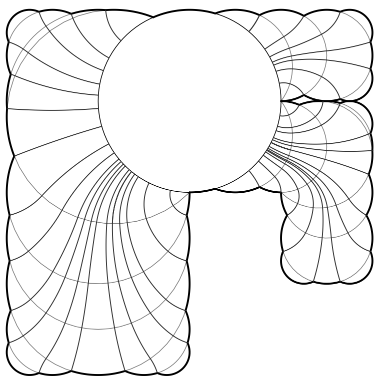
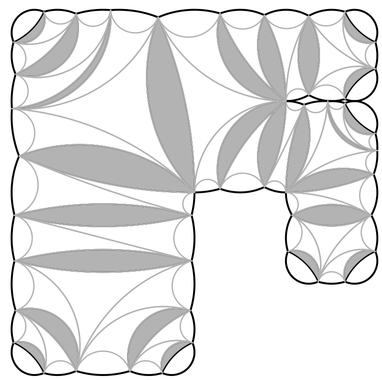
Conformal Mapping in Linear Time

Christopher J. Bishop
SUNY Stony Brook





copies of lecture slides available at

`www.math.sunysb.edu/~bishop/lectures`

Riemann Mapping Theorem: If Ω is a simply connected, proper subdomain of the plane, then there is a conformal map $f : \Omega \to \mathbb{D}$.

• conformal = 1-1, onto, homomorphic.

• Usual proof is to first show there is some 1-1 from Ω into $\mathbb{D}$, then show that maximizing $|f'(z_0)|$ at some point $z_0 \in \Omega$ gives onto map.

• There are various numerical methods to compute f in practice: Schwarz-Christoffel, integral equations, circle packing, "zipper".

- What is the complexity of computing f?

- Assume Ω is given by n parameters, e.g., $\partial\Omega$ is an n-gon.

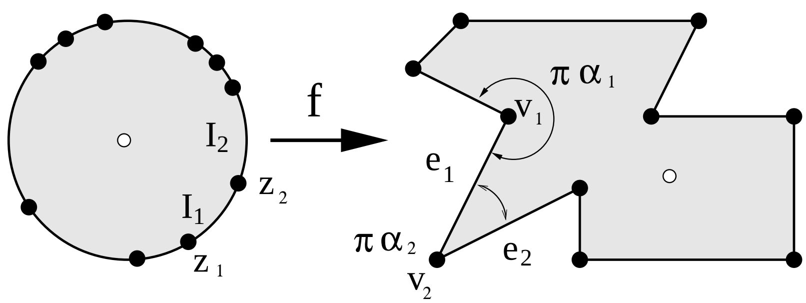
- What is an ϵ-approximation of a conformal map? Define a metric on maps.

- Bound the time $\leq C(n, \epsilon)$ need to find an ϵ-approximation onto any n-gon.

- My interest was sparked by a 1998 paper of T. Driscoll and S. Vavasis, "Numerical conformal mappings using cross-ratios and Delaunay triangulation".

The Schwarz-Christoffel formula gives the Riemann map onto a polygonal:

$$f(z) = A + C \int^{z} \prod_{k=1}^{n} (1 - \frac{w}{z_k})^{\alpha_k - 1} dw.$$



Must solve nonlinear system

$$f(z_k) = v_k, \quad k = 1, \ldots, n$$

Davis's algorithm: Suppose P has edge lengths

$$\mathbf{e} = \{e_1, \dots, e_n\} \in \mathbb{C}^n$$

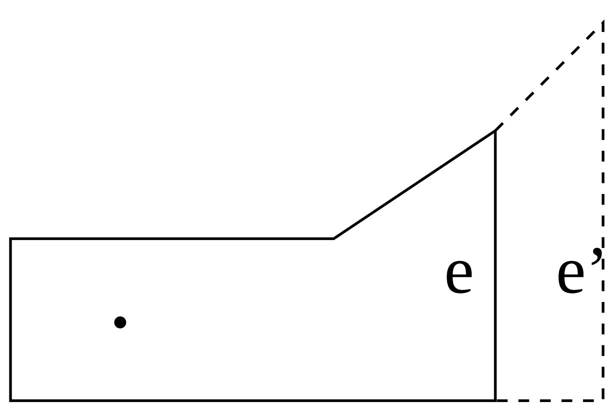
and we guess the SC-parameter gaps

$$\mathbf{I} = \{I_1, \dots, I_n\} \in [0, 2\pi]^n, \sum I_k = 2\pi$$

Plug these into SC-formula and get polygon P' with edges $\mathbf{e}'$. Define new parameter gaps by

$$I'_k = \lambda I_k \frac{e_k}{e'_k}$$

where λ is a normalizing factor. This sometimes works, but . . .



Theorem: Supppose $\partial\Omega = P$ is an n-gon and $\mathbf{z} = f^{-1}(\mathbf{v})$ are conformal prevertices. We can construct points $\mathbf{w} = \{w_1, \dots, w_n\} \subset \mathbb{T}$ so that:

1. $\mathbf{w}$ can be computed in at most $C(\epsilon)n$ steps.
2. $d_{QC}(\mathbf{w}, \mathbf{z}) < \epsilon$.

$d_{QC}(\mathbf{w}, \mathbf{z}) = \inf\{\log K : \exists h \in \mathrm{QC_K}, \mathrm{h}(\mathbf{w}) = \mathbf{z}\}$.

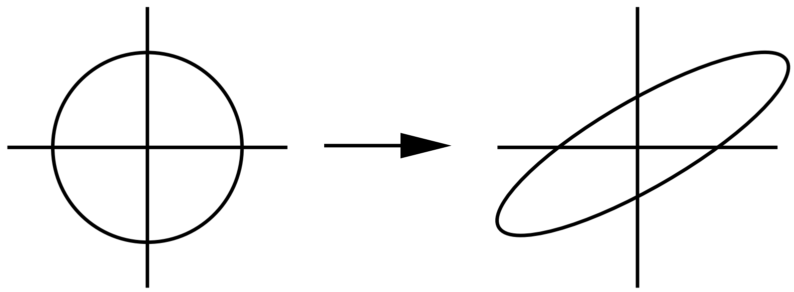
$\mathrm{QC_K} = K$-quasiconformal maps.

Can take $C(\epsilon) = C + C \log^2 \frac{1}{\epsilon} \log\log \frac{1}{\epsilon}$

Theorem: If $\partial\Omega$ is an n-gon we can compute a $(1+\epsilon)$-quasiconformal map between Ω and $\mathbb{D}$ in time $O(n \log^2 \frac{1}{\epsilon} \log\log \frac{1}{\epsilon})$.

A mapping is K-quasiconformal if either:

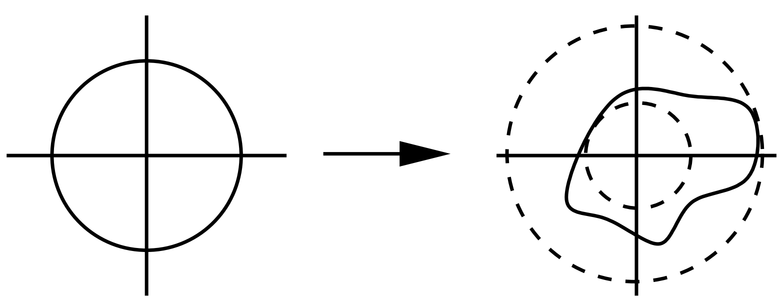
Analytic definition: $|f_{\bar{z}}| \leq \frac{K-1}{K+1}|f_z|$



$$f_z = \frac{1}{2}(f_x - if_y),\ f_{\bar{z}} = \frac{1}{2}(f_x + if_y).$$

Metric definition: For every $x \in \Omega$, $\epsilon > 0$ and small enough $r > 0$, there is $s > 0$ so that

$$D(f(x), s) \subset f(D(x, r)) \subset D(f(x), s(K+\epsilon)).$$



• The map is determined (up to Möbius maps) by

$$\mu_f = f_{\bar{z}}/f_z,$$

For μ with $\|\mu\|_\infty < 1$, there is a f with $\mu_f = \mu$.

• $\mu = 0$ iff f is conformal.

• K-QC maps form a compact family.

• f is a **quasi-isometry** (or rough isometry) if

$$\frac{1}{A}\rho(x,y) - B \leq \rho(f(x), f(y)) \leq A\rho(x,y) + B.$$

Theorem: $f : \mathbb{T} \to \mathbb{T}$ has a QC-extension to interior iff it has QI-extension (hyperbolic metric).

• Time estimate is clearly optimal in n.

• If given power series for f, we need $p = O(\log \frac{1}{\epsilon})$ terms to get accuracy ϵ.

• Given such series we can invert using Newton's method. Error squares each iteration, so need $O(\log p)$ iterations to reach accuracy ϵ; takes work p per iteration. Work to invert at n points is $O(np \log p) = O(n \log \frac{1}{\epsilon} \log \log \frac{1}{\epsilon})$. Our estimate is only slightly worse.

Proof of theorem is in three steps:

Step 1: Find $\mathbf{w}_0$ so that $d_{QC}(\mathbf{w}_0, \mathbf{z}) \leq K$. Takes $O(n)$ time and is motivated by 3-D hyperbolic geometry and computational geometry.

Step 2: If $d_{QC}(\mathbf{w}_n, \mathbf{z}) < \epsilon < \epsilon_0$ then construct $\mathbf{w}_{n+1}$ so that $d_{QC}(\mathbf{w}_{n+1}, \mathbf{z}) < C\epsilon^2$. Takes $O(n \log^2 \frac{1}{\epsilon} \log \log \frac{1}{\epsilon})$ per step. Uses fast multipole method to approximate solutions of Beltrami equation. Is a version of Newton's method.

If $K < \epsilon_0$ then done. Otherwise need to bridge gap between w_0 and ball of convergence for Newton's method.

Step 3: Construct a chain of domains connecting disk to Ω,

$$\mathbb{D} = \Omega_0, \ldots, \Omega_N = \Omega$$

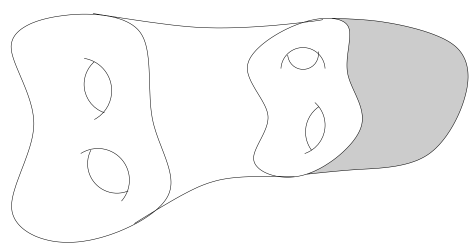
and explicit QC maps $\Omega_k \to \Omega_{k+1}$ with QC constant $< 1 + \epsilon_0/2$.

Define vertex sets V_k that correspond under these maps. Find preimages of V_k under conformal map $f_k : \mathbb{D} \to \Omega_k$. $k = 0$ is trival, $k = N$ is problem we want to solve.

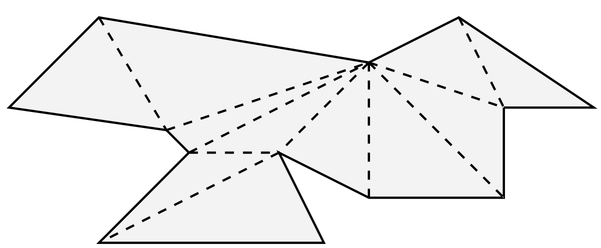
Use $f_0 = \mathrm{Id}$ as starting point in Step 2, to iterate to f_1. When within $\epsilon_0/2$ of f_1, start iterating to f_2. Continue until reach ϵ_0-ball around f_N.

First step comes from two theorems that seem unrelated to conformal mappings:

Theorem (Sullivan '81, Epstein-Marden '85): If M is a hyperbolic 3-manifold and $C(M)$ is the convex core of M, then there is a biLipschitz map between $\partial_\infty M$ and $\partial C(M)$.



