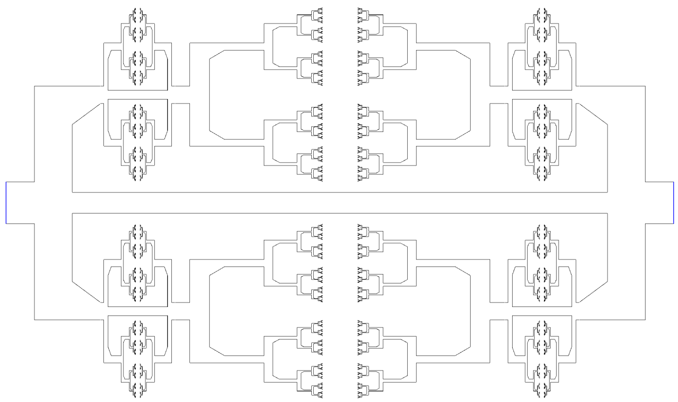
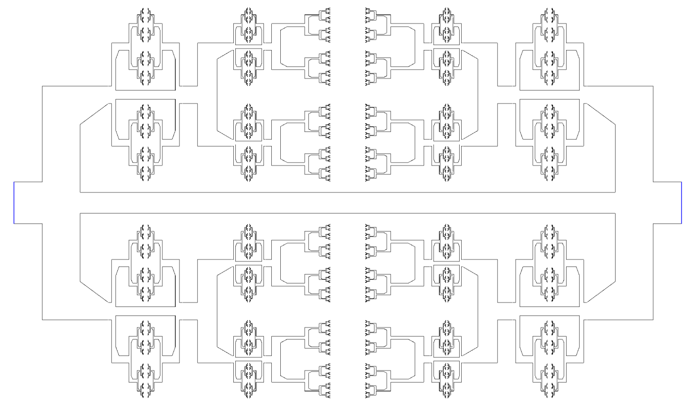
CONFORMAL REMOVABILITY IS HARD

Christopher Bishop, Stony Brook University

Guest Lecture, MAT 626 (Ntalampekos)
on Monday, November 29 at 9:45am





THE IDEA

Given a certain property of holomorphic functions, such as “boundedness” or “bijective”, there is a corresponding collection of compact sets called the **removable sets** for this property.

A few of these collections have been completely characterized (sometimes after decades of work), but others remain quite mysterious.

We will view this via descriptive set theory: how complicated are these collections as subsets of the space of all compact sets?

Definition: A planar compact set E is called **removable** for a property P if every function with property P on $\Omega = E^c = \mathbb{C} \setminus E$ is the restriction of a function on $\mathbb{C}$ with this property.

Otherwise it is **non-removable**.

removable = "small"

non-removable = "large"

Most attention has been devoted to the following properties:

- H^∞-removable: $P =$ bounded,
- A-removable: $P =$ bounded and uniformly continuous,
- S-removable: 1-to-1 (= conformal or schlicht),
- CH-removable: $P =$ extends to a homeomorphism of $\mathbb{C}$.

See Malik Younsi's 2015 paper in European Math. Surveys.

Xavier Tolsa has given a characterization of H^∞-removable sets in terms of the types of positive measures supported on the set:

Thm: E is H^∞-non-removable iff it supports $\mu \geq 0$ of linear growth

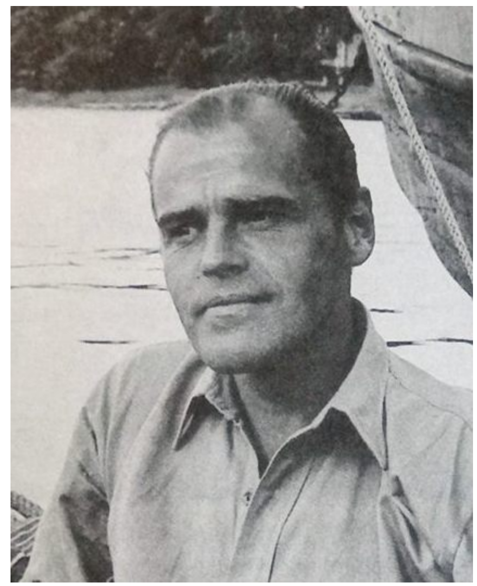
$$\mu(D(x,r)) \leq Mr, \tag{1}$$

and finite Menger curvature

$$c^2(\mu) = \int\int\int c^2(x,y,z)d\mu(x)d\mu(y)d\mu(z) < \infty. \tag{2}$$

$c(x,y,z)$ is 1/radius of circle containing (x,y,z).

Alhfors and Beurling gave a characterization of S-removable sets in terms of a quantity called "absolute area zero" (the complement of every conformal image of Ω has zero area).



If S is compact metric space, let 2^S denote all compact subsets of S.

The Hausdorff distance between sets E and F is the infimum of $\epsilon > 0$ so that each set is contained in a ϵ-neighborhood of the other.

Fact: 2^S with the Hausdorff metric is a compact metric space.

Defn: G_δ set $=$ countable intersection of open sets.

Defn: F_σ set $=$ countable union of closed sets.

Defn: Borel sets $=$ smallest σ-algebra containing all open sets.

Theorem: Let $S = [0,1]^2$. Inside 2^S, the collection of

1. H^∞-removable subsets is a G_δ,
2. S-removable subsets is a G_δ,
3. A-removable sets is not Borel,
4. CH-removable sets is not Borel,
5. A-removable closed Jordan curves is not Borel.

