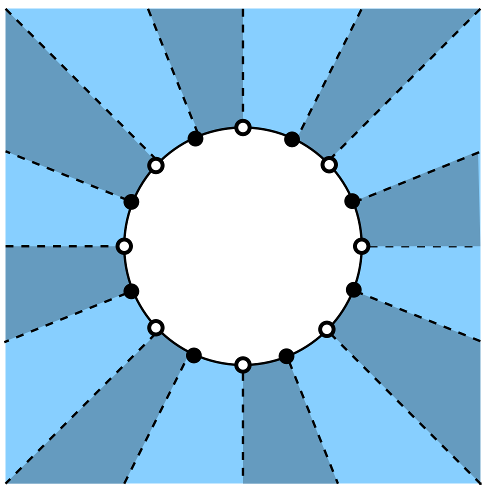
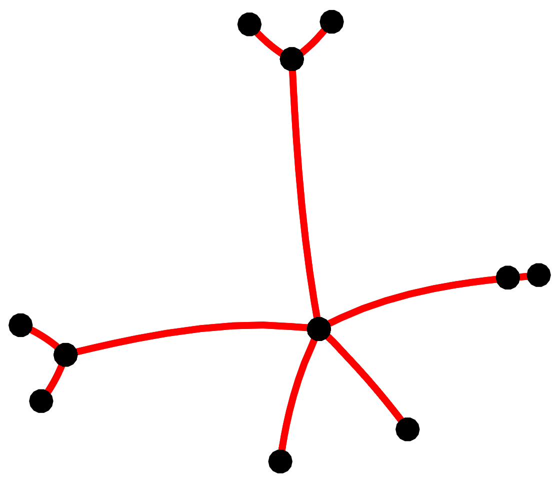
Quasiconformal Folding: trees, triangles and tracts

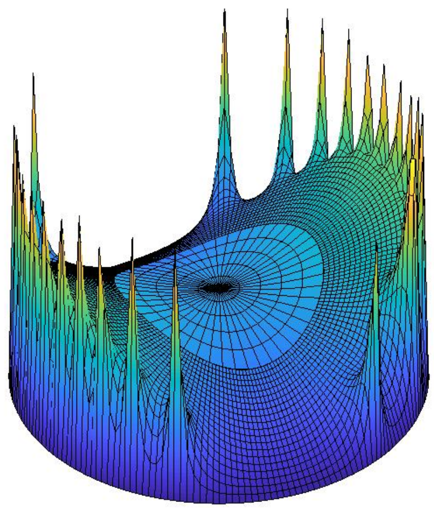
Christopher Bishop, Stony Brook

Complex Analysis, Geometry and Dynamics
Portorož, Slovenia, June 20-24, 2022

www.math.stonybrook.edu/~bishop/lectures

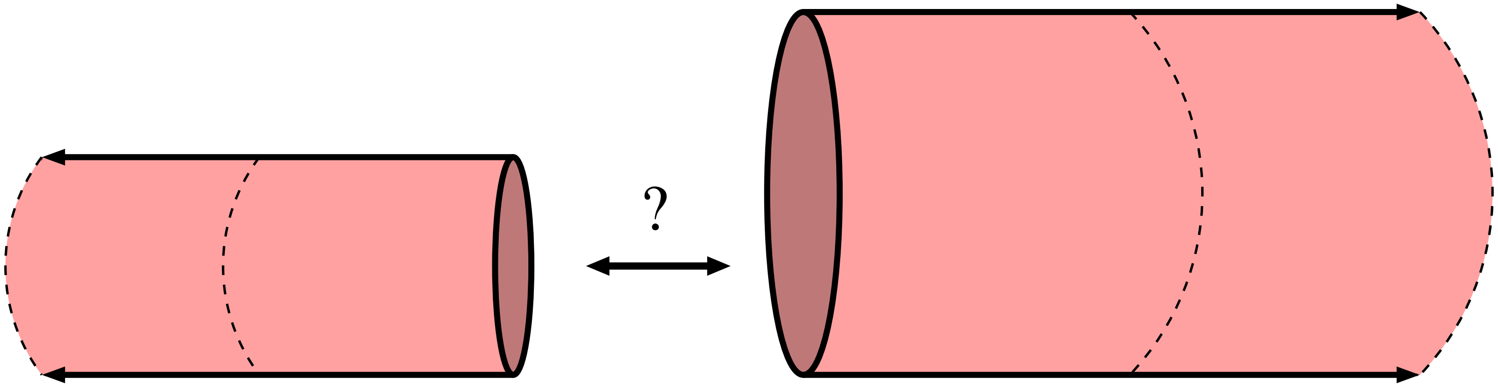




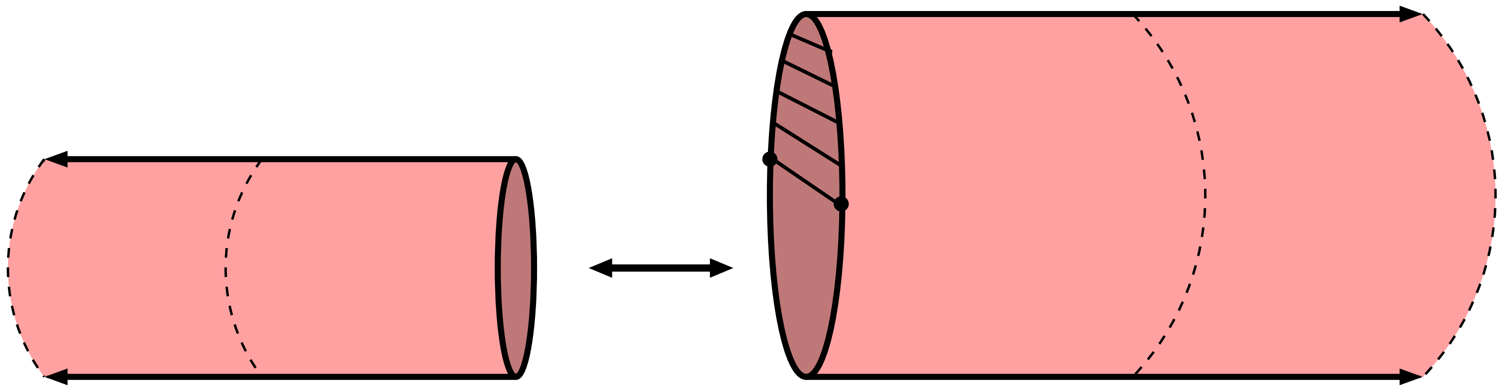


THE PLAN

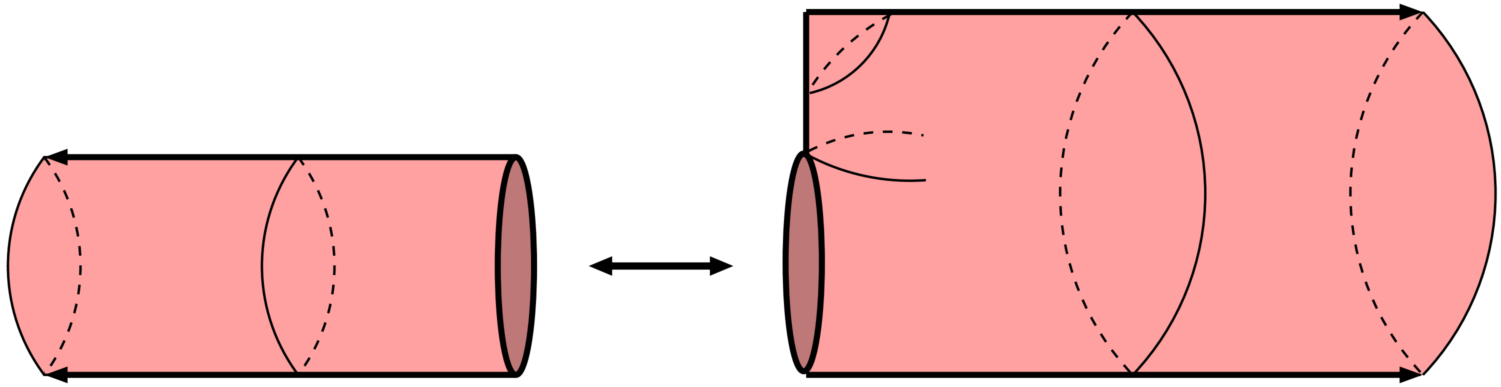
- Harmonic measure
- Finite trees and Shabat polynomials
- Runge's theorem
- Infinite trees and entire functions
- Equilateral triangulations (if time permits)



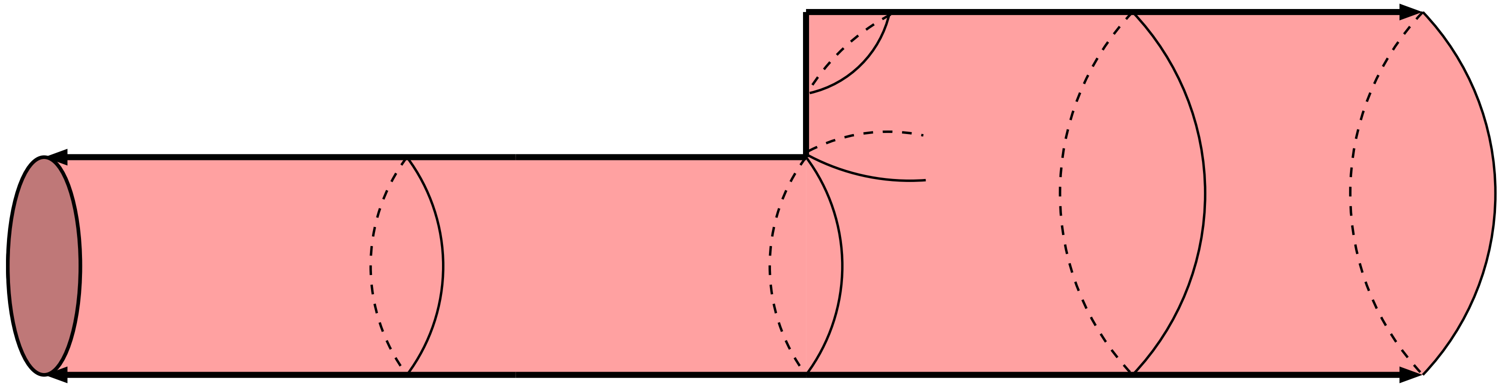
How to join different sized tubes?



Identify points on larger tube?



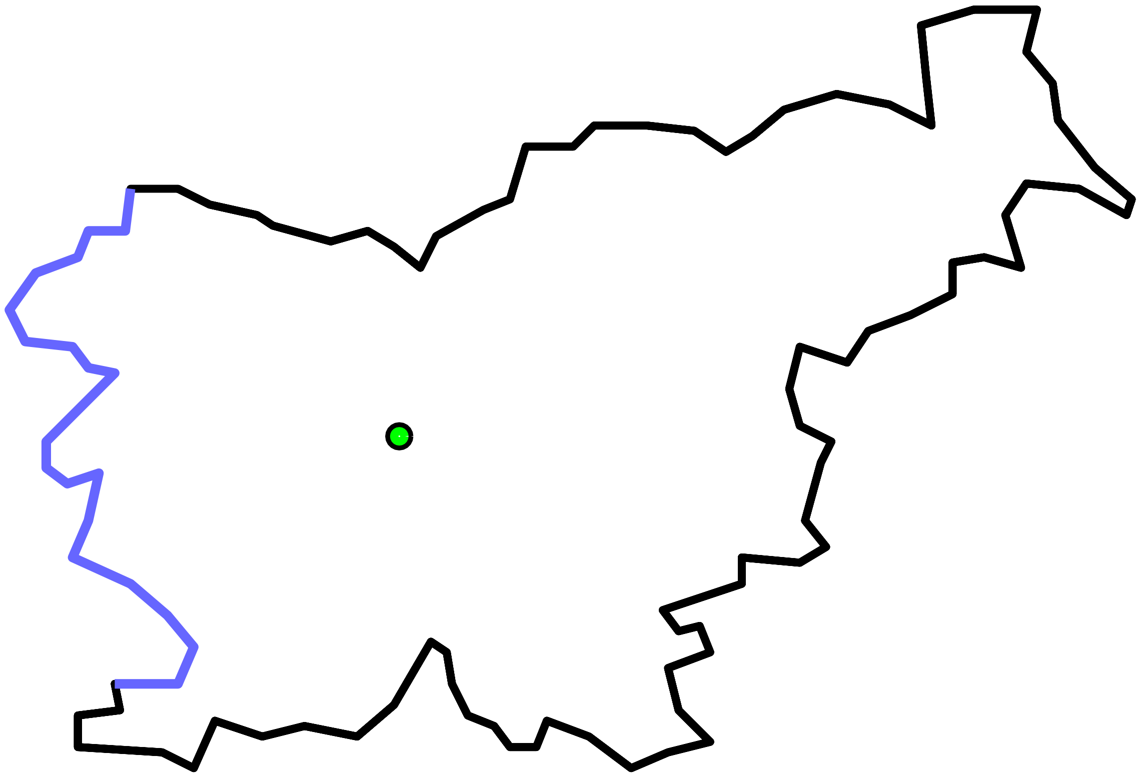
Pinch tube to create smaller opening



Now glue the tubes together

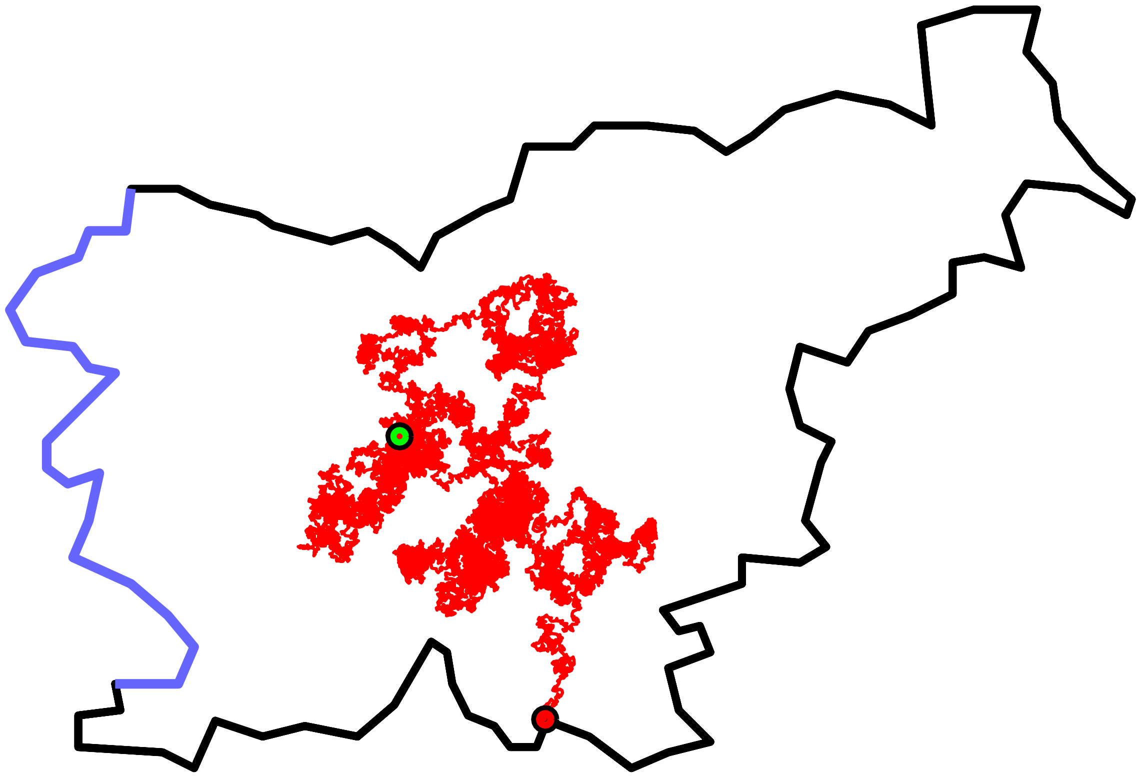
Harmonic measure and conformal maps

Harmonic measure = hitting distribution of Brownian motion



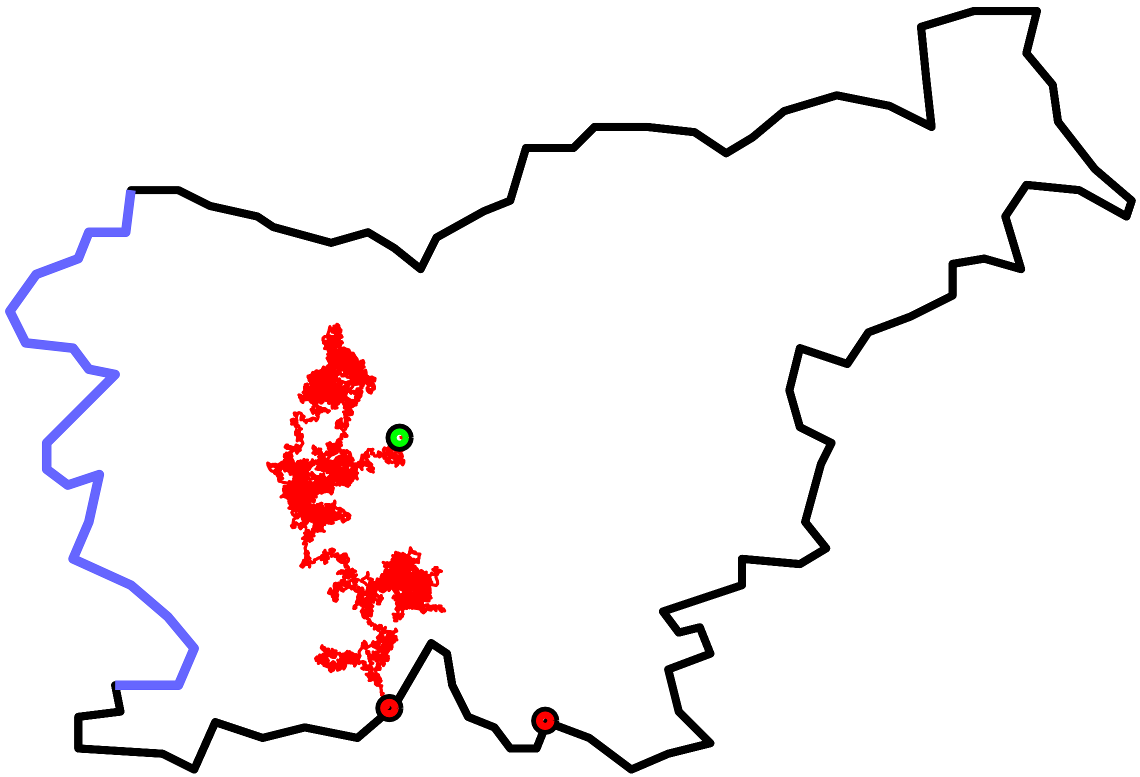
Choose Jordan domain Ω, a boundary set E and an interior point z.

Harmonic measure = hitting distribution of Brownian motion



$\omega(z, E, \Omega)$ = probability a particle started at z first hits $\partial\Omega$ in E.

Harmonic measure = hitting distribution of Brownian motion



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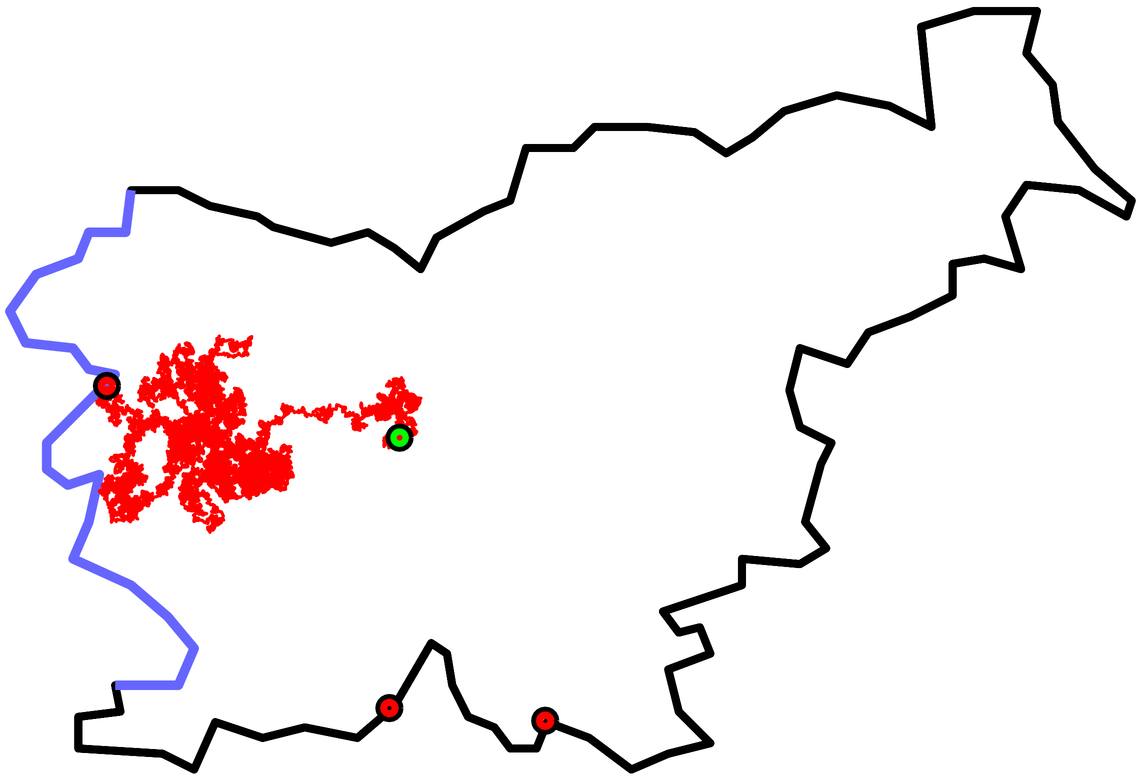

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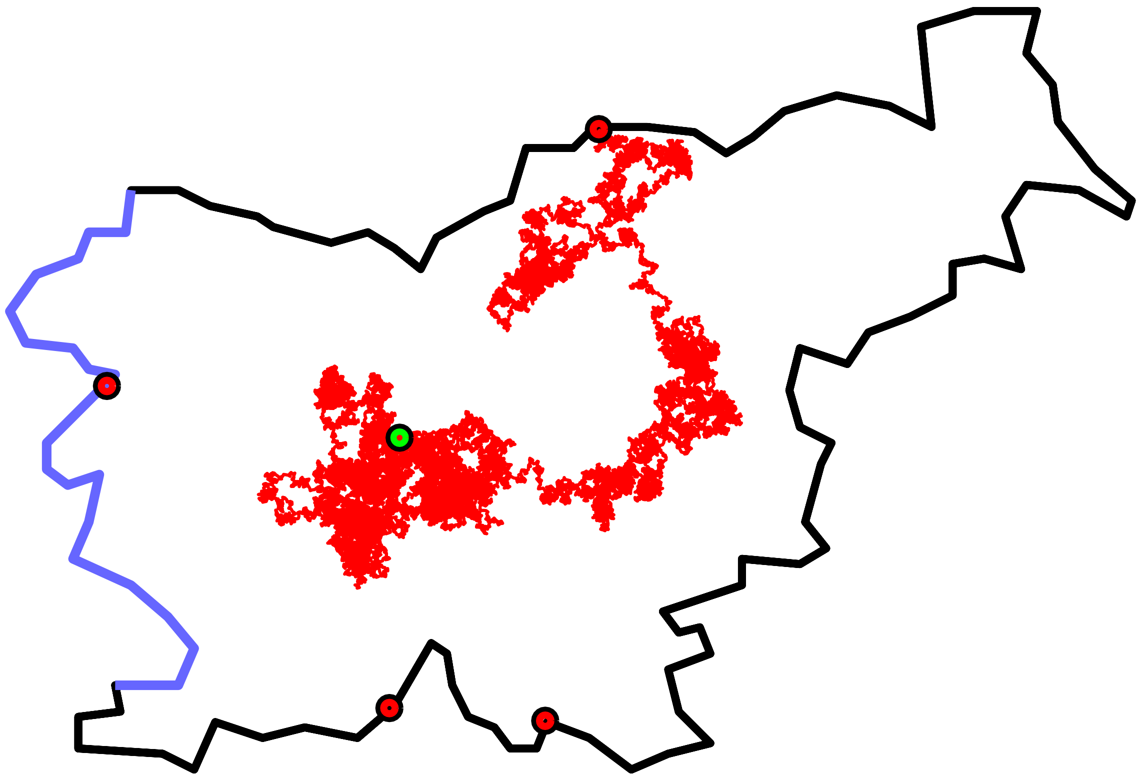

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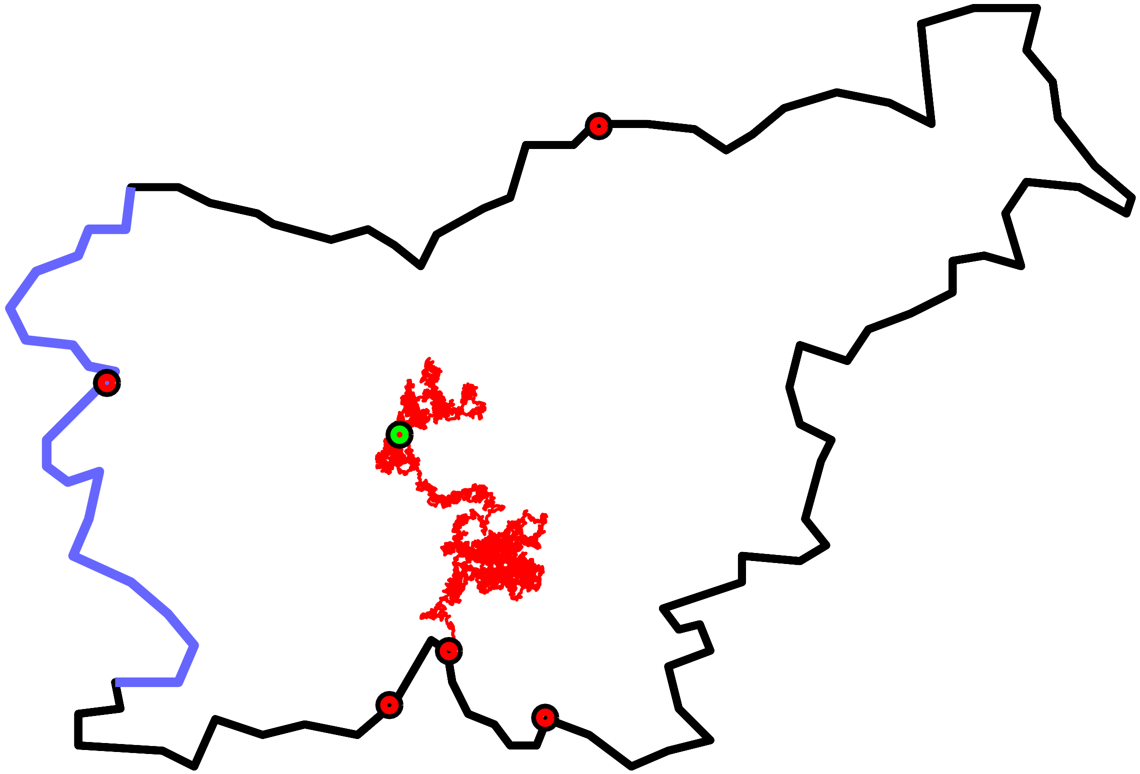

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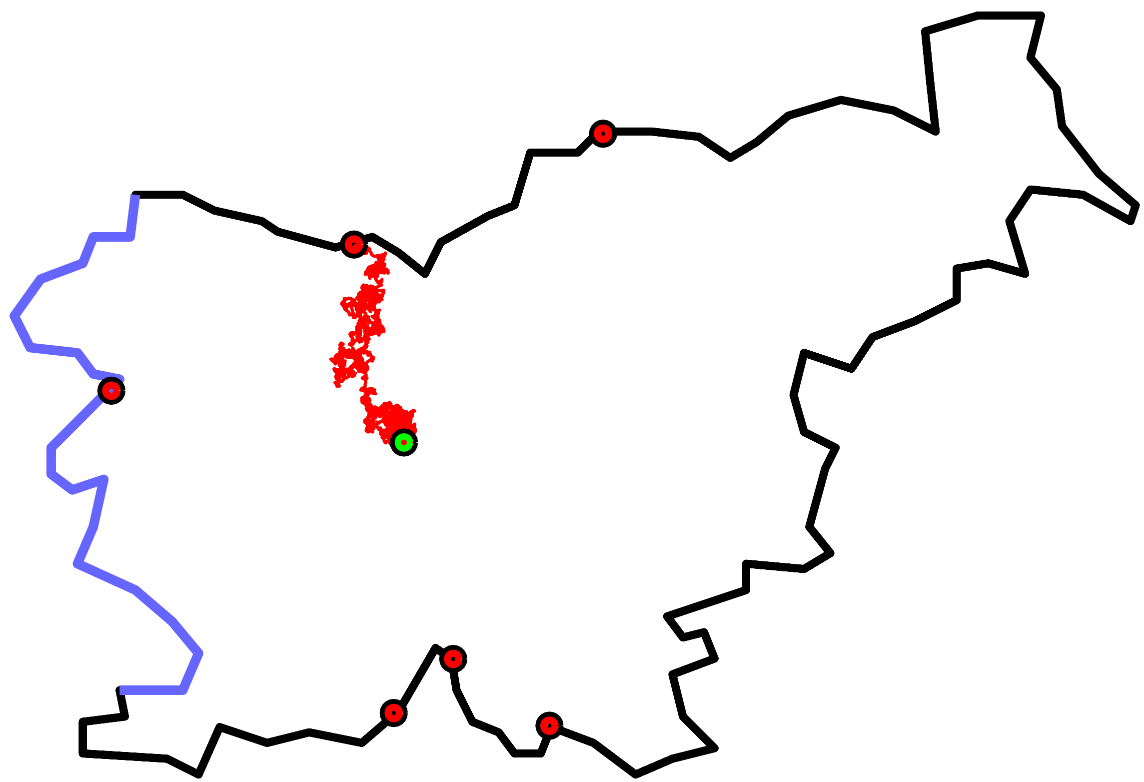

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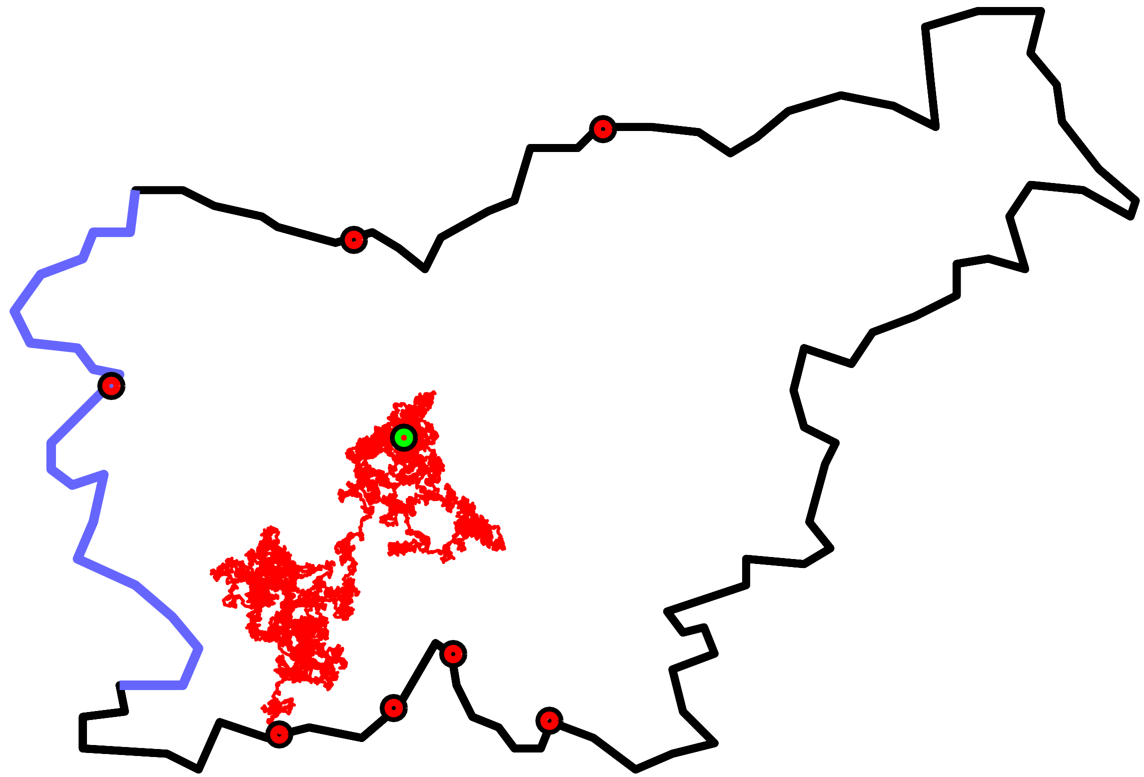

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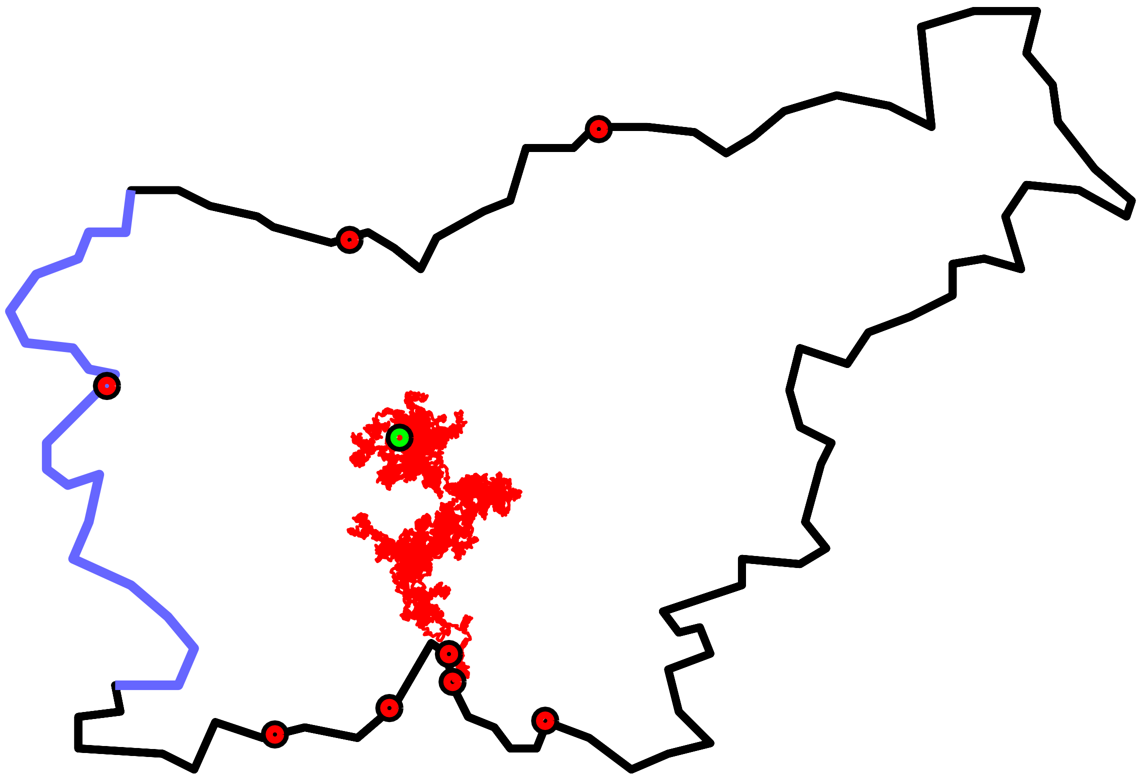

