y|^2}.$$

= energy needed to place arclength charge on Γ under inverse-cube force.

(inverse cube law = electrostatic repulsion in 4 dimensions.)

Möbius energy is Möbius invariant (hence the name).

The **Möbius energy** of a curve $\Gamma \in \mathbb{R}^n$ is

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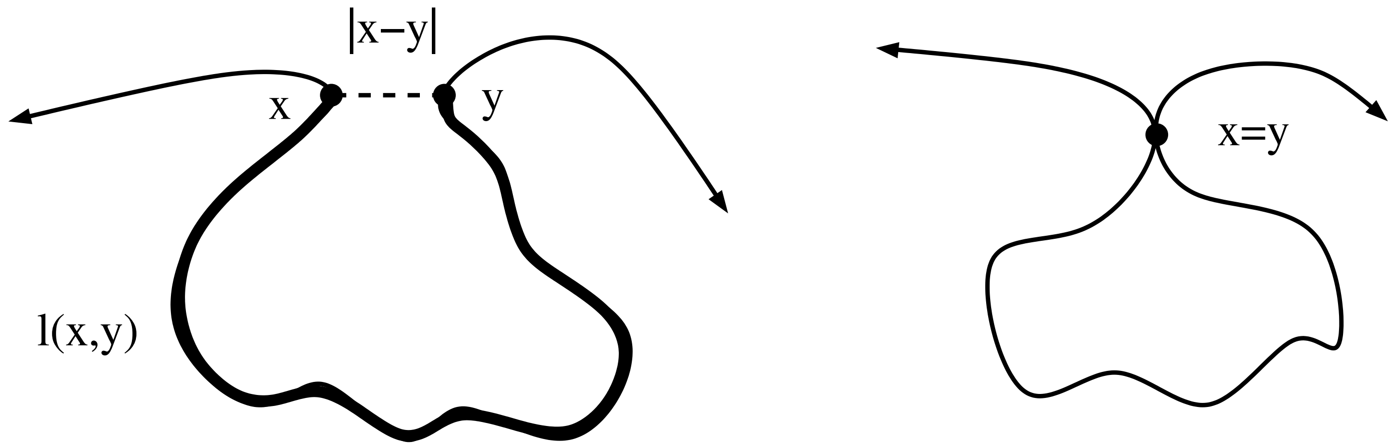


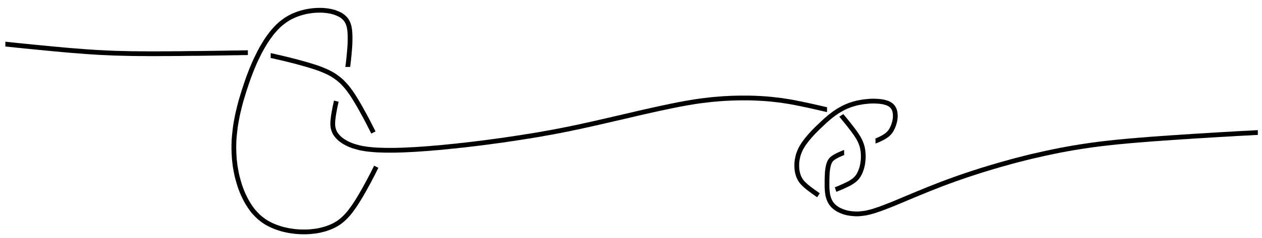
Möbius energy blows up if curve self-intersects.

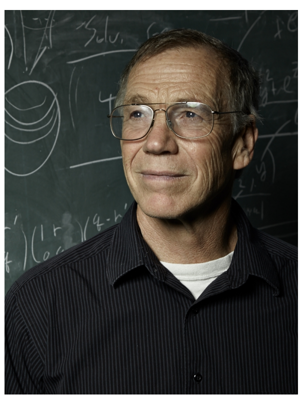
$\Rightarrow$ deforming Γ to lower energy doesn't change topology.

$\Rightarrow$ minimizing should give canonical representation of a knot.

Möbius energy is one of several “knot energies” due to Jun O’Hara.

Studied by Freedman, He and Wang in 1990’s. They showed:
- Möb(Γ) is Möbius invariant (hence the name),
- that finite energy curves are chord-arc,
- and in $\mathbb{R}^3$ they are topologically tame.



Theorem (Blatt, 2012): $\text{Möb}(\Gamma) < \infty$ iff arclength parameterization is $H^{3/2}$.

$H^{3/2}$ = Sobolev space = $\frac{3}{2}$-derivative in L^2.

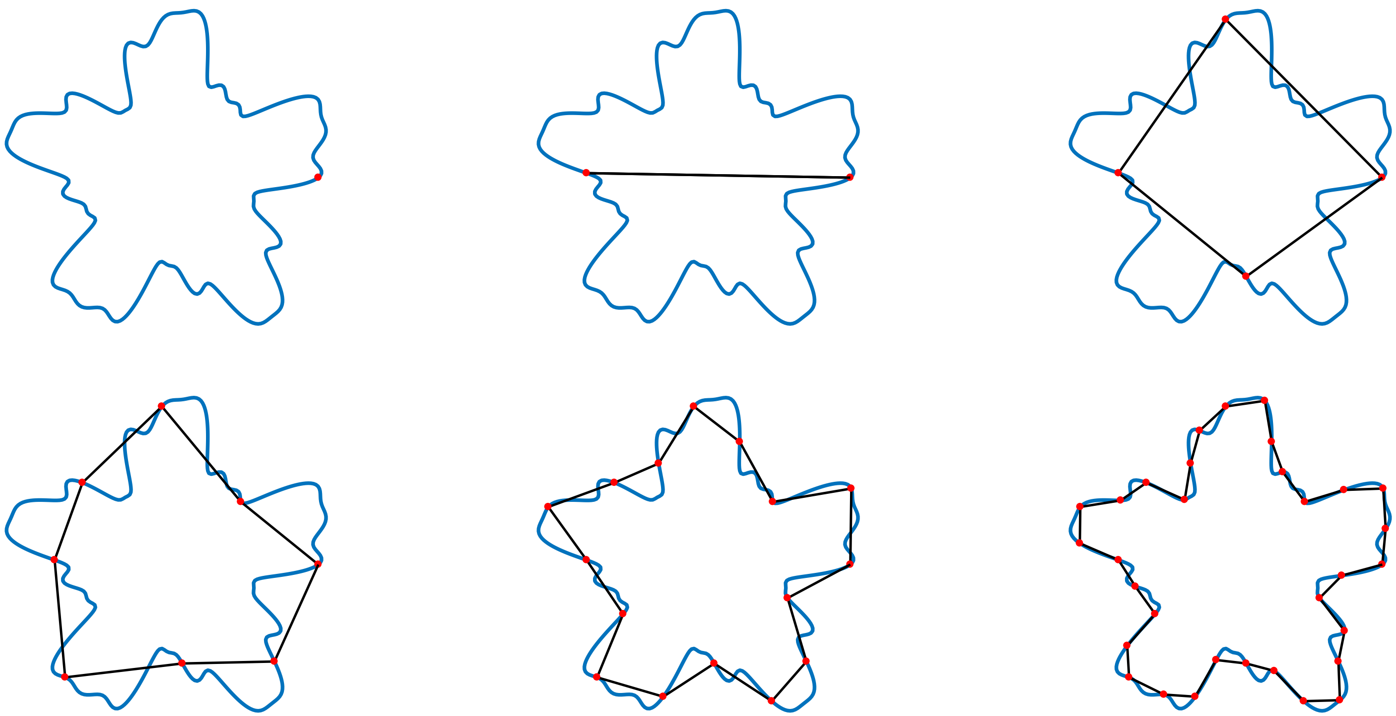
Cor: Γ is WP iff arclength parameterization is in $H^{3/2}$.

Quasiconformal maps and $H^{3/2}$ and are pretty sophisticated.

How can you describe WP curves to a calculus student?

Dyadic decomposition.

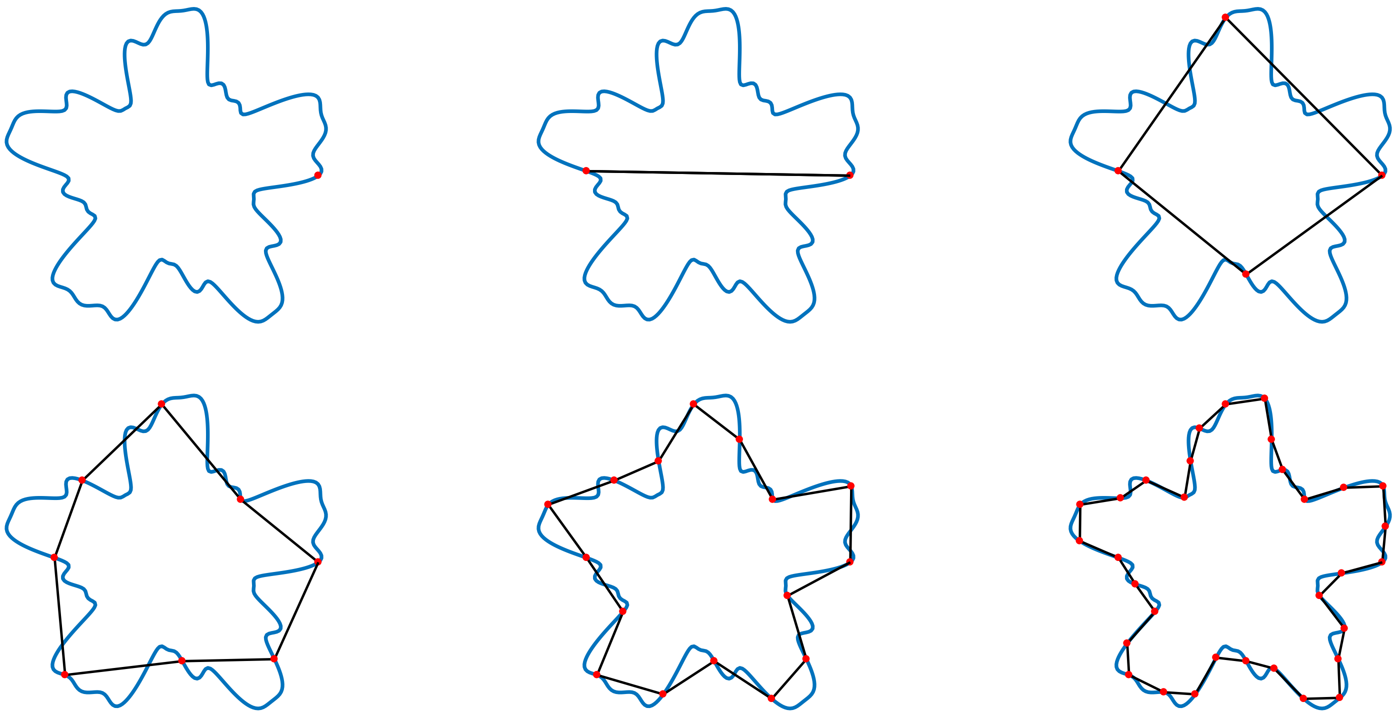
- Divide Γ into nested families of 2^n equal length arcs.
- Inscribe a polygon Γ_n at these points.
- Clearly $\ell(\Gamma_n) \nearrow \ell(\Gamma)$.



Theorem: Γ is Weil-Petersson if and only if

$$\sum_{n=1}^{\infty} 2^n \left[\ell(\Gamma) - \ell(\Gamma_n)\right] < \infty$$

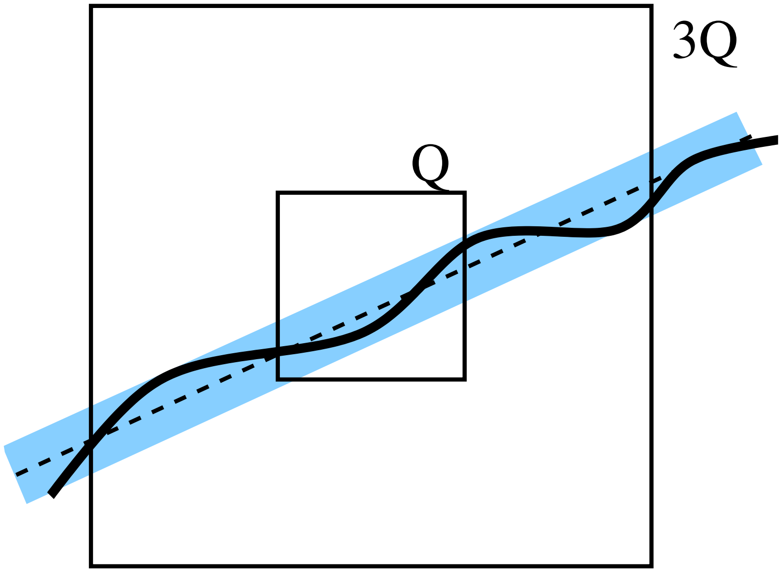
with a bound that is independent of the dyadic family.



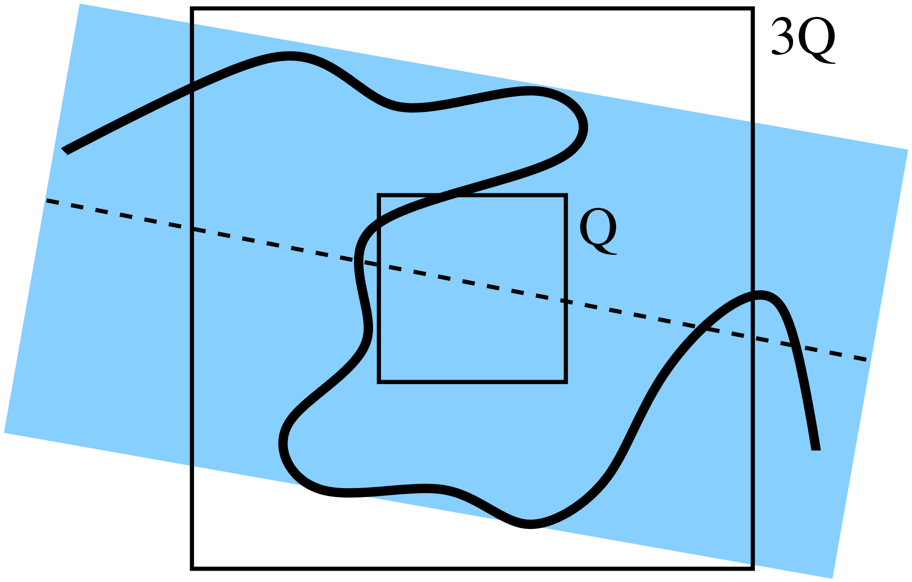
Peter Jones's β-numbers:

$$\beta_\Gamma(Q) = \inf_L \sup\{\frac{\mathrm{dist}(z,L)}{\mathrm{diam}(Q)} : z \in 3Q \cap \Gamma\},$$

where the infimum is over all lines L that hit $3Q$.



$\beta \approx 0$

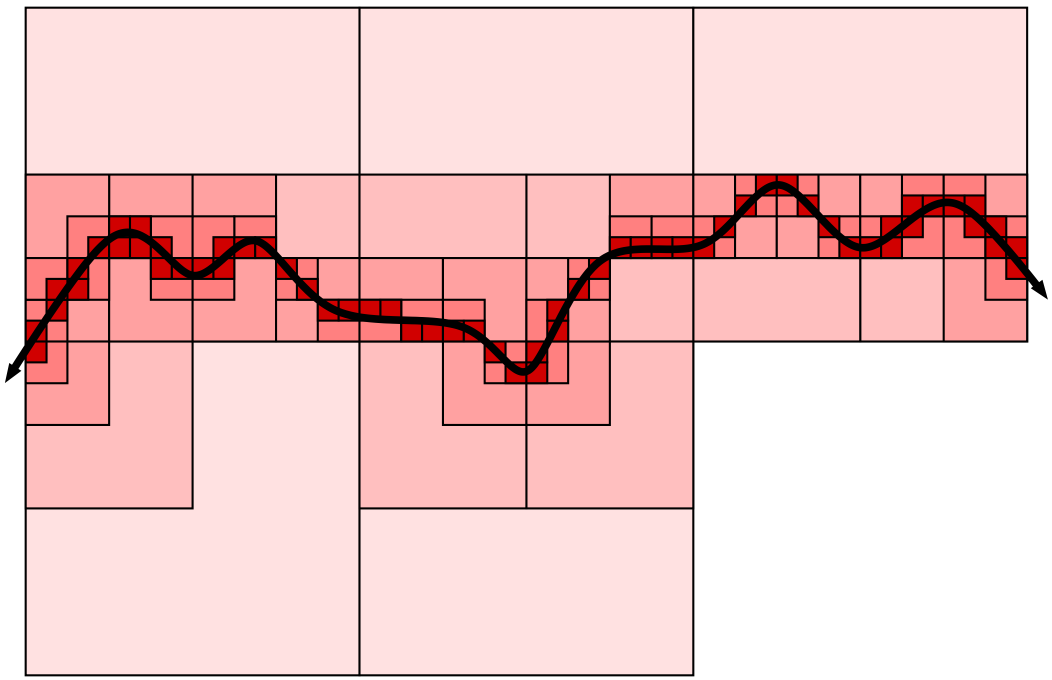


$\beta \approx 1$

Jones invented the β-numbers for his traveling salesman theorem:

$$\ell(\Gamma) \simeq \mathrm{diam}(\Gamma) + \sum_Q \beta_\Gamma(Q)^2 \mathrm{diam}(Q),$$

where the sum is over all dyadic cubes Q in $\mathbb{R}^n$ hitting Γ.



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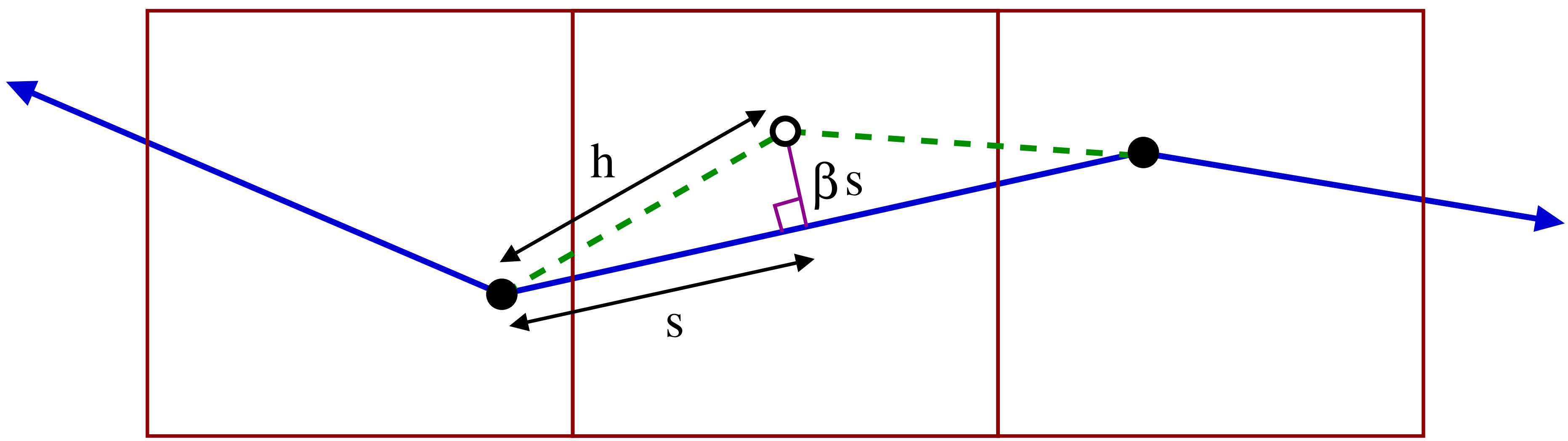
Idea of proof is just the Pythagorean theorem:



$$h = \sqrt{s^2 + (\beta s)^2} = s + O(s\beta^2)$$

Jones invented the β-numbers for his traveling salesman theorem:

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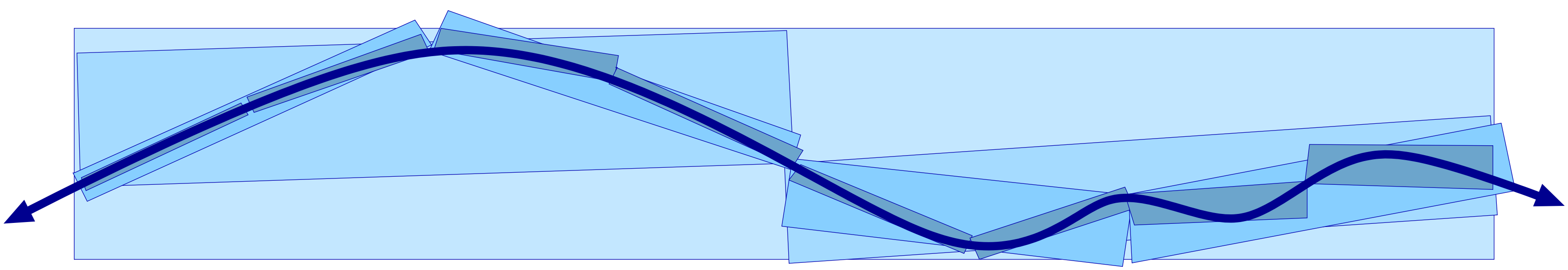
where the sum is over all dyadic cubes Q in $\mathbb{R}^n$ hitting Γ.

