How To Draw a Conformal Map

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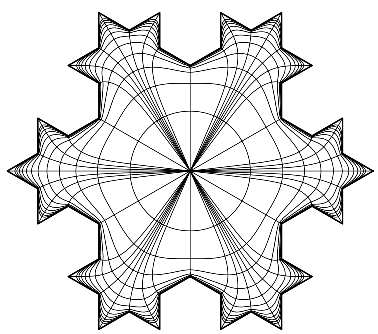
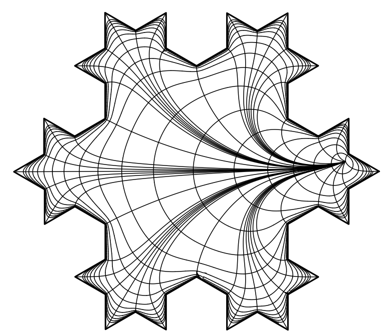
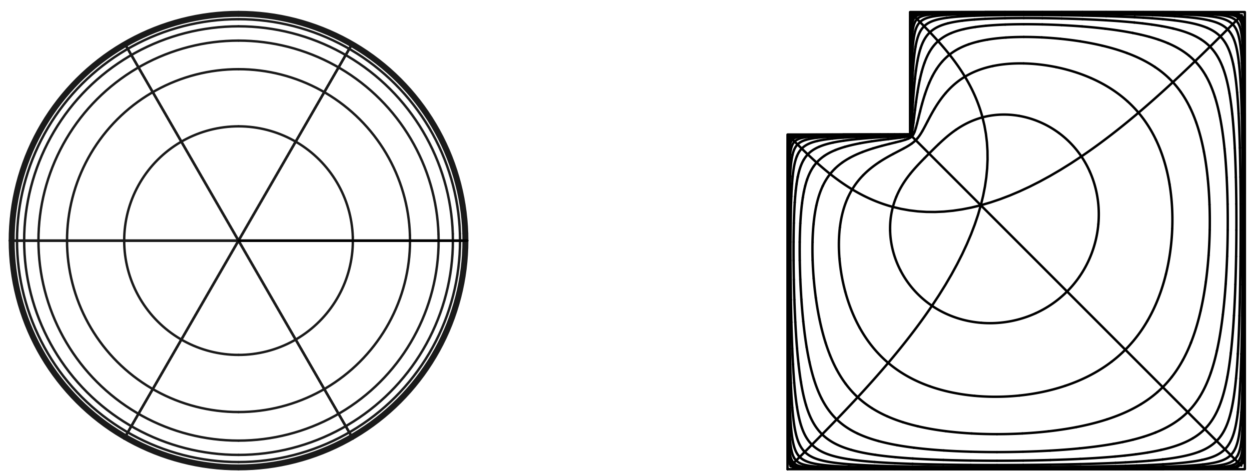
Math Club, Fall 2011



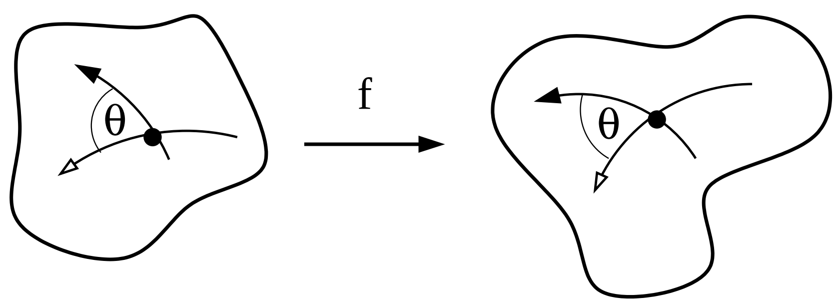


www.math.sunysb.edu/~bishop/lectures

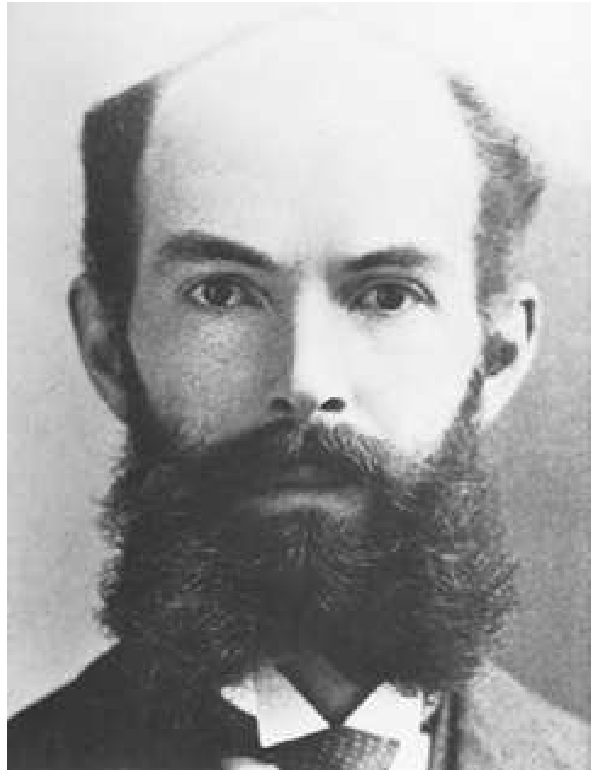
Riemann Mapping Theorem: If Ω is a simply connected, proper subdomain of the plane, then there is a conformal map $f : \mathbb{D} \to \Omega$.



Conformal = angle preserving



Our Founder



William Fogg Osgood
First proof of RMT, 1900

Paul Koebe
Modern Proof of RMT,

- Assume Ω is bounded, choose $z_0 \in \Omega$.
- $\mathcal{F}$ = conformal maps into $\mathbb{D}$, $z_0 \to 0$
- $\mathcal{F} \neq \emptyset$: $(z \to z - z_0)/\text{diam}(\Omega) \in \mathcal{F}$.
- There is $f \in \mathcal{F}$ that maximizes $|f'(z_0)|$.
- This f is onto (hence Riemann map).

Last two steps are hard parts.

Next to last step uses compactness (normal families).

Last step is proof by contradiction.

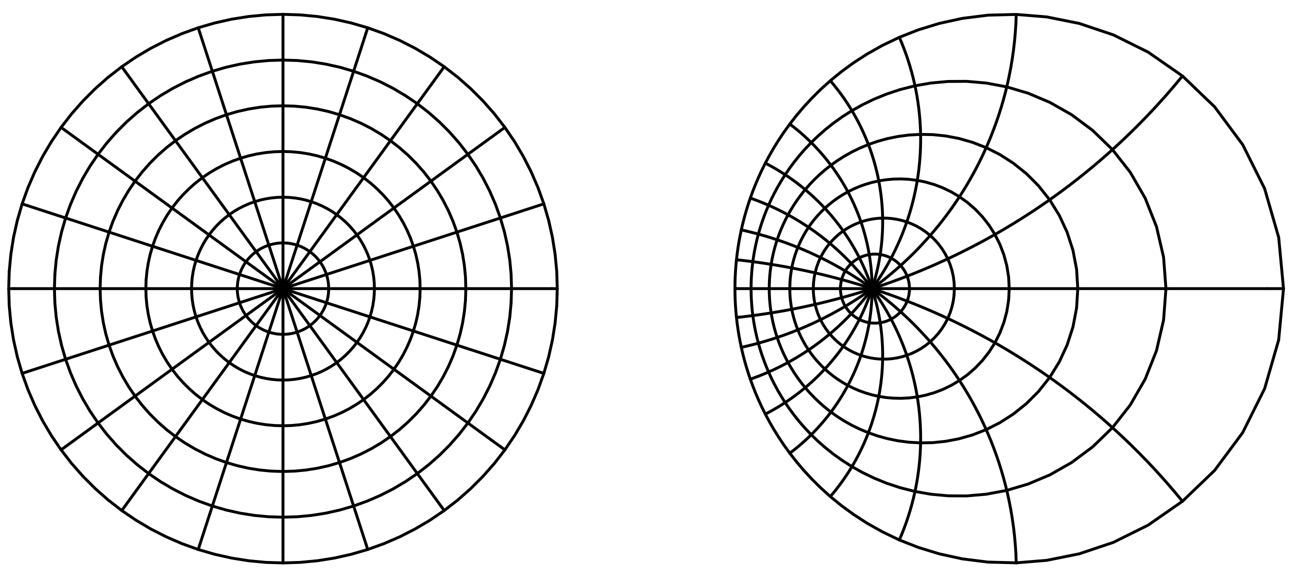
General Möbius transformations (conformal, 1-1, group):

$$z \to \frac{az+b}{cz+d}$$

Special case:

$$z \to \frac{z-w}{1-\bar{w}z}$$

maps unit disk to itself and maps $w \to 0$ and $0 \to -w$.



Assume $|f'(z_0)|$ is maximal, but $f(\Omega) \neq \mathbb{D}$.

Choose $w \in \mathbb{D} \setminus f(\Omega)$.

Lemma: If $0 \in \Omega \subset \mathbb{D}$ is simply connected but not the whole disk then there is conformal map $g : \Omega \to \Omega' \subset \mathbb{D}$ with $|g'(0)| > 1$.

$g \circ f : \Omega \to \mathbb{D}$ is conformal and by chain rule

$$\frac{d}{dz}|g \circ f(z_0)| > |f'(z_0)|.$$

Contradiction. Hence f is onto $\mathbb{D}$.

Proof of lemma:

Choose Möbius maps σ, τ of disk so

$$\sigma(w) = 0, \qquad \tau(\sqrt{\sigma(0)}) = 0.$$

Compute derivative at 0 of $g = \tau(\sqrt{\sigma})$

$$|\frac{d}{dz}\tau(\sqrt{\sigma(z)})|_0 \geq \frac{1+|w|}{2\sqrt{|w|}} > 1.$$

Proof of RMT gives an algorithm:

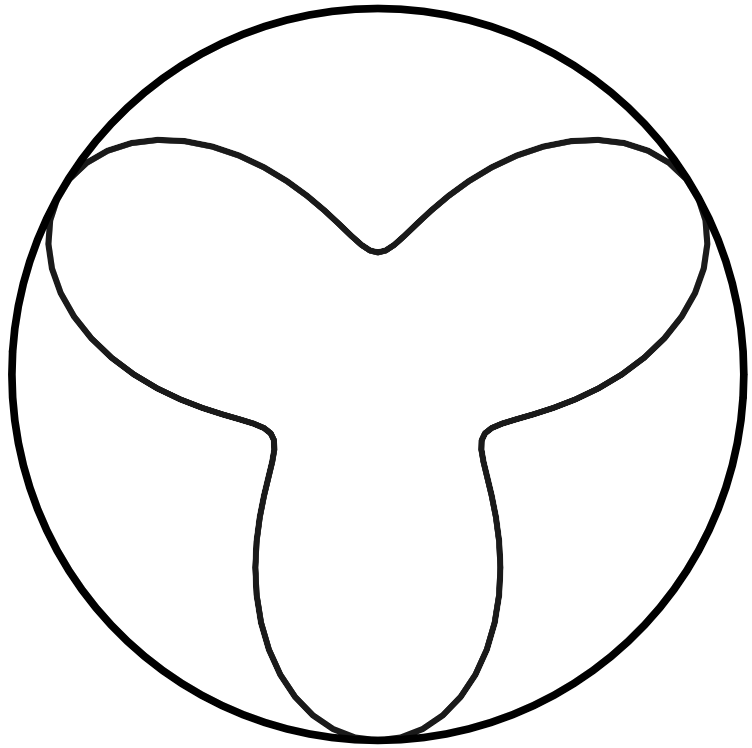
- Choose linear map of Ω into $\mathbb{D}$.
- Choose image boundary point closest to origin.
- Compose f with $g = \tau(\sqrt{\sigma}) : \mathbb{D} \to \mathbb{D}$.
- Repeat.

$d_n = \mathrm{dist}(\partial\Omega_n, 0)$

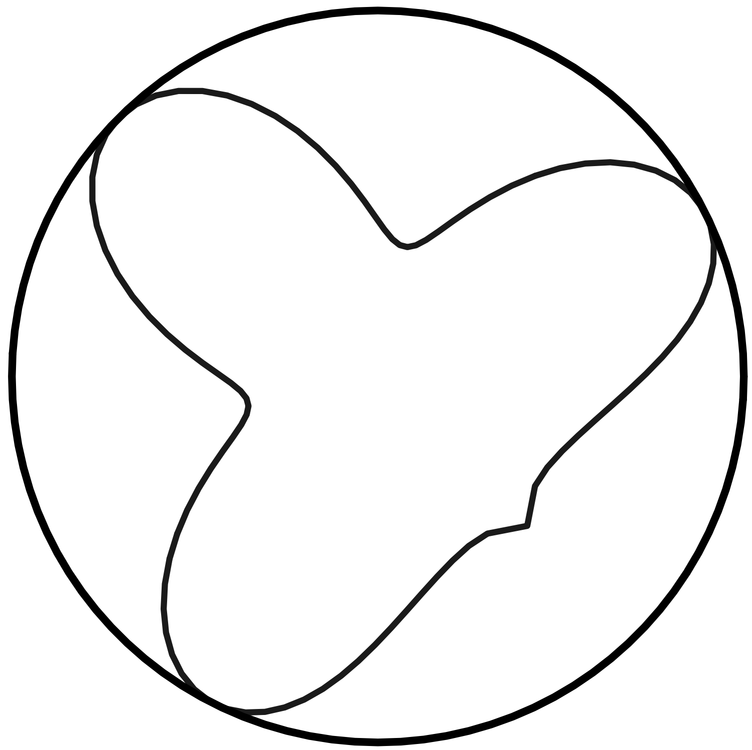
Lemma: If $k \geq 4/(1-\sqrt{d_n})$, then $d_{n+k} > \sqrt{d_n}$.

Cor: If $d_0 \geq 1/2$ then $d_n = 1 - O(1/n)$.

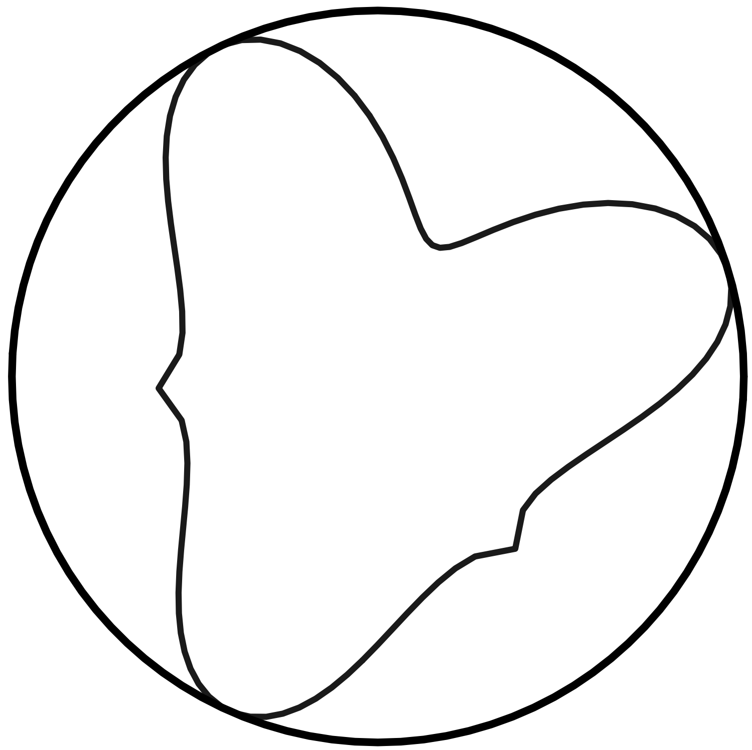
Takes 1,000,000 iterations to get $d_n = .000001$.