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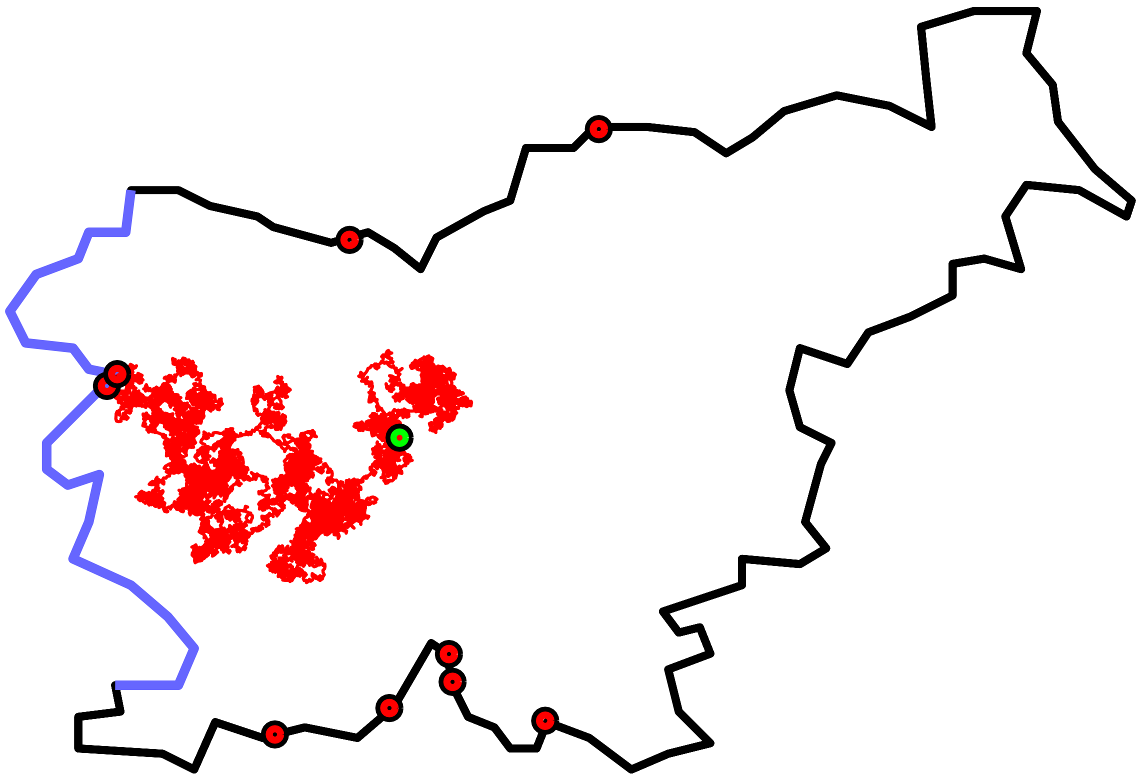

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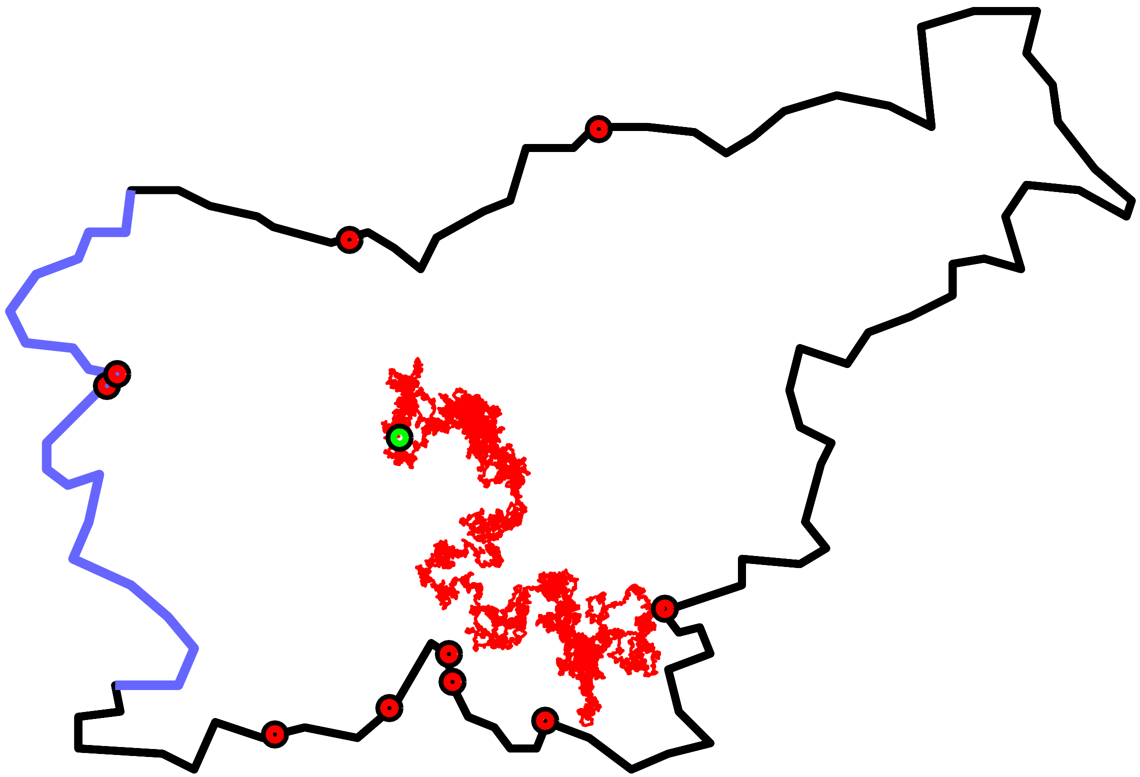
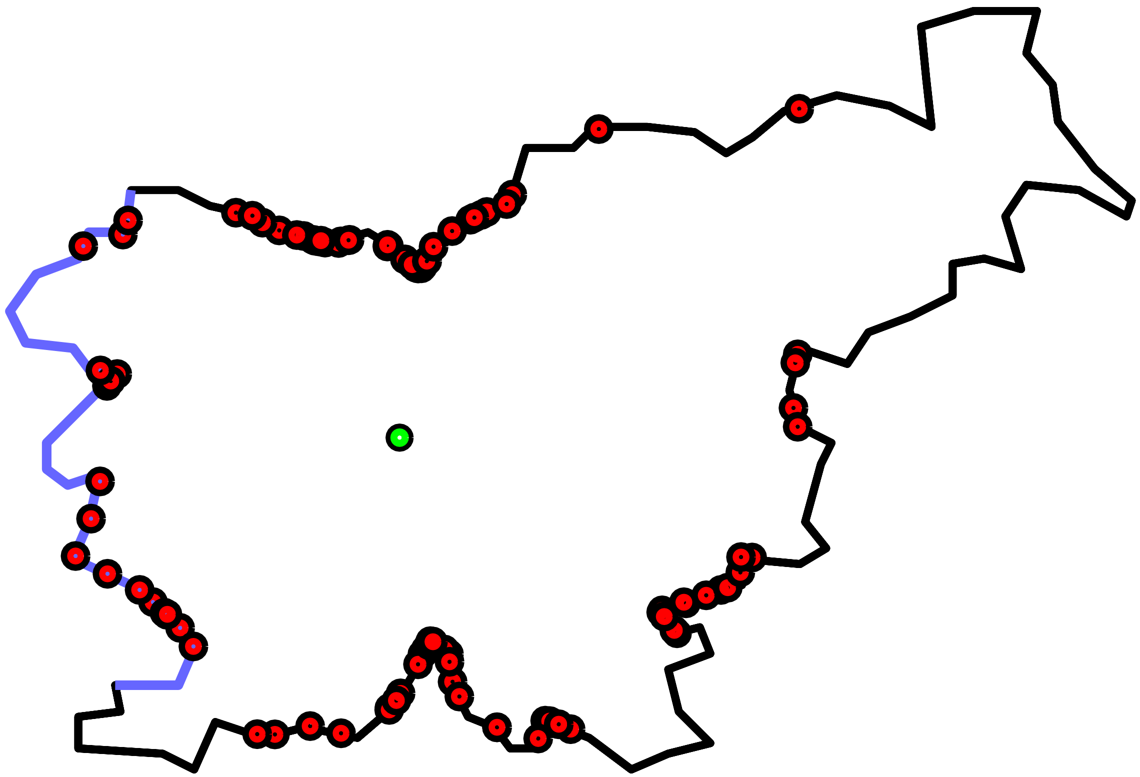


$$\omega(z, E, \Omega) \approx 2/10.$$

Harmonic measure = hitting distribution of Brownian motion

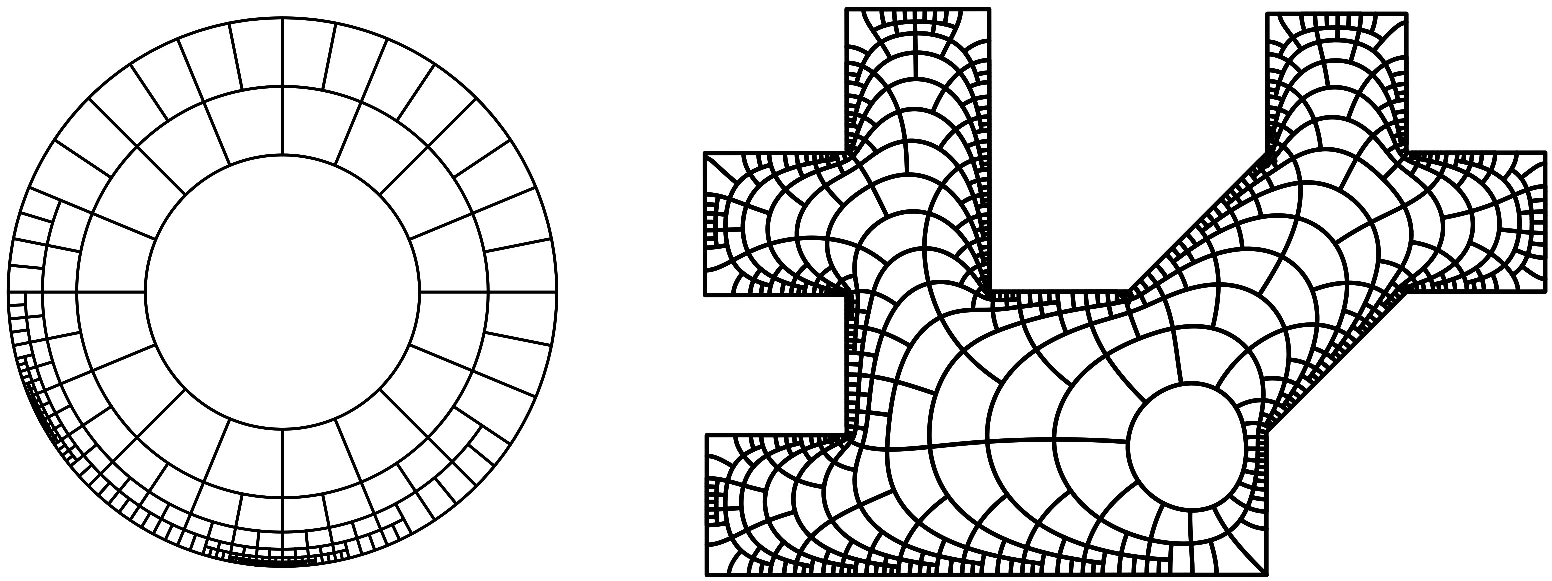


$$\omega(z, E, \Omega) \approx 18/100.$$

This is a harmonic function of base point z,
with boundary values 1 on E an and 0 elsewhere.

Riemann Mapping Theorem: If $\Omega \subsetneq \mathbb{R}^2$ is simply connected and $z \in \Omega$, then there is a conformal map $f : \mathbb{D} \to \Omega$ with $f(0) = z$.

conformal = 1-1, holomorphic= angle and orientation preserving



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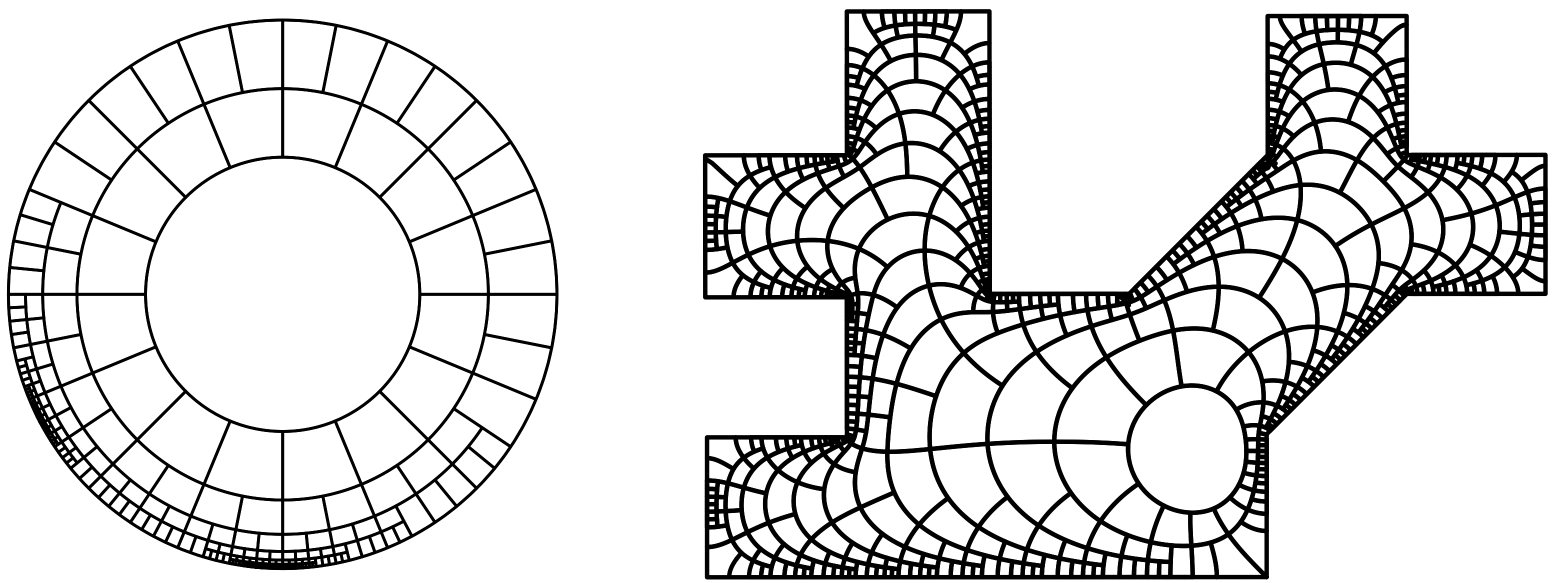
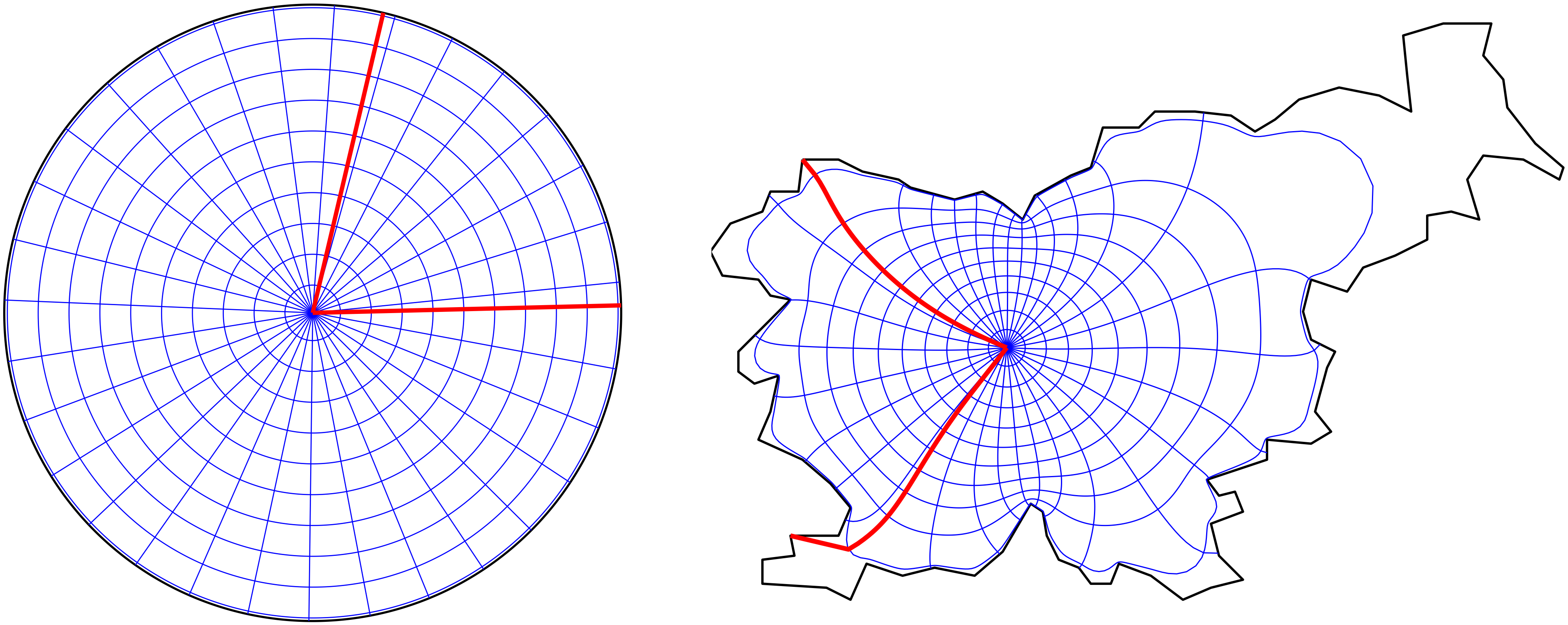


If ω is harmonic on Ω, then $\omega \circ f$ is harmonic on $\mathbb{D}$.

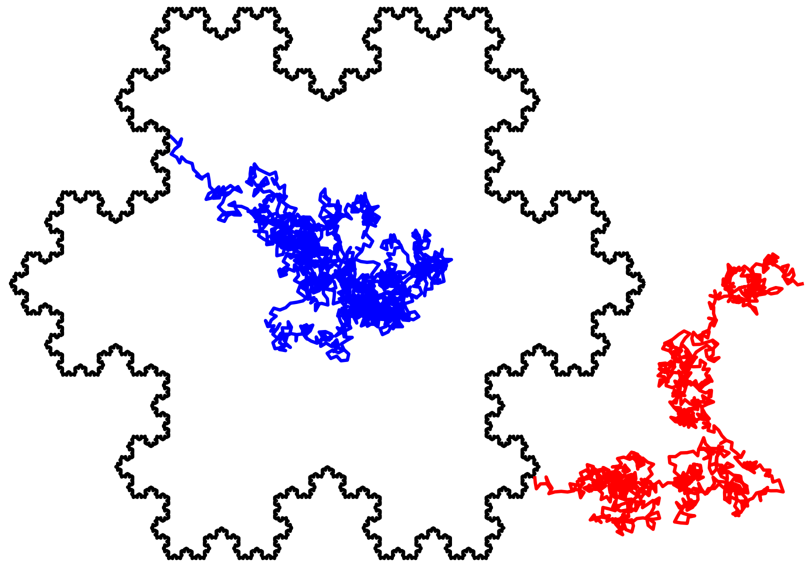
$\Rightarrow \omega(z, E, \Omega) = \omega(0, f^{-1}(E), \mathbb{D}) = |E|/2\pi.$

$\Rightarrow \omega$ on $\partial\Omega$ = conformal image of (normalized) length on $\mathbb{T}$

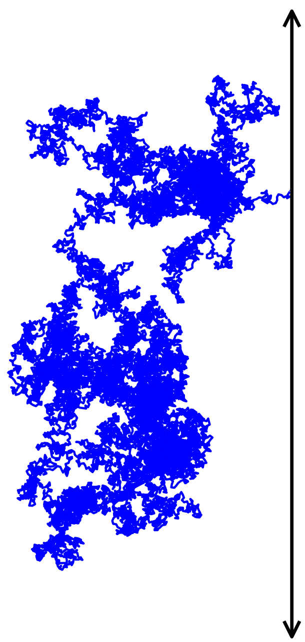


Harmonic measure ≈ .20938

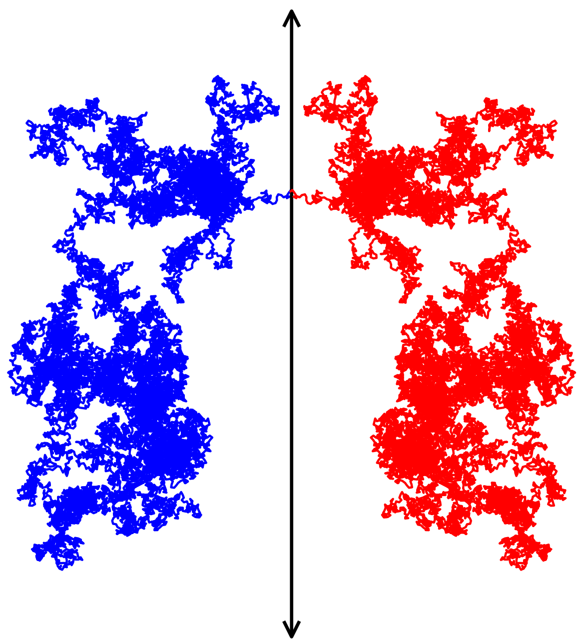
For starting points on opposites sides, can we get the same measure?



Usually not. For fractals inside/outside measures can be singular.

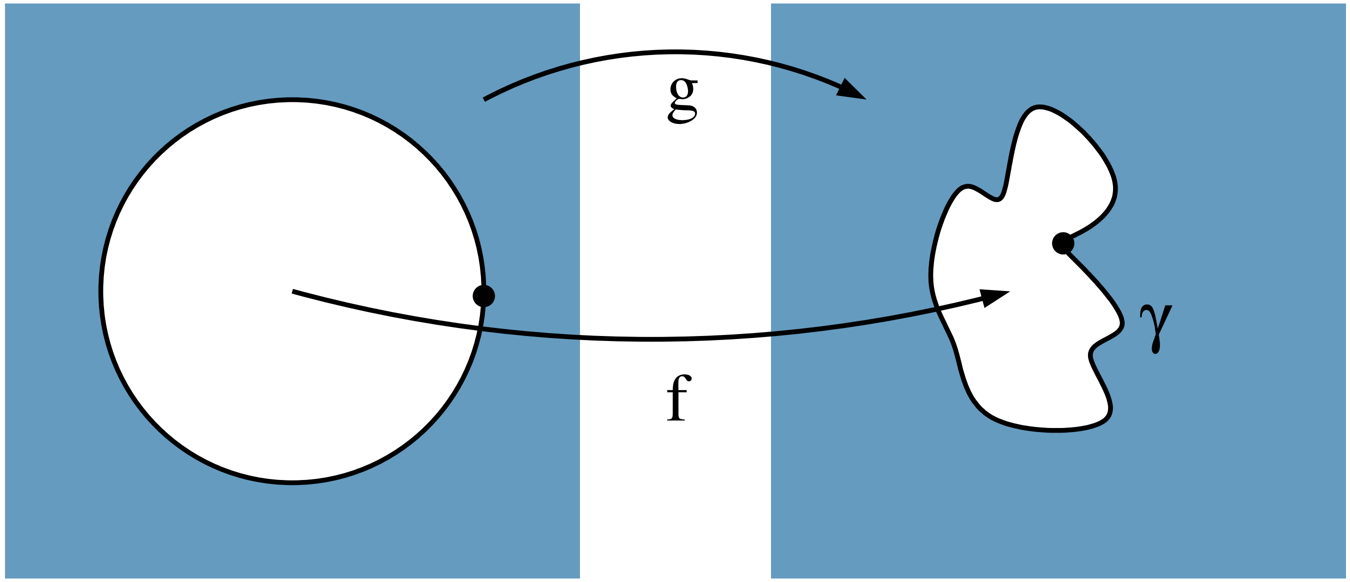


True for lines and circles.



True for lines and circles.

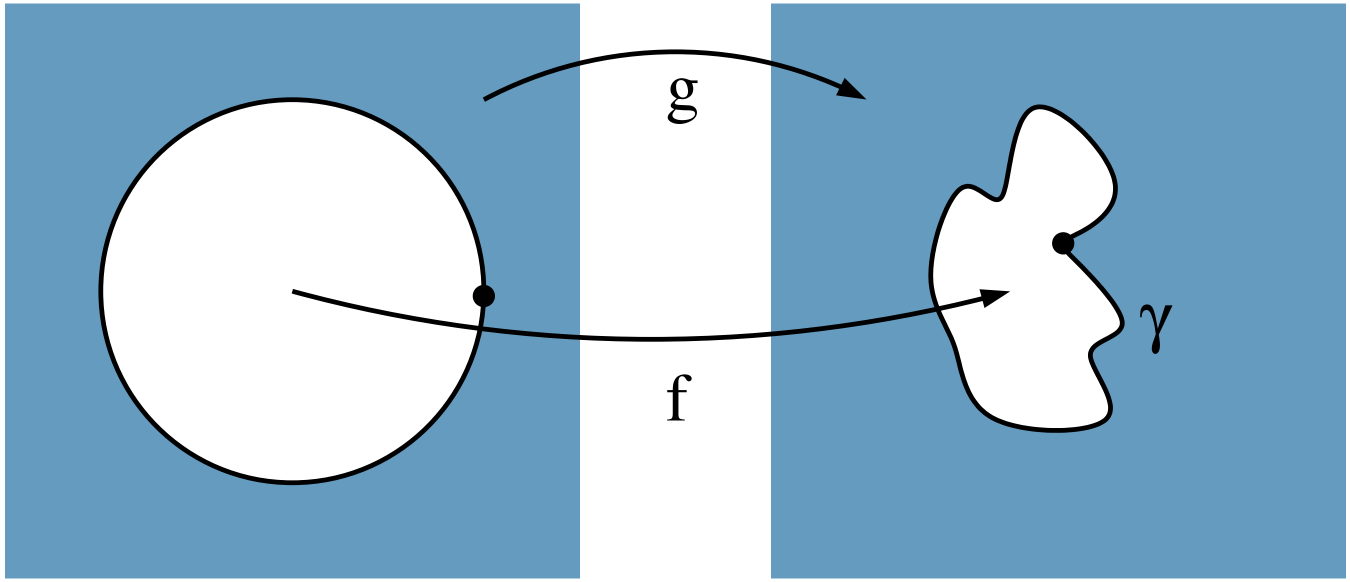
Are these the only cases?



Suppose $\omega_1 = \omega_2$ for a curve γ.

Conformally map two sides of circle to two sides of γ so $f(1) = g(1)$.

$\omega_1 = \omega_2$ implies maps agree on whole boundary.



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Conformally map two sides of circle to two sides of γ so $f(1) = g(1)$.

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So f, g define homeomorphism h of plane holomorphic off circle.

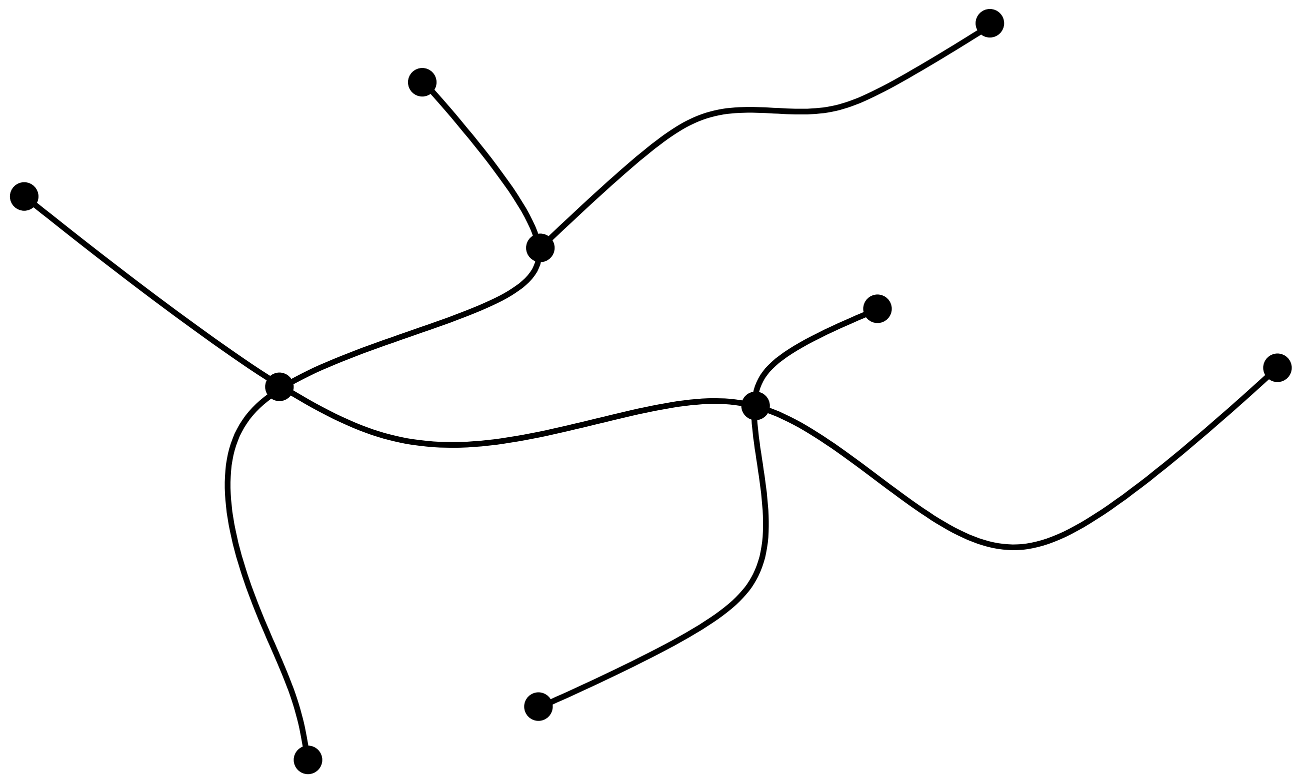
Then h is entire by Morera's theorem.

Entire and 1-1 implies h is linear (Liouville's thm), so γ is a circle.

Circles are only closed curves with equal harmonic measures on both sides.

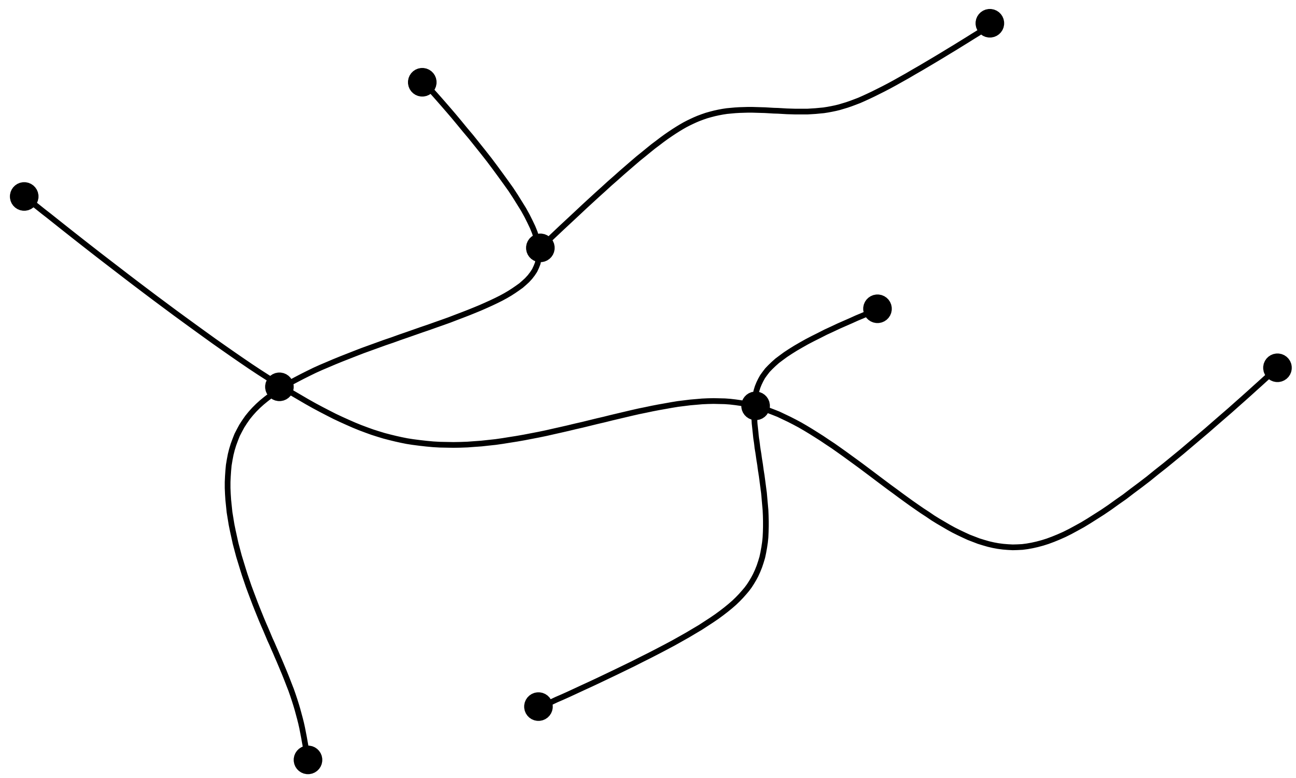
What about other kinds of 2-sided objects?

Conformally balanced trees and dessins d'enfants



A planar graph is a finite set of points connected by non-crossing edges.

It is a tree if there are no closed loops.



A planar tree is **conformally balanced** if

- every edge has equal harmonic measure from ∞
- edge subsets have same measure from both sides

This is also called a "**true tree**". A line segment is an example.