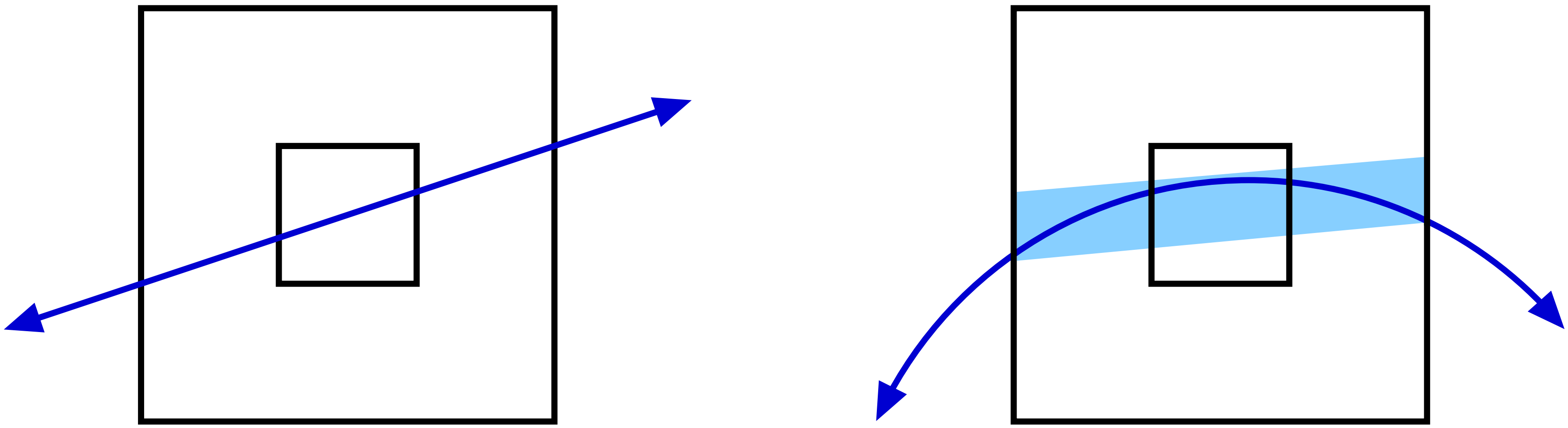
Theorem: Γ is Weil-Petersson iff $\sum_Q \beta_\Gamma(Q)^2 < \infty$.



WP = "curvature in L^2, summed over all positions and scales".
= "rectifiable in scale invariant way".

The Weil-Petersson class is Möbius invariant.

β-numbers are not: lines ($\beta = 0$) can map to circles ($\beta > 0$).

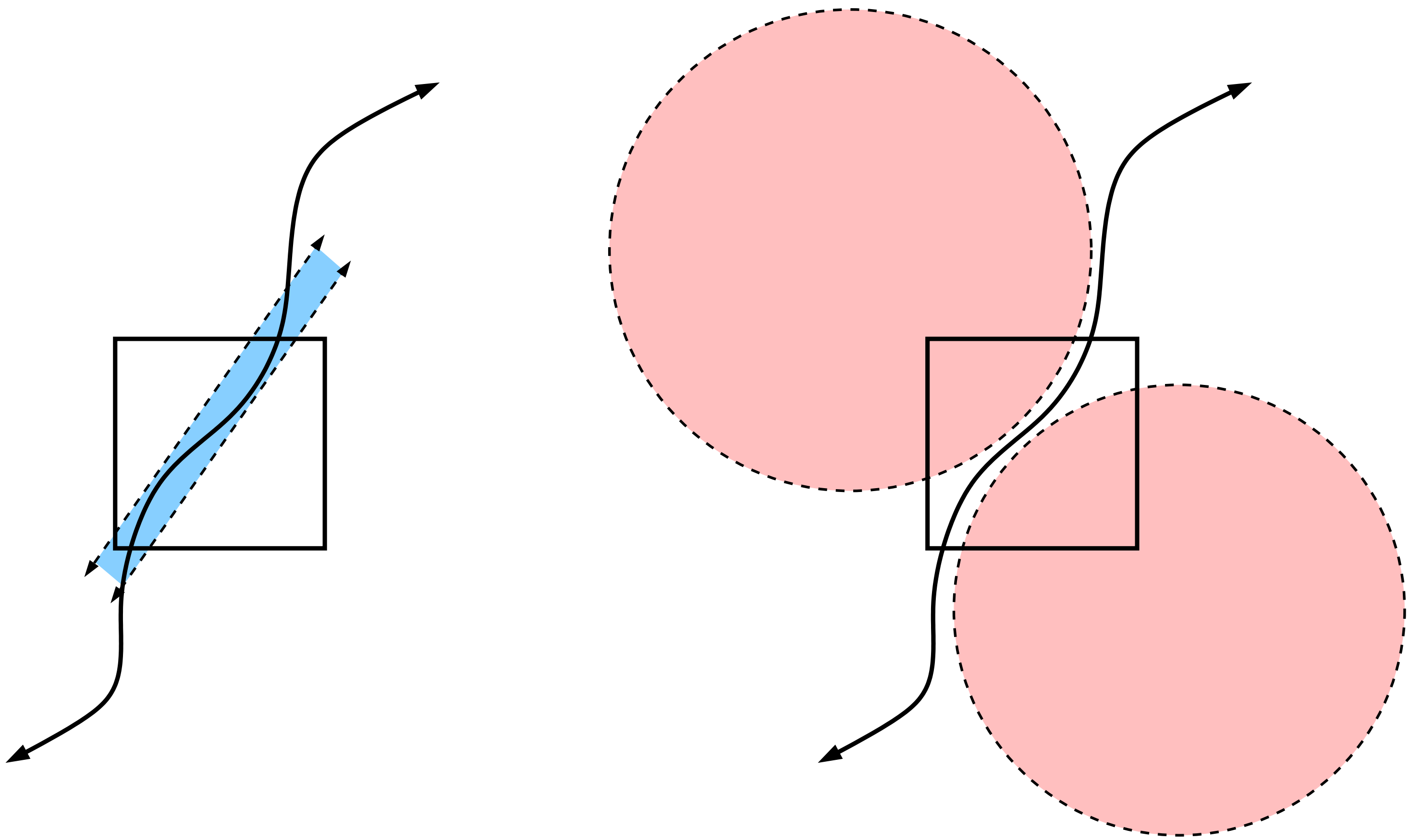


What is a Möbius invariant version of the β-numbers?

Möbius = linear fractional = $\frac{az+b}{cz+d}$ = conformal self-maps of sphere

Möbius transformations preserve lines/circles.

β-numbers trap curve between lines. Trap curve between disks instead.



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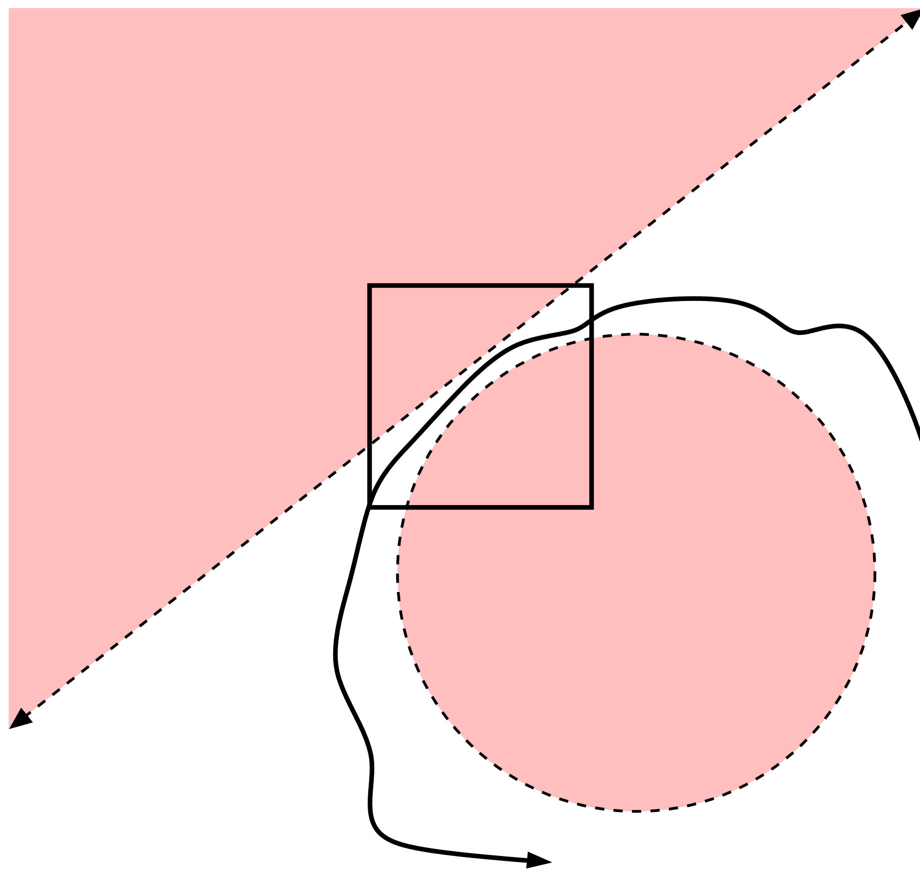

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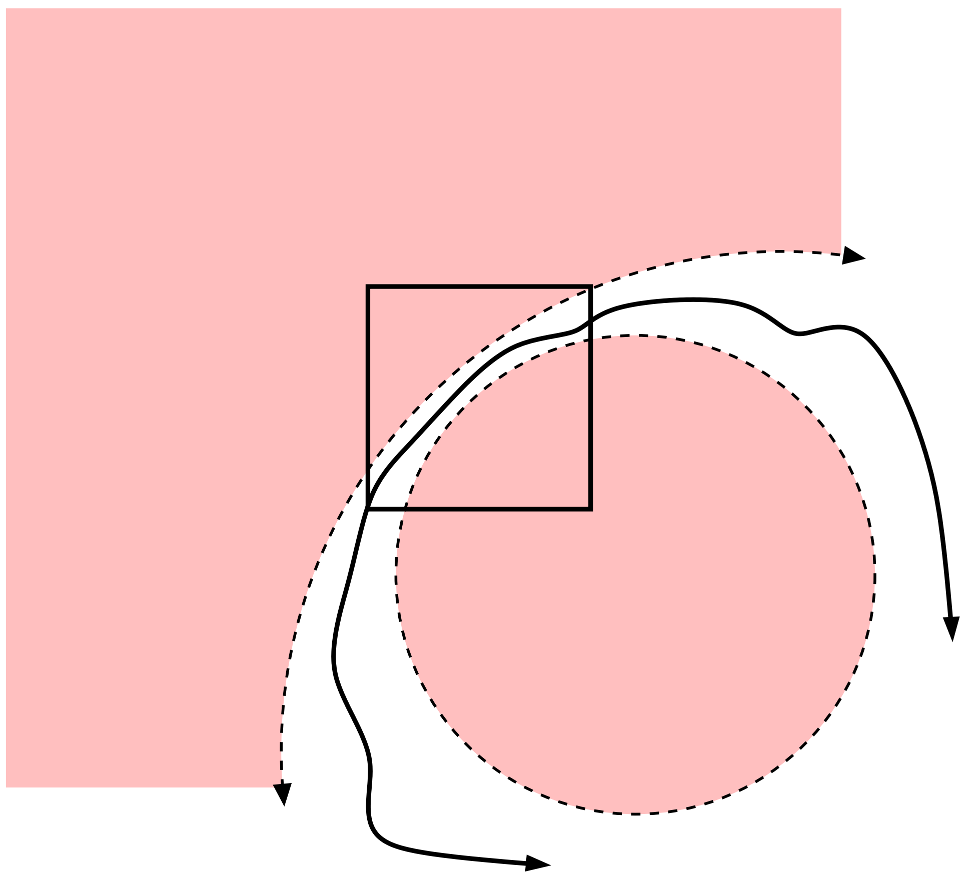
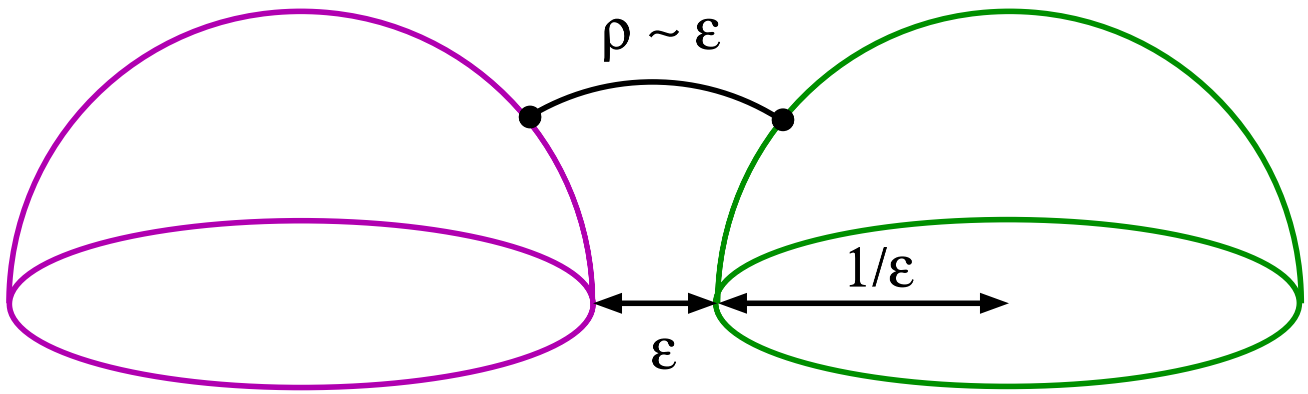


Each disk is the base of a hemisphere in the upper half-space $\mathbb{H}^3 = \mathbb{R}^3_+$.

The hyperbolic distance between these hemispheres is $\lesssim \varepsilon(Q)$.



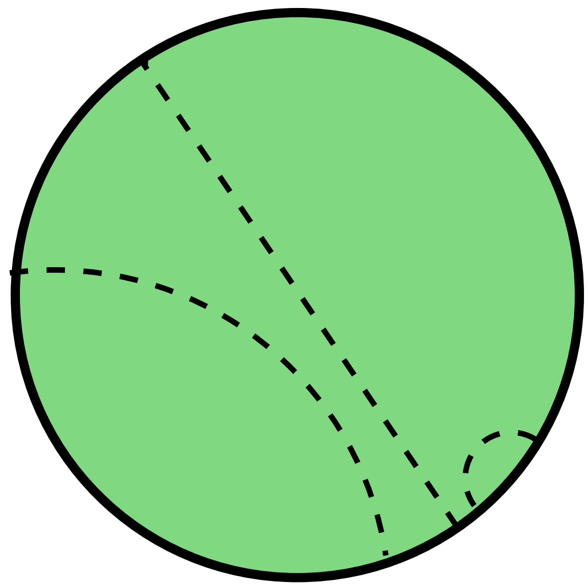
These Euclidean hemispheres are hyperbolic half-spaces.

Möbius transformation of plane extends to isometry of upper half-space.

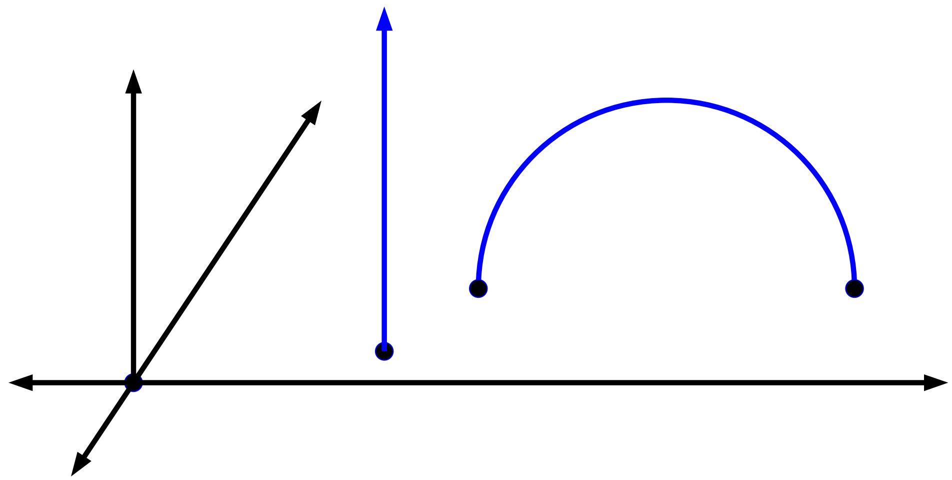
Hyperbolic metric on disk given by

$$d\rho = \frac{ds}{1-|z|^2} \simeq \frac{ds}{\mathrm{dist}(z, \partial D)}.$$

Geodesics are circles perpendicular to boundary (or diameters).

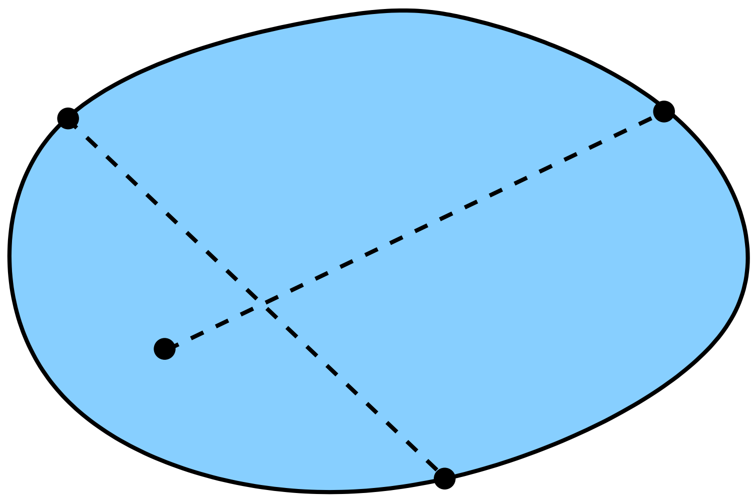


In the upper half-space $\mathbb{R}^3_+ = \{(x, y, t) : t > 0\}$, metric is $d\rho = ds/2t$.

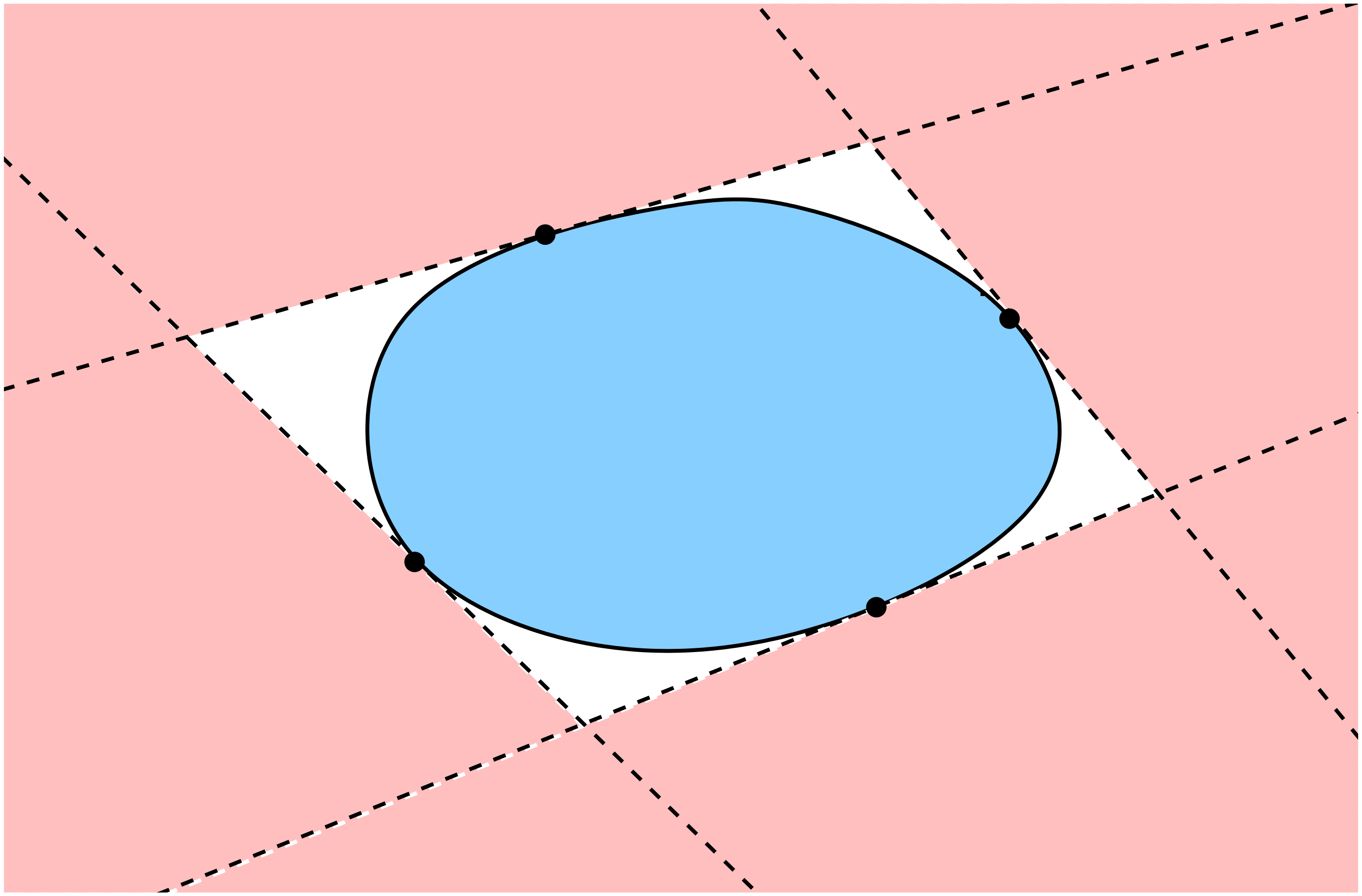


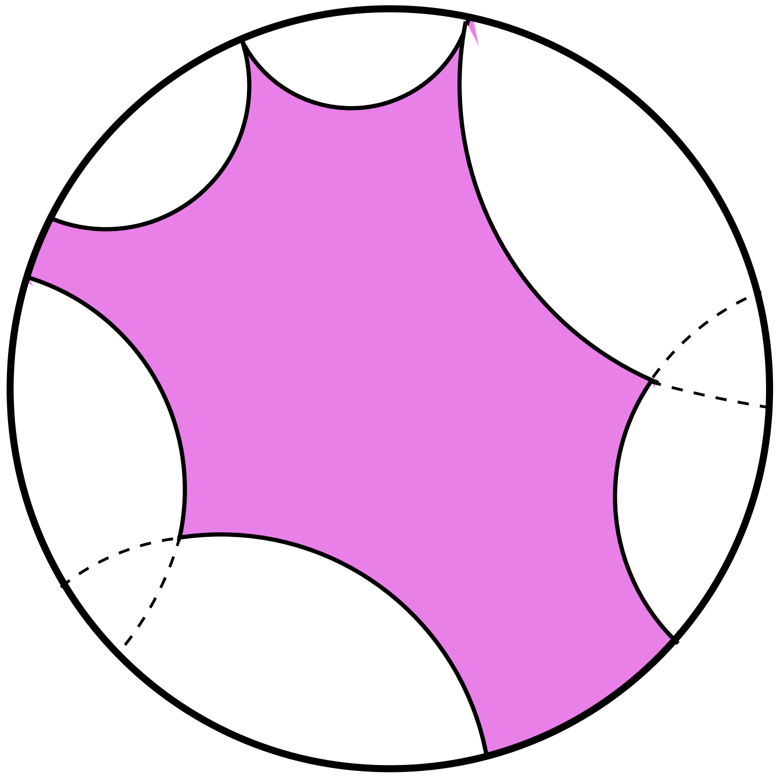
Geodesics in $\mathbb{R}^3_+$ are vertical rays or semi-circles perpendicular to $\mathbb{R}^2$.