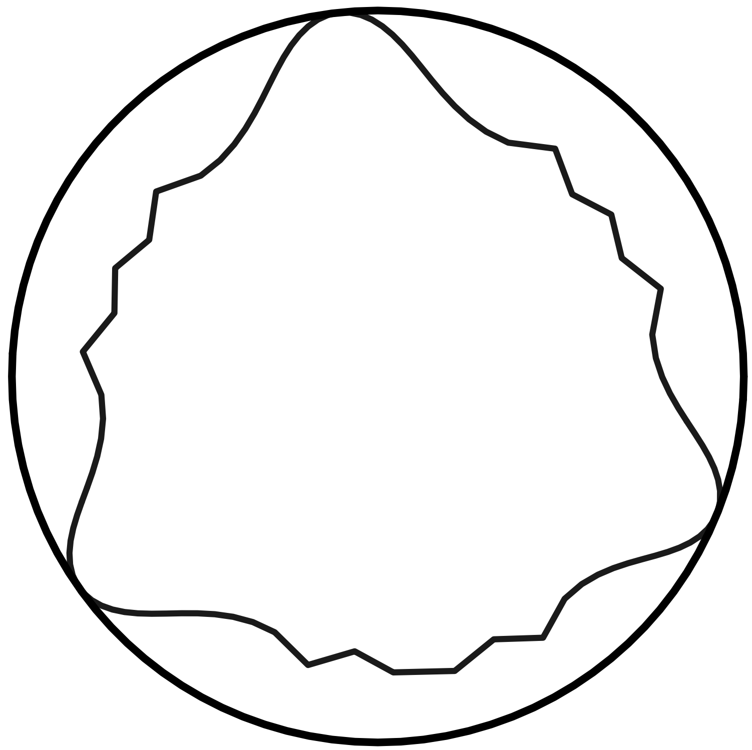
0th iteration



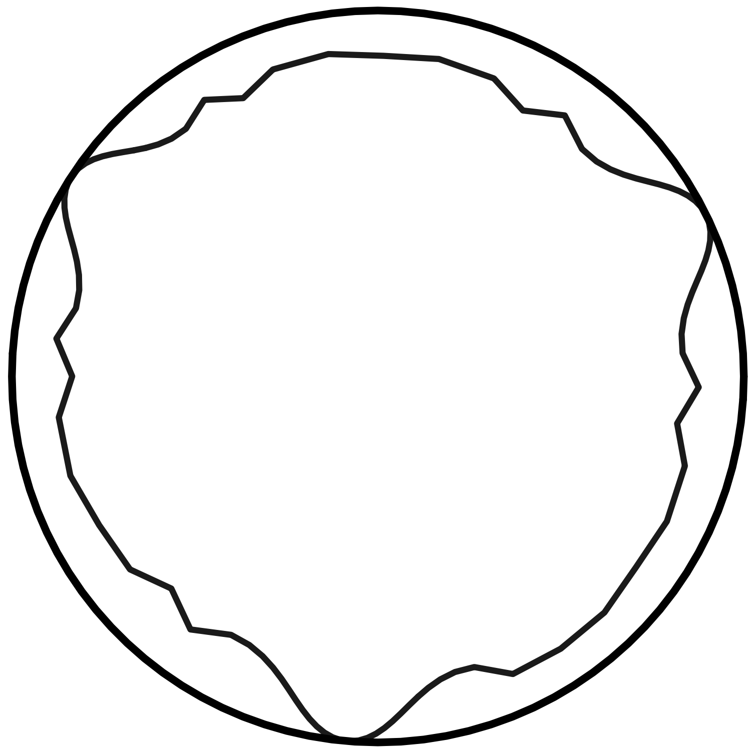
1th iteration



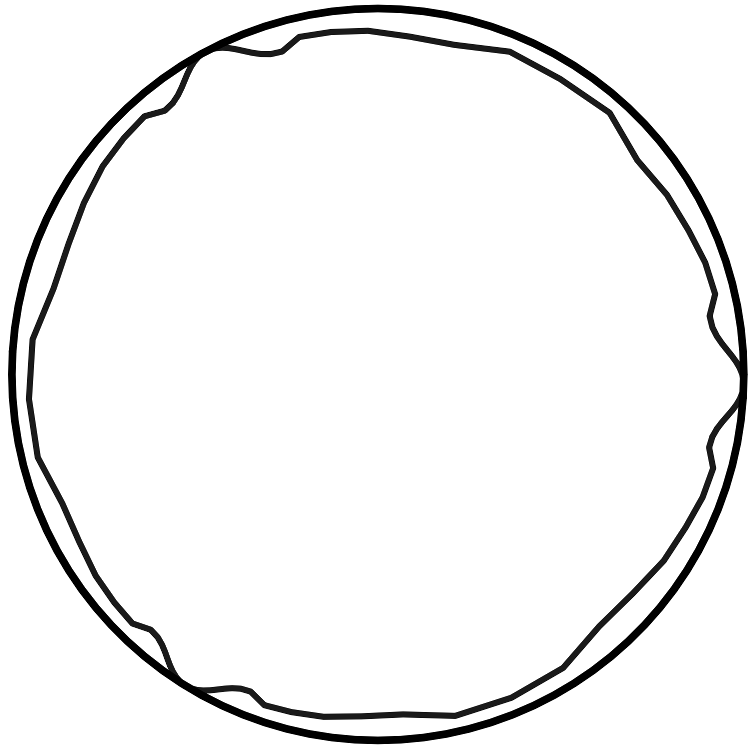
2nd iteration



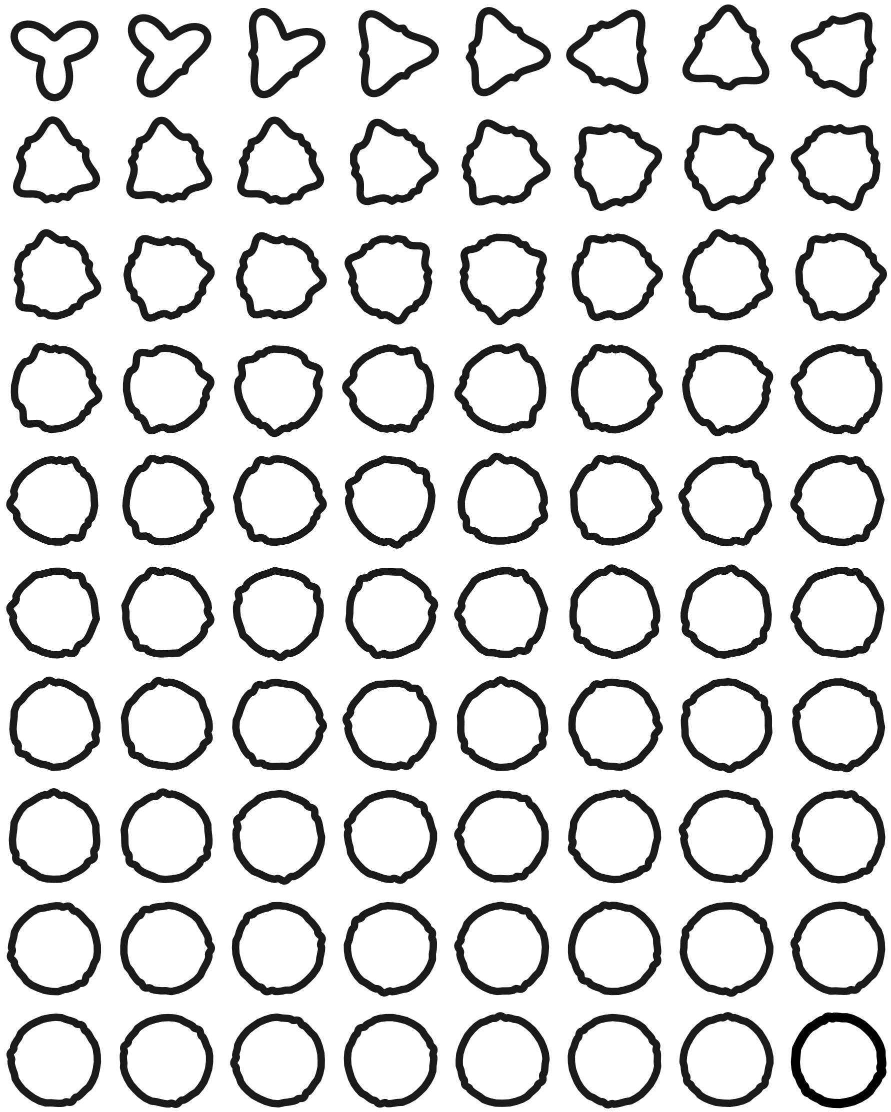
10th iteration



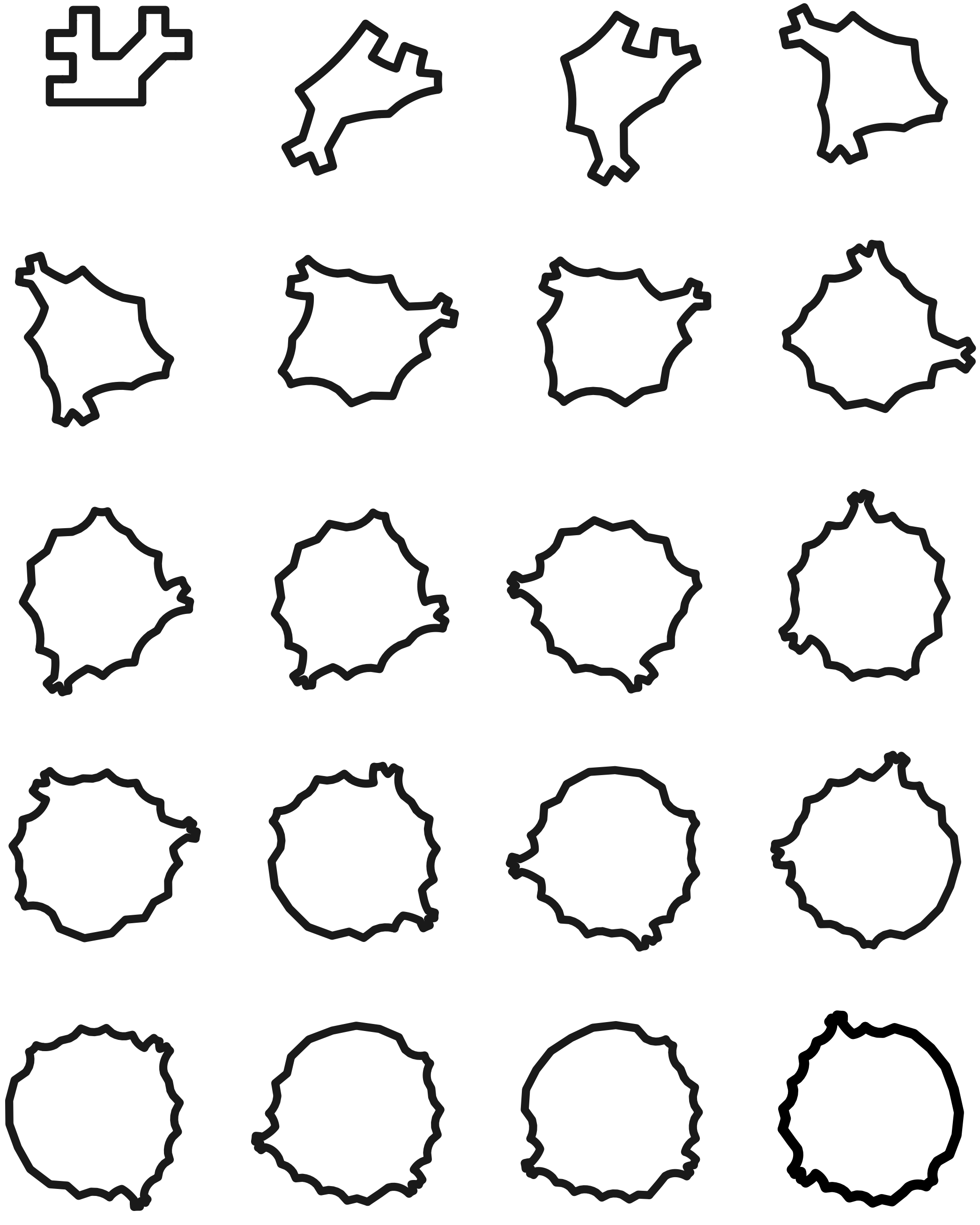
20th iteration

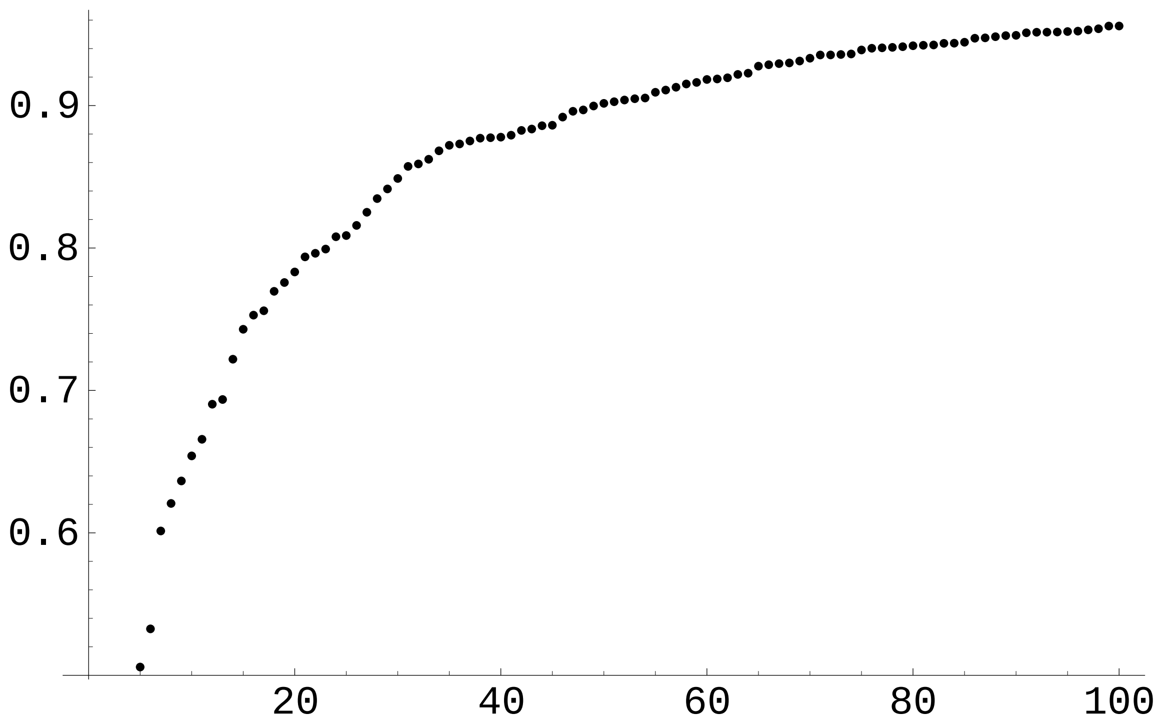


50th iteration



First 80 iterations





0.9
0.8
0.7
0.6
20
40
60
80
100

Koebe's iteration: it works, but is slow.

Numerous faster methods.

Schwarz-Christoffel formula (1867):

$$f(z) = A + C \int^{z} \prod_{k=1}^{n} (1 - \frac{w}{z_k})^{\alpha_k - 1} dw,$$

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Christoffel

Schwarz

Schwarz-Christoffel formula (1867):

$$f(z) = A + C \int^{z} \prod_{k=1}^{n} (1 - \frac{w}{z_k})^{\alpha_k - 1} dw,$$

$\{\alpha_1\pi, \ldots, \alpha_n\pi\}$, are interior angles of polygon.
$\{z_1, \ldots, z_n\}$ are points on circle mapping to vertices.

α's are known.

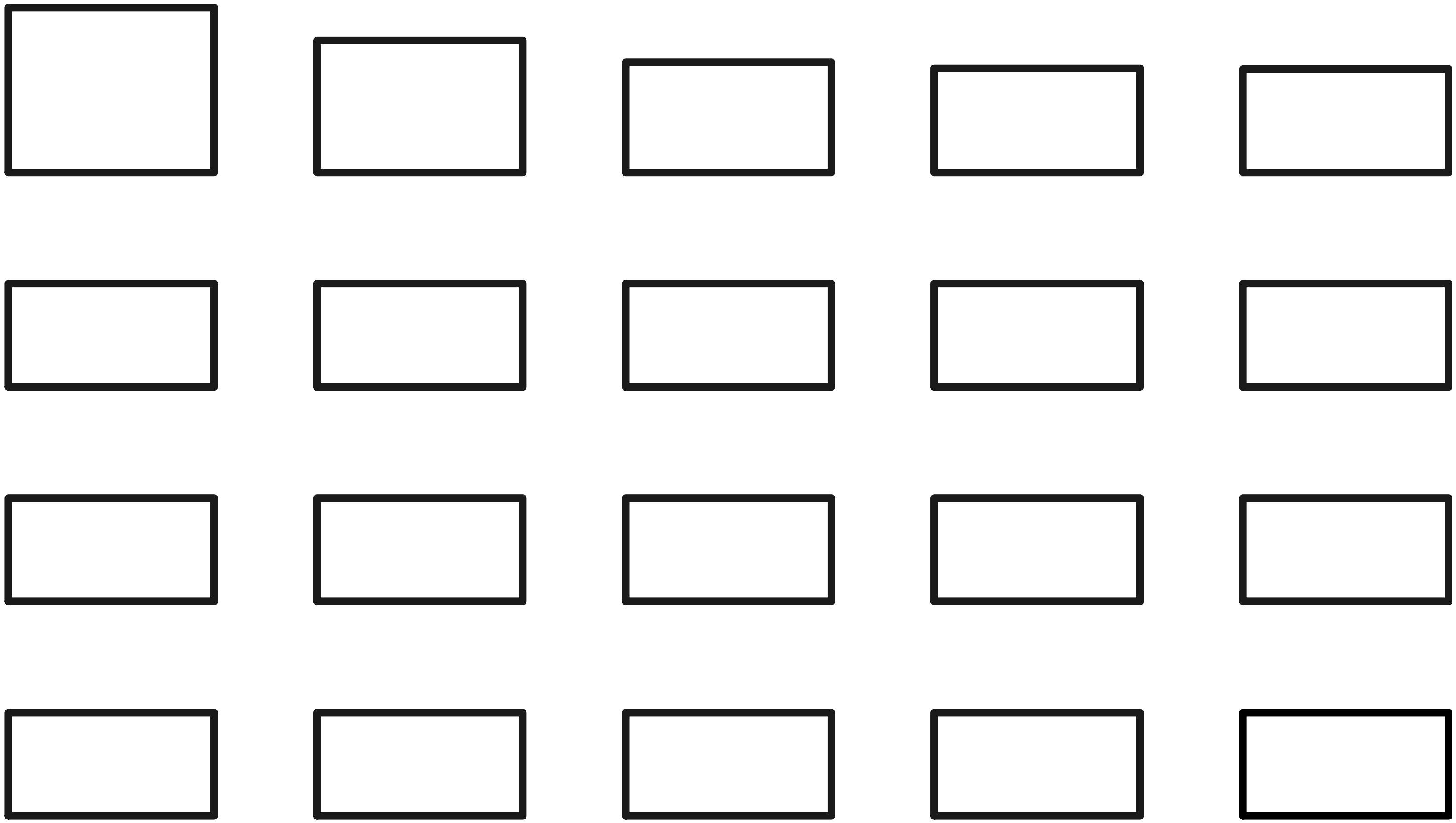
z's must be solved for.

Basic idea: guess some parameters. Use formula to draw the corresponding polygon. Compare to target polygon and revise guesses.

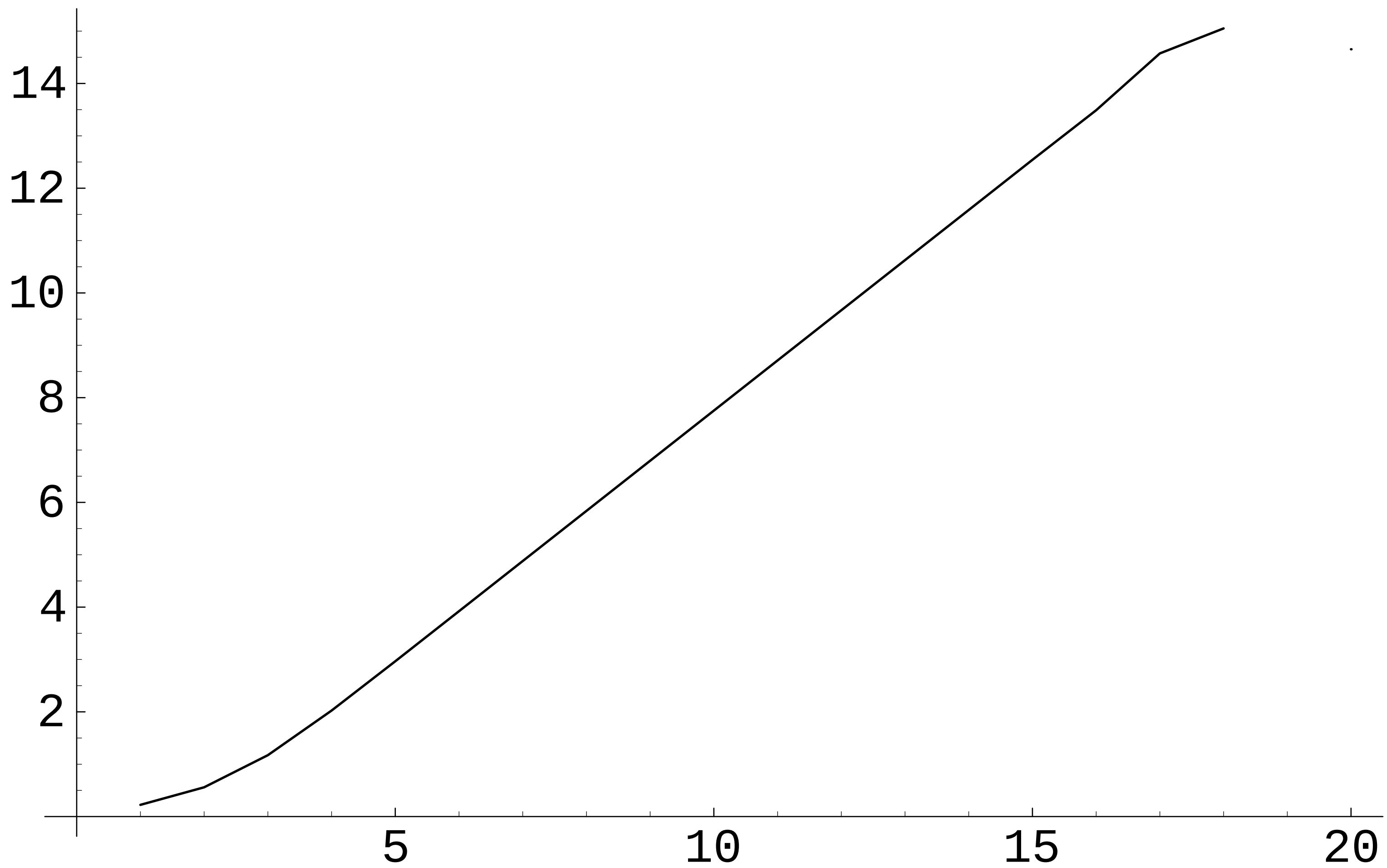
Davis' method:

- Compare guessed polygon to target polygon.
- If an edge is too long, shorten corresponding parameter arc.
- If too long, lengthen the gap.

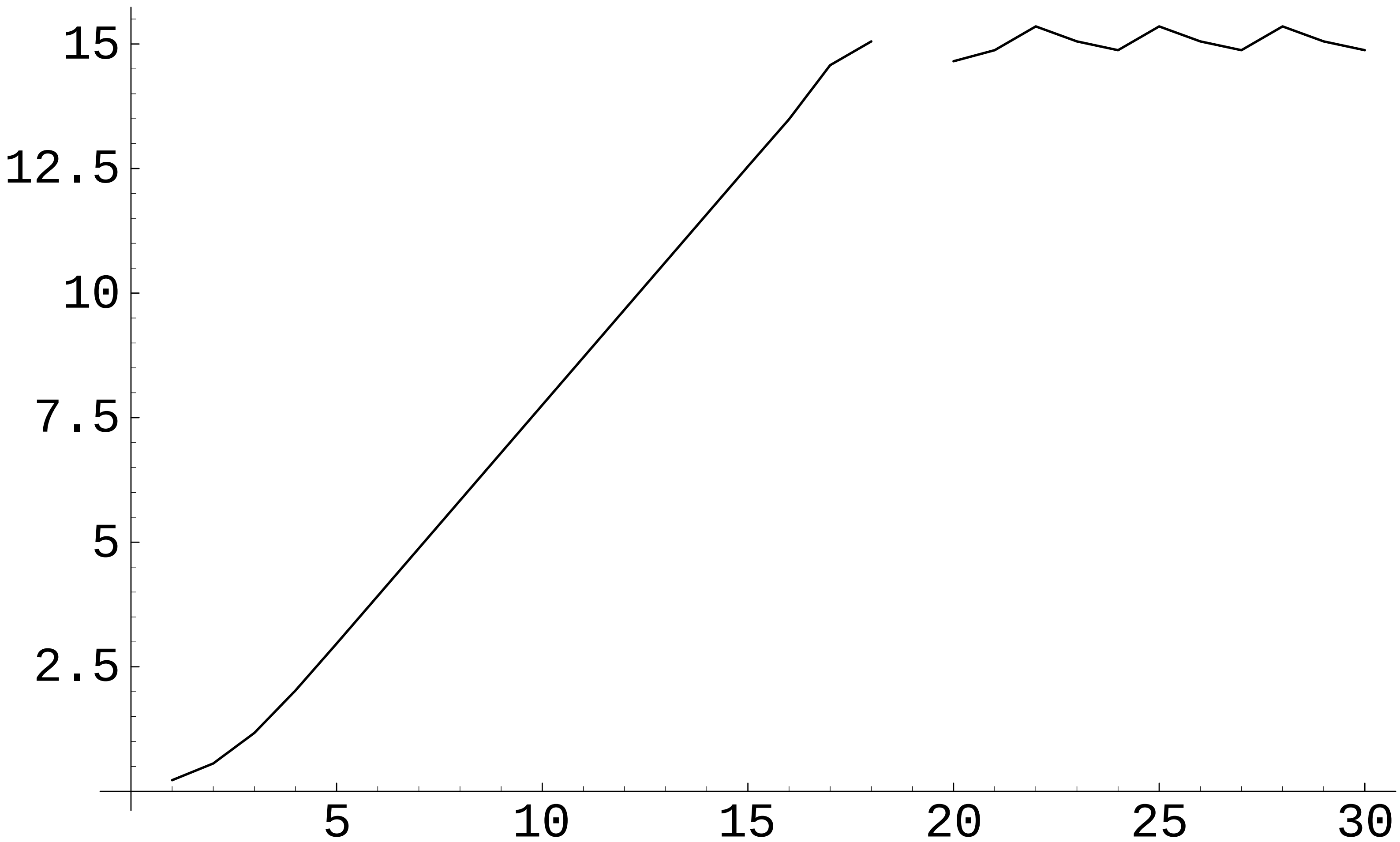
$$\text{new gap} = \text{old gap} \times \frac{\text{target side}}{\text{old side}}$$



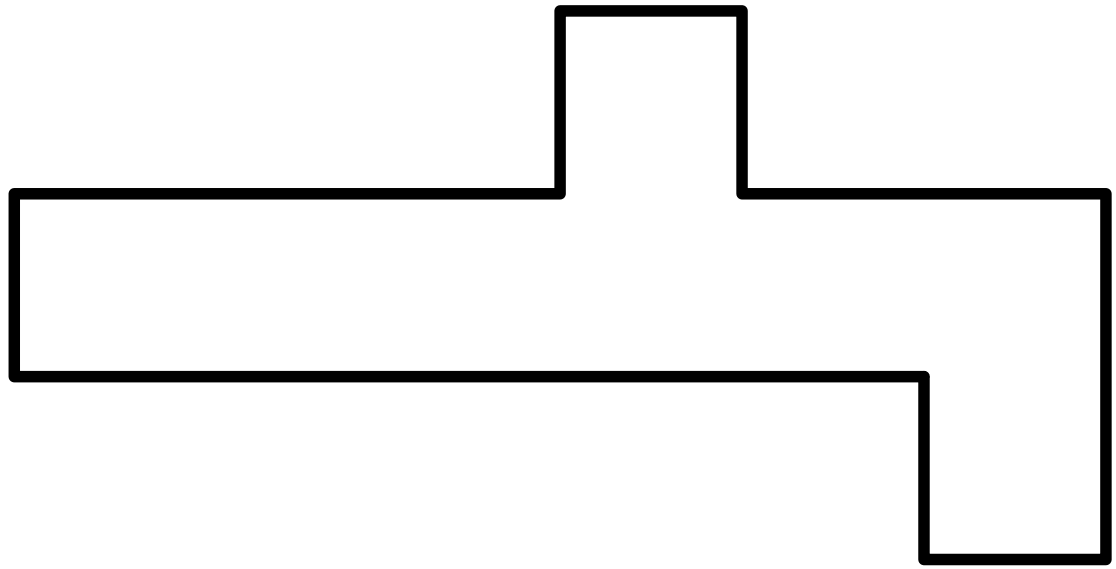
20 iterations of Davis' method for a rectangle.

