Conformal Mapping in Linear Time

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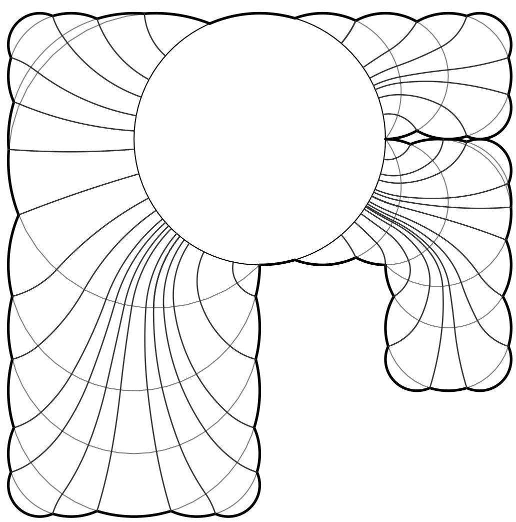
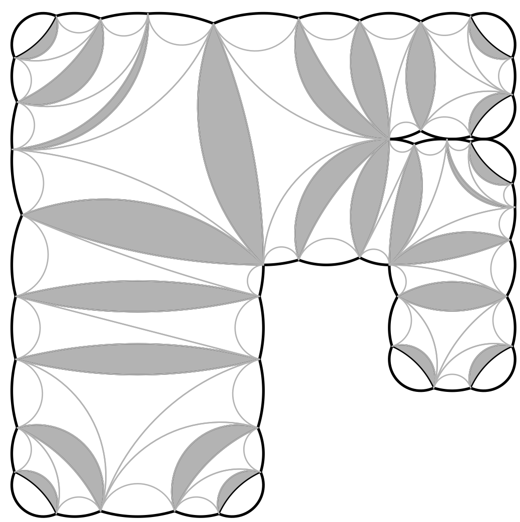




copies of lecture slides available at

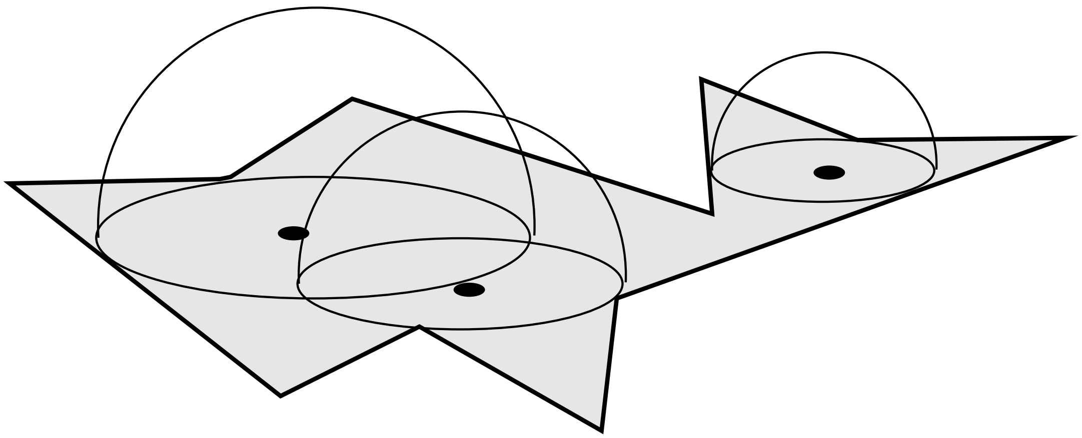
`www.math.sunysb.edu/~bishop/lectures`

“But Holmes, how did you know that any simple n-gon has a quadrilateral mesh with $O(n)$ pieces and all angles between 60° and 120°?”

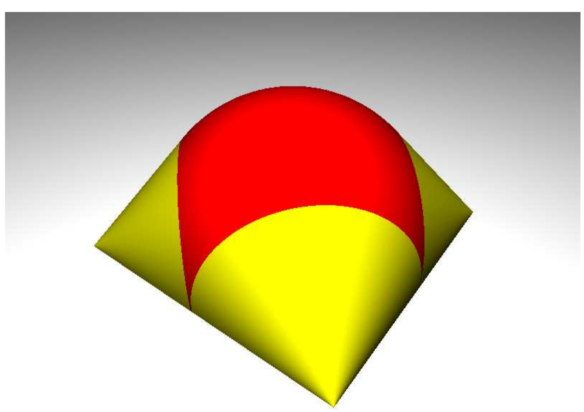
“Surely you recall from *The Case of the Kleinian Groups* that the boundary of a hyperbolic 3-manifold is bi-Lipschitz equivlent to the boundary of its convex hull. I deduced that the Riemann map from a polygon to the disk can be computed in linear time and the rest is quite elementary my dear Watson.”

(My talk, in the style of Arthur Connan Doyle.)

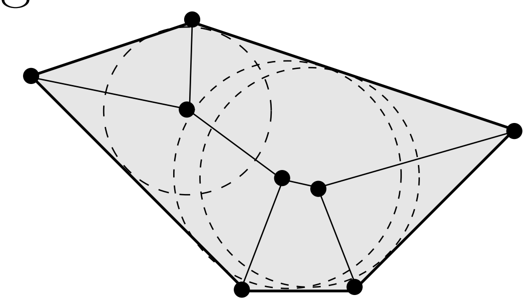
The **dome** of Ω is boundary of union of all hemispheres with bases contained in Ω.

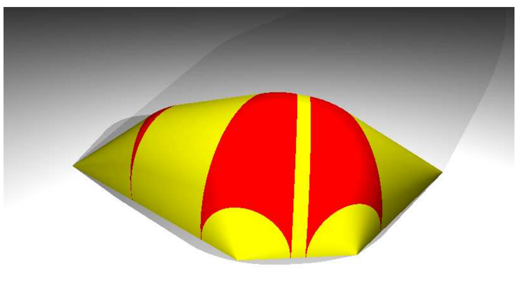


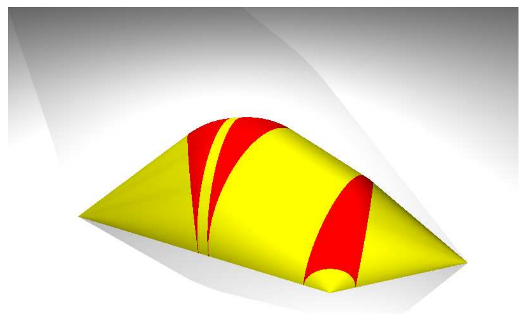
Equals boundary of hyperbolic convex hull of Ω^c. Similar to Euclidean space where complement of closed convex set is a union of half-spaces.



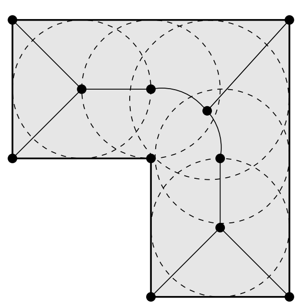
A convex polygon:

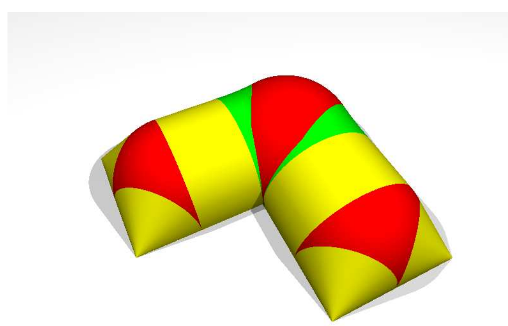






A non-convex polygon:





Hyperbolic half-plane: Metric on $\mathbb{R}^2_+$,

$$d\rho = |dz|/\mathrm{dist}(z, \mathbb{R}^2).$$

Geodesics are circles or lines orthogonal to $\mathbb{R}$.
Hyperbolic disk: Metric on $\mathbb{D}$,

$$d\rho = |dz|/1 - |z|^2.$$

Geodesics are circles or lines orthogonal to $\partial\mathbb{D}$.

The hyperbolic metric on a simply connected domain plane Ω is defined by transferring the metric on the disk by the Riemann map.

Important Fact: $\rho \simeq \tilde{\rho}$ where

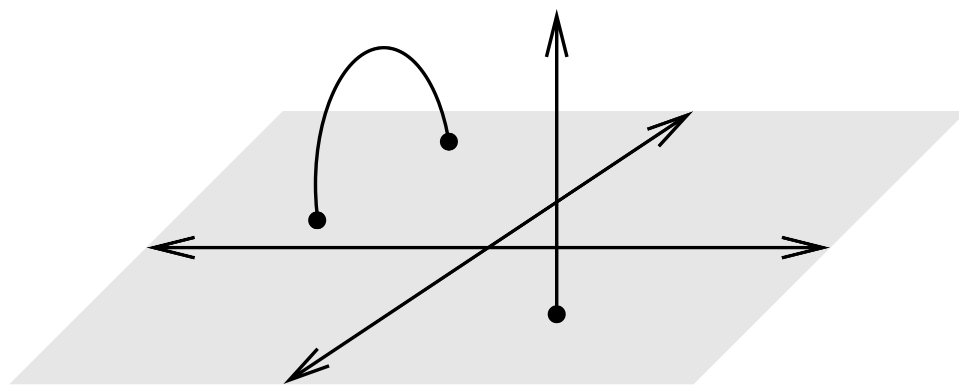
$$d\tilde{\rho} = \frac{|dz|}{\mathrm{dist}(z, \partial\Omega)},$$

is pseudo-hyperbolic metric.

Hyperbolic space: Metric on $\mathbb{R}^3_+$,

$$d\rho = |dz|/\text{dist}(z, \mathbb{R}^2).$$

Geodesics are circles or lines orthogonal to $\mathbb{R}^2$.



The dome of Ω is boundary of hyperbolic convex hull of Ω^c.

Sullivan-Epstein-Marden found bi-Lipschitz map from base to dome, fixes boundary.

Used to be hard; now is easy.

Each point on Dome(Ω) is on dome of a maximal disk D in Ω. Must have $|\partial D \cap \partial \Omega| \geq 2$. The centers of these disks form the **medial axis**.

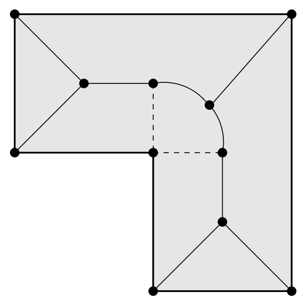
For polygons is a finite tree with 3 types of edges:
- point-point bisectors (straight)
- edge-edge bisectors (straight)
- point-edge bisector (parabolic arc)

For applications see:
www.ics.uci.edu/ eppstein/gina/medial.html+

In CS is attributed to Blum (1967), but Erdös proved dim(MA) = 1 in 1945.