Usual definition of convex: contains geodesic between any two points.



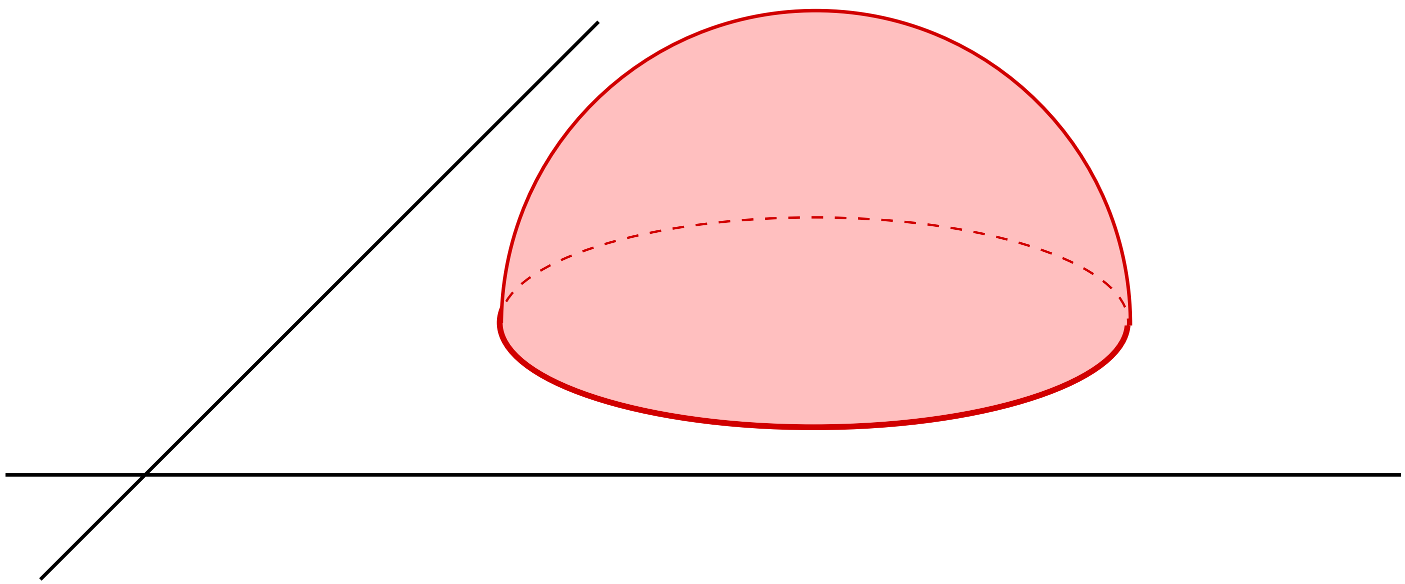
More useful for us: complement is a union of half-spaces.





Convex set in hyperbolic disk

Complement = union of half-spaces



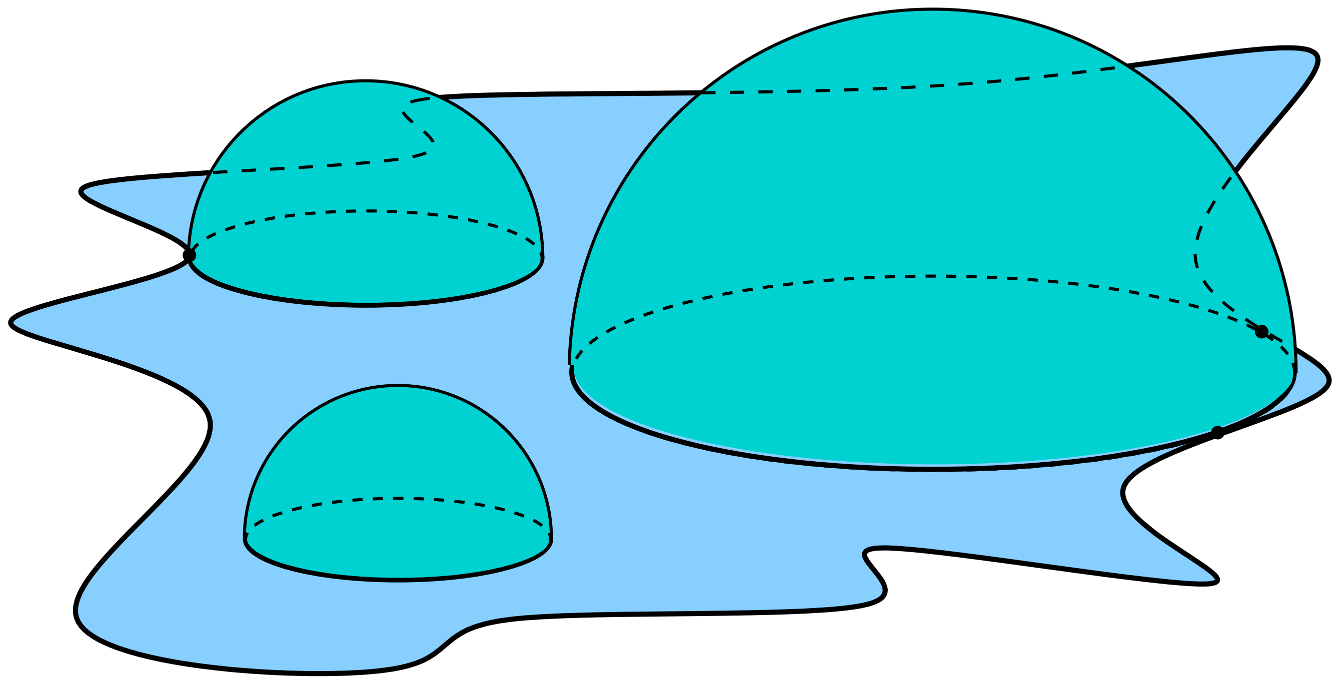
In $\mathbb{R}^3_+$, a hyperbolic half-space = hemisphere.

$\mathrm{CH}(\Gamma)$ = complement of all open half-spaces that miss Γ.

In general, $\mathrm{CH}(\Gamma)$ has non-empty interior and 2 boundary components.

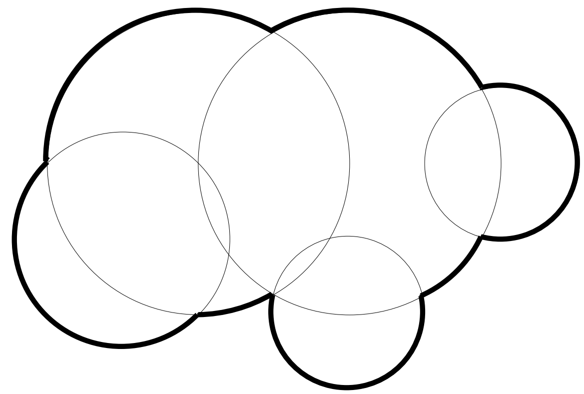
A hyperbolic half-space missing CH(Γ) has boundary disk missing Γ.

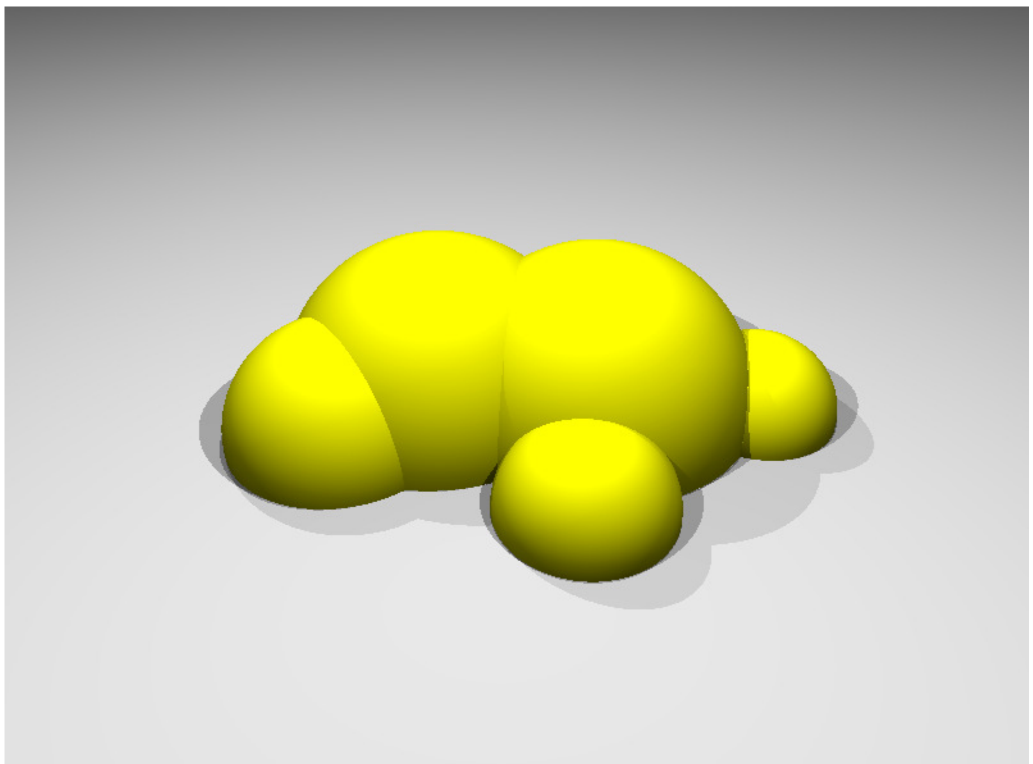
This disk is inside or outside Γ. Dome(Ω) is union over "inside" disks.

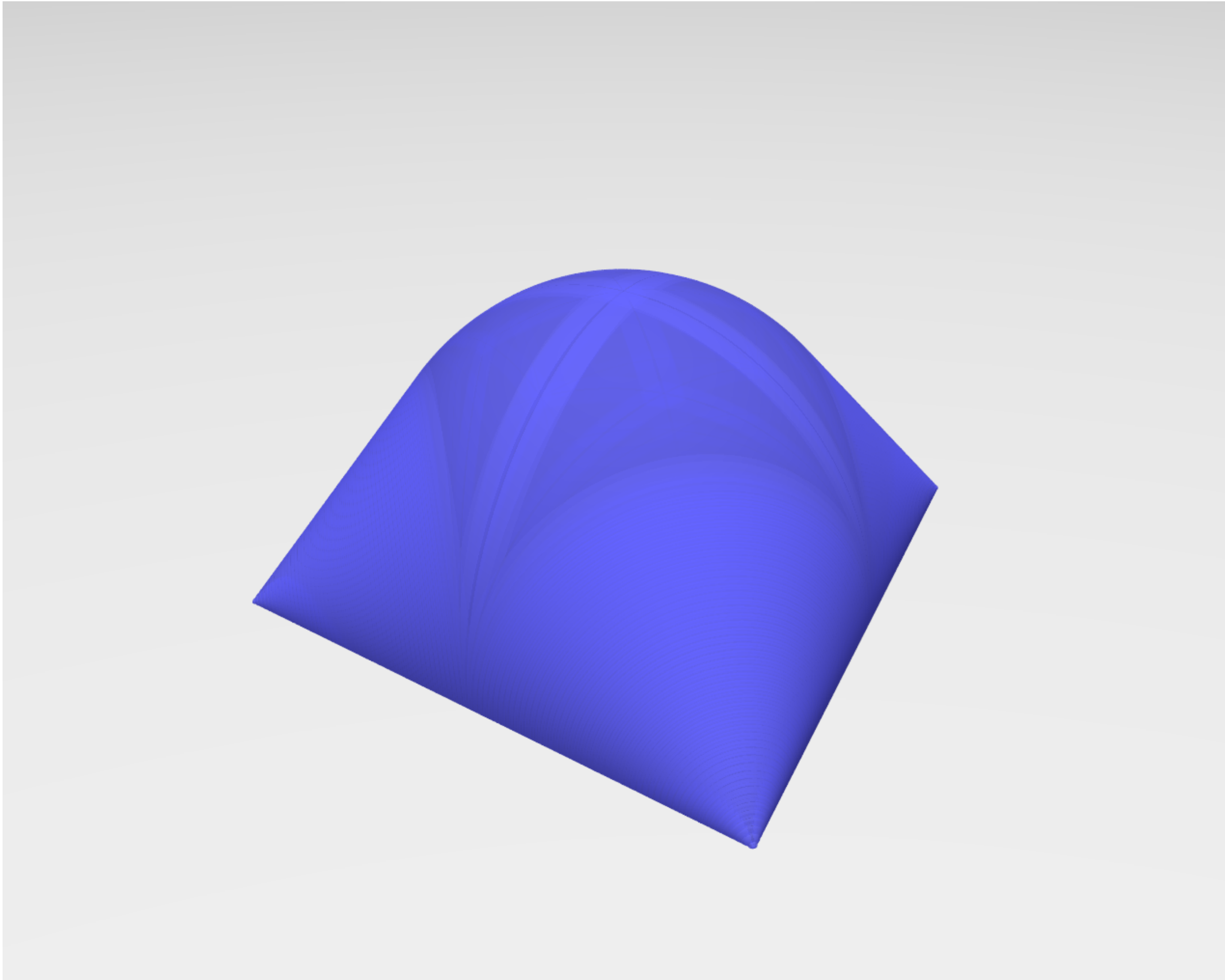


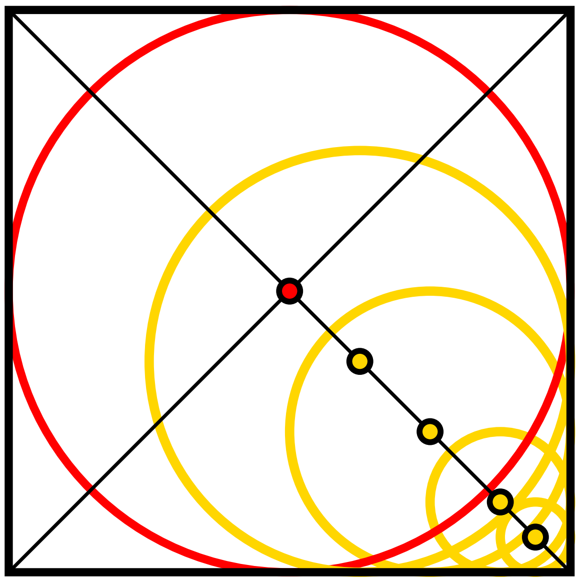
Region above dome is intersection of half-spaces, hence convex.

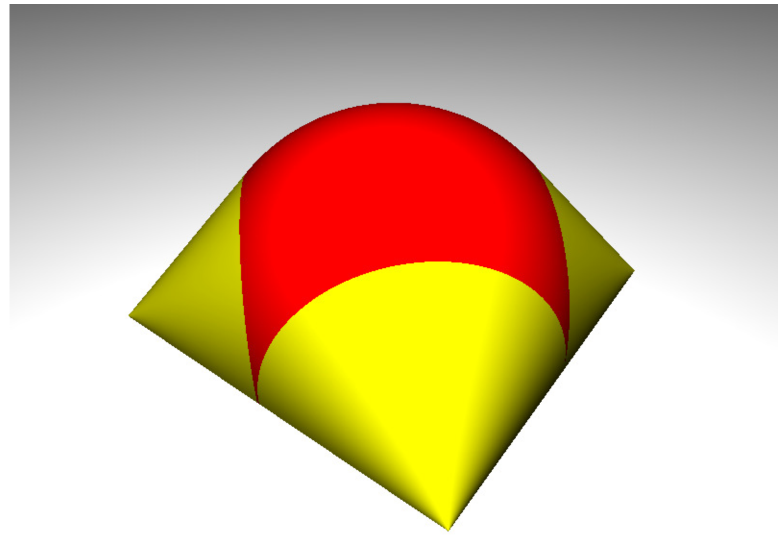
CH(Γ) is region between domes for "inside" and "outside" of Γ.