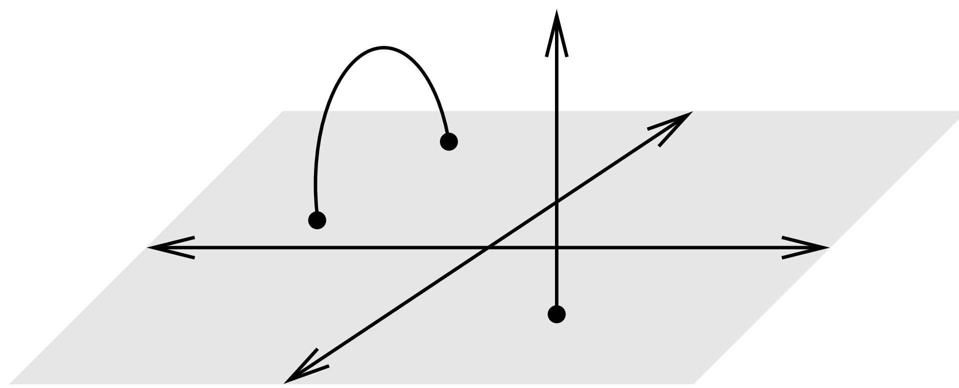
Theorem (Chazelle '91): A simple n-gon can be triangulated in time $O(n)$.



Hyperbolic space: Metric on $\mathbb{R}^3_+$,

$$d\rho = |dz|/\mathrm{dist}(z, \mathbb{R}^2).$$

Geodesics are circles or lines orthogonal to $\mathbb{R}^2$.

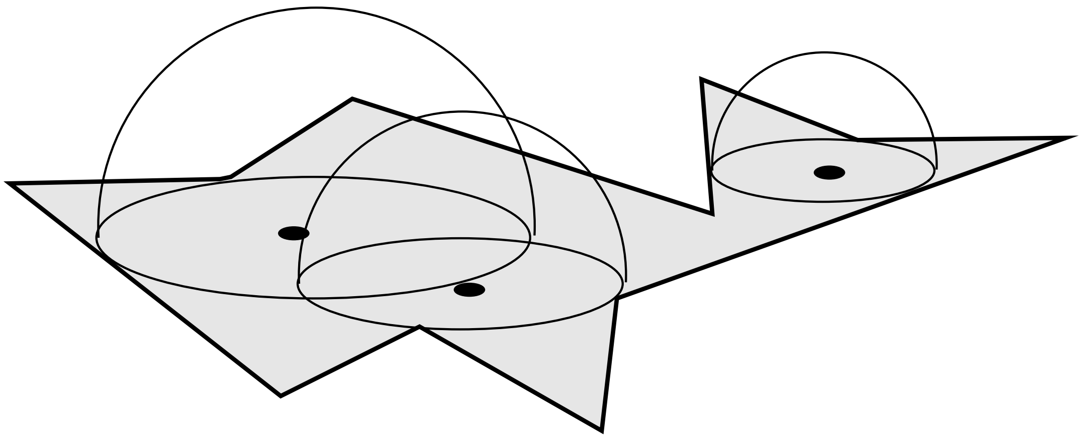


The hyperbolic metric on the disk or ball is

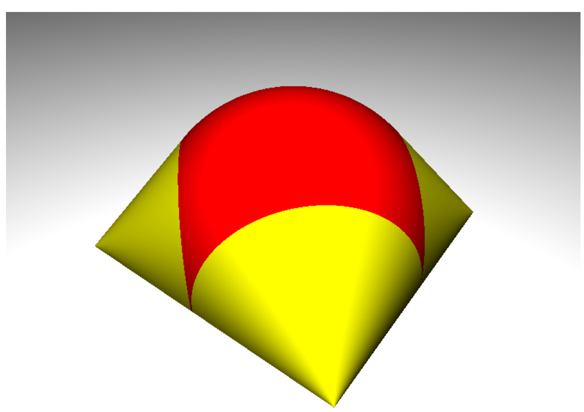
$$d\rho = 2|dz|/(1 - |z|^2).$$

The hyperbolic metric on a simply connected domain plane Ω is defined by transferring the metric on the disk by the Riemann map.

The **dome** of Ω is boundary of union of all hemispheres with bases contained in Ω.



Equals boundary of hyperbolic convex hull of Ω^c. Similar to Euclidean space where complement of closed convex set is a union of half-spaces.



Each point on Dome(Ω) is on dome of a maximal disk D in Ω. Must have $|\partial D \cap \partial\Omega| \geq 2$. The centers of these disks form the **medial axis**.

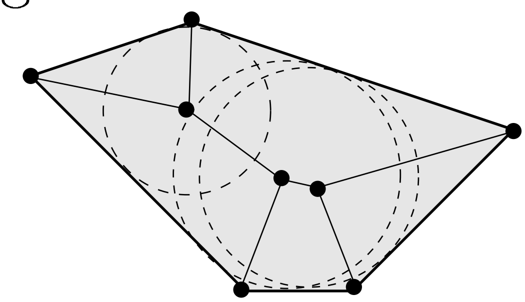
For polygons the medial axis is a finite tree. Three type of edges:

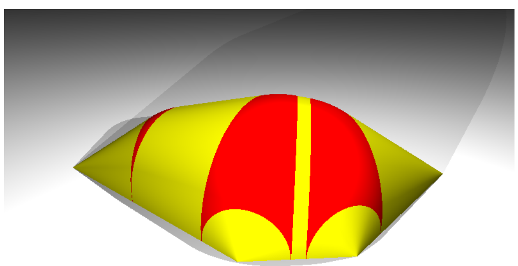
- point-point bisctors (straight)
- edge-edge bisectors (straight)
- point-edge bisector (parabolic arc)

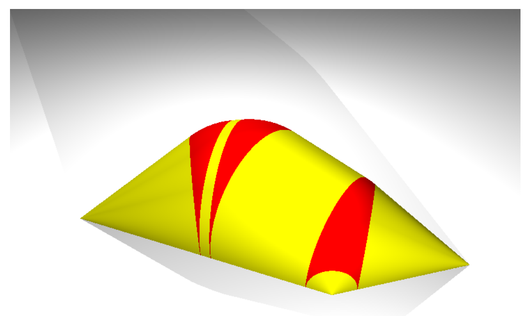
For general simply connected domains it is $\mathbb{R}$-tree.

The intersection Dome(D)$\cap$Dome(Ω) is either an infinite geodesic (**bending line**) or a **geodesic face** bounded by geodesics.

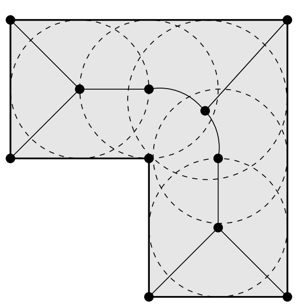
A convex polygon:

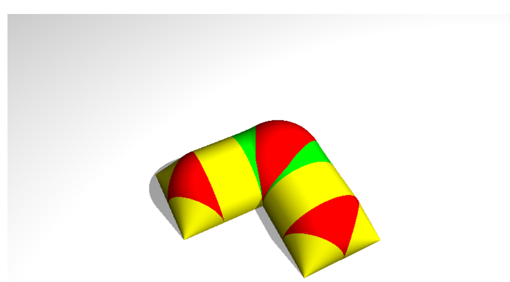






A non-convex polygon:



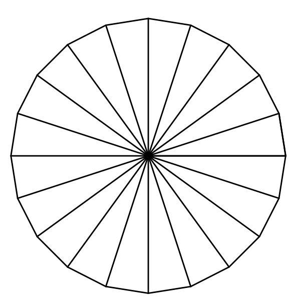


Applications of the medial axis:

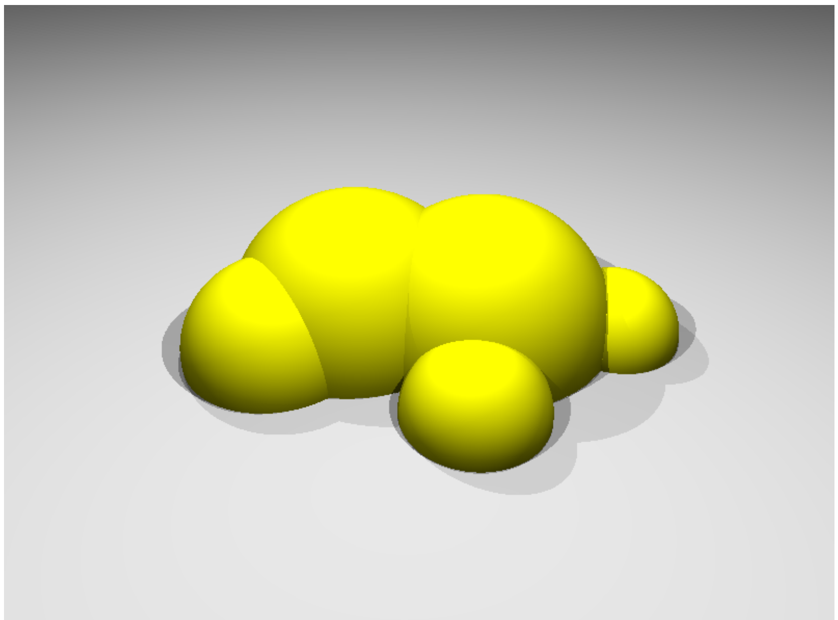
- Analysis of chromosones
- Designs of type fonts
- Describe statisical features of porus materials
- Shape recognition
- Time critical collision detection
- Robotic motion
- Biological description of shape
- Mesh generation
- Computer vision
- Radiosurgery

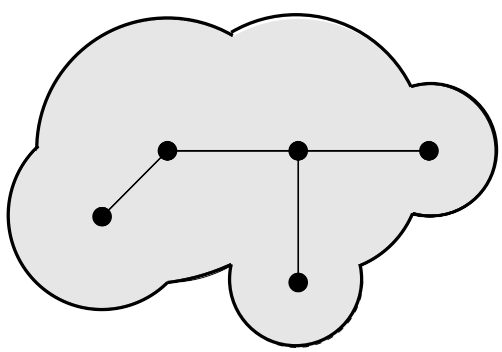
`http://www.ics.uci.edu/~eppstein/gina/medial.html`

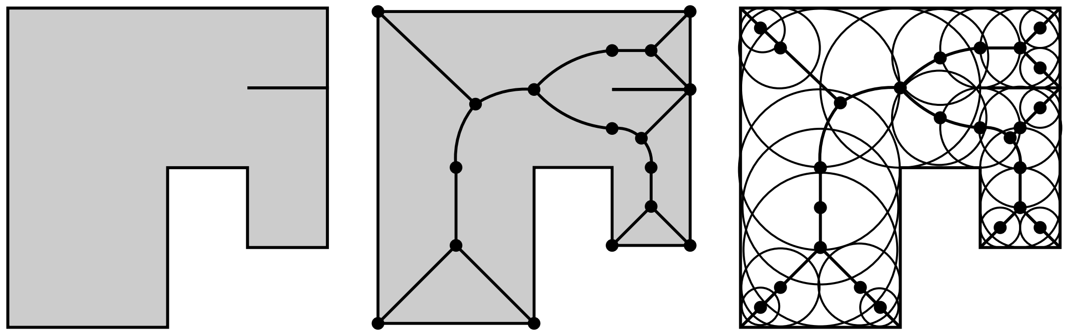
But medial axis is unstable, e.g., perturb disk to regular n-gon.