Goggle("medial axis")= 26,300
Goggle("hyperbolic convex hull")= 71

Medial axis is boundary of Voronoi cells:

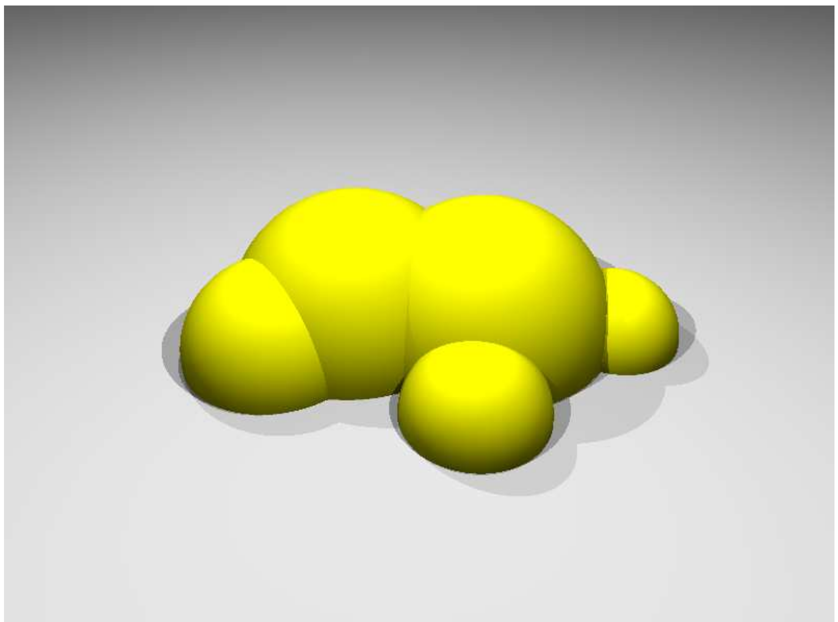


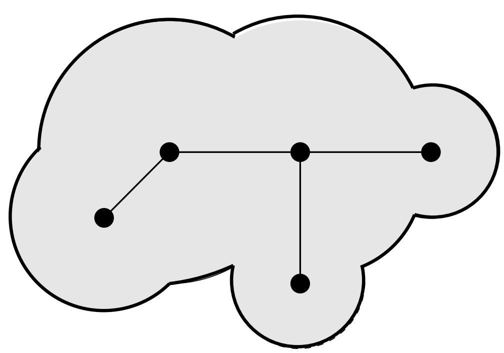
Chin-Snoeyink-Wang (1998) gave $O(n)$ algorithm. Uses Chazelle' theorem (1991): an n-gon can be triangulated in $O(n)$ time.

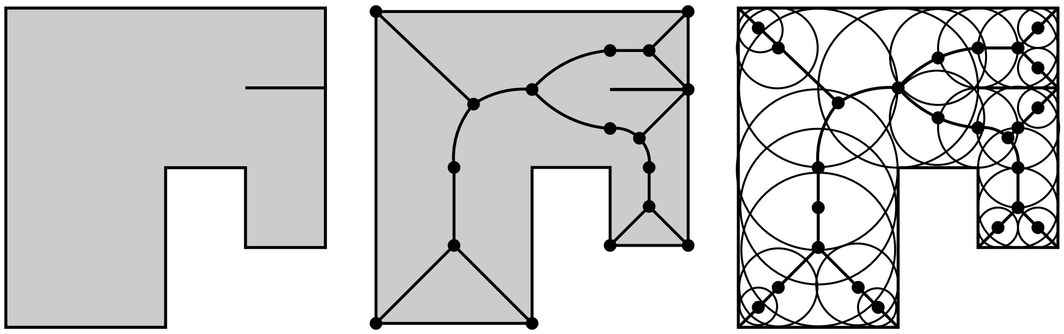
They use this to divide polygon into almost convex regions ("monotone histograms"); compute for each piece (Aggarwal-Guibas-Saxe-Shor, 1989) and merge results.

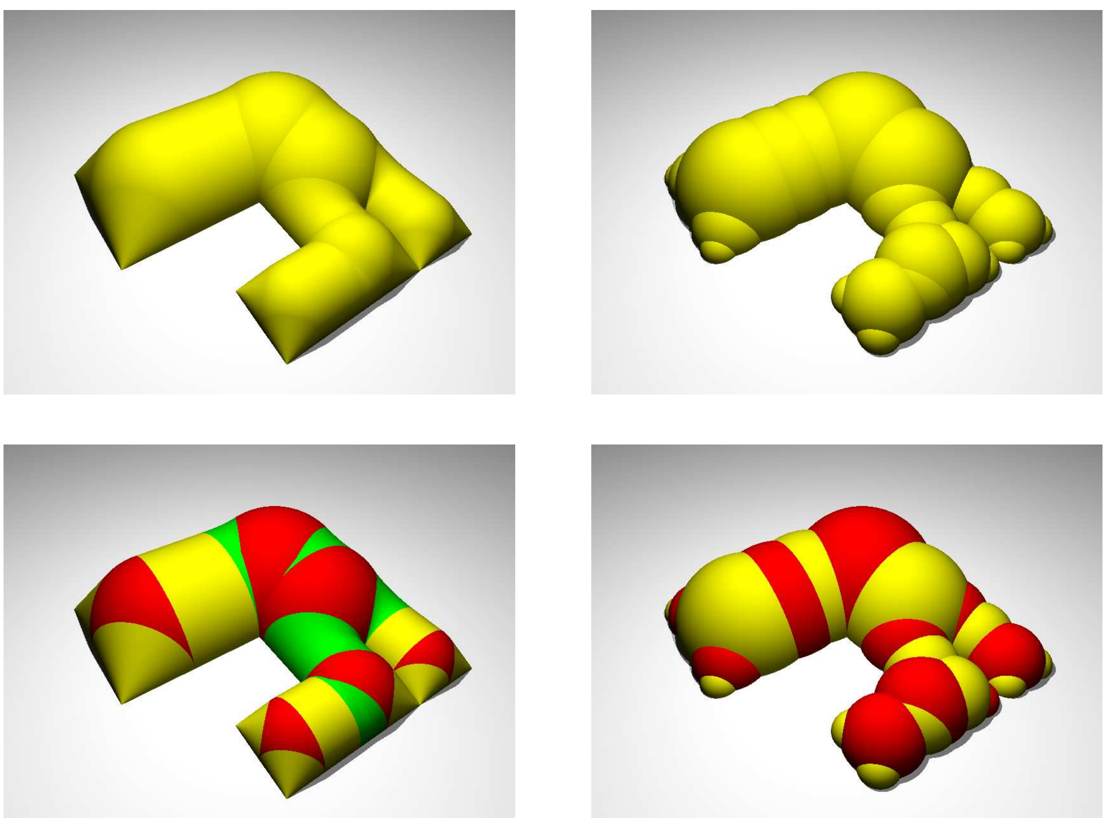
Merge Lemma: Suppose n sites $S = S_1 \cup S_2$ are divided by a line. Then diagram for S can be built from diagrams for S_1, S_2 in time $O(n)$.

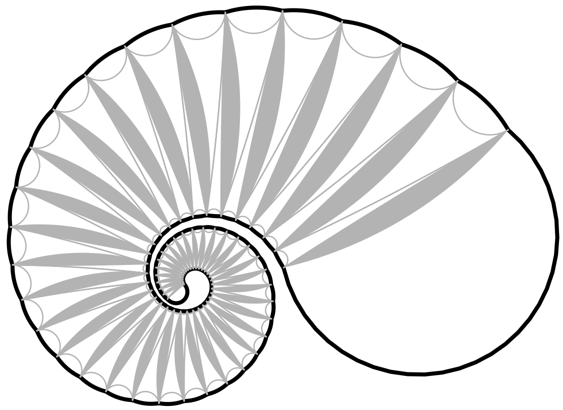
Finitely bent domain (= finite union of disks).

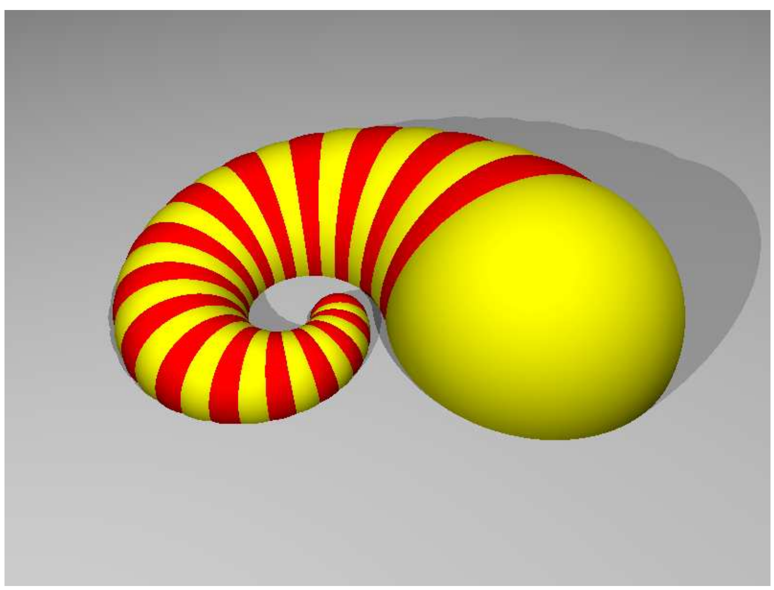








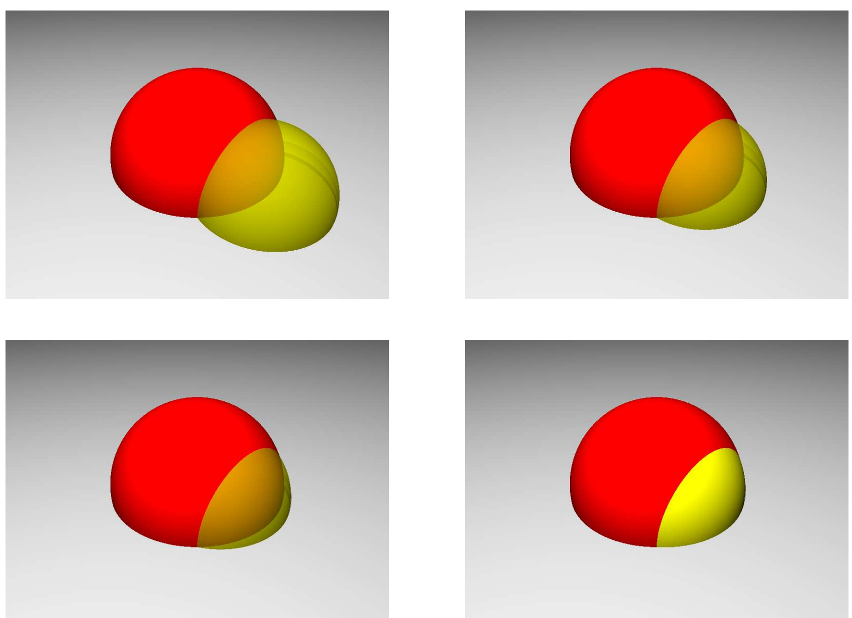




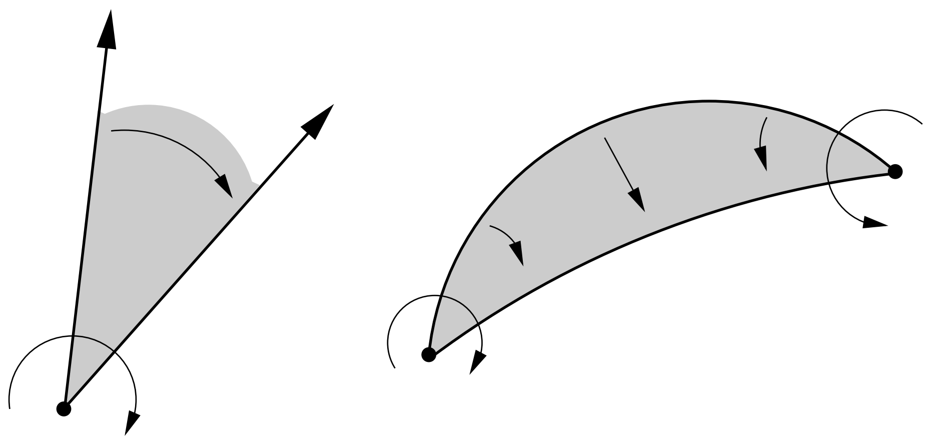
Let ρ_S be the hyperbolic path metric on S.