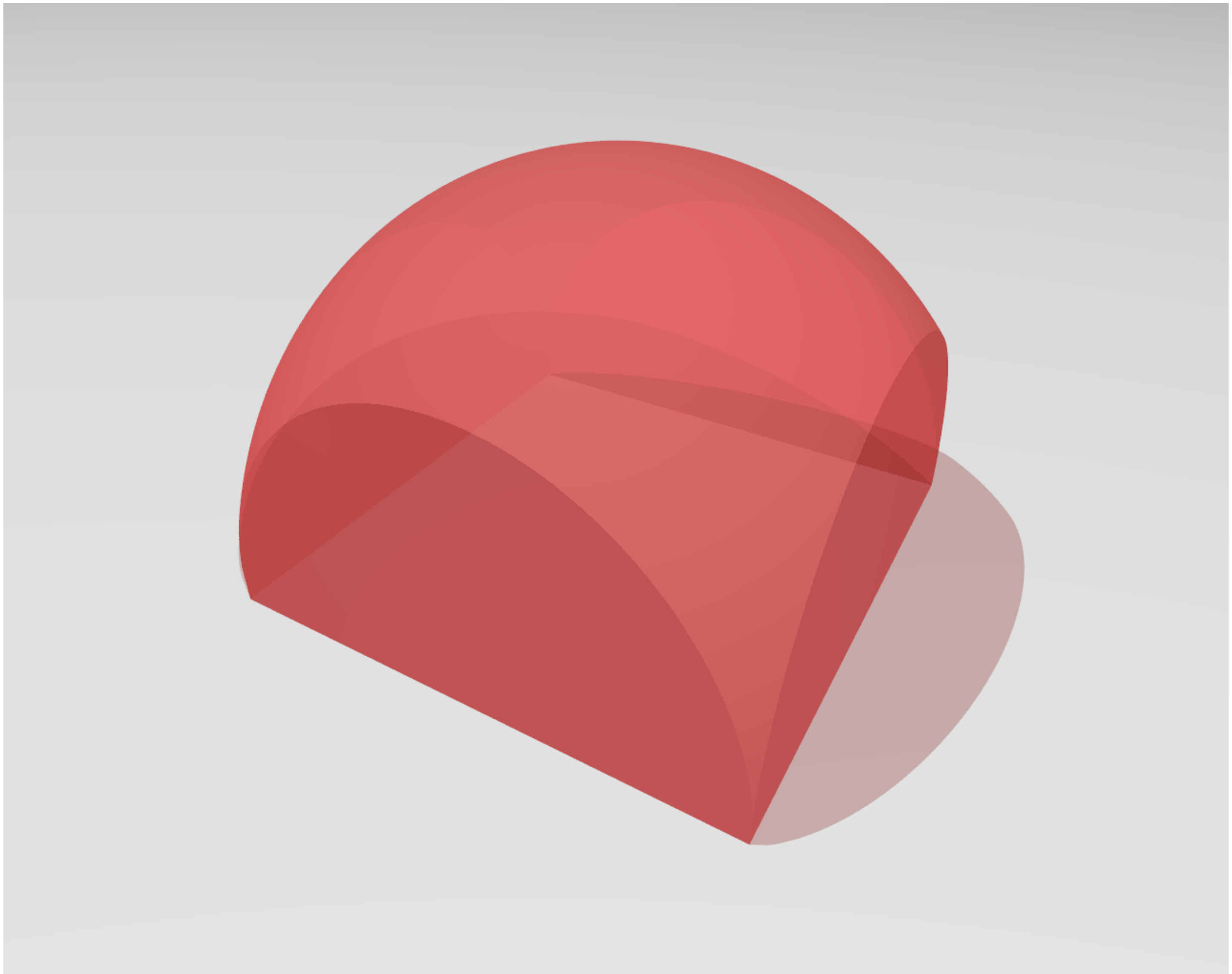


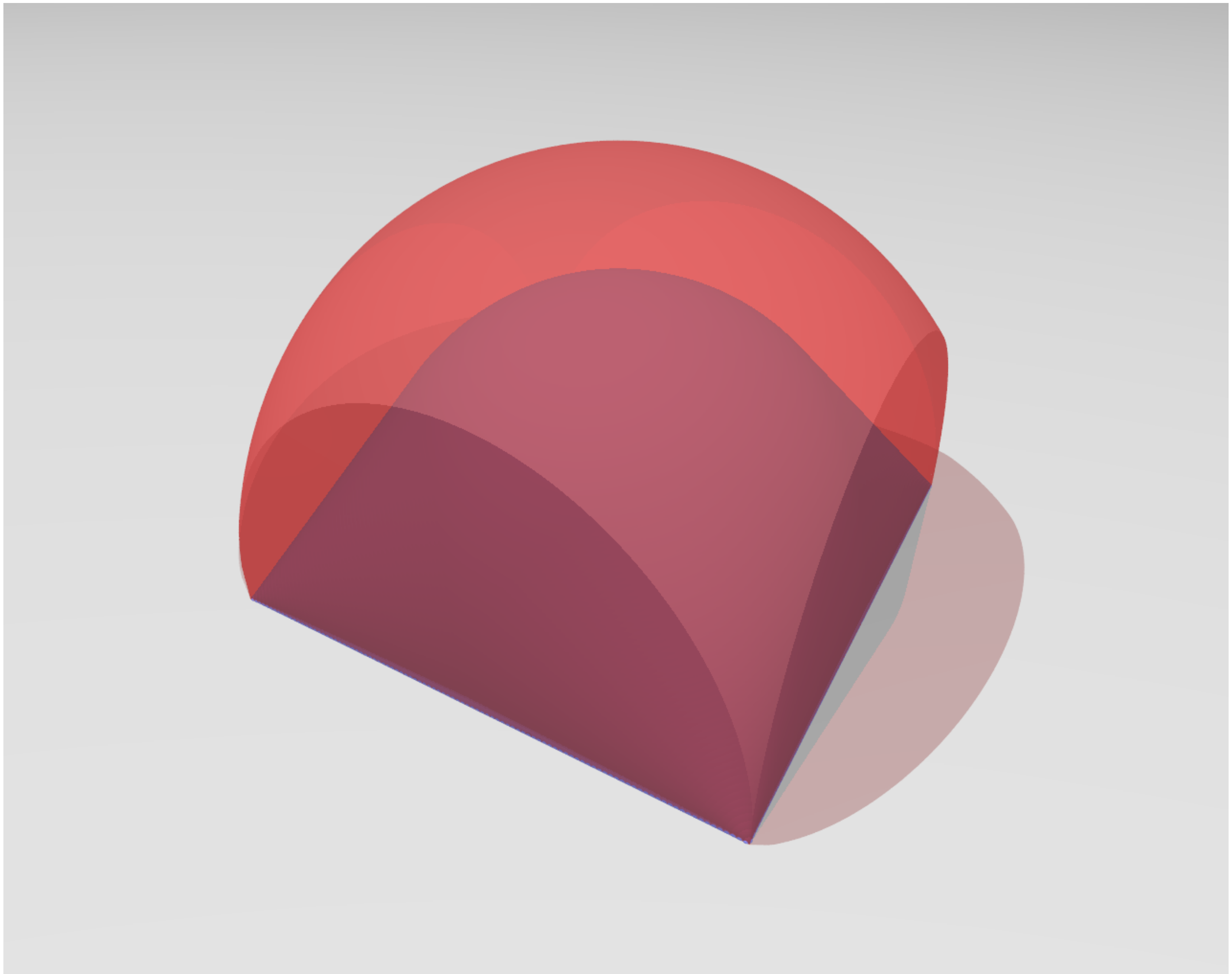


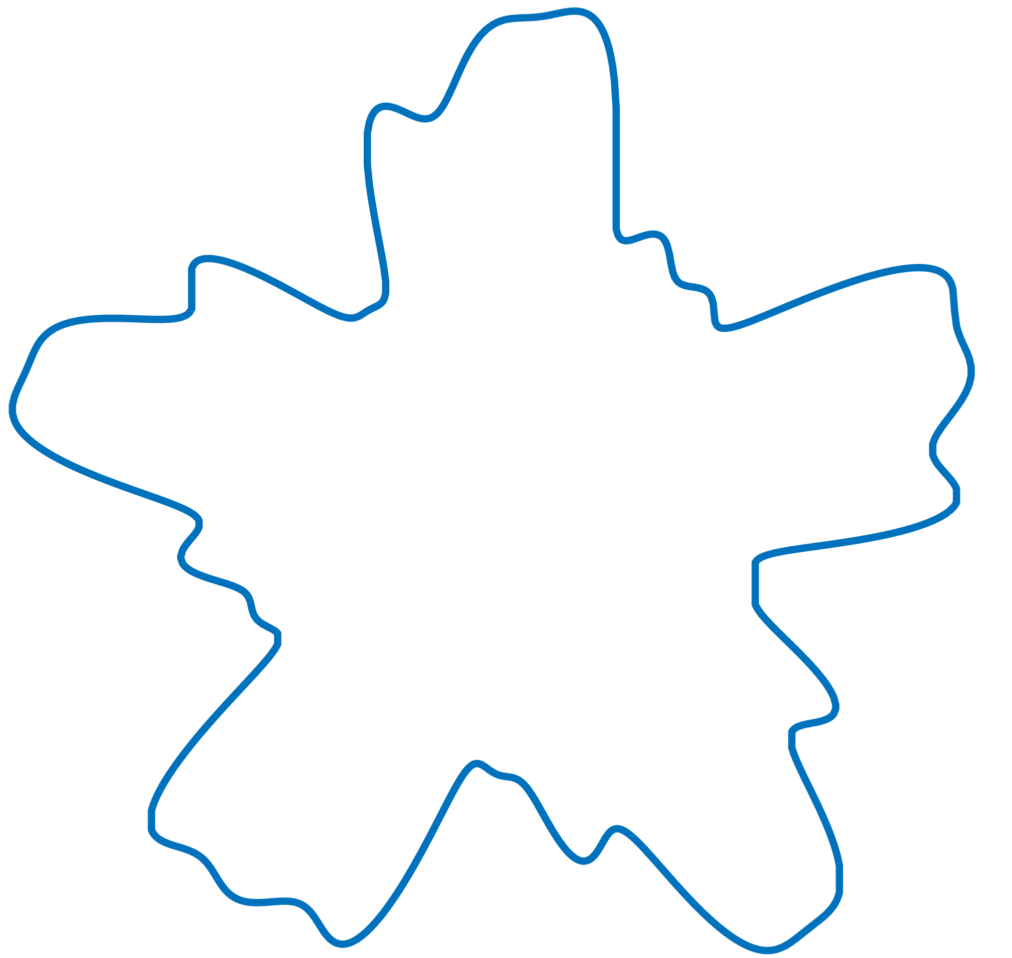


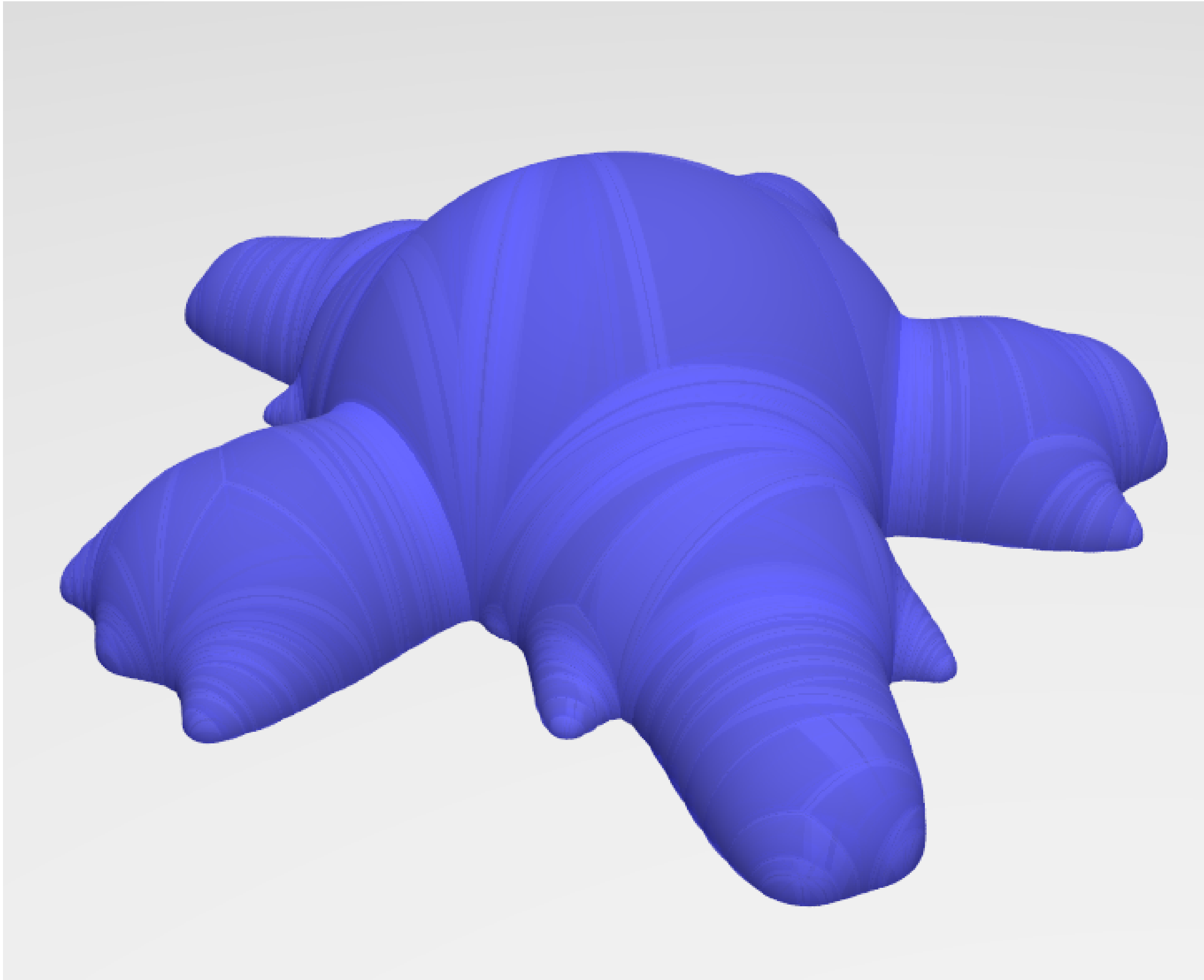


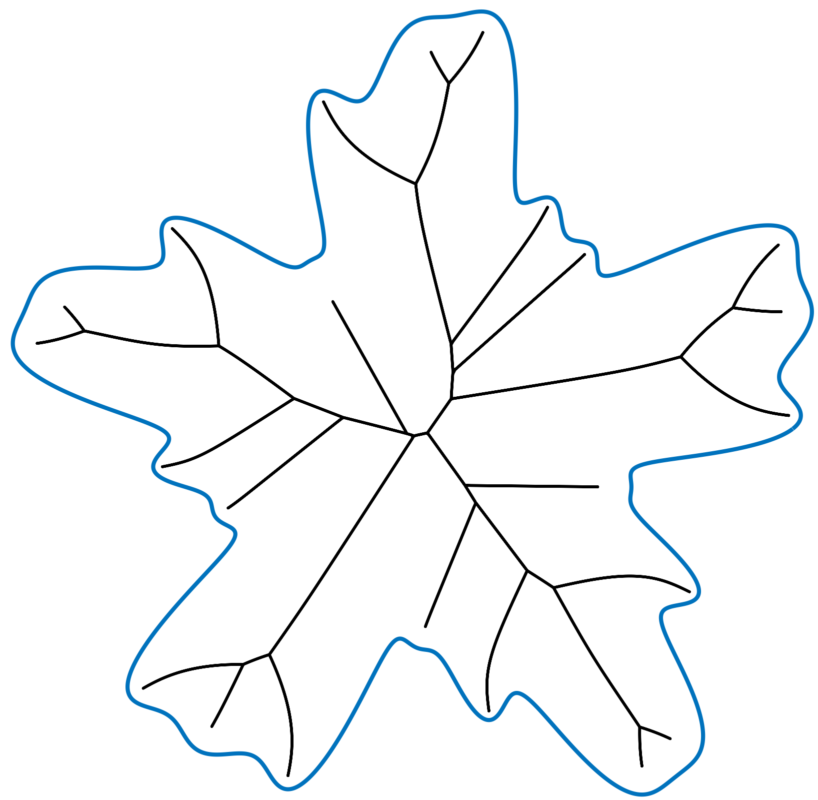




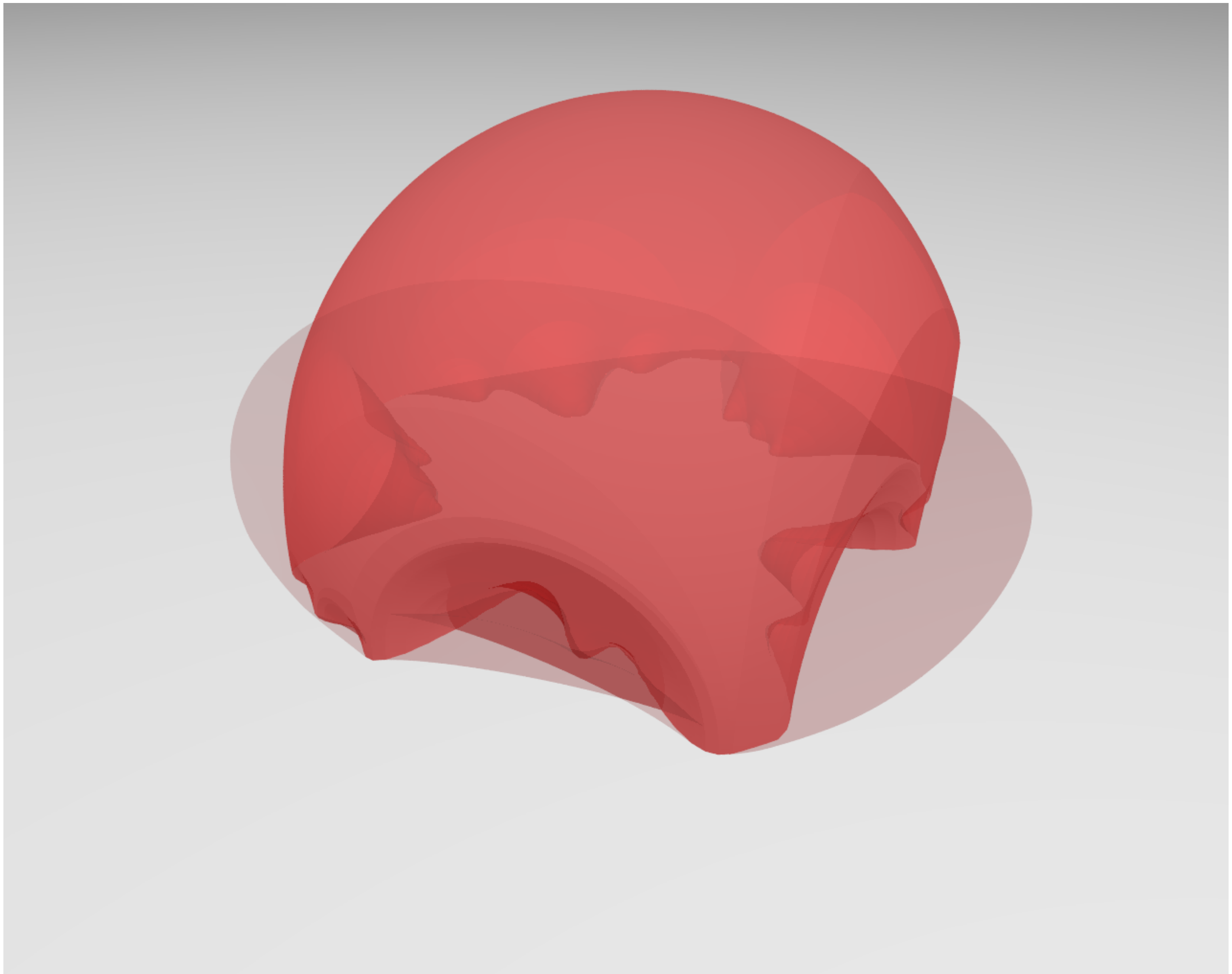


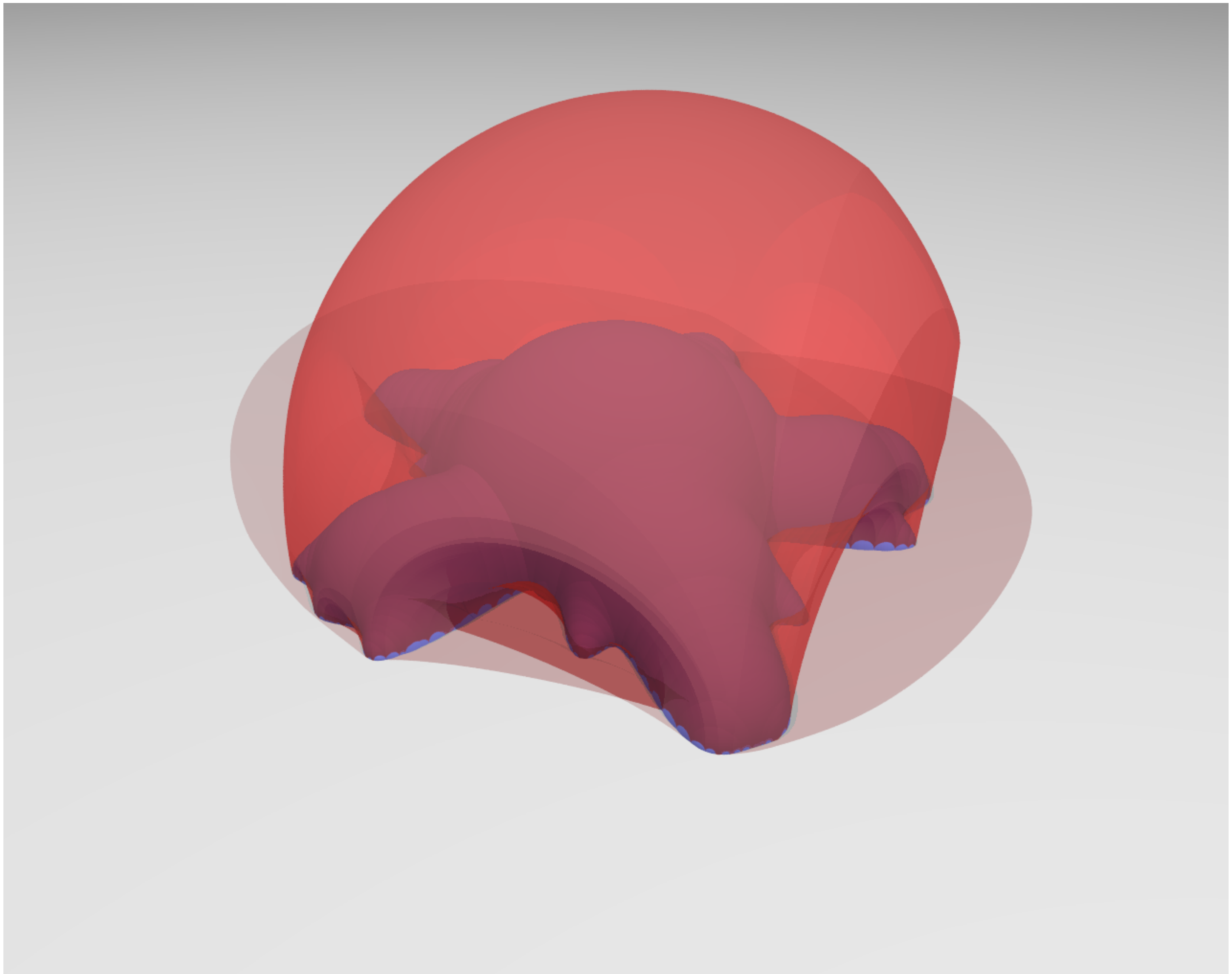


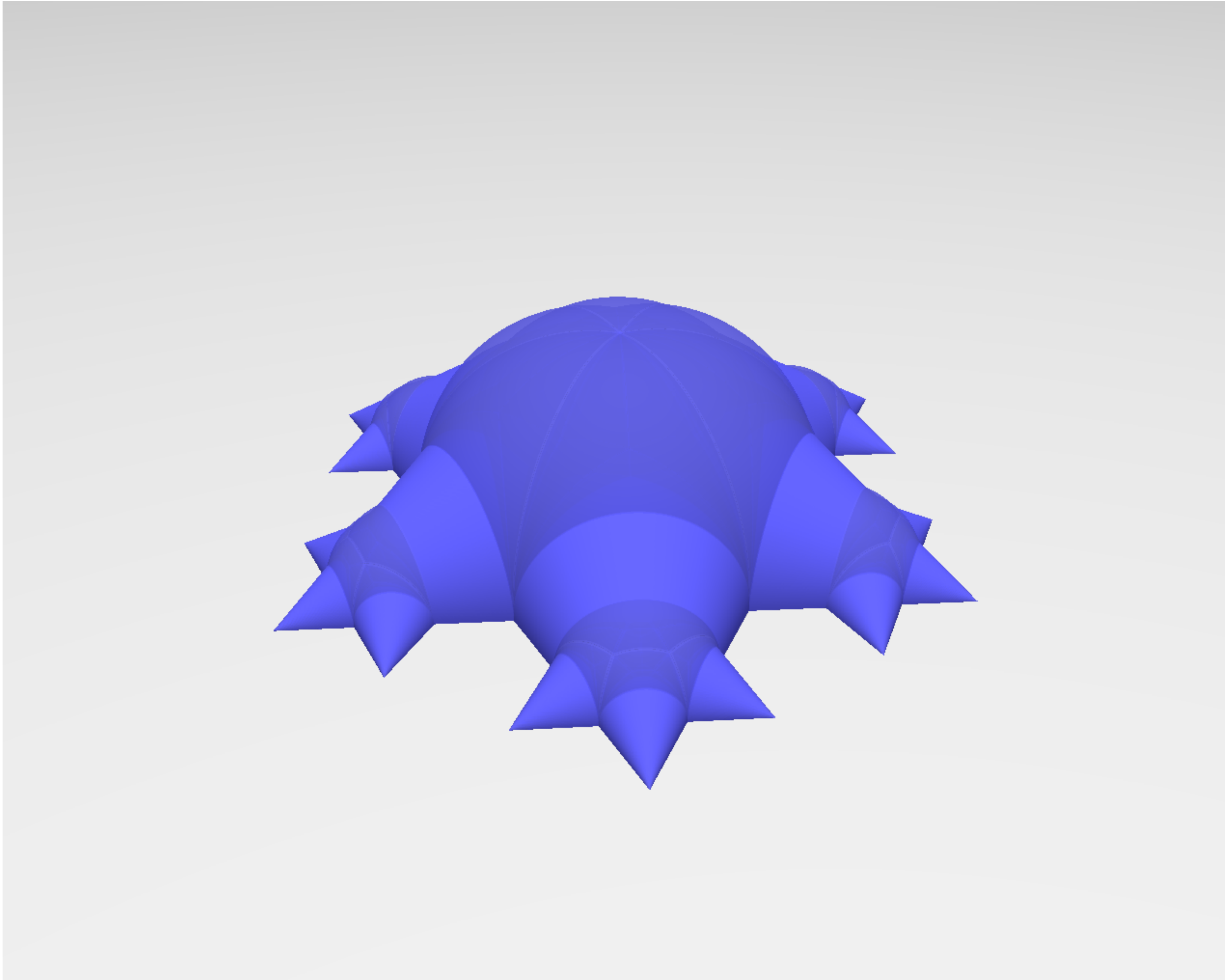


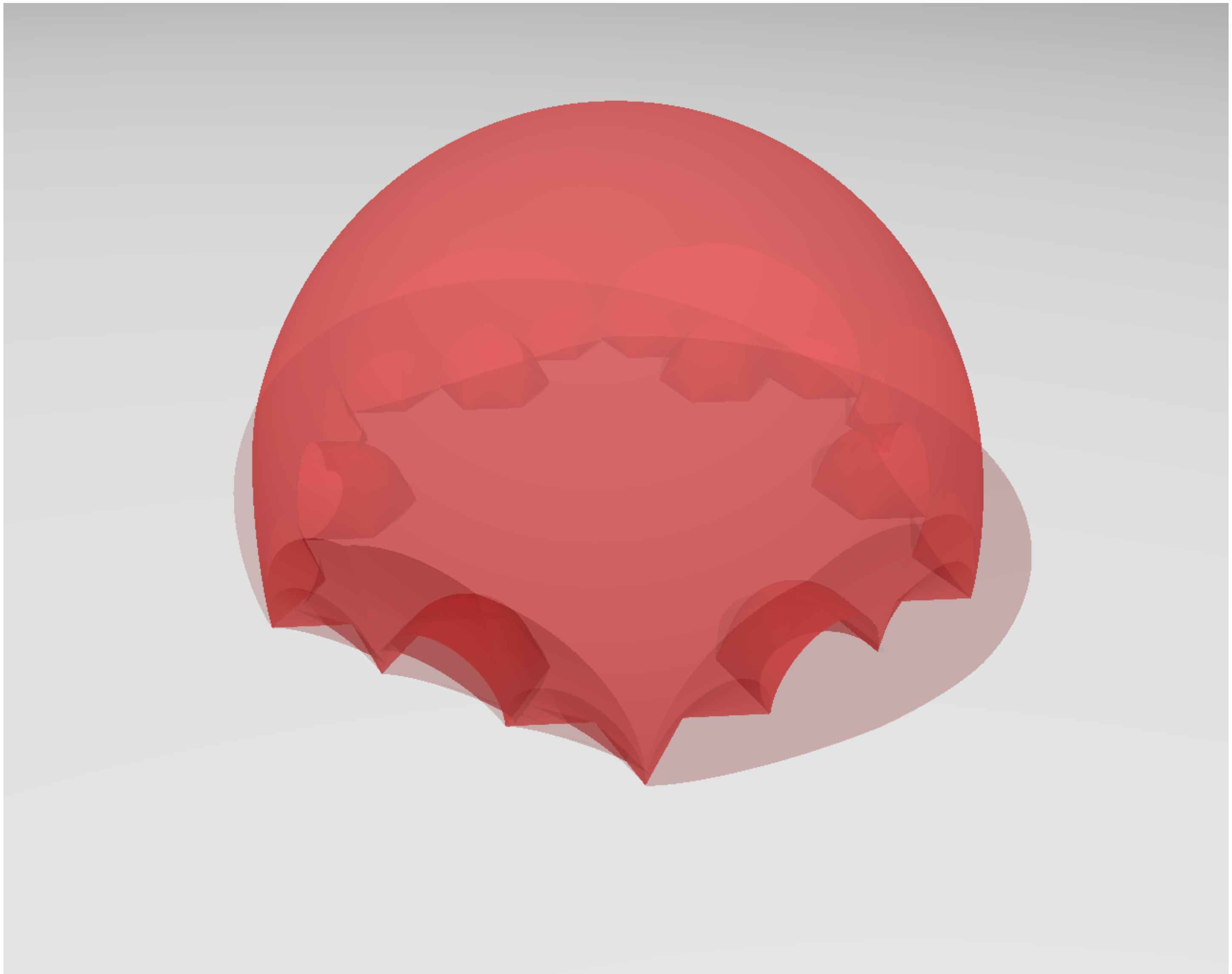


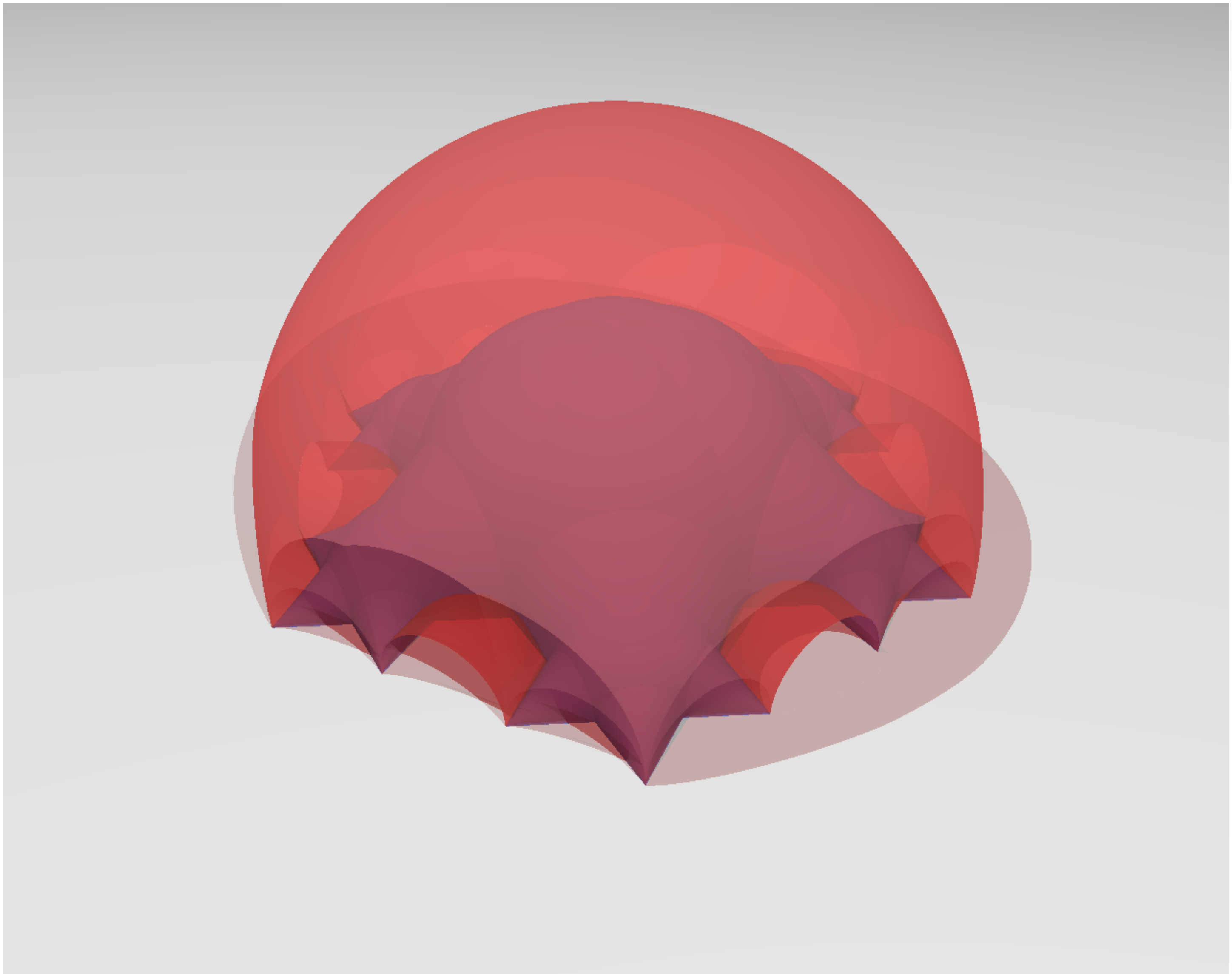
The medial axis. Equidistant from at least two boundary points.
Corresponding hemispheres give the dome.
Medial axis is widely studied in computational geometry.



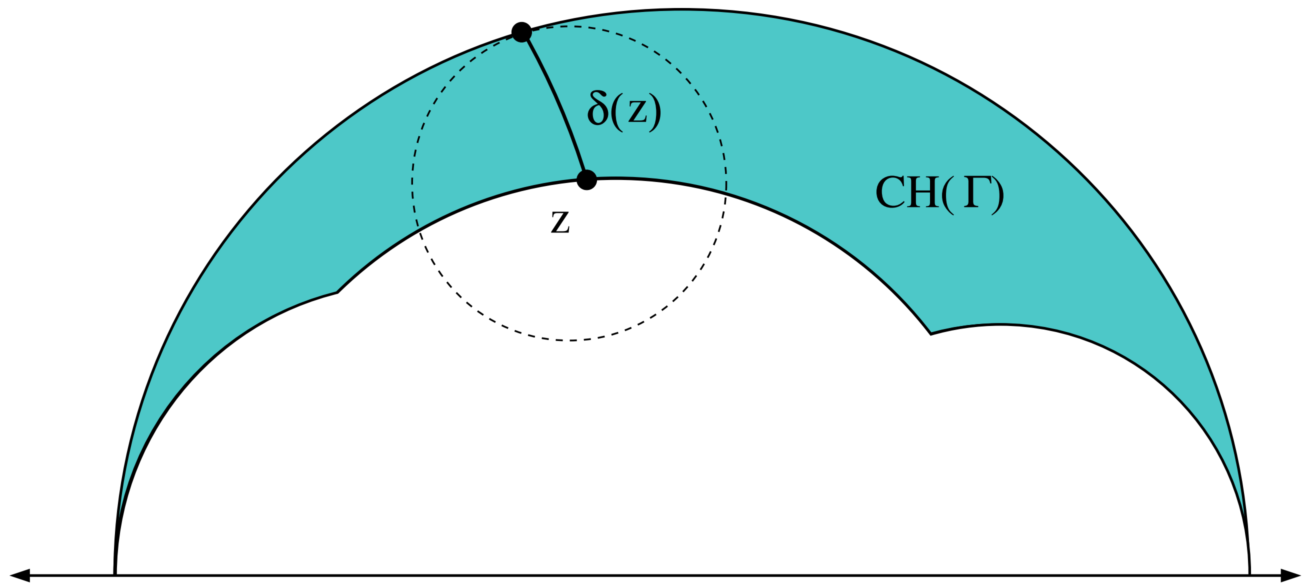








Let $\delta(z)$ be the maximum distance from z to the components of $\partial \mathrm{CH}(\Gamma)$.
Can show $\delta(z) \lesssim \varepsilon_\Gamma(Q)$, for Q "near" z.



Theorem: Γ is Weil-Petersson implies $\int_{\partial \mathrm{CH}(\Gamma)} \delta^2(z) dA_\rho < \infty$.