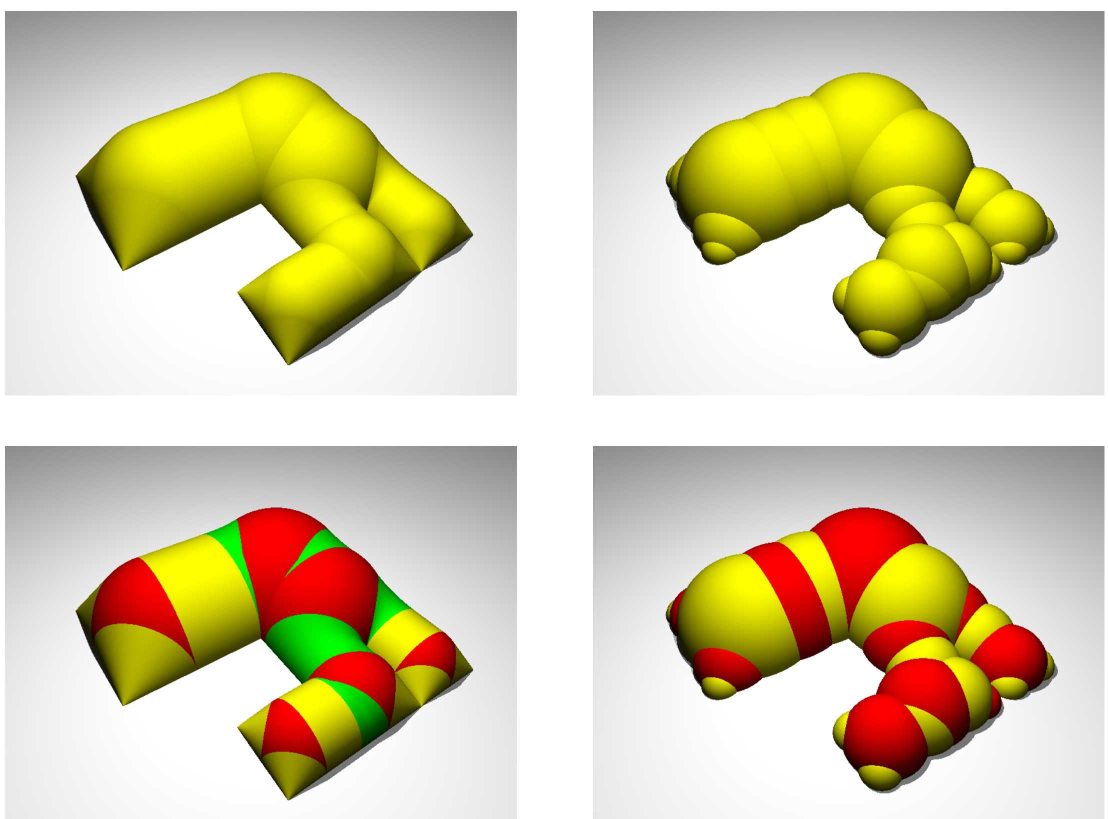


Finitely bent domain (= finite union of disks).





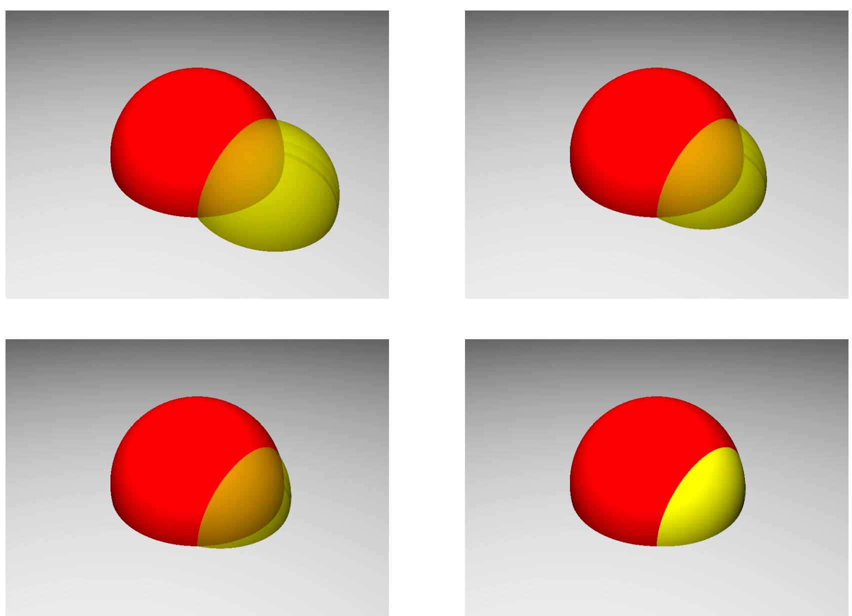




Let ρ_S be the hyperbolic path metric on S.

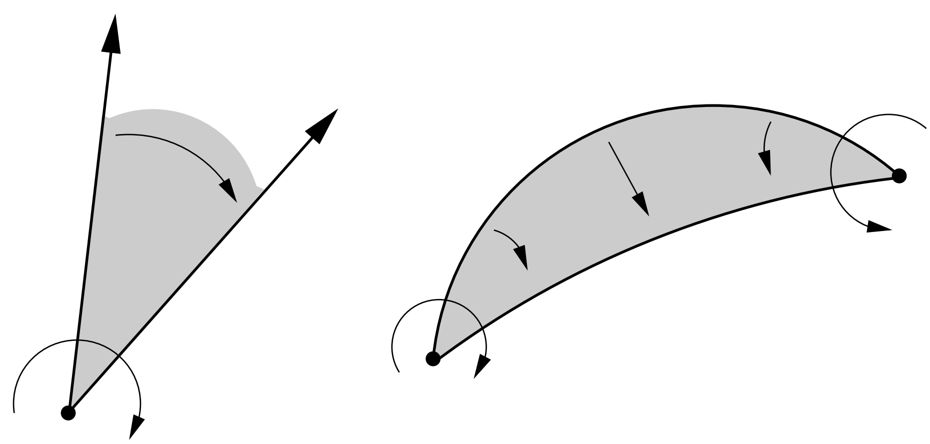
Theorem (Thurston): There is an isometry ι from (S, ρ_S) to the hyperbolic disk.

For finitely bent domains rotate around each bending geodesic by an isometry to remove the bending (more obvious if vertices are 0 and ∞).



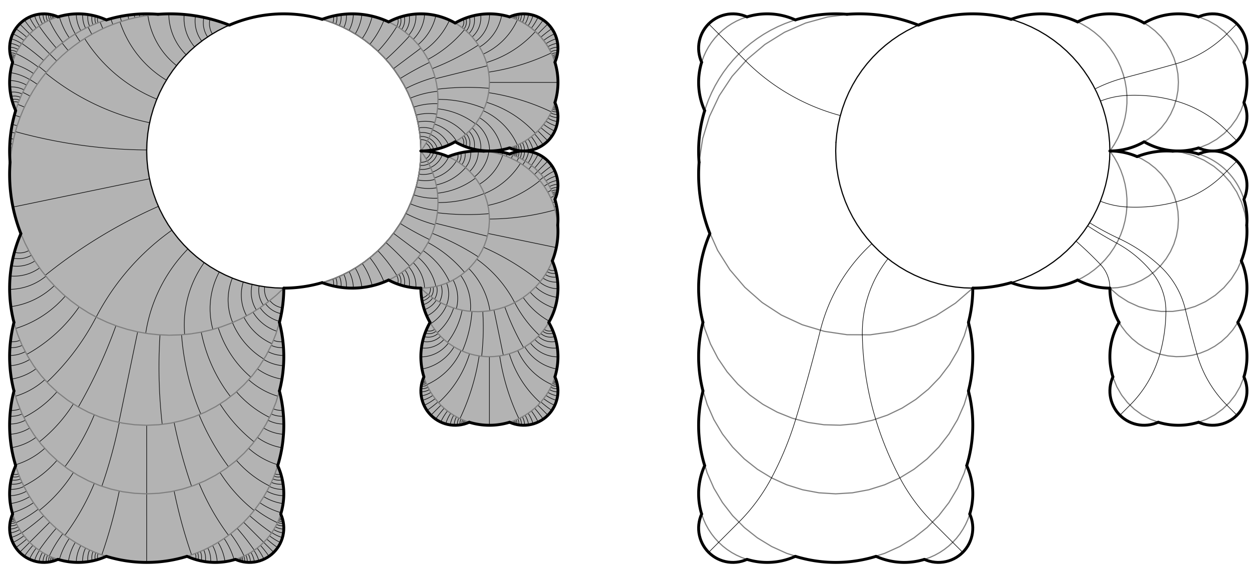
The map $\iota : \partial\Omega \to \partial\mathbb{D}$ gives **Step 1** of our proof.

Elliptic Möbius transformation is conjugate to a rotation.



Elliptic transformation determined by fixed points and angle of rotation θ. It identifies sides of a crescent of angle θ: think of flow along circles orthogonal to boundary arcs.

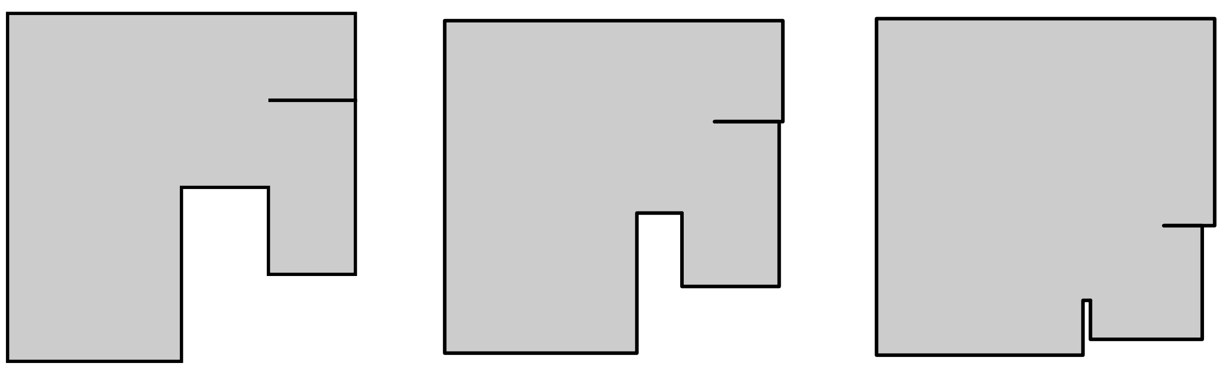
Visualize ι as a flow: For a finitely bent domain, write Ω as a disk D and a union of crescents. Foliate crescents by orthogonal circles. Following leaves of foliation in $\Omega \setminus D$ gives $\iota : \partial\Omega \to \partial D$.



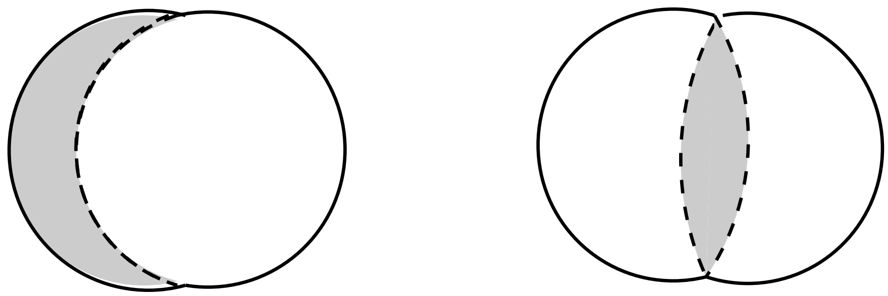
Has continuous extension to interior: identity on disk and collapses orthogonal arcs to points.

- ι has K-QC extension to interior.
- ι can be evaluated at n points in time $O(n)$.

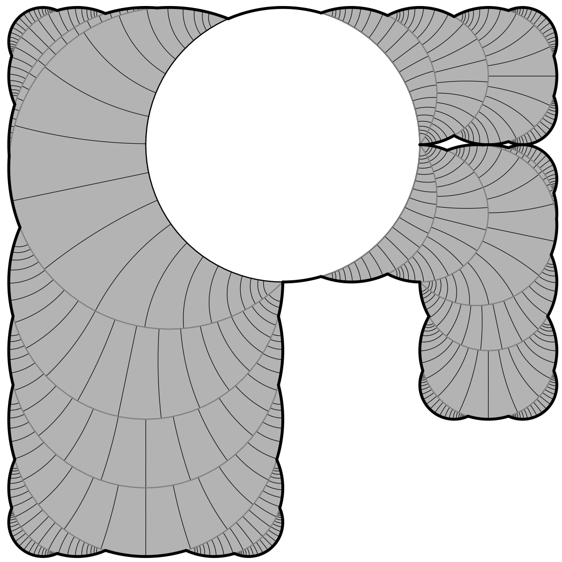
If we plug these parameters in Schwarz-Christoffel formula we almost get the correct polygon (center). Using uniformly spaced points is clearly worse (right).