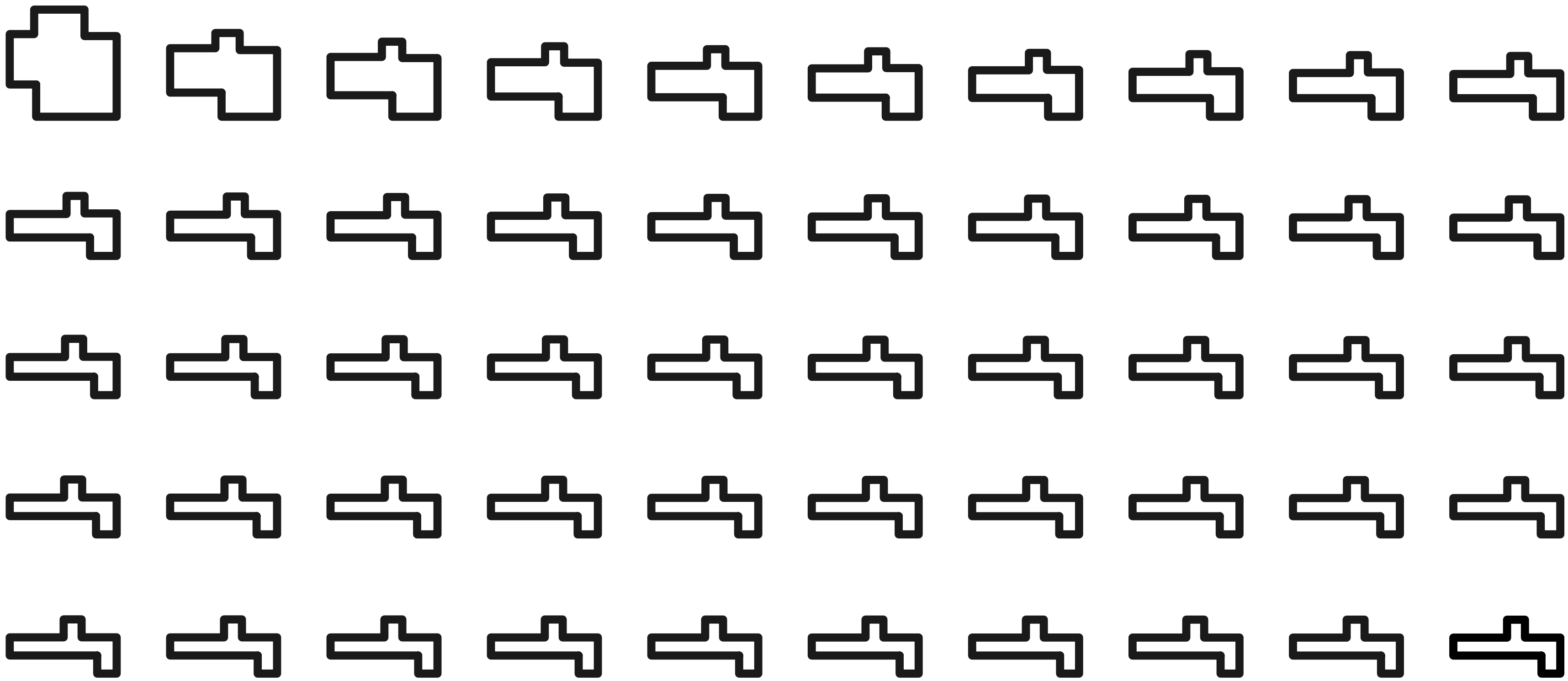


20 iterations of Davis' method - QC error.

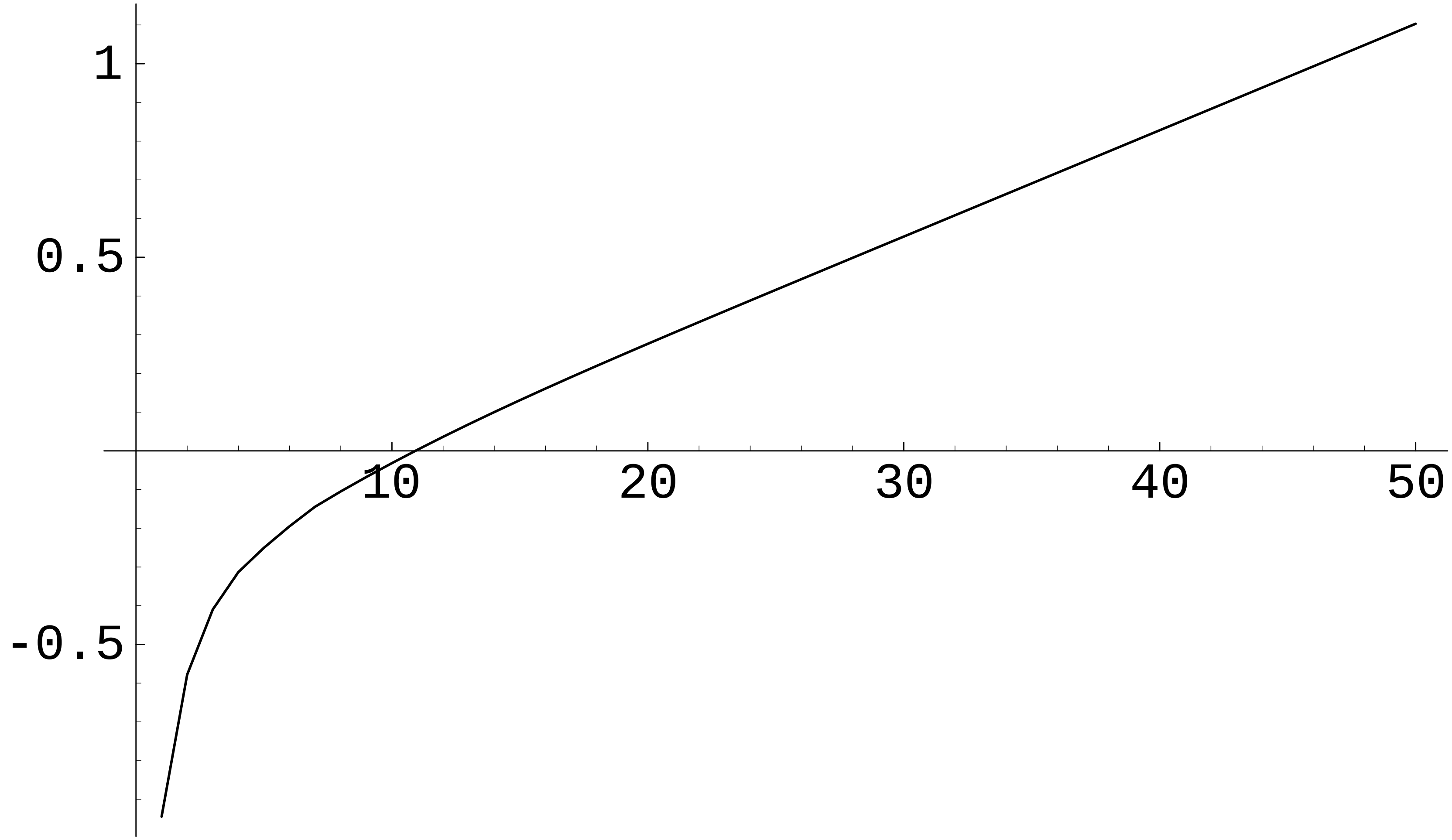


30 iterations of Davis' method - QC error.
We hit machine error $\approx 10^{-15}$.

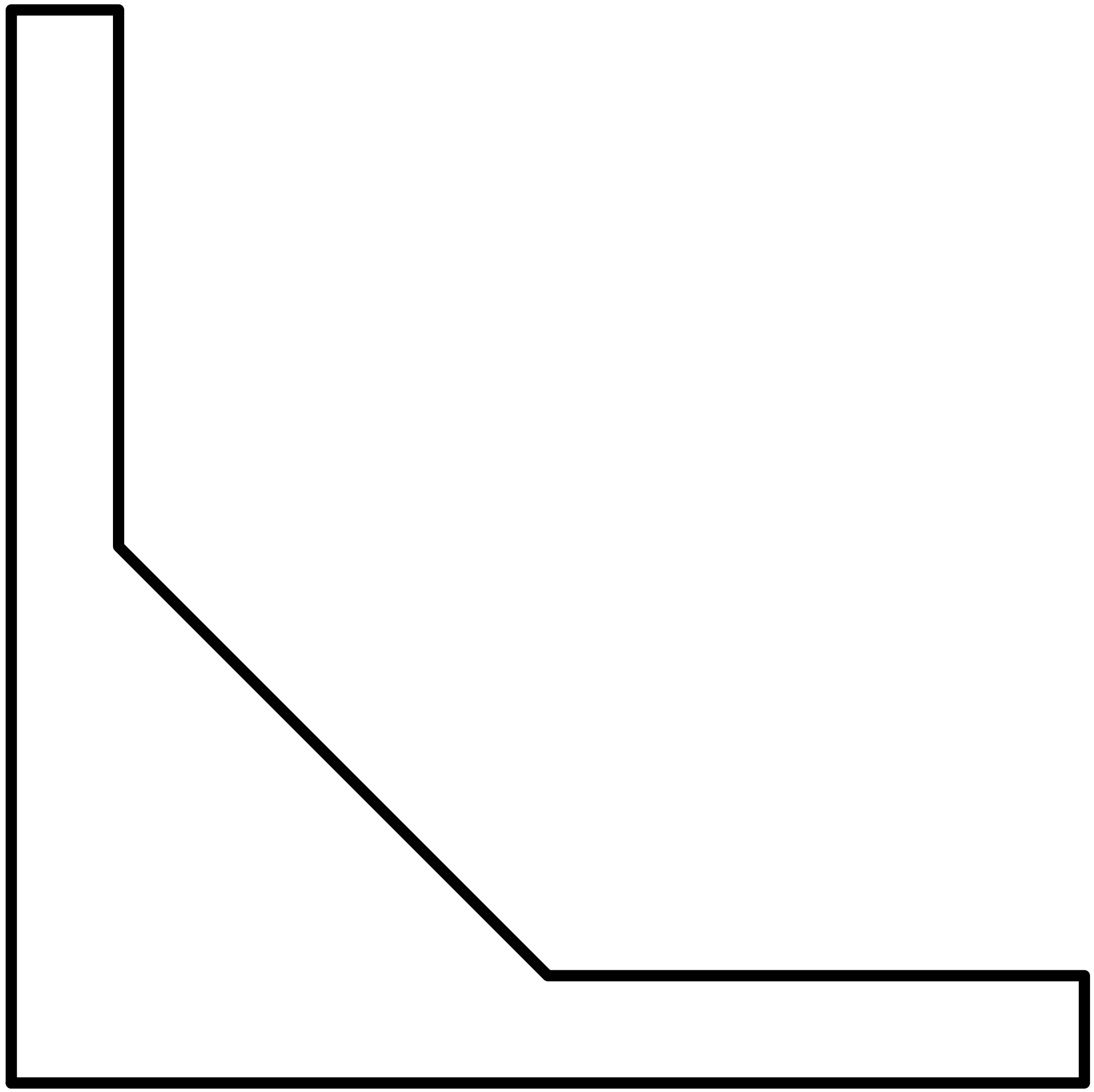


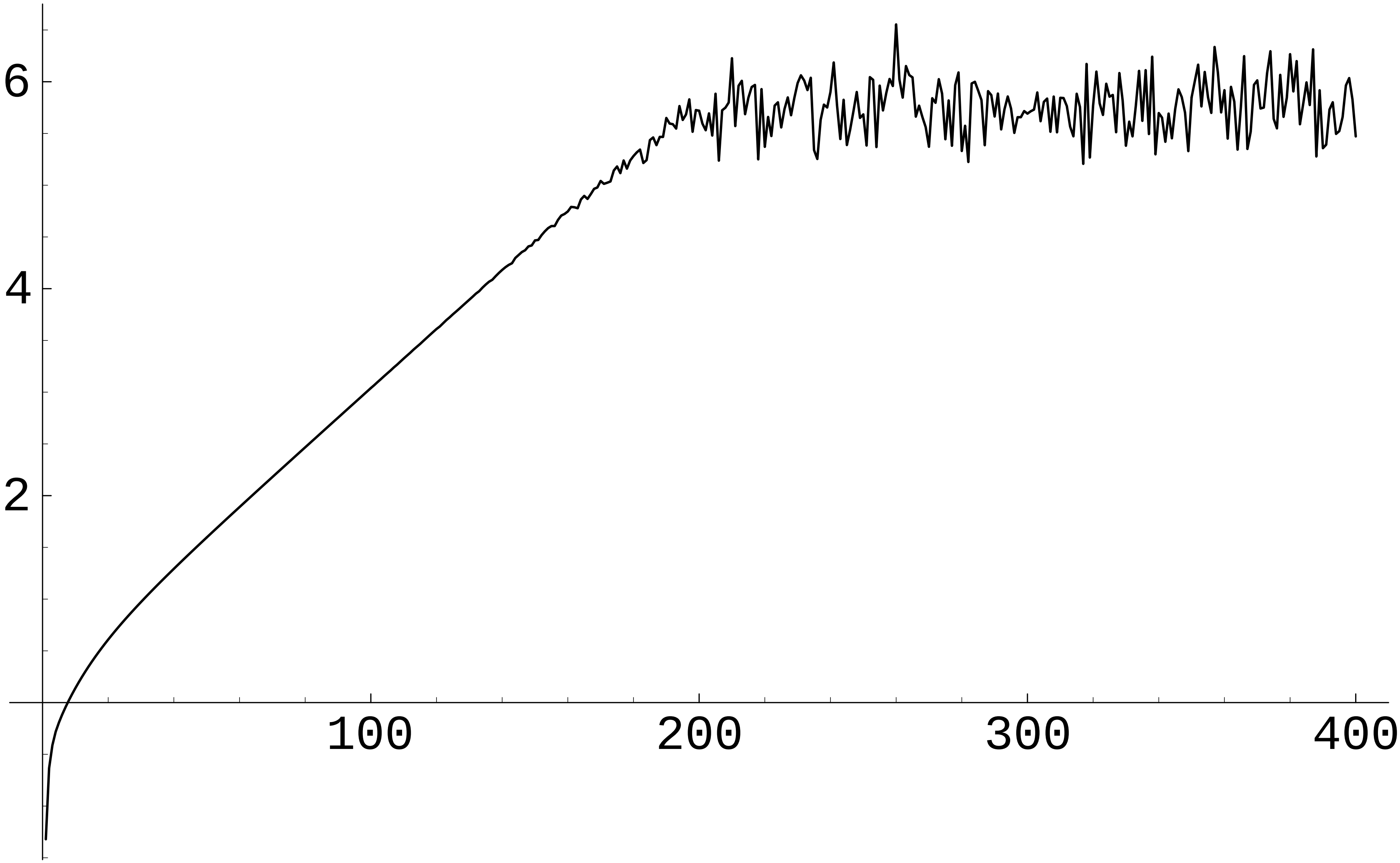


50 iterations of Davis' method.



QC error for 50 iterations of Davis' method.





QC error for 400 iterations of Davis' method.

Conformal Crowding:

Riemann map can dramatically shrink distances.

For a $1 \times R$ rectangle, two parameters are $\leq e^{-\pi R}$ apart.

If $e^{-\pi R} \leq 10^{-16}$ = machine precision, they are the same point to the computer. $R \approx 11$.

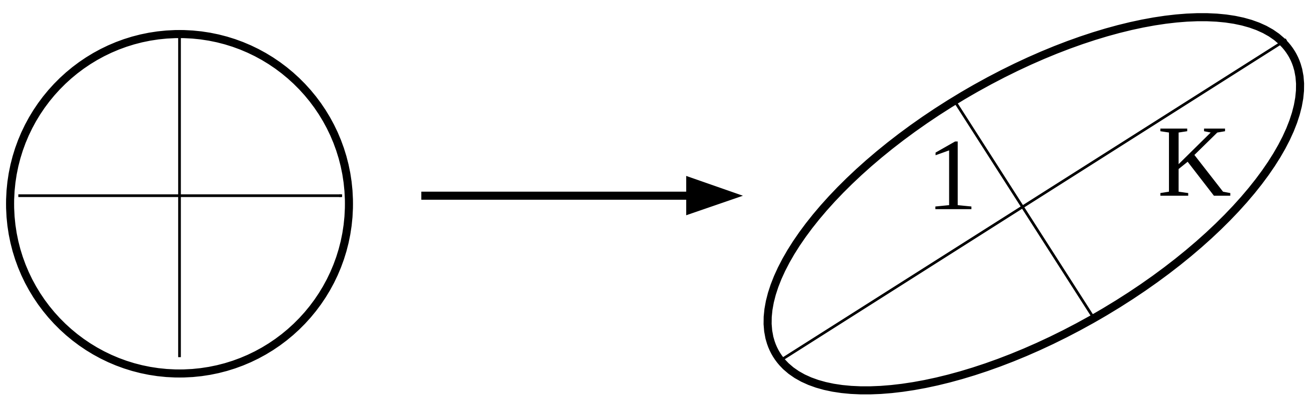
QC mappings: distort angles by bounded amount.

$$\partial f = \frac{1}{2}(f_x - i f_y), \quad \overline{\partial} f = \frac{1}{2i}(f_x + i f_y).$$

Conformal $f : \mathbb{D} \to \Omega$ with $\overline{\partial} f = 0$ (Cauchy-Riemann).

We measure distance to conformality by dilatation

$$\|f\| = \sup |\mu_f| \equiv \sup |\overline{\partial} f / \partial f|.$$



Affine map between triangles $\{0, 1, a\}$ and $\{0, 1, b\}$ is

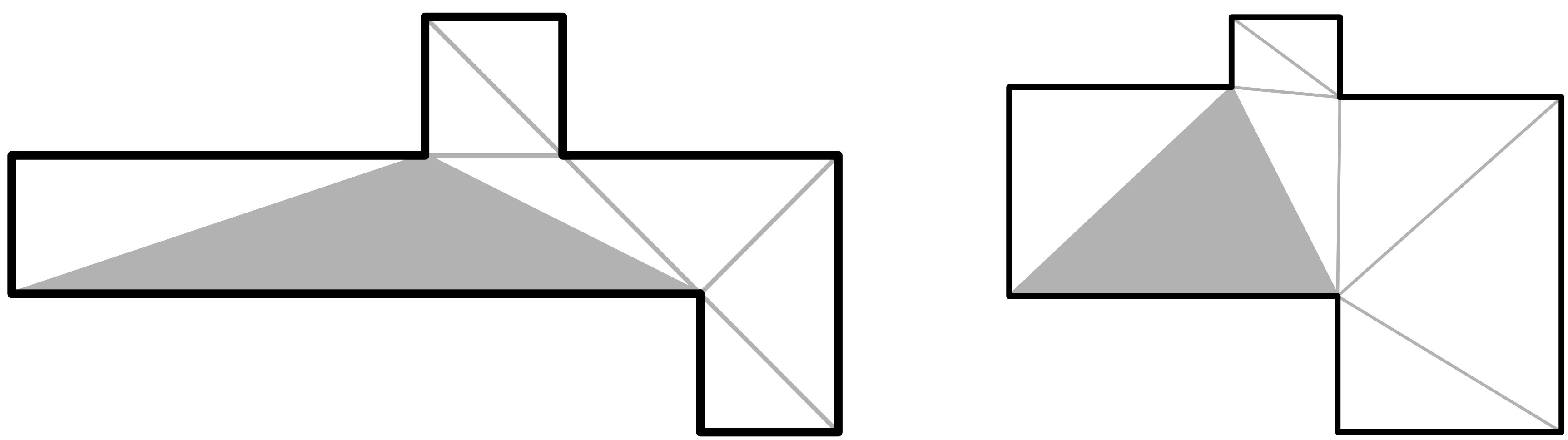
$$f(z) \to \alpha z + \beta \bar{z}$$

where $\alpha + \beta = 1$ and $\beta = (b - a)/(a - \bar{a})$. Then

$$K_f = \frac{1 + |\mu_f|}{1 - |\mu_f|},$$

where

$$\mu_f = \frac{f_{\bar{z}}}{f_z} = \frac{\beta}{\alpha} = \frac{b - a}{b - \bar{a}},$$



How to compute integrals in SC-formula (error $< 10^{-16}$)?

$$\int_a^b f(x)dx \approx \sum_{k=1}^{n} a_n f(x_n)$$

- Right-hand-rule: error $= O(n^{-1})$.
- Midpoint rule: error $= O(n^{-2})$.
- Simpson's rule: error $= O(n^{-4})$.
- Gauss Quadrature: error $= O(n^{-2n})$.

Evaluation points $\{x_k\}$ and weights $\{a_k\}$ are given in terms of orthogonal polynomials.

SC-parameters by Newton's method.

$\mathbb{T}$ = unit circle

$\mathbb{T}_n$ = ordered n-tuple on circle (= SC parameter guess)

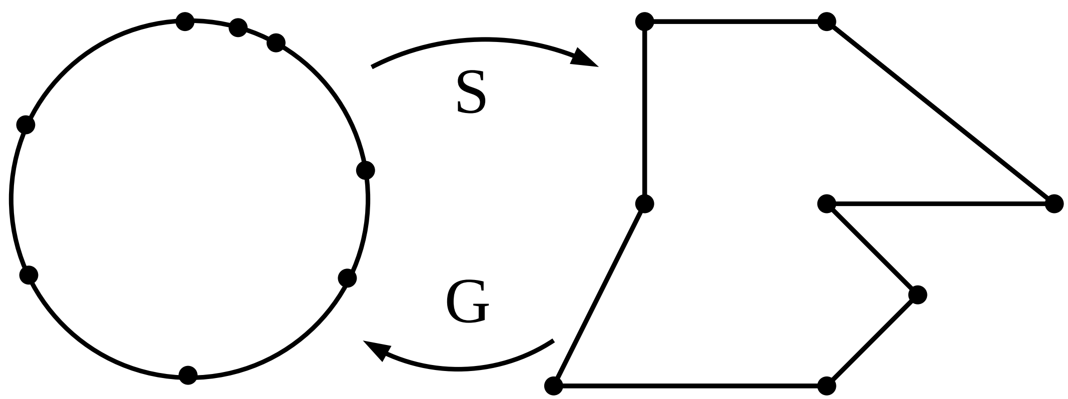
$\mathbb{C}$ = complex numbers

$\mathbb{C}_n$ = n-tuple of complex numbers (= polygons)

Fix angles in SC-formula. Then we get map:
$S : \mathbb{T}_n \to \mathbb{C}_n$ (parameters $\to$ polygons)

Guessing map:
$G : \mathbb{C}_n \to \mathbb{T}_n$ (polygons $\to$ parameters)

Compose $H = G \circ S : \mathbb{T}_n \to \mathbb{T}_n$ (para $\to$ para)



If P is target polygon, let $z_0 = G(P)$.