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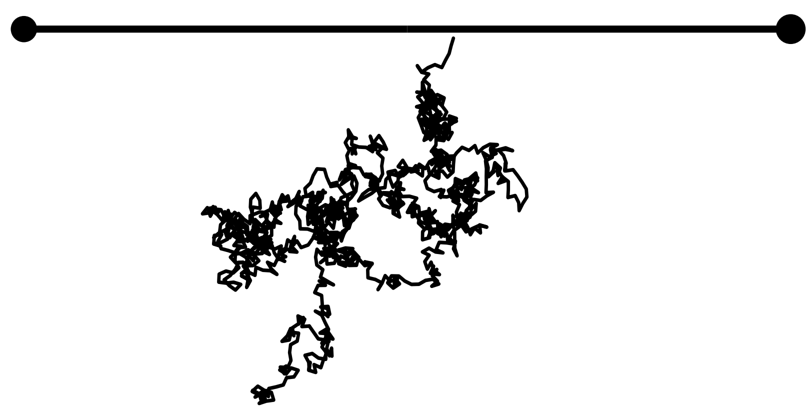
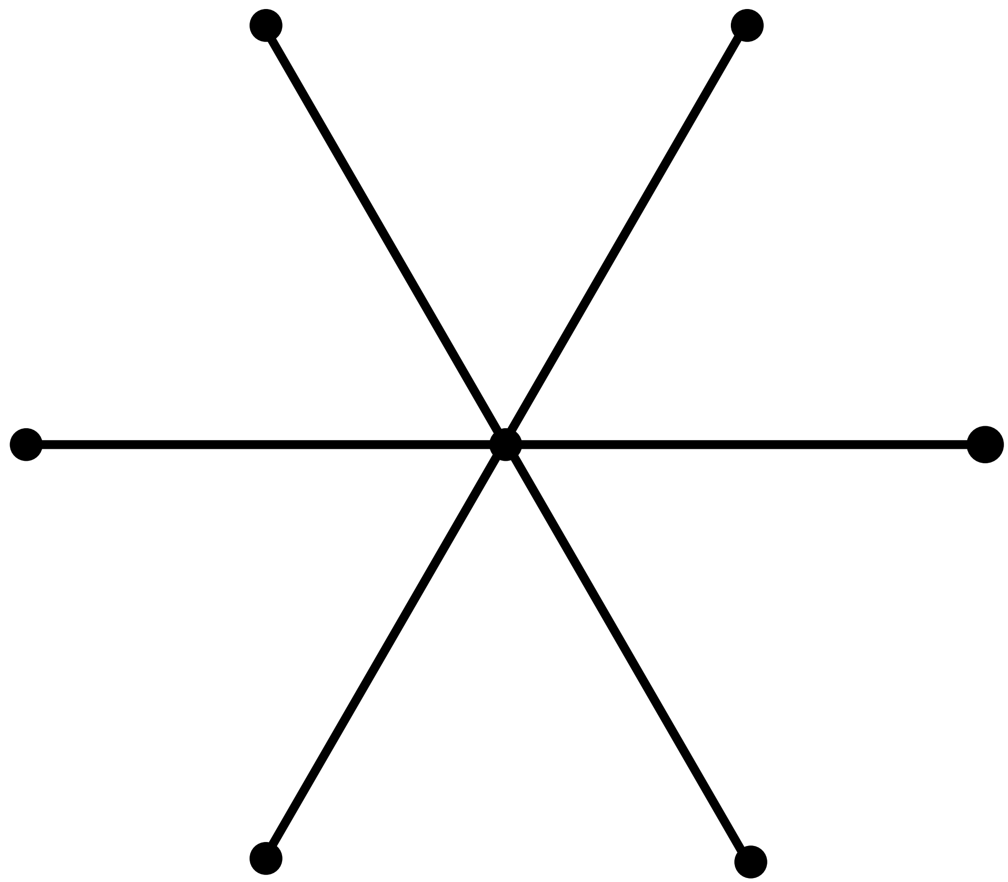
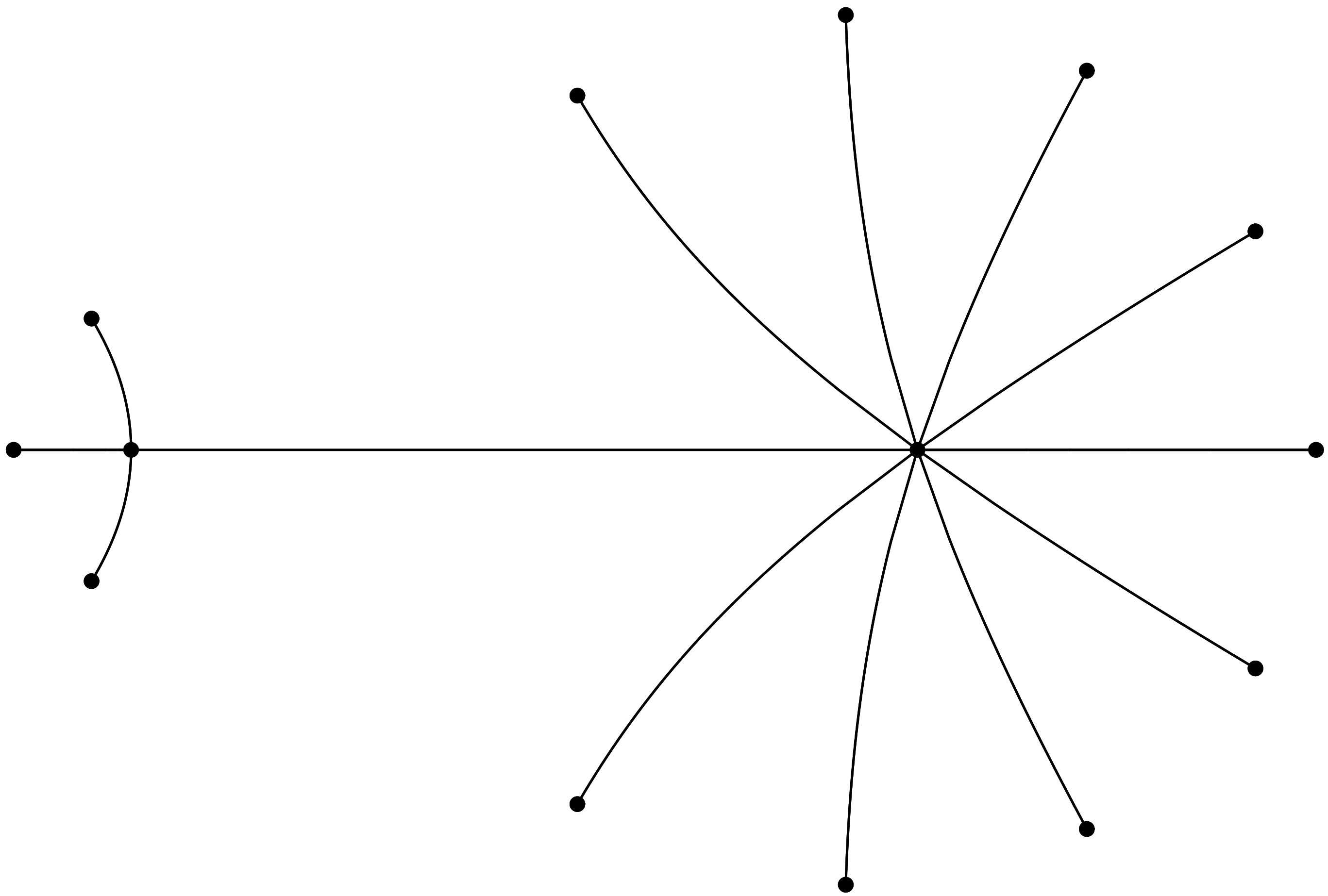




Trivially true by symmetry



Non-obvious true tree

Definition of critical value: if p = polynomial, then

$$\mathrm{CV}(p) = \{p(z) : p'(z) = 0\} = \text{critical values}$$

If $\mathrm{CV}(p) = \pm 1$, p is called **generalized Chebyshev** or **Shabat**.

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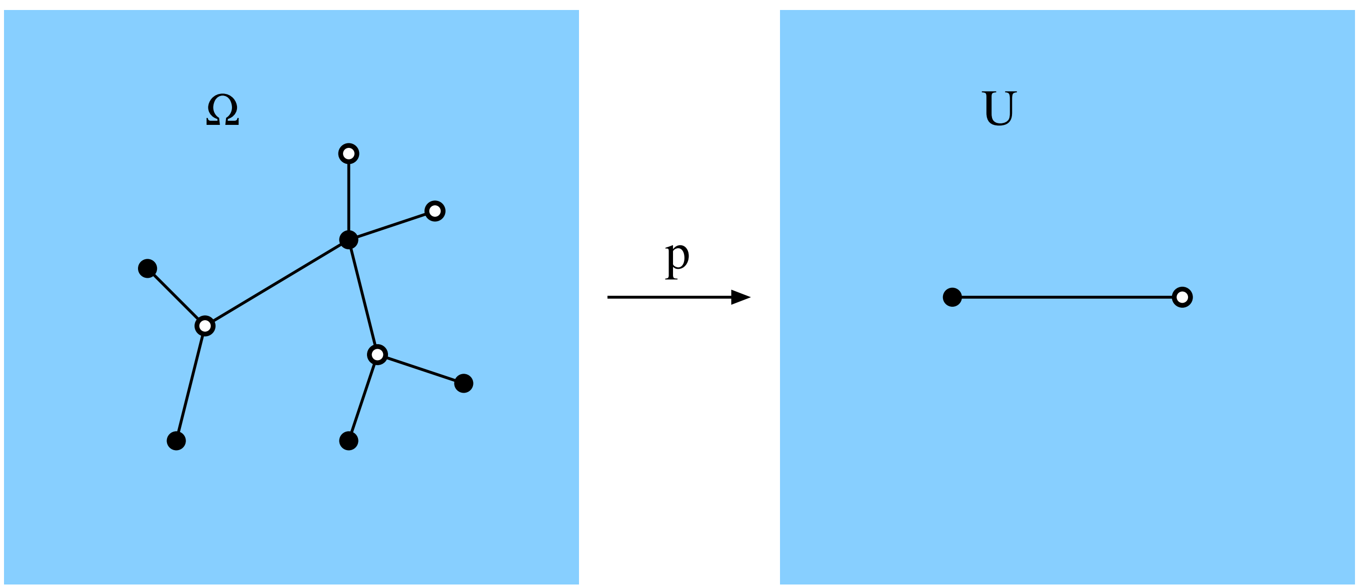
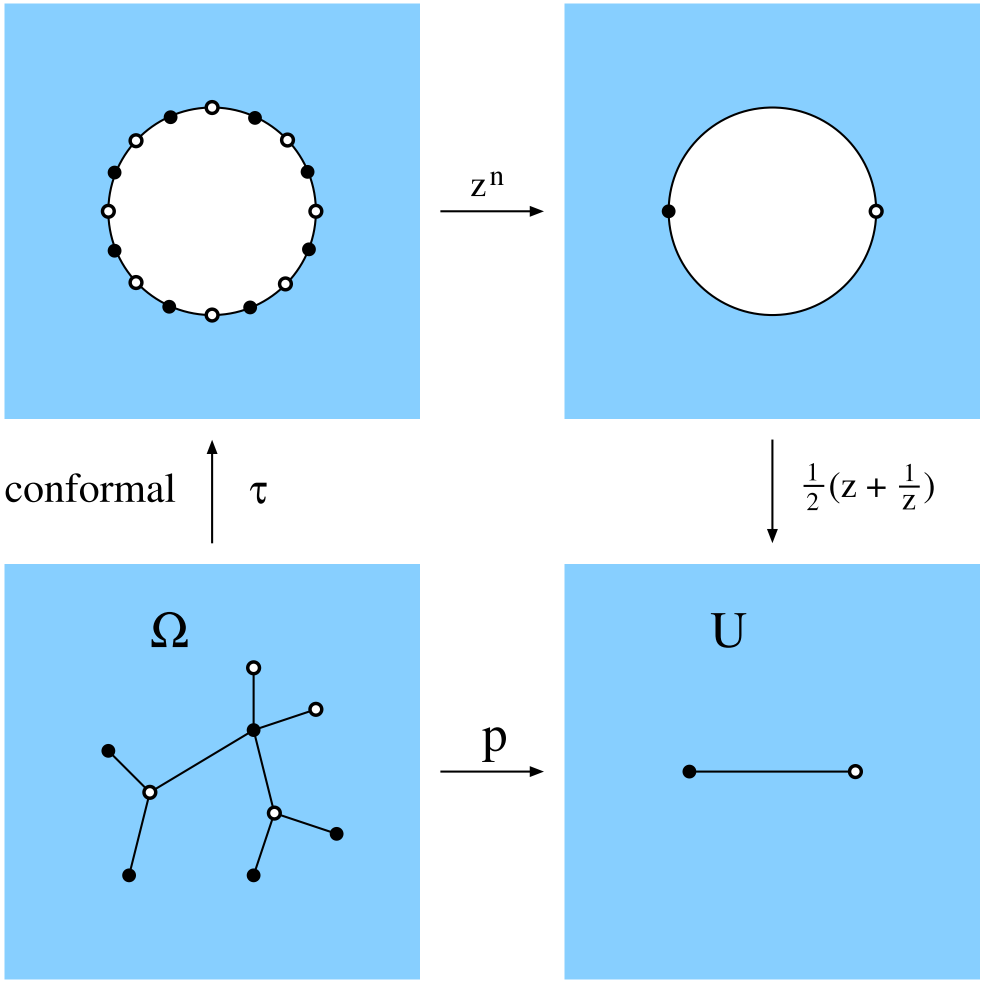
Thm: T is balanced iff $T = p^{-1}([-1, 1])$, $p =$ Shabat.



$$\Omega = \mathbb{C} \setminus T \qquad\qquad U = \mathbb{C} \setminus [-1, 1]$$

T conformally balanced $\Leftrightarrow$ p Shabat.



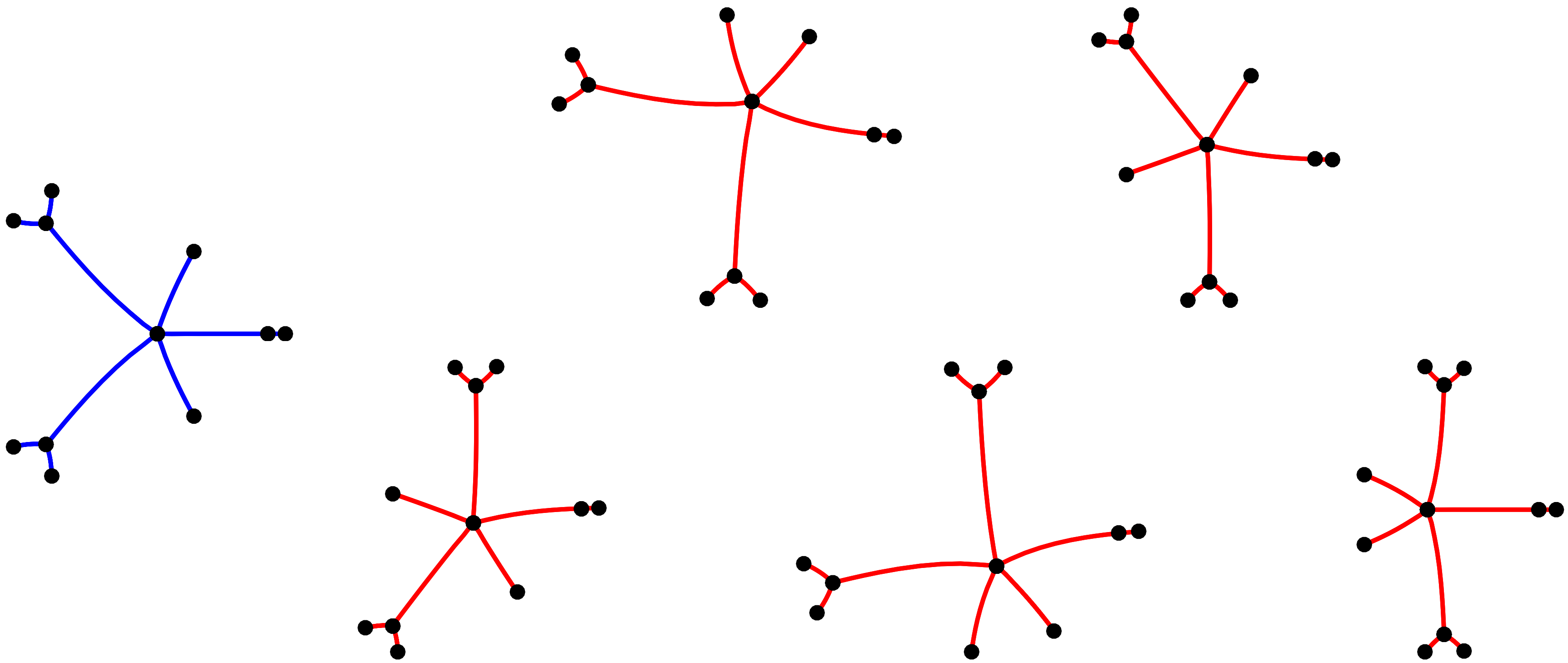
p is entire and n-to-1 $\Leftrightarrow$ $p =$ polynomial.

$\mathrm{CV}(p) \notin U \leftrightarrow p : \Omega \to U$ is covering map.

Algebraic aside:

True trees are examples of Grothendieck's *dessins d'enfants* on sphere.

Normalized polynomials are algebraic, so trees correspond to number fields. Computing number field from tree is difficult.



Six true trees - same abstract tree, different planar trees.

Which planar trees have a true form?

What can that true form look like?

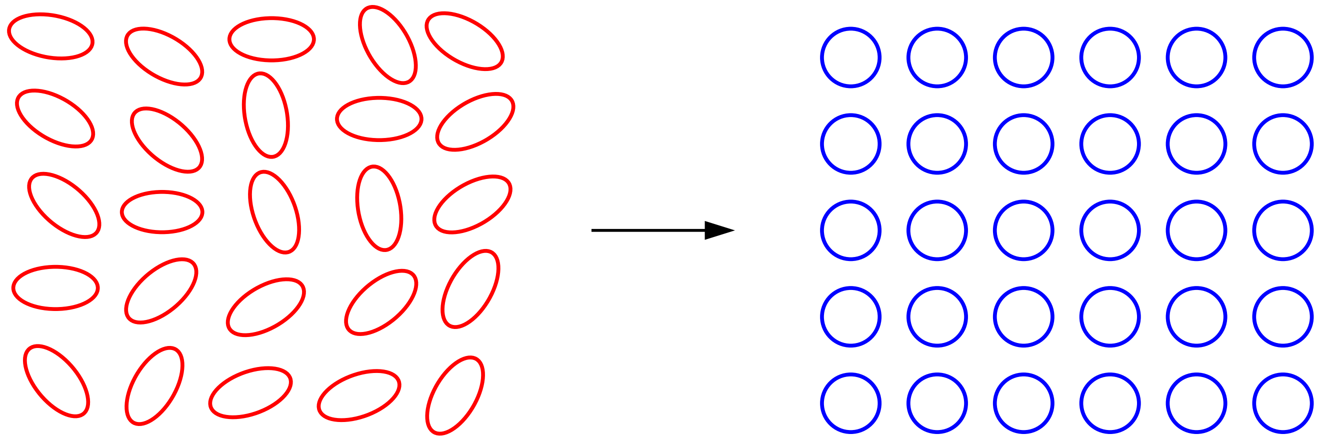
Theorem: Every finite planar tree has a true form.

Unique up to similarities (Morera + Liouville).

Standard proof uses the uniformization theorem.

I will describe an alternate proof using quasiconformal maps.

Diffeomorphisms send infinitesimal ellipses to circles.



Eccentricity = ratio of major to minor axis of ellipse.

K-quasiconformal = ellipses have eccentricity $\leq K$ almost everywhere

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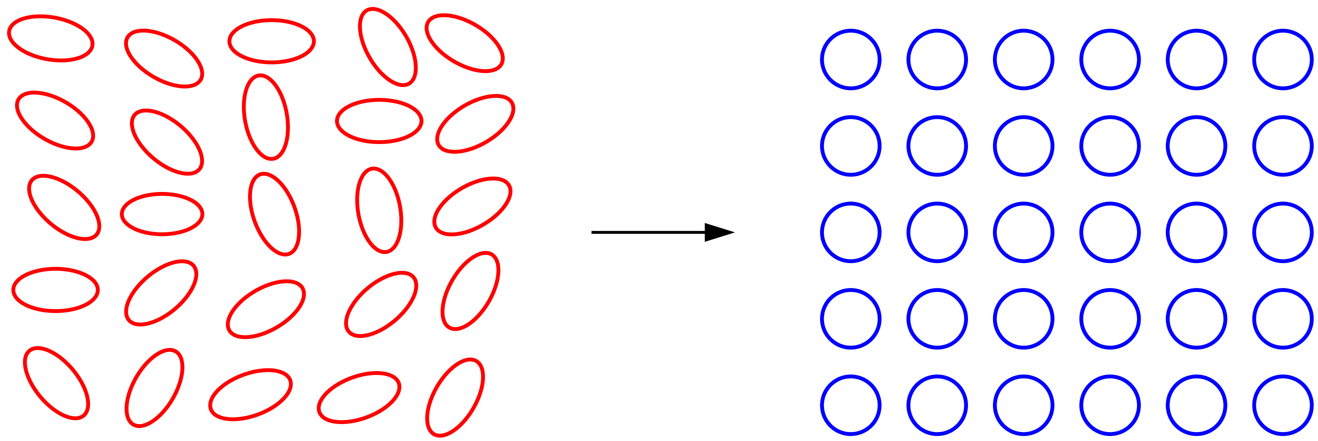

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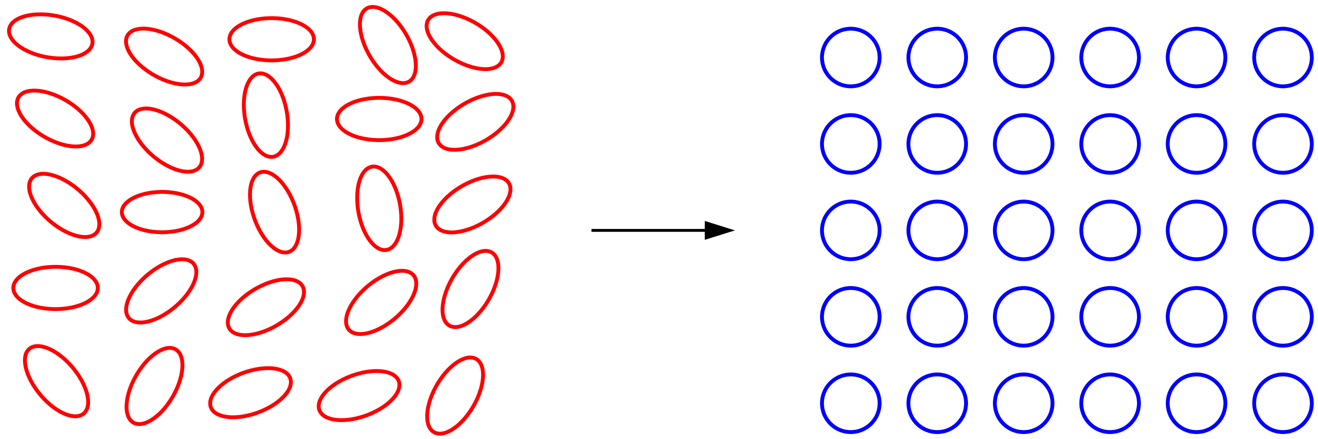
Ellipses determined a.e. by measurable dilatation $\mu = f_{\overline{z}}/f_z$ with

$$|\mu| \leq \frac{K-1}{K+1} < 1.$$

Conversely, ...

Diffeomorphisms send infinitesimal ellipses to circles.



Mapping theorem: any such μ comes from some QC map f.

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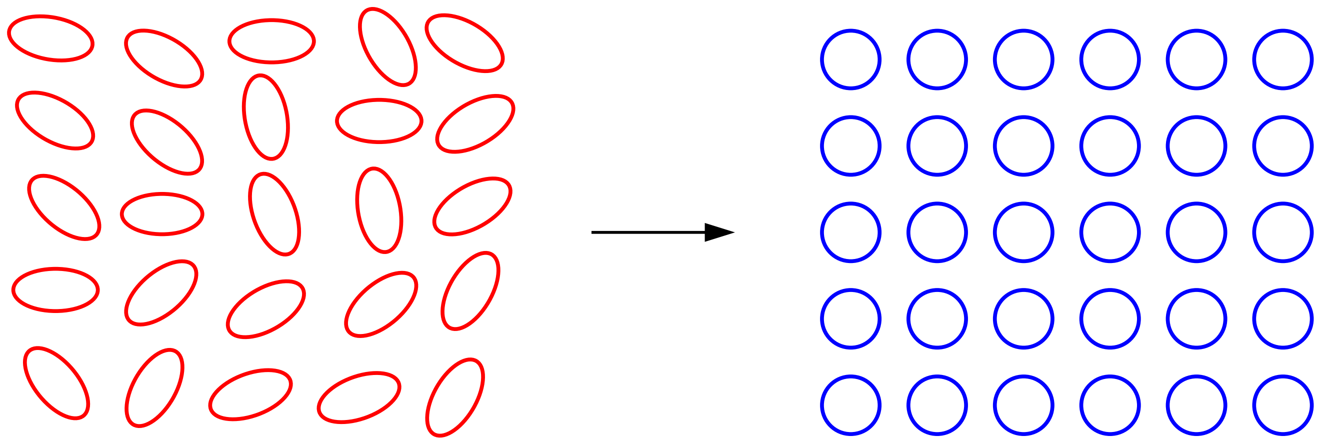
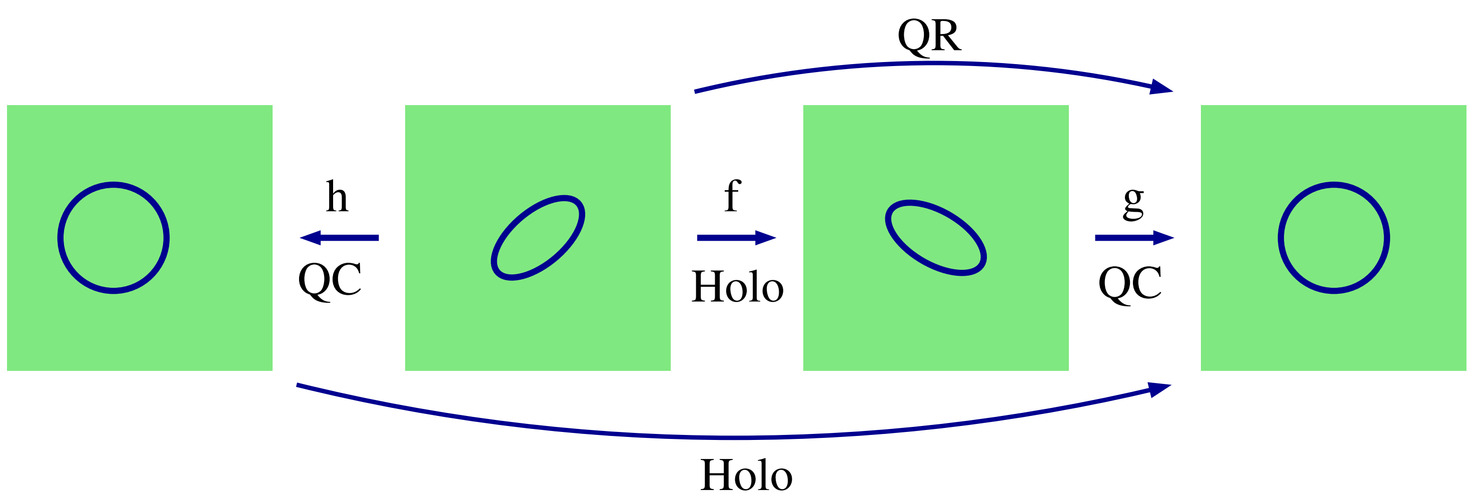


Mapping theorem: any such μ comes from some QC map f.

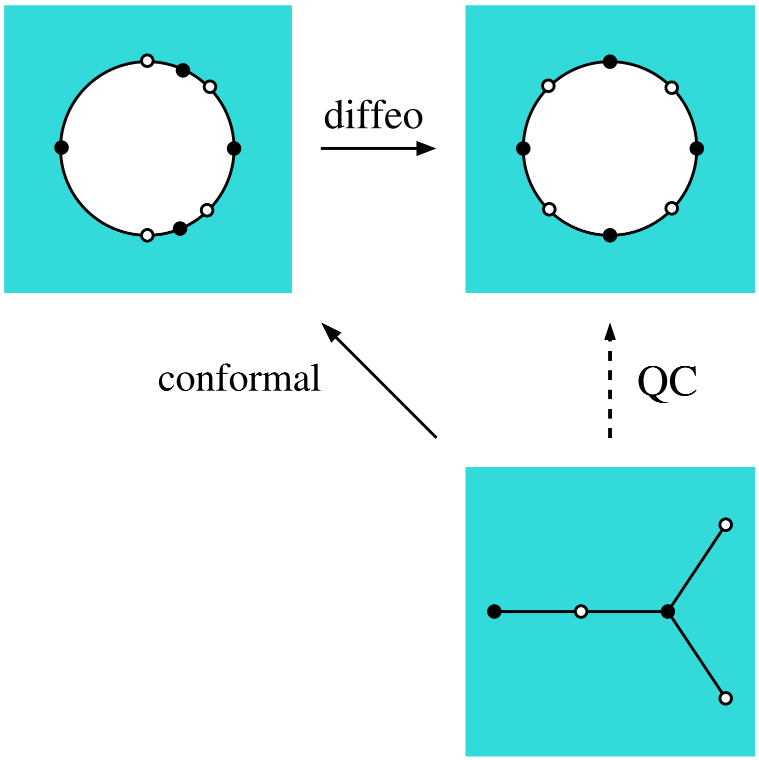
Cor: If f is holomorphic and g is QC, then there is a QC map h so that $F = g \circ f \circ h^{-1}$ is also holomorphic. ($g \circ f$ is called QR = quasiregular)



QC proof that every finite tree has a true form:

Map $\Omega = \mathbb{C} \setminus T$ to $\{|z| > 1\}$ conformally.

"Equalize intervals" by diffeomorphism. Composition is quasiconformal.



Dilatation of QC map depends degree of "imbalance" of harmonic measure.

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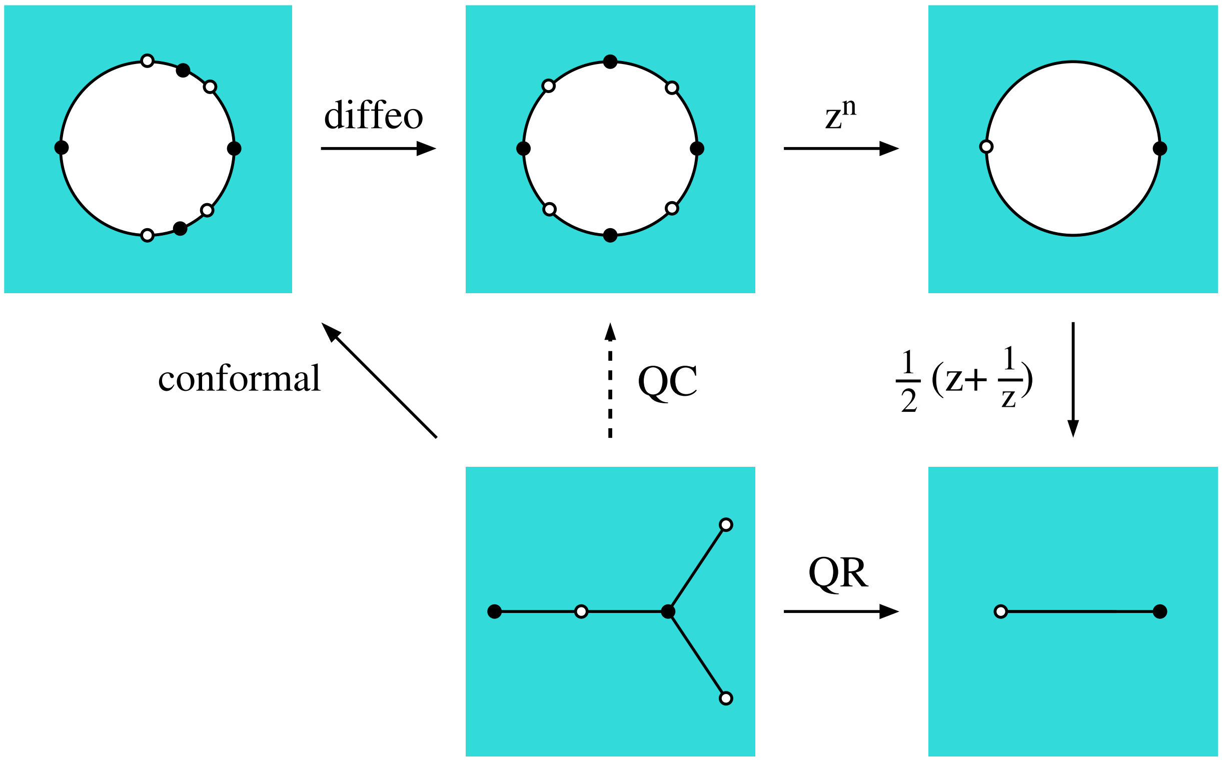

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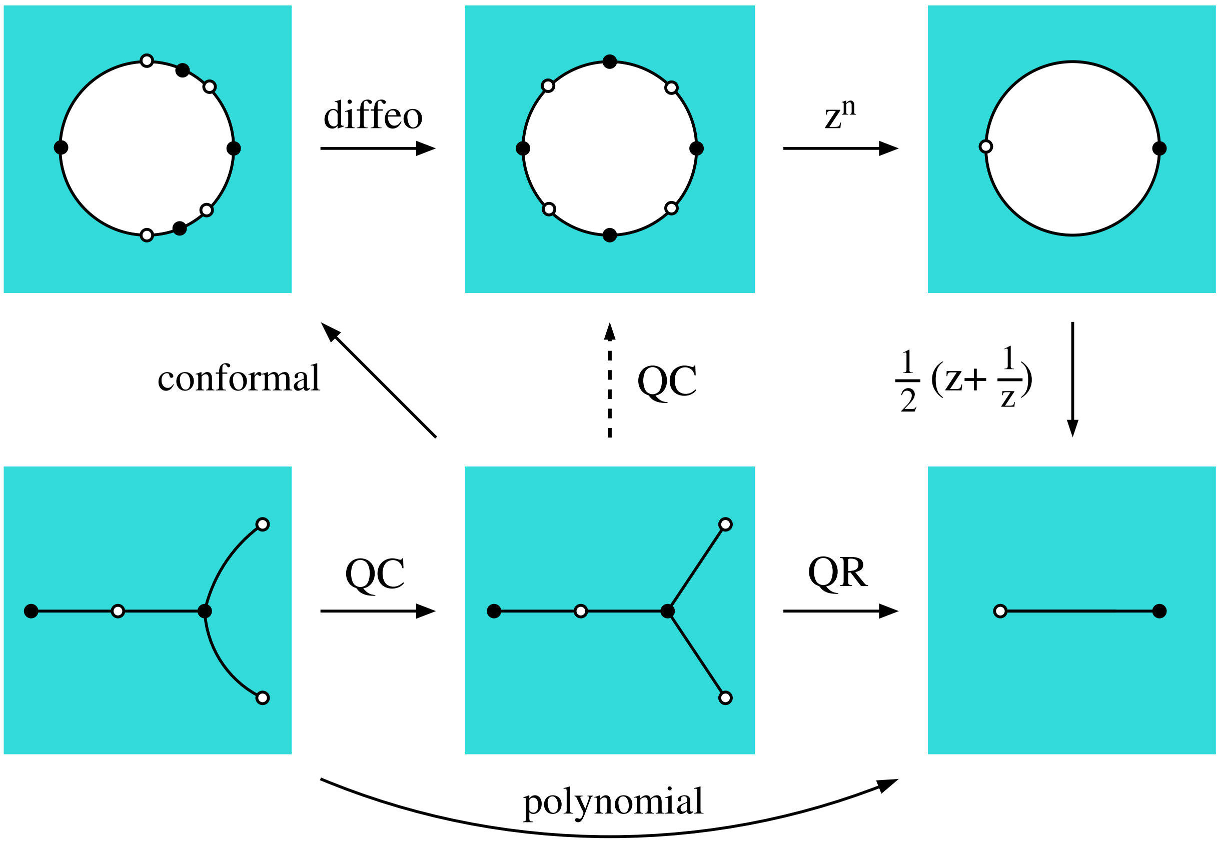

Mapping theorem implies there is a QC φ so $p = q \circ \varphi$ is a polynomial.