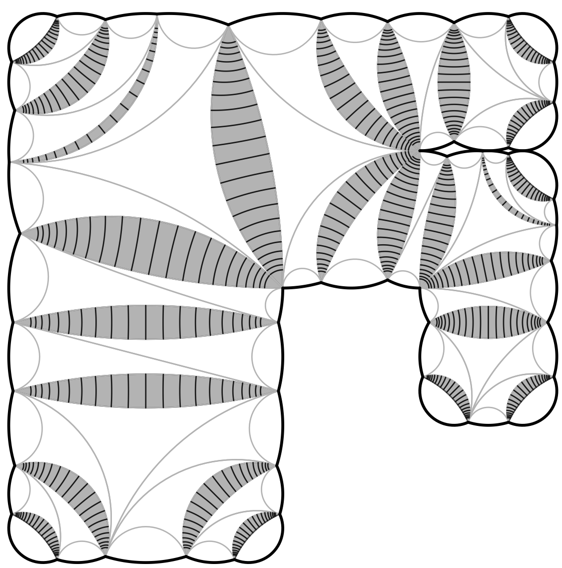
There are at least two ways to decompose a finite union of disks using crescents (with same angles and vertices in both cases).

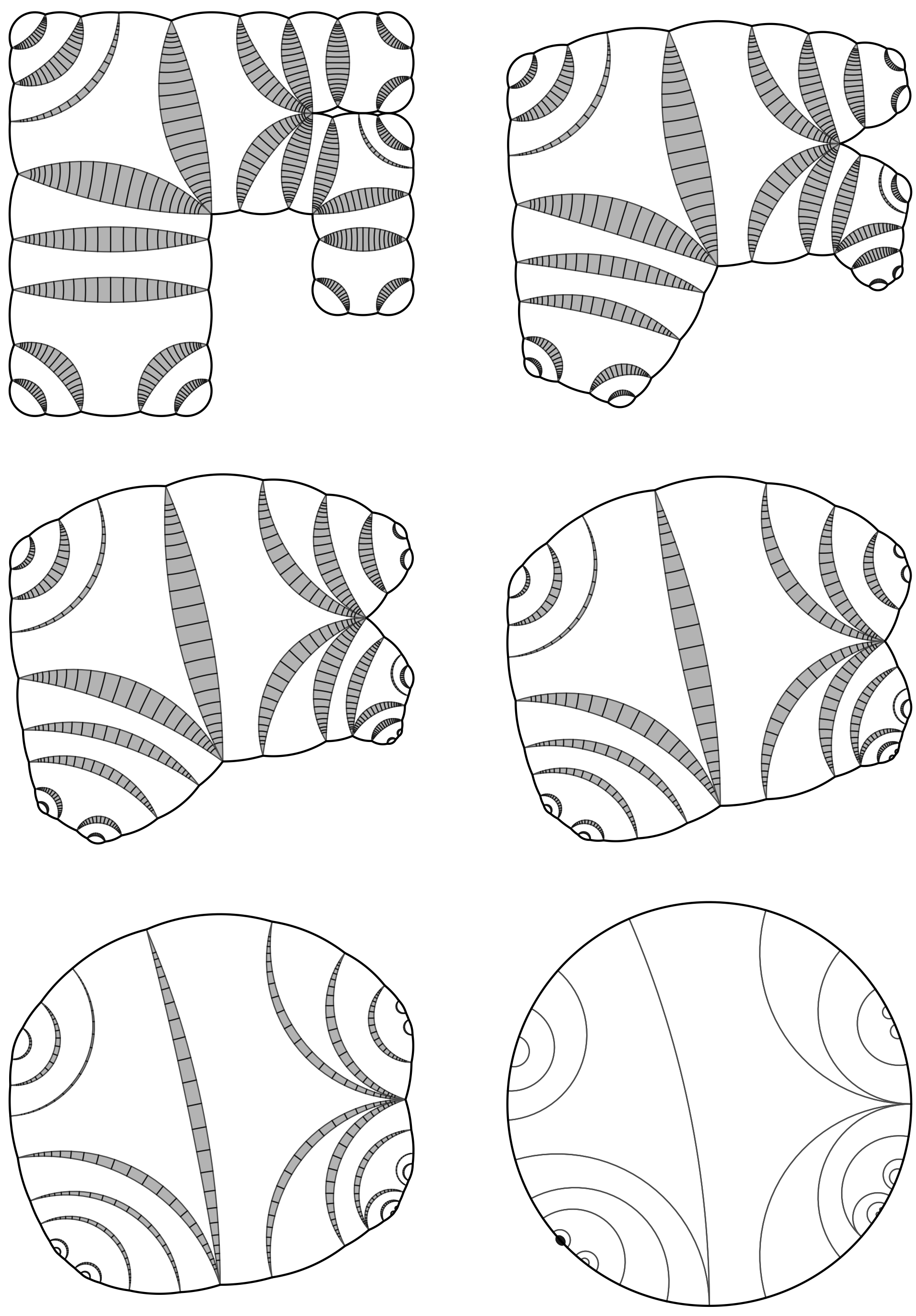


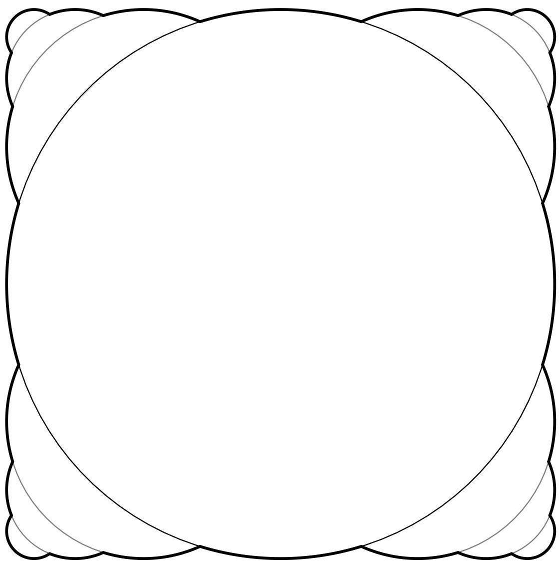
Decomposition with tangential crescents

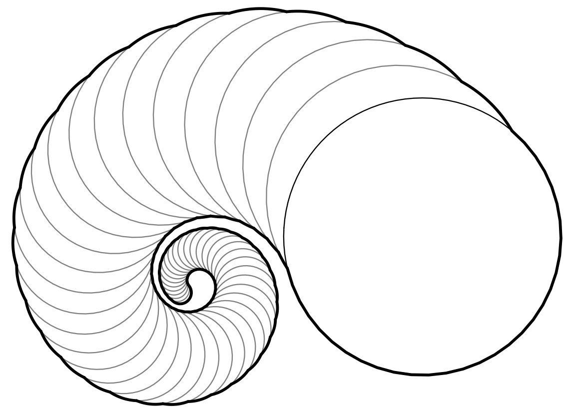


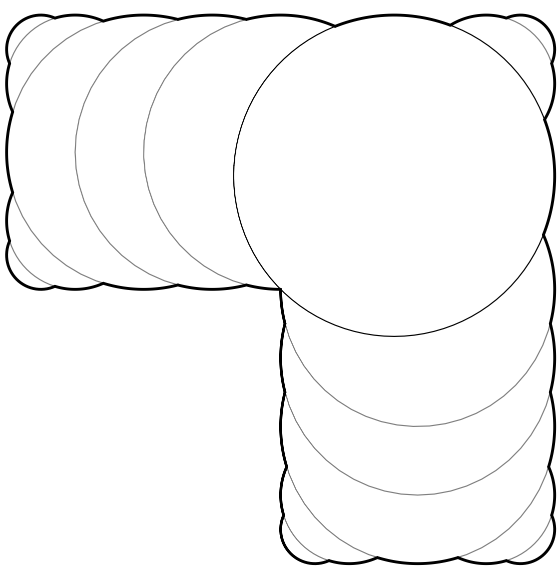
Decomposition with normal crescents

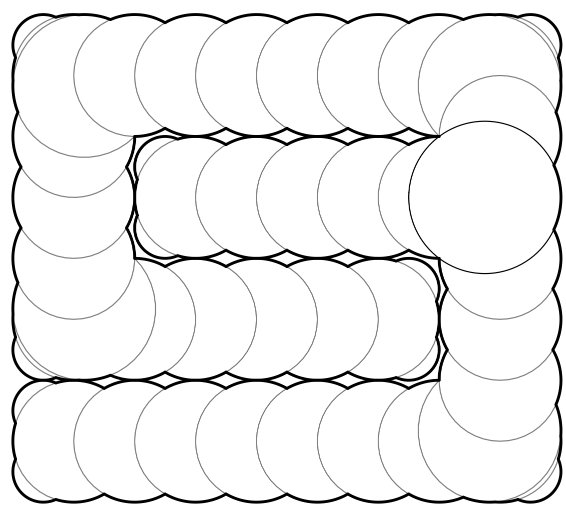


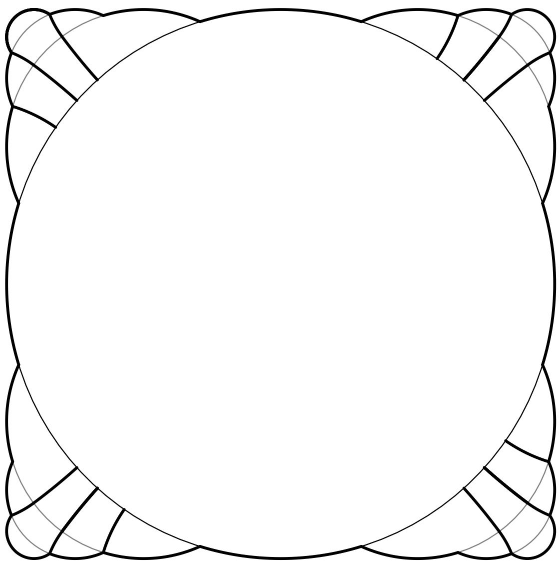


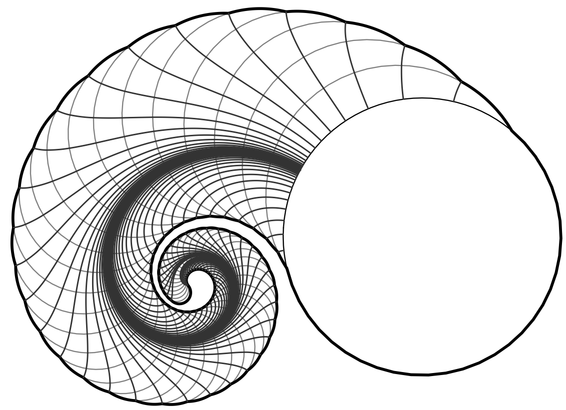


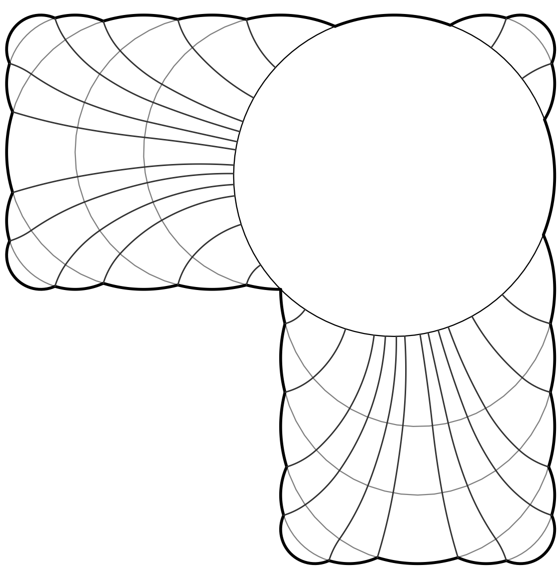