Theorem (Thurston): There is an isometry ι from (S, ρ_S) to the hyperbolic disk.

For finitely bent domains rotate around each bending geodesic by an isometry to remove the bending (more obvious if vertices are 0 and ∞).

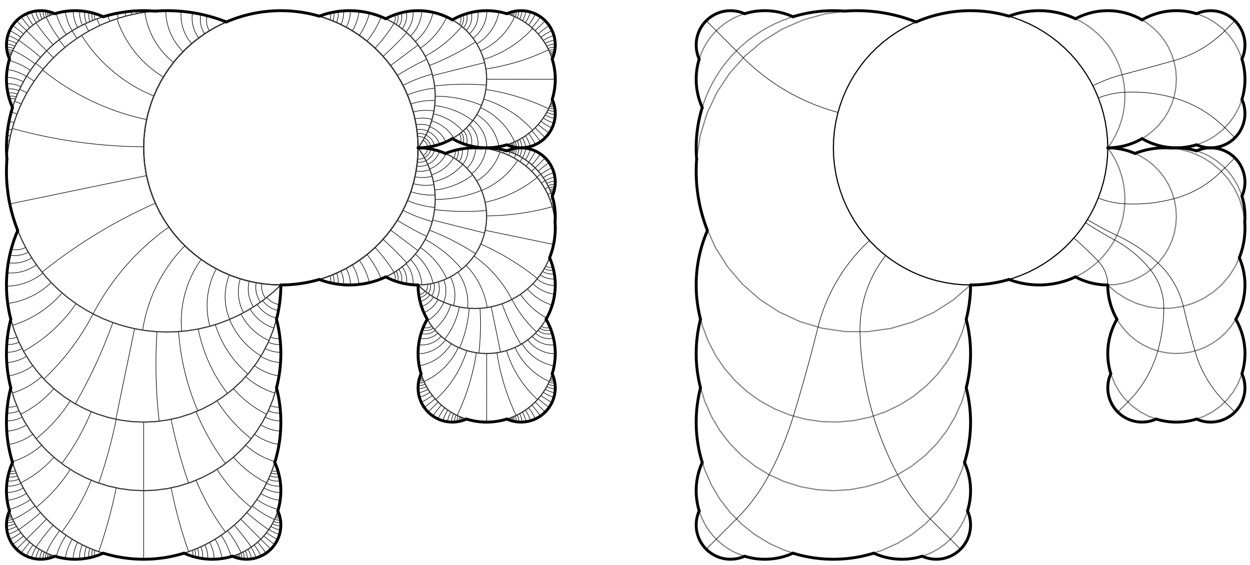


Elliptic Möbius transformation is conjugate to a rotation.



Elliptic transformation determined by fixed points and angle of rotation θ. It identifies sides of a crescent of angle θ: think of flow along circles orthogonal to boundary arcs.

Visualize ι as a flow: Write finitely bent Ω as a disk D and a union of crescents. Foliate crescents by orthogonal circles. Following leaves of foliation in $\Omega \setminus D$ gives $\iota : \partial\Omega \to \partial D$.



Has continuous extension to interior: identity on disk and collapses orthogonal arcs to points.

- ι can be evaluated at n points in time $O(n)$.

Theorem: ι has a K-QC extension to interior.

Corollary (Sullivan, Epstein-Marden): There is a K-QC map $\sigma : \Omega \to \text{Dome}$ so that $\sigma = \text{Id}$ on $\partial\Omega = \partial S$.

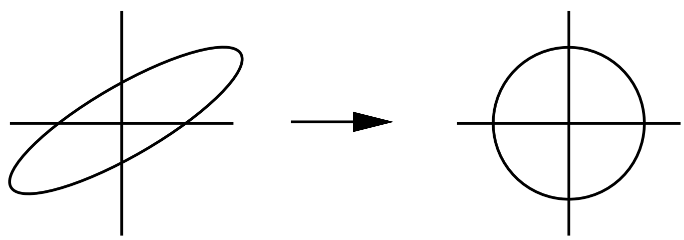
Result comes from hyperbolic 3-manifolds. If Ω is invariant under Möbius group G, $M = \mathbb{R}^3_+/G$ is hyperbolic manifold,

$$\partial_\infty M = \Omega/G, \quad \partial C(M) = \text{Dome}(\Omega)/G.$$

Thurston conjectured $K = 2$ is possible. Best known upper bound is $K < 7.82$. Epstein and Markovic showed $K > 2.1$ for some example.

A mapping is K-quasiconformal if either:

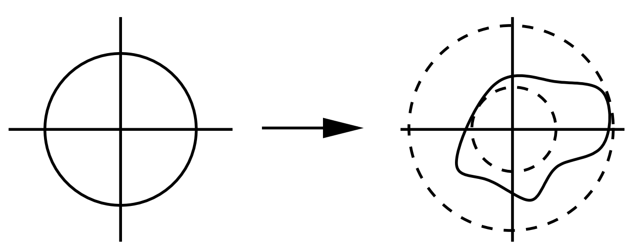
Analytic definition: $|f_{\bar{z}}| \le \frac{K-1}{K+1}|f_z|$



$$f_z = \tfrac{1}{2}(f_x - i f_y),\ f_{\bar{z}} = \tfrac{1}{2}(f_x + i f_y).$$

Metric definition: For every $x \in \Omega$, $\epsilon > 0$ and small enough $r > 0$, there is $s > 0$ so that

$$D(f(x), s) \subset f(D(x, r)) \subset D(f(x), s(K+\epsilon)).$$



Notation for today: ϵ-conformal $= e^{\epsilon}$-quasiconformal.

- The map is determined (up to Möbius maps) by
$$\mu_f = f_{\bar z}/f_z,$$
For μ with $\|\mu\|_\infty < 1$, there is a f with $\mu_f = \mu$.

- $\|\mu\|_\infty \leq k$, $k = (K-1)/(K+1)$ iff f is K-QC.

- $\mu = 0$ iff f is conformal.

- K-QC maps form a compact family.

- f is a **bi-Lipschitz** if
$$\frac{1}{A}\rho(x,y) \le \rho(f(x),f(y)) \le A\rho(x,y).$$

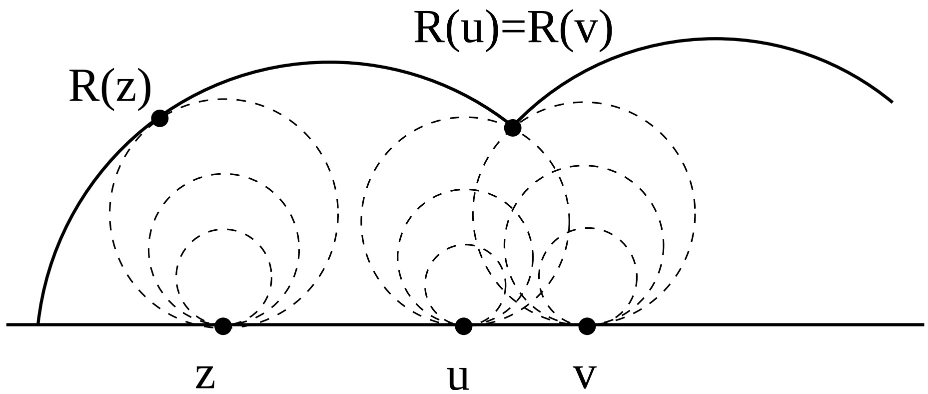
- f is a **quasi-isometry** if
$$\frac{1}{A}\rho(x,y) - B \le \rho(f(x),f(y)) \le A\rho(x,y) + B.$$

- QI=BL at "large scales".

- On hyperbolic disk, BL $\Rightarrow$ QC $\Rightarrow$ QI.

Theorem: $f : \mathbb{T} \to \mathbb{T}$ has a QC-extension to interior iff it has QI-extension (hyperbolic metric) iff it has a BL-extension.

Nearest point retraction $R : \Omega \to \mathrm{Dome}(\Omega)$**:** Expand ball tangent at $z \in \Omega$ until it hits a point $R(z)$ of the dome.



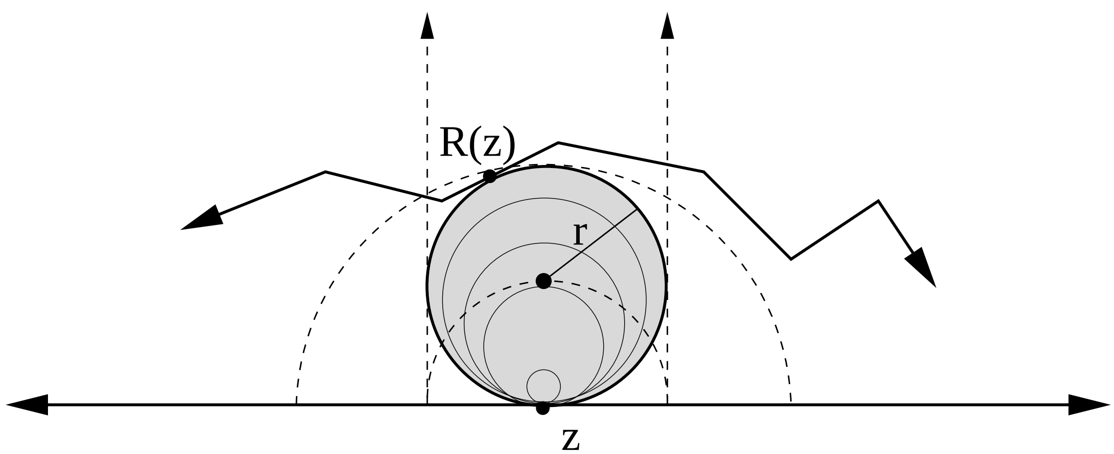
$$\text{normal crescents} = R^{-1}(\text{bending lines})$$
$$\text{gaps} = R^{-1}(\text{faces})$$

collapsing crescents = nearest point retraction

Suffices to show nearest point retraction is a quasi-isometry. This follows from three easy facts.

Fact 1: If $z \in \Omega$, $\infty \notin \Omega$,

$$r \simeq \operatorname{dist}(z, \partial\Omega) \simeq \operatorname{dist}(R(z), \mathbb{R}^2) \simeq |z - R(z)|.$$



Fact 2: R is Lipschitz.