Let $F(z) = H(z) - z_0$.

Find solution of $F(z) = 0$ by Newton's method.

Then $H(z) = z_0 \Rightarrow G \circ S(z) = G(P) \Rightarrow S(z) = P$ (if G is 1-1).

We need:

- G to be 1-1.
- G to be computable.

How to solve $F(z) = 0$?

Define an iteration by

$$z_{n+1} = z_n - D_F^{-1}(F(z_n)).$$

where D_F is the derivative matrix of $F = (F_1, \ldots, F_d)$

$$(D_F)_{jk} = \frac{dF_k}{dx_j}.$$

In one dimension, this is

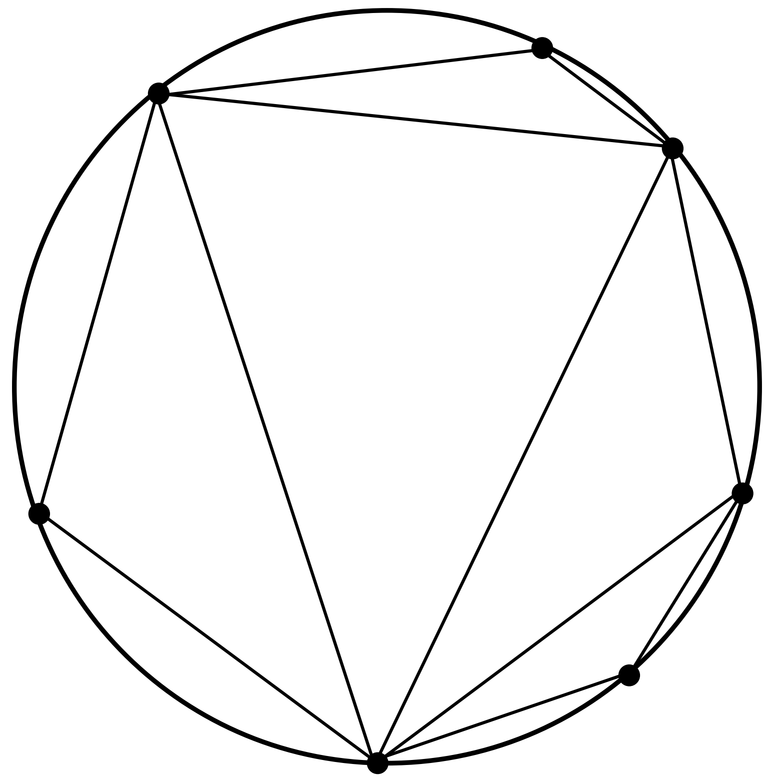
$$z_{n+1} = z_n - \frac{F(z_n)}{F'(z_n)}.$$

A couple of problems with this:

- What is a good G to choose?
- How do we compute D_F?
- $z_n, F(z_n) \in \mathbb{T}$. How do we do linear algebra?

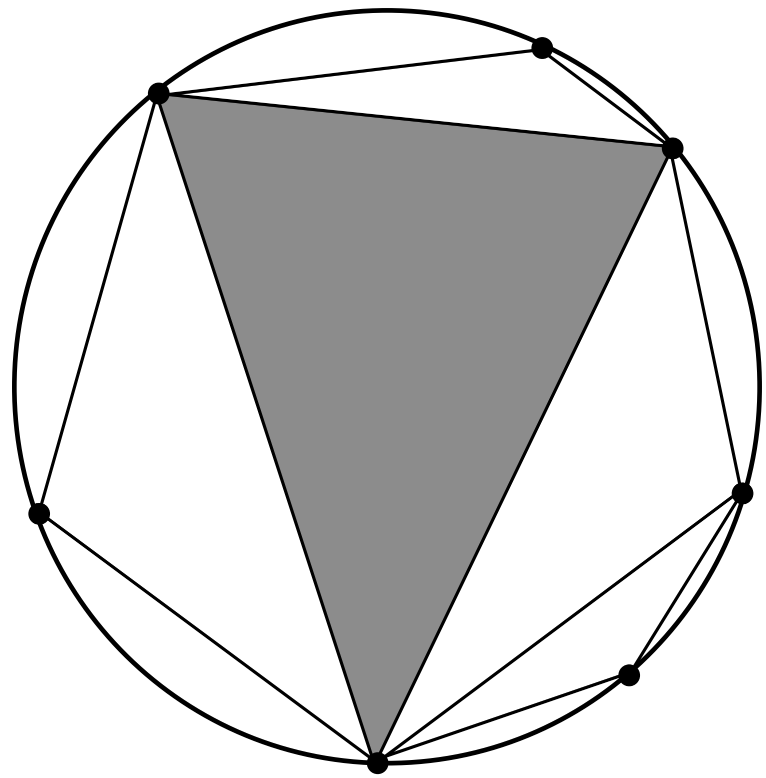
We will deal with these in reverse order.

How to make n-tuples on circle into a vector space?



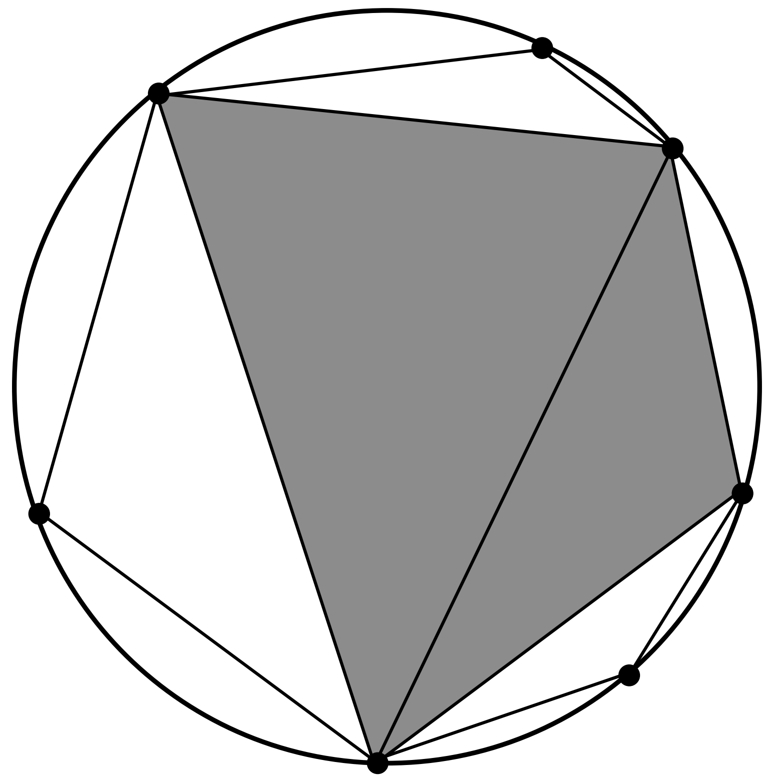
Triangulate the points.

How to make n-tuples on circle into a vector space?



Choose a root triangle.

How to make n-tuples on circle into a vector space?



For each triangle adjacent to root, form quadrilateral by union of the two triangles

How to make n-tuples on circle into a vector space?

Record the cross ratio of the four points

$$\mathrm{cr}(a,b,c,d) = \frac{(d-a)(b-c)}{(c-d)(a-b)}.$$

Invariant under Möbius transformations.

Points on circle $\Rightarrow$ cross ratio real valued.

How to make n-tuples on circle into a vector space?

The $n-2$ numbers $\log|\rho|$ determine n-tuple on circle up to a Möbius transformation of disk (free to place vertices of root triangle where we please).

If two n-tuples differ by a Möbius transformation, Schwarz-Christoffel gives similar polygons.

$\mathbb{T}_n^* = \mathbb{R}^{n-3}$= equivalence classes of ordered n-tuples on circle identified via Möbius transformations.

$\mathbb{C}_n^* = n$-tuples om complex numbers modulo similiarities

We can think of

$$G : \mathbb{C}_n^* \to \mathbb{T}_n^*, \qquad S : \mathbb{T}_n^* \to \mathbb{C}_n^*,$$

and

$$F : \mathbb{T}_n^* \to \mathbb{T}_n^*,$$

or

$$F : \mathbb{R}^{n-3} \to \mathbb{R}^{n-3}.$$

So now we can do linear algebra.

How do we compute derivative of $F = G \circ S$?

(1) Use a discrete approximation

$$\partial_j F_k(x_1, \ldots, x_m) = \frac{1}{h}[(F_k(x_1, \ldots, x_j + he_j, \ldots, x_m) - F_k(x_1, \ldots, x_j, \ldots x_m)].$$

Gives good result but slow ($m+1$ evaluations of F).

(2) Assume $DF = \mathrm{Id}$. Easy, fast, often works.

(3) Broyden updates. Start by assuming $DF = \mathrm{Id}$, but update DF after each evaluation of F. Often best compromise between speed and accuracy.

What is a good choice for G, the guessing function?

Davis's method: based on edge lengths.

CRDT : based on triangulation and cross ratios.

Iota: based on hyperbolic geometry.

CRDT

Cross Ratios and Delaunay Triangulations

Toby Driscoll and Stephen Vavasis, 1998

Triangulate polygon

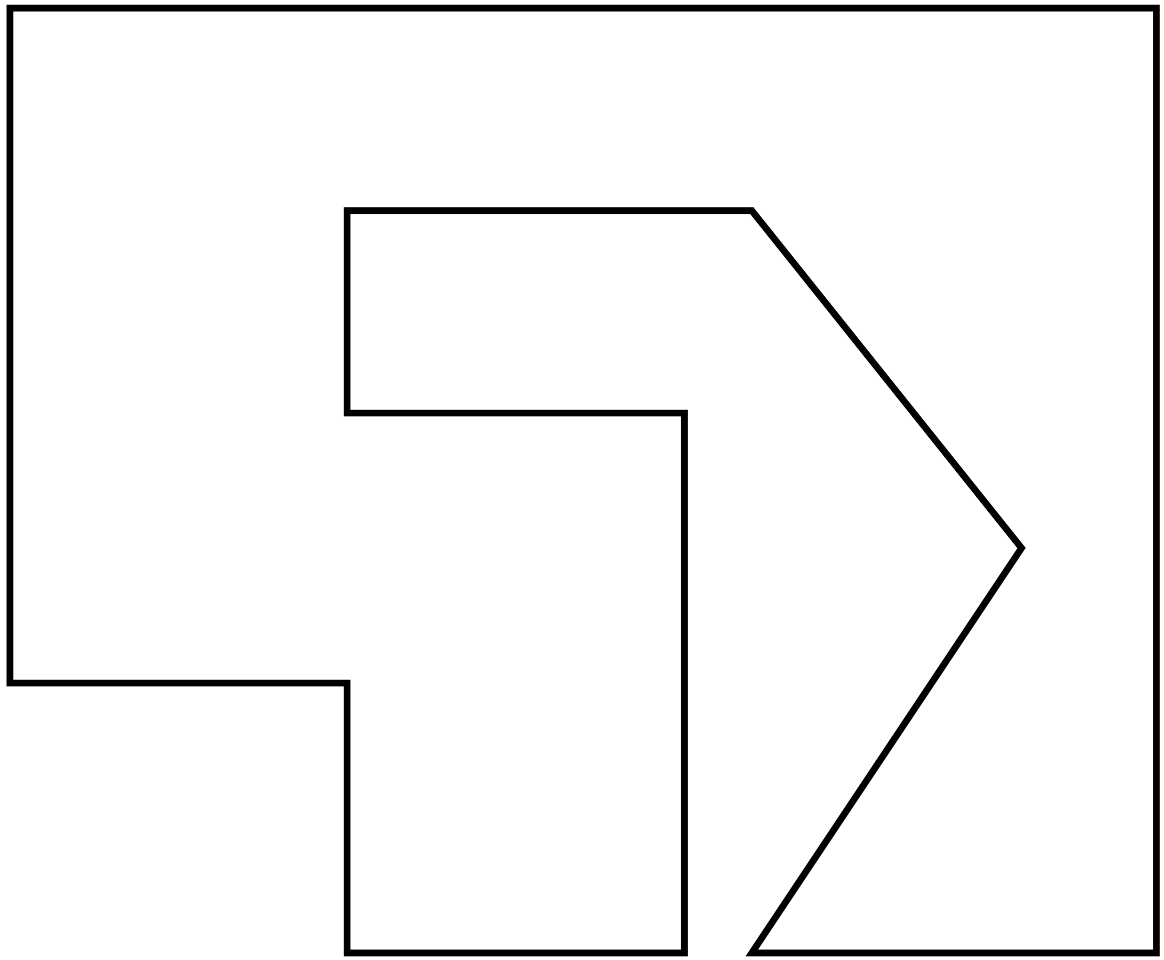
Choose root triangle

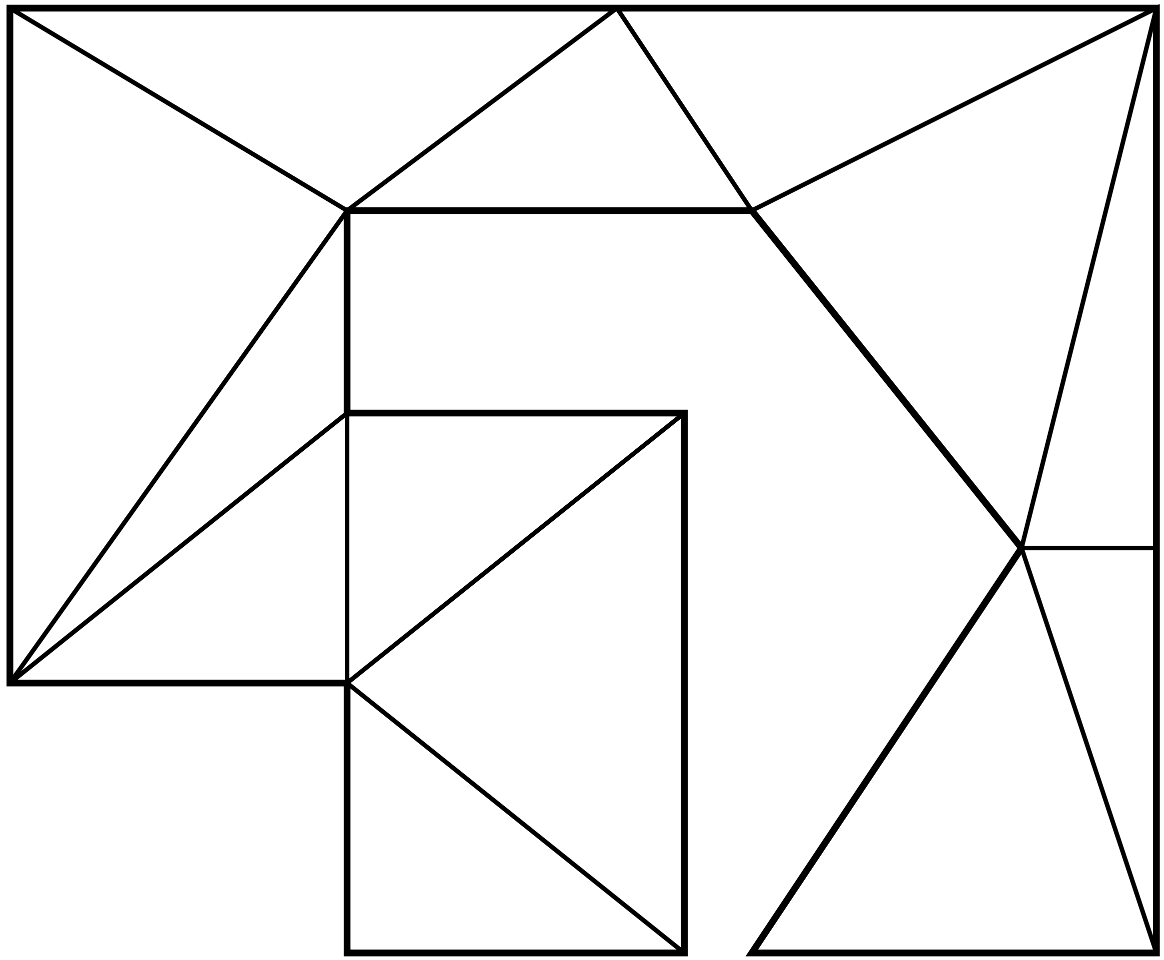
For non-roots form quadrilateral of triangle and parent

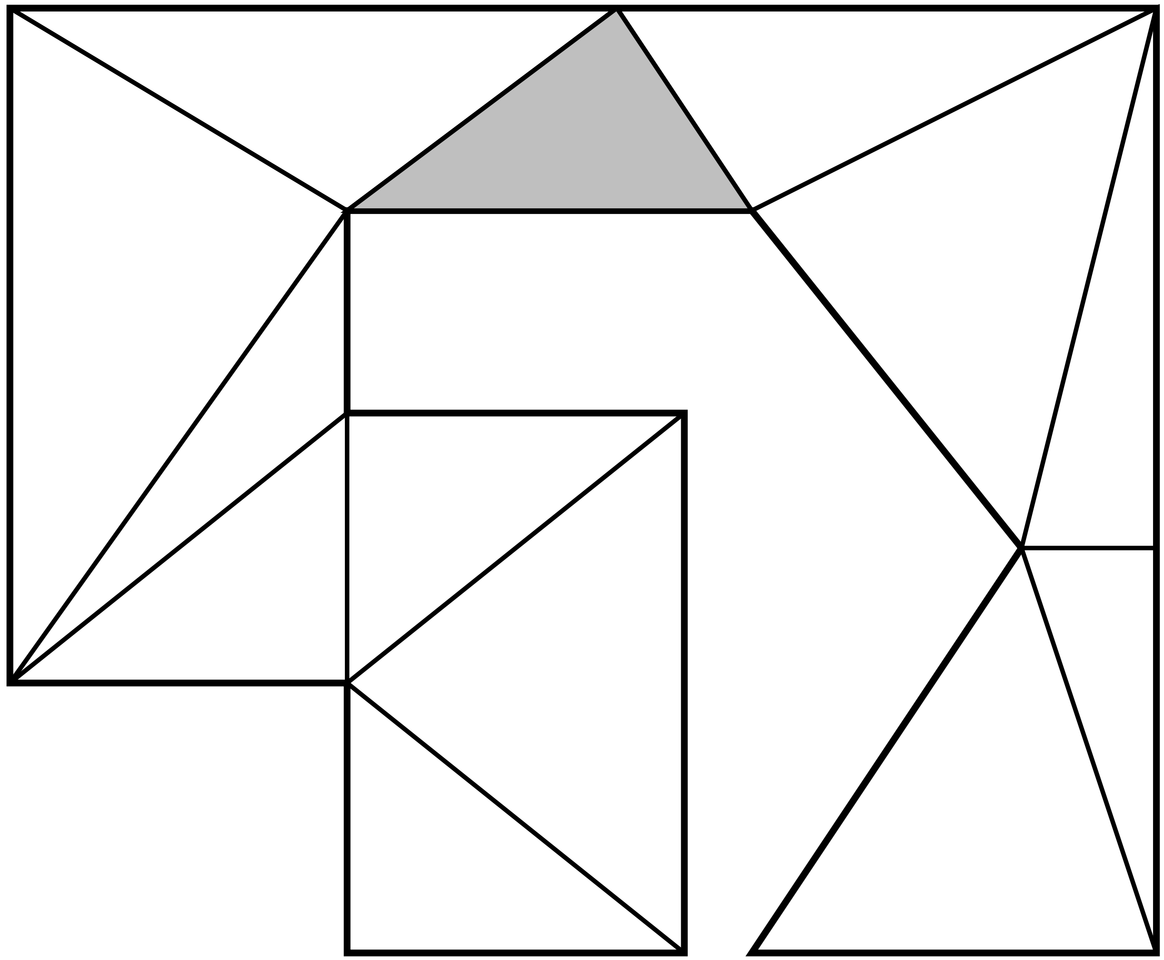
Compute cross ratio ρ of 4 vertices (complex number).

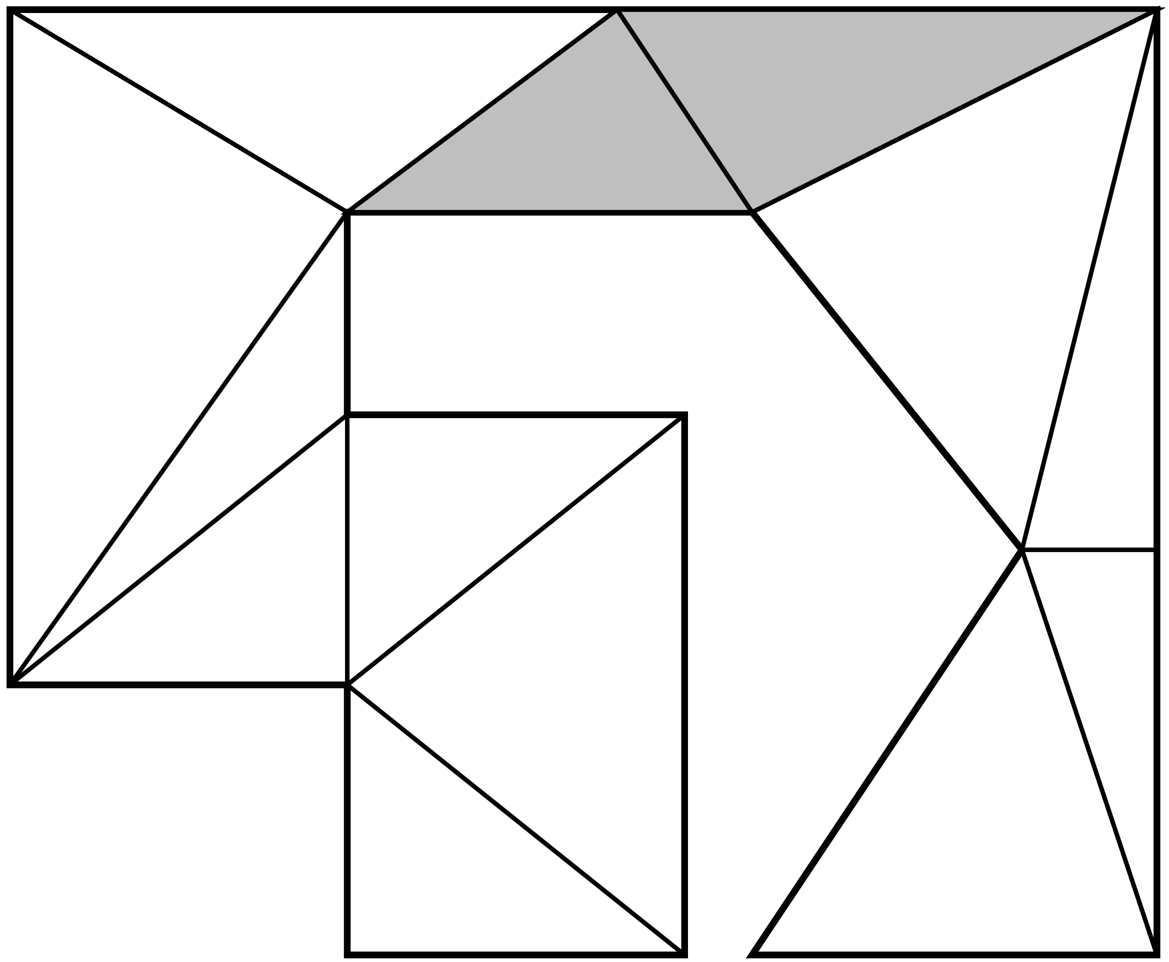
Record $\log|\rho|$.