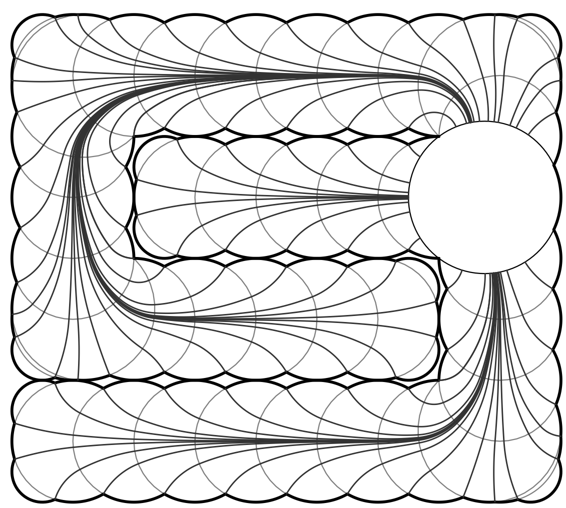


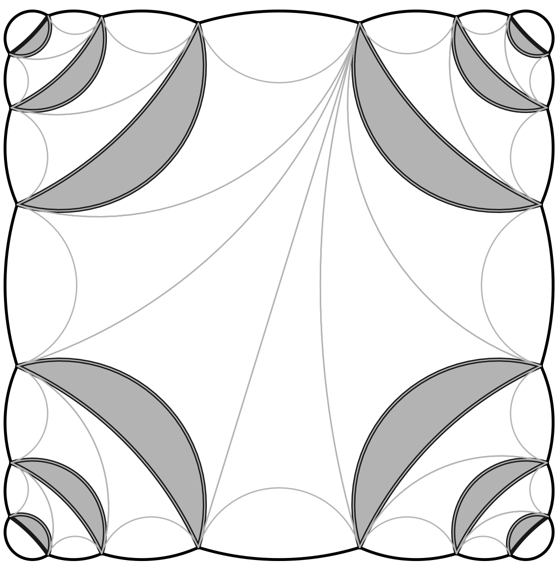


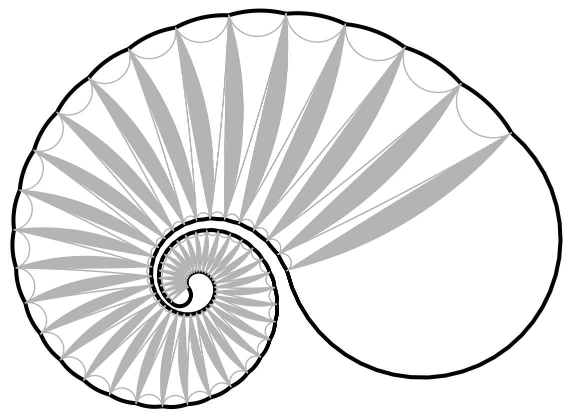


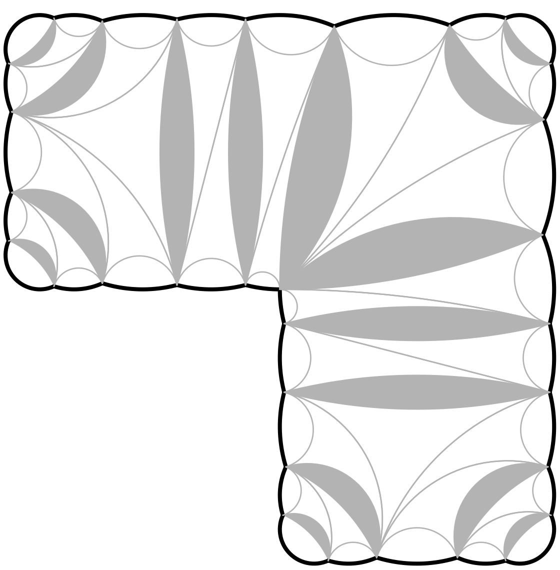


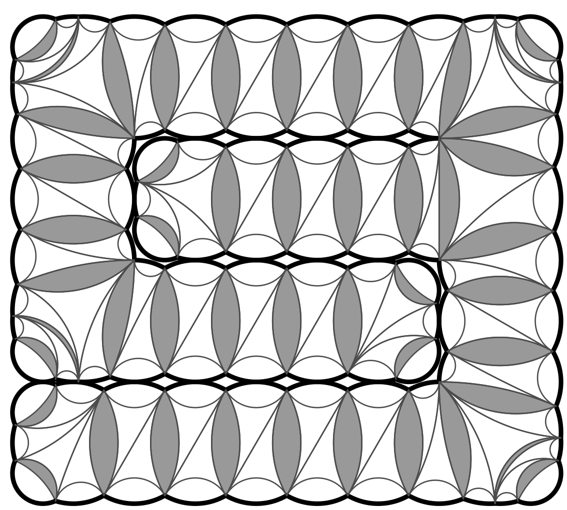








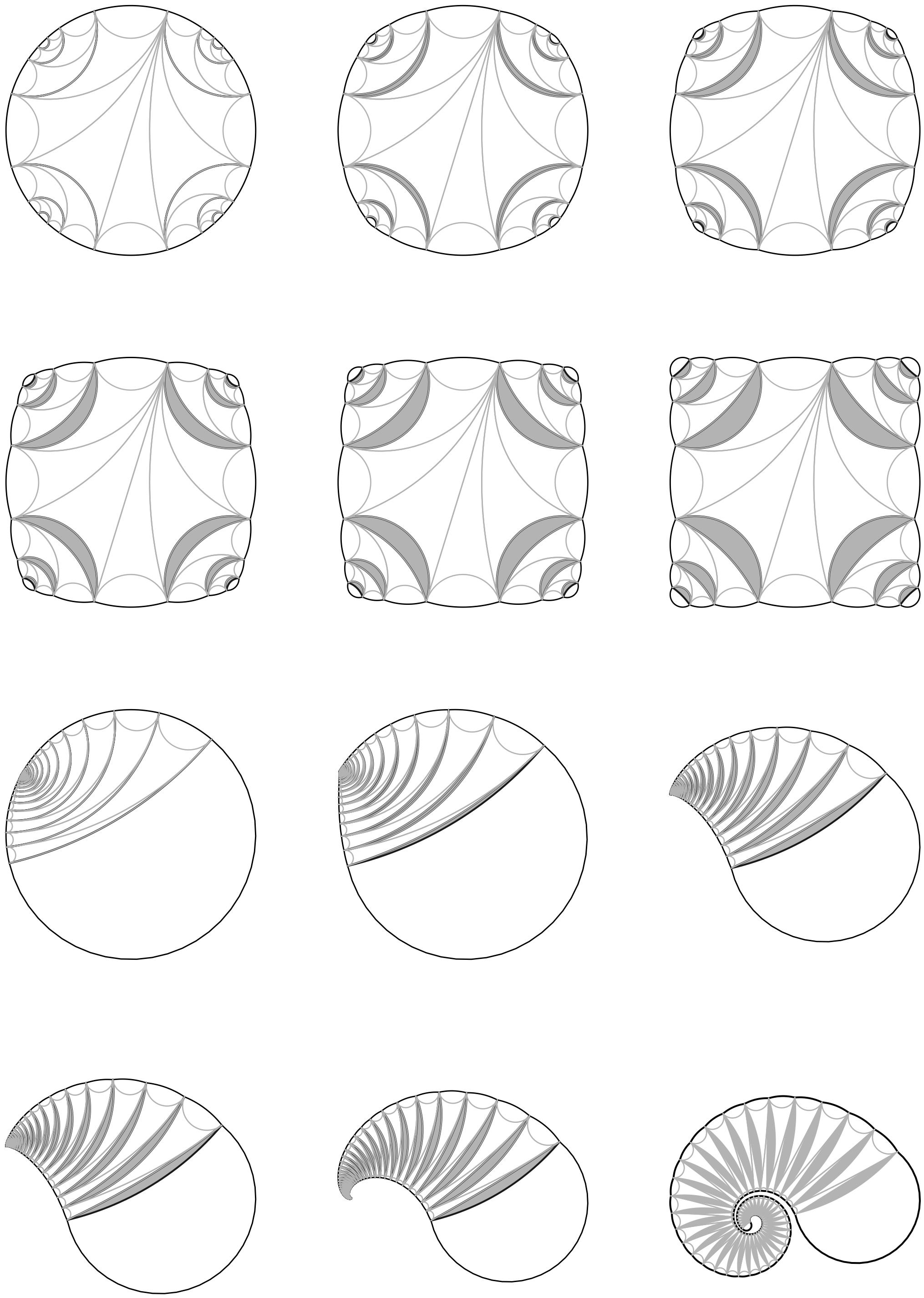


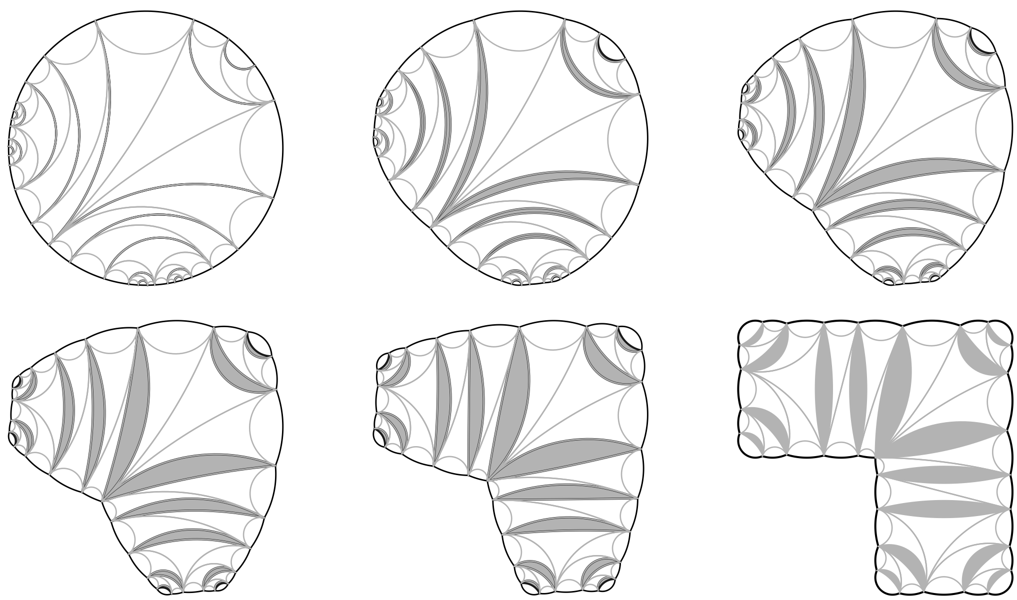


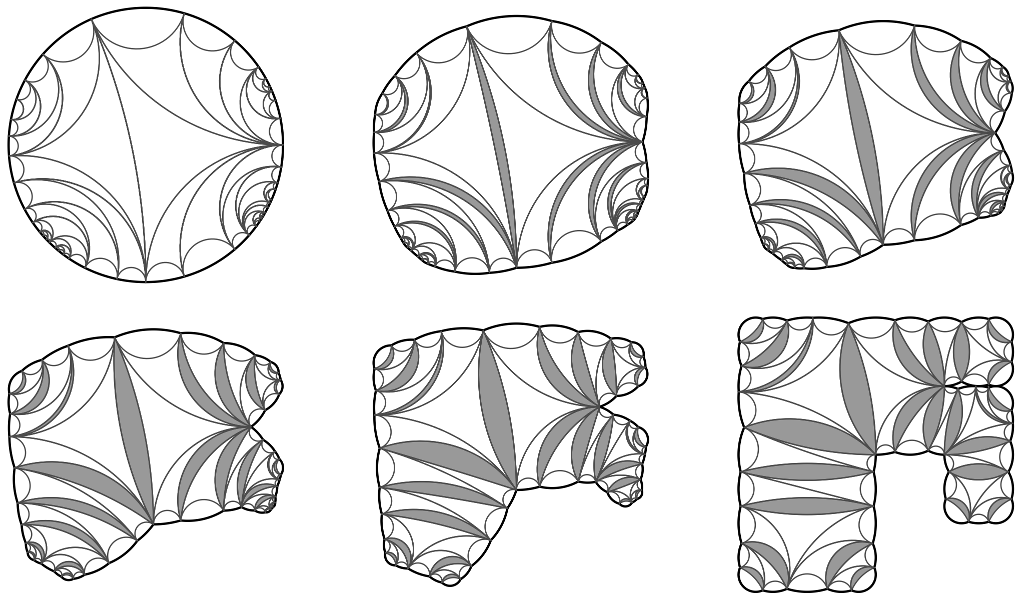
We can visualize the map ι by deforming Ω into $\mathbb{D}$ by **angle scaling**. Multiply the angle of each crescent by a factor $t \in [0, 1]$ and apply Möbius transformations on gaps. Then $t = 1$ gives Ω and $t = 0$ is disk (every crescent collapsed to an arc).

