Conformal Maps, Optimal Meshes and Sullivan’s Convex Hull Theorem

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Colloquium, March 3, 2011

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Suppose $M$ is a hyperbolic 3-manifold.

Convex hull of closed geodesics is the convex core, $C(M)$.

Boundary at infinity is a Riemann surface, $\partial M$.

**Sullivan’s convex hull theorem:** $\partial C(M)$ and $\partial M$ are bi-Lipschitz equivalent (hyperbolic metrics).

BiLipschitz: $A|x - y| \leq |f(x) - f(y)| \leq B|x - y|$. 
Pass to universal cover: upper half-space.

- $\partial M$ becomes planar $\Omega$.
- $C(M)$ (shaded) is convex hull of $\Omega^c$.
- $M \setminus C(M)$ is union of hemispheres with base in $\Omega$.
- Boundary is called **dome** of $\Omega$.

David Epstein and Al Marden: CHT is true for any simply connected $\Omega$. ($\Omega$ and dome are biLipschitz equiv.)
Nearest point retraction in hyperbolic space extends to map $R : \Omega \to S = \text{Dome}$ and is a quasi-isometry

$$\frac{1}{A} \rho_\Omega(x, y) - B \leq \rho_S(R(x), R(y)) \leq A \rho_\Omega(x, y).$$

Standard techniques improve to biLipschitz or QC.
The dome of a geometrically infinite Kleinian group.
It is easy to map any dome conformally to a disk.

Iota map = isometry from a dome to hyperbolic disk.
A dome is a hinged surface. We map it to a hemisphere by making all faces flush with each other. More interesting in hyperbolic space than Euclidean space because parallel postulate fails (more non-intersecting lines).
Flattening dome collapses crescent in base by collapsing orthogonal arcs.
Instead of collapsing all crescents at once, we may do one at a time, from leaves towards root.

Defines a flow from boundary to disk along foliation of crescents by orthogonal arcs.

Taking limits, this flow exists for any domain.
Iota is closed related to medial axis in computer science.

**Medial axis** is set of centers of subdisks of $\Omega$ that hit boundary in at least two points.

For $n$-gon it is computable in $O(n)$ time.

Iota is computable from Medial Axis in linear time.
Iota $\approx$ Riemann Mapping by Sullivan’s CHT.

Iota is computable in $O(n)$ time for $n$-gons.

Iota is locally Lipschitz (decreases boundary length).
Factorization Thm: A conformal map $f : \Omega \rightarrow \mathbb{D}$ can be written as $f = h \circ g$ where
- $g : \Omega \rightarrow \mathbb{D}$ is locally Lipschitz (Euclidean metrics)
- $h : \mathbb{D} \rightarrow \mathbb{D}$ is biLipschitz (hyperbolic metric)
Both maps are 8-quasiconformal.
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This has an application to conformal dynamics.
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- Bowen: true if \( R = \Omega/G \) is compact.
- Sullivan: true if \( R \) has finite area.
- B: true if \( R \) is recurrent for Brownian motion.
- Astala & Zinsmeister: false otherwise.
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Bowen’s Dichotomy: If $\Omega$ is simply connected and invariant under a discrete group $G$ of Möbius transformations then $\partial \Omega$ is either a circle or $\dim(\partial \Omega) > 1$.

Idea of proof is to replace conformal map $\mathbb{D} \to \Omega$ (which both expands and contracts) by a QC map that only expands. Instead of discussing the technical details I will give some more recent applications the CHT.
Riemann Mapping Theorem: If $\Omega$ is a simply connected, proper subdomain of the plane, then there is a conformal map $f : \mathbb{D} \to \Omega$.

Conformal = angle preserving
Our Founder
Stated RMT in 1851 thesis
William Fogg Osgood
First proof of RMT, 1900
How much time is needed to compute a conformal map?
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**Theorem (DCG Sept 2010):** We can compute a $\epsilon$-conformal map onto an $n$-gon in $O(n \log \frac{1}{\epsilon} \log \log \frac{1}{\epsilon})$. 
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$\epsilon$-conformal = $\epsilon$-distortion of angles = $(1 + \epsilon)$-QC.

**Goal:** Find good representation of map in time $O(n)$.
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**Goal:** Find good representation of map in time $O(n)$.

Proof of the fast mapping theorem:
- Local representation of maps
- Newton’s method for Beltrami’s equation
- Use Iota for initial guess
Conformal maps have power series, but corners of polygon create singularities on circle. Convergence is slow.

20 terms
Conformal maps have power series, but corners of polygon create singularities on circle. Convergence is slow.

100 terms
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2500 terms

$1 \times 20$ rectangle would require about $10^{15}$ terms.

Need more efficient representation.
Schwarz-Christoffel formula (1867):

\[ f(z) = A + C \int \frac{\prod_{k=1}^{n}(1 - \frac{w}{z_k})^{\alpha_k - 1}}{z} \, dw, \]
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\{ \alpha_1 \pi, \ldots, \alpha_n \pi \}, are interior angles of polygon.
\{ z_1, \ldots, z_n \} are points on circle mapping to vertices.