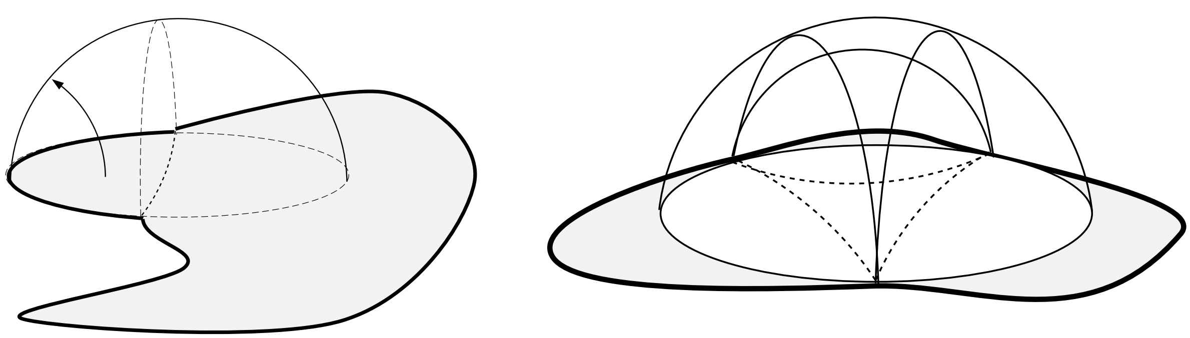
Ω simply connected $\Rightarrow$

$$d\rho \simeq \frac{|dz|}{\mathrm{dist}(z, \partial\Omega)}.$$

$z \in D \subset \Omega$ and $R(z) \in \mathrm{Dome(D)} \Rightarrow$

$$\mathrm{dist}(z, \partial\Omega)/\sqrt{2} \le \mathrm{dist}(z, \partial D) \le \mathrm{dist}(z, \partial\Omega)$$

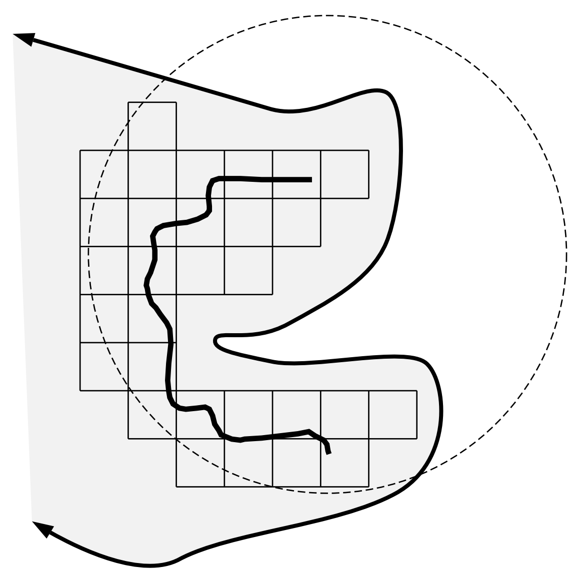
$$\Rightarrow \quad \rho_\Omega(z) \simeq \rho_D(z) = \rho_{\mathrm{Dome}}(R(z)).$$



Fact 3: $\rho_S(R(z), R(w)) \leq 1 \Rightarrow \rho_\Omega(z, w) \leq C$.

Suppose $\mathrm{dist}(R(z), \mathbb{R}^2) = r$ and γ is geodesic from z to w.

$$\begin{aligned}
\Rightarrow \qquad & \mathrm{dist}(\gamma, \mathbb{R}^2) \simeq r \\
\Rightarrow \qquad & \mathrm{dist}(R^{-1}(\gamma), \partial\Omega) \simeq r, \\
& \qquad R^{-1}(\gamma) \subset D(z, Cr) \\
\Rightarrow \qquad & \rho_\Omega(z, w) \leq C
\end{aligned}$$

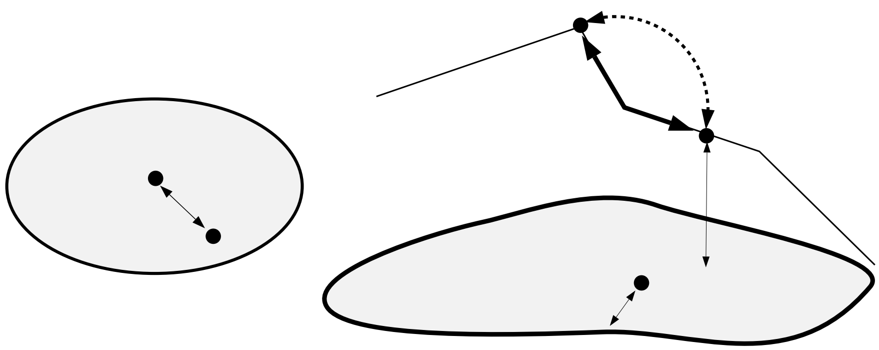


Moreover, $g = \iota \circ \sigma : \Omega \to \mathbb{D}$ is locally Lipschitz. Standard estimates show

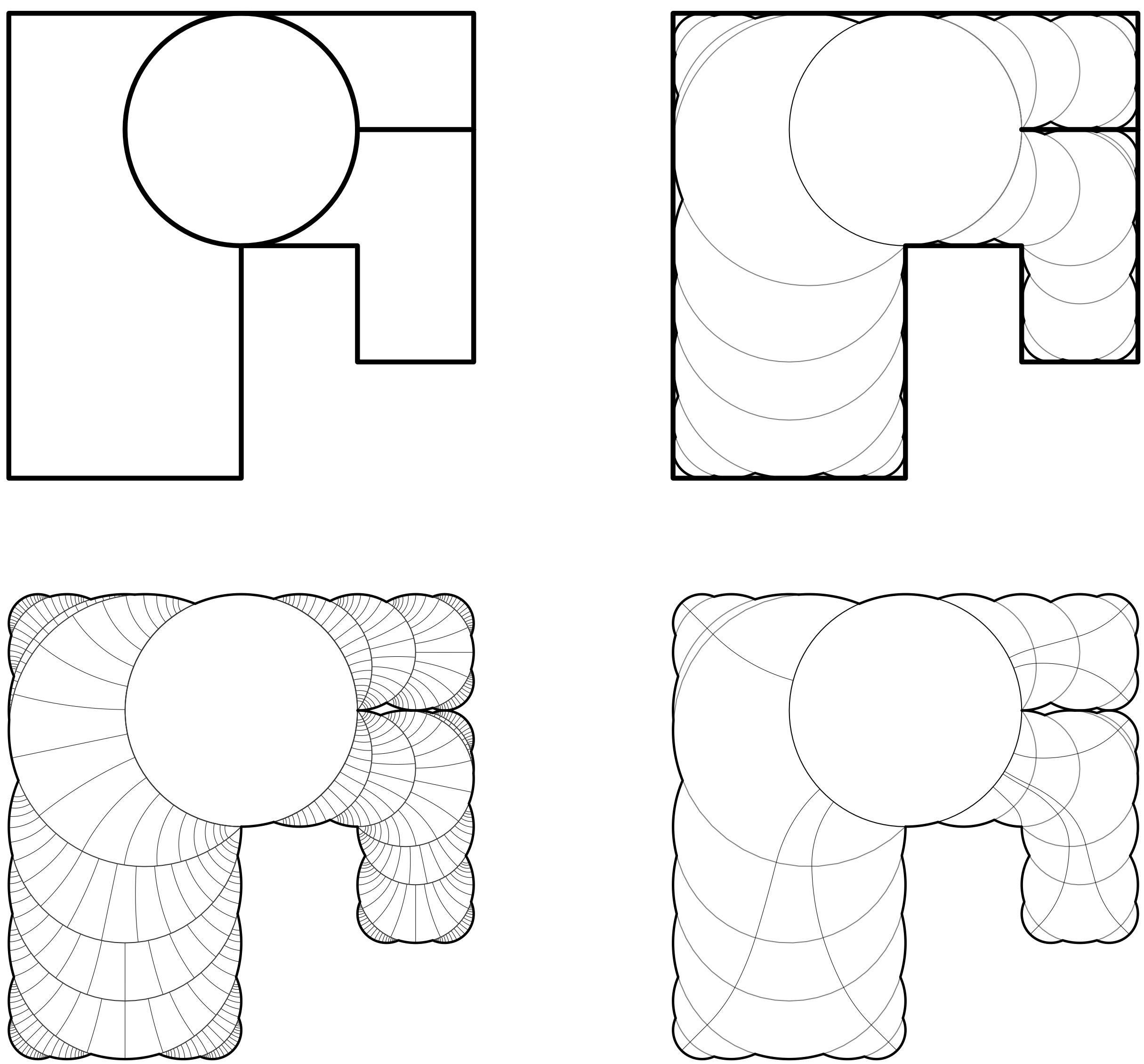
$$|g'(z)| \simeq \frac{\mathrm{dist}(g(z), \partial \mathbb{D})}{\mathrm{dist}(z, \partial\Omega)}.$$

Use Fact 1

$$\begin{aligned}
\mathrm{dist}(z, \partial\Omega) &\simeq \mathrm{dist}(\sigma(z), \mathbb{R}^2) \\
&\simeq \exp(-\rho_{\mathbb{R}^3_+}(\sigma(z), z_0)) \\
&\gtrsim \exp(-\rho_S(\sigma(z), z_0)) \\
&= \exp(-\rho_D(g(z), 0)) \\
&\simeq \mathrm{dist}(g(z), \partial D)
\end{aligned}$$

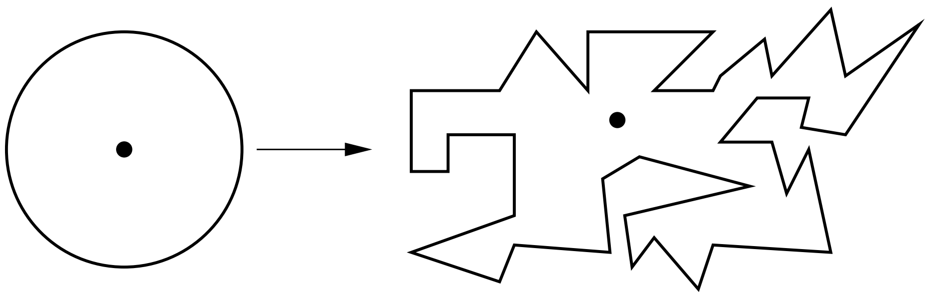


Fast Almost Riemann Mapping Theorem: Can construct a K-QC map from n-gon Ω to disk in $O(n)$ time, and K independent of n and Ω.



- Has simple geometric definition
- Only requires a "tree-of-disks" to define.
- Is stable; limit exists as disks fill in polygon.
- Fast to compute using medial axis.
- Is uniformly close to Riemann map.
- Can be used to compute Riemann map quickly.
- Definition motivated by hyperbolic 3-manifolds.
- Extends to Lipschitz map of interiors.

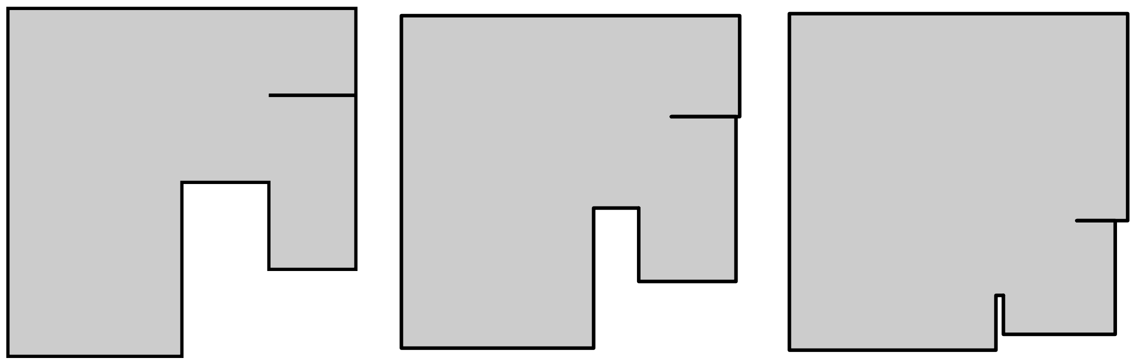
Riemann Mapping Theorem: If Ω is a simply connected, proper subdomain of the plane, then there is a conformal map $f : \Omega \to \mathbb{D}$.