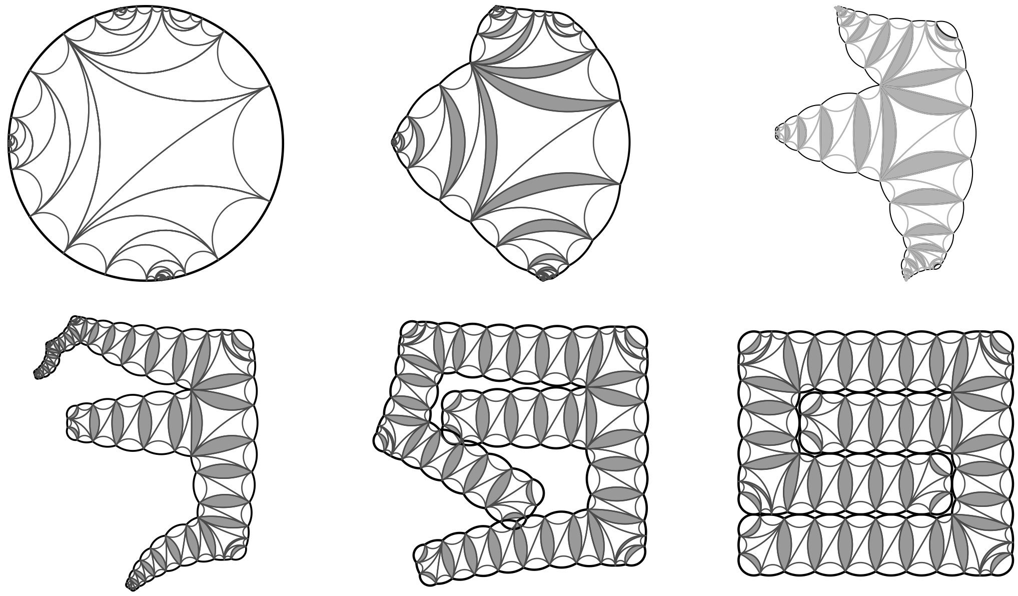
Discretizing this one-parameter family gives chain of domains needed in Step 3 of algorithm. Can show each domain is QC-close to adjacent ones. Hard part is to construct angle scaling map (or approximation) in linear time.

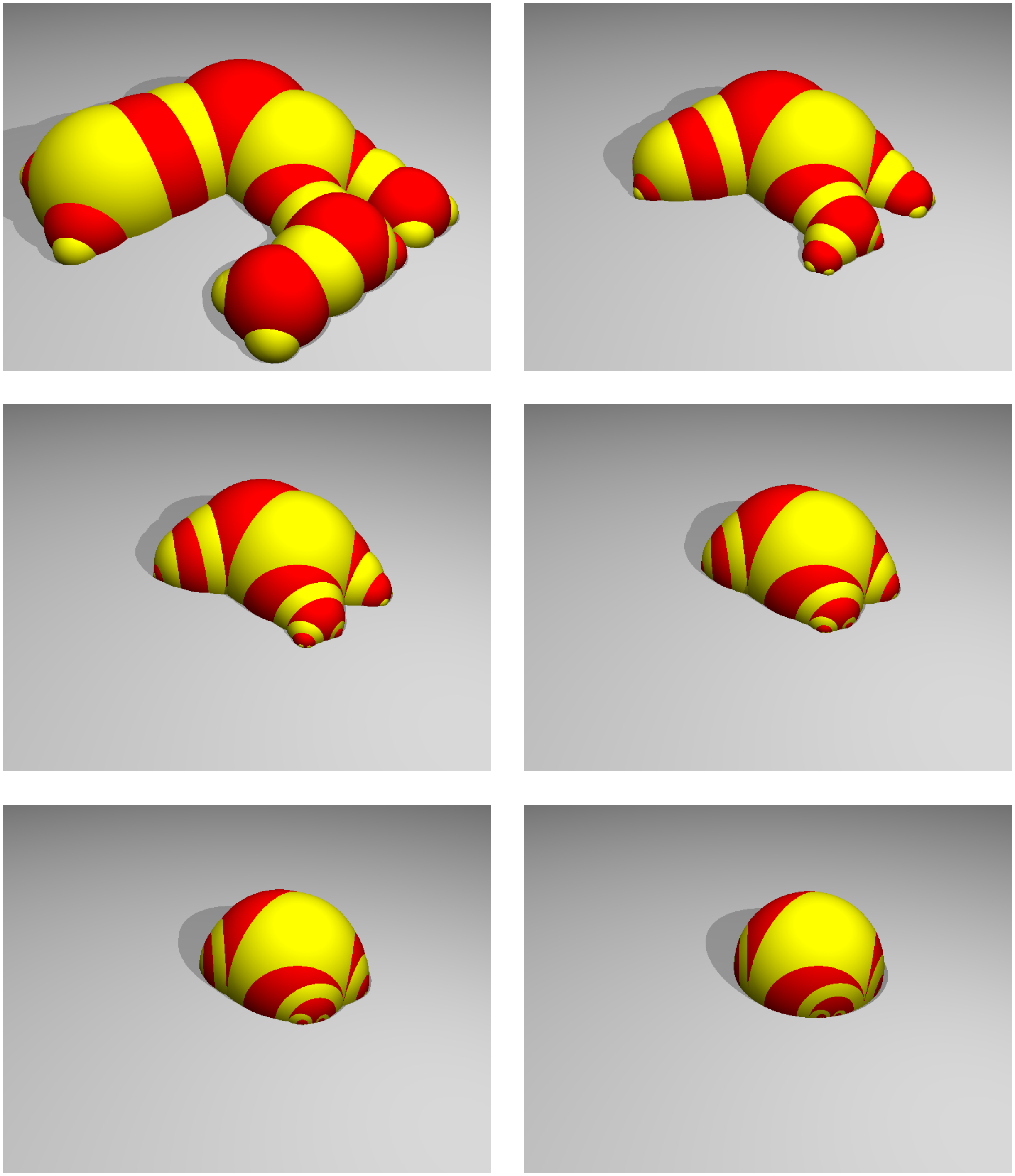
One can also angle scale with complex parameters, but we don't need this in current problem. Angle scaling is an example of a **holomorphic motion** of $\partial\Omega$. Can that theory be applied to computational problems?