Thus true trees can have any combinatorics.

Alex Eremenko: can they have any shape?

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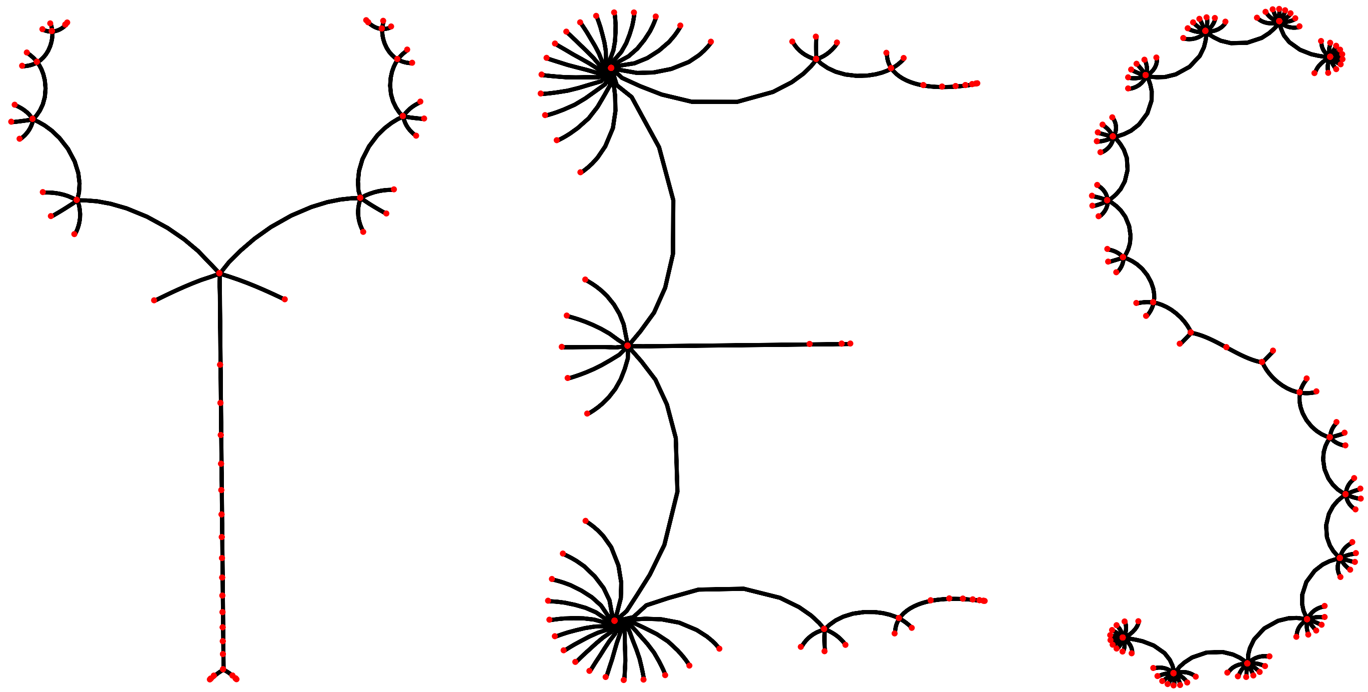
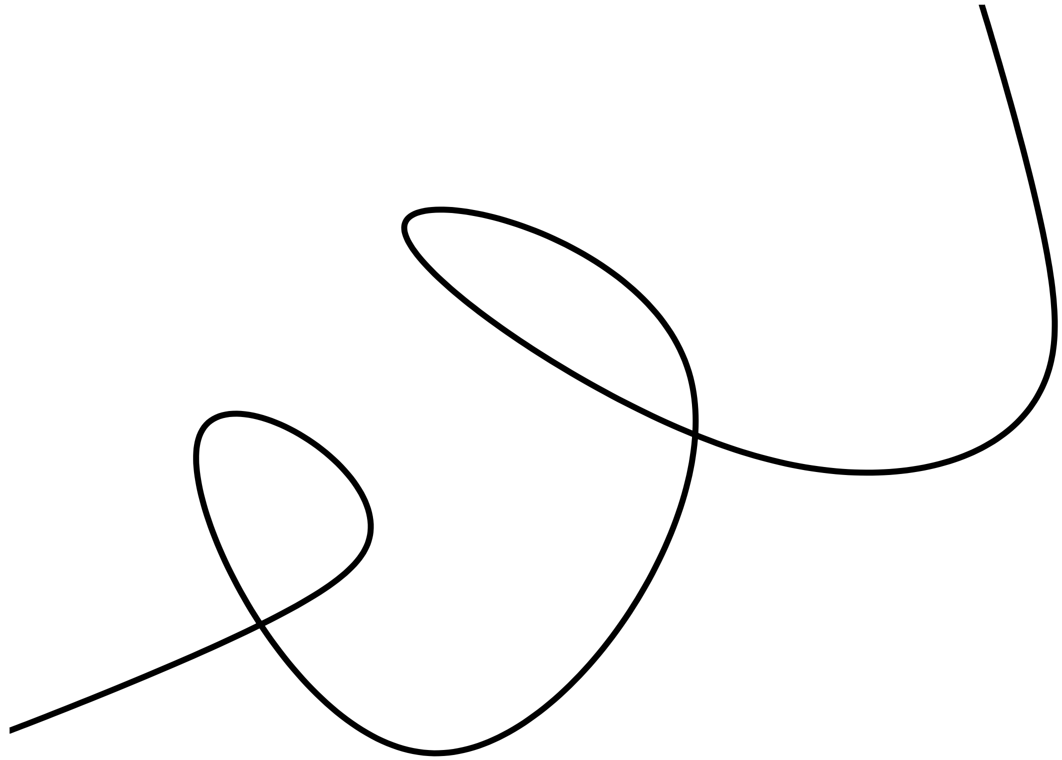
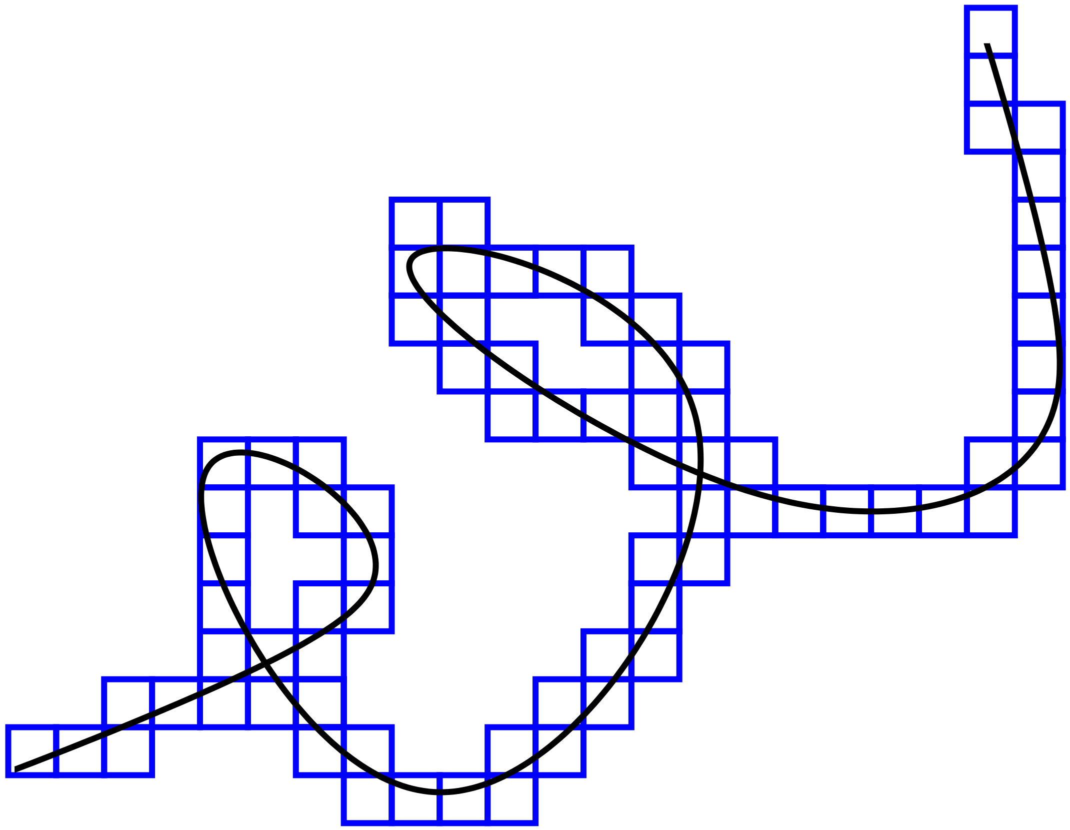


Thm: Every planar continuum is limit of true trees.



Suffices to approximate subtrees of a grid.



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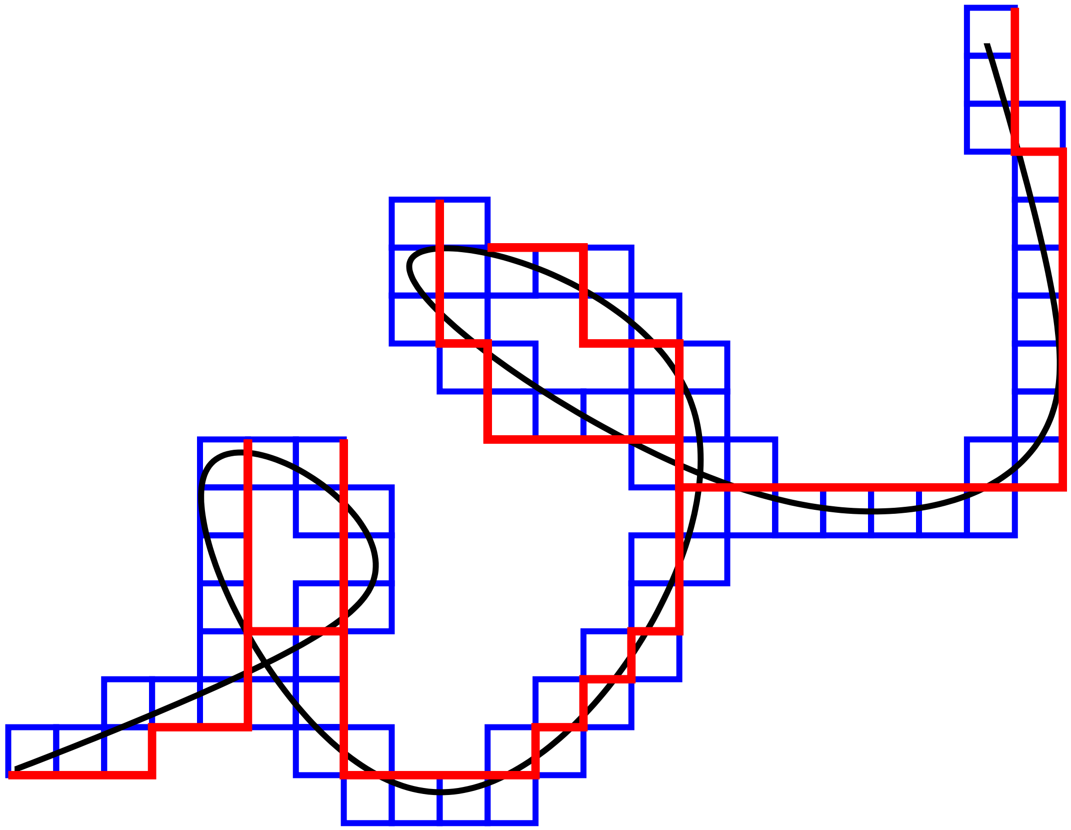

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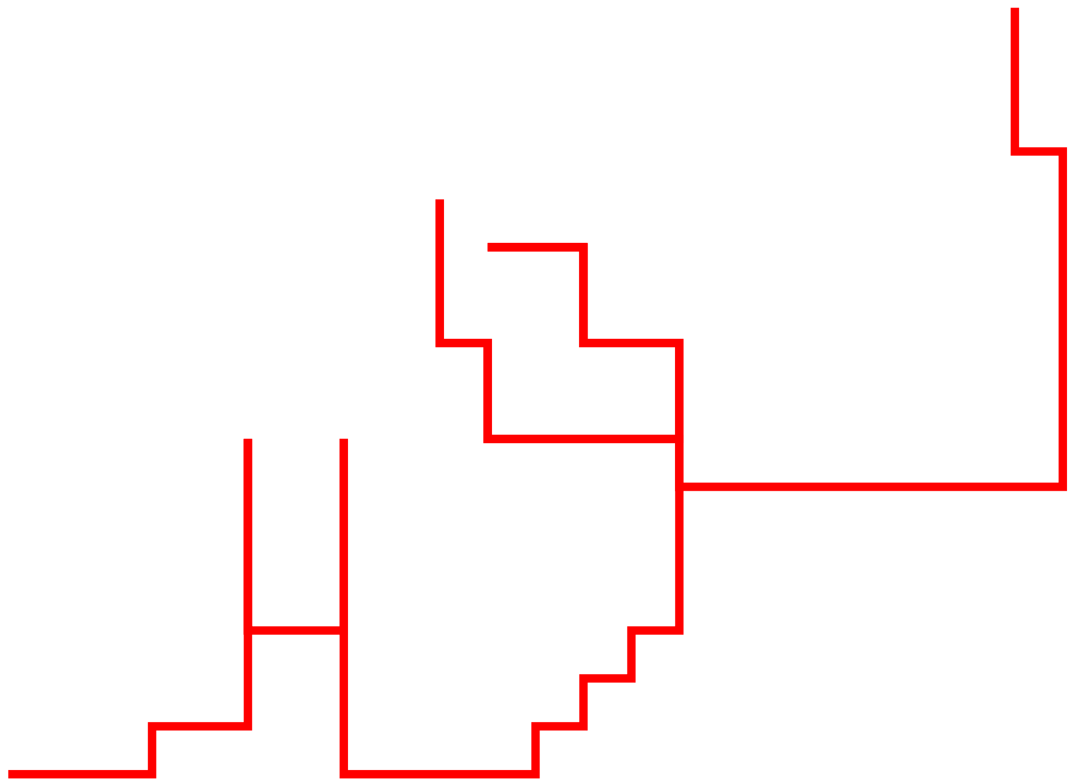
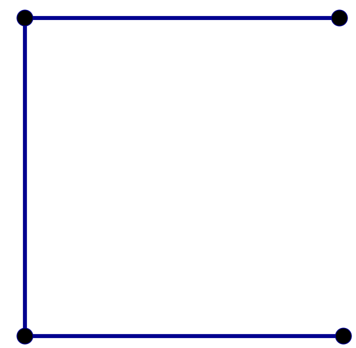


Suffices to approximate subtrees of a grid.

Theorem: Every planar continuum is a limit of true trees.

Idea of Proof: reduce harmonic measure ratio by adding edges.



Vertical side has much larger harmonic measure from left.

Add edges ($\Rightarrow$ change combinatorics) to "balance" harmonic measure.

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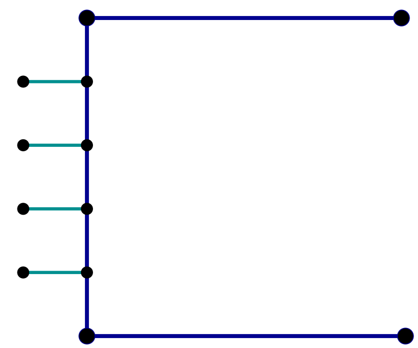

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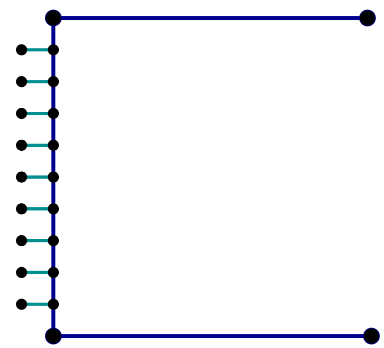


"Left" harmonic measure is reduced (roughly 3-to-1).

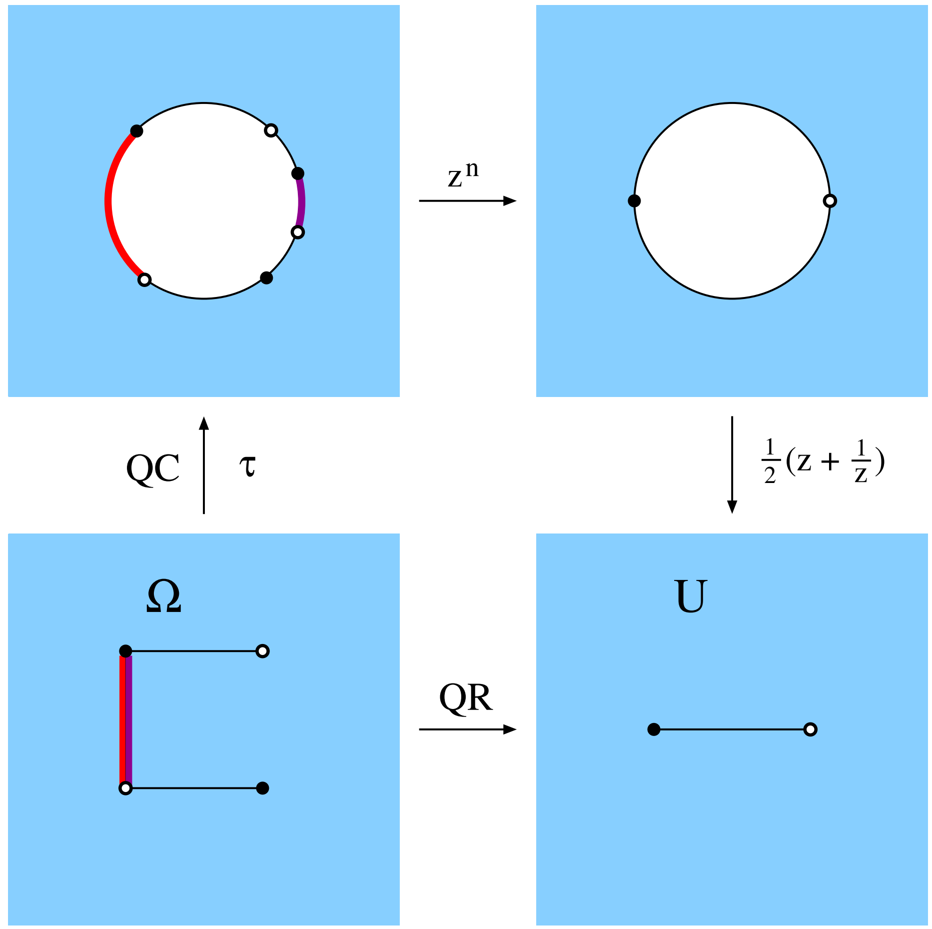
New edges are approximately balanced (universal constant).

Mapping theorem gives exactly balanced.

QC correction map is near identity if "spikes" are short.

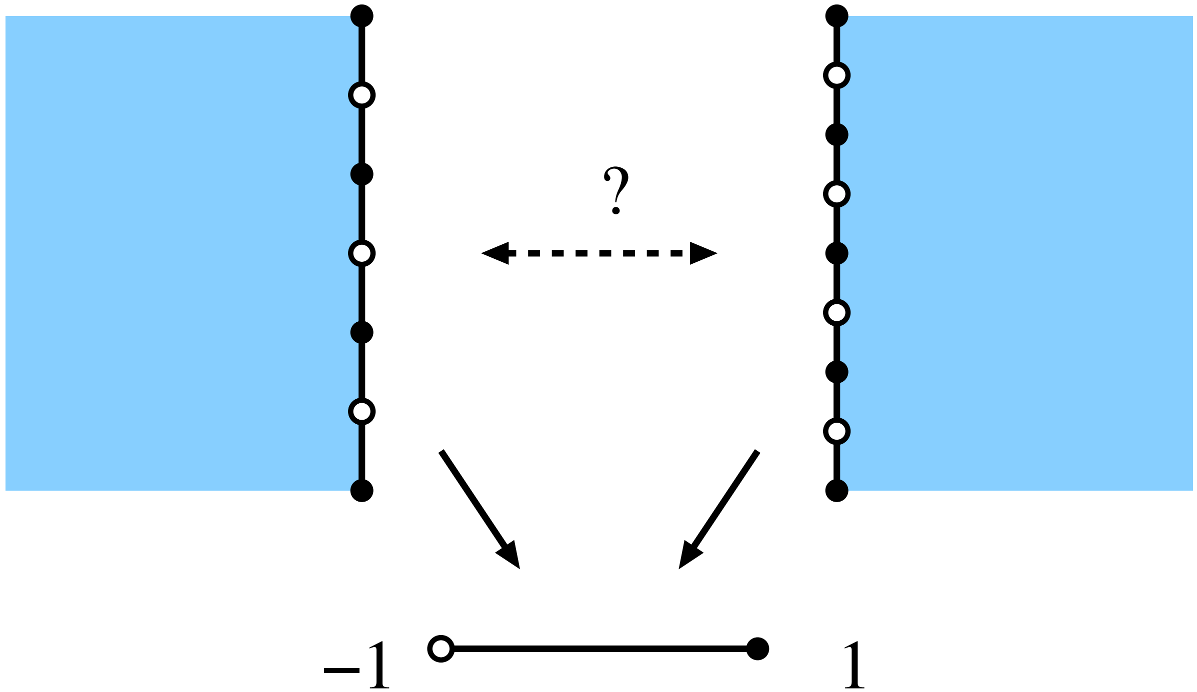
New tree approximates shape of old tree; different combinatorics.

Summary of folding procedure



Start with unbalanced tree. Different harmonic measures.
Conformal map sends edges to different lengths.
Opposite sides map to $[-1, 1]$ with different degrees.

A mathematical version of joining the pipes:

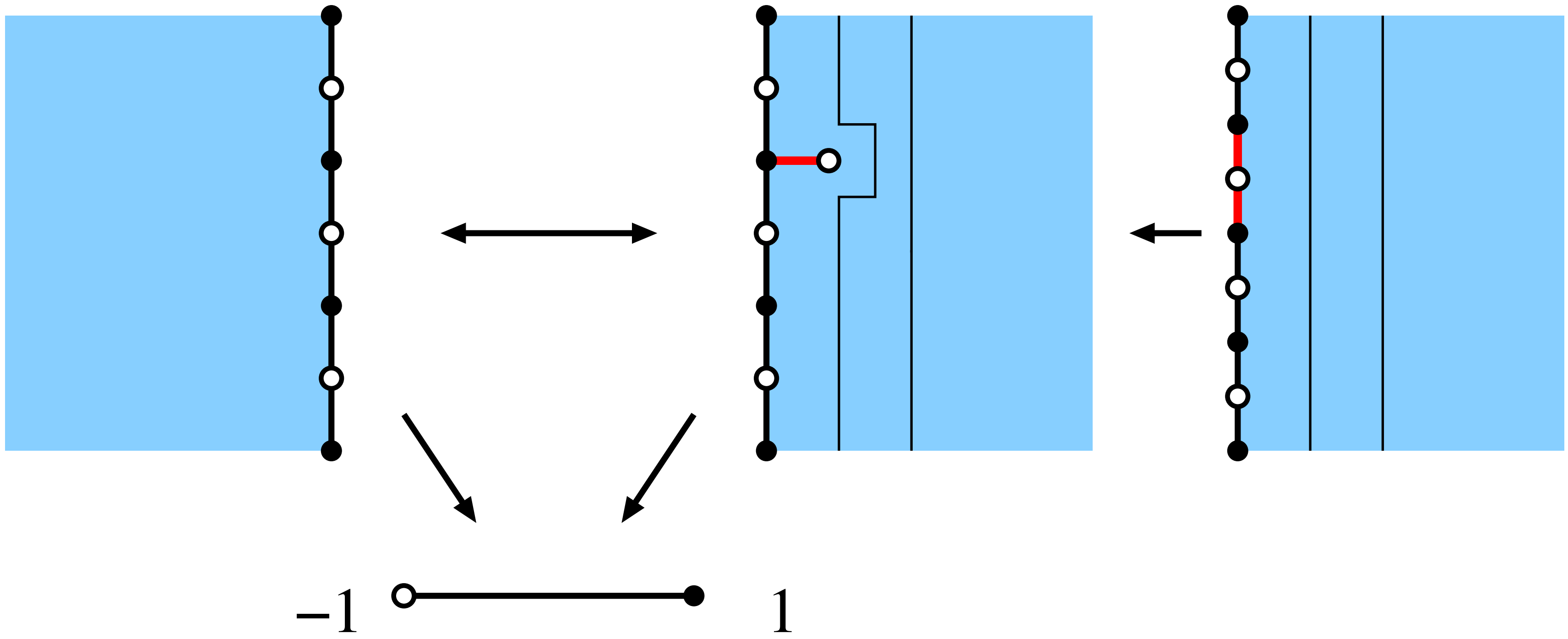


Functions on two domains, mapping boundaries to $[-1, 1]$.

Can we identify boundaries so functions match up?

There is a problem if degrees differ.

A mathematical version of joining the pipes:



Fold larger boundary so "extra" edges attach to each other.

Then glue domains along the remaining free edges.

This is simplest type of "quasiconformal folding".

Runge's theorem and critical values

Runge's theorem: if $\epsilon > 0$, K is compact, and f is holomorphic on a simply connected neighborhood W of K, then there is a polynomial p so that $|f(z) - p(z)| \leq \epsilon$ for $z \in K$.

Theorem (B.-Lazebnik): in addition, all critical values of p have modulus $\leq \sup_K |f|$.

Finite Blaschke products: for $|\lambda| = 1$, $\{z_k\} \subset \mathbb{D}$,

$$B(z) = \lambda \cdot \prod_{k=1}^{n} \frac{z - z_k}{1 - \overline{z_k} z}, \qquad z \in \mathbb{D}.$$

These are n-to-1, holomorphic, proper $\mathbb{D} \to \mathbb{D}$.

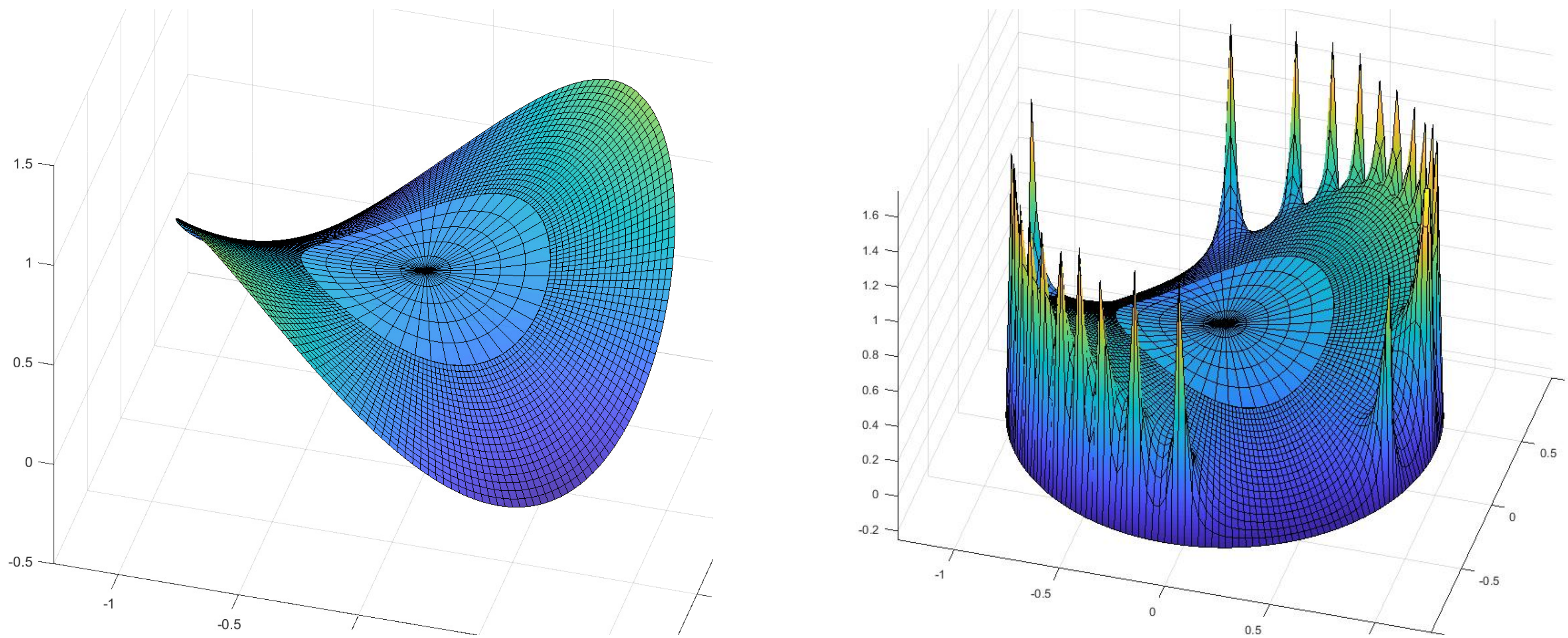
Degree n covering map $\mathbb{T} \to \mathbb{T}$.

Contour plot of BP with 5 random zeros.

Carathéodory's Theorem: Any holomorphic $f : \mathbb{D} \to \mathbb{D}$ can be uniformly approximated on compact $K \subset \mathbb{D}$ by finite Blaschke products.

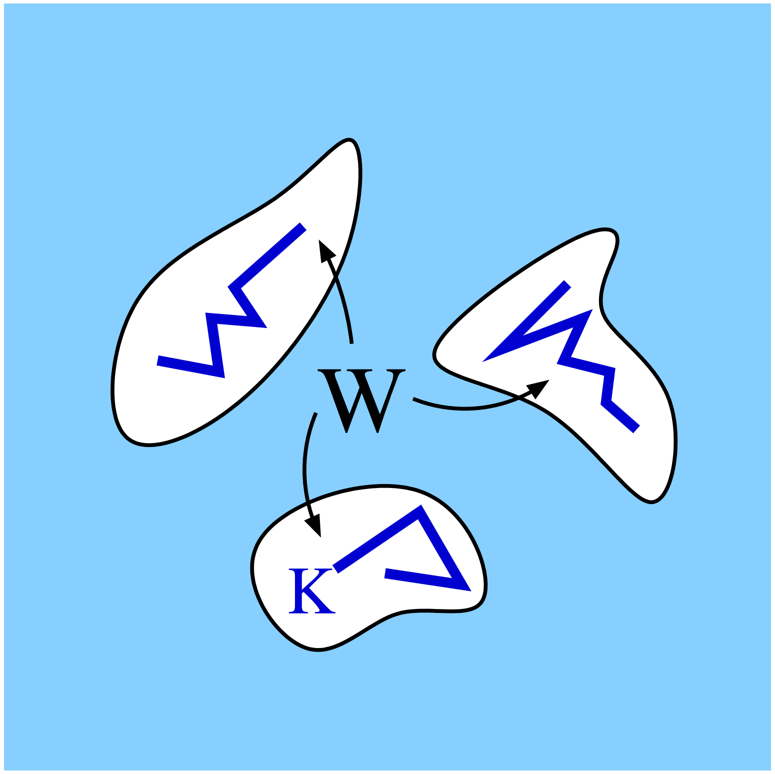
Idea of proof:

- $\log|B(z)|$ is zero on $\mathbb{T}$, harmonic except for log poles at $\{z_n\}$.
- $-\log|B(z)| =$ sum of Green's functions
- Choose poles so sum $\approx$ discretized Poisson integral of $\log|f|$.
- $\Rightarrow -\log|B| \approx \log|f|$ on compact sets.
- $\Rightarrow B \approx f$ on compact sets.



Approximation of $u(x, y) = \frac{1}{2} + xy$.

To prove Runge, we may assume f is proper holomorphic map $W \to \mathbb{D}$ (analog of finite Blaschke product). Maps $\partial W \to \mathbb{T}$, finite-to-1.