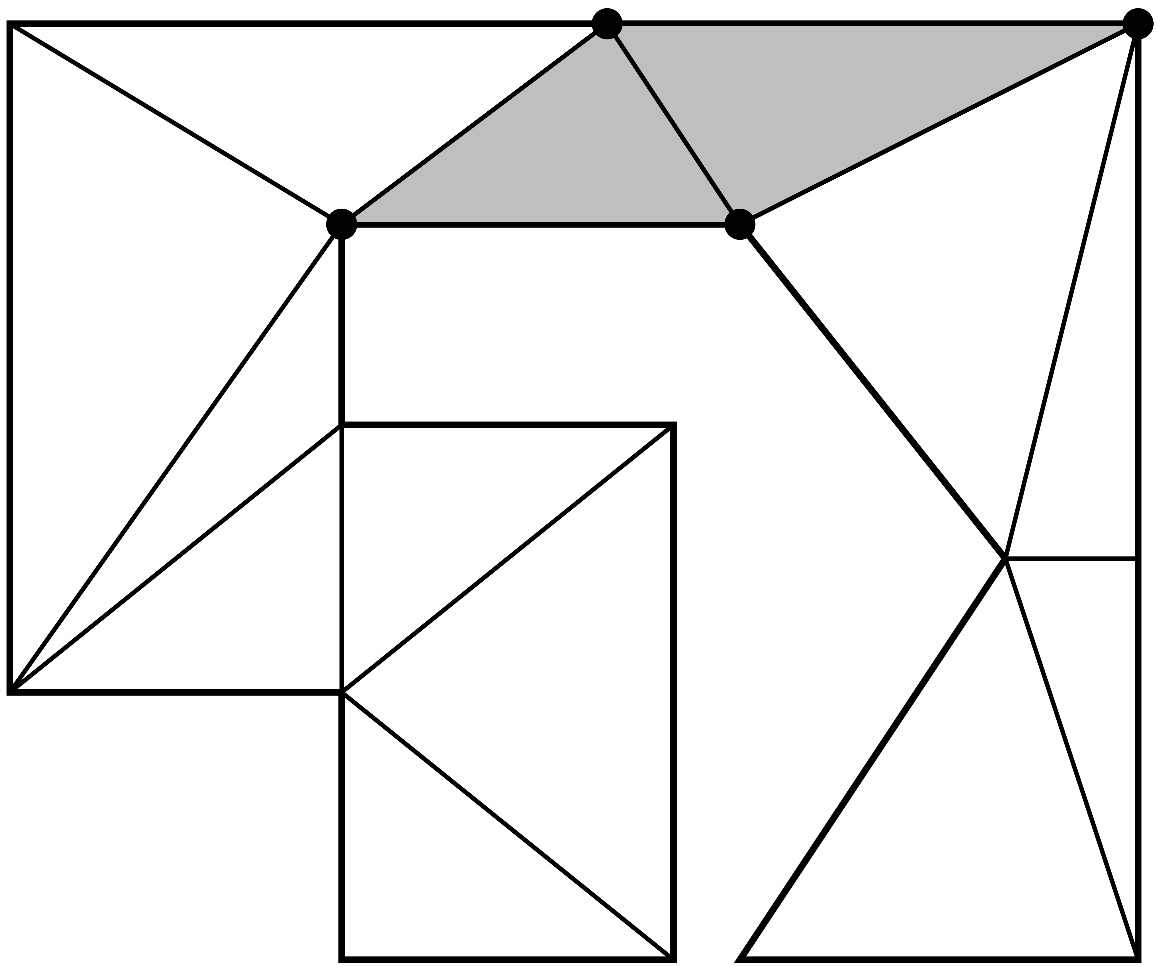
Identify with n-tuple (modulo Möbius) as before.
  Full CRFT: DF = discrete approximation
  Simple CRDT: DF = Identity
  Shortcut CRDT: DF using Broyden updates

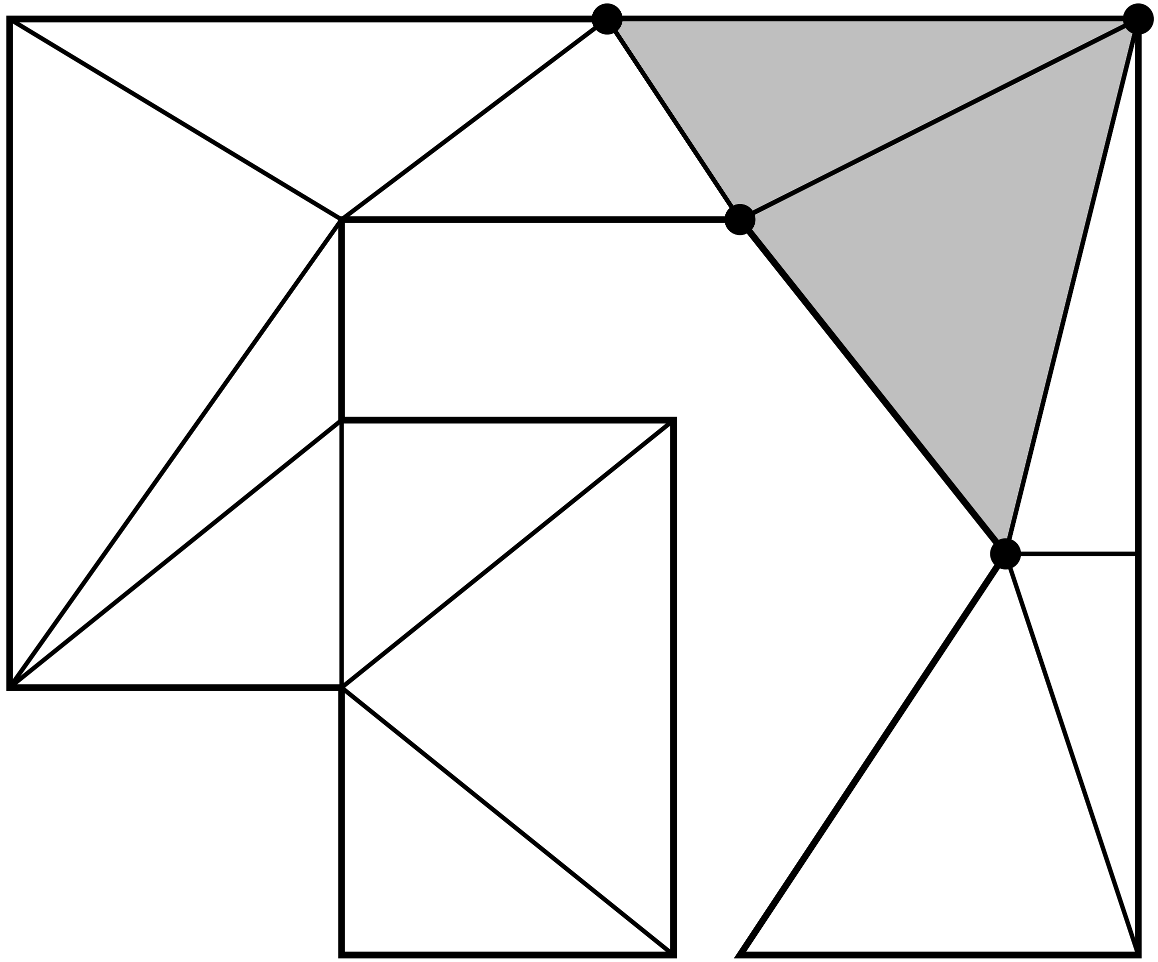


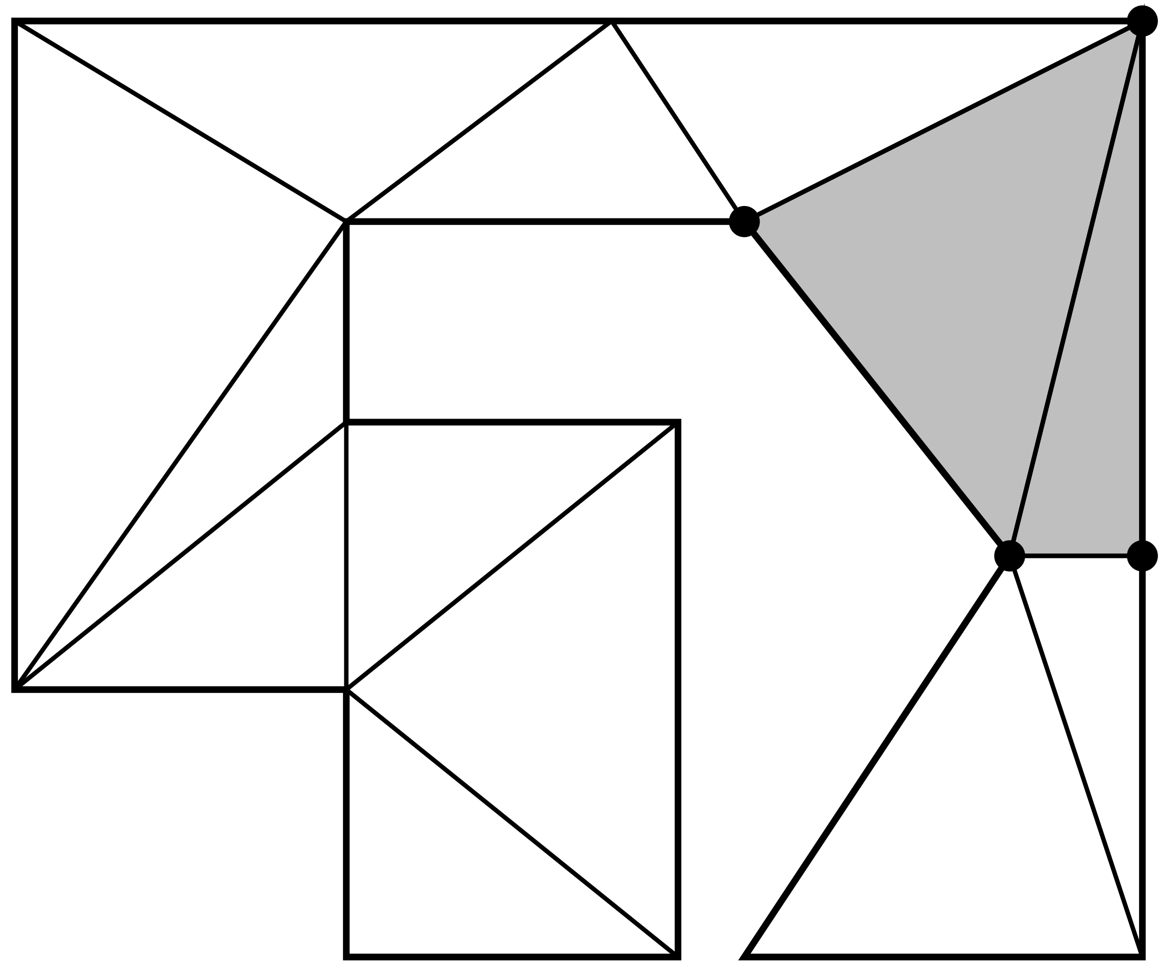










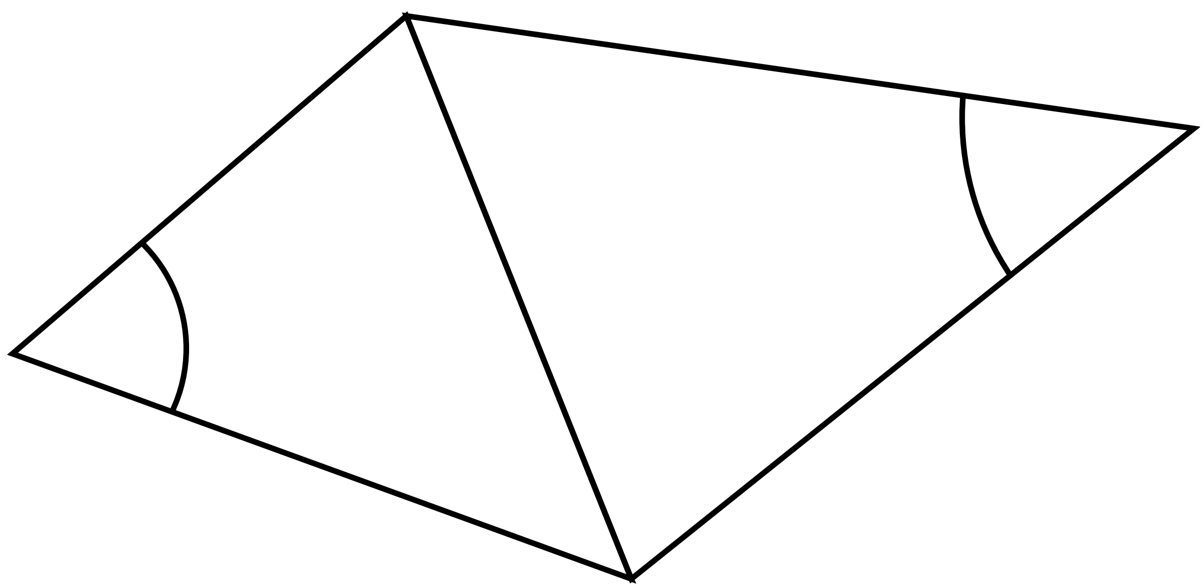


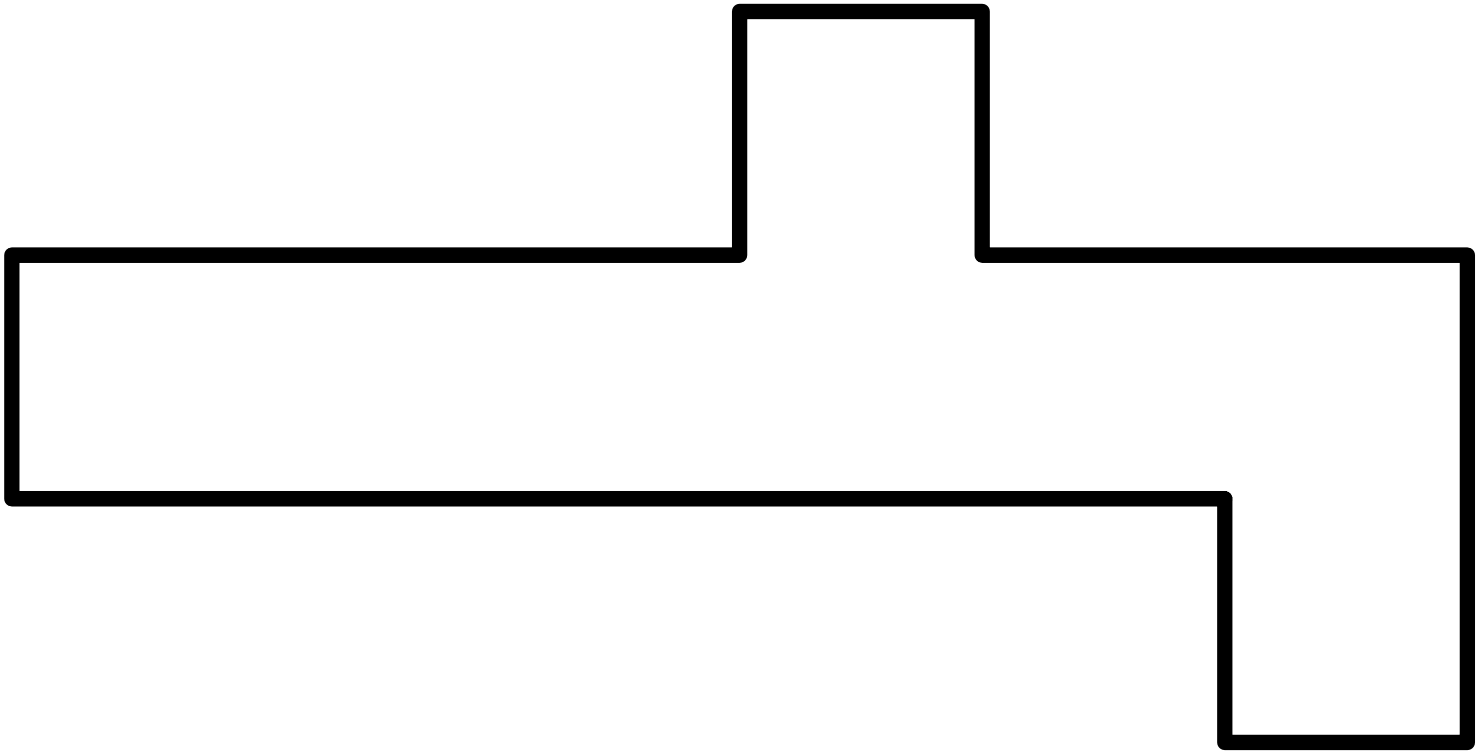
What is Delaunay Triangulation?

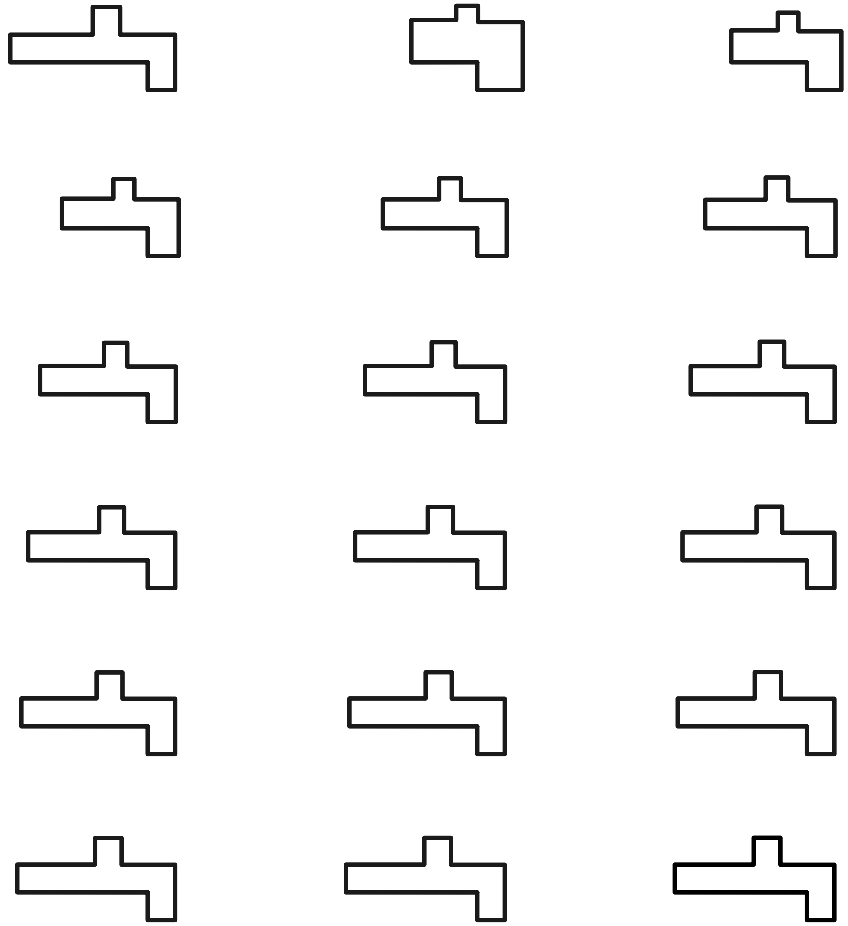
A triangulation is Delaunay if whenever triangles share and edge, the opposite angles sum to $\leq \pi$.