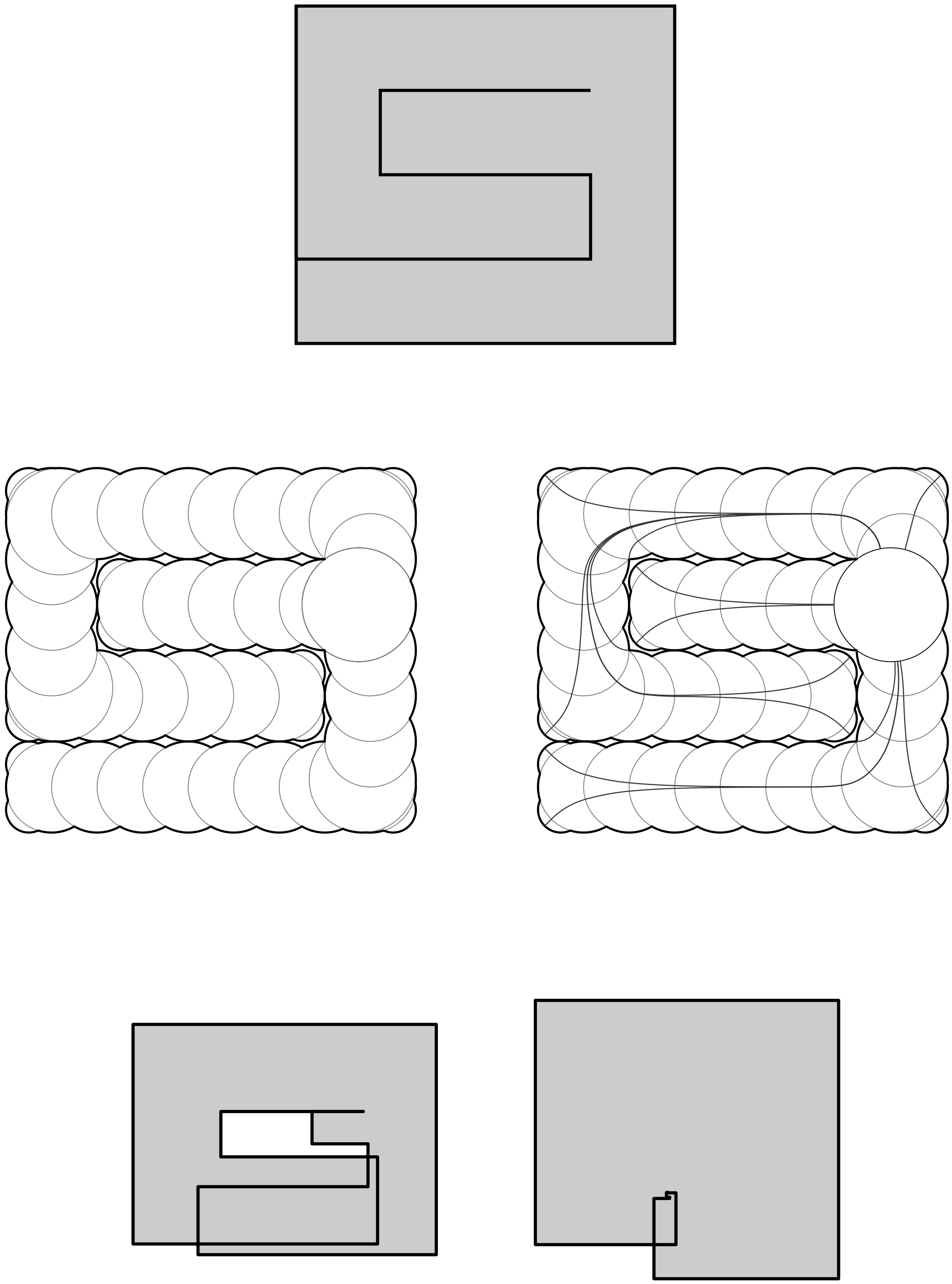
The Schwarz-Christoffel formula gives the Riemann map onto a polygonal:

$$f(z) = A + C \int^z \prod_{k=1}^{n} (1 - \frac{w}{z_k})^{\alpha_k - 1} dw.$$

α's are known (interior angles) but z's are not (preimages of vertices).



If we plug in ι-images of vertices we almost get the correct polygon (center). Using uniformly spaced points is clearly worse (right).



Theorem: If $\partial\Omega$ is an n-gon we can compute a $(1+\epsilon)$-quasiconformal map between Ω and $\mathbb{D}$ in time $O(n \log \frac{1}{\epsilon} \log \log \frac{1}{\epsilon})$.

Maps are stored as $O(n)$ power series. Need $p = O(|\log \epsilon|)$ terms to get accuracy ϵ. Need time $O(p \log p)$ to multiply, p-long series.

Theorem allows $O(1)$ such operations per vertex of polygon.

Proof of theorem is in two steps:

Step 1: Given $\epsilon < \epsilon_0$ and ϵ-QC $f_n : \Omega \to \mathbb{D}$ construct $C\epsilon^2$-QC map $f_{n+1} : \Omega \to \mathbb{D}$. Construction takes time $C(\epsilon) = C + C \log^2 \frac{1}{\epsilon} \log \log \frac{1}{\epsilon}$.

Step 2: Build domains and finite boundary sets

$$(\Omega_0, V_0), \ldots, (\Omega_N, V_N)$$

so that

- $\Omega_0 = \mathbb{D}$,
- $\Omega_N = \Omega$, $V_N = V$,
- δ-QC maps $g_k : \Omega_k \to \Omega_{k+1}$, $V_k \to V_{k+1}$.

If $\delta < \epsilon_0/2$ then find conformal maps by induction (use previous map as starting point of Step 1 to find next map).

Amazing Fact 1: Can take ϵ_0 independent of Ω and n.

Amazing Fact 2: Can take N independent of Ω and n.

Consequence: Get ϵ_0 approximation in time $O(n)$ (independent of Ω). Then just repeat Step 1 until get desired accuracy :

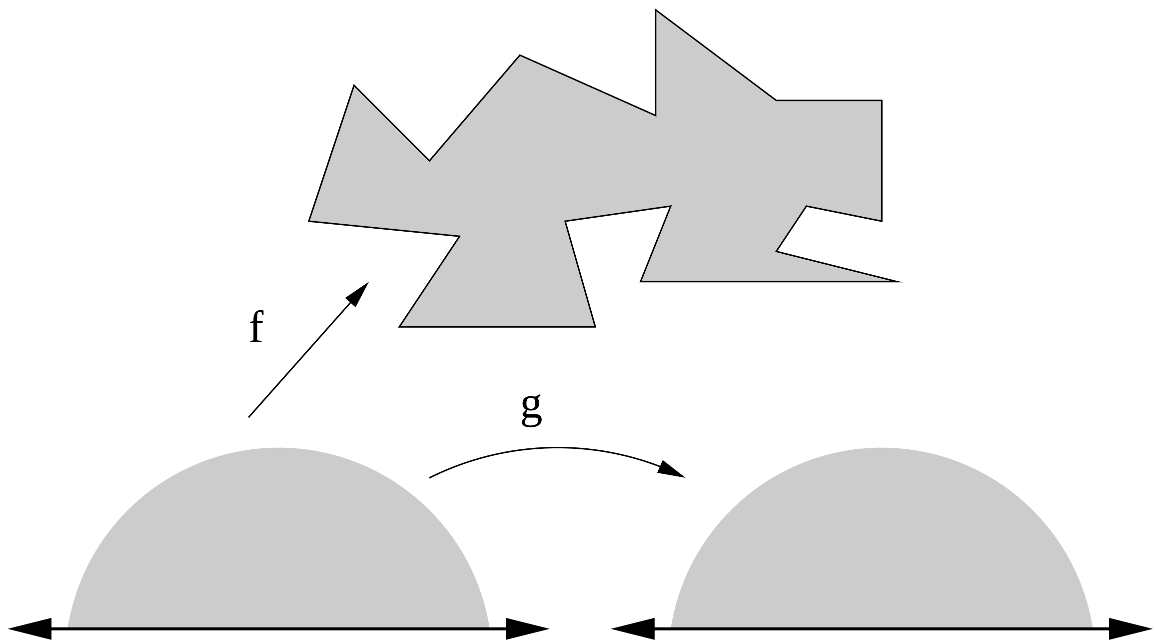
$$\epsilon_0, C\epsilon_0^2, \ldots C^k \epsilon_0^{2^k}.$$

About $\log \log \epsilon$ iterations suffice and time for kth iteration is $O(k2^{2k})$, so work dominated by final step.

Idea for Step 1: Suppose

$$f : \mathbb{H} \to \Omega, \qquad g : \mathbb{H} \to \mathbb{H}, \qquad \mu_f = \mu_g.$$

Then $f \circ g^{-1} : \mathbb{H} \to \Omega$ is conformal.

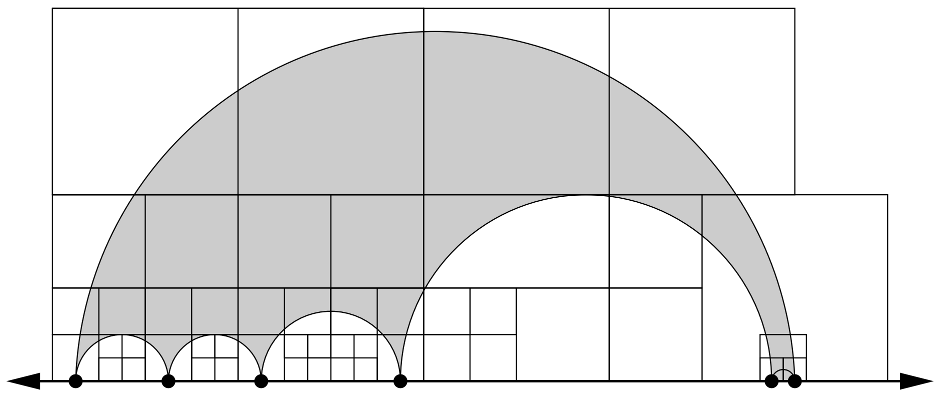


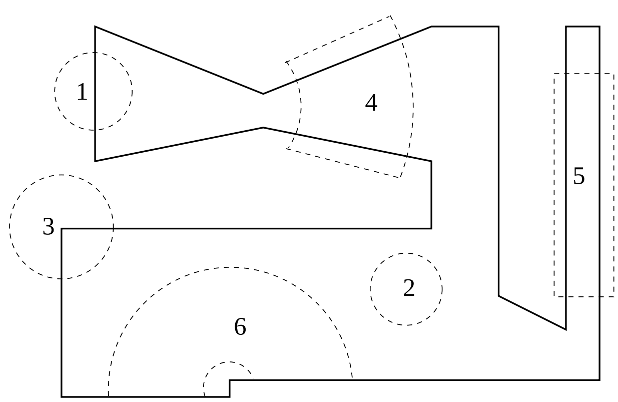
Can't solve Beltrami equation $g_{\bar{z}} = \mu g_z$ exactly in finite time, but can quickly solve

$$g_{\bar{z}} = (\mu + O(\|\mu\|^2))g_z.$$

Then $f \circ g^{-1}$ is $(1 + C\|\mu\|^2)$-QC.

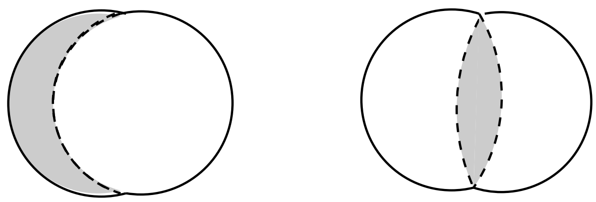
Cut $\mathbb{H}$ into $O(n)$ pieces on which f, f^α or $\log f$ has nice series representation. Need $p = O(|\log \epsilon|)$ terms on each piece to get ϵ accuracy.