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Suppose X, Y complete metric spaces and $f : X \to Y$ continuous.

If $E \subset Y$ is Borel, so is $f^{-1}(E) \subset X$.

So if $f^{-1}(E) \subset X$ is not Borel, E is not Borel.

Idea: find known non-Borel set that maps continuously onto E.

Side Remark: continuous images of Borel sets need not be Borel.

Lebesgue famously claimed otherwise in a 1905 paper. Cooke refers to this as "one of the most fruitful mistakes in all the history of analysis"

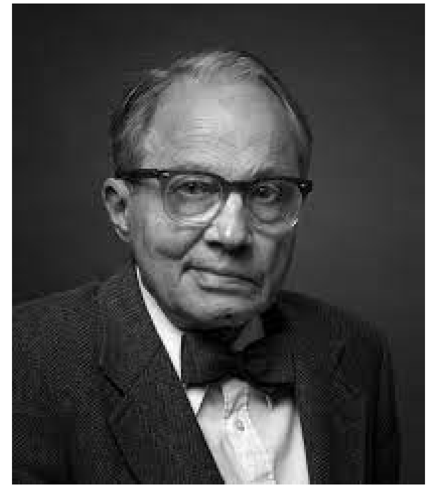
His mistake was discovered in 1917 by Suslin, a student of Lusin, and led to the founding of field known today as descriptive set theory.

Continuous images of Borel sets are called **Suslin sets** or **analytic sets**.

Thm: closed, countable subsets of $[0,1]$ are not Borel in $2^{[0,1]}$.

This is standard, not too hard to prove. We will take it as a fact.

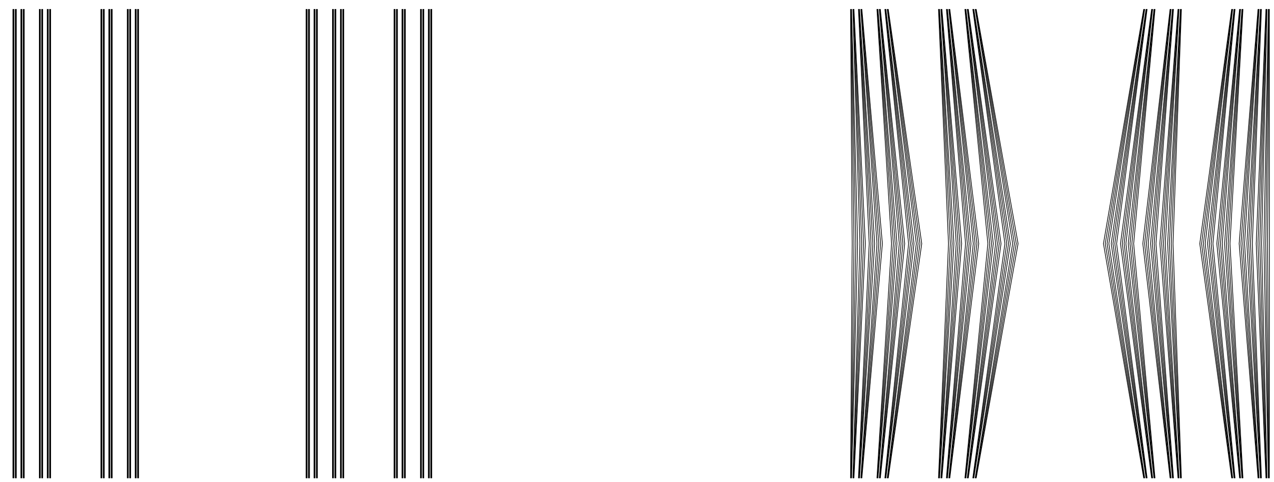
Thm (Gehring, 1960): If $E \subset [0,1]$ is compact, then $E \times [0,1]$ is CH-removable iff E is countable.



Cor: CH-removable sets are not Borel in $2^{[0,1]^2}$.

Proof: $E \to E \times [0,1]$ is continuous from $2^{[0,1]}$ to $2^{[0,1]^2}$.

E is countable iff the image is removable.



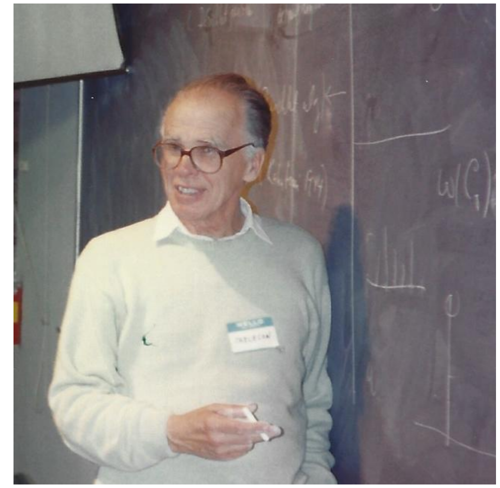
Proof of Gehring's theorem:

If E is a Cantor set there is homeomorphism h of $\mathbb{C}$ that is quasiconformal off $E \times [0,1]$ and maps $E \times [0,1]$ to a set of positive area.

f can't extend QC to $\mathbb{C}$ since QC maps are absolutely continuous.

Measurable Riemann mapping thm gives non-linear conformal g on E^c. Proves E is CH-non-removable.

Thm (Carleson, 1951): If $E_1, E_2 \subset [0,1]$ are compact and E_2 has positive Lebesgue measure then $E = E_1 \times E_2$ is A-removable iff E_1 is countable.