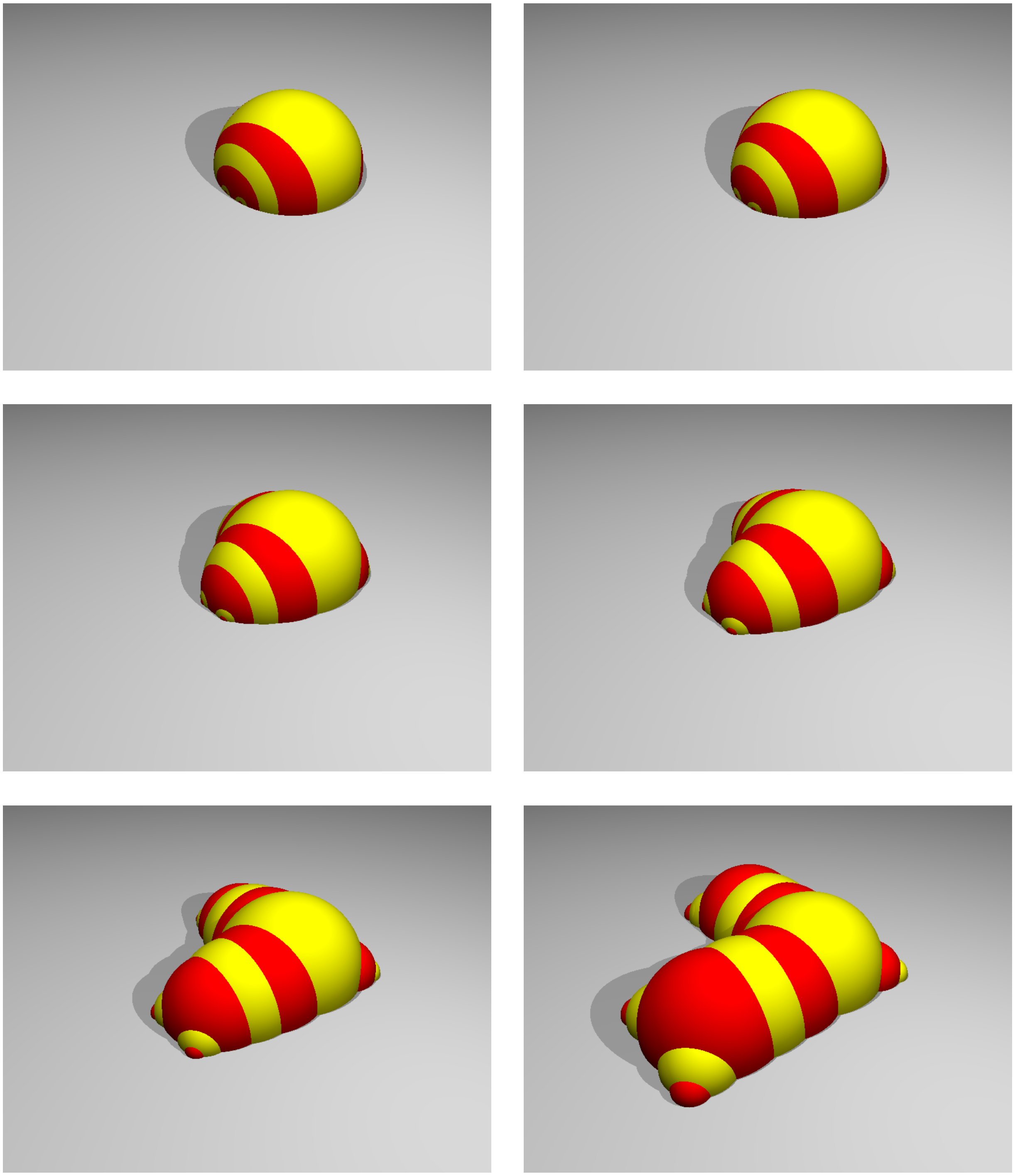




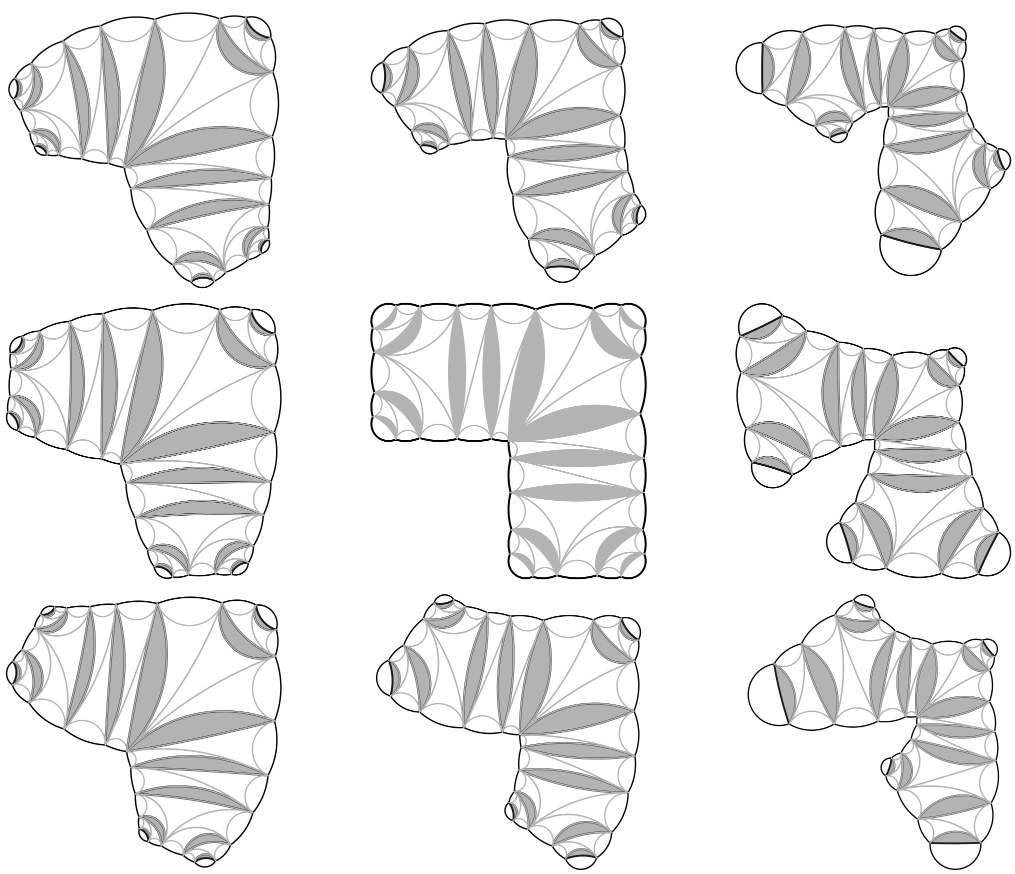








Example of complex angle scaling.



Theorem: Collapsing normal crescents gives hyperbolic quasi-isometry $R : \Omega \to \mathbb{D}$.

Corollary: ι has a K-QC extension to interior.

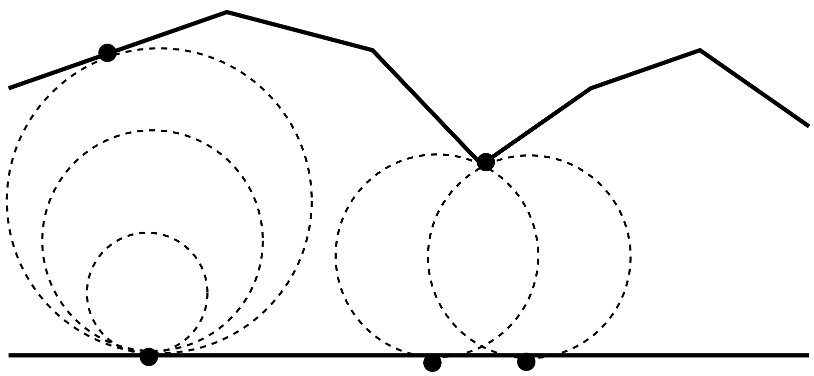
Corollary (Sullivan, Epstein-Marden): There is a K-QC map $\sigma : \Omega \to S_\Omega$ so that $\sigma = \mathrm{Id}$ on $\partial\Omega = \partial S$.

If Ω is invariant under Möbius group G, $M = \mathbb{R}^3_+/G$ is hyperbolic manifold,

$$\partial_\infty M = \Omega/G, \quad \partial C(M) = \mathrm{Dome}/G.$$

This gives theorem about hyperbolic 3-manifolds mentioned earlier. Best current bounds are $K \leq 7.82$ (B) and $K > 2.1$ (Epstein-Markovic).

Nearest point retraction $R : \Omega \to \mathrm{Dome}(\Omega)$: Expand ball tangent at $z \in \Omega$ until it hits a point $R(z)$ of the dome.



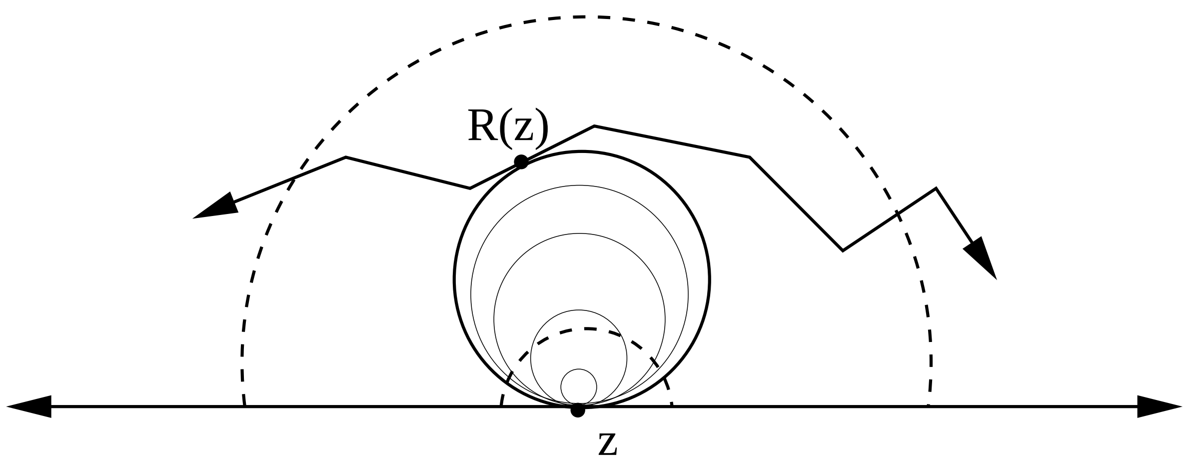
$$\text{normal crescents} = R^{-1}(\text{bending lines})$$
$$\text{gaps} = R^{-1}(\text{faces})$$

collapsing crescents = nearest point retraction

Suffices to show nearest point retraction is a quasi-isometry. This follows from three easy facts.

Fact 1: If $z \in \Omega$,

$$\operatorname{dist}(z, \partial\Omega) \simeq \operatorname{dist}(R(z), \mathbb{R}^2) \simeq |z - R(z)|.$$



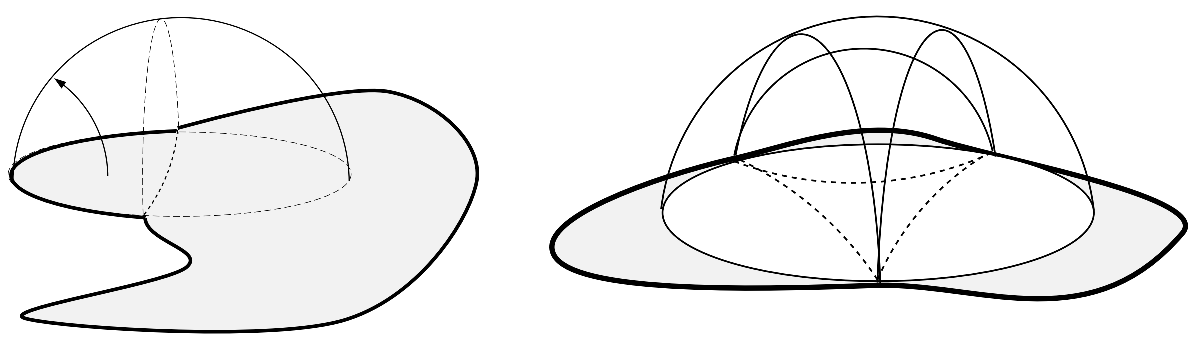
Fact 2: R is Lipschitz. Ω simply connected $\Rightarrow$

$$d\rho \simeq \frac{|dz|}{\mathrm{dist}(z, \partial\Omega)}.$$

$z \in D \subset \Omega$ and $R(z) \in \mathrm{Dome(D)} \Rightarrow$

$$\mathrm{dist}(z, \partial\Omega) \lesssim \mathrm{dist}(z, \partial D) \leq \mathrm{dist}(z, \partial\Omega)$$

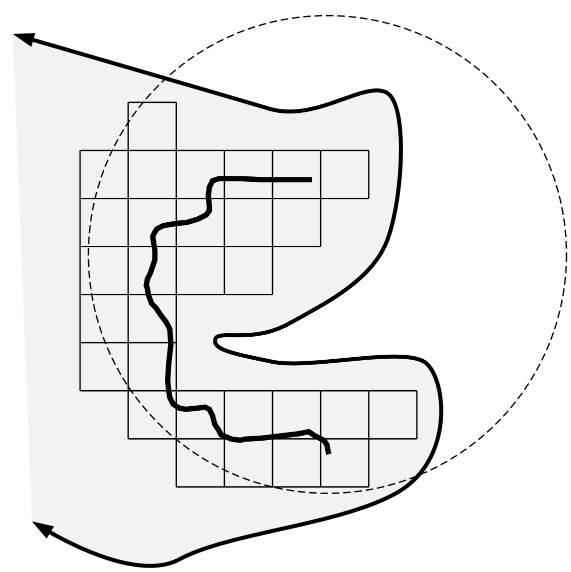
$$\Rightarrow \quad \rho_\Omega(z) \simeq \rho_D(z) = \rho_{\mathrm{Dome}}(R(z)).$$



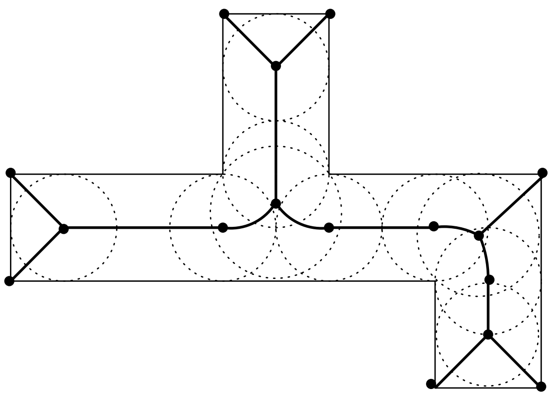
Fact 3: $\rho_S(R(z), R(w)) \le 1 \Rightarrow \rho_\Omega(z,w) \le C$.

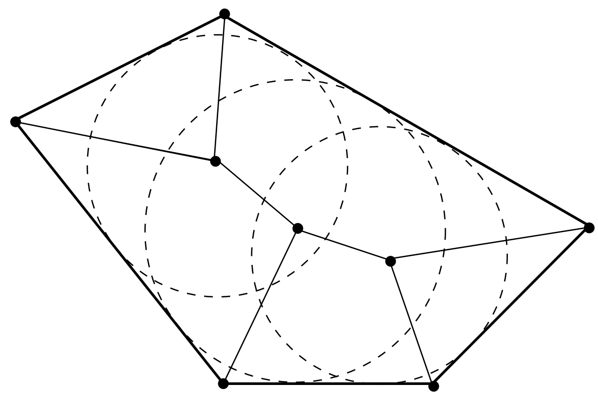
Suppose $\mathrm{dist}(R(z), \mathbb{R}^2) = r$ and γ is geodesic from z to w.

$$\begin{aligned} \Rightarrow \qquad & \mathrm{dist}(\gamma, \mathbb{R}^2) \simeq r \\ \Rightarrow \qquad & \mathrm{dist}(R^{-1}(\gamma), \partial\Omega) \simeq r, \\ & \qquad R^{-1}(\gamma) \subset D(z, Cr) \\ \Rightarrow \qquad & \rho_\Omega(z,w) \le C \end{aligned}$$

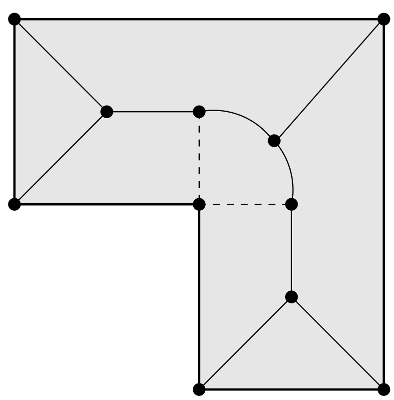


The medial axis: recall MA is the set of points which are equidistant from two or more boundary points. For polygons = centers of maximal disks.