$\delta =$ "conformally invariant β"

$\partial \mathrm{CH}(\Gamma)$ has two components. Nicer to have single surface with $\partial S = \Gamma$.

Let S be a surface in $\mathbb{H}^3$ that has asymptotic boundary Γ.

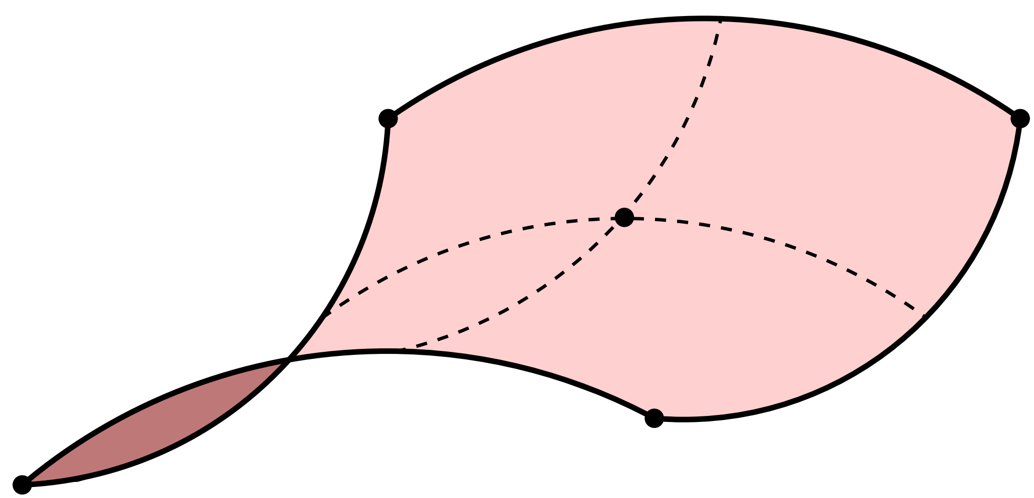
$K(z) =$ Gauss curvature of S at z.

$\kappa_1, \kappa_2 =$ principle curvatures.

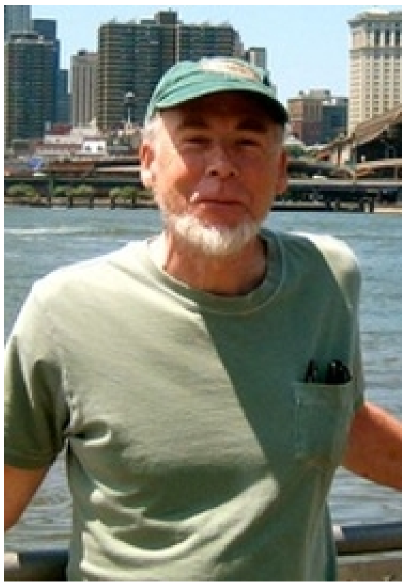
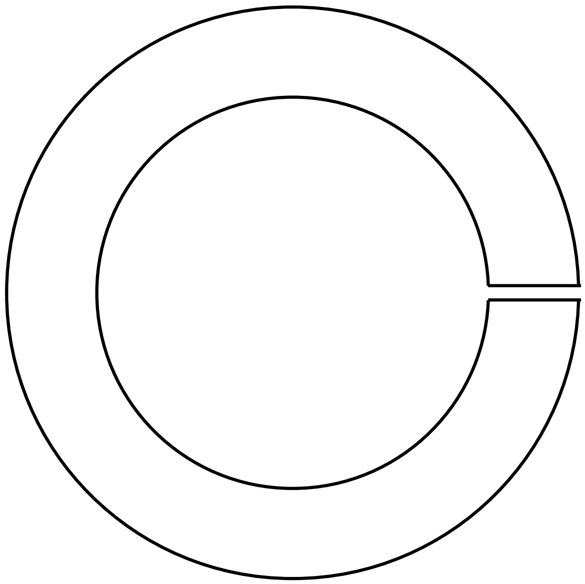
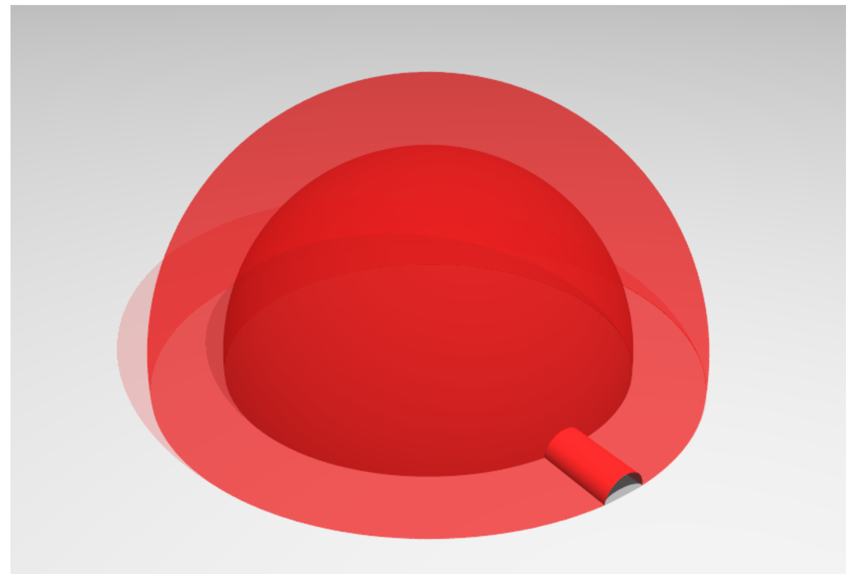
Gauss equation: $K(z) = -1 + \kappa_1(z)\kappa_2(z)$.

S is a **minimal surface** if $\kappa_1 = -\kappa_2$ (the mean curvature is zero).

In that case, $K(z) = -1 - \kappa^2(z) \leq -1$.



Theorem (Anderson, 1983): Every closed Jordan curve $\Gamma \subset \mathbb{R}^2$ bounds a minimal disk $S \subset \mathrm{CH}(\Gamma) \subset \mathbb{H}^3$.

Minimal surface with boundary Γ is contained in convex hull of Γ.