Recall A-removable = removable for continuous extension

Recall CH-removable = removable for homeomorphic extension

Cor: A-removable sets are not Borel in $2^{[0,1]^2}$.

Thm: A-removable **Jordan curves** are not Borel in $2^{[0,1]^2}$.

First idea for proof: Run a curve γ_E through the Cantor sets $E \times F$.

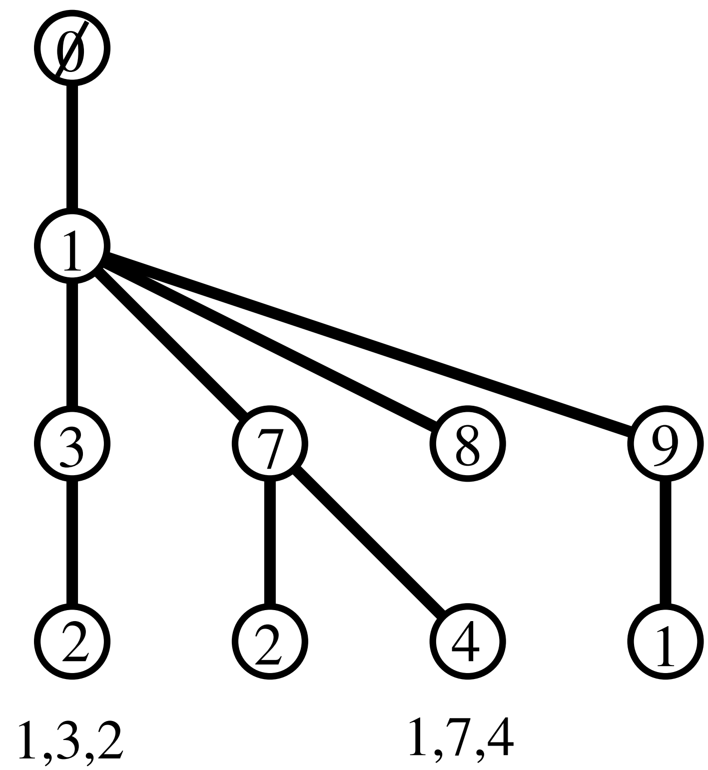
- γ_E contains the Cantor set $E \times F$.
- the map $E \mapsto \gamma_E$ is continuous in Hausdorff metric.
- show γ_E is A-removable iff E is countable.

However, there are technical difficulties.

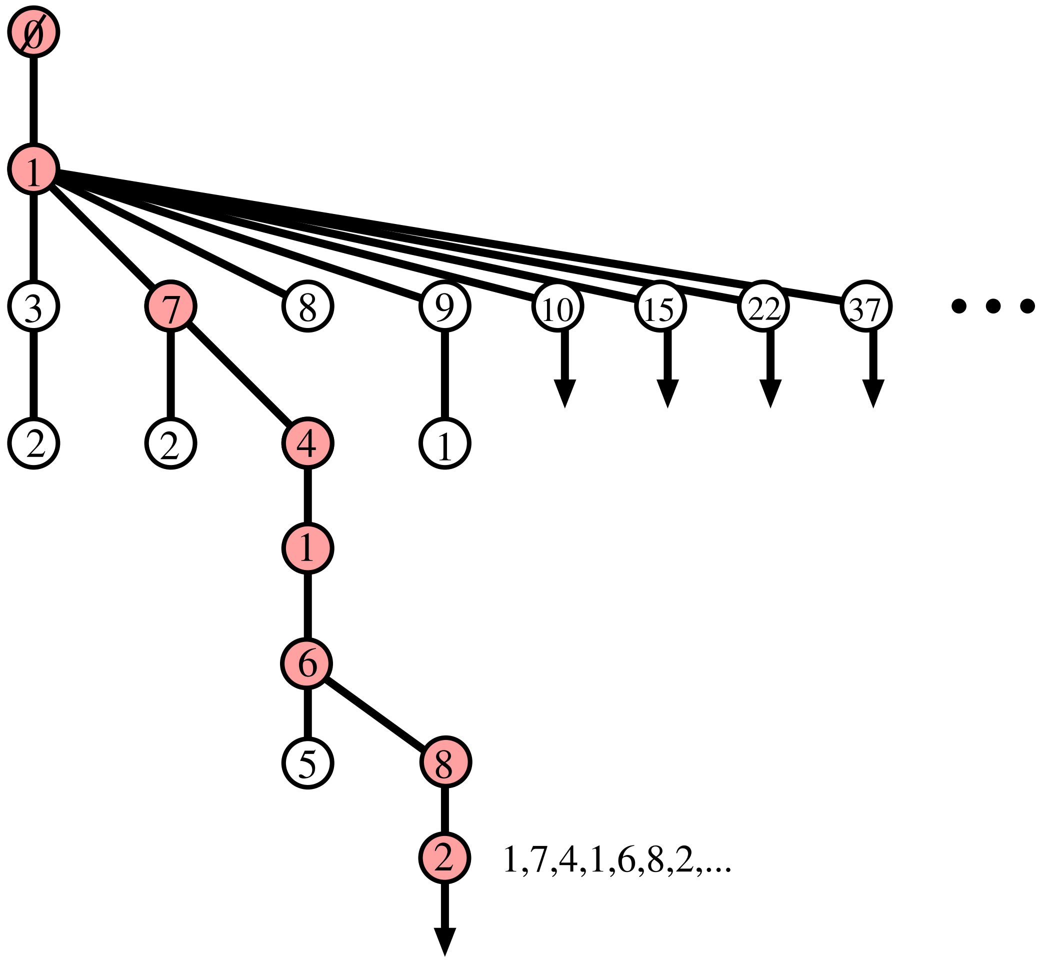
Easier to work with a different non-Borel example.