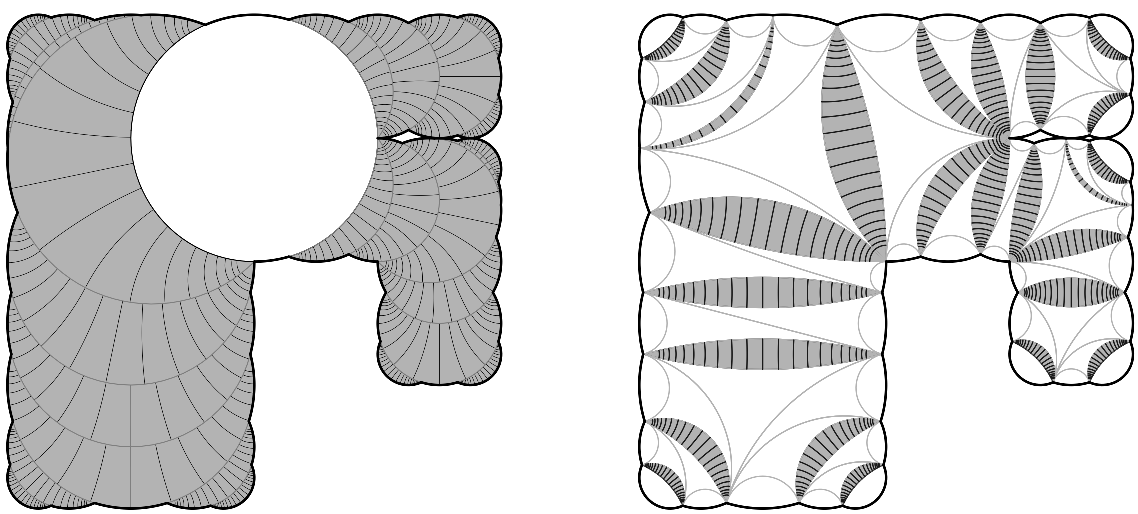


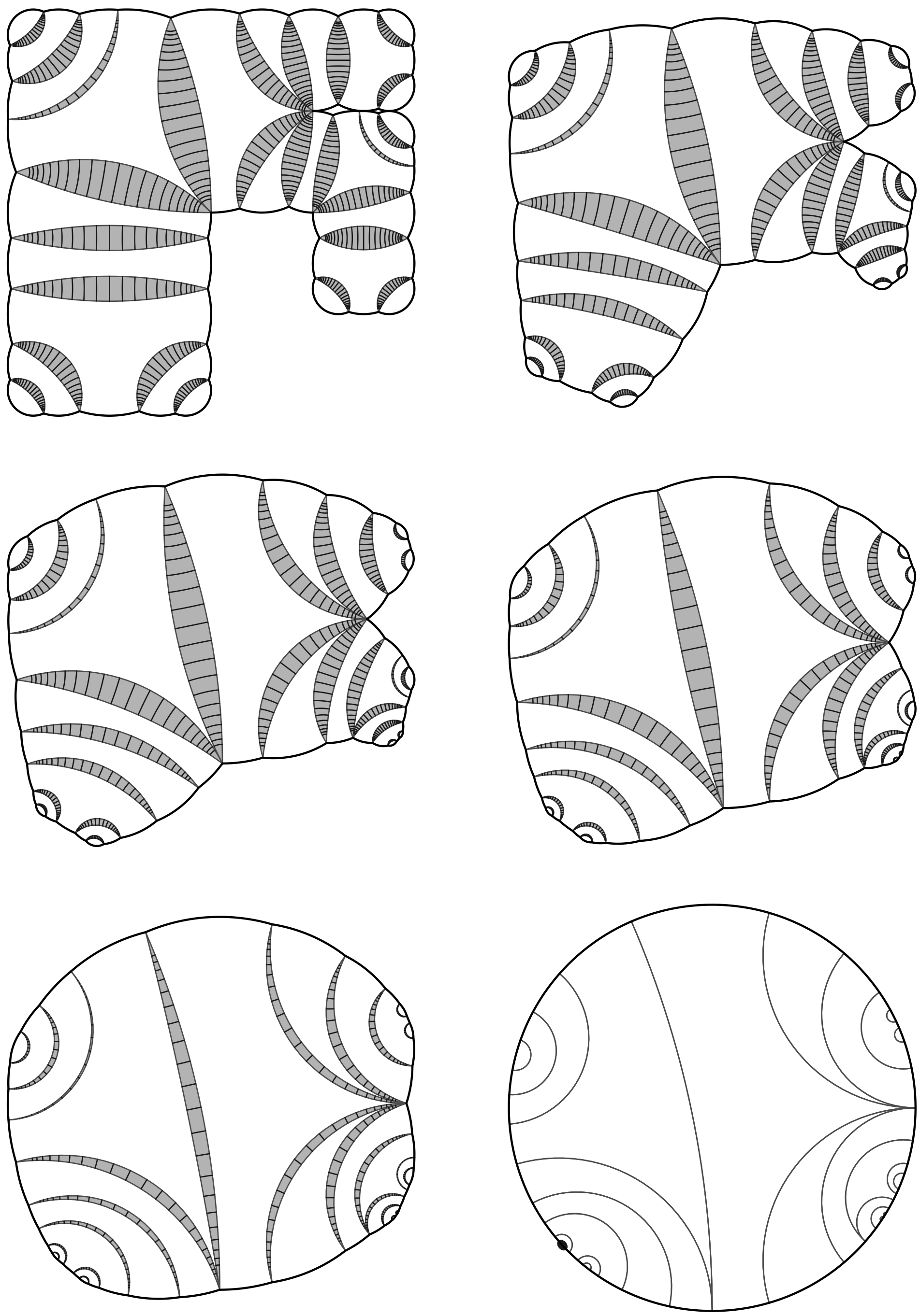
Idea for step 2: Use angle scaling.

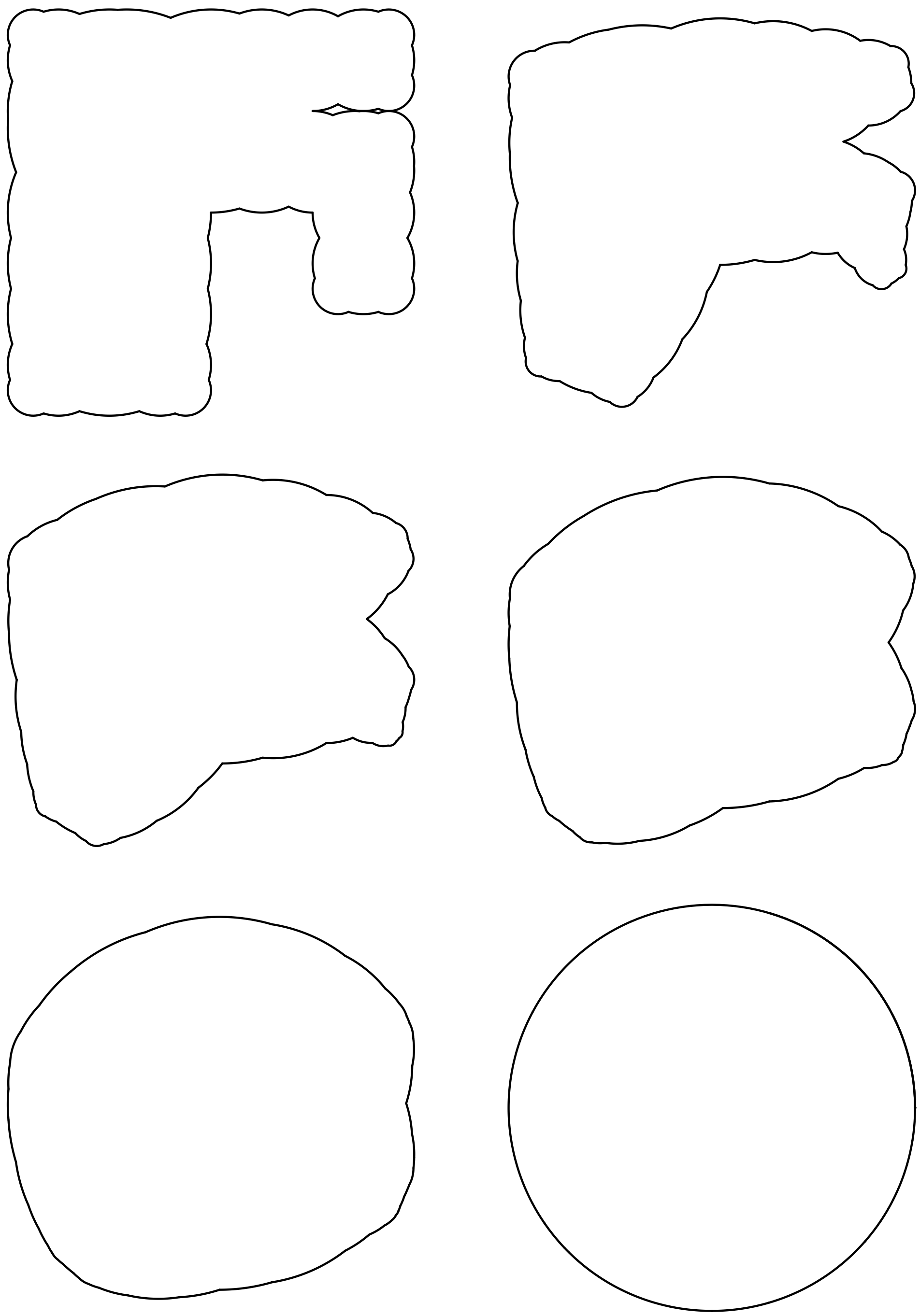
There are at least two ways to decompose a finite union of disks using crescents.