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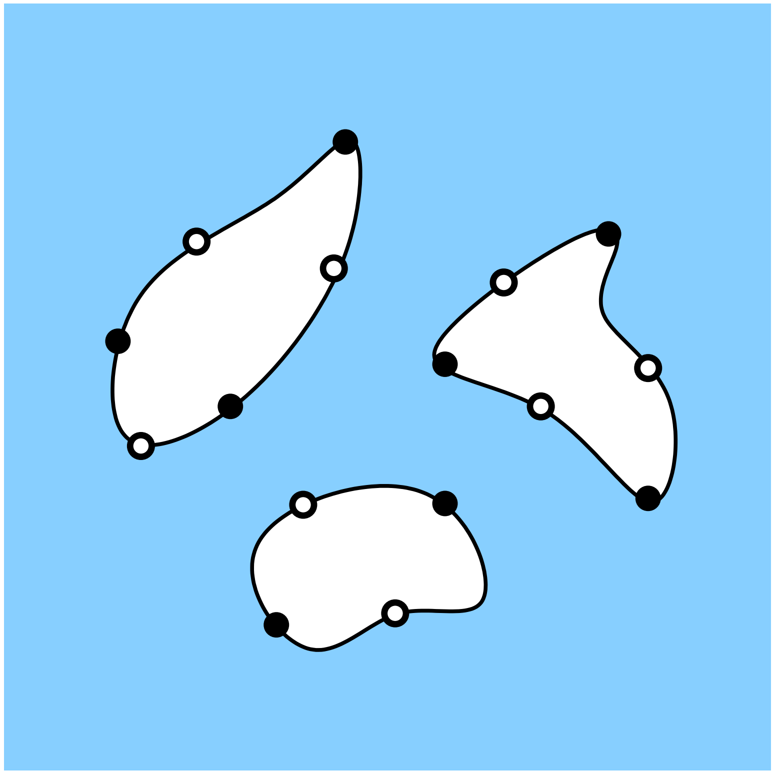
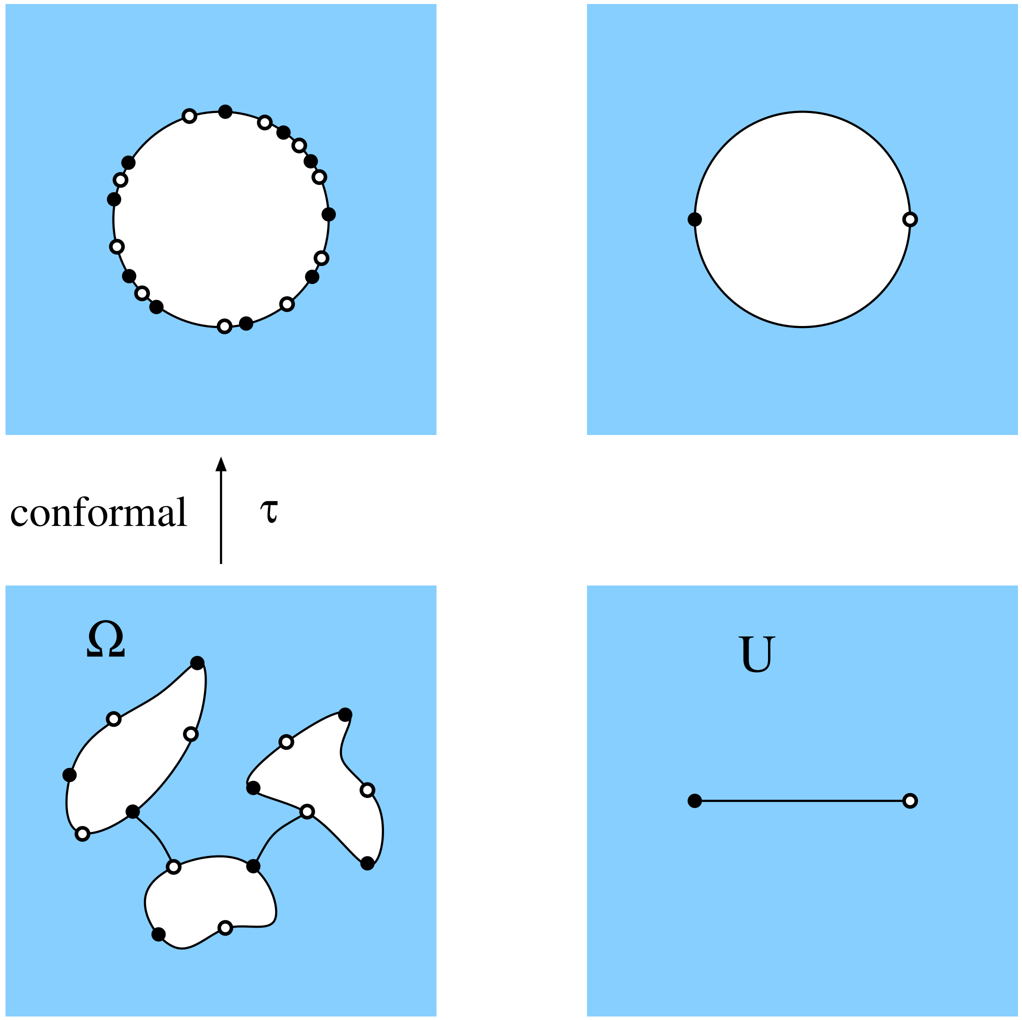


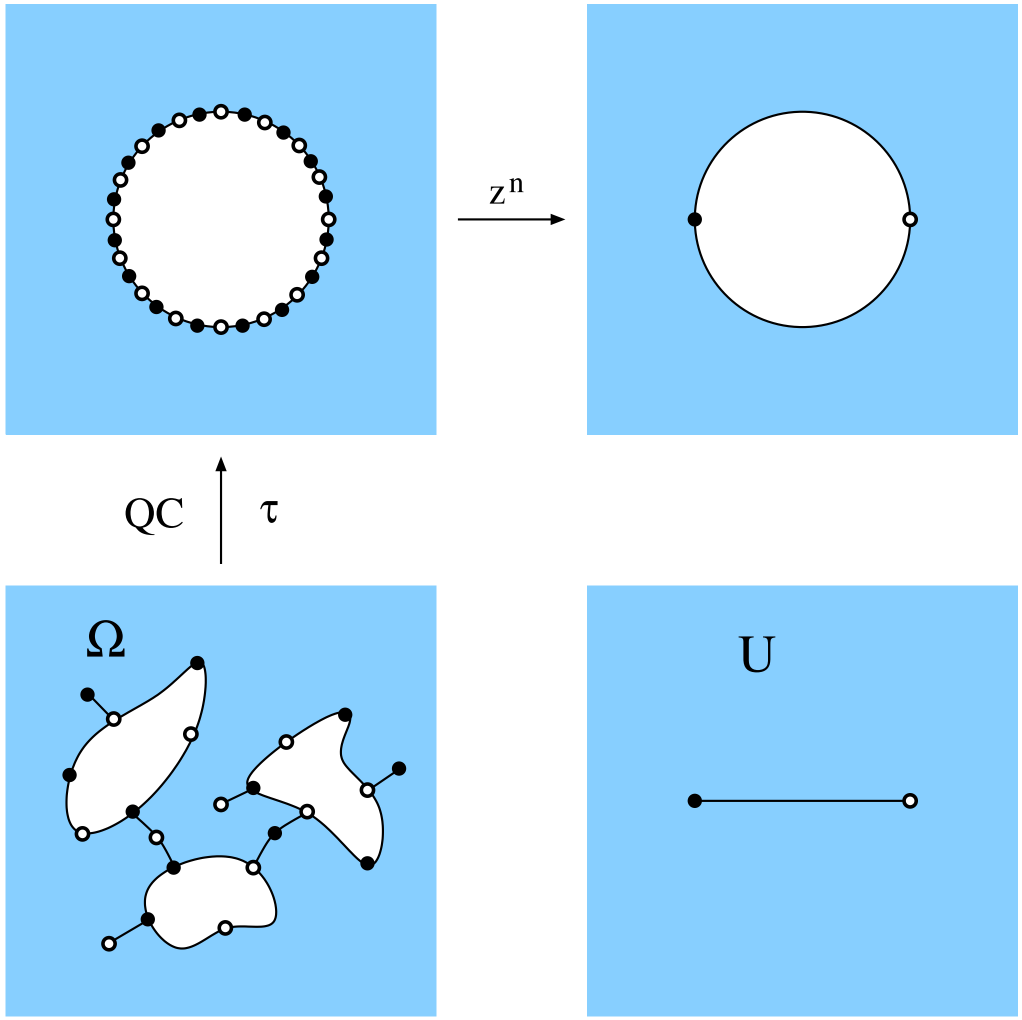
Inverse images of ± 1 under Blaschke products partition ∂W.

Proof of Runge's theorem



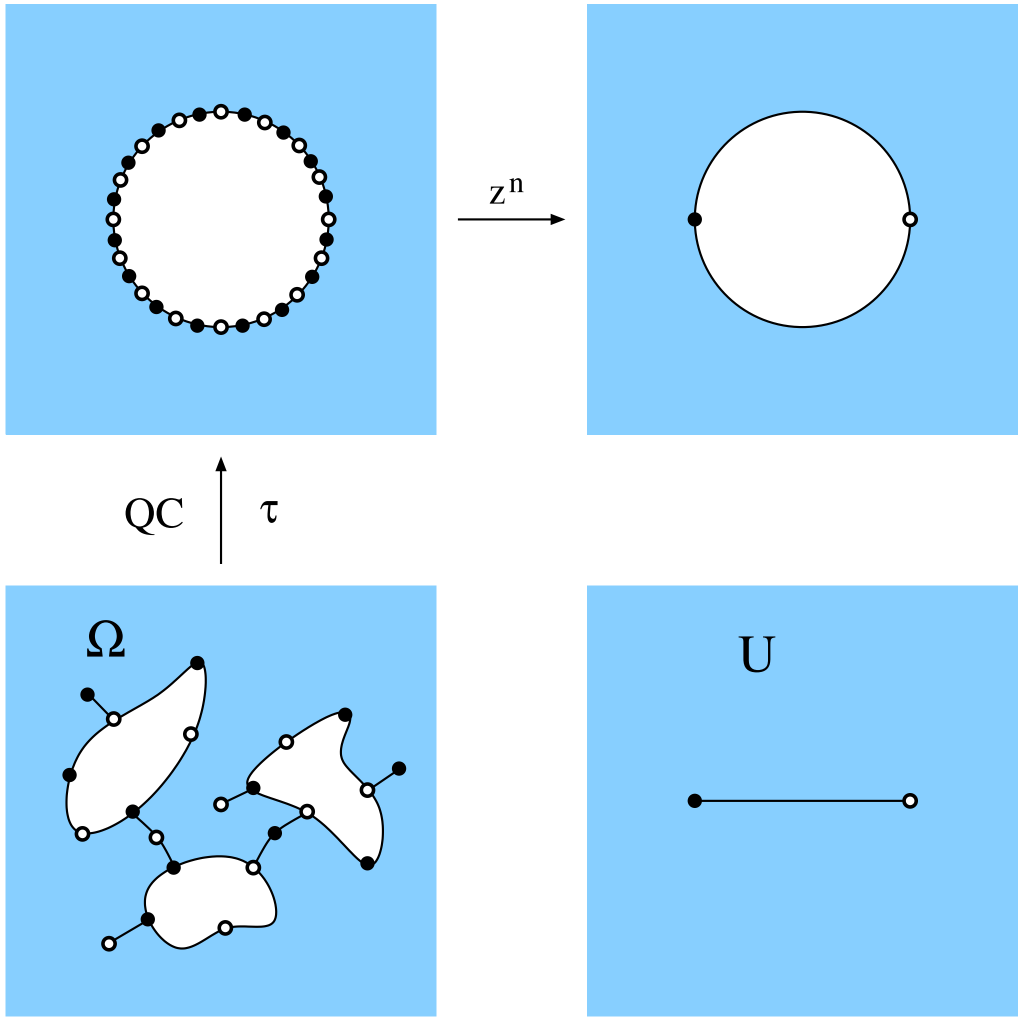
Join components by arcs (tree with blobs).
Harmonic measures are probably unbalanced.

Proof of Runge's theorem



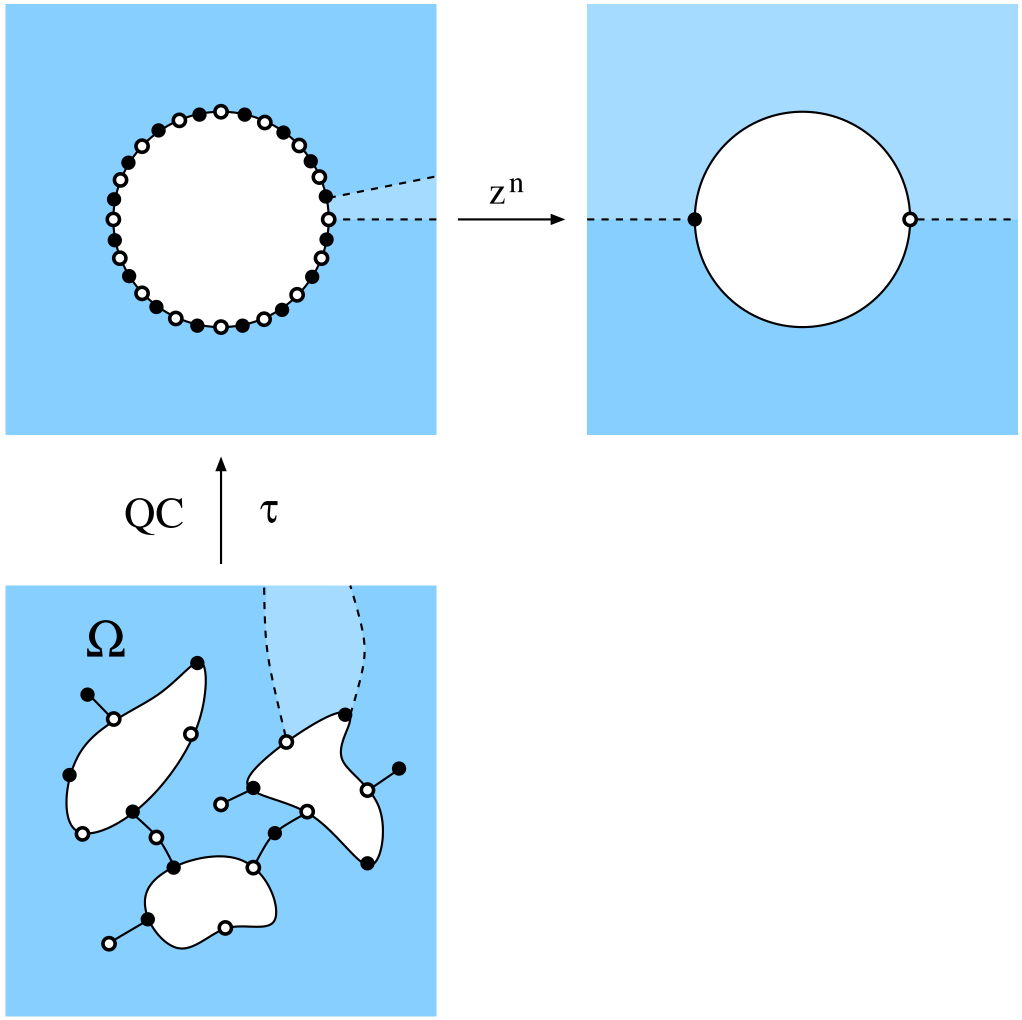
Add decorations to make edges almost balanced.
Get QC map to $\{|z| > 1\}$ with even spacing.

Proof of Runge's theorem



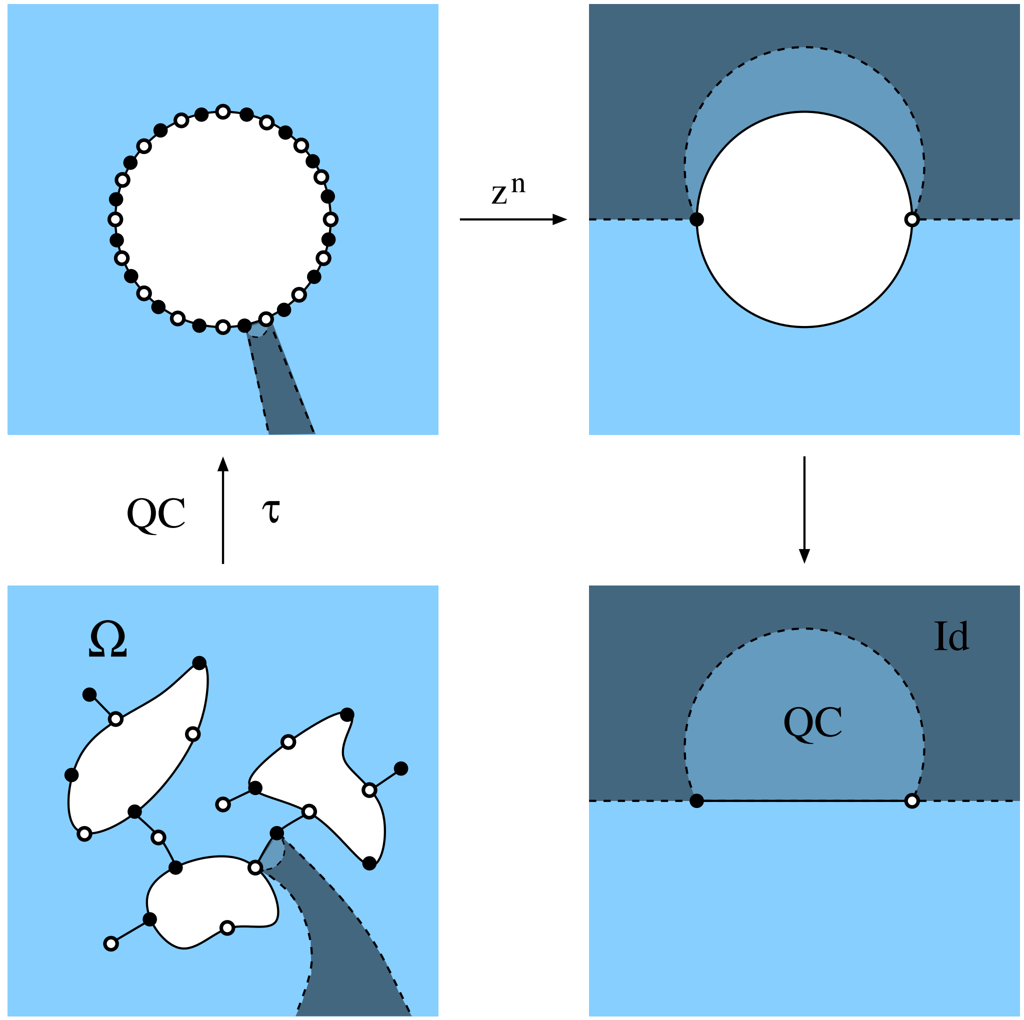
Two types of edges: 1-sided and 2-sided.
Each type is treated differently.

Proof of Runge's theorem



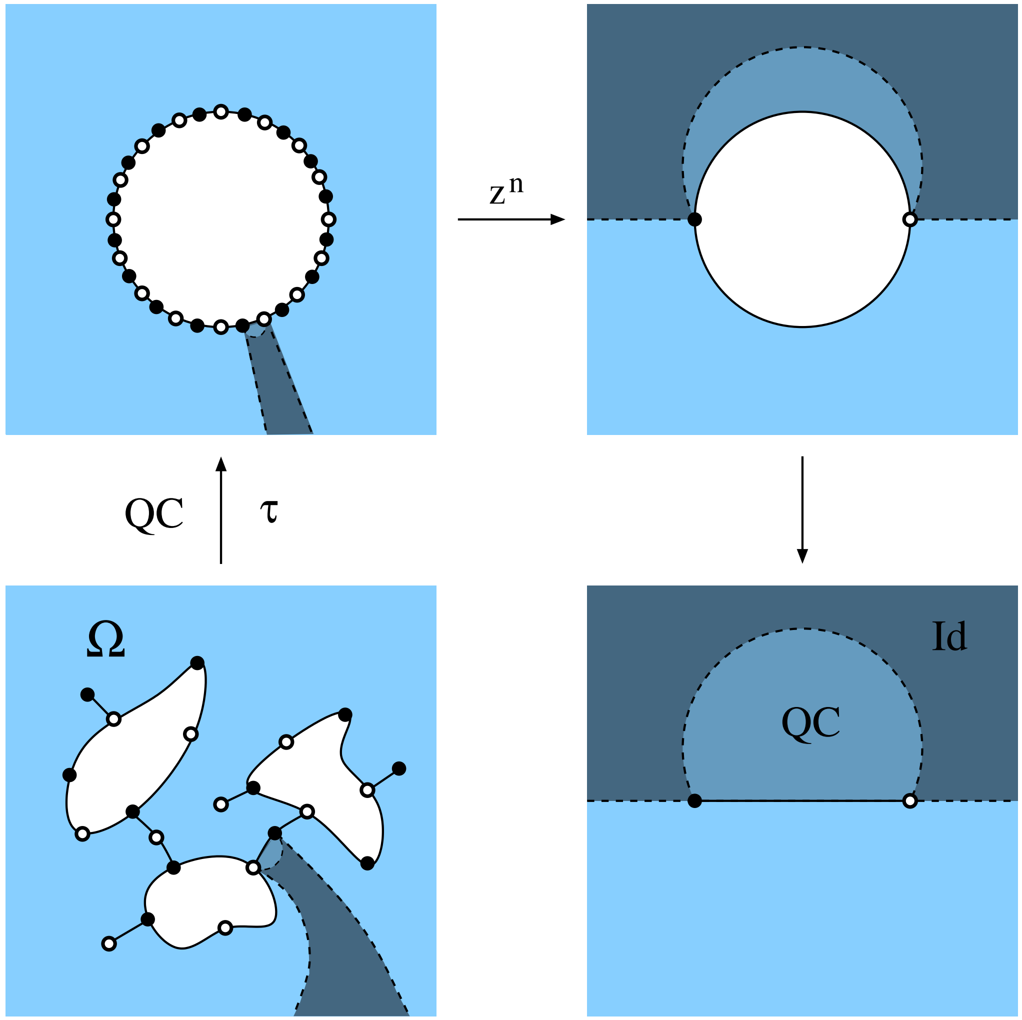
1-sided edges are mapped to upper/lower half-circle.
These exterior edges are glued to Blaschke products.

Proof of Runge's theorem



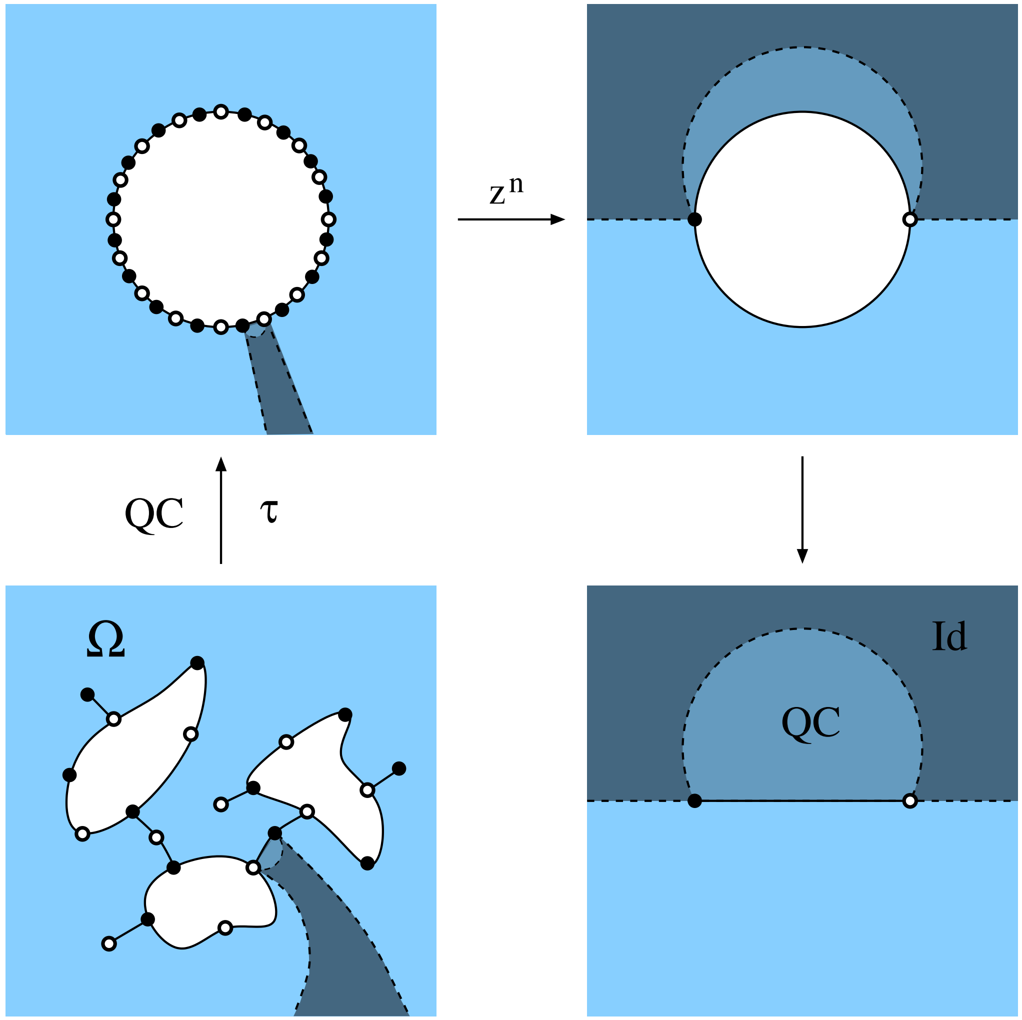
2-sided edges are mapped to $[0, 1]$.
Along these edges, Ω is glued to itself.

Proof of Runge's theorem



Gives n-to-1 quasiregular approximation to f on K.
Solving Beltrami gives polynomial approximation to f on K.

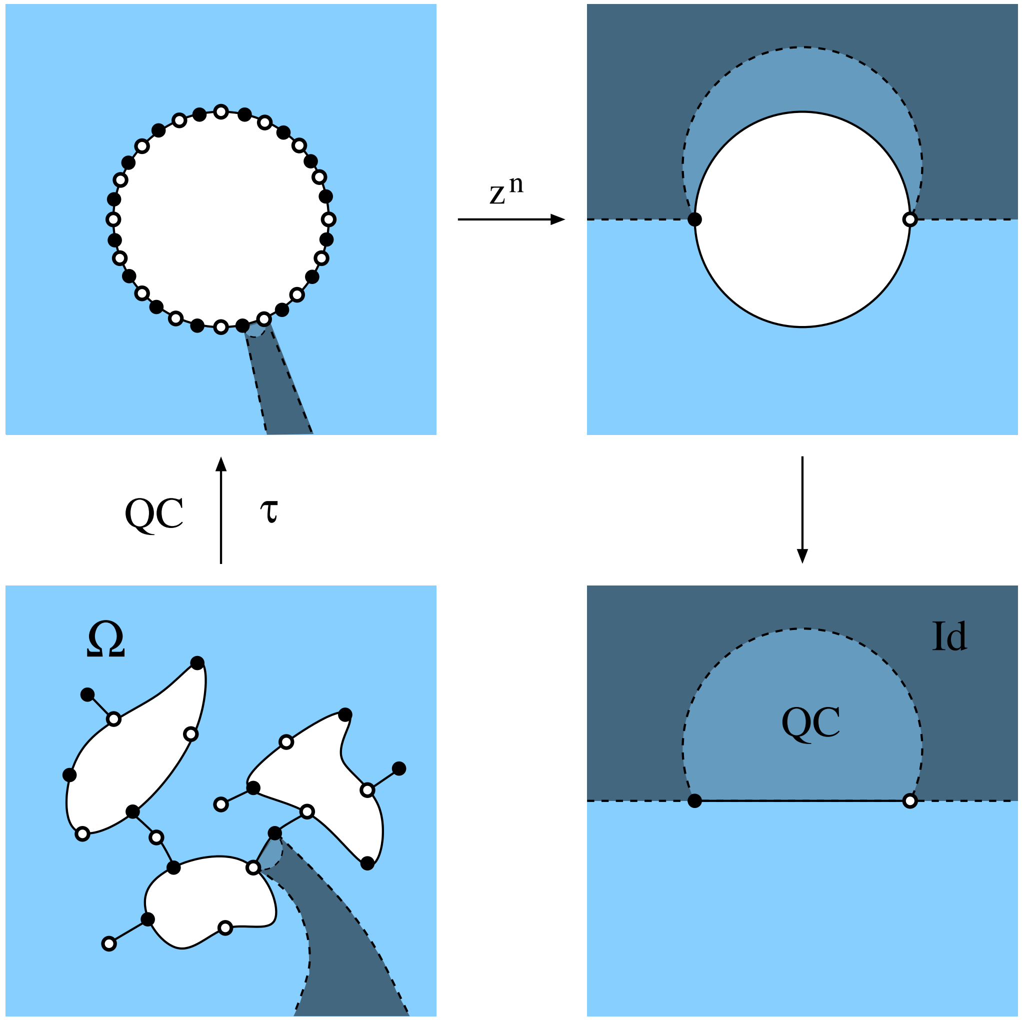
Proof of Runge's theorem



Critical values of Blashcke products are all < 1.

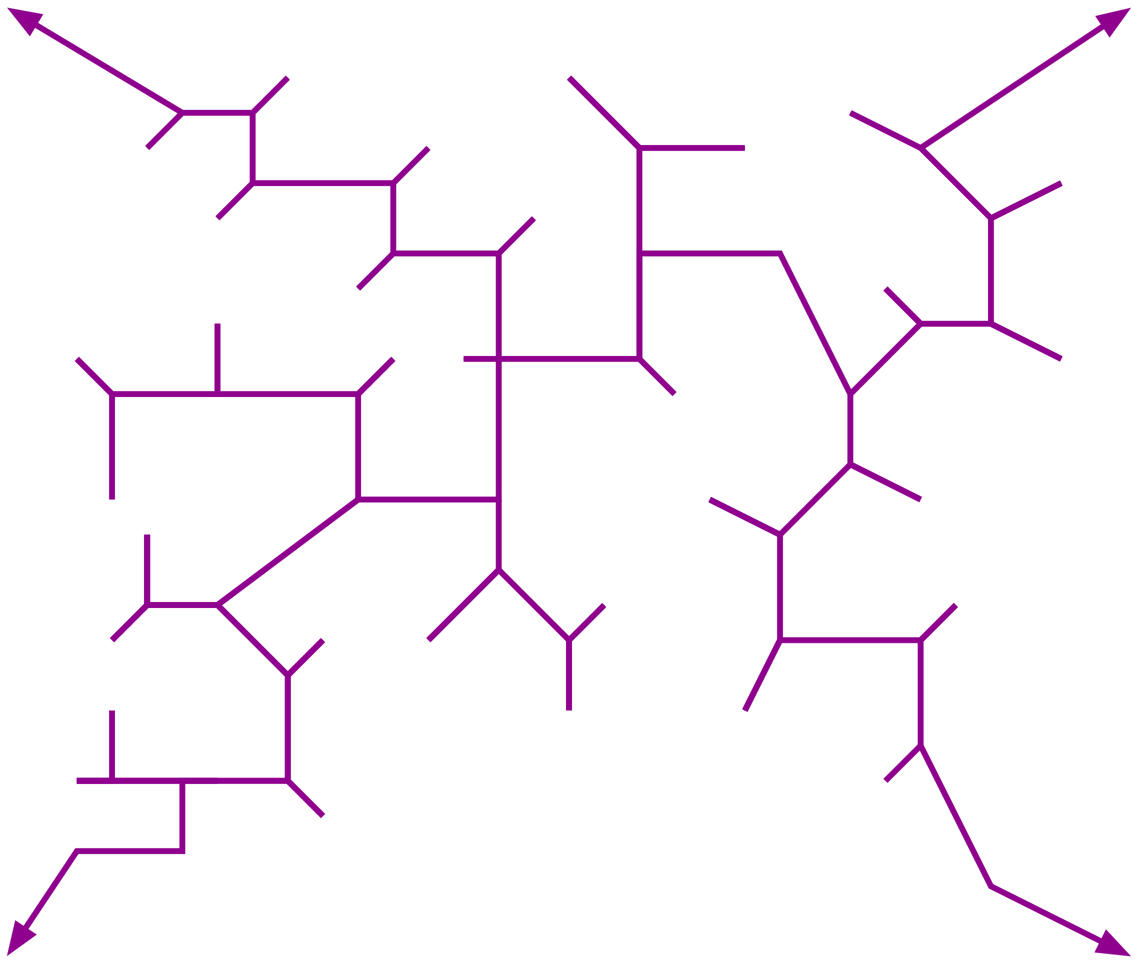
Critical values from folding are all ± 1.

Proof of Runge's theorem



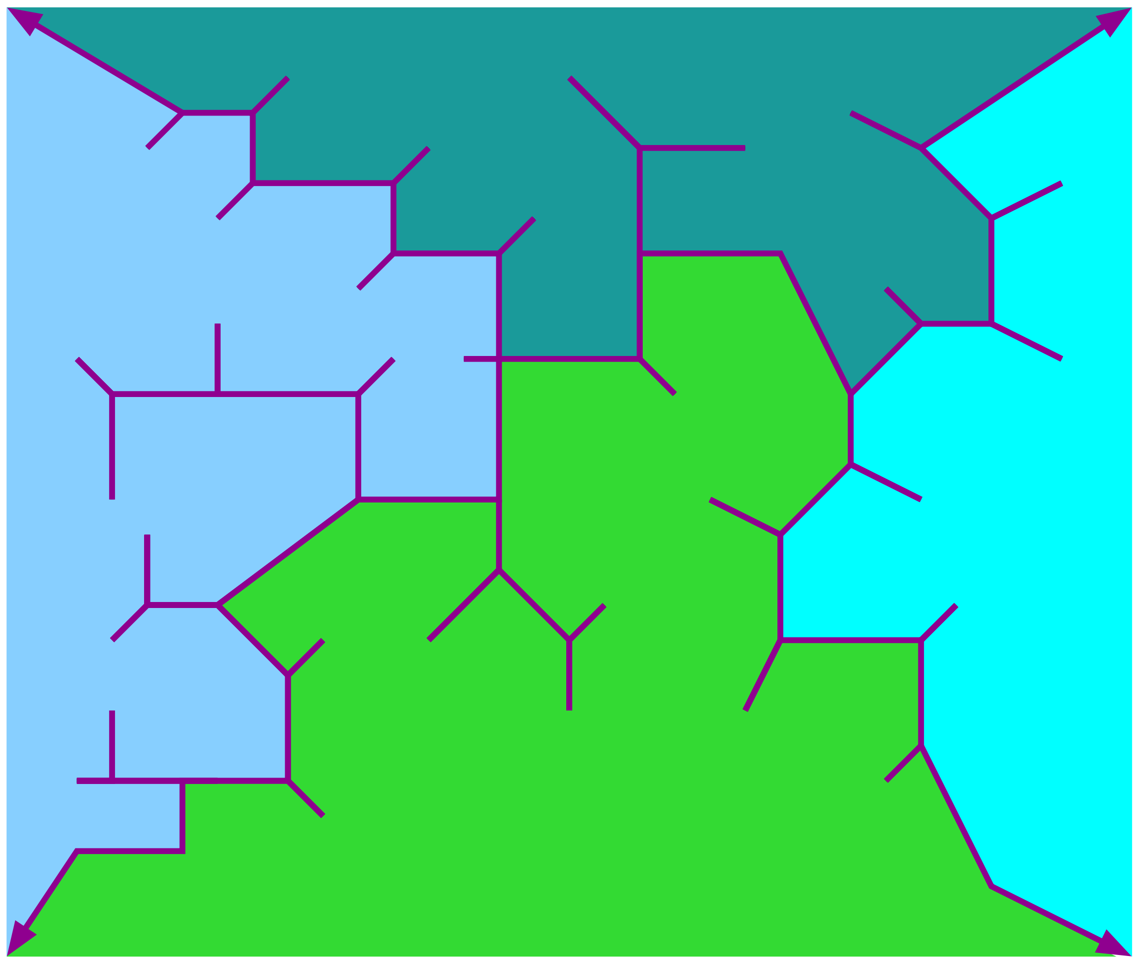
Further modifications can give rational approximations.
Estimate degree using harmonic measure of smallest edge.

Infinite trees and entire functions



Finite planar trees $\Leftrightarrow$ polynomials with 2 critical values.

What about infinite planar trees? Shabat entire functions?