$\mathbb{N} = \{0, 1, 2, 3, \dots\}$. $\mathbb{N}^* =$ finite sequences in $\mathbb{N}$.

A tree T is a subset of $\mathbb{N}^*$ that is closed under removing the final element.



Infinite branch of T = element of $\mathbb{N}^{\mathbb{N}}$, all finite initial segments in T.



A tree is **wellfounded** if it has no infinite branches.

Finite trees are obviously wellfounded.

Lemma: The wellfounded trees are not Borel in $\mathbb{N}^{\mathbb{N}}$.

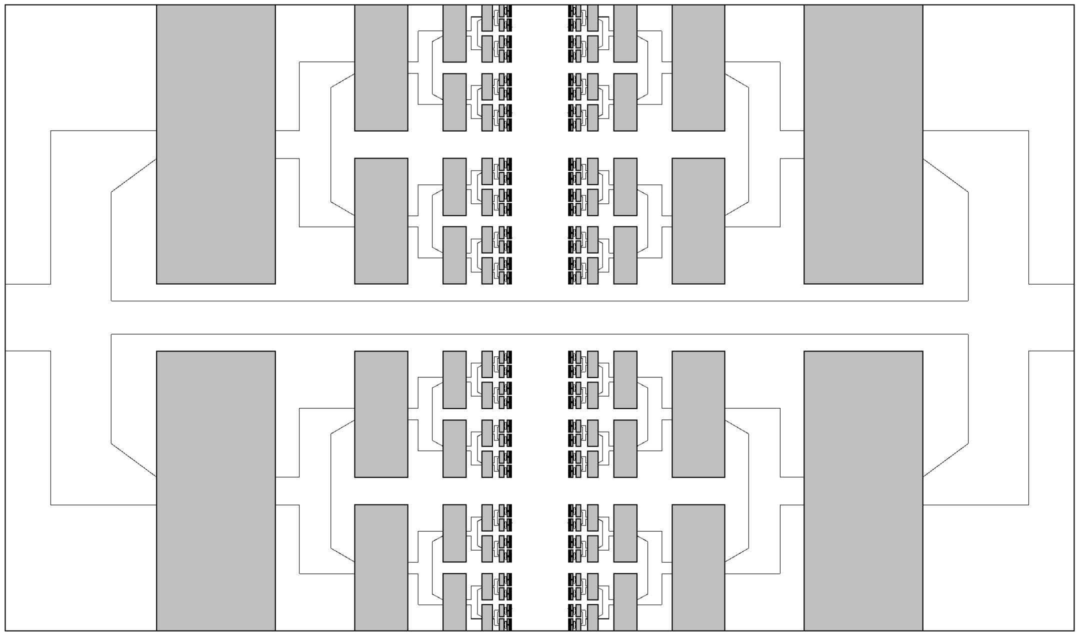
Proof that A-removable Jordan curves are not Borel.

Construct a continuous map

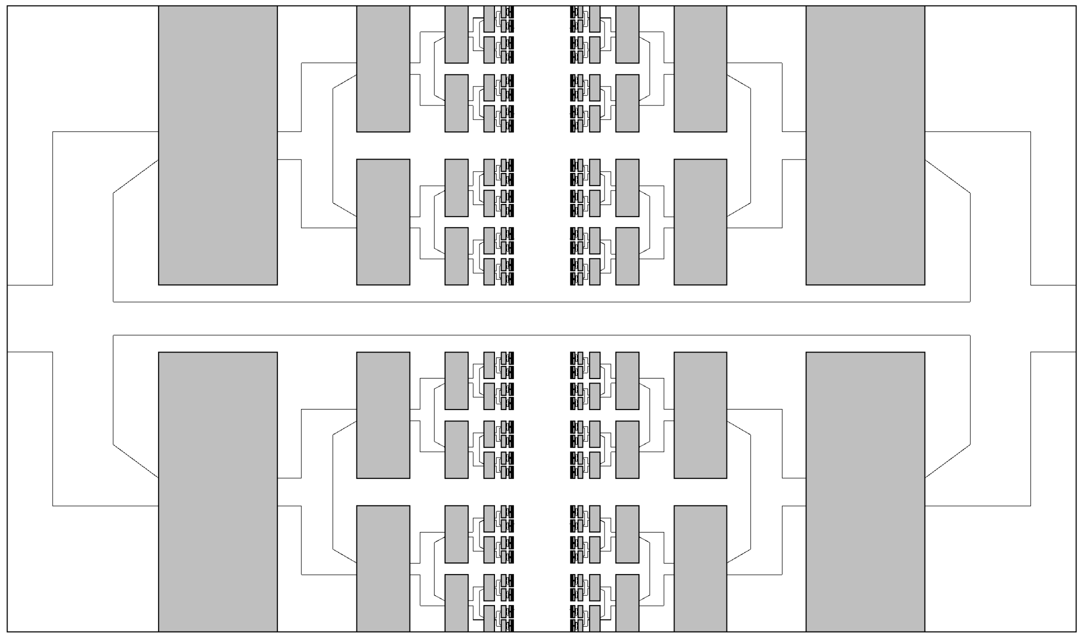
$$\text{trees} \quad \to \quad \text{curves}$$