\( \alpha \)'s are known.

\( z \)'s must be solved for.

Approximate \( z \)'s give correct angles, wrong side lengths.

Each evaluation of integrand is \( n \)-fold product.
Local series representation: We cut disk into $O(n)$ regions and use a $p$-term series on each piece to approximate map with accuracy $\epsilon \approx 2^{-p}$ in hyperbolic metric.

Use partition of unity to get global map.

Easy to evaluate; just plug in.

Also more subtle advantage.
Schwarz-Christoffel always gives conformal map, but onto wrong polygon if $z$-parameters are wrong.

**Hard** to understand relationship between parameters and image domain, so **hard** to update parameters in provably correct way (OK in practice, e.g., CRDT).

Local series map is always onto correct domain, but is not conformal if series are only approximate.

**Easy** to improve conformality and preserve image.
\[ \partial f = \frac{1}{2}(fx - ify), \quad \bar{\partial} f = \frac{1}{2i}(fx + ify). \]

We want \( f : \mathbb{D} \rightarrow \Omega \) with \( \bar{\partial} f = 0 \) (Cauchy-Riemann).

We measure distance to conformality by dilatation

\[ \|f\| = \sup |\mu_f| \equiv \sup |\bar{\partial}f/\partial f|. \]

**Main point:** If \( f : \mathbb{D} \rightarrow \Omega, \ g : \mathbb{D} \rightarrow \mathbb{D} \) and \( \mu_g = \mu_f \), then \( h = f \circ g^{-1} : \mathbb{D} \rightarrow \Omega \) is conformal.
Beltrami equation: given $\mu$ find $g$ with $\mu g = \mu$, 
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Set \( g = P[\mu(h + 1)] + z \), where

\[
h = T\mu + T\mu T\mu + T\mu T\mu T\mu + \ldots,
\]
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$T$ is the Beurling transform 

$$T\varphi(w) = \lim_{r \to 0} \frac{1}{\pi} \iint_{|z-w|>r} \frac{\varphi(z)}{(z-w)^2} dxdy,$$

$P$ is the Cauchy transform 

$$P\varphi(w) = -\frac{1}{\pi} \iint \varphi(z)\left(\frac{1}{z-w} - \frac{1}{z}\right) dxdy.$$
If $f$ has local representation by $O(n)$ $p$-term series, we can compute a $g$ in time $O(np \log p)$ so that

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Iteration gives quadratic convergence to conformal map.

Uses fast multipole method and FFT

For $O(n)$ bound iteration needs a starting map $\mathbb{D} \to \Omega$ that is close to conformal (independent of $\Omega$).
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For $O(n)$ bound iteration needs a starting map $\mathbb{D} \rightarrow \Omega$ that is close to conformal (independent of $\Omega$).

Use Iota map for initial guess. By CHT it is bounded distance to correct answer. Only requires $O(\log|\log \epsilon|)$ steps to reach accuracy $\epsilon$. 
How good is the iota approximation in practice?
Use “iota parameters” in Schwarz-Christoffel formula.
Triangulate polygons and form piecewise linear maps. Max $|\mu|$ gives upper bound for distance to conformal.

The most distorted triangle is shaded. Here $|\mu| = .108$.

We can bound conformal distance to true SC parameters even though we don’t know what the are.
Triangulate polygons and form piecewise linear maps. Max $|\mu|$ gives upper bound for distance to conformal.

The most distorted triangle is shaded. Here $|\mu| = .108$.

Chris Green: Use $\mu$ from piecewise linear map and linear approximation to Beltrami equation to recompute SC parameters. Seems to work well in theory and practice.
Quadrilateral meshes
Does every $n$-gon have a good mesh?
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Good geometry = angles bounded away from 0, 180.

Good complexity = # of elements polynomial in $n$.

Common element types = triangles, quadrilaterals.

Many more results for triangulations.

**Theorem**: Every simple $n$-gon has $O(n)$ quad mesh with angles $\leq 120^\circ$.

Sharp: Any quad mesh of hexagon has angle $\geq 120^\circ$. 

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$P = \text{hyperbolic geometry, University of Warwick}$

$P^2 = \text{computational geometry, UC Irvine}$

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**Theorem (DCG 2010)**: Every $n$-gon has $O(n)$ quad mesh with all angles $\leq 120^\circ$ and new angles $\geq 60^\circ$.

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**Idea of proof:**
- Decompose polygon into thick and thin parts.
- Mesh thin parts by explicit construction.
- Thick parts use hyperbolic geometry.
Thick and Thin parts

An $\epsilon$-thin part of a surface is a union of non-trivial loops of length $\leq \epsilon$ (parabolic/hyperbolic).