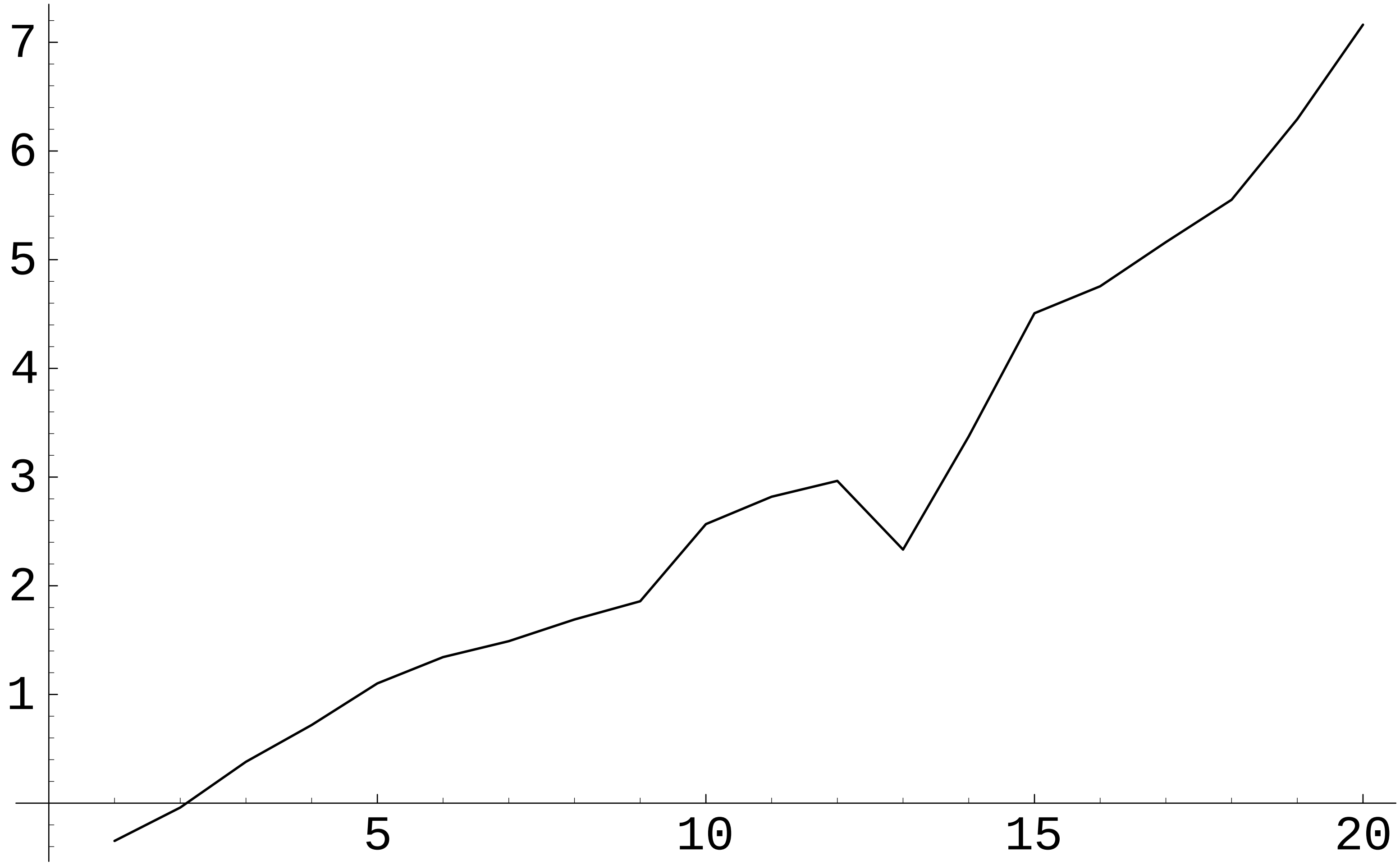
A DT always exists and minimizes the maximum angle.

Not needed to define CRDT, but makes it work better.

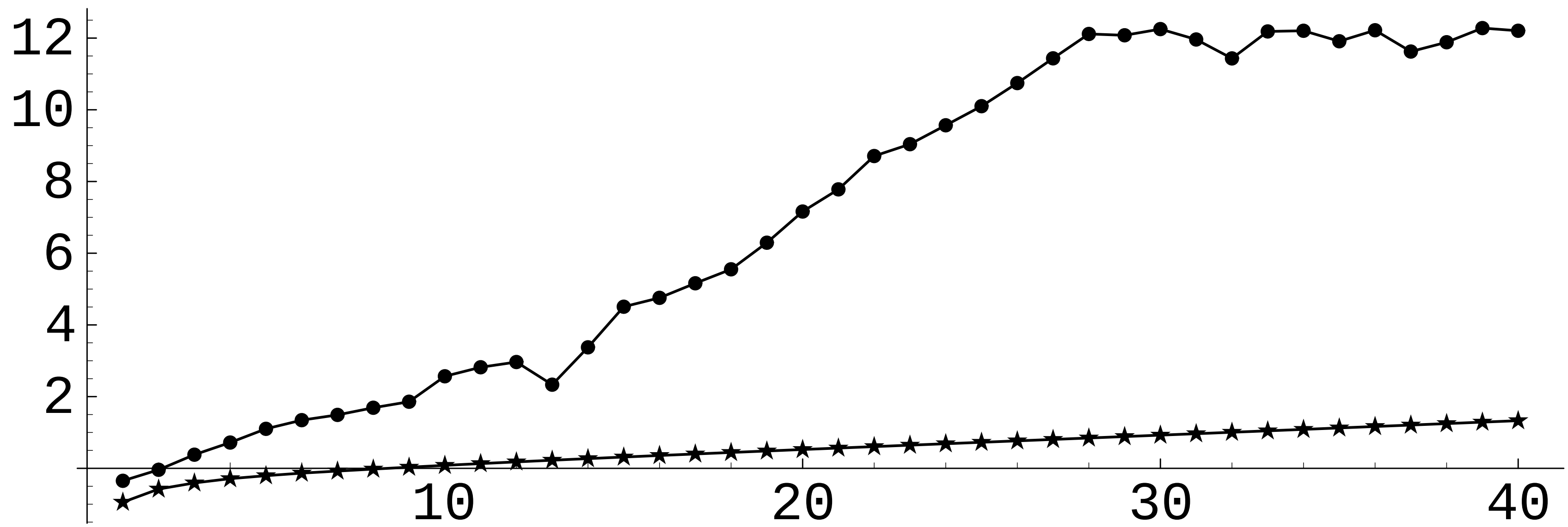




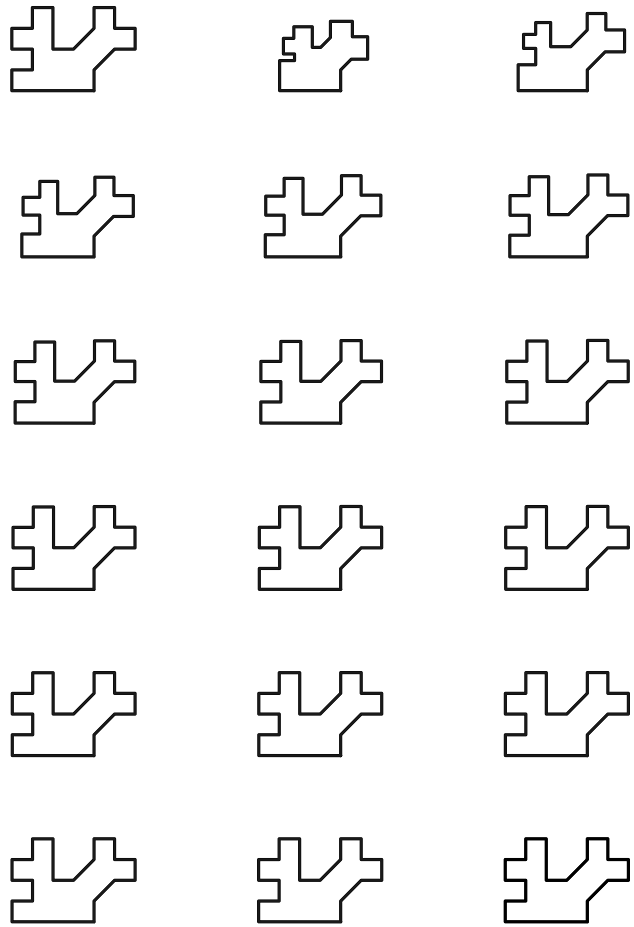


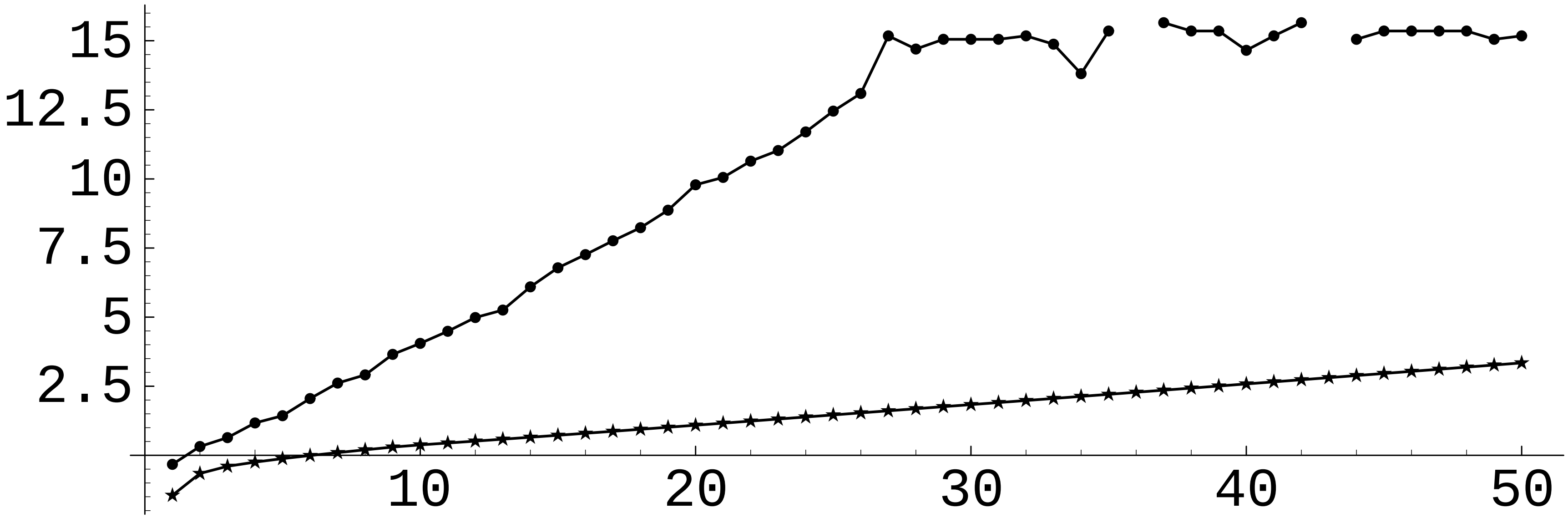


QC error for CRDT



Comparison of Davis and CRDT




15
12.5
10
7.5
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2.5
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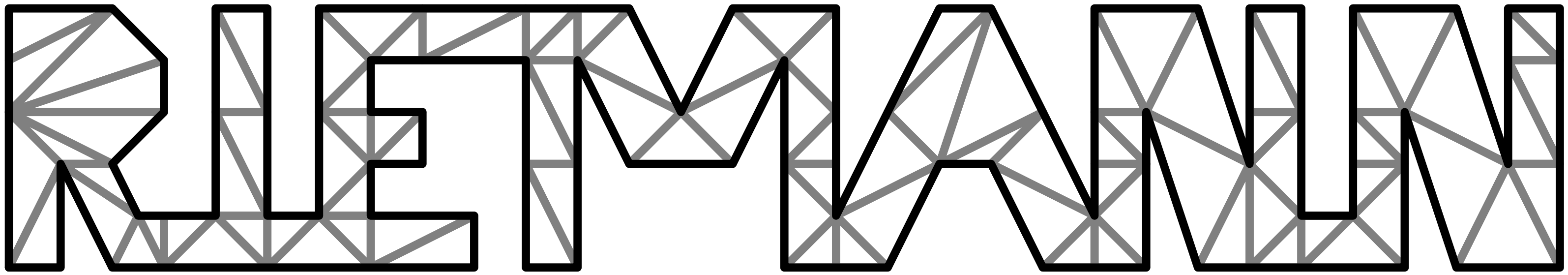
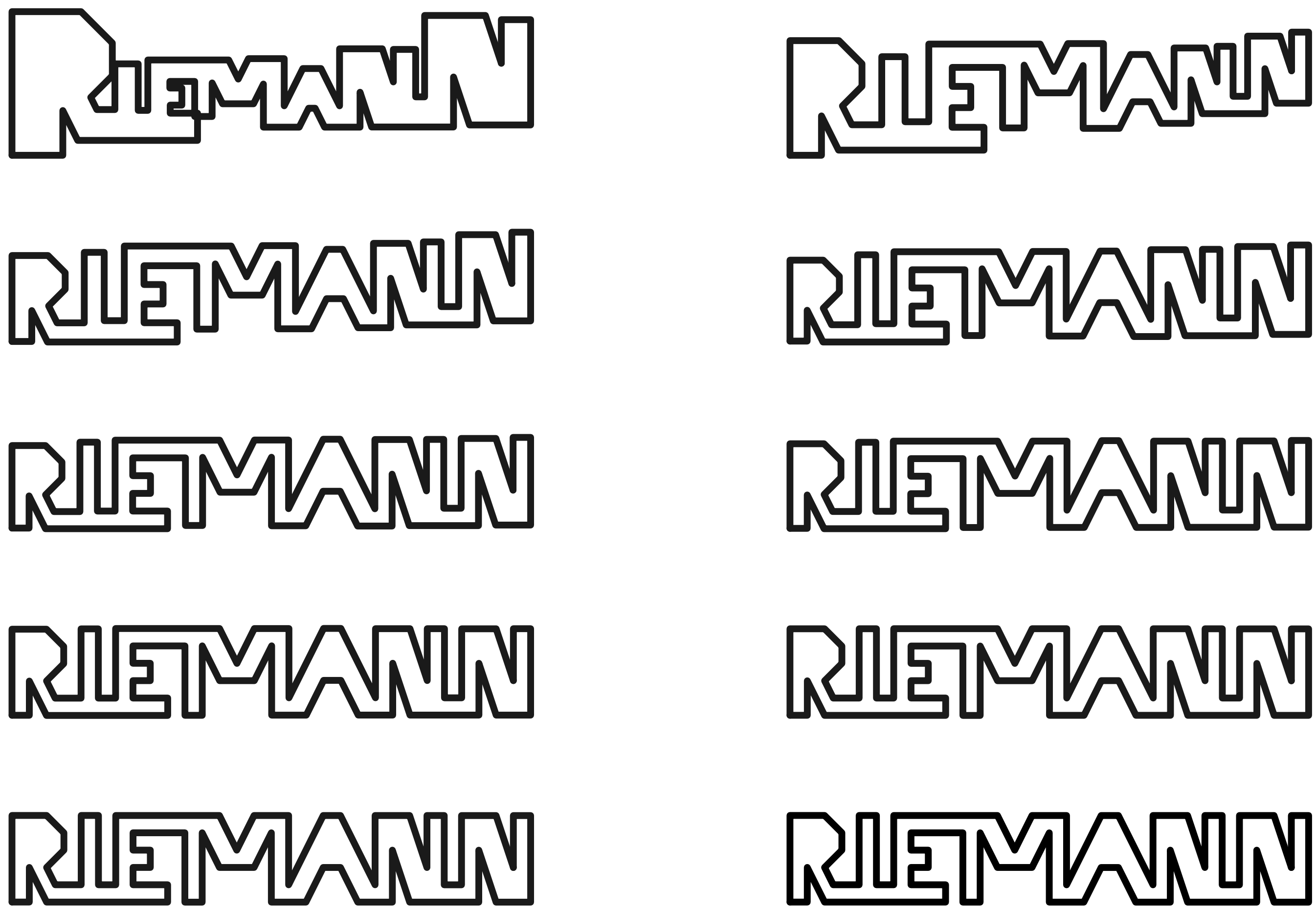
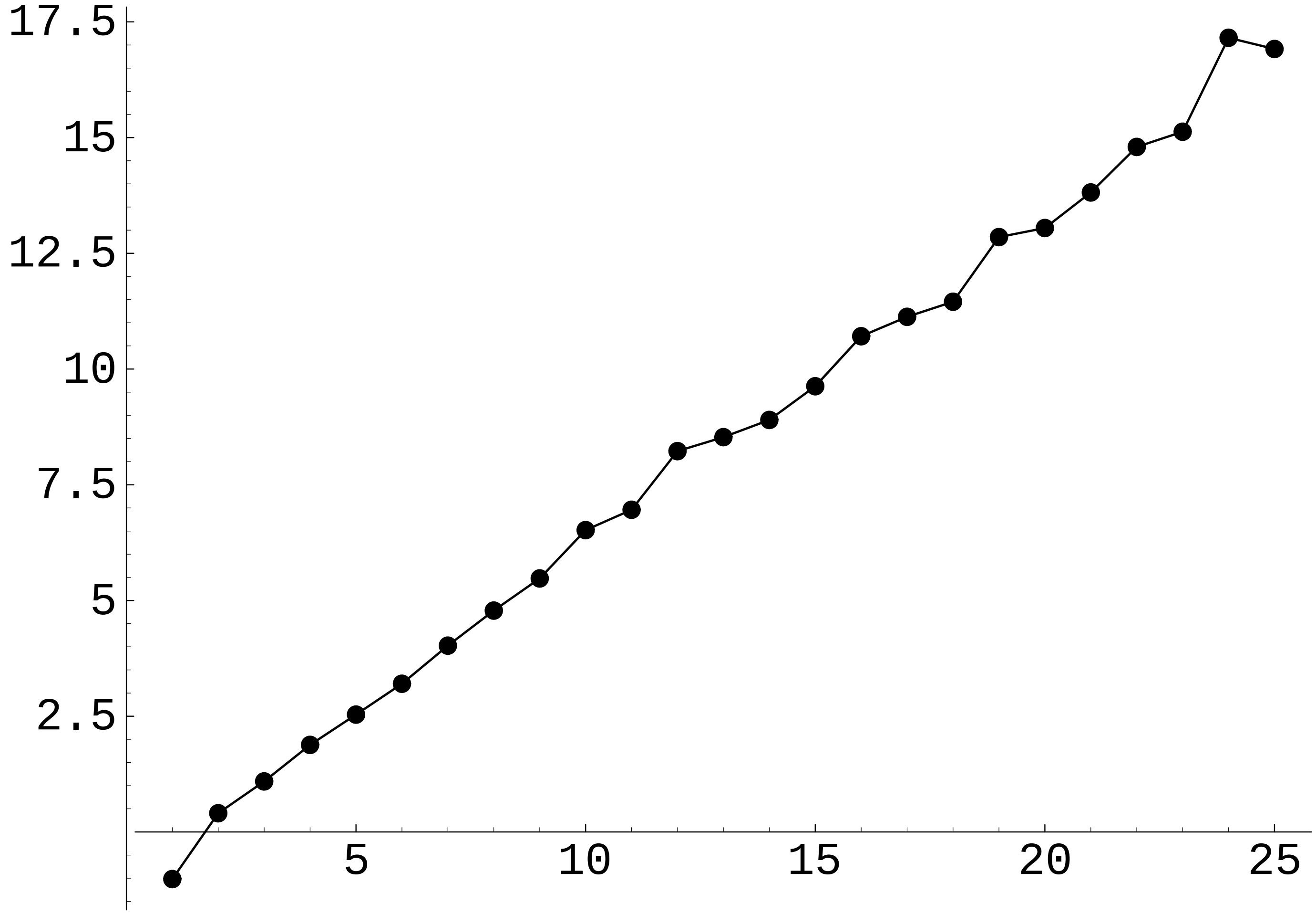
50



A 98-gon and its Delaunay triangulation.



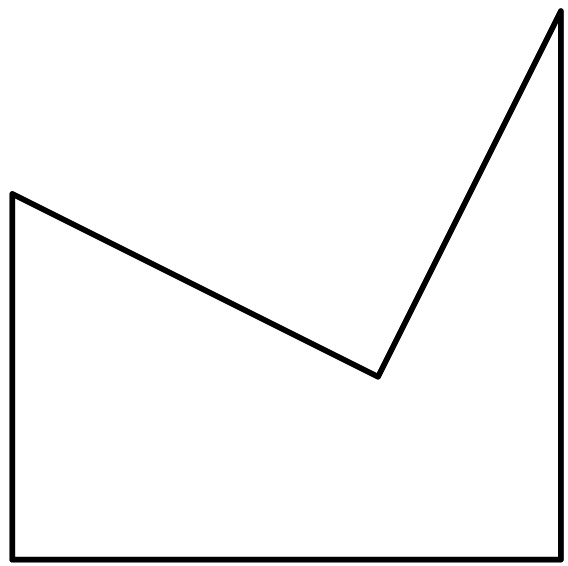
10 iterations of shortcut CRDT applied to a 98-gon.

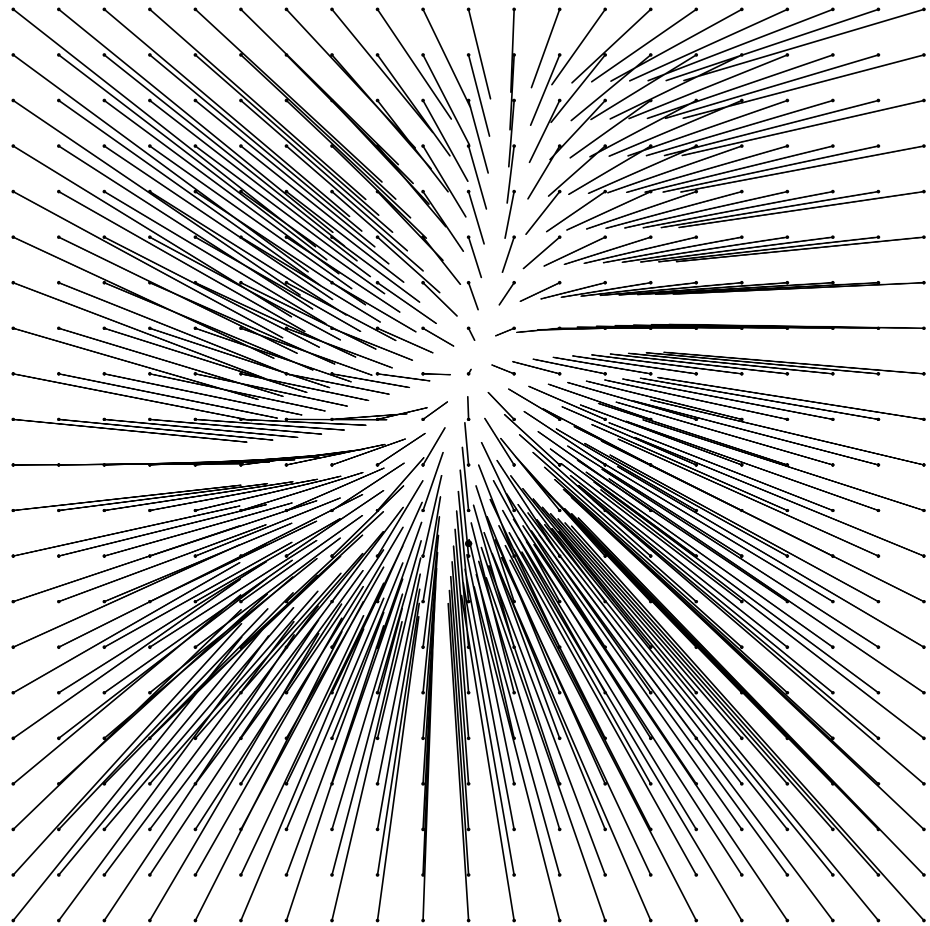


QC error of shortcut CRDT applied to the 98-gon.