so that

$$\text{wellfounded trees} \quad \to \quad \text{removable curves.}$$

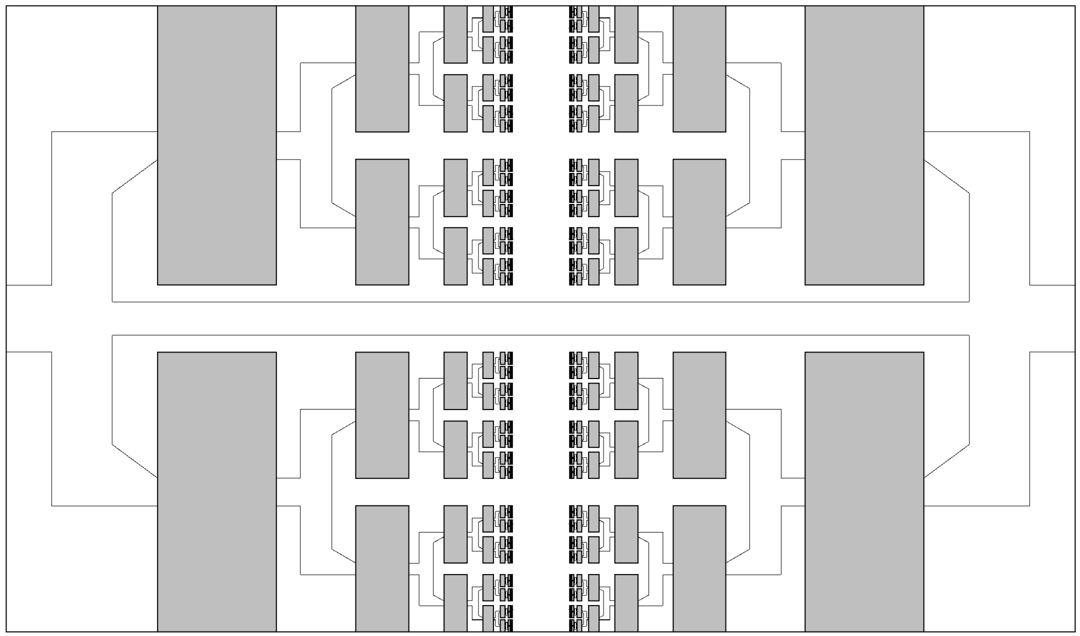


This it the basic building block of the construction.



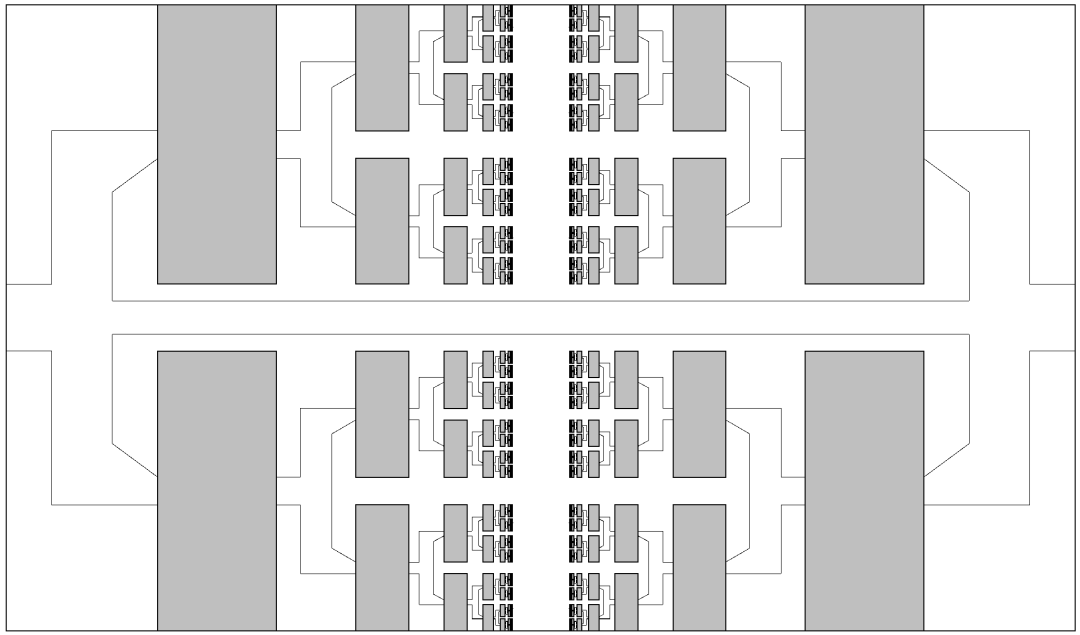
An infinite collection of boxes connected by polygonal arcs.

Boxes accumulate on two vertical Cantor sets (positive length).