Thin piece is a sector whose two straight sides satisfy
\[ \text{dist}(I, J) \ll \min(|I|, |J|). \]

Precise definition: extremal distance in $\Omega$ is $< \epsilon$. 
Parabolic and hyperbolic thin parts correspond to thin parts of “doubled” polygon = Riemann surface.
Thin parts computable in $O(n)$ using conformal map.

- Map polygon conformally to half-plane. Vertices map to points on line.
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Find clusters $T \subset S$ so distances inside cluster are much smaller than connecting cluster to complement.

$$\max\{|x-y| : x, y \in T\} \leq \delta \min\{|x-z| : x \in T, z \in S \setminus T\}$$
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- Finds all thin parts in linear time.
- By CHT enough to use Iota instead of conformal map.
Thin parts are meshed by explicit construction (easy).
Basic idea for meshing thick parts: Conformal map from disk preserves angles except near vertices.

Transfer mesh on disk to mesh of polygon.

Need to be careful with tiles and timing.
Euclidean plane can be tesselated by squares
Hyperbolic disk can be tesselated by right pentagons.
Conformal map from polygon to disk takes thick and thin parts to disk as shown.
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Analog of Whitney or quadtree construction.
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Draw (hyperbolic) convex hull of thin regions.
Take pentagons from tessellation hitting convex hull but missing thin parts. Extend pentagon edges to boundary. Pentagons, quadrilaterals, triangles and half-annuli.
Shapes can be meshed to match along common edges.
This completes sketch of quad meshing of polygons.

Theorem can be extended from polygons to PSLGs.
A Planar Straight Line Graph (PSLG) is a finite point set plus a set of disjoint edges between them.
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Mesh must cover the edges of the PSLG.
May be necessary to add Steiner points.
Fills convex hull.
More PSLGs
Theorem (B, 2011): Every PSLG has a quadrilateral mesh with $O(n^2)$ elements, all angles less than 120° and all new angles greater than 60°.
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Angles and complexity sharp.

All but $O(n)$ vertices have angles in $[89°, 91°]$.

All but $O(n)$ vertices are degree 4.

Mesh has $O(n)$ sub-meshes, each a rectangular grid.
Any bound $\theta < 180^\circ$ sometimes requires $n^2$ vertices.
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Proof quad meshing for PSLGs requires new ideas:

- connect PSLG without small angles
- thick/thin decompose complementary components
- thick parts meshed by polygon method
- foliation of thin parts
- bending foliation paths
- traps, sinks
Convert quadrilaterals to triangles by adding diagonals.

**Corollary:** Every PSLG has a $O(n^2)$ triangulation with maximum angle $\leq 120^\circ$.

Compare S. Mitchell 1993 (157.5°) and Tan 1996 (132°).
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Can we get a positive lower angle bound?
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Can the 120° upper bound be improved? **Yes**

Can we get a positive lower angle bound? **No**
No lower angle bound. For $1 \times R$ rectangle, the number of triangles $\gtrsim R \times \text{(smallest angle)}$.

So uniform complexity $\Rightarrow$ no lower angle bound.
No upper bound $< 90^\circ$:

If angles are $\leq 90^\circ - \epsilon$ then all angles are $\geq 2\epsilon$.

$$\gamma = 180 - \alpha - \beta \geq 180 - (90 - \epsilon) - (90 - \epsilon) \geq 2\epsilon.$$  

So **nonobtuse** triangulation is best we can hope for.
Brief history of nonobtuse triangulation:

- $O(n)$ for points sets: Bern, Eppstein, Gilbert 1990
- $O(n^2)$ for polygons: Bern, Eppstein 1991
- $O(n)$ for polygons: Bern, Mitchell, Ruppert 1994

Numerous applications, heuristics: discrete maximum principle, condition numbers for finite element method, fast marching method, computer learning, …

**Open problem:** does every PSLG have a polynomial sized nonobtuse triangulation?
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Cor (of proof): Every PSLG has a triangulation with all angles $\leq 90^\circ + \epsilon$ and $O(n^2/\epsilon^2)$ elements.
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Cor: For any PSLG there is a set of size $O(n^{2.5})$ whose Voronoi diagram covers the PSLG.

Proofs have no conformal maps or hyperbolic geometry.

CHT motivates proof, but is not explicitly used.
Applications of Sullivan’s CHT:

- Dimension of geometrically infinite limit sets.
- Conformal factorization into Lipschitz maps.
- Bowen’s dichotomy for divergence type groups.
- Linear time algorithm for conformal mapping.
- Optimal quadrilateral meshing of polygons and PSLGs.
- Polynomial algorithm for nonobtuse triangulation.
Questions:

- Can we replace 2.5 by 2?
- Best QC constant for iota map (2.1 < K < 7.82)?
- Can we do better than iota?
- 2-QC + Lipschitz \(\Rightarrow\) Brennan’s Conj.
- 3-D meshes? Thick/Thin? Convex hull? Ricci flow?
- Applications of Mumford-Bers compactness?
- Applications of Kahn-Markovic results?