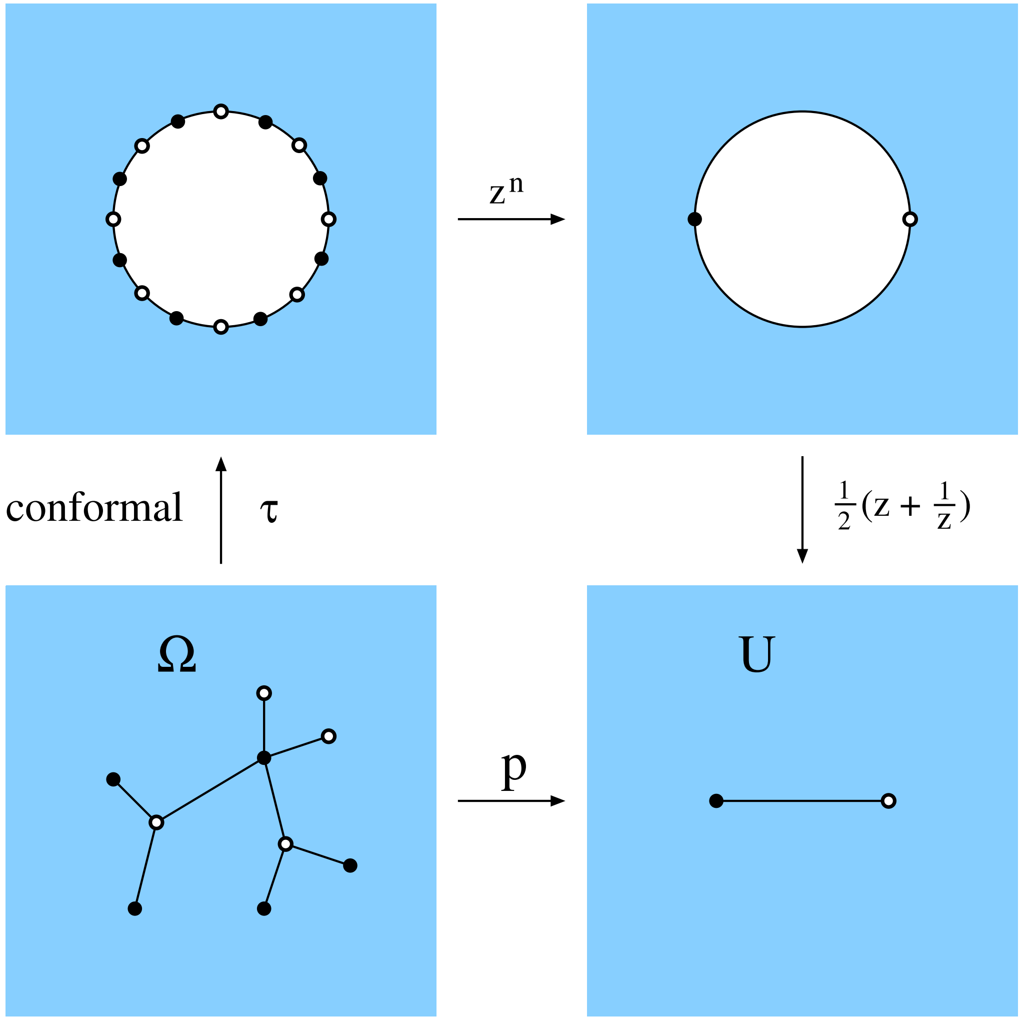


Main difference:

$\mathbb{C} \setminus$ finite tree = one topological annulus

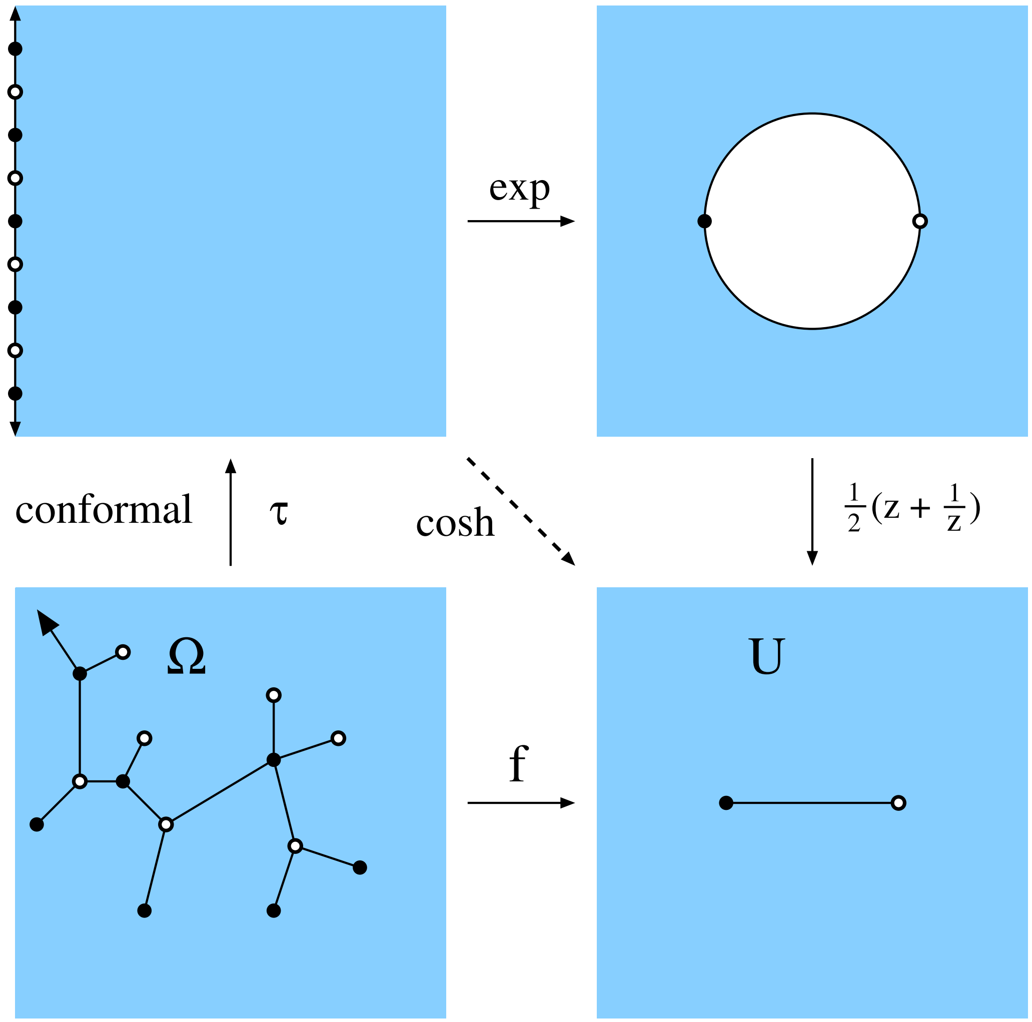
$\mathbb{C} \setminus$ infinite tree = many simply connected components

Recall finite case



T is true tree $\Leftrightarrow p = \frac{1}{2}(\tau^n + 1/\tau^n)$ is continuous across T.

Infinite case



Infinite true tree $\Leftrightarrow$ $f = \cosh \circ \tau$ is continuous across T.

Not every infinite planar tree has a true form (e.g., infinite 3-regular).

However,

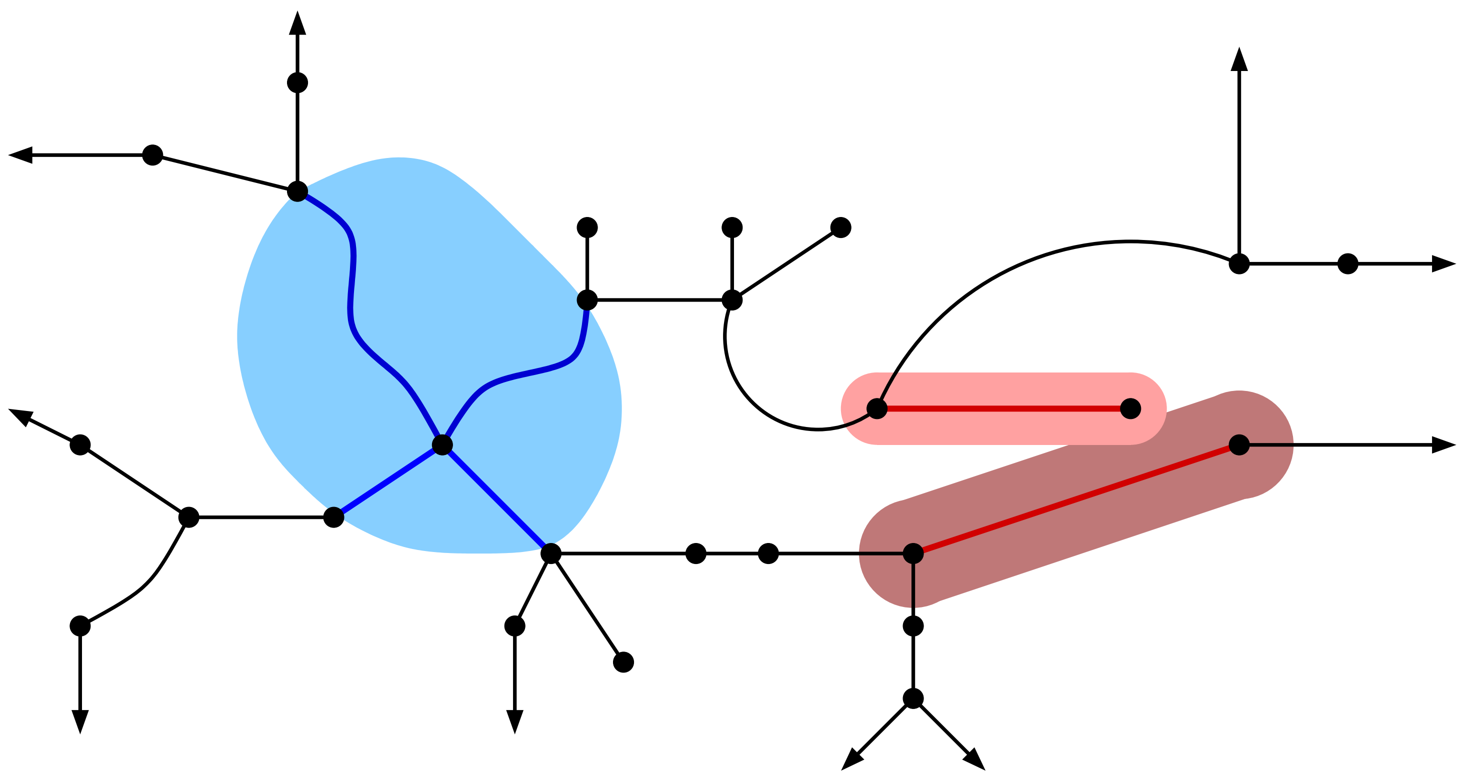
Thm: every "nice" infinite planar tree T is approximated by $f^{-1}([-1,1])$ for some entire function f with $\mathrm{CV} = \{\pm 1\}$.

"nice" $=$ two conditions that are automatic for finite trees.

(1) Bounded Geometry (local condition; easy to verify):

- edges are uniformly smooth.
- adjacent edges form bi-Lipschitz image of a star $= \{z^n \in [0, r]\}$
- non-adjacent edges are well separated,

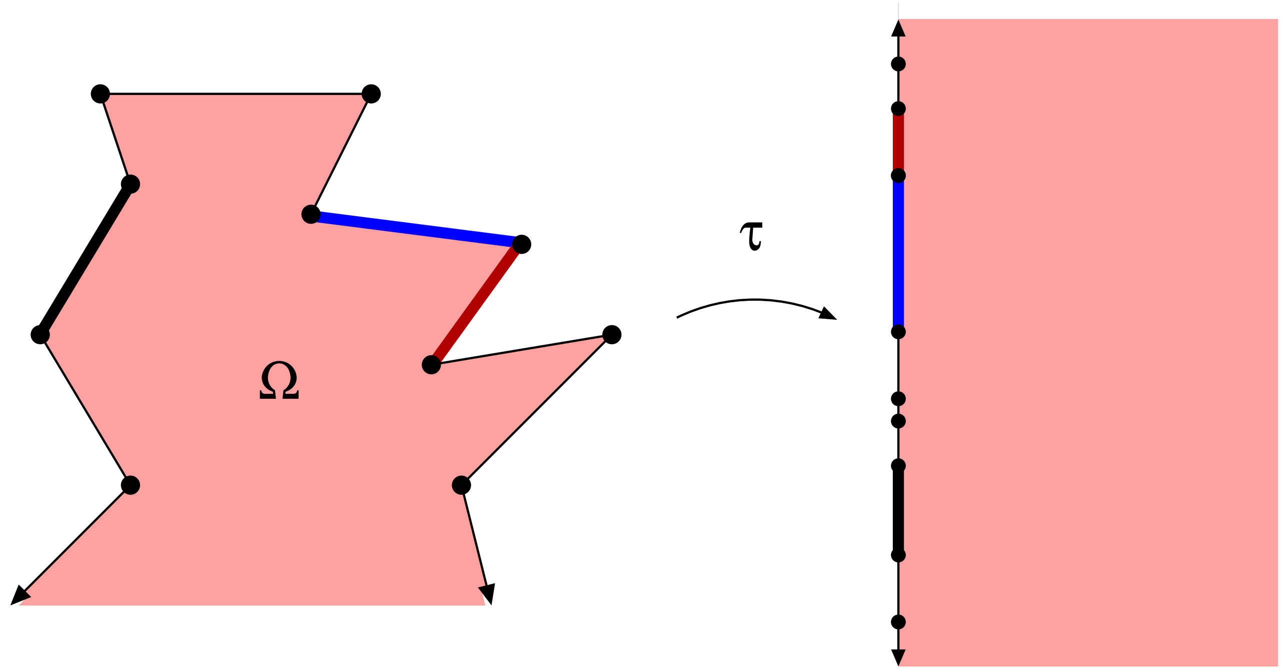
$$\mathrm{dist}(e, f) \geq \epsilon \cdot \min(\mathrm{diam}(e), \mathrm{diam}(f)).$$



(2) τ-Lower Bound (global condition; harder to check):

Complementary components of tree are simply connected.

Each can be conformally mapped to right half-plane. Call map τ.

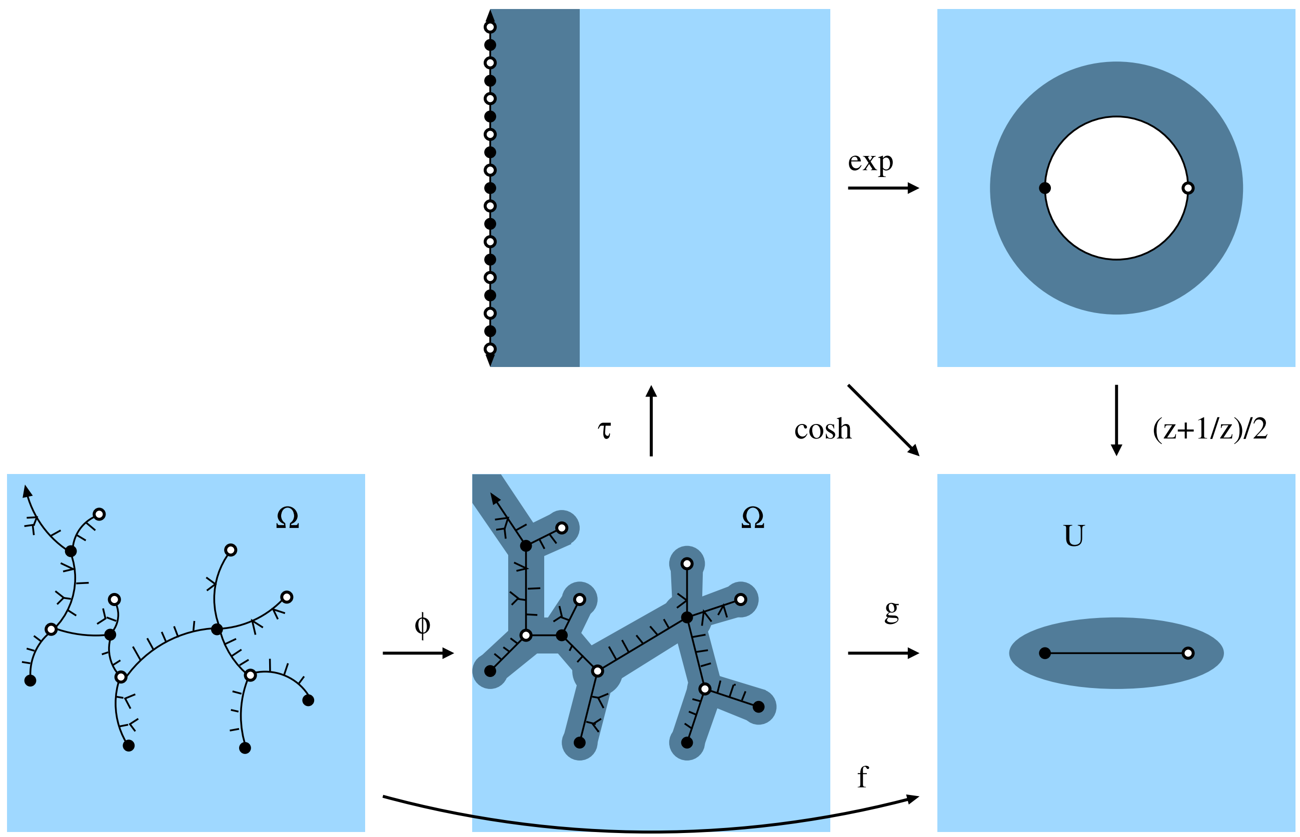


We assume all images have length $\geq \pi$.

Need positive lower bound; actual value usually not important.

Components are "thinner" than half-plane near ∞.

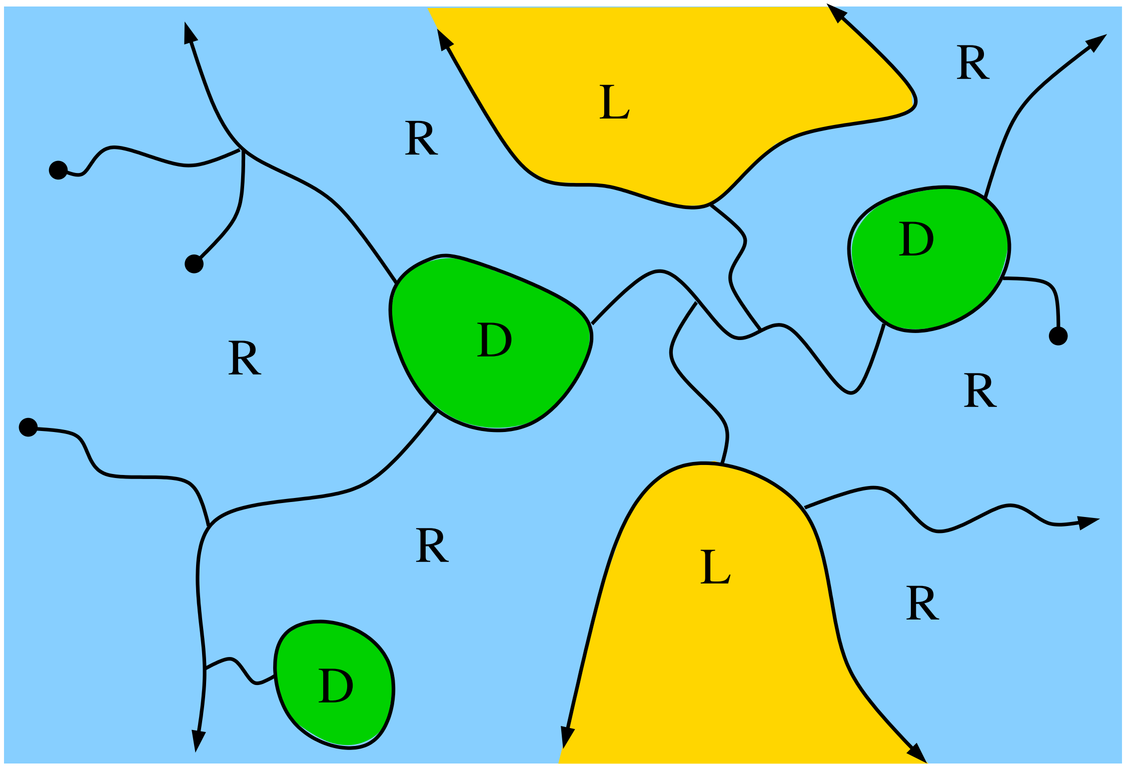
Thm: If T has bounded geometry and satisfies a τ-lower bound, then there is a QR g with $g = \cosh \circ \tau$ off nbhd of T and $\mathrm{CV}(g) = \pm 1$.



Add decorations, convert QR g to holomorphic $f = g \circ \phi$.

Often can prove $\phi(z) \approx z$, so $T \approx f^{-1}([-1, 1])$

More generally, replace tree by graph.



R = unbounded domains ($F = e^{\tau(z)}$, previous case)

D = bounded Jordan domains (F = finite Blaschke product)

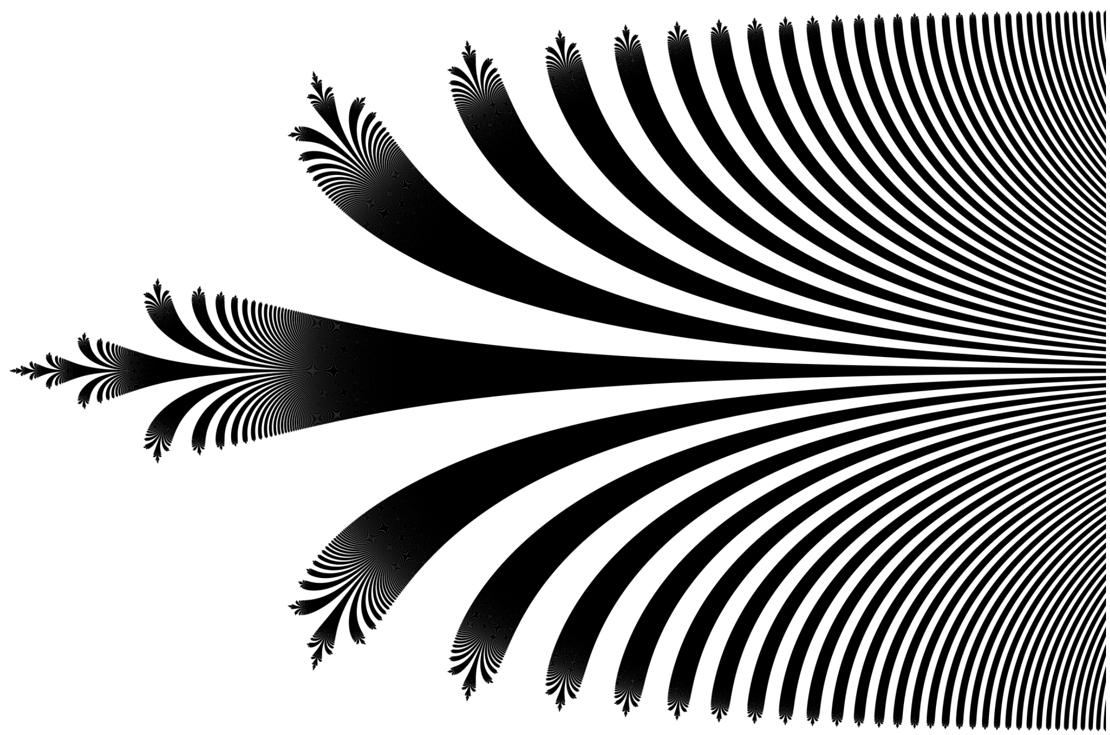
L = unbounded Jordan domains ($F = e^{-\tau(z)}$, finite asymptotic values)

Given an entire function f,

Fatou set $= \mathcal{F}(f) =$ open set where iterates are normal family.

Julia set $= \mathcal{J}(f) =$ complement of Fatou set.

f maps Fatou components into other Fatou components.



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f maps Fatou components into other Fatou components.

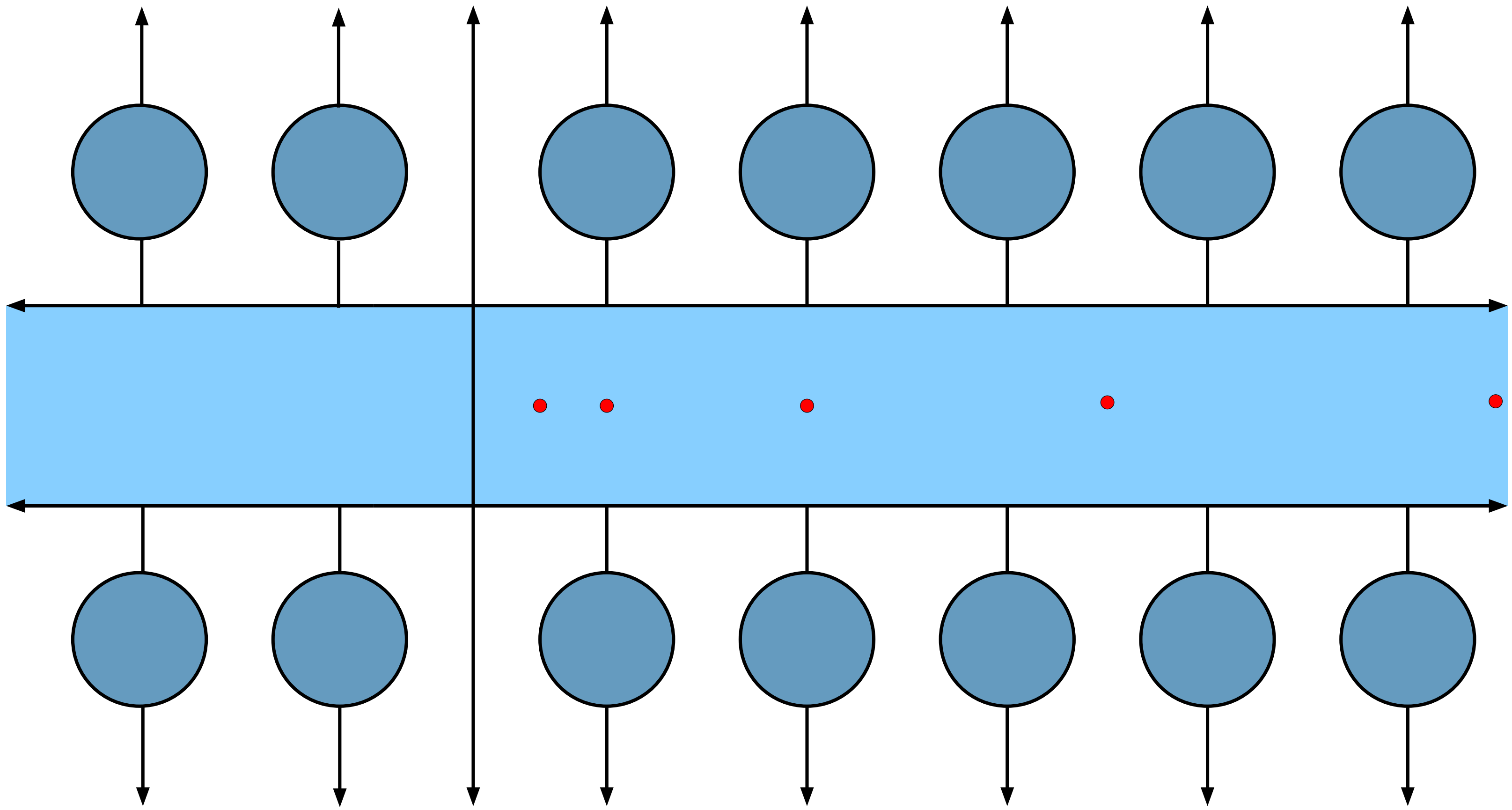
Wandering domain = Fatou component with infinite orbit.

- No wandering domains for rational functions (Sullivan 1985).

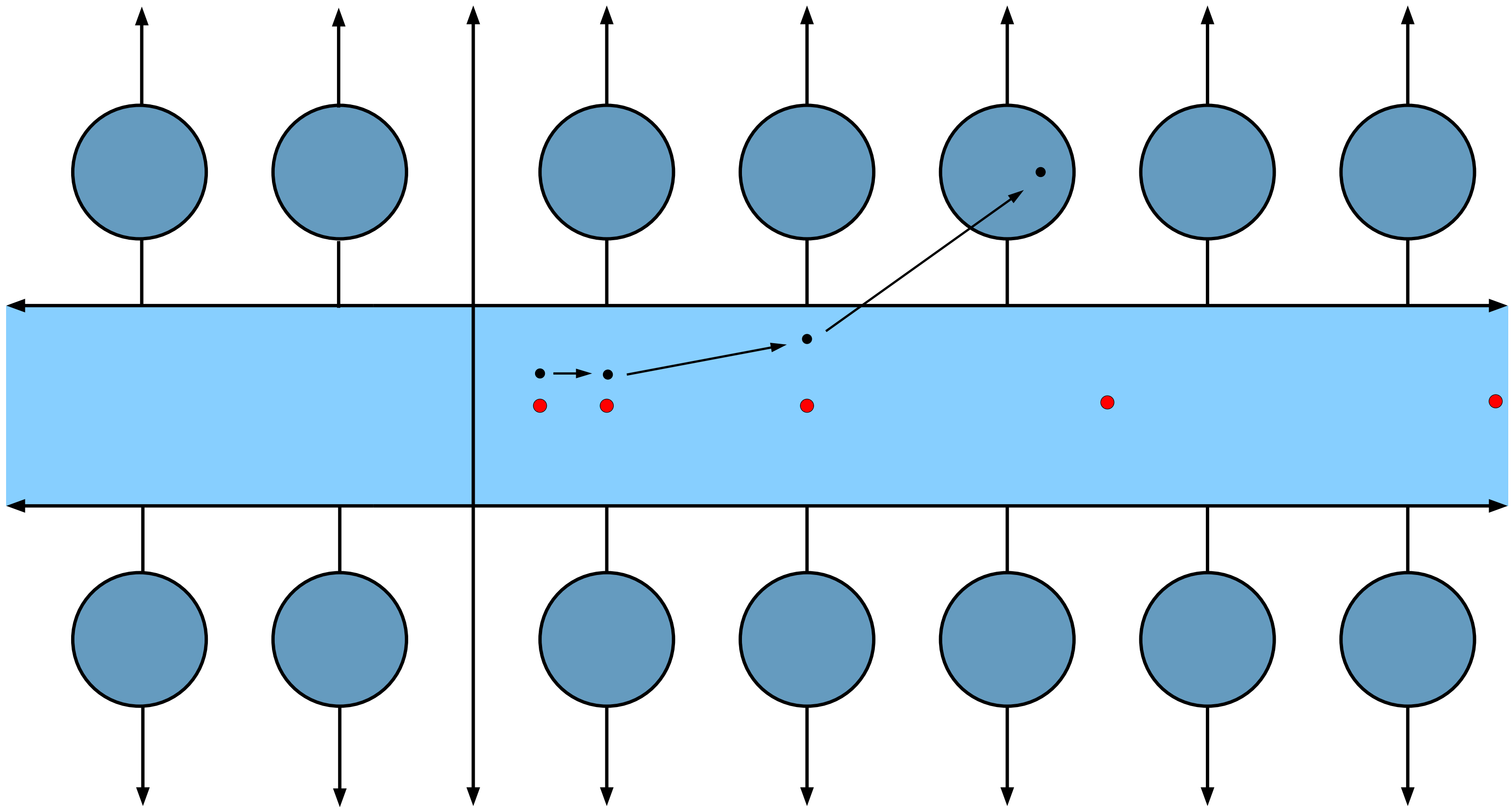
- Entire functions can have wandering domains (Baker 1975).

- None if finite singular set (Eremenko-Lyubich, Goldberg-Keen).

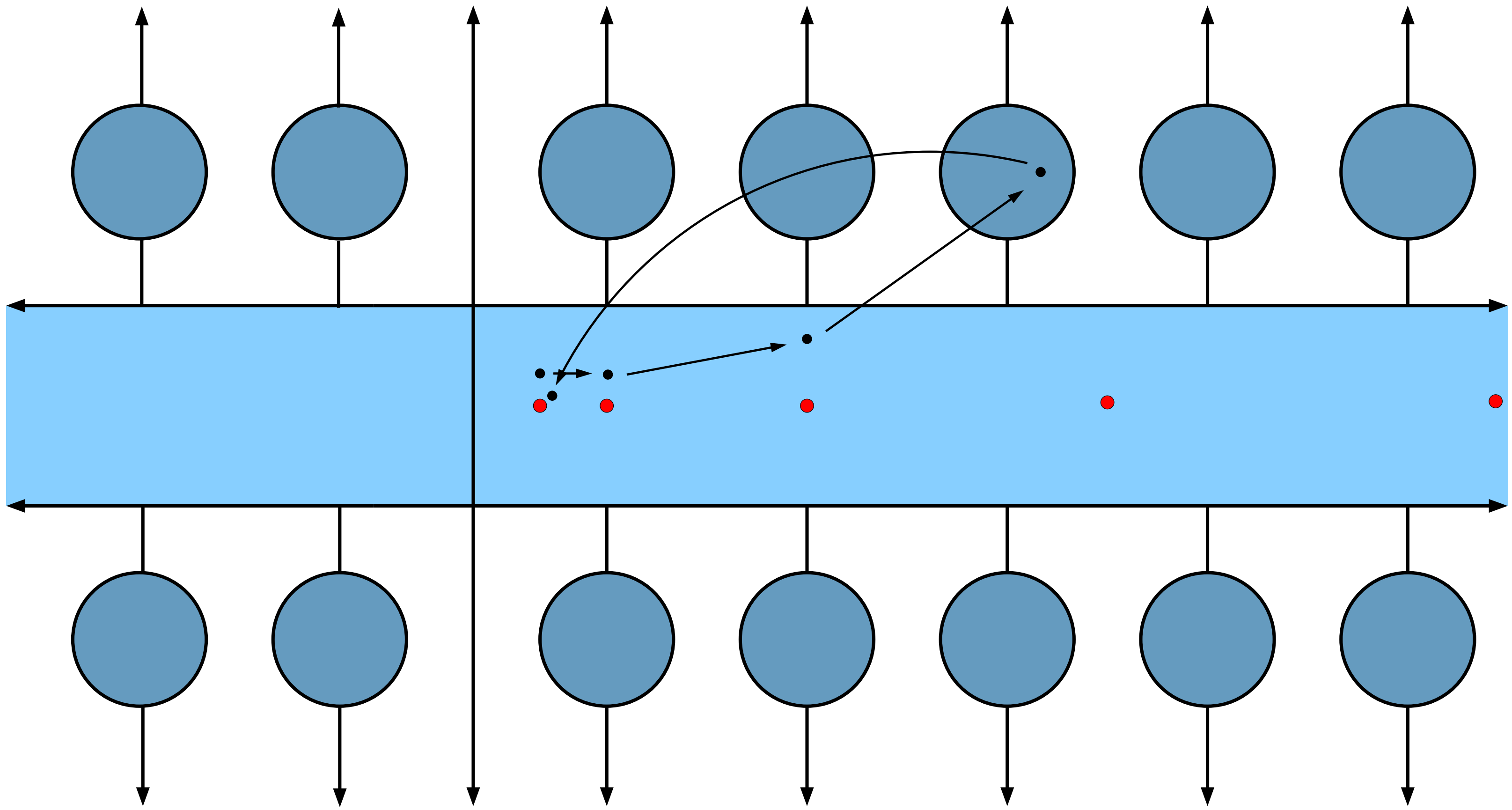
Bounded singular set? Open since 1985.