Medial axis is a type of Voronoi diagram:



Given medial axis we can compute ι in time $O(n)$.

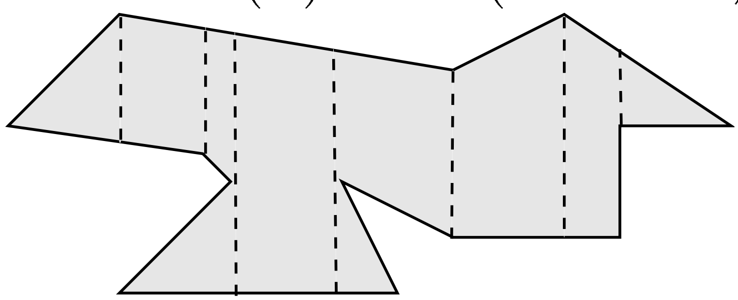
In 1998, Chin-Snoeyink-Wang gave $O(n)$ algorithm to compute the medial axis. There are much simplier $O(n \log n)$ methods which are faster in practice. Key step is:

Merge Lemma: Suppose n sites S are divided into S_1 and S_2 by a line and that the Voronoi diagrams of S_1 and S_2 are given. Then the Voronoi diagram of S can be found in at most $O(n)$ additional time.

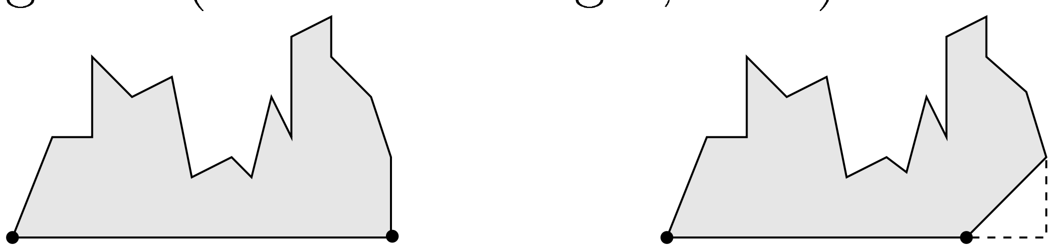
Sort x-coordinates of vertices and group into vertical slabs. Recursively dividing P into two almost equal sized pieces gives a $O(n \log n)$ algorithim (Yap 1993).

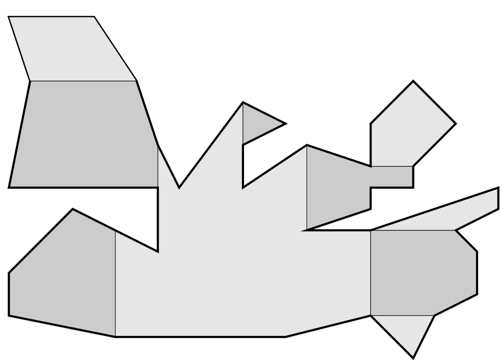
The $O(n)$ estimate is more involved.

• Cut interior of P into trapedzoids with vertical sides. Possible in $O(n)$ time (Chazelle, 1991).

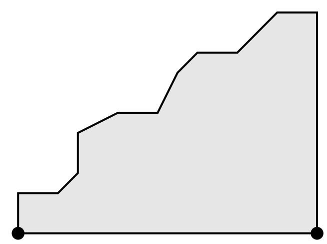


• Use trapezoids to divide P into pseudo-normal histograms (Klein and Lingas, 1993).





• Cut each pseudo-normal histogram into monotone histograms.



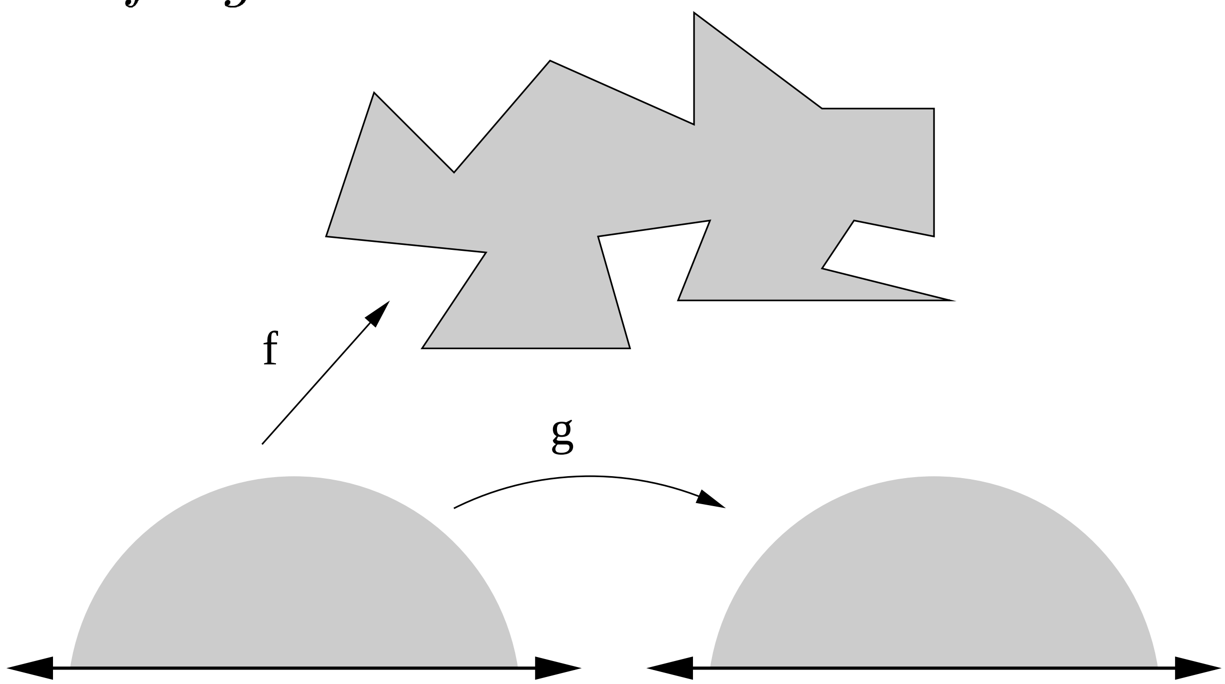
• Compute Voronoi diagrams of monotone histograms (Aggarwal, Guibas, Saxe, Shor, 1989 and Djidev, Lingas 1991)

• Merge monotone diagrams into diagrams for pseudo-normal histograms and merge the results into diagram for P.

Idea for Step 2: Suppose $f_{\bar{z}}/f_z = \mu_f = \mu_g$ and

$$f : \mathbb{H} \to \Omega, \qquad g : \mathbb{H} \to \mathbb{H}.$$

Then $f \circ g^{-1} : \mathbb{H} \to \Omega$ is conformal.

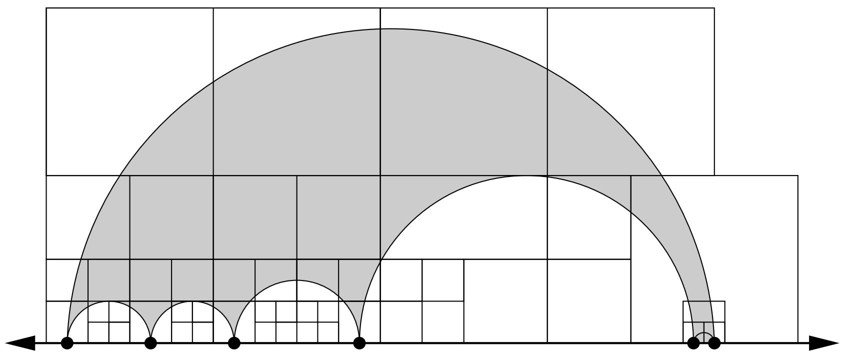


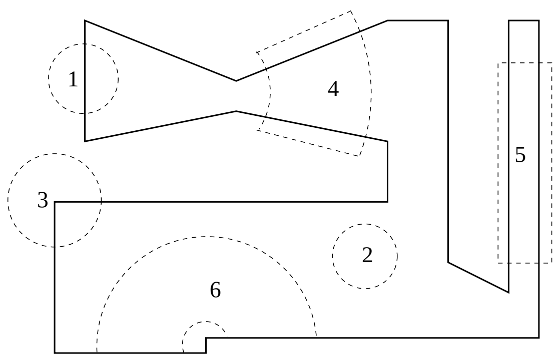
Given μ can't solve $g_{\bar{z}} = \mu g_z$ exactly in finite time, but can quickly solve

$$g_{\bar{z}} = (\mu + O(\|\mu\|^2))g_z.$$

Then $f \circ g^{-1}$ is $(1 + C\|\mu\|^2)$-QC.

Cut $\mathbb{H}$ into $O(n)$ pieces on which f, f^{α} or $\log f$ has nice series representation. Need $p = O(|\log \epsilon|)$ terms on each piece to get ϵ accuracy.





Use partition of unity supported near partition edges to combine expansions. Can compute μ explicitly.

If μ_f is small and f fixes ∞, then f_z is close to constant. Hence Beltrami equation is similar to $f_{\bar{z}} = C\mu$. Solve this by convolving μ with $1/z$.

$$f(z) = \int \frac{d\mu(w)}{z - w},$$

where μ is piecewise defined on each of $O(n)$ squares.

We can't evaluate integral exactly in finite time. But we can approximate it quickly using the fast multipole method of Greengard and Rohklin.

Idea of Multipole method: Suppose we are given n points $x_k \in [-1, 1]$ with weights w_k such that $\sum_k w_k = 1$ and n points $y_k \in [2, 4]$. How long does it take to evaluate

$$F(y) = \sum_{k=1}^{n} \frac{w_k}{y - x_k},$$

at all n points $\{y_k\}$?

Exact evaluation takes $\approx n^2$ steps.

Fast multipole method gives approximate evaluation in $O(n)$ steps!

$$\frac{1}{y-x} = \frac{1}{y}\sum_{j=0}^{\infty}(\frac{x}{y})^j, \quad |x| \leq 1, |y| \geq 2$$

$$\sum_{k=1}^{n} \frac{w_k}{y-x_k} = \sum_{j=0}^{p}(\sum_{k=0}^{n} w_k x_k^j)y^{-j-1} + O(2^{-p})$$

Work required is $O(pn) = O(n \log \frac{1}{\epsilon})$.

General method groups points into clusters and computes interactions between clusters as above.

Named one of top ten algorithms of 20th century.

1. Monte Carlo method,
2. Simplex method,
3. Krylov Subspace Iteration method,
4. Householder matrix decomposition,
5. Fortran compiler,
6. QR algorithm for eigenvalue calculation,
7. Quicksort algorithm,
8. Fast Fourier Transform,
9. Integer Relation Detection Algorithm,
10. Fast Multipole algorithm,

`http://orion.math.iastate.edu/burkardt/misc/algorithms_dongarra.html`

SIAM News, Volume 33, Number 4, May 2000.

I wouldn't even think of playing music if I was born in these times... I'd probably turn to something like mathematics. That would interest me.

Bob Dylan

"Ah!" replied Pooh. He'd found that pretending a thing was understood was sometimes very close to actually understanding it. Then it could easily be forgotton with no one the wiser...

Winnie-the-Pooh