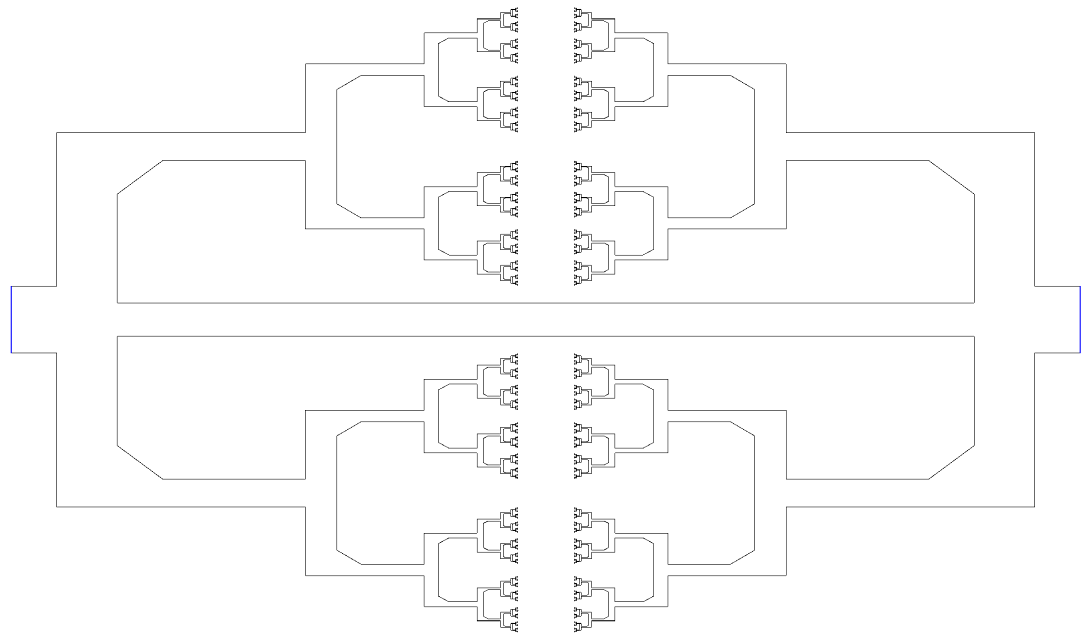
Each integer n corresponds to a finite collection of rectangles.

All of our Jordan curves are subsets of this compact set.



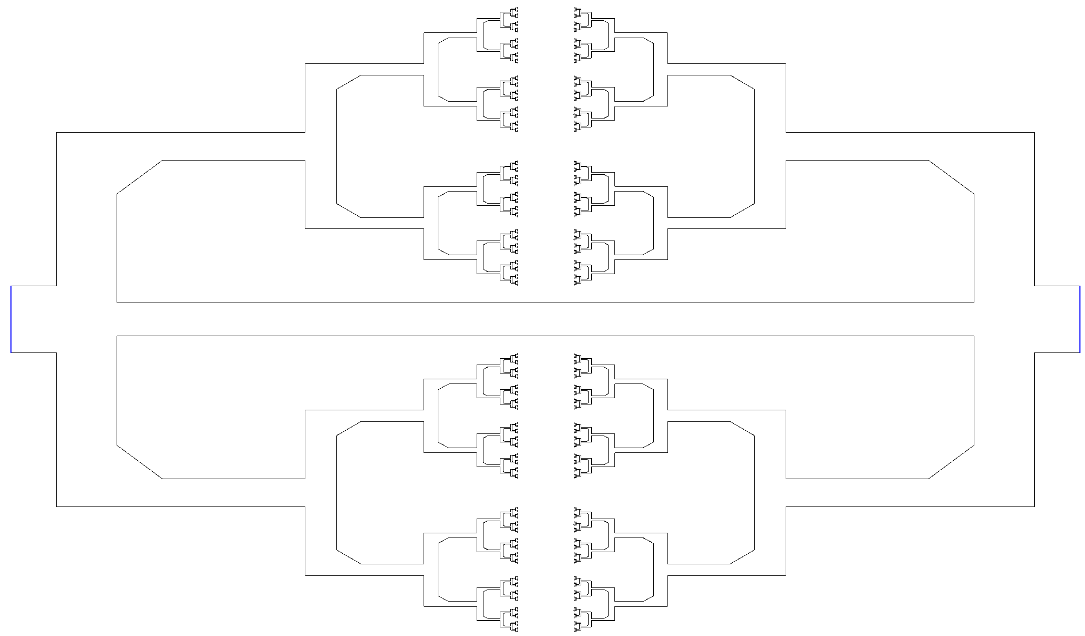
If n starts string, n-boxes get copy of the template.

Otherwise the box is replaced by a horizontal segment.



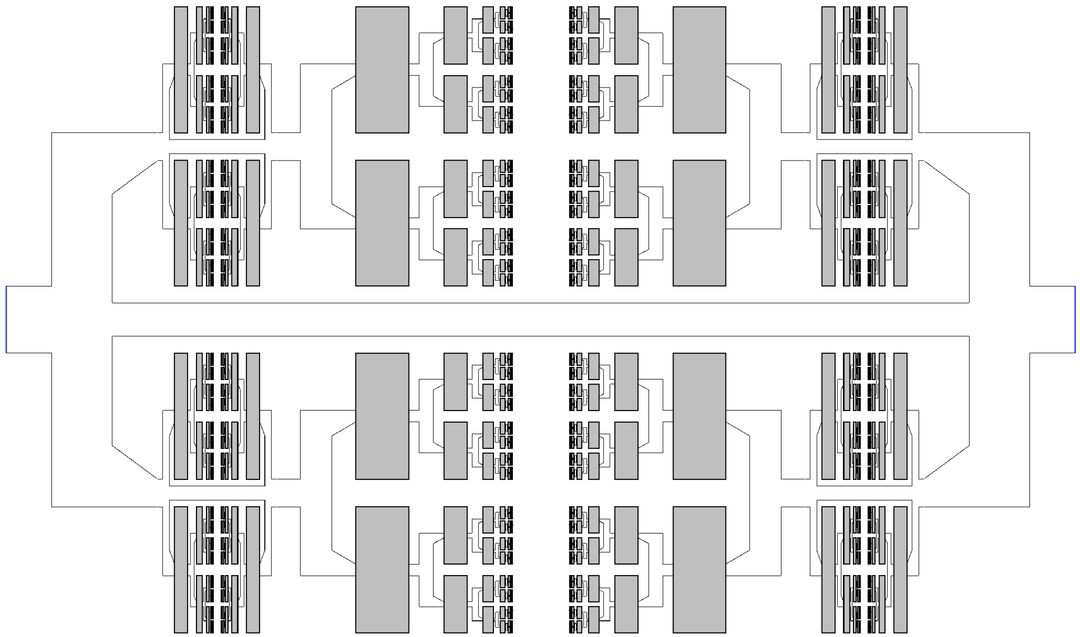
Tree represented by empty string. Simplest curve in our collection.