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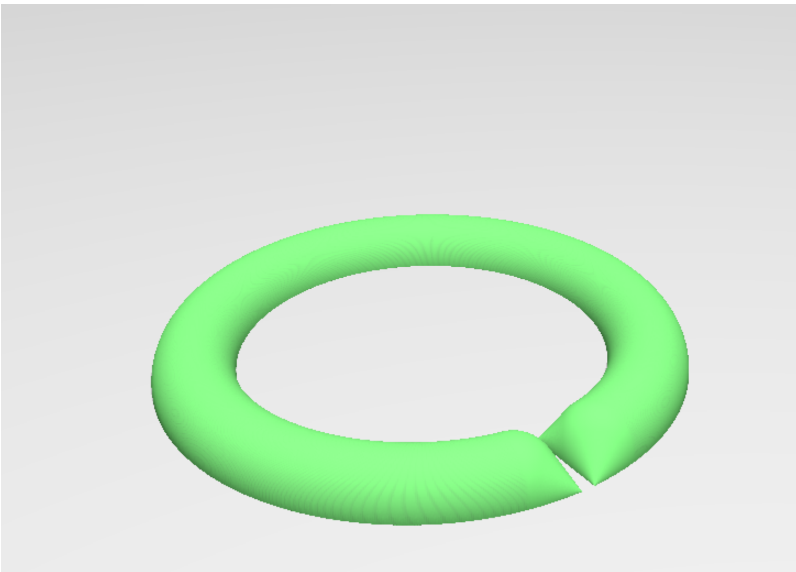
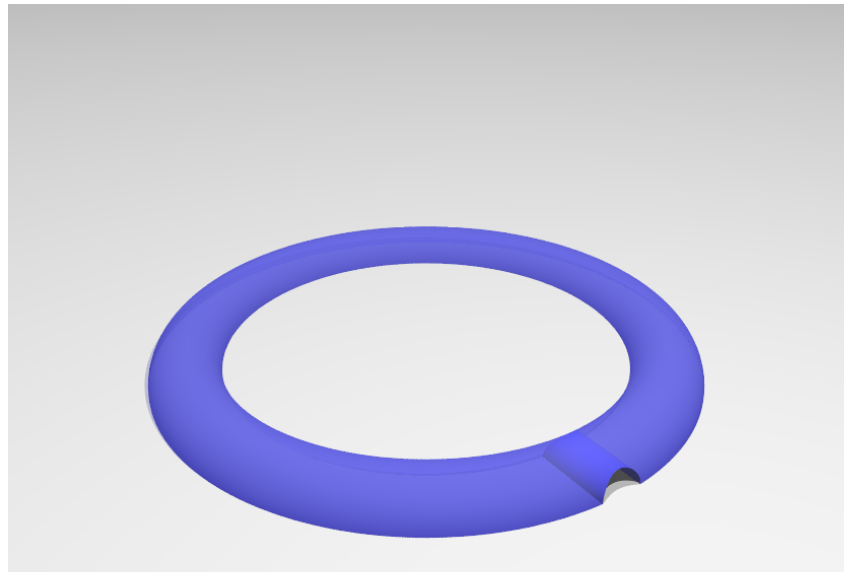
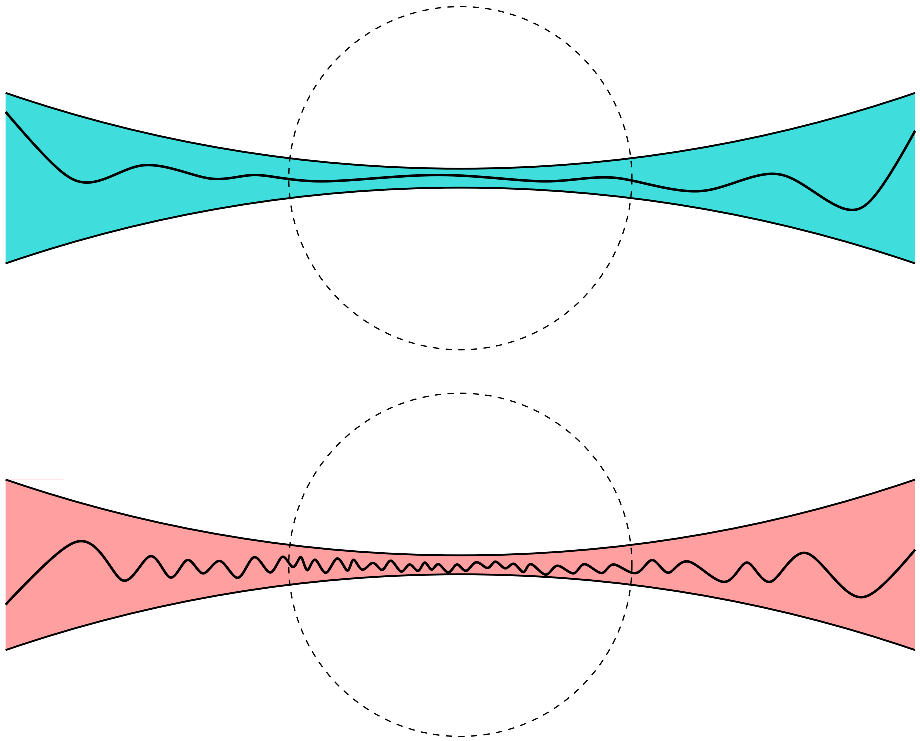



Minimal surface with boundary Γ is contained in convex hull of Γ.

Minimal surface with boundary Γ need not be unique.

Theorem (Anderson, 1983): Every closed Jordan curve $\Gamma \subset \mathbb{R}^2$ bounds a minimal disk $S \subset \mathrm{CH}(\Gamma) \subset \mathbb{H}^3$.

Theorem (Seppi, 2016): Principle curvatures satisfies $\kappa(z) = O(\delta(z))$.



$$u(z) = \sinh(\mathrm{dist}(z, P)) \text{ satisfies } \Delta_S u - 2u = 0.$$
$$\text{Use Schauder estimate } \|\nabla^2 u\|_\infty \leq C\|u\|_\infty = O(\delta).$$

Seppi's estimate + "$\int \delta^2 < \infty$" $\Rightarrow$

Theorem: If Γ is WP then it bounds a minimal disk with

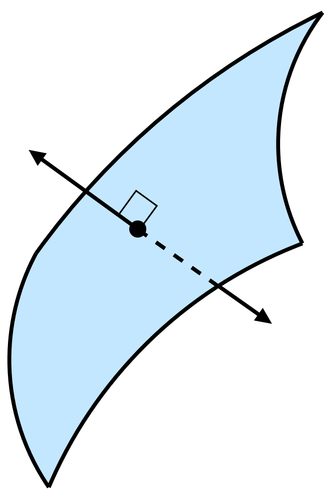
$$\int_S |K+1| dA_\rho = \int_S \kappa^2(z) dA_\rho < \infty.$$

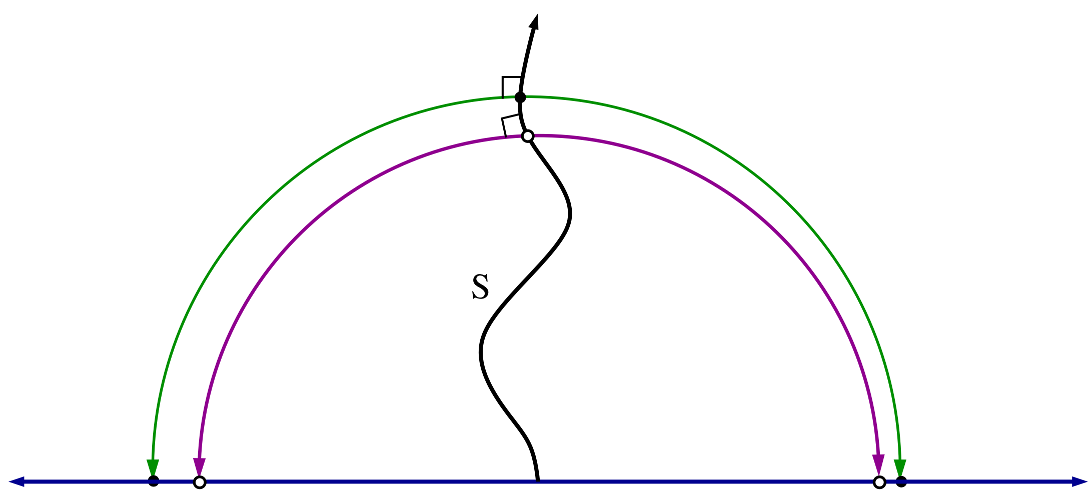
We say such a surface has **finite total curvature**.

Cor: Boundary of surface of finite total curvature need not be C^1.

Gauss map: follow normal geodesic from surface S to $\mathbb{R}^2 = \partial\mathbb{H}^3$.

Two directions. Defines reflection across Γ.





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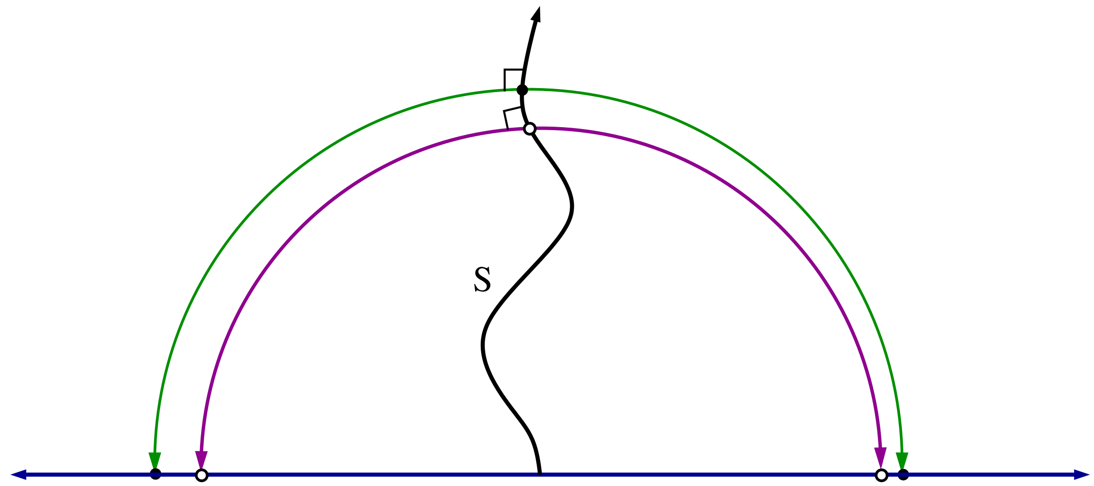
Two directions. Defines reflection across Γ.



Theorem (C. Epstein, 1986): If $|\kappa_1|, |\kappa_2| < 1$, then the Gauss maps define a quasiconformal reflection across Γ. Moreover, if S has finite total curvature, then $\int_{\mathbb{C}\setminus\Gamma} |\mu|^2 d\mathrm{A}_\rho < \infty$.

$\Rightarrow$ Γ is fixed by a QC involution with $\mu \in L^2(dA_\rho)$ $\Rightarrow$ Weil-Petersson.

Weil-Petersson

$\Rightarrow \log f'$ in $W^{1,2}$

$\Rightarrow$ finite Möbius energy

$\Rightarrow$ parameterization in $H^{3/2}$

$\Rightarrow$ inscribed polygons

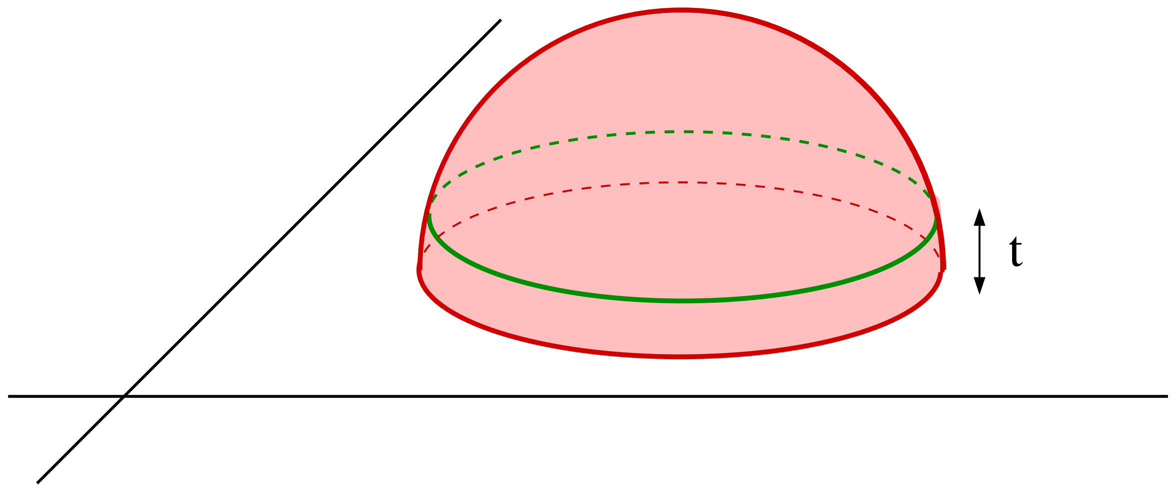
$\Rightarrow \sum \beta^2 < \infty$

$\Rightarrow \int_S \delta^2 dA_\rho < \infty$

$\Rightarrow \int_S \kappa^2 dA_\rho < \infty$

$\Rightarrow$ fixed by "nice" involution of S^2

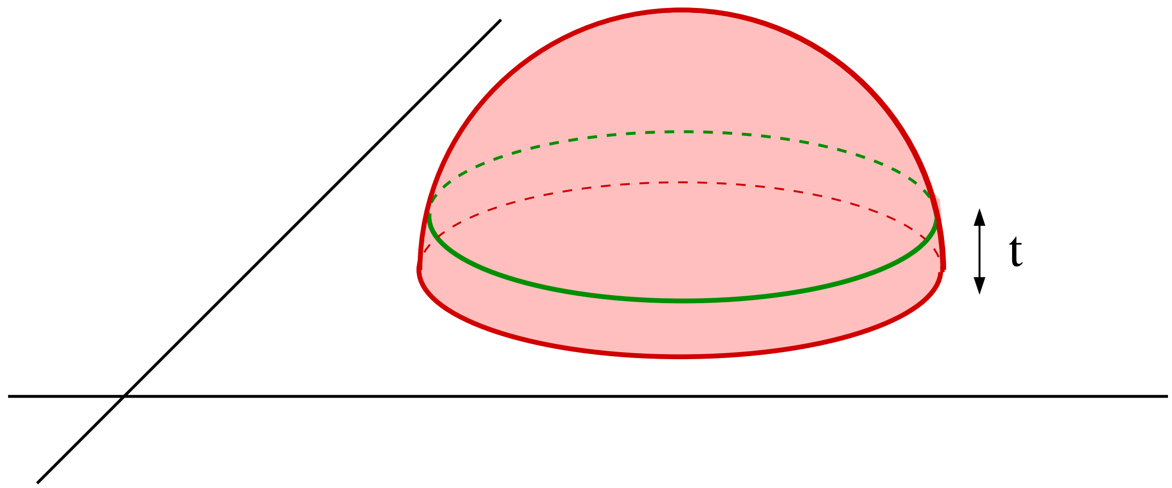
$\Rightarrow$ Weil-Petersson



Truncate $S \subset \mathbb{R}^3_+$ at a fixed height above the boundary, i.e.,

$$S_t = S \cap \{(x, y, s) \in \mathbb{R}^3_+ : s > t\},$$

Boundary length $\ell(\partial S_t)$ and interior area $A(S_t)$ both grow to ∞.



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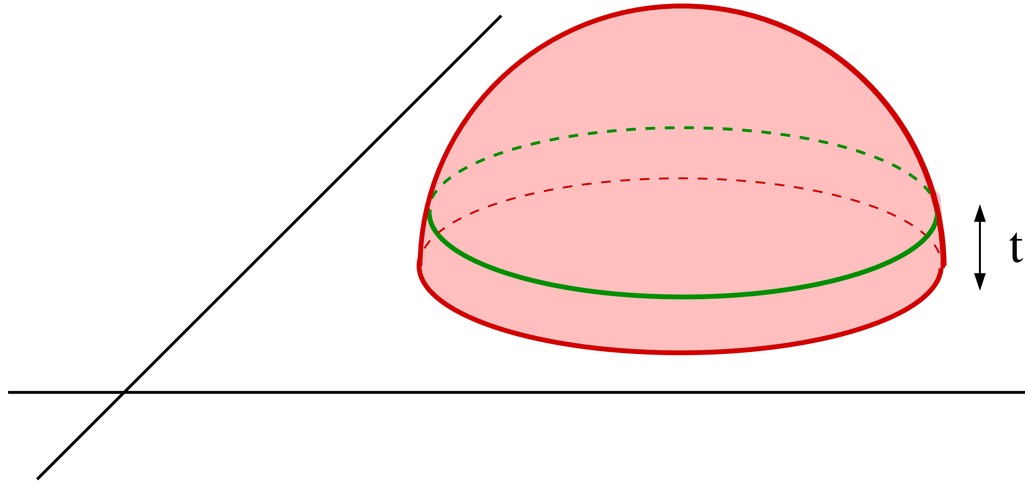
Isoperimetric inequality: if $K(z) \leq -1$, then

$$\ell(\partial S_t) \geq A(S_t) + 4\pi\chi(S_t).$$

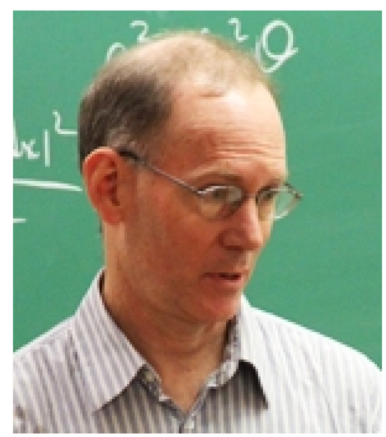
Does the gap $\ell(\partial S_t) - A(S_t)$ stay bounded or grow to ∞?



Renormalized area: $\mathcal{A}_R(S) = \lim_{t \searrow 0} \left[\mathrm{A}_\rho(S_t) - \ell_\rho(\partial S_t) \right].$

Graham and Witten proved well defined.

Related to quantum entanglement, AdS/CFT correspondence.



Theorem: S has finite renormalized area iff Γ is Weil-Petersson.

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Hard direction: use isoperimetric inequalities to show

$$\mathcal{A}_R(S) < \infty \quad \Rightarrow \quad \int_S \kappa^2 dA_\rho < \infty \quad \Rightarrow \quad \text{WP}$$

Easier converse uses Seppi's estimate and the Gauss-Bonnet formula:

$$\int_M K dA + \int_{\partial M} \kappa_g ds = 2\pi\chi(M).$$

Using the Gauss-Bonnet theorem

$$\begin{aligned}
\mathrm{A}_\rho(S_t) - \ell_\rho(\partial S_t) &= \int_{S_t} 1 d\mathrm{A}_\rho - \int_{\partial S_t} 1 d\ell_\rho \\
&= \int_{S_t} (1+\kappa^2) d\mathrm{A}_\rho - \int_{S_t} \kappa^2 d\mathrm{A}_\rho - \int_{\partial S_t} 1 d\ell_\rho \\
&= -\int_{S_t} K d\mathrm{A}_\rho - \int_{S_t} \kappa^2 d\mathrm{A}_\rho - \int_{\partial S_t} 1 d\ell_\rho \\
&= -2\pi\chi(S_t) + \int_{\partial S_t} \kappa_g d\ell_\rho - \int_{S_t} \kappa^2 d\mathrm{A}_\rho - \int_{\partial S_t} 1 d\ell_\rho \\
&= -2\pi\chi(S_t) - \int_{S_t} \kappa^2 d\mathrm{A}_\rho + \int_{\partial S_t} (\kappa_g - 1) d\ell_\rho
\end{aligned}$$

Prove $\kappa_g(z) = 1 + O(\delta^2(z))$.

Then WP implies last term $\to 0$.

Theorem: For any closed curve $\Gamma \subset \mathbb{R}^2$ and for any minimal surface $S \subset \mathbb{R}^3_+$ with finite Euler characteristic and asymptotic boundary Γ,

$$\mathcal{A}_R(S) = -2\pi\chi(S) - \int_S \kappa^2(z) d\mathrm{A}_\rho.$$

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Due to Alexakis and Mazzeo (2010) assuming that Γ is $C^{3,\alpha}$.