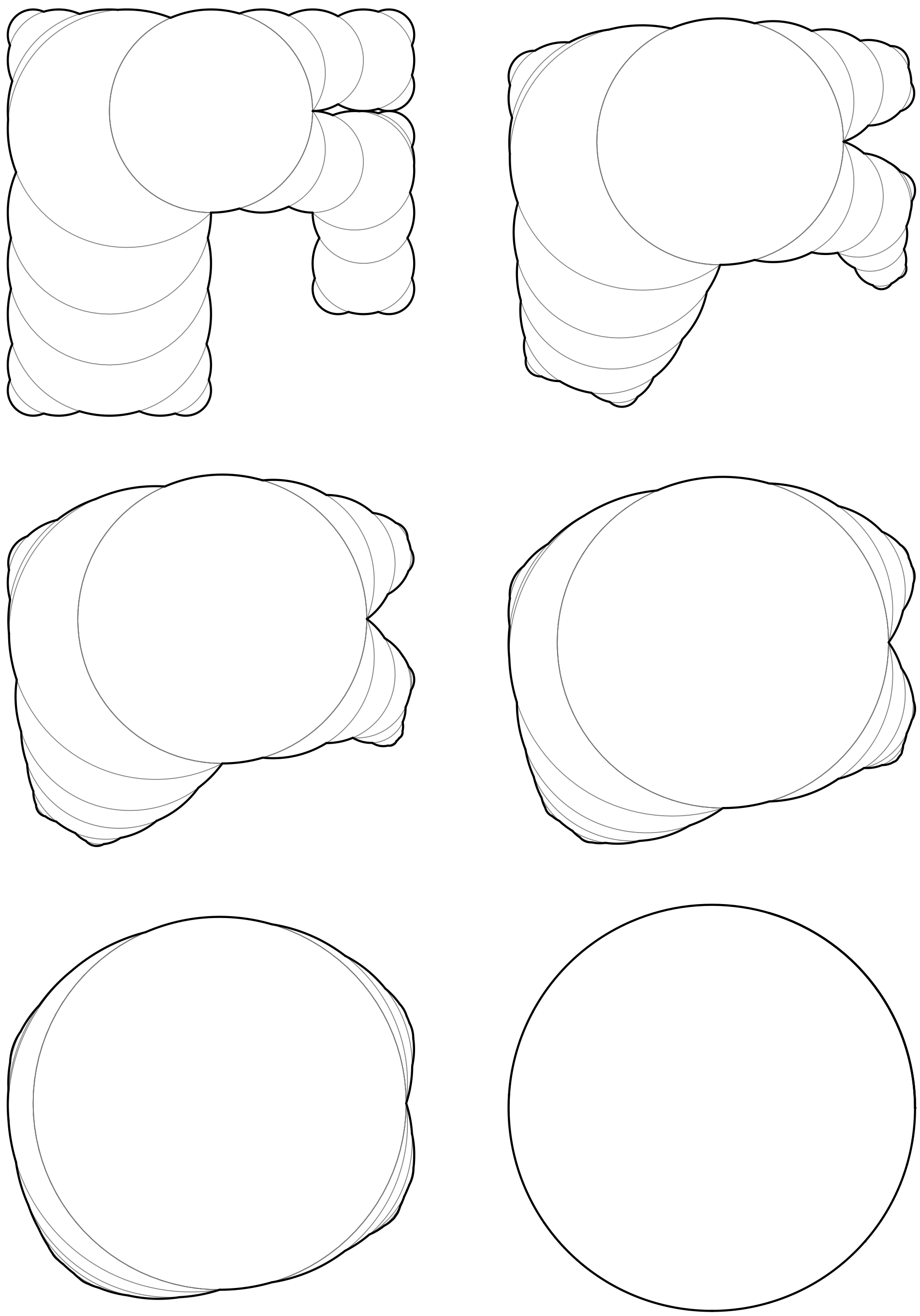


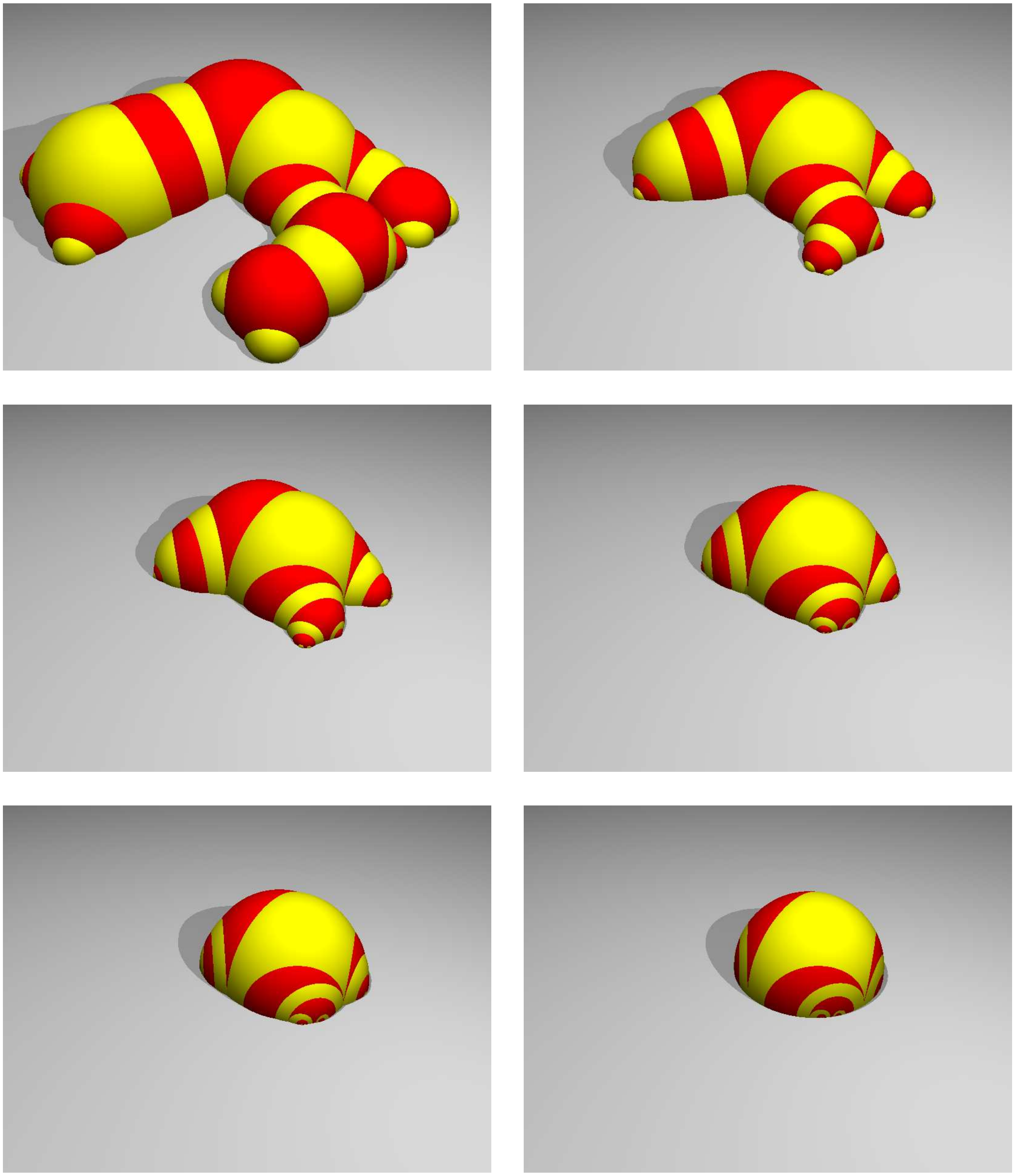
We call these **tangential** and **normal** crescents. A finitely bent domain can be decomposed with either kind of crescent.