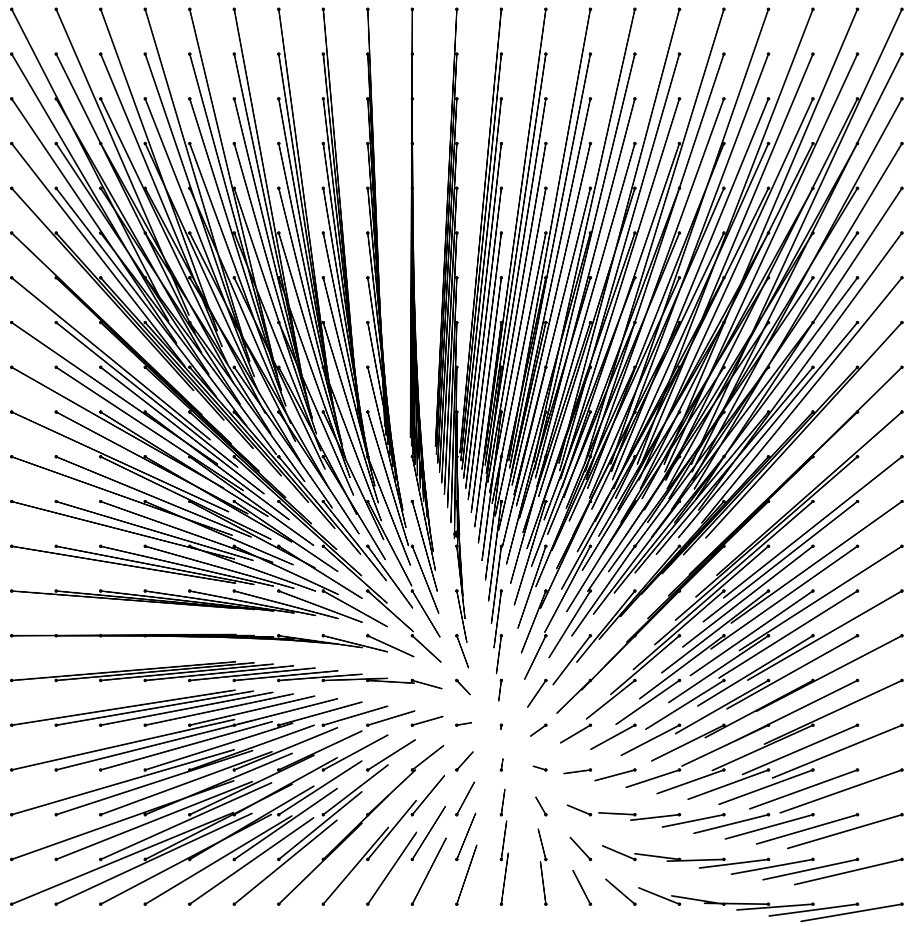
For a pentagon, the iteration is on $\mathbb{R}^2$.

We can draw a picture: connect z to $F(z)$ by a segment.



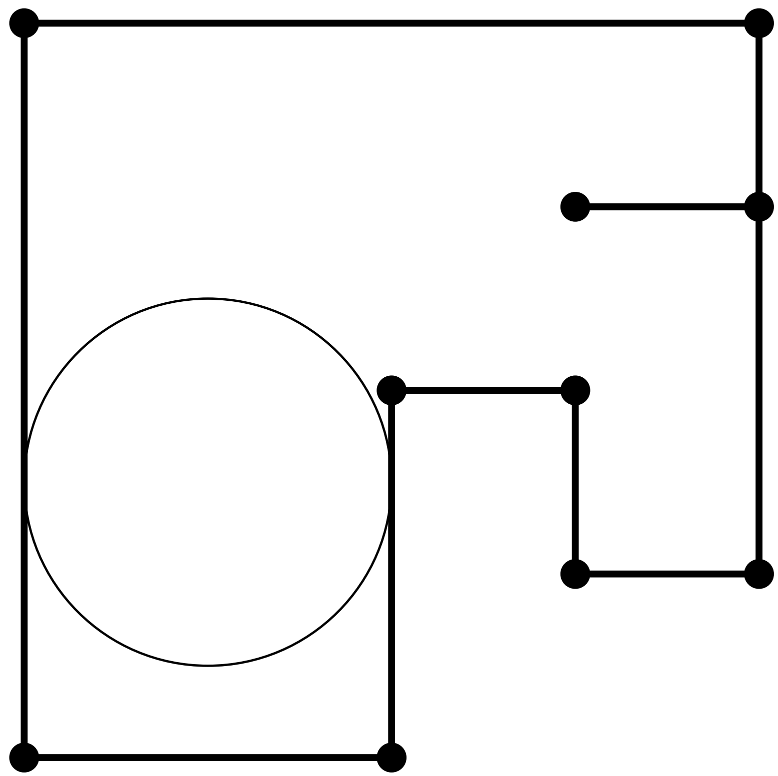


The map $F = G \circ S$. We want to solve $F(z) = z_0$.



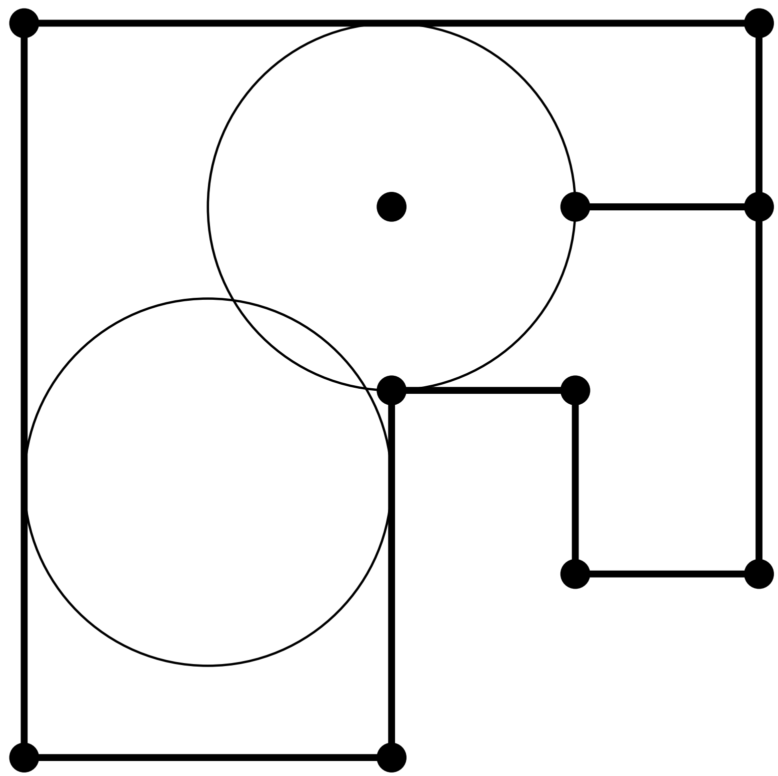
Iteration $\mathbf{z} \to \mathbf{z} - (F(\mathbf{z}) - \mathbf{z}_0)$. We want fixed point.

Another guessing map: iota



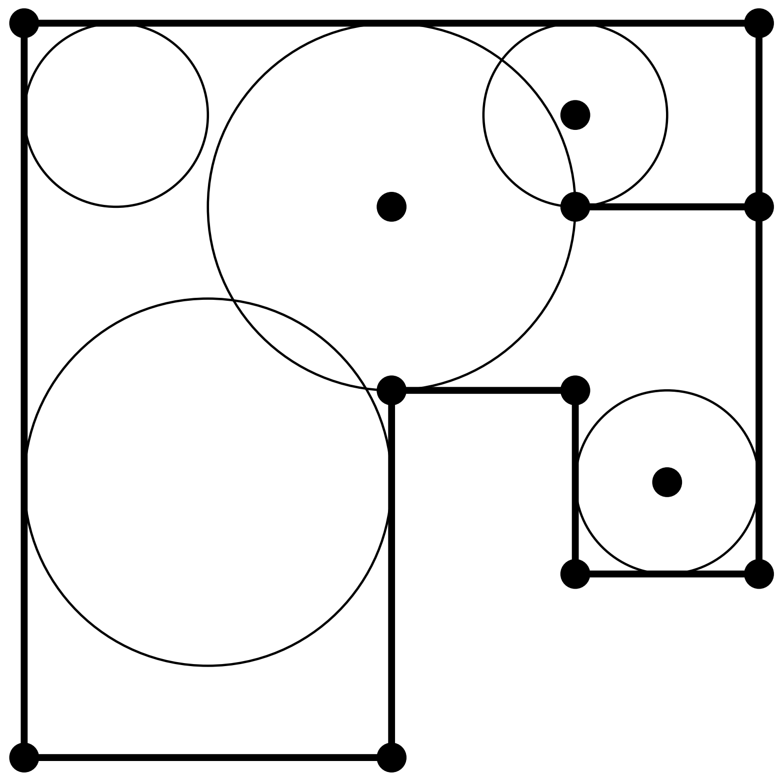
Consider interior disks with ≥ 2 contacts on boundary.

Another guessing map: iota



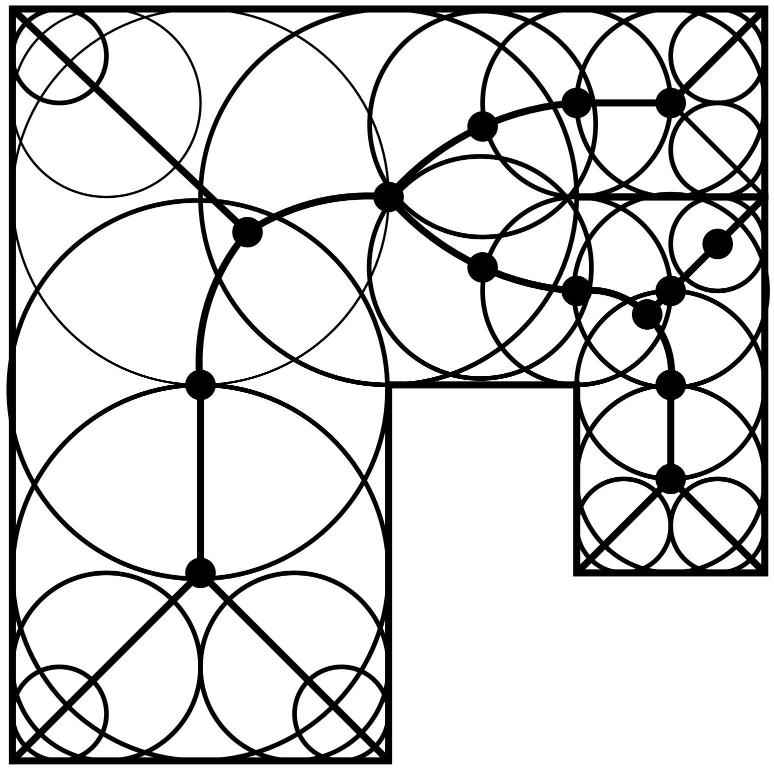
Consider interior disks with ≥ 2 contacts on boundary.

Another guessing map: iota



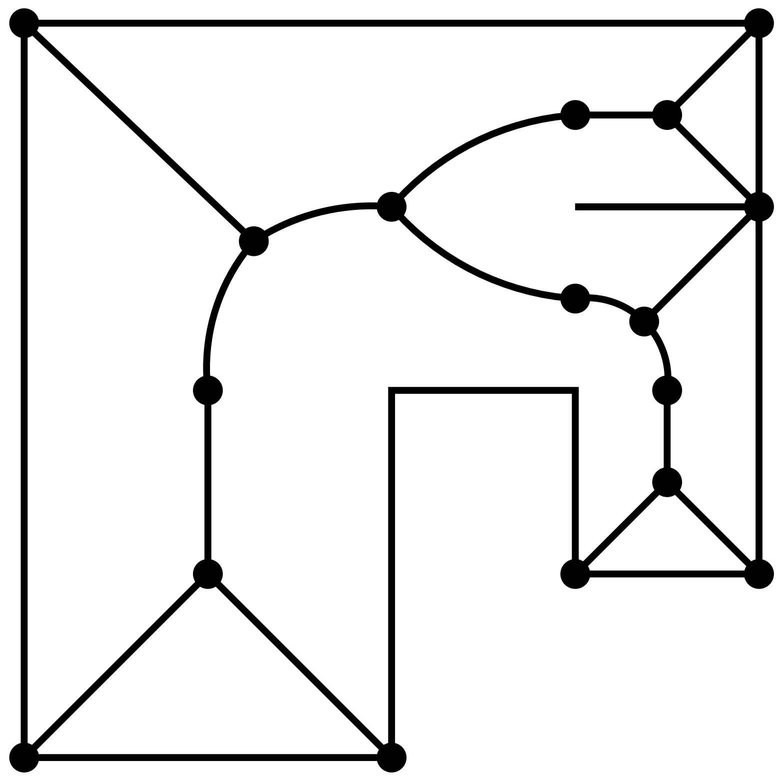
Consider interior disks with ≥ 2 contacts on boundary.

Another guessing map: iota



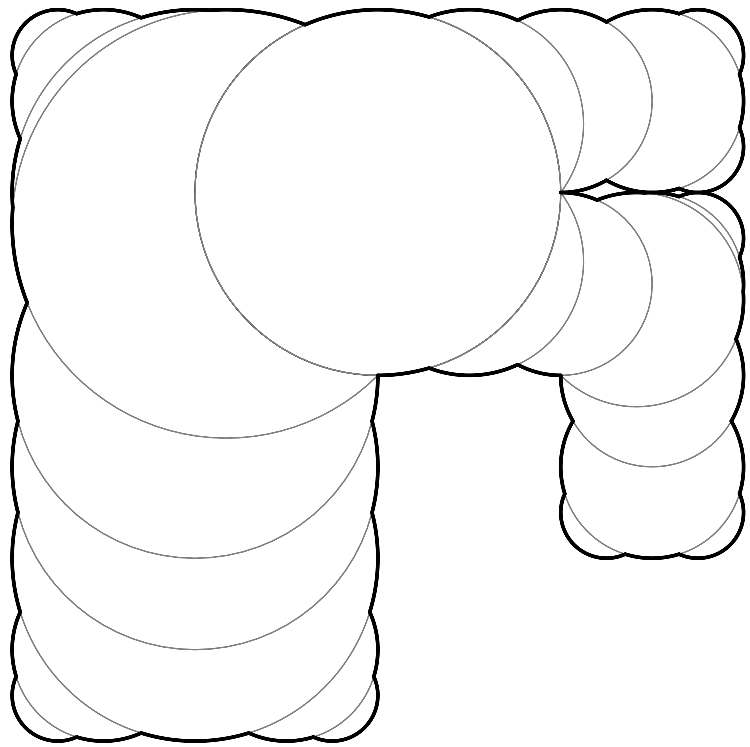
Centers of all such disks define **medial axis**.

Another guessing map: iota



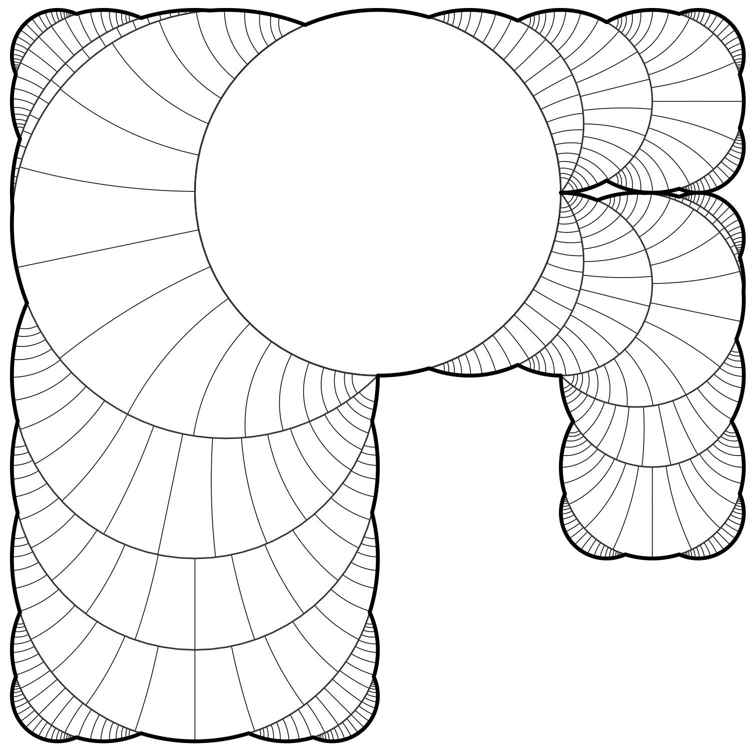
Centers of all such disks define **medial axis**.

Another guessing map: iota



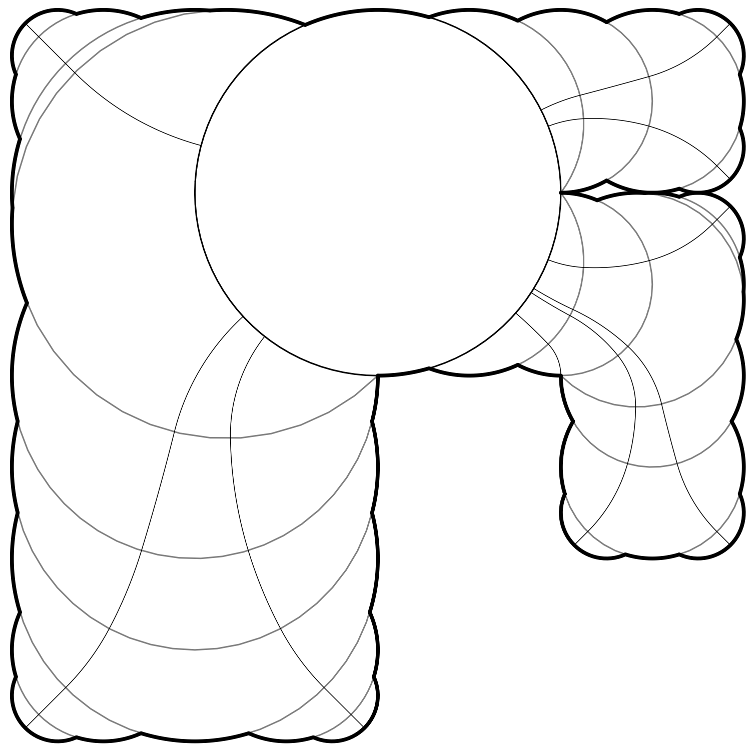
Take a finite set of medial axis disks. Choose a root.

Another guessing map: iota



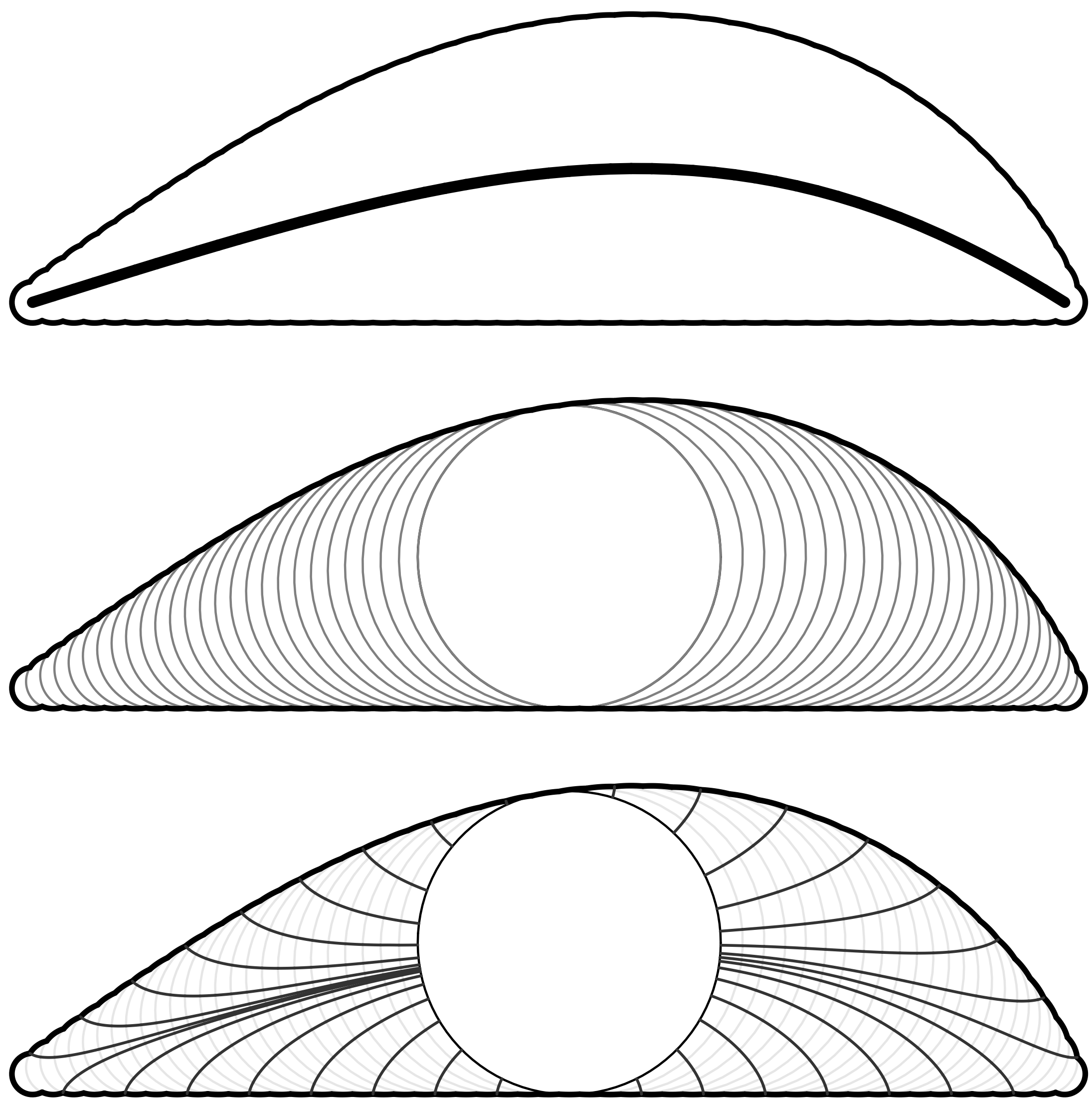
Foliate crescents by orthogonal arcs.