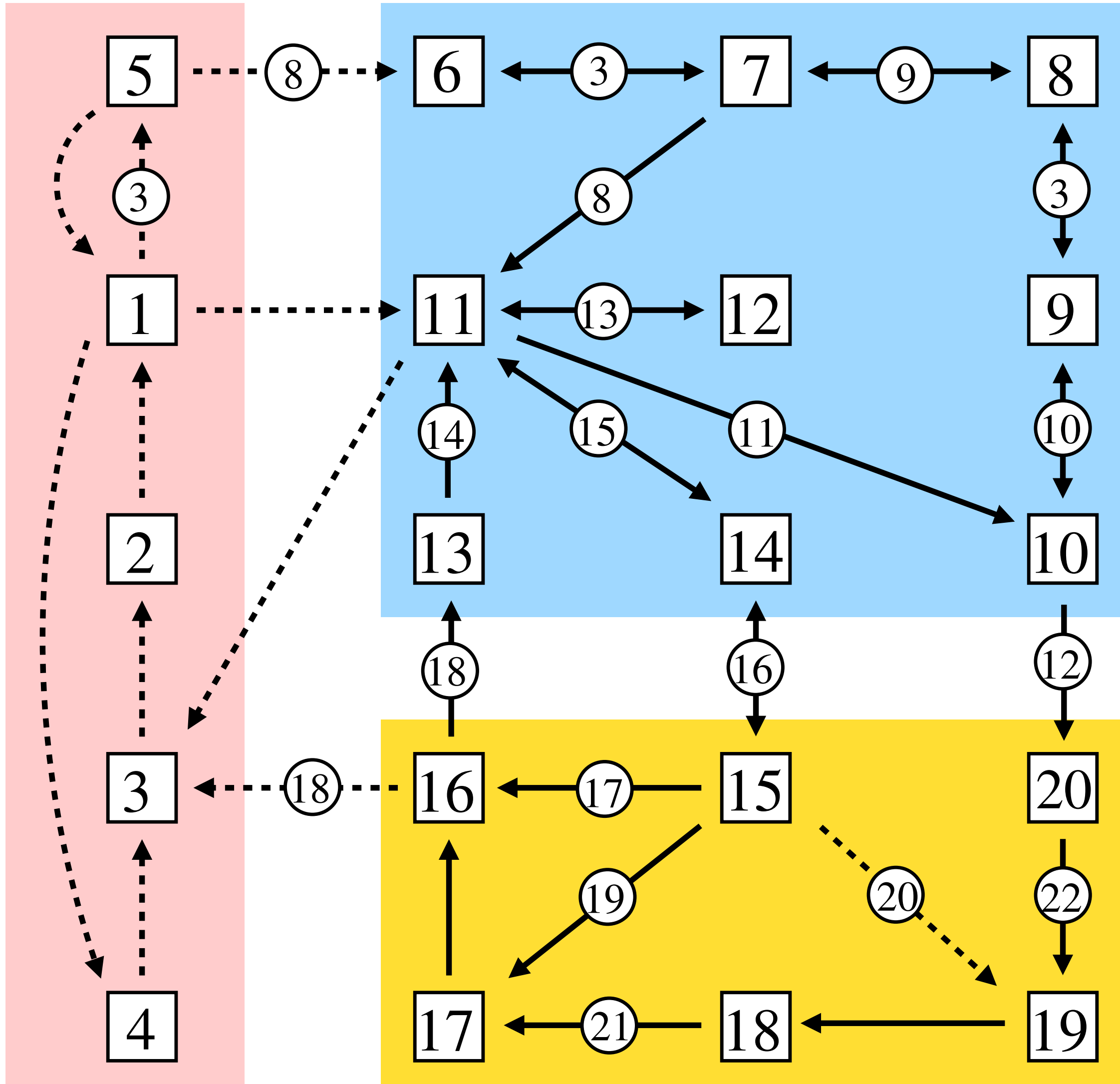
Definition	Description
1	$\log f'$ in Dirichlet class
2	Schwarzian derivative
3	QC dilatation in L^2
4	conformal welding midpoints
5	$\exp(i \log f')$ in $H^{1/2}$
6	arclength parameterization in $H^{3/2}$
7	tangents in $H^{1/2}$
8	finite Möbius energy
9	Jones conjecture
10	good polygonal approximations
11	β^2-sum is finite
12	Menger curvature
13	biLipschitz involutions
14	between disjoint disks
15	thickness of convex hull
16	finite total curvature surface
17	minimal surface of finite curvature
18	additive isoperimetric bound
19	finite renormalized area
20	dyadic cylinder

Weil-Petersson curves

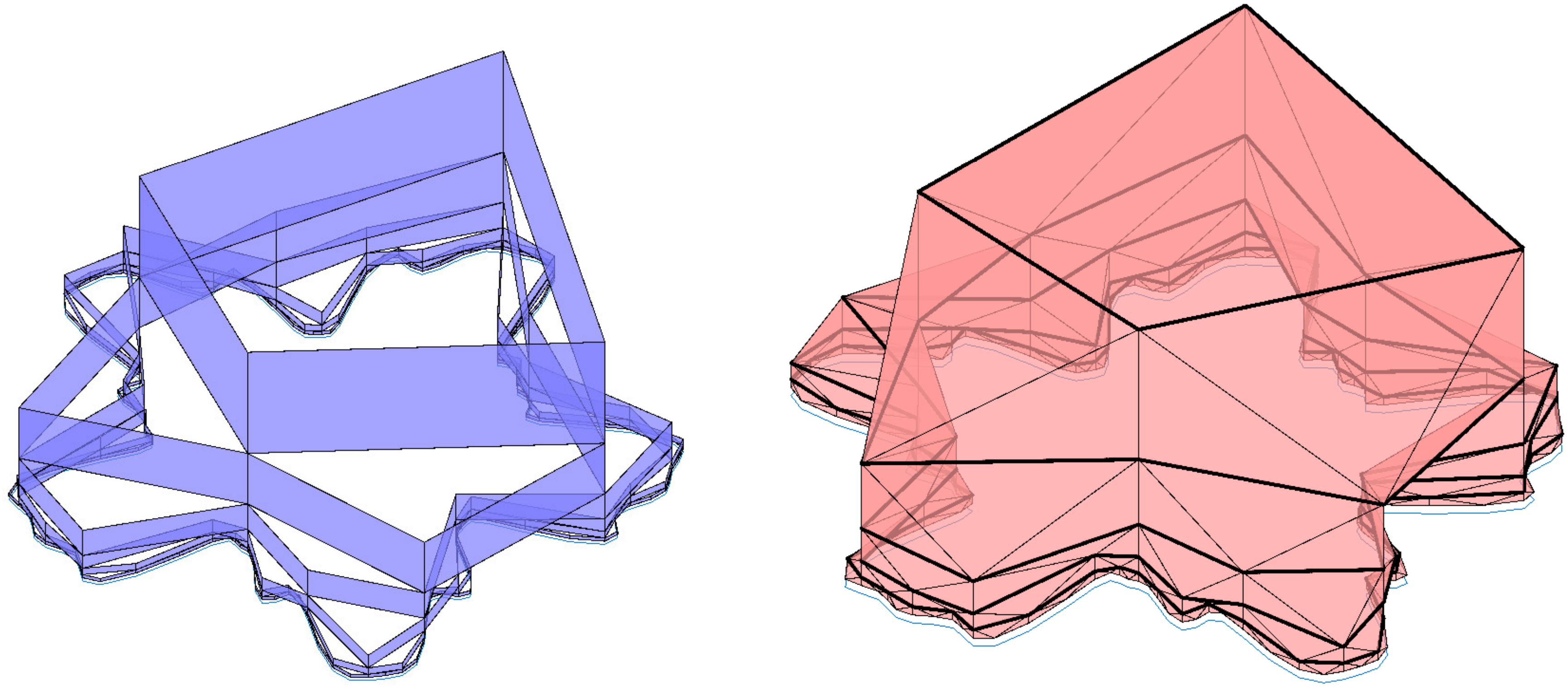
André Weil

Hans Petersson

THE THEOREM



THANKS FOR LISTENING. QUESTIONS?



The dyadic dome.
Polyhedral approximation to minimal surface.
Intermediary between Euclidean and hyperbolic regimes.

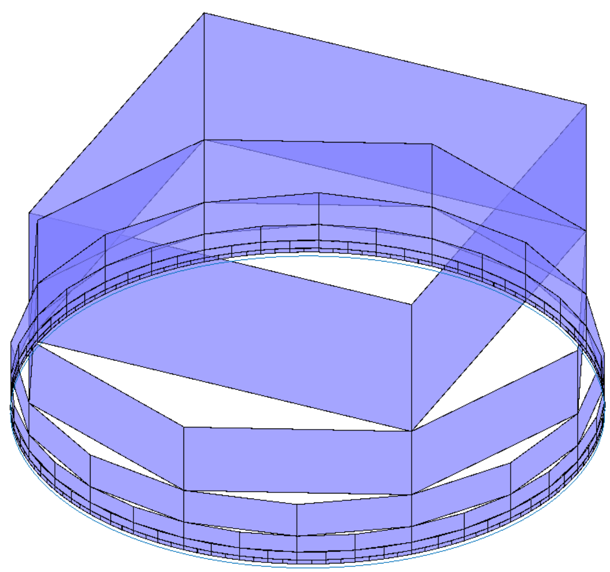
An idea connecting Euclidean and hyperbolic results.

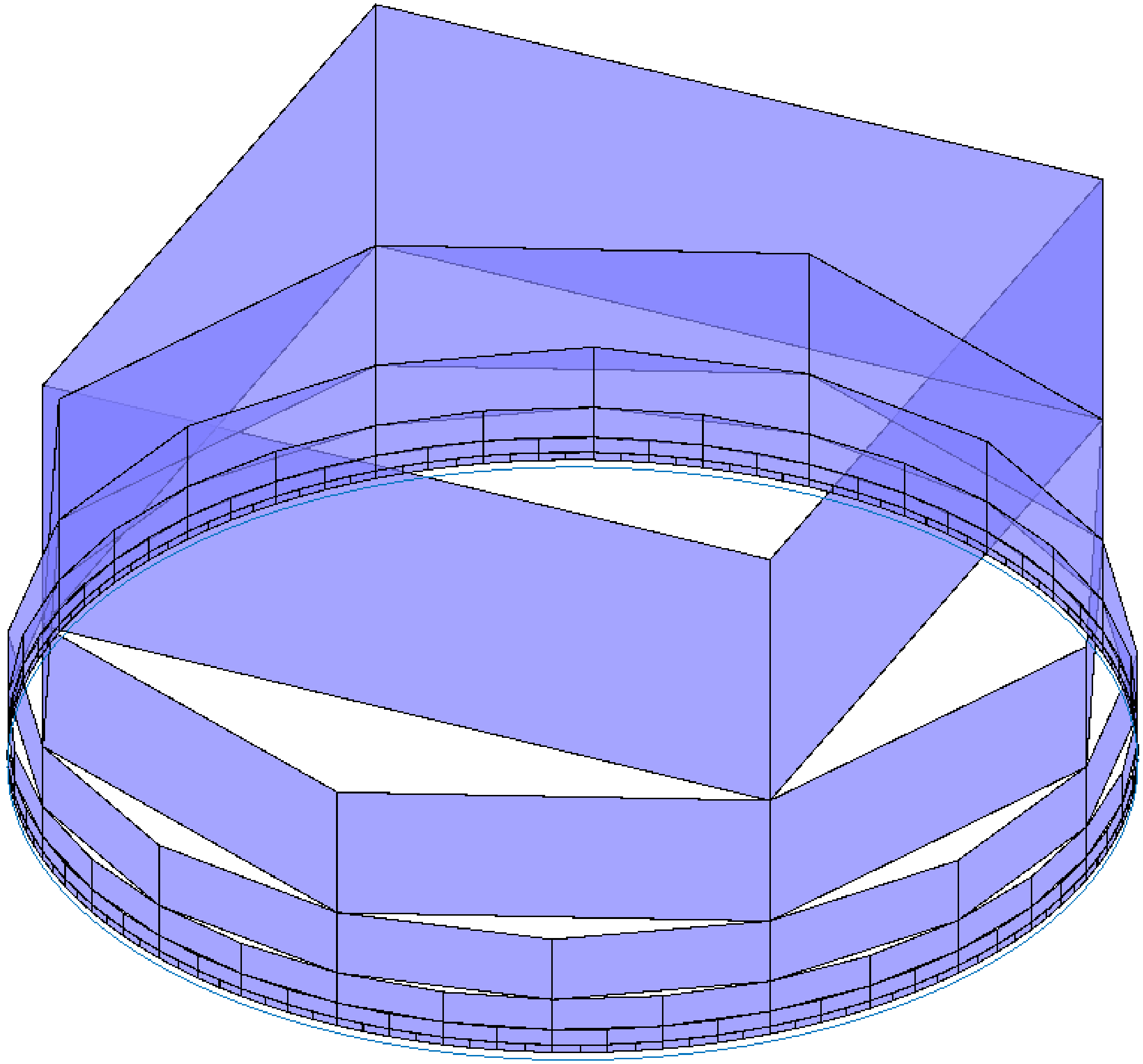
Define a dyadic cylinder in the upper half-space:

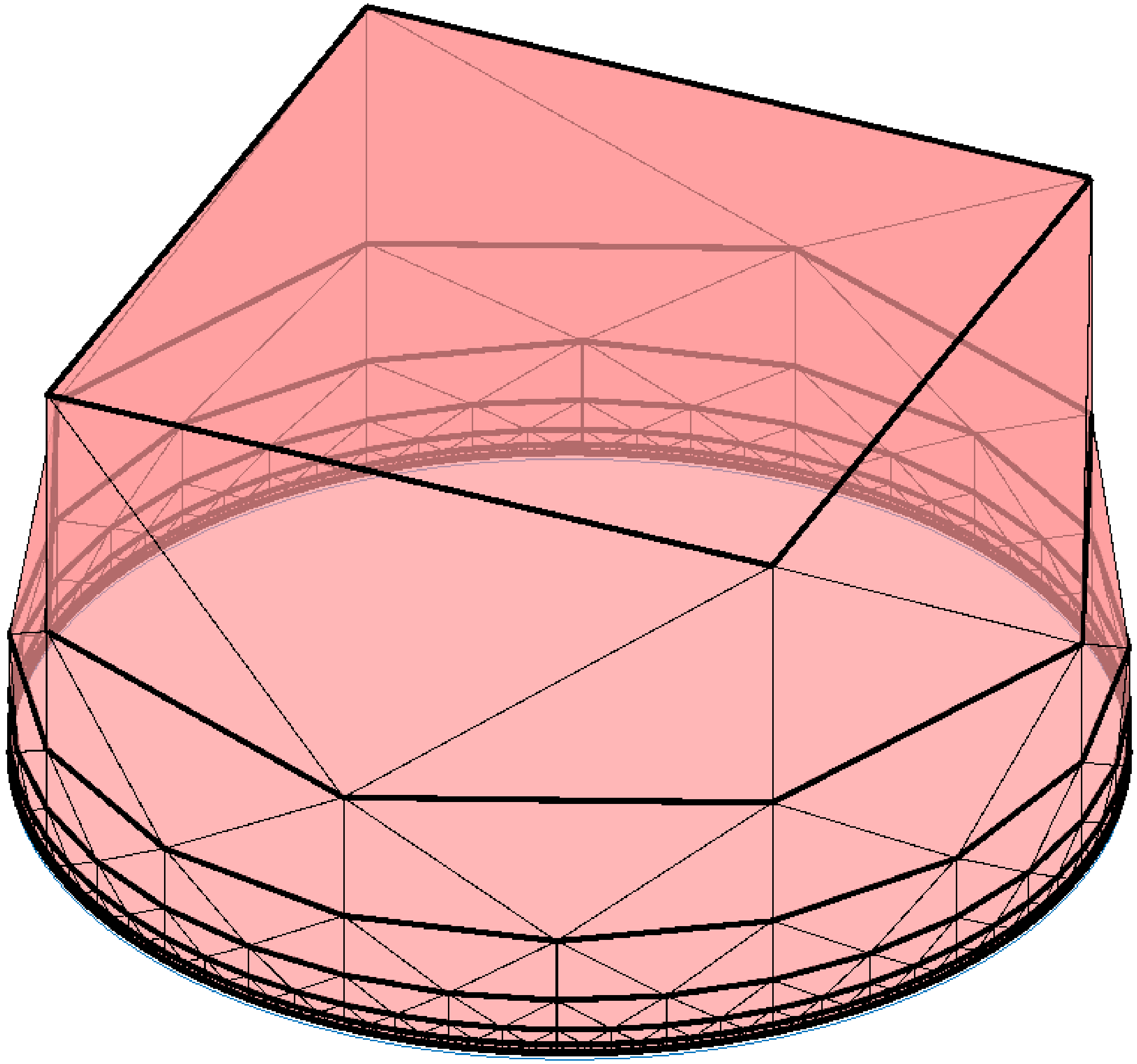
$$X = \bigcup_{n=1}^{\infty} \Gamma_n \times [2^{-n}, 2^{-n+1}),$$

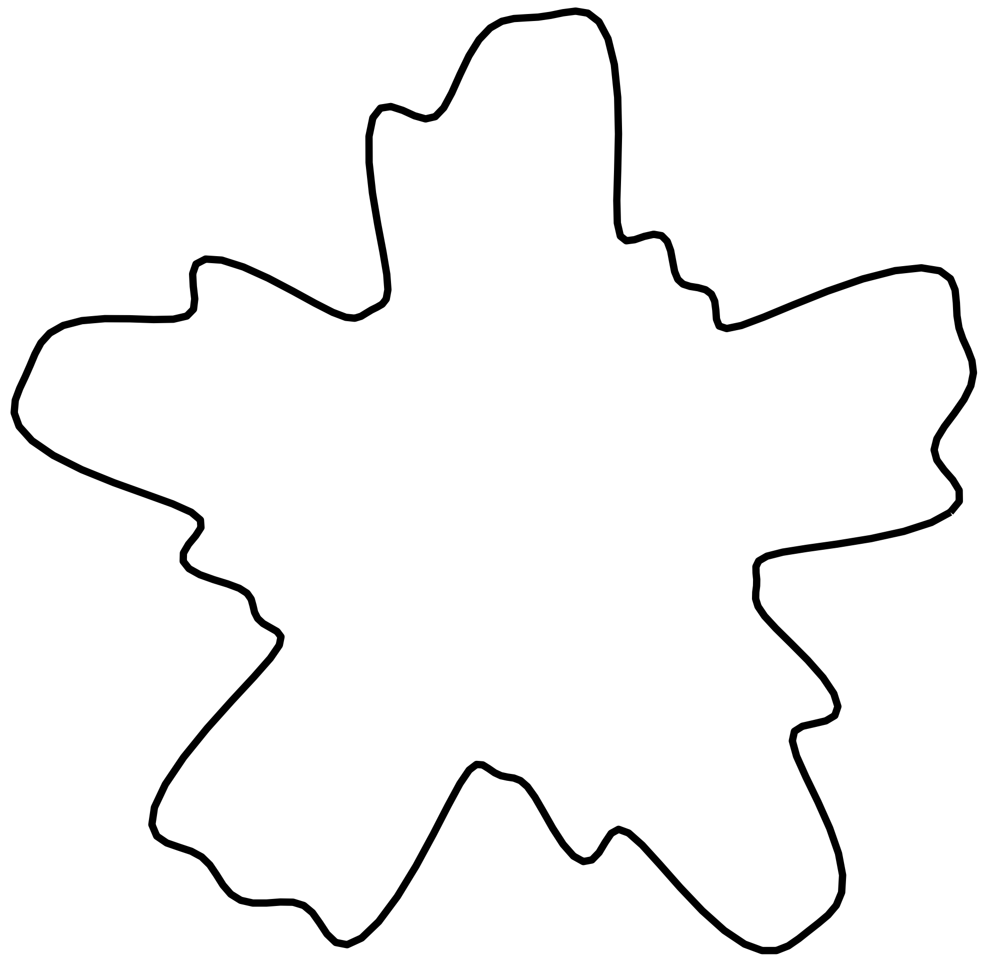
where $\{\Gamma_n\}$ are inscribed dyadic polygons in Γ.

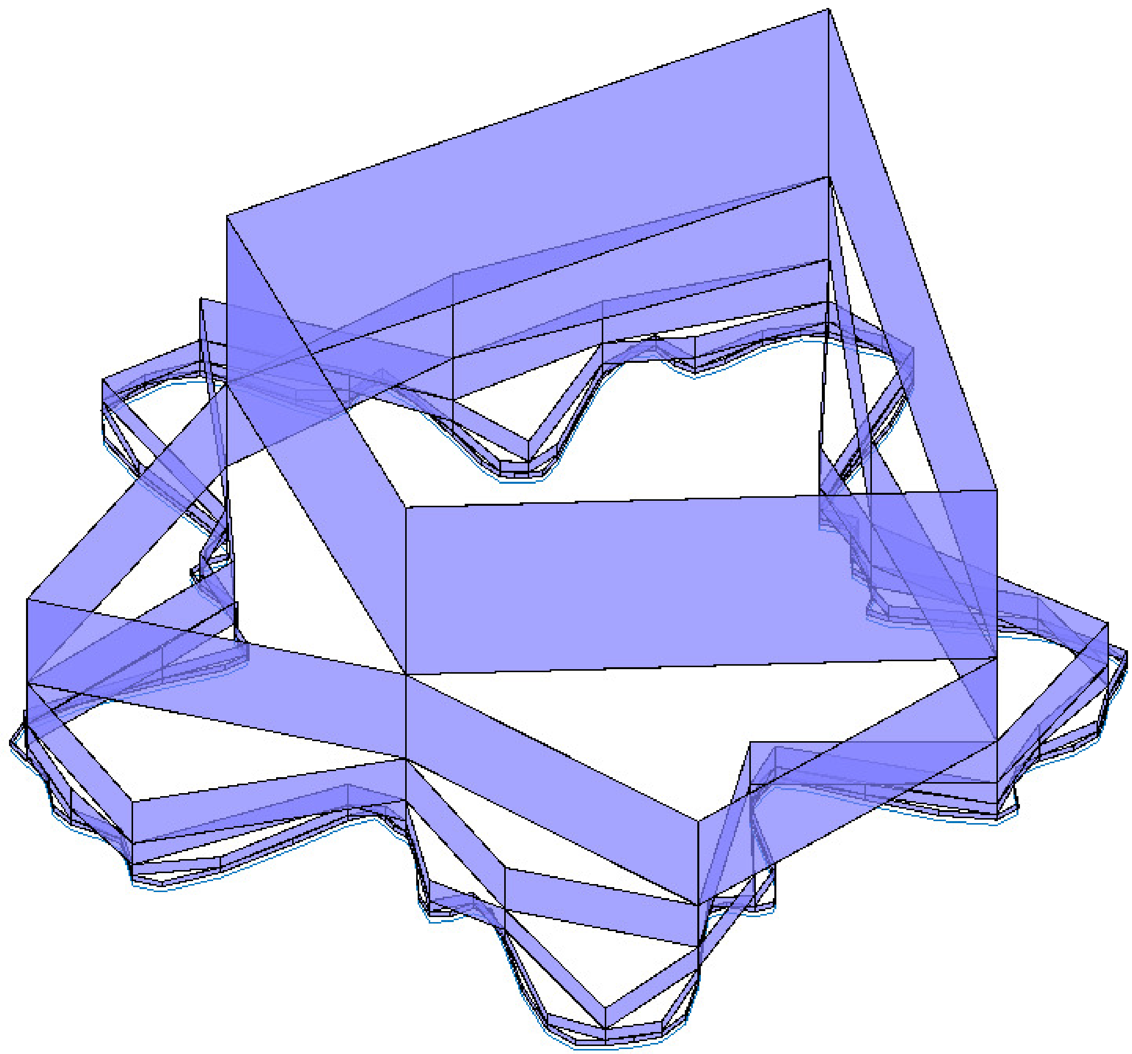
Discrete analog of minimal surface with boundary Γ.