Every box is replaced by a segment; no replication of template.

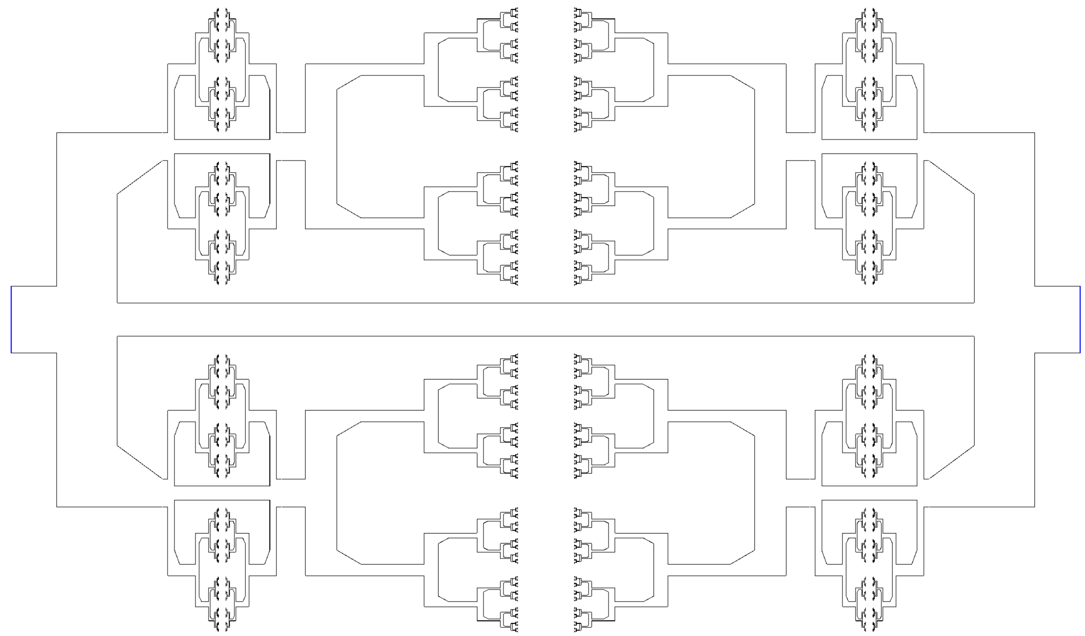


This curve is countable union of finite length sets.

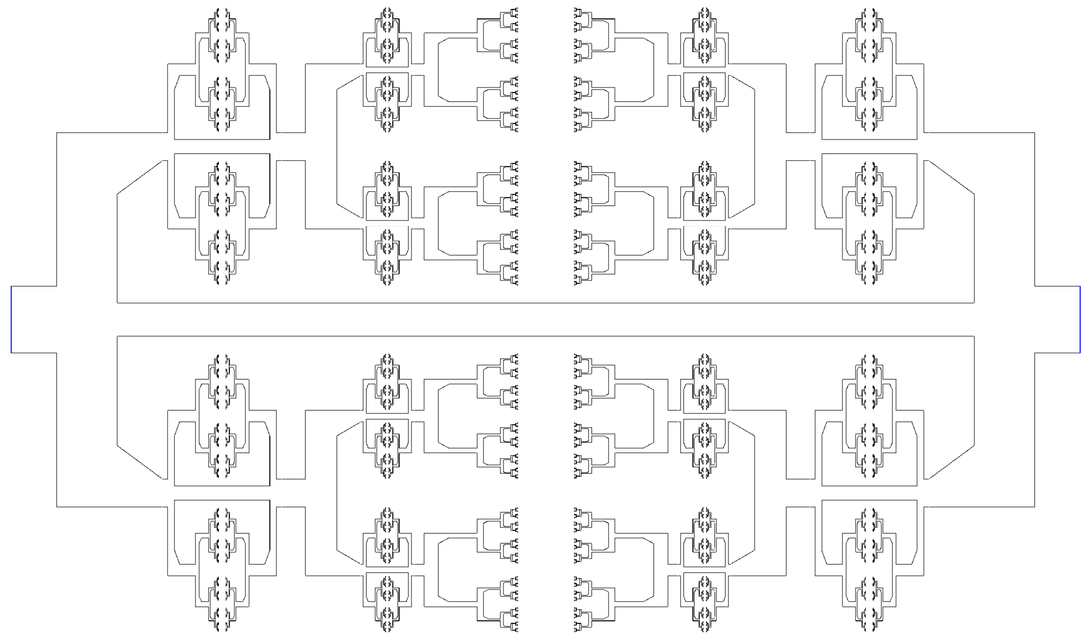
$\Rightarrow$ A-removable.



If a string starts with 1, the four 1-boxes are replaced by a template.