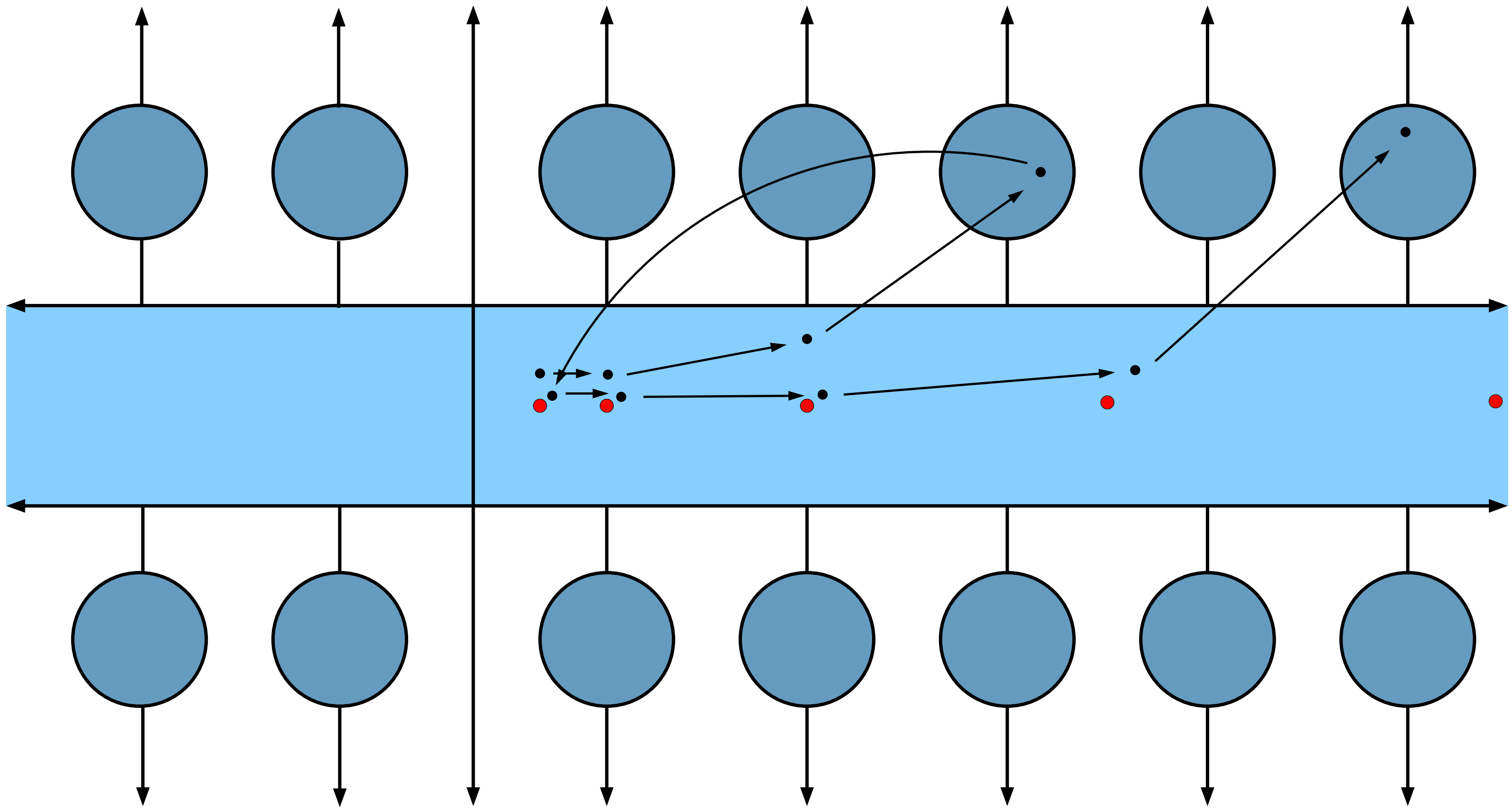
Graph giving EL wandering domain via folding (B 2015).
Symmetry $\Rightarrow x = \frac{1}{2}$ iterates to $+\infty$ on $\mathbb{R}$.



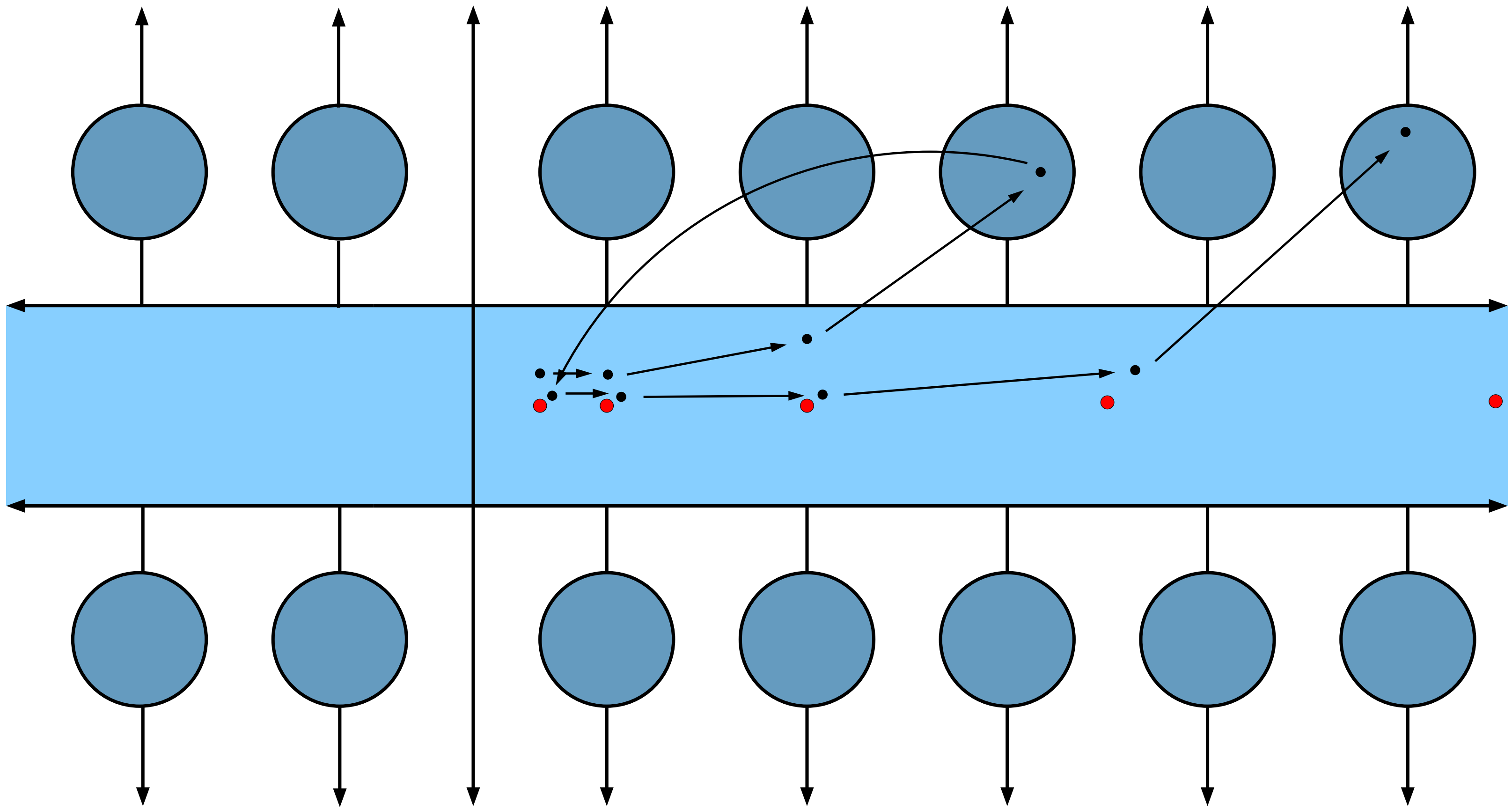
Iterates of $\frac{1}{2} + i\epsilon$ follow orbit of $\frac{1}{2}$ for a time, eventually diverge.



Orbit lands in D-component.
High degree critical point compresses.
Orbit as close to $\frac{1}{2}$ as desired.



Orbit follows orbit of $\frac{1}{2}$-orbit longer than before.
Eventually lands in a D-component and returns near $\frac{1}{2}$.



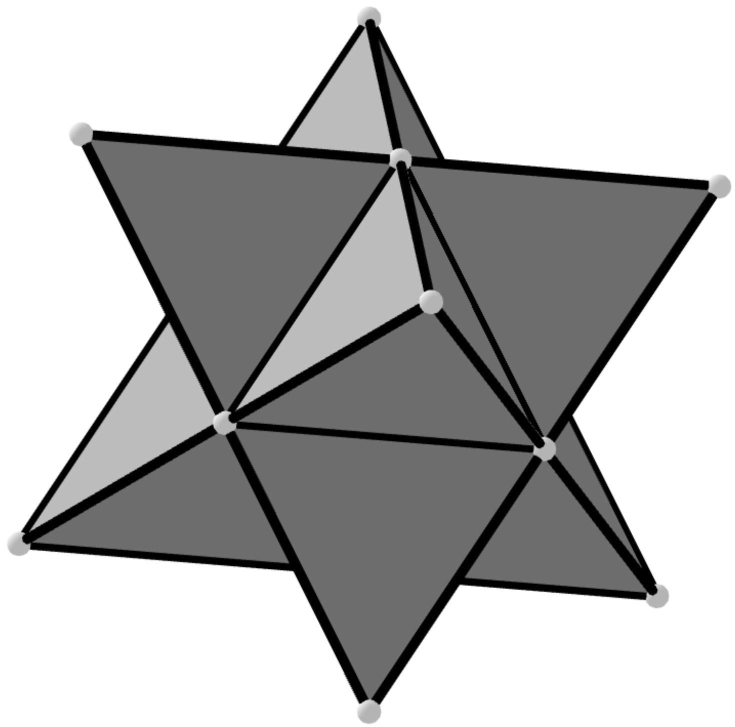
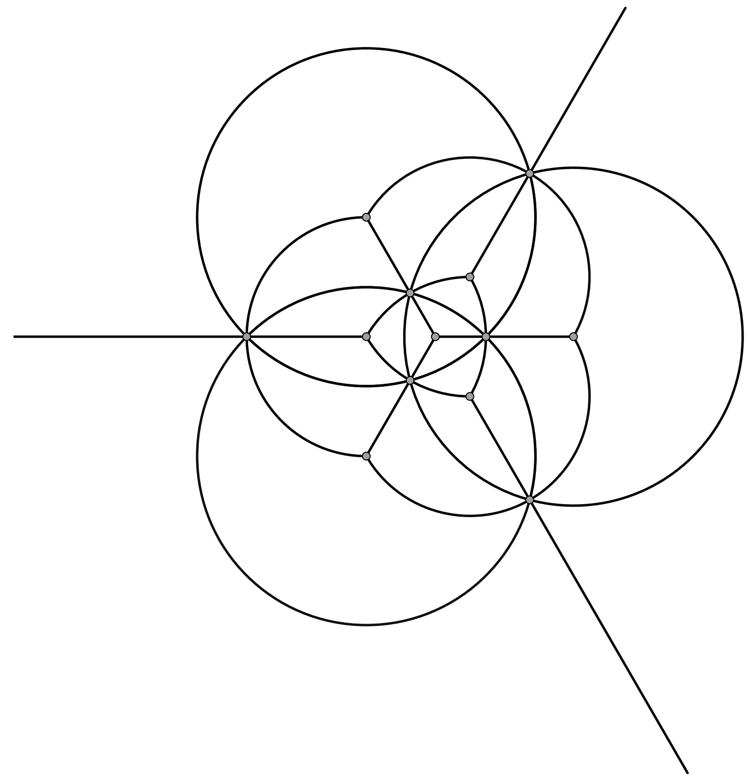
Compression $\Rightarrow$ orbit in Fatou component.

Oscillation between $\frac{1}{2}$ and ∞ $\Rightarrow$ wandering domain.

(Schwarz lemma: iteration decreases hyperbolic metric.)

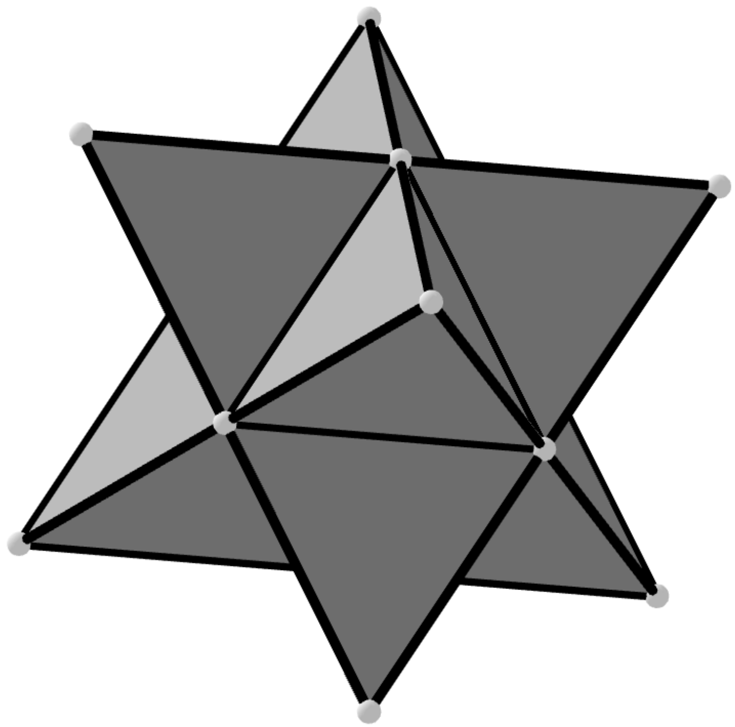
THANKS

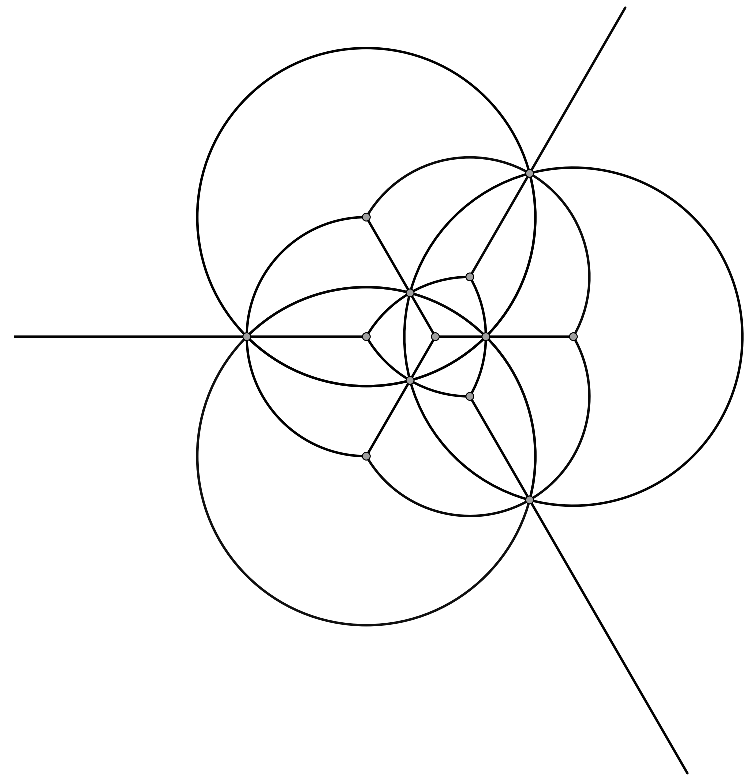
Equilateral triangulations of Riemann surfaces

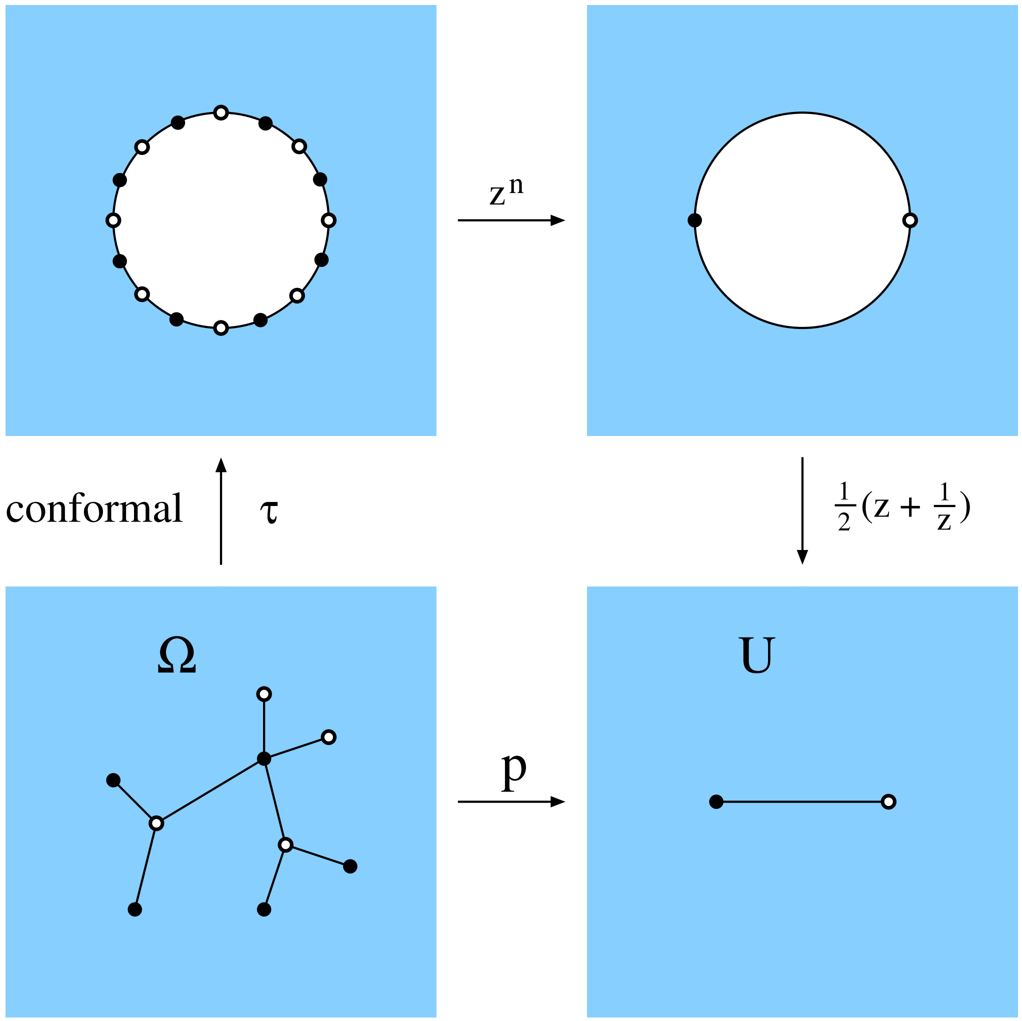
Given a finite or countable collection of equilateral triangles, identify edges pairwise so each vertex is identified with finitely many others.

Resulting surface has a conformal structure = is a Riemann surface.

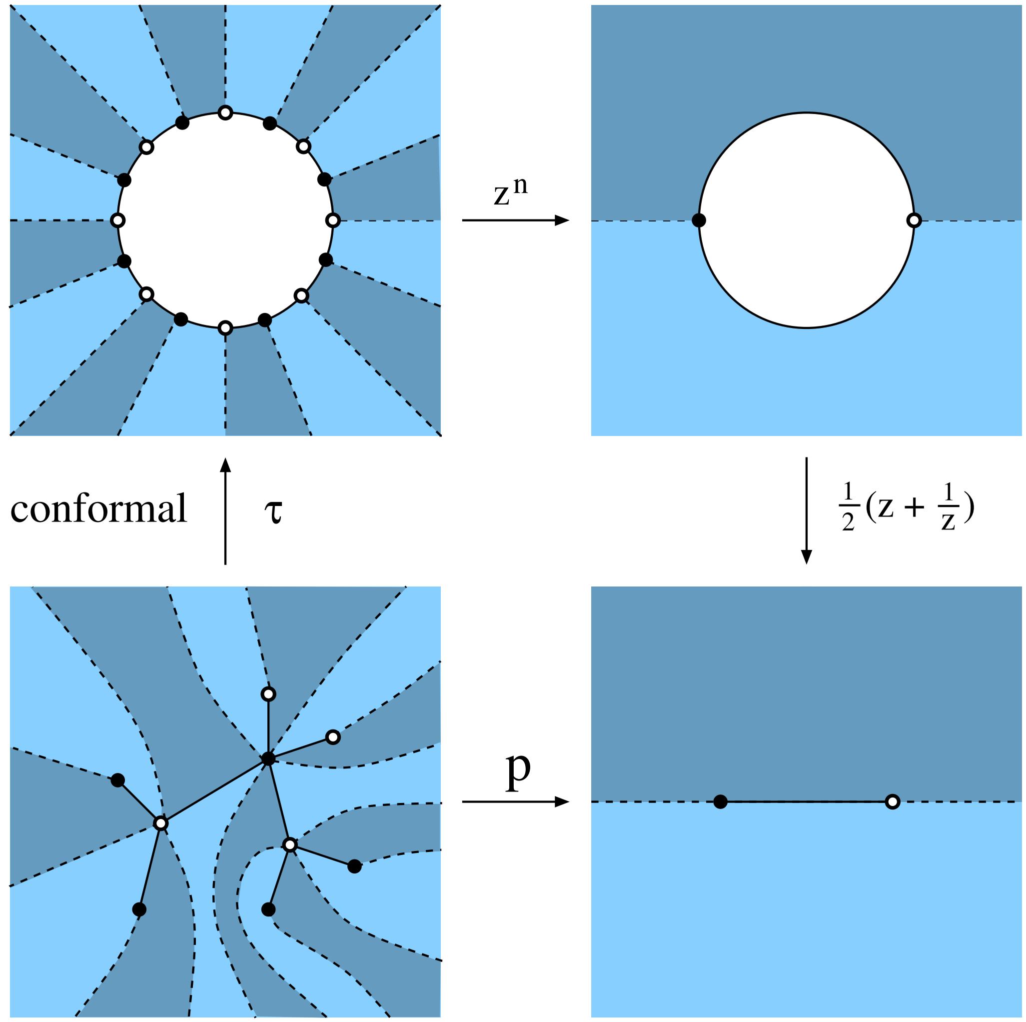




Fact: A triangulation of a surface is equilateral iff any two triangles sharing an edge have an anti-holomorphic reflection across that edge.



Shabat polynomials give equilateral triangulations.



Upper/lower halfplanes give equilateral triangulation of plane.
Inverse image under covering map is also equilateral triangulation.

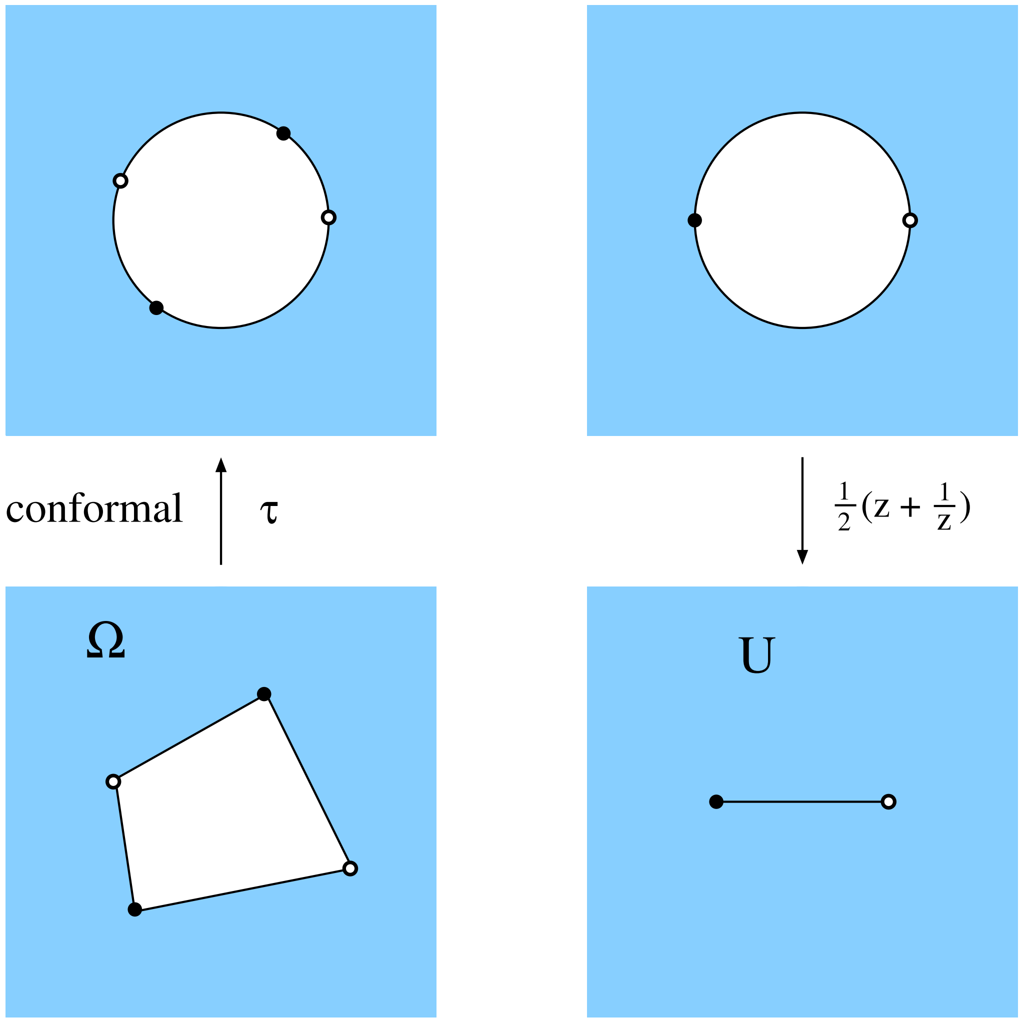
Belyi function on Riemann surface $R =$

holomorphic map from R to S^2 branched over $-1, 1, \infty$.

Shabat polynomial is a Belyi function on sphere.

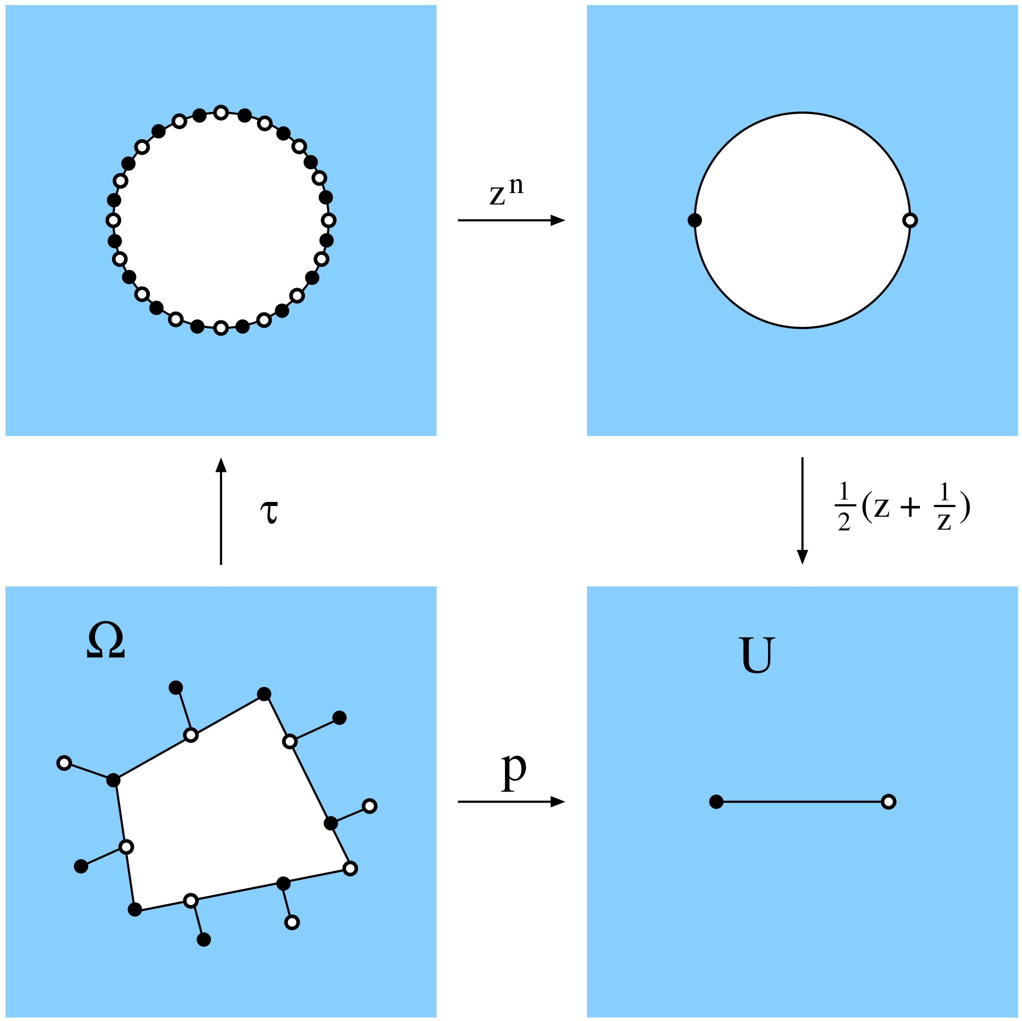
Theorem (Voevodsky-Shabat): A surface has a Belyi function iff it has an equilateral triangulation.

Which Riemann surfaces have a Belyi function?

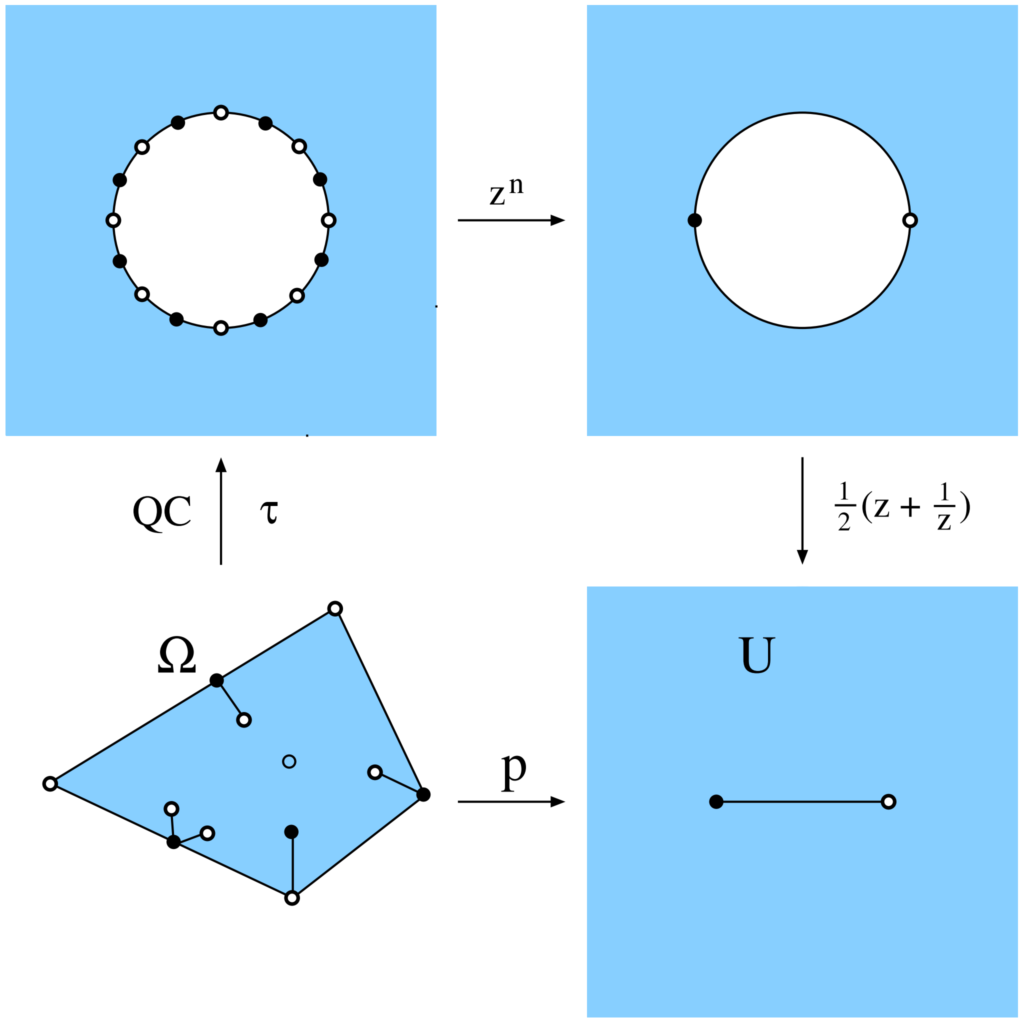


Apply folding idea to a polygon instead of tree.

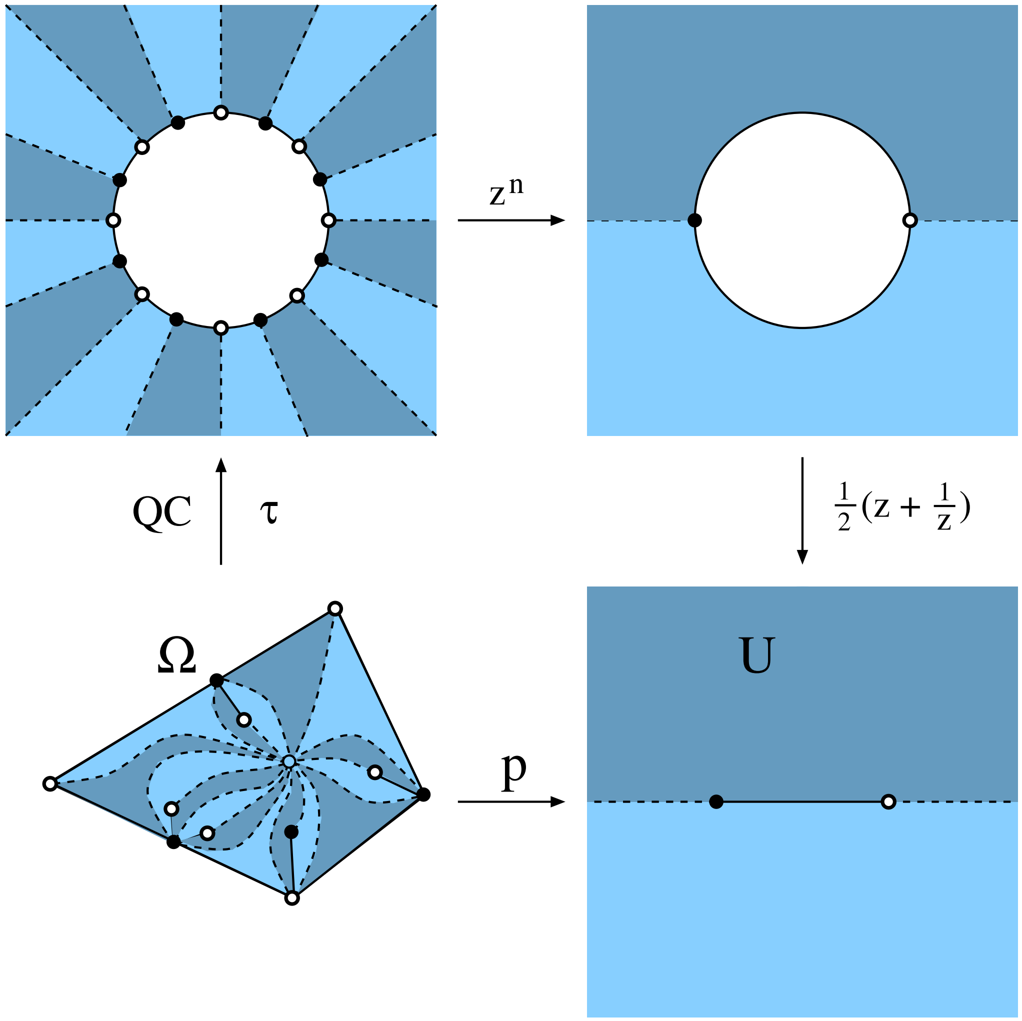
For general polygon, edges have different harmonic measures.