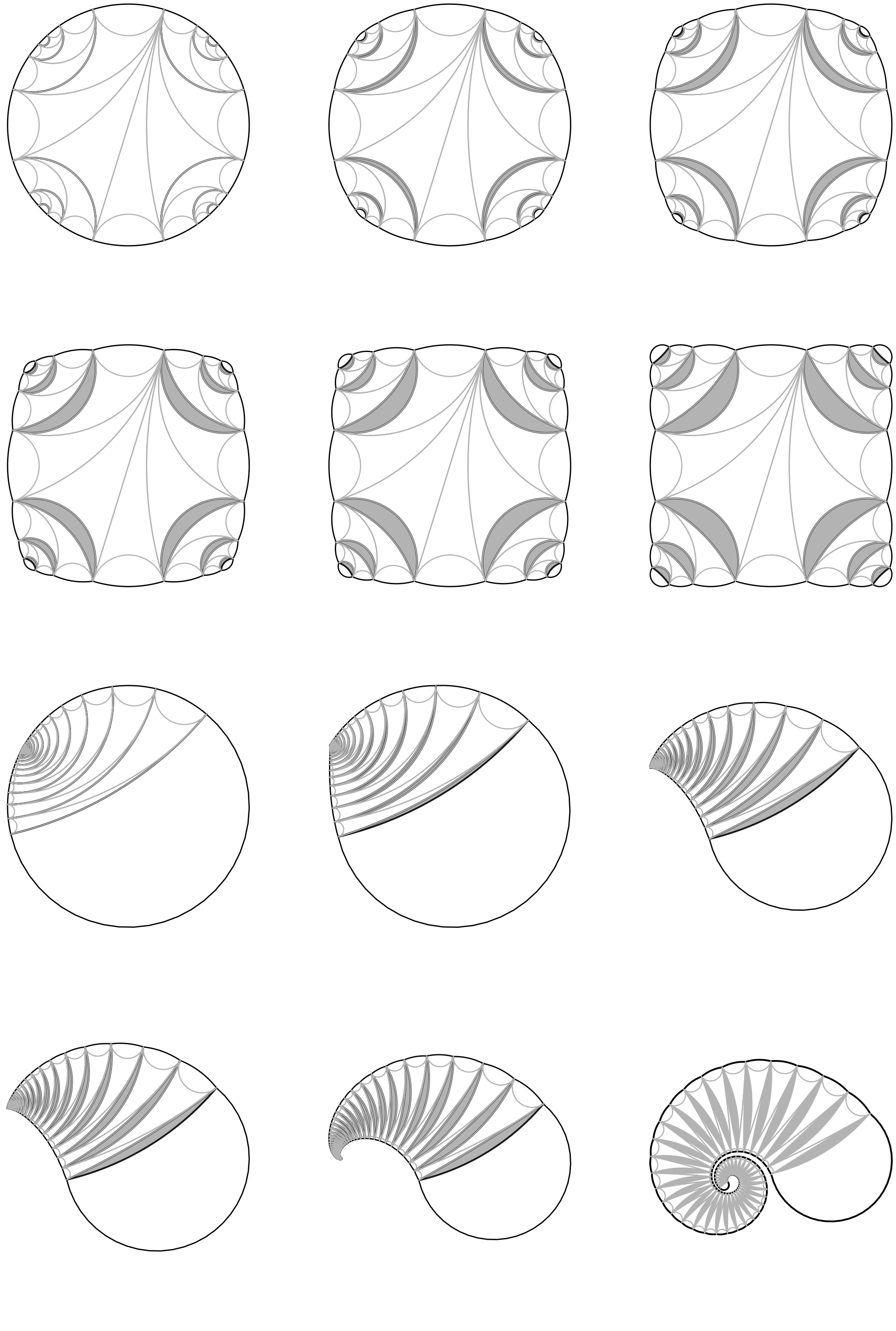


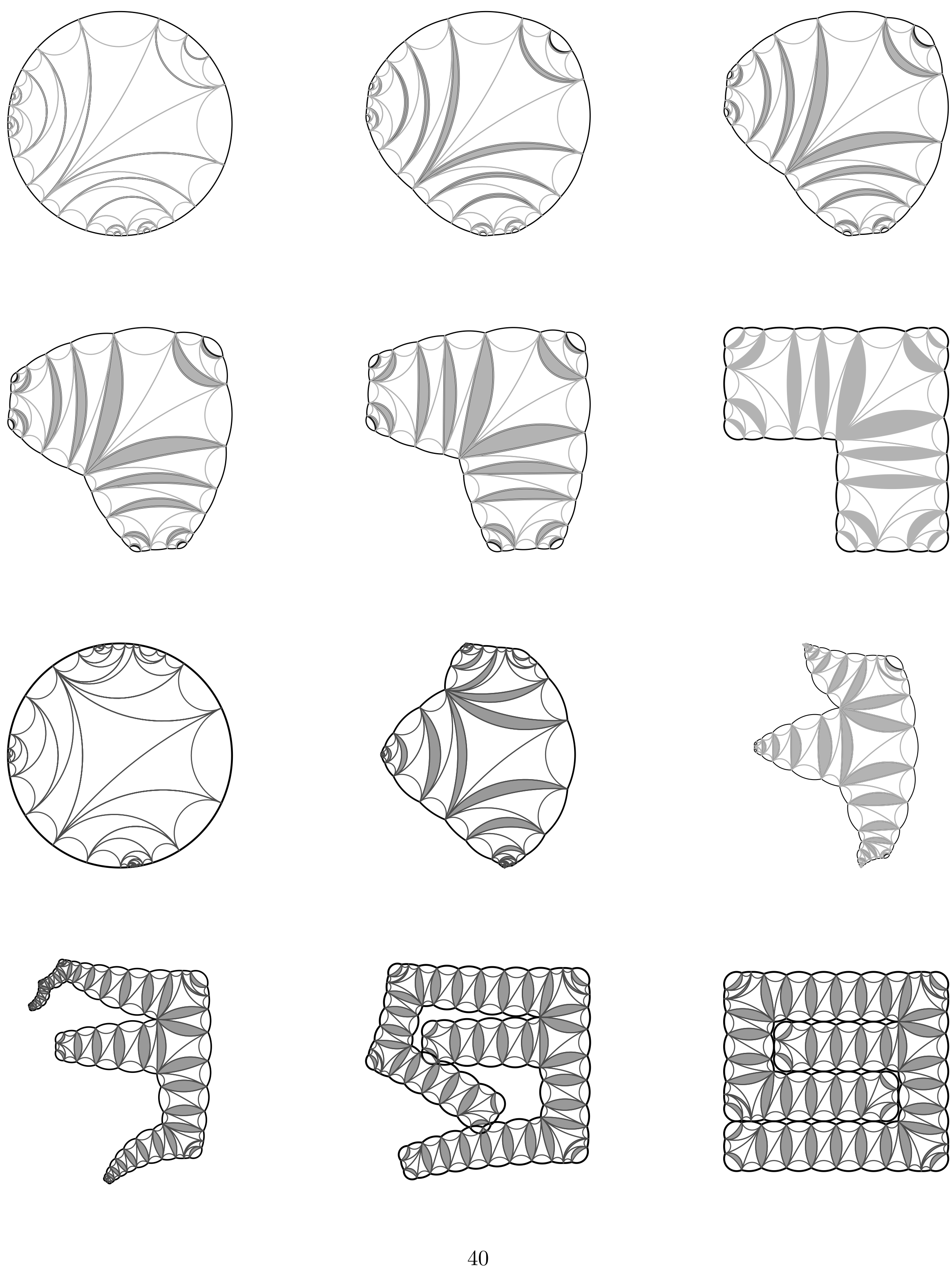


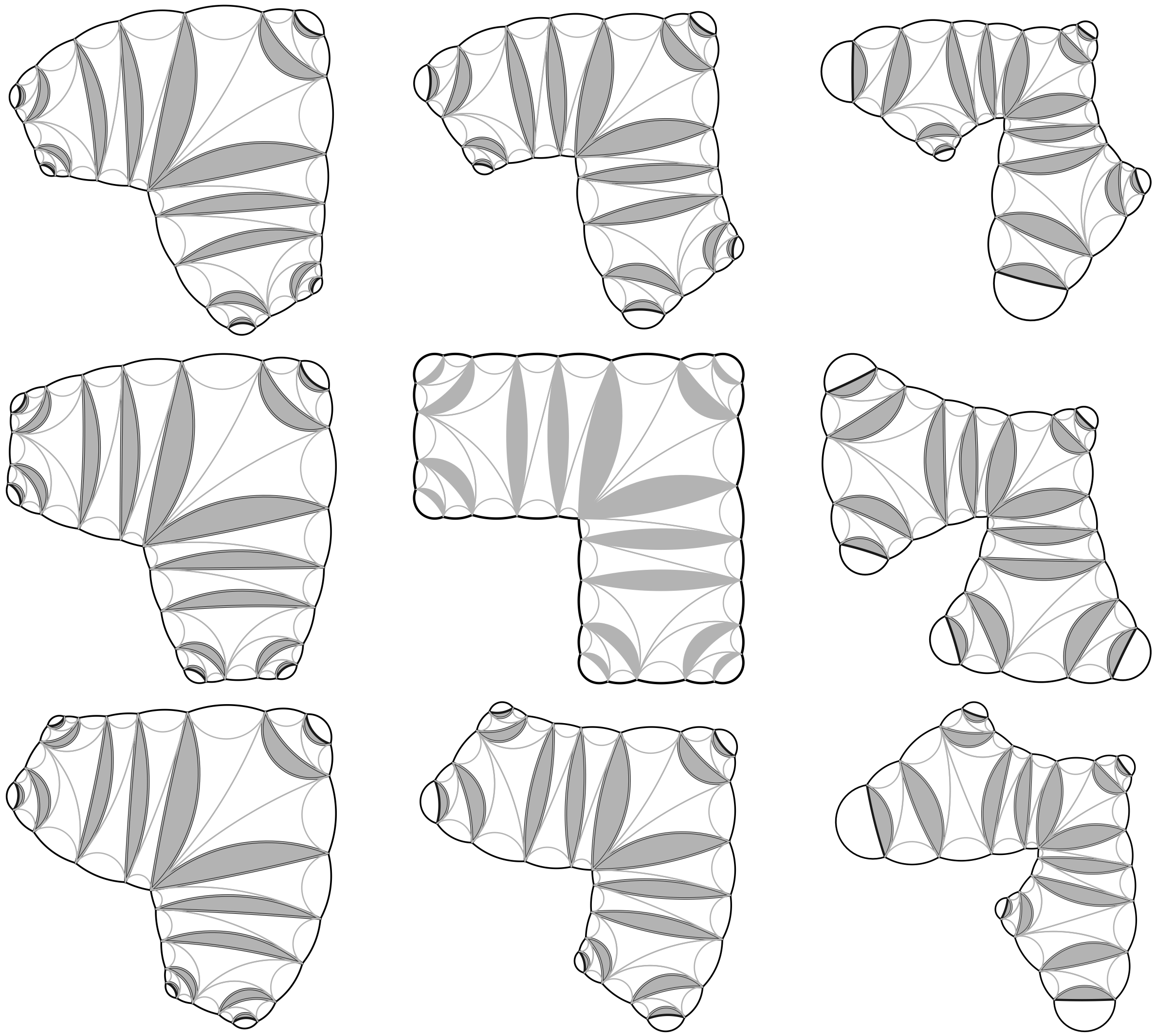








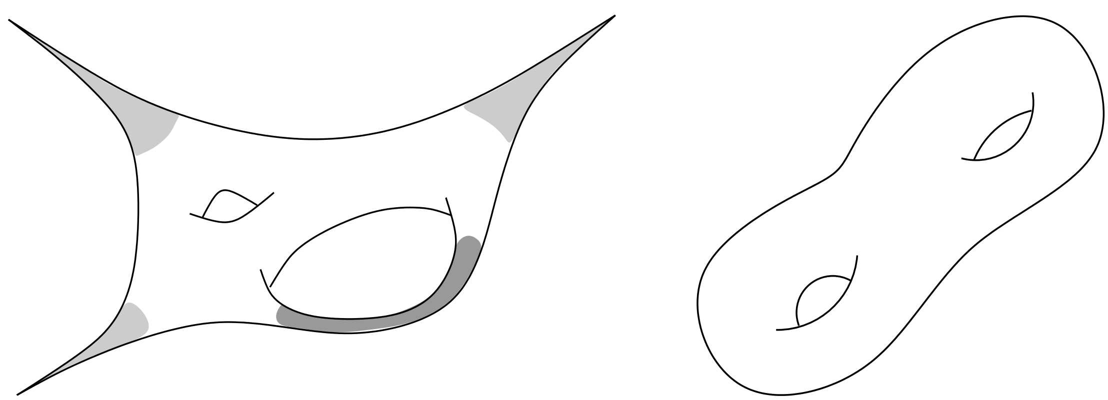




Another idea inspired by hyperbolic geometry: Thick/Thin decompositions.

Standard technique in hyperbolic manifolds is to partition the manifold based on the size of the injectivity radius. Thin parts often cause problems, there are only a few possible types and each has a well understood shape.

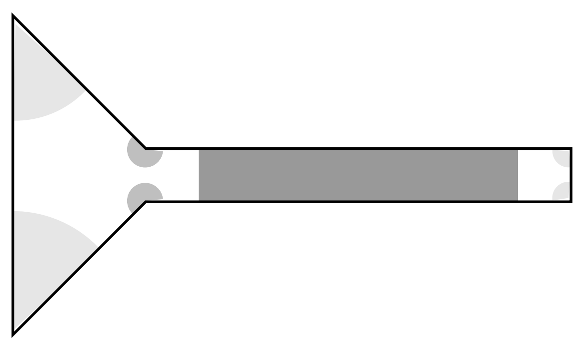
M = interesting thick parts + annoying thin parts

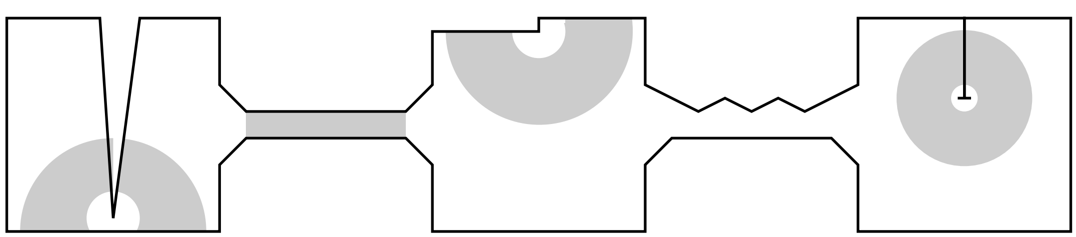


There is analogous decomposition of polygons.

An ϵ-thin part corresponds to two edges whose extremal distance in Ω is $< \epsilon$.

Parabolic thin parts occur at every vertex. Hyperbolic thins parts correspond to non-adjacent edges.





• At most $O(n)$ thin parts.

• Can be located in linear time using iota map.

• Conformal maps onto thin parts "explicitly known".

• Remaining thick components have good approximations by $O(n)$ disks.

• Can mesh thick part into $O(n)$ pieces Q_j so map is conformal on $100Q_j$. Hence small angle distortion on thick parts.

Application to meshing:
Marshall Bern and David Eppstein showed any n-gon has quadrilateral mesh with all angles ≤ 120 which can be found in time $O(n \log n)$.

They asked if lower bound on angles is possible. Fast Riemann mapping theorem implies

Theorem: Any n-gon has quadrilateral mesh with all new angles between 60° and 120° which can be found in time $O(n)$.

Both angle bounds are sharp.

Idea of proof

- Decompose polygon into thick and thin parts.

- Find explicit meshes in thin parts (known shapes).

- Find preimages on unit circle of vertices under conformal map.

- Remove disks around prevertices, tile remainder by hyperbolic pentagons, quadrilaterals, triangles.

- Mesh each hyperbolic polygon using angles in $[60, 120]$.

- Map mesh forward to Ω by conformal map. Straighten sides.

- Gives $60-\epsilon$, $120+\epsilon$. Extra work to remove $\pm\epsilon$.

If you understand the figures, you understand the book.

John Garnett,
Bounded Analytic Functions, 1981

"Ah!" replied Pooh. He'd found that pretending a thing was understood was sometimes very close to actually understanding it. Then it could easily be forgotton with no one the wiser...

Winnie-the-Pooh

I wouldn't even think of playing music if I was born in these times... I'd probably turn to something like mathematics. That would interest me.

Bob Dylan, 2005