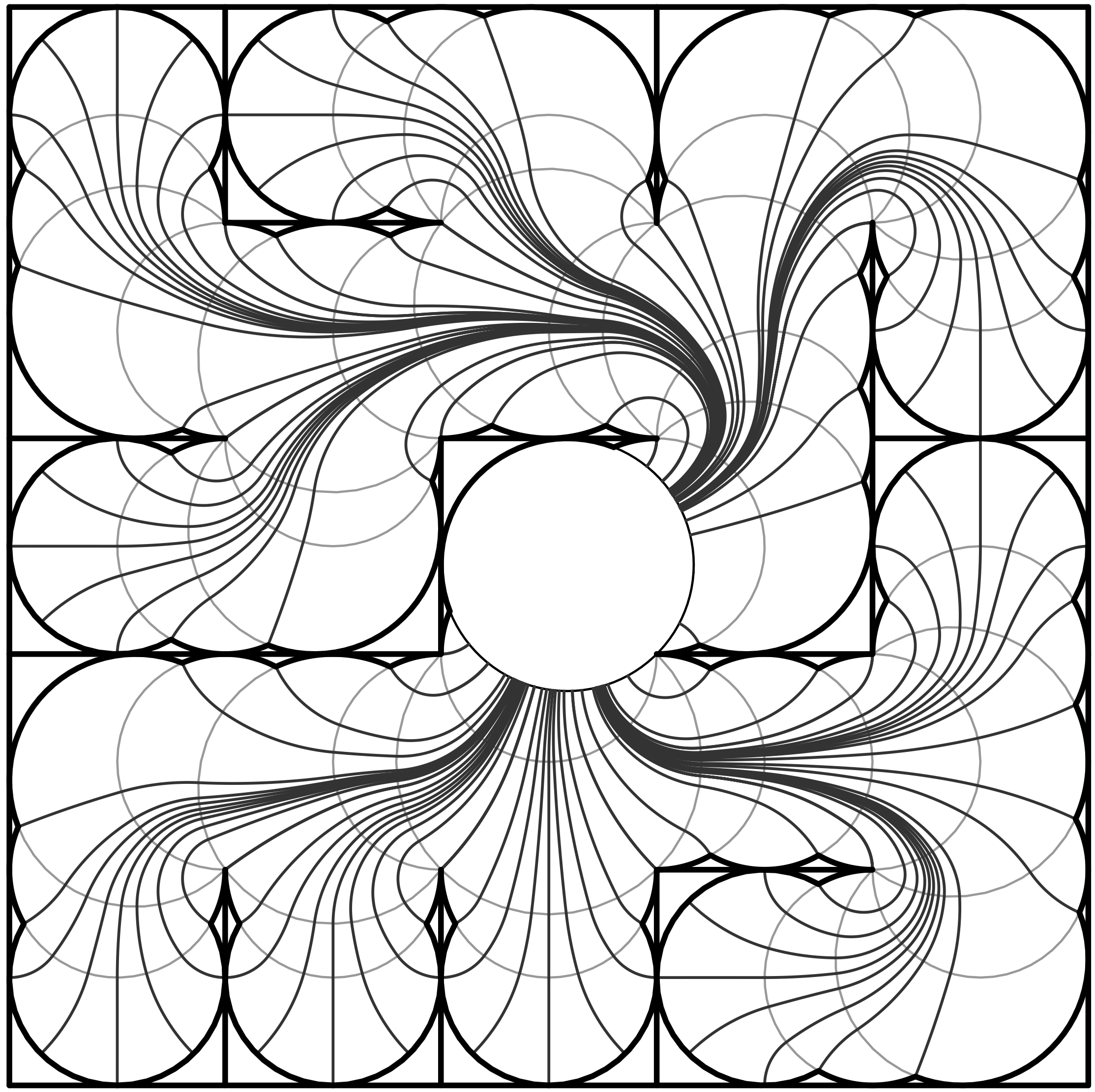
Another guessing map: iota

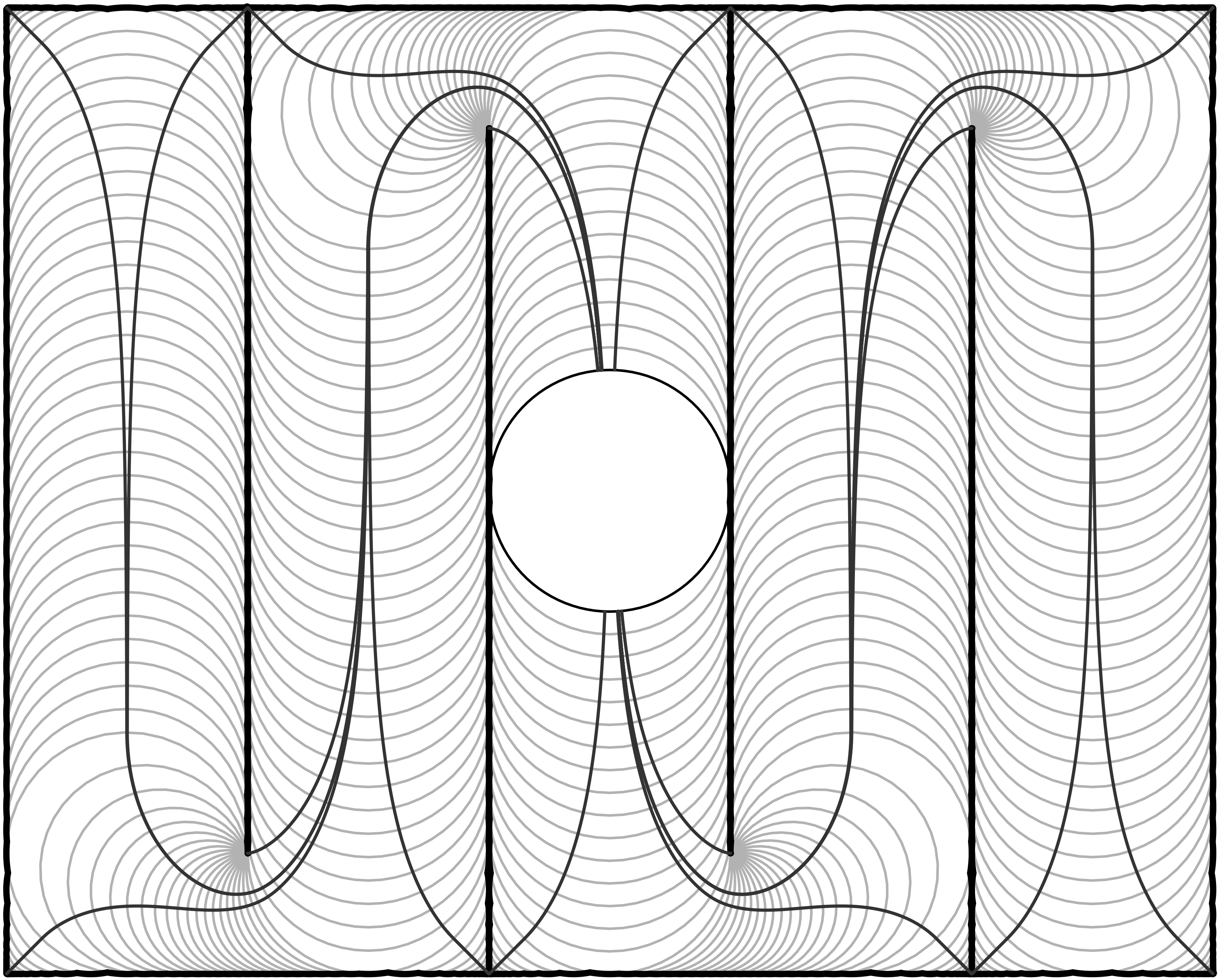


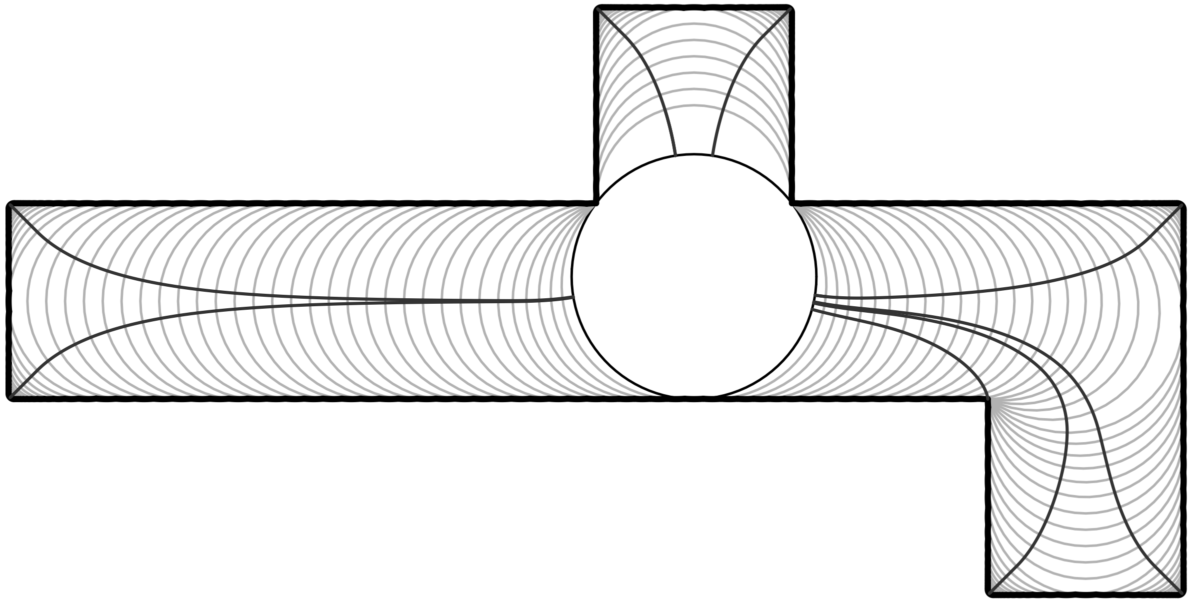
Follow arcs to define map of boundary to circle.

Similar flow for any simply connected domain.



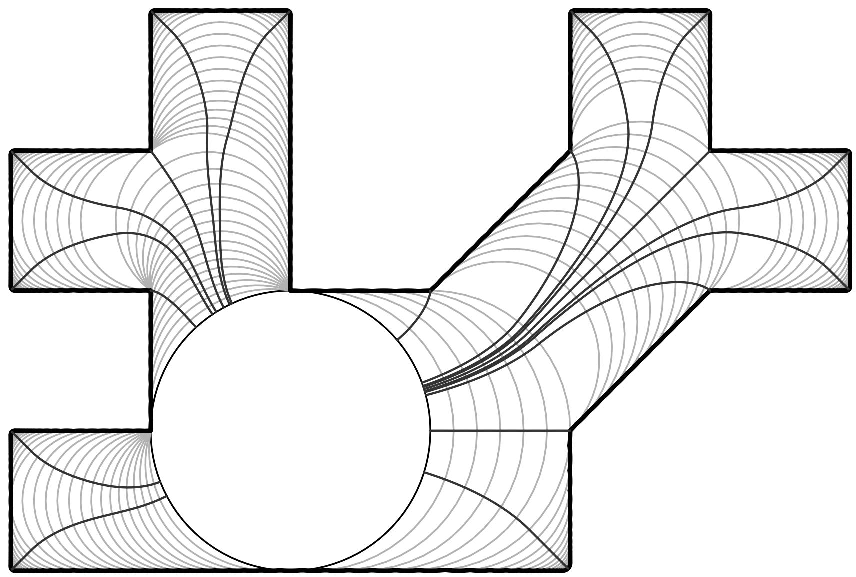




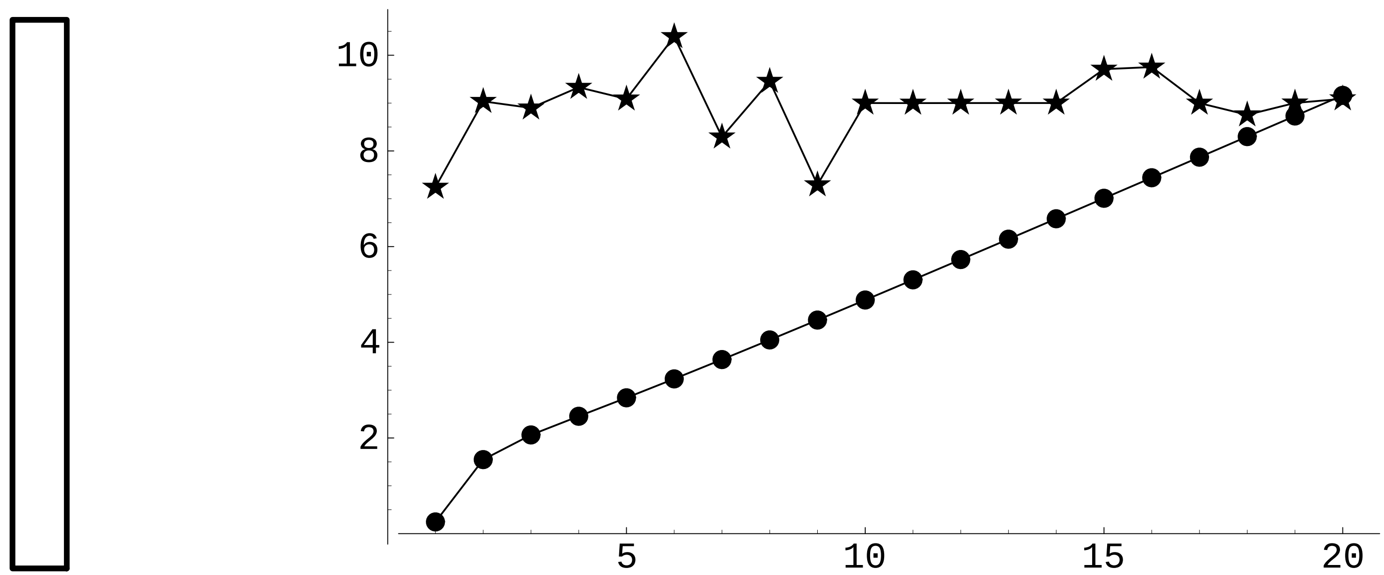


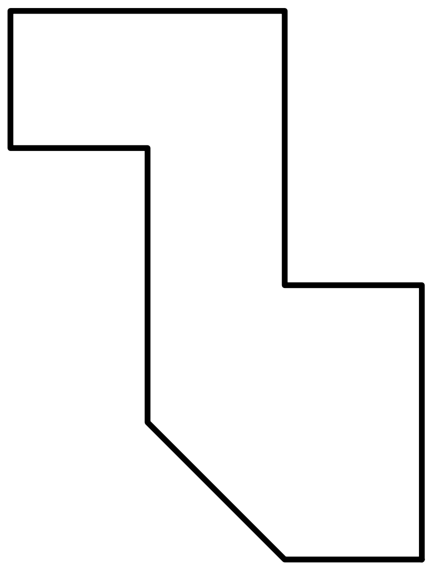
Thm: Iota is 8-QC close to conformal.

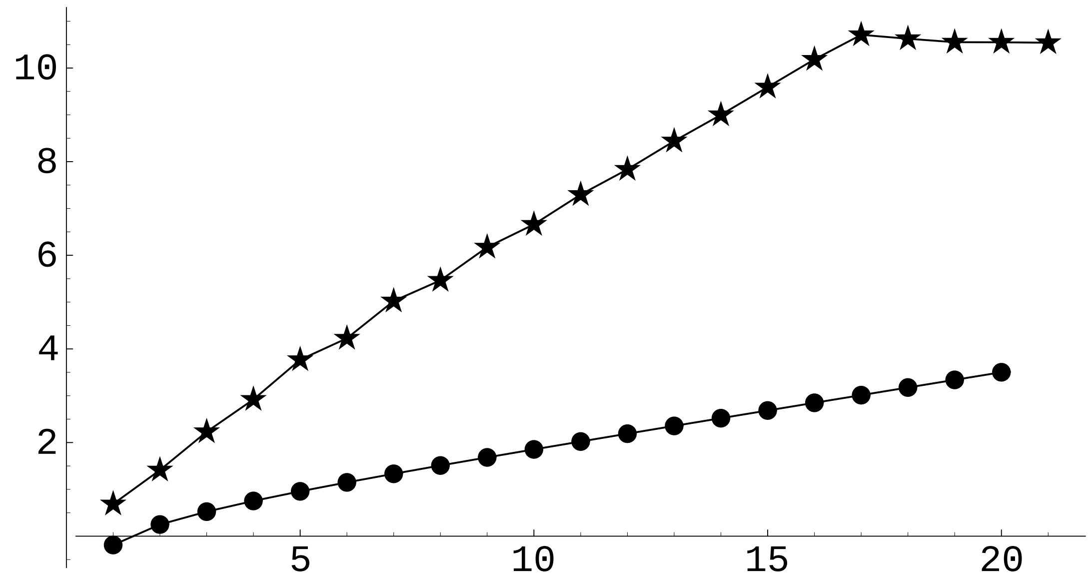
Thm: Iota is computable in $O(n)$ time.



How does iota compare to CRDT?







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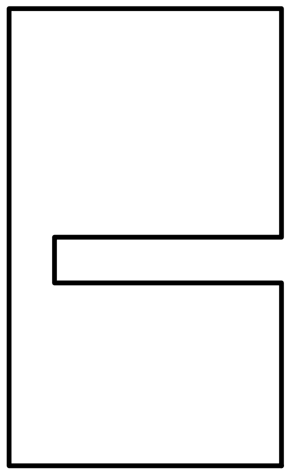

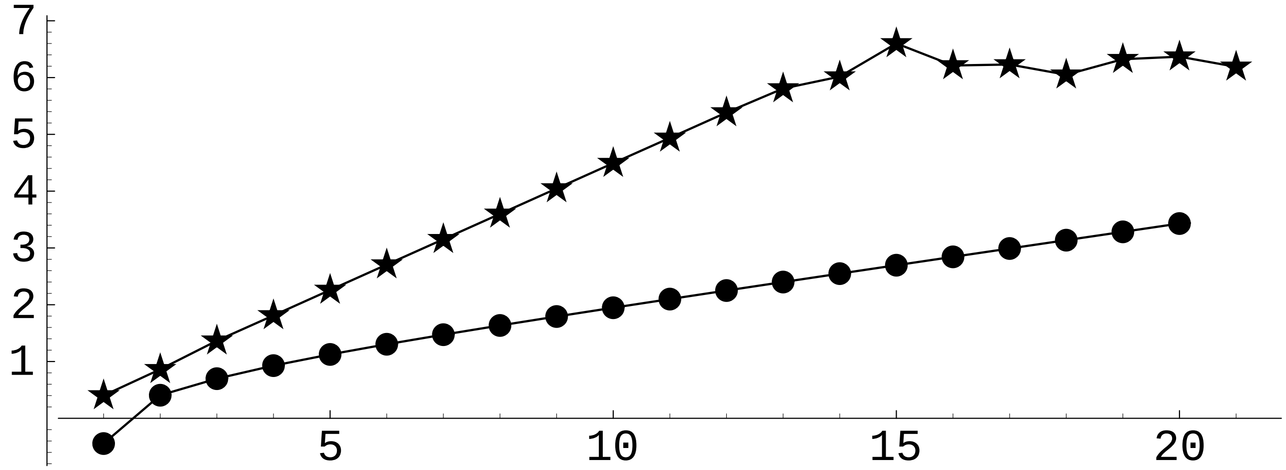

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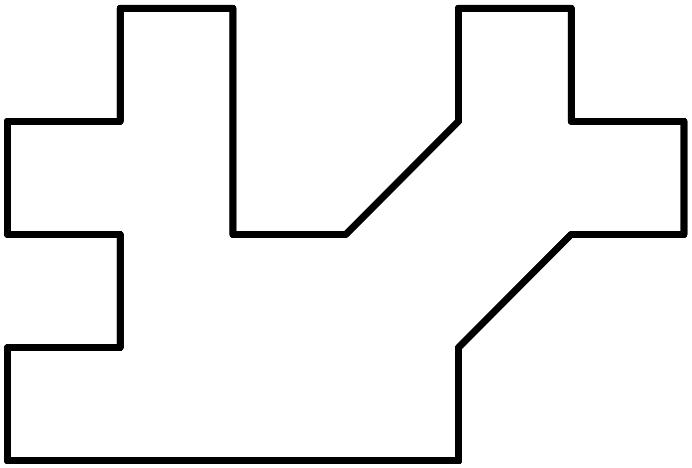

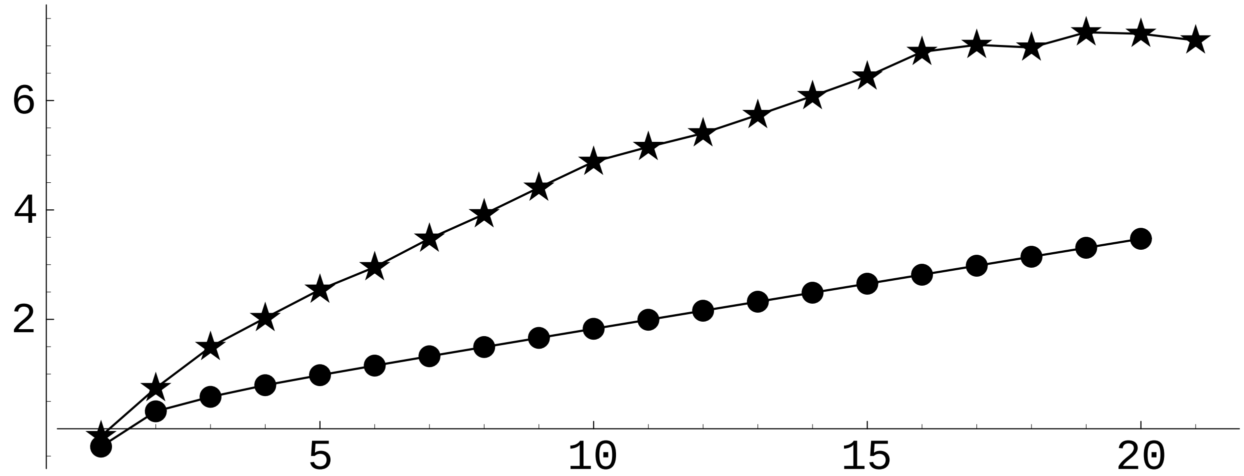

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