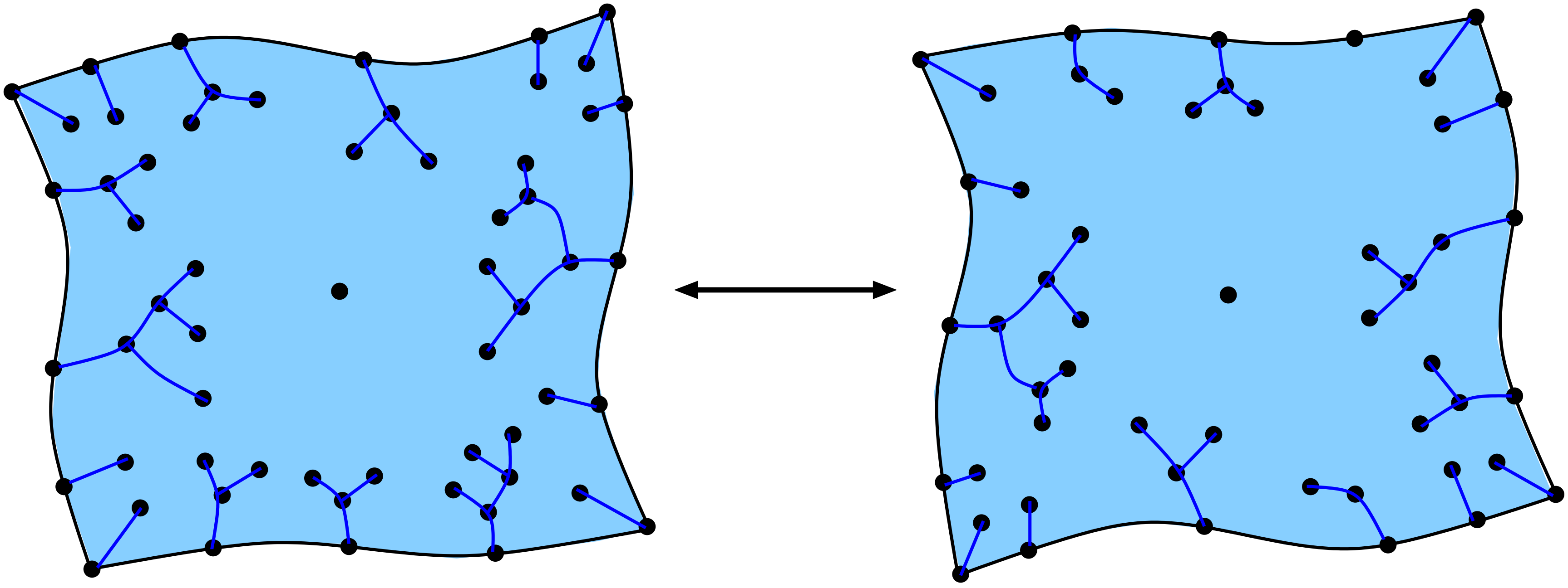
Add decorations to nearly equalize harmonic measures.
Length measure maps to multiple of length measure.



Inverting allows same construction for interior domains.
Now p has a degree n pole.



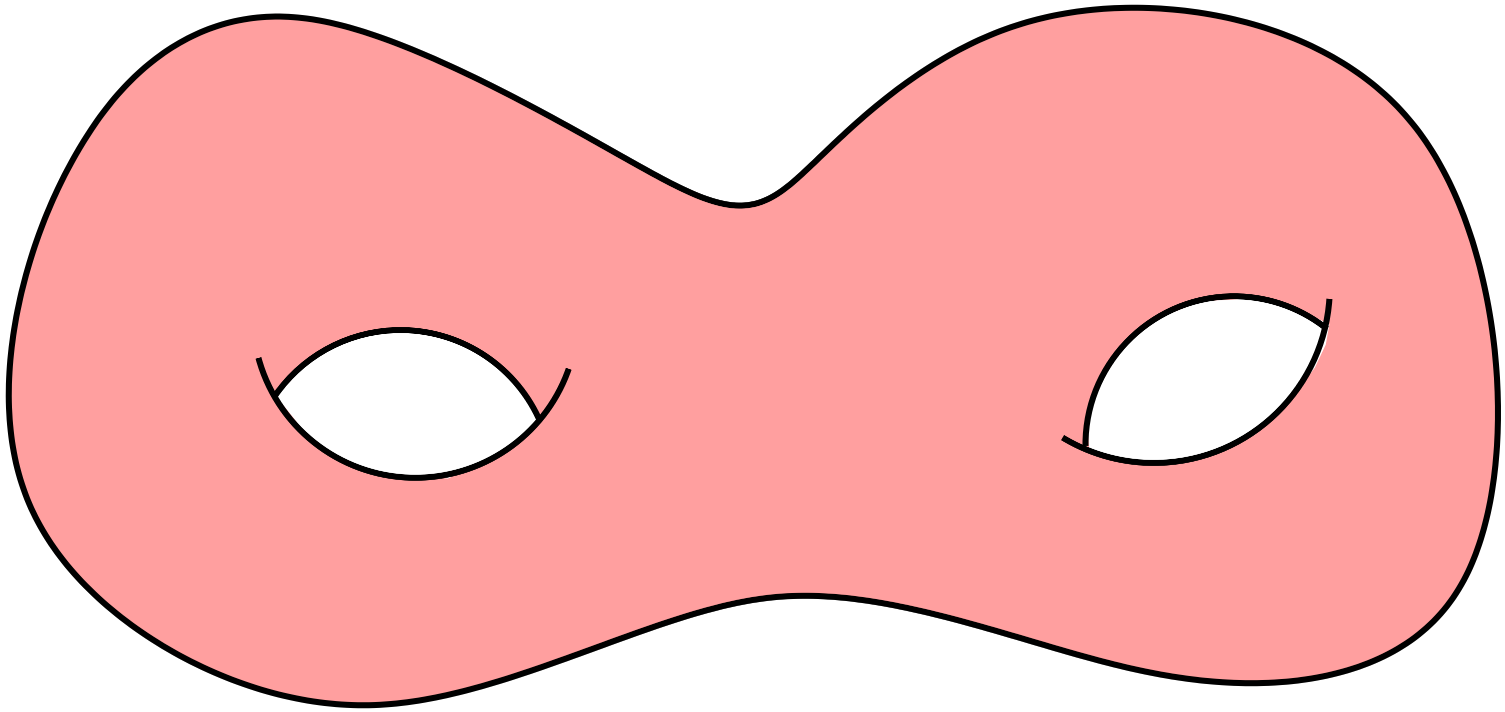
Preimages of half-planes gives equilateral triangulation.
Normalized arclength on boundary edges maps to arclength on $[-1, 1]$.



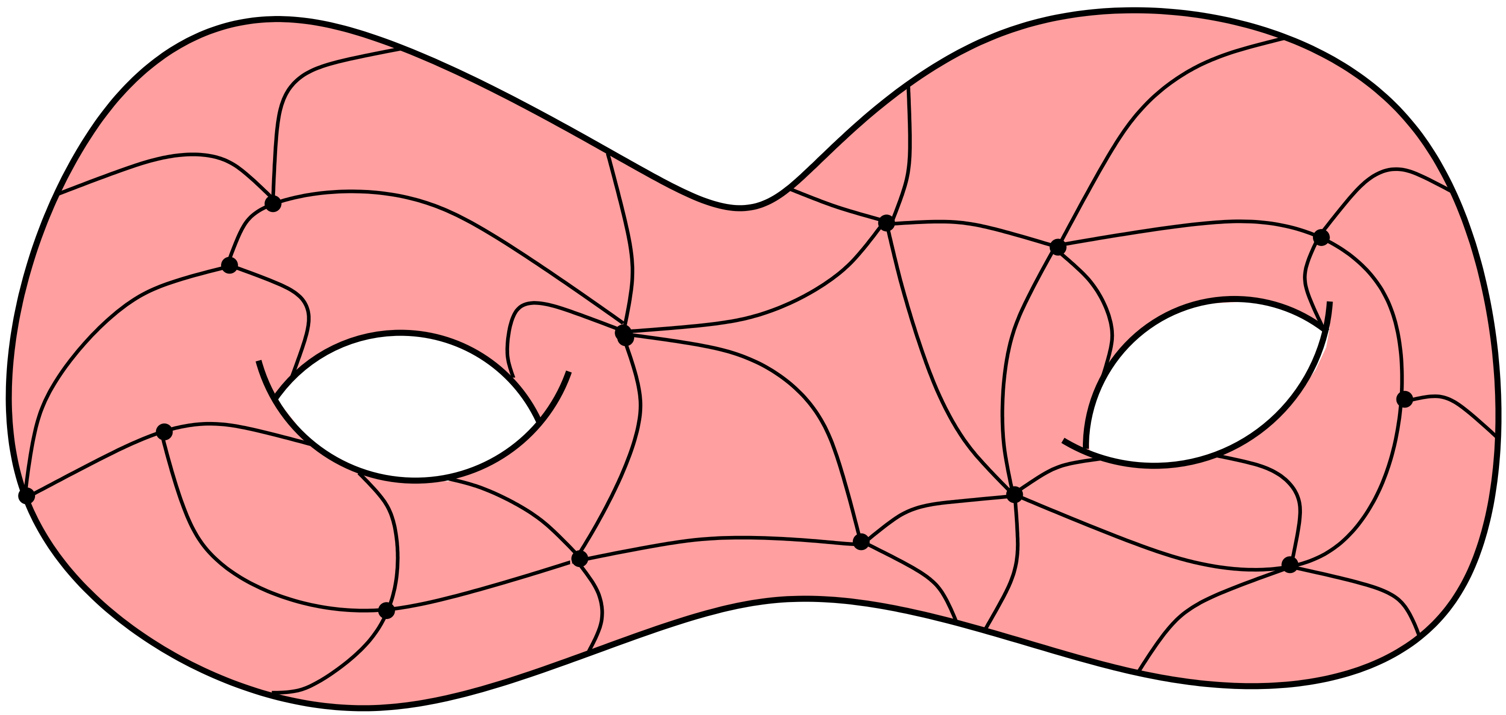
Boundary conditions allow triangulated pieces to be glued together.

We get a well defined, QR function on union.

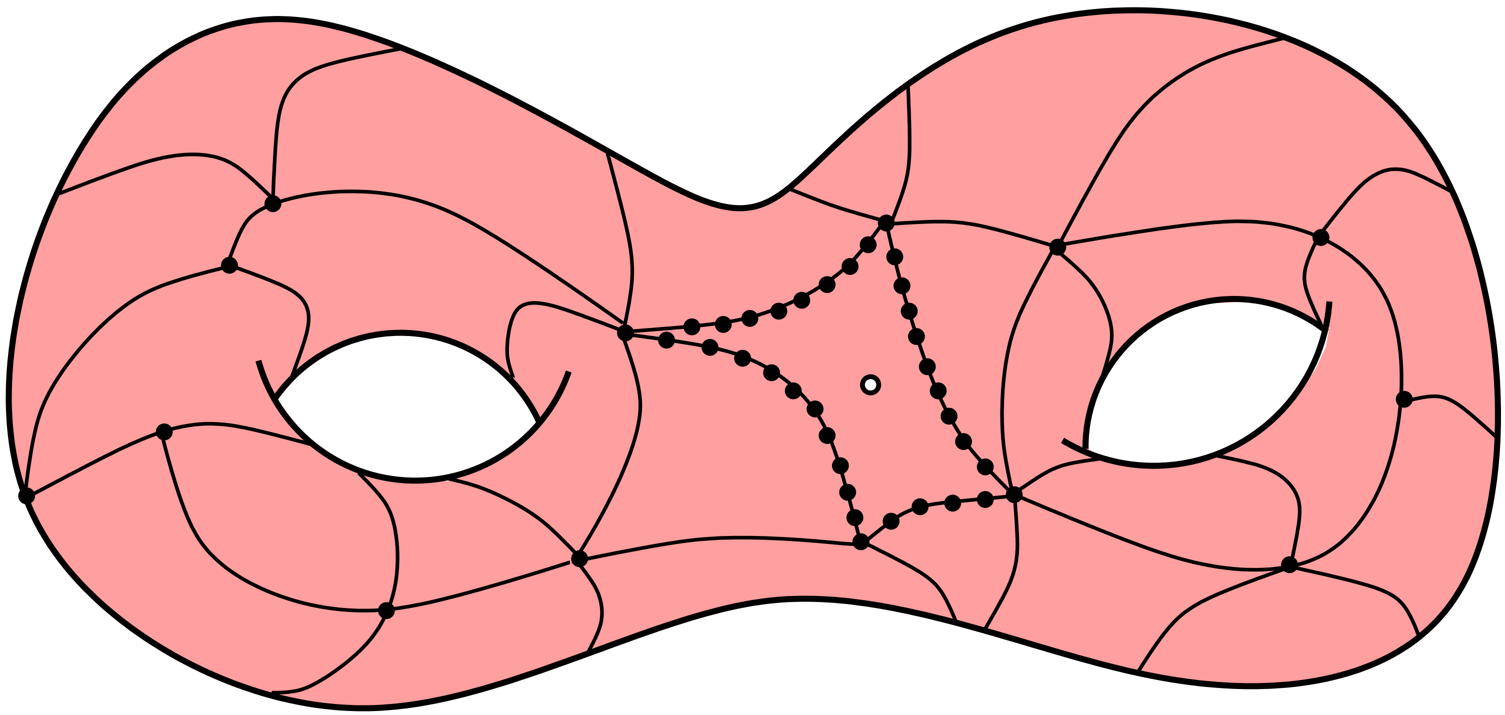
Holomorphic except on small area. QC correction close to identity.



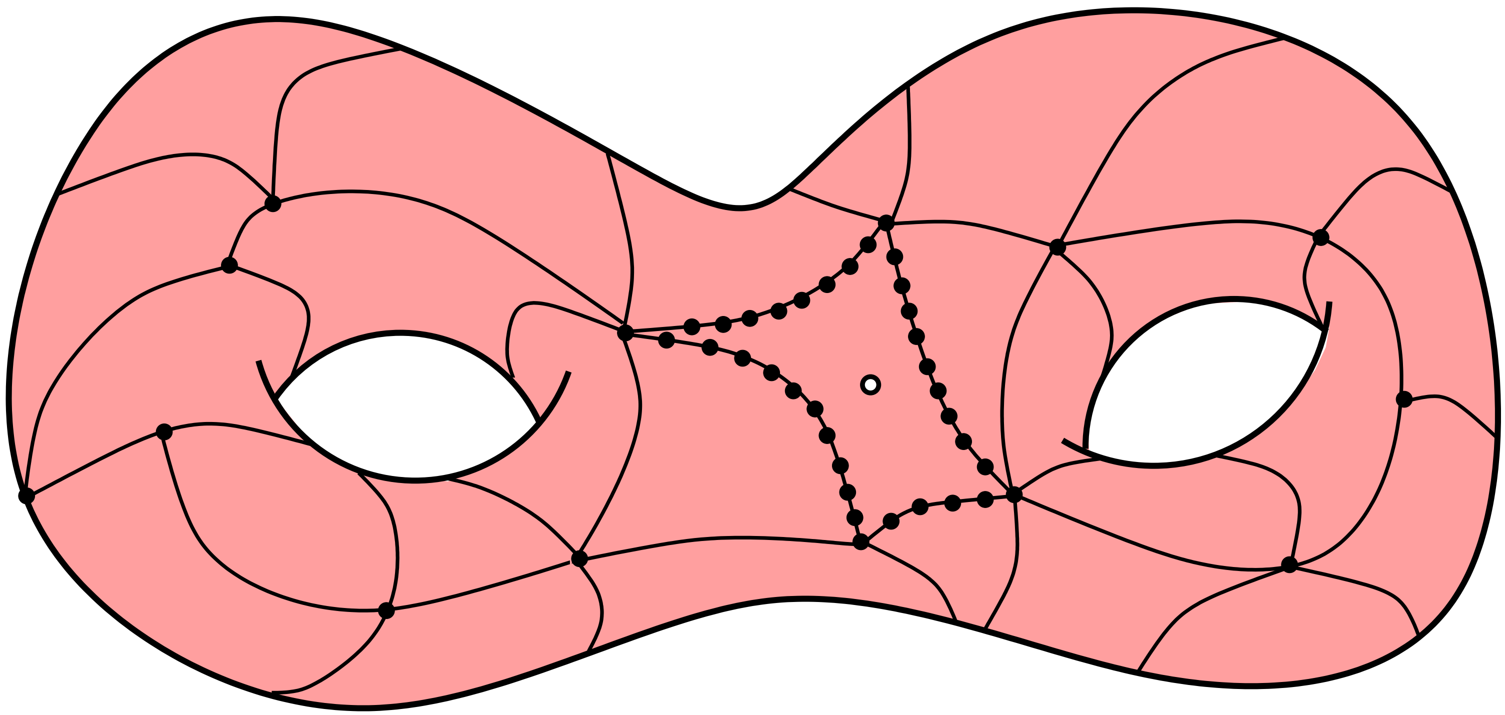
Now do this on a Riemann surface.



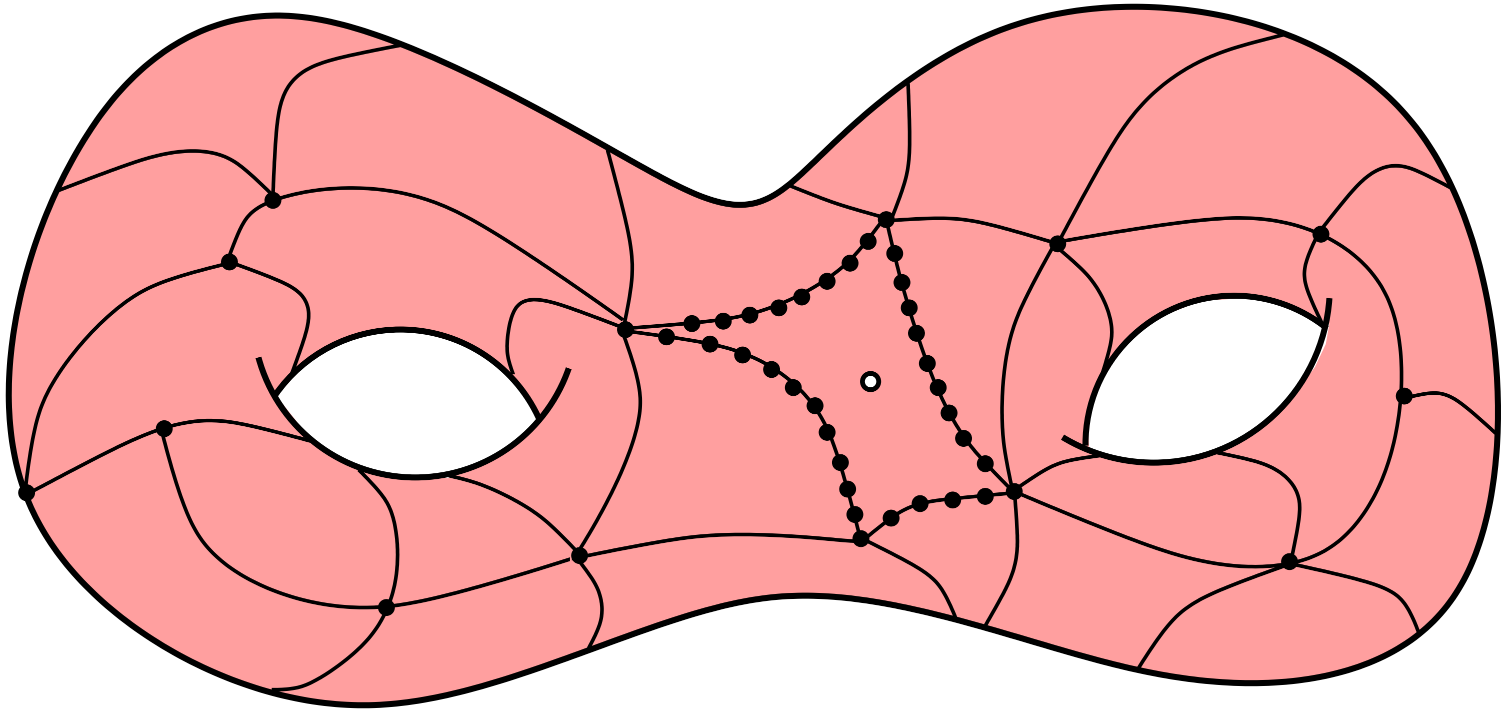
Cut the surface into nice pieces.



QC folding gives QR Belyi maps on each piece.
Boundary edges map to $[-1, 1]$ and match up between pieces.
We get a QR Belyi map on any compact surface surface.



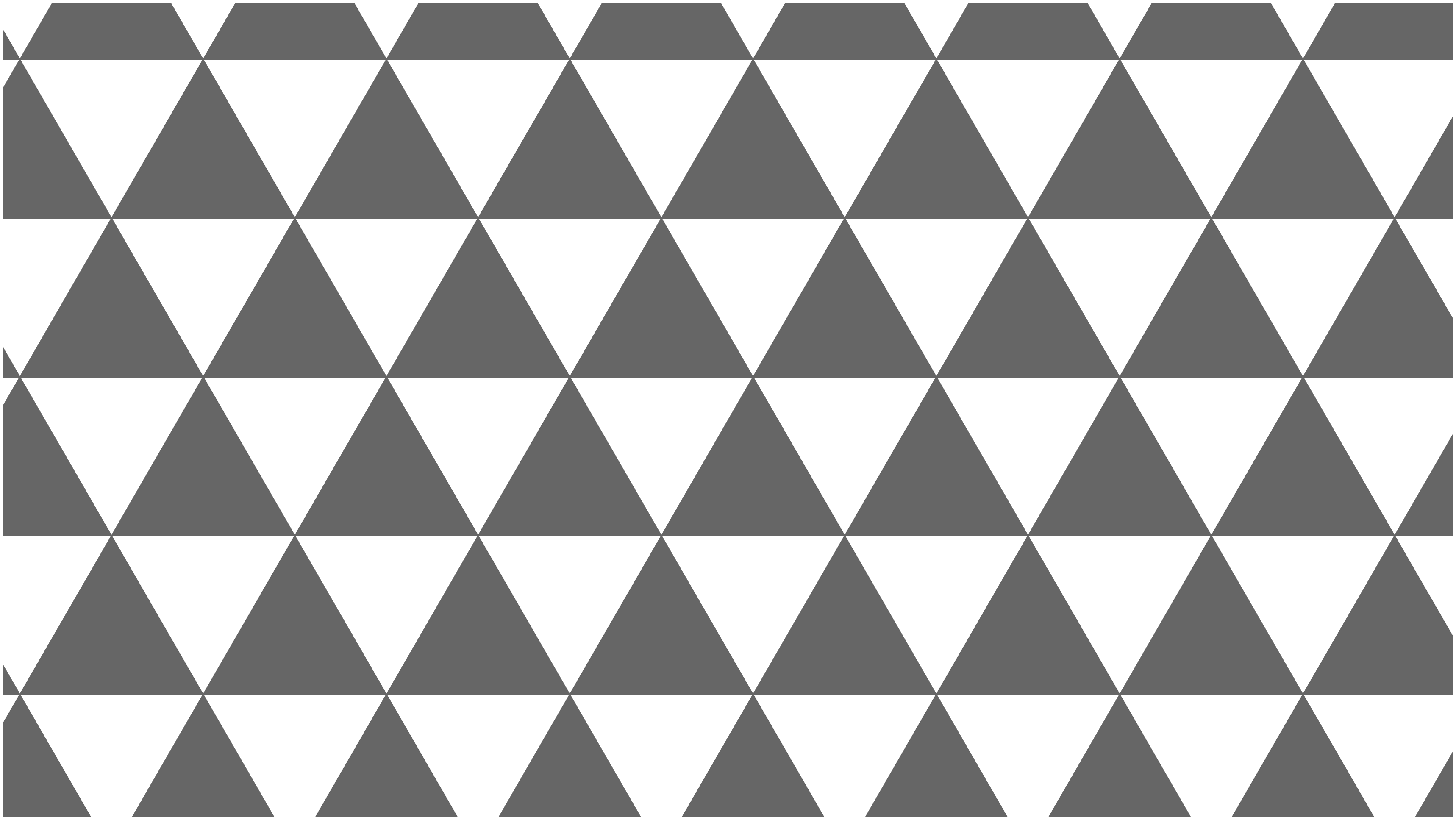
Solving Beltrami equation gives holomorphic Belyi function,
but defined on slightly different Riemann surface.
(Only countably many Riemann surfaces are Belyi.)



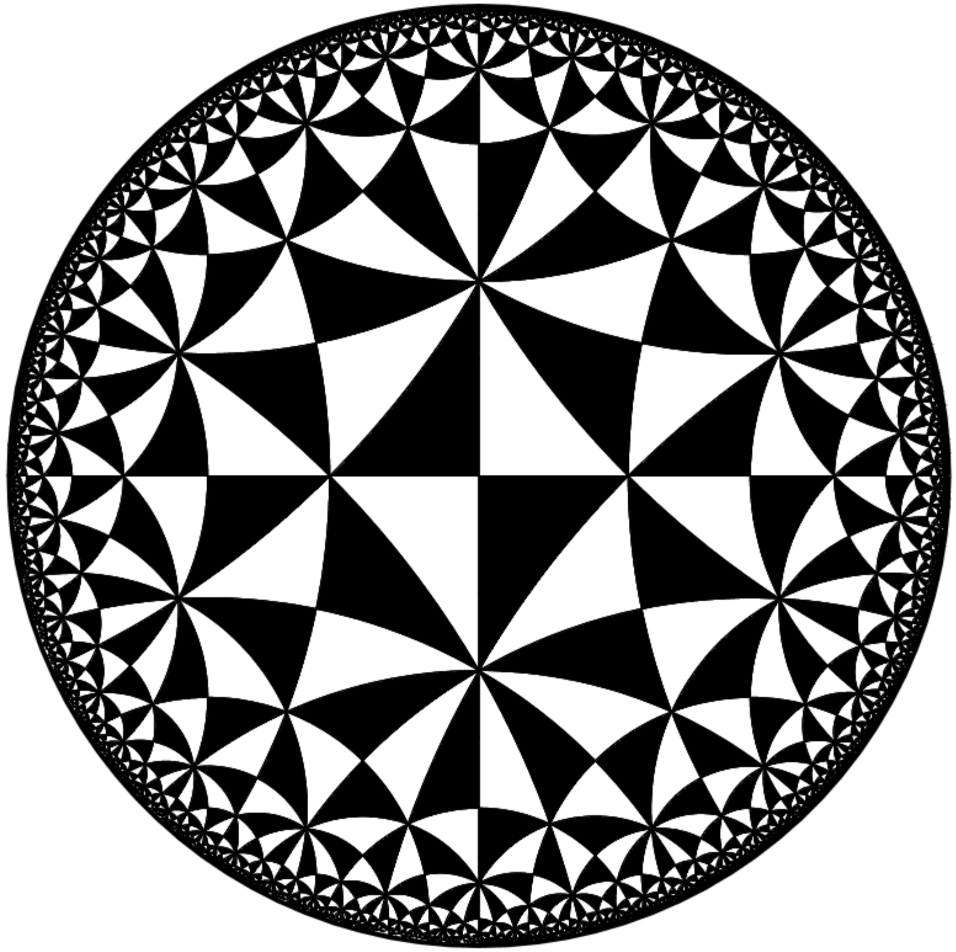
Belyi's theorem: A compact surface is Belyi iff it is algebraic.

Algebraic = zero variety of algebraic polynomial.

Which non-compact surfaces have equilateral triangulations?



Equilateral triangulation of the plane



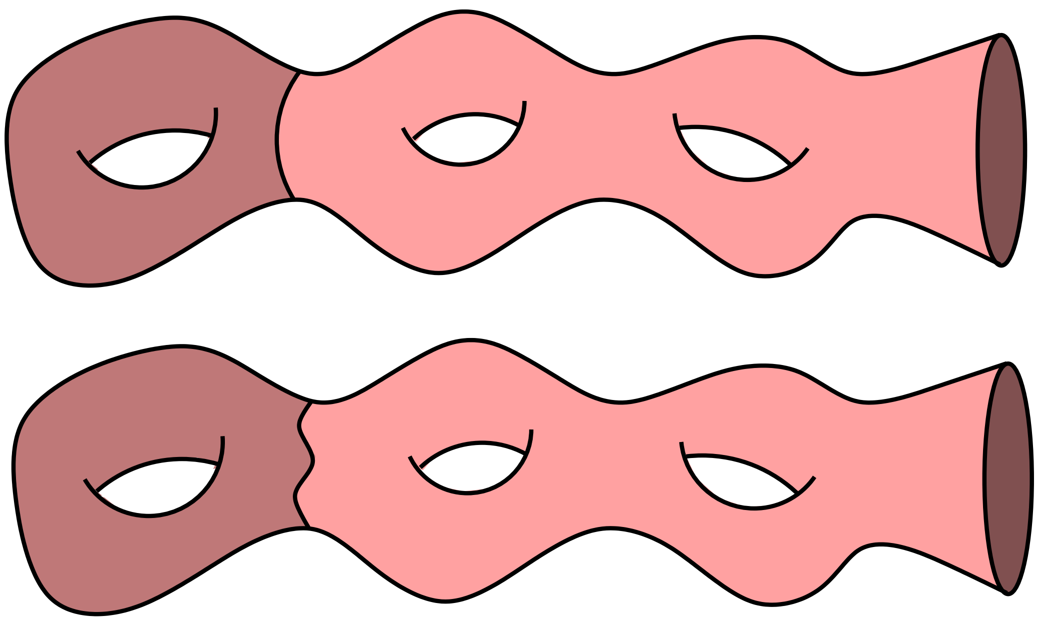
Equilateral triangulation of the disk

Thm (B-Rempe): Every non-compact surface has a Belyi function.

Thm (B-Rempe): Every non-compact surface has a Belyi function.

Idea of proof:

- Compact pieces can be approximated by triangulated surfaces.
- Conformal structure is changed, but as little as we wish.
- **Key fact:** small perturbation $\Rightarrow$ triangulated pieces re-embed in S.
- Triangulate a compact exhaustion of S. Take limit.

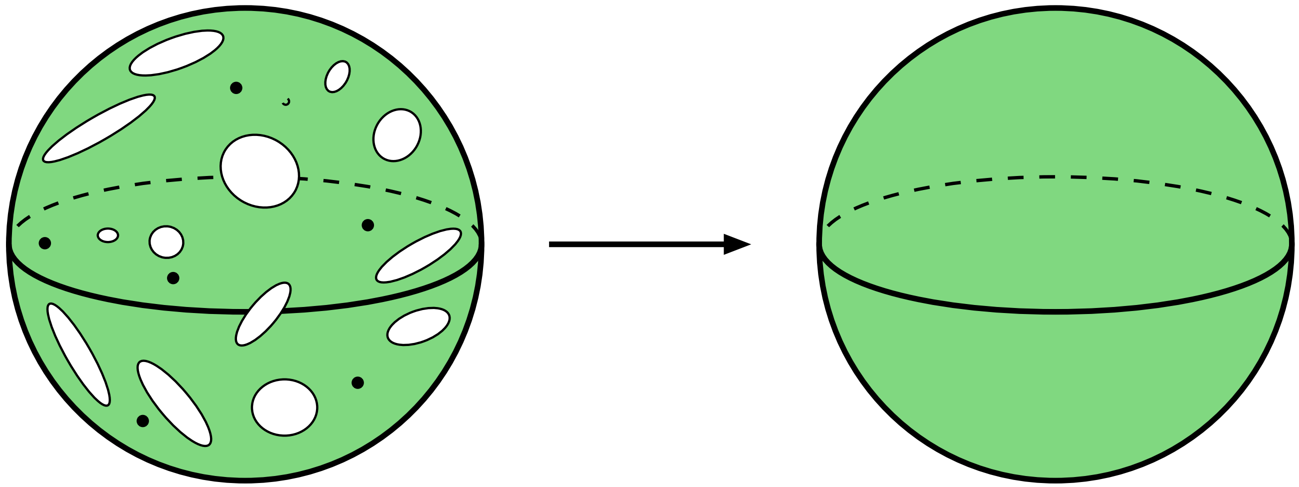


Thm (B-Rempe): Every non-compact surface has a Belyi function.

Corollary: Every Riemann surface is a branched cover of the sphere, branched over finitely many points.

- For compact surfaces, this is Riemann-Roch.
- Compact, genus g sometimes needs $3g$ branch points.
- 3 branch points suffice for all non-compact surfaces.

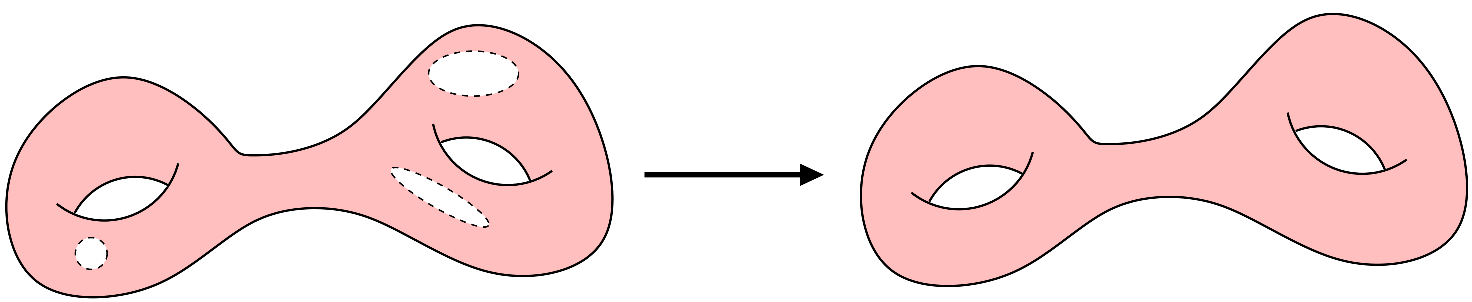
Corollary: Any open $U \subset \mathbb{C}$ is 3-branched cover of the sphere.



Gives many new dynamical systems of finite type (after A. Epstein).

Previous: rational maps on sphere, Speiser class on plane.

Also some new higher genus examples.



THANKS