Curve for 2-vertex tree: strings $\{\emptyset, 1\}$.



The curve corresponding to the tree with vertices $\{\emptyset, 1, 2\}$.

This curve is also A-removable. Same for all wellfounded trees.

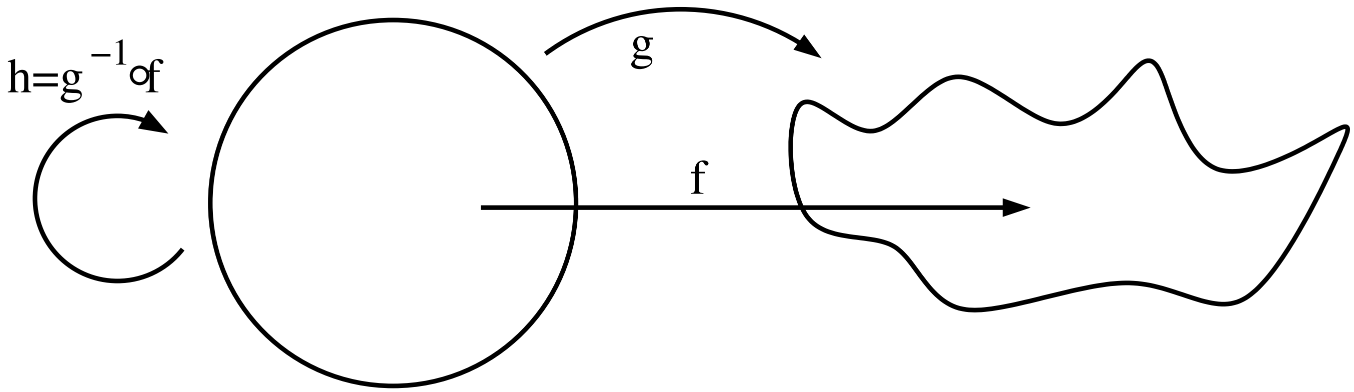
An infinite branch in the tree creates uncountably many copies of the linear Cantor set, hence the Jordan curve is not removable.

Thus tree wellfounded iff corresponds to A-removable the Jordan curves.

This construction does not work for CH-removable Jordan curves.

Question: do CH-removable Jordan curves form a Borel subset of $2^{[0,1]^2}$?

Given a closed Jordan curve $\Gamma \subset \mathbb{C}$, let f, g be conformal maps from the inside/outside or the unit circle to the inside/outside of Γ.



f, g are homeomorphisms $\mathbb{T} \to \Gamma$, so $h = f \circ g$ is a circle homeomorphism.

Such an h is called a **conformal welding**.

Not all circle homeomorphisms are conformal weldings.

Characterizing conformal weldings seems intractable.

Question: are conformal weldings Borel in all circle homeomorphisms?

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We know they are a residual set in space of circle homeomorphisms.

Use my thm on “flexible curves” and a construction of Pugh and Wu.

(For experts: quasisymmetric homeomorphisms are first category.)

Continuous images of Borel sets need not be Borel, but ...

1-1 continuous images of Borel sets are Borel.

Jordan curves are complete metric space (not obvious, due to Thurston).

We have a map from Jordan curves to conformal weldings.

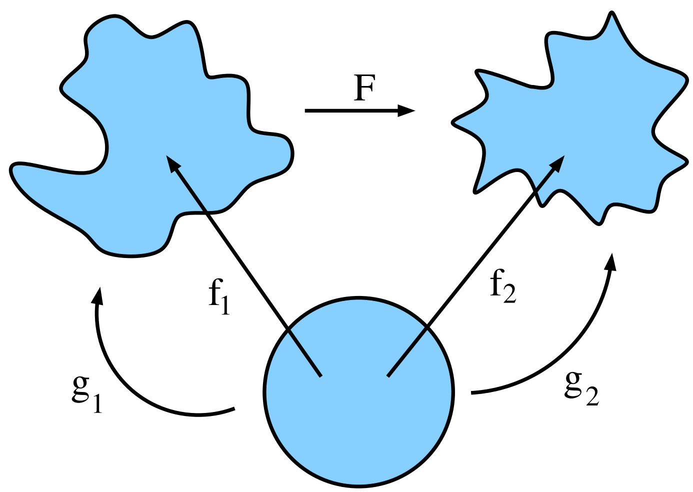
More precisely, from curves modulo similarities into homeomorphisms modulo Möbius transformations.

Is map from curves to weldings 1-1?

No.

If γ is not CH-removable and f is a homeomorphism of S^2 conformal off γ, then $\gamma' = f(\gamma)$ has same welding.

We call such a γ' a **CH-image** of γ (if f not Möbius)..



If γ' is not a Möbius image of γ then curves $\to$ weldings is not 1-1.

Curves of positive area give examples.

There are weldings whose preimage curves are dense in all Jordan curves.

So map curves $\to$ weldings is not 1-1.

“Conformal welding is hard because CH-removable sets are hard.”

Ques: Is map curves $\to$ weldings 1-1 exactly on CH-removable curves?

In other words, does every CH-non-removable curve have a CH-image that is not a Möbius image?

It was obvious to experts (including myself) that CH-images are never Möbius images

Ques: Is map curves $\to$ weldings 1-1 exactly on CH-removable curves?

In other words, does every CH-non-removable curve have a CH-image that is not a Möbius image?

It was obvious to experts (including myself) that CH-images are never Möbius images until Malik Younsi recently found a counterexample.

So question above is still open.

Thanks for listening.

Questions?