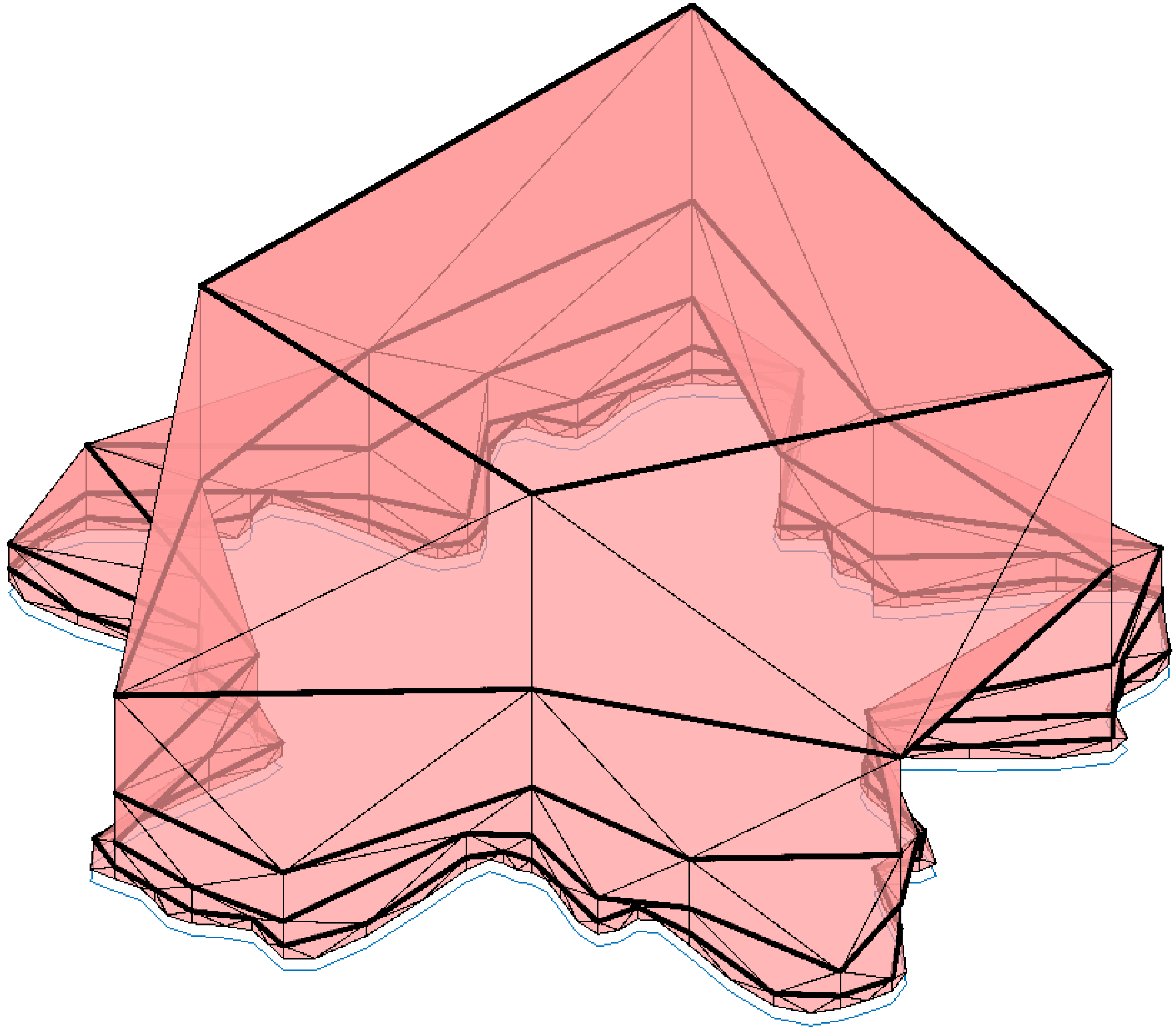












Our earlier estimate

$$\sum_n 2^n(\ell(\Gamma) - \ell(\Gamma_n)) < \infty$$

is equivalent to the dyadic cylinder having finite renormalized area.

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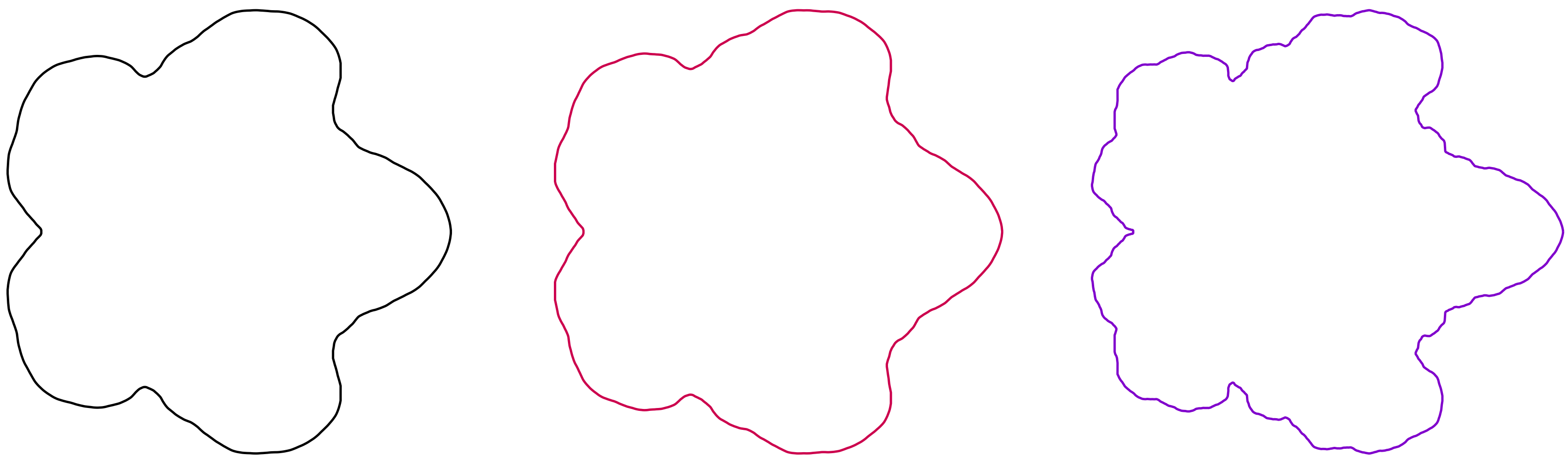
is equivalent to the dyadic cylinder having finite renormalized area.

Obvious "normal projection" from the dyadic cylinder to minimal surface, distorts length and area each by a bounded additive error.

We can deduce finite renormalized area for the minimal surface from the same result for the dyadic cylinder.

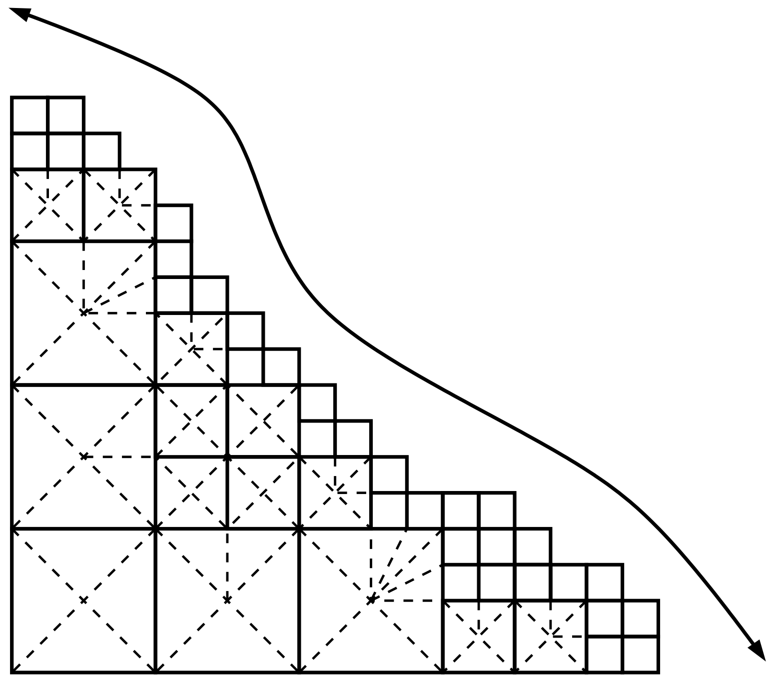
$F(z) = \sum_1^\infty a_n z^n$ is Dirichlet class iff $\sum n|a_n|^2 < \infty$.

If $\log f' = \sum \sqrt{\frac{b_k}{\lambda^k}}\, z^{\lambda^k}$ then $\Gamma = f(\mathbb{T})$ is WP iff $\sum b_k < \infty$.



$\sum \frac{1}{k \log^2 k} < \infty$ $\qquad$ $\sum \frac{1}{k \log k} = \infty$ $\qquad$ $\sum 1 = \infty$

Easy to see $\sum \beta^2 < \infty$ implies Weil-Petersson.



- Triangulate one side of Γ (e.g., triangulate Whitney squares).

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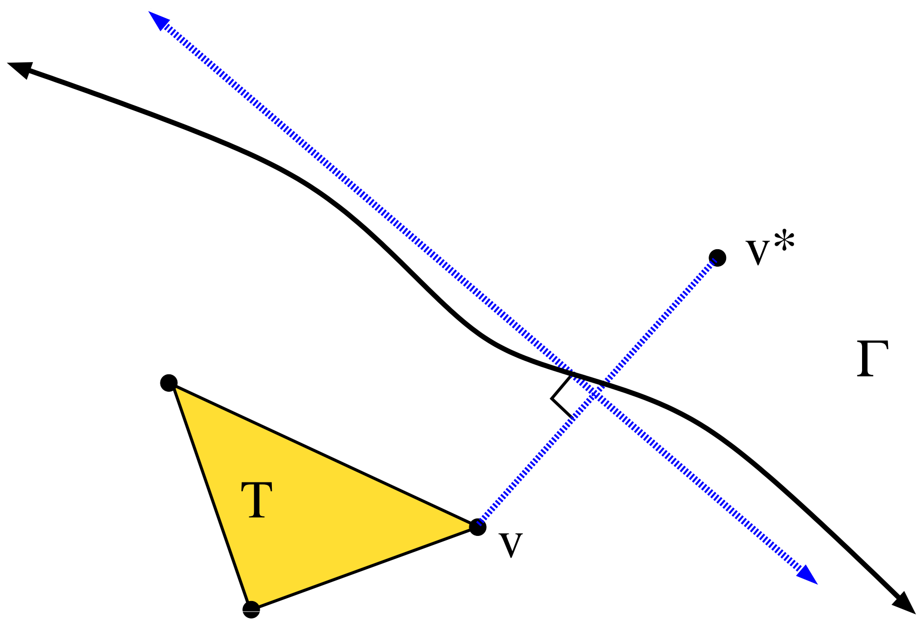

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- Use approximating lines to reflect vertices.

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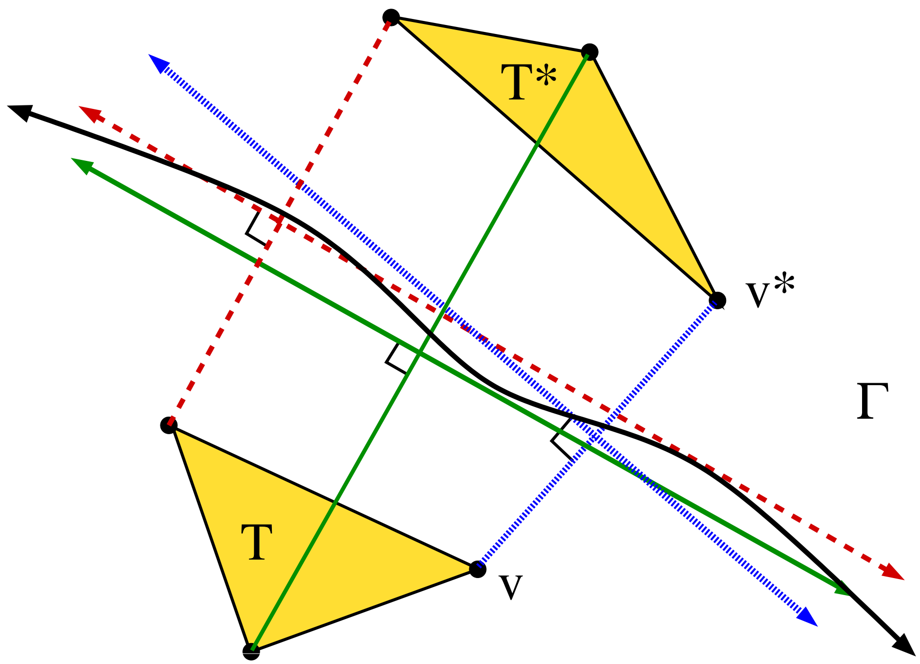
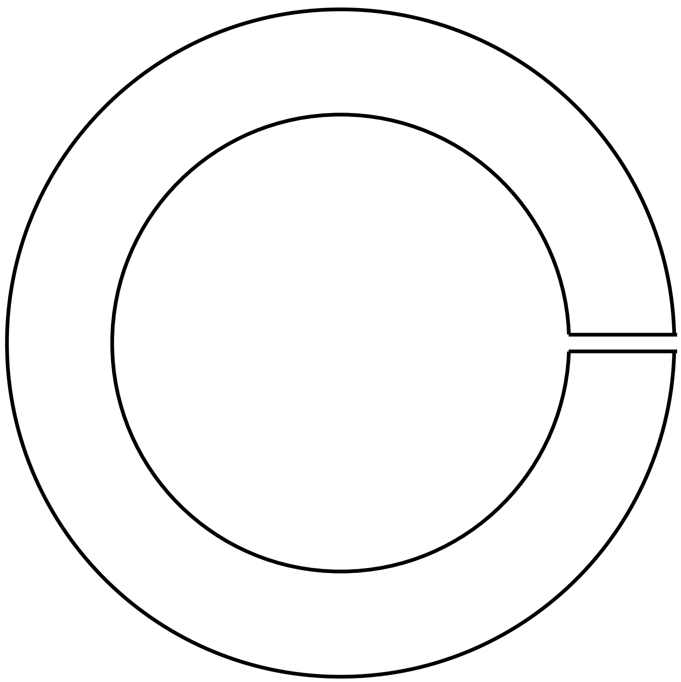
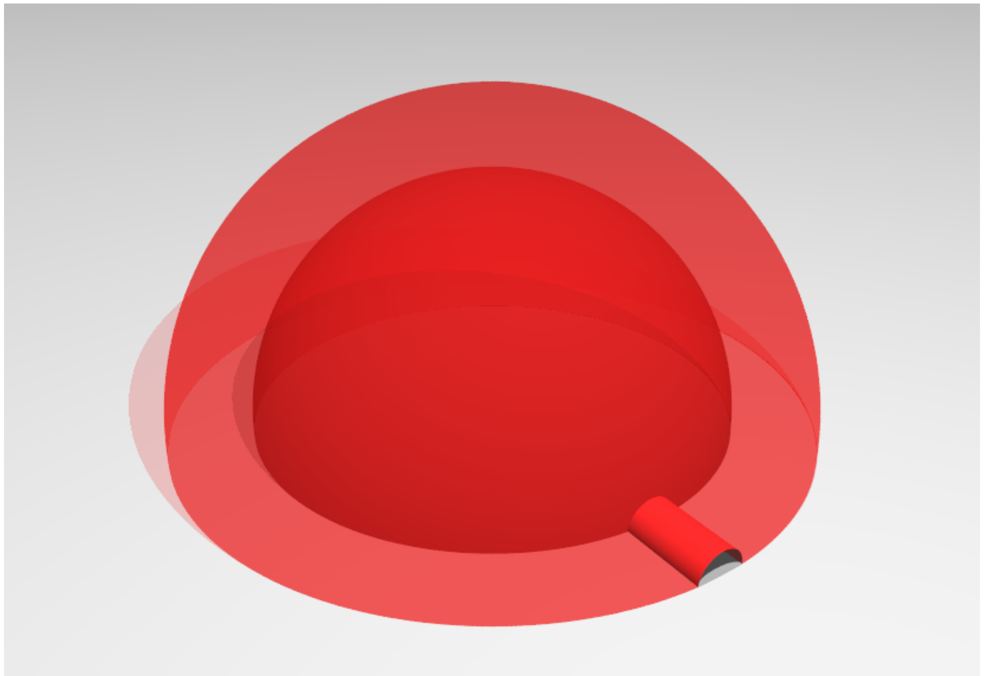


- Triangulate one side of Γ (e.g., triangulate Whitney squares).
- Use approximating lines to reflect vertices.
- Define piecewise linear map.
- $|\mu| = O(\beta)$.
- Get involution fixing Γ with $|\mu| \in L^2(dA_\rho) \Rightarrow$ Weil-Petersson.

Theorem (Anderson, 1983): Every closed Jordan curve $\Gamma \subset \mathbb{R}^2$ bounds a minimal disk $S \subset \mathrm{CH}(\Gamma) \subset \mathbb{H}^3$.





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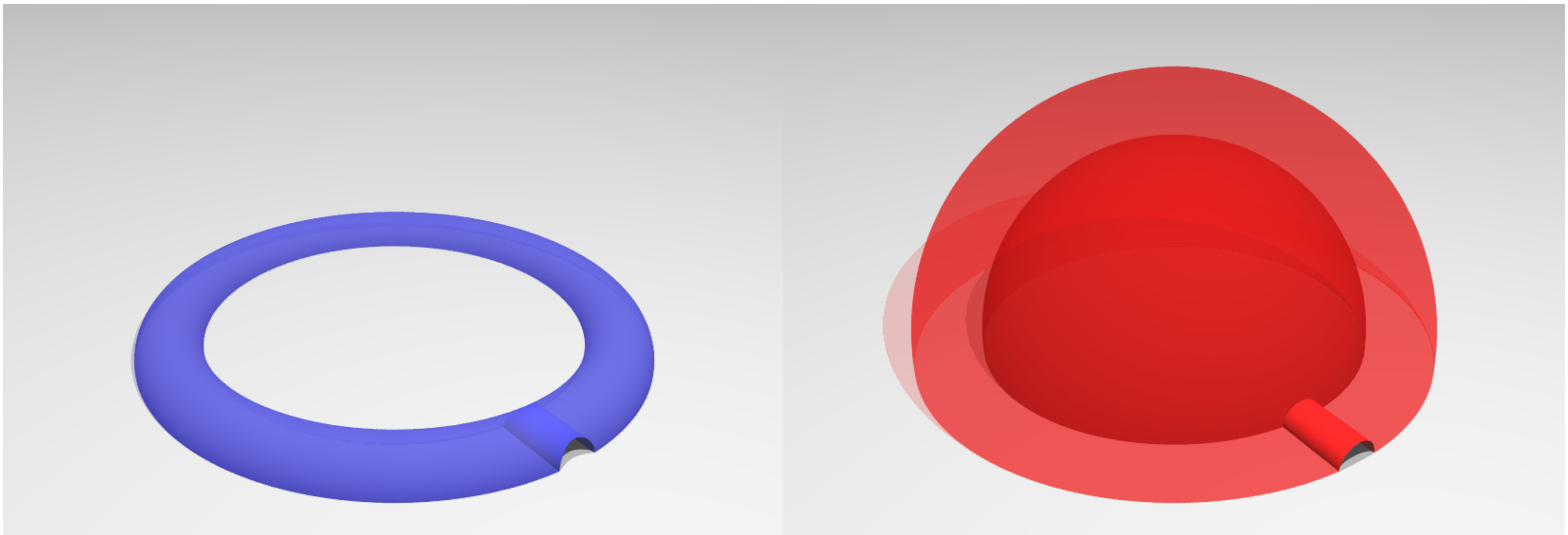

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