Riemann Mapping Theorem: If $\Omega$ is a simply connected, proper subdomain of the plane, then there is a conformal map $f : \mathbb{D} \to \Omega$.

Conformal = angle preserving
Our Founder
William Fogg Osgood
First proof of RMT, 1900
Paul Koebe
Modern Proof of RMT,
• Assume \( \Omega \) is bounded, choose \( z_0 \in \Omega \).
• \( \mathcal{F} = \) conformal maps into \( \mathbb{D} \), \( z_0 \to 0 \)
• \( \mathcal{F} \neq \emptyset \): \( (z \to z - z_0)/\text{diam}(\Omega) \in \mathcal{F} \).
• There is \( f \in \mathcal{F} \) that maximizes \( |f'(z_0)| \).
• This \( f \) is onto (hence Riemann map).

Last two steps are hard parts.

Next to last step uses compactness (normal families).

Last step is proof by contradiction.
General Möbius transformations (conformal, 1-1, group):

\[ z \rightarrow \frac{az + b}{cz + d} \]

Special case:

\[ z \rightarrow \frac{z - w}{1 - \bar{w}z} \]

maps unit disk to itself and maps \( w \rightarrow 0 \) and \( 0 \rightarrow -w \).
Assume $|f'(z_0)|$ is maximal, but $f(\Omega) \neq \mathbb{D}$.

Choose $w \in \mathbb{D} \setminus f(\Omega)$.

**Lemma:** If $0 \in \Omega \subset \mathbb{D}$ is simply connected but not the whole disk then there is conformal map $g : \Omega \to \Omega' \subset \mathbb{D}$ with $|g'(0)| > 1$.

$g \circ f : \Omega \to \mathbb{D}$ is conformal and by chain rule

$$\frac{d}{dz} |g \circ f(z_0)| > |f'(z_0)|.$$

Contradiction. Hence $f$ is onto $\mathbb{D}$. 
Proof of lemma:
Choose Möbius maps $\sigma, \tau$ of disk so
\[ \sigma(w) = 0, \quad \tau(\sqrt{\sigma(0)}) = 0. \]

Compute derivative at 0 of $g = \tau(\sqrt{\sigma})$
\[ |d\frac{d}{dz}\tau(\sqrt{\sigma(z)})|_0 \geq \frac{1 + |w|}{2\sqrt{|w|}} > 1. \]
Proof of RMT gives an algorithm:
• Choose linear map of $\Omega$ into $\mathbb{D}$.
• Choose image boundary point closest to origin.
• Compose $f$ with $g = \tau(\sqrt{\sigma}) : \mathbb{D} \to \mathbb{D}$.
• Repeat.

\[ d_n = \text{dist}(\partial\Omega_n, 0) \]

**Lemma:** If \( k \geq 4/(1 - \sqrt{d_n}) \), then \( d_{n+k} > \sqrt{d_n} \).

**Cor:** If \( d_0 \geq 1/2 \) then \( d_n = 1 - O(1/n) \).

Takes 1,000,000 iterations to get \( d_n = .000001 \).
0th iteration
1th iteration
2nd iteration
10th iteration
20th iteration
50th iteration
Koebe’s iteration: it works, but is slow.

Numerous faster methods.
Schwarz-Christoffel formula (1867):

\[ f(z) = A + C \int_{z}^{\tilde{z}} \prod_{k=1}^{n} (1 - \frac{w}{z_k})^{\alpha_k - 1} dw, \]
Schwarz-Christoffel formula (1867):

\[ f(z) = A + C \int_{z}^{\bar{z}} \prod_{k=1}^{n} \left(1 - \frac{w}{z_k}\right)^{\alpha_k-1} dw, \]
Schwarz-Christoffel formula (1867):

\[ f(z) = A + C \int \prod_{k=1}^{n} \left(1 - \frac{w}{z_k}\right)^{\alpha_k-1} dw, \]

\( \{\alpha_1 \pi, \ldots, \alpha_n \pi\} \), are interior angles of polygon.
\( \{z_1, \ldots, z_n\} \) are points on circle mapping to vertices.

\( \alpha \)'s are known.

\( z \)'s must be solved for.

**Basic idea:** guess some parameters. Use formula to draw the corresponding polygon. Compare to target polygon and revise guesses.
Davis’ method:

- Compare guessed polygon to target polygon.
- If an edge is too long, shorten corresponding parameter arc.
- If too long, lengthen the gap.

\[
\text{new gap} = \text{old gap} \times \frac{\text{target side}}{\text{old side}}
\]
20 iterations of Davis’ method for a rectangle.
20 iterations of Davis’ method - QC error.
30 iterations of Davis’ method - QC error.
We hit machine error $\approx 10^{-15}$.
50 iterations of Davis’ method.
QC error for 50 iterations of Davis’ method.
QC error for 400 iterations of Davis’ method.
Conformal Crowding:

Riemann map can dramatically shrink distances.

For a $1 \times R$ rectangle, two parameters are $\leq e^{-\pi R}$ apart.

If $e^{-\pi R} \leq 10^{-16} = \text{machine precision}$, they are the same point to the computer. $R \approx 11$. 
QC mappings: distort angles by bounded amount.

\[ \partial f = \frac{1}{2}(f_x - if_y), \quad \overline{\partial f} = \frac{1}{2i}(f_x + if_y). \]

Conformal \( f : \mathbb{D} \to \Omega \) with \( \overline{\partial f} = 0 \) (Cauchy-Riemann).

We measure distance to conformality by dilatation

\[ \|f\| = \sup |\mu_f| \equiv \sup \left| \frac{\overline{\partial f}}{\partial f} \right|. \]
Affine map between triangles \{0, 1, a\} and \{0, 1, b\} is
\[ f(z) \rightarrow \alpha z + \beta \bar{z} \]
where \(\alpha + \beta = 1\) and \(\beta = (b - a)/(a - \bar{a})\). Then
\[ K_f = \frac{1 + |\mu_f|}{1 - |\mu_f|}, \]
where
\[ \mu_f = \frac{f_{\bar{z}}}{f_z} = \frac{\beta}{\alpha} = \frac{b - a}{b - \bar{a}}, \]
How to compute integrals in SC-formula (error $< 10^{-16}$)?

$$\int_{a}^{b} f(x) dx \approx \sum_{k=1}^{n} a_n f(x_n)$$

- Right-hand-rule: error $= O(n^{-1})$.
- Midpoint rule: error $= O(n^{-2})$.
- Simpson's rule: error $= O(n^{-4})$.
- Gauss Quadrature: error $= O(n^{-2n})$.

Evaluation points $\{x_k\}$ and weights $\{a_k\}$ are given in terms of orthogonal polynomials.
SC-parameters by Newton’s method.

$T = \text{unit circle}$

$T_n = \text{ordered } n\text{-tuple on circle } (= \text{SC parameter guess})$

$\mathbb{C} = \text{complex numbers}$

$\mathbb{C}_n = \text{ } n\text{-tuple of complex numbers } (= \text{polygons})$
Fix angles in SC-formula. Then we get map:
$S : \mathbb{T}_n \rightarrow \mathbb{C}_n$ (parameters $\rightarrow$ polygons)

Guessing map:
$G : \mathbb{C}_n \rightarrow \mathbb{T}_n$ (polygons $\rightarrow$ parameters)

Compose $H = G \circ S : \mathbb{T}_n \rightarrow \mathbb{T}_n$ (para $\rightarrow$ para)
If $P$ is target polygon, let $z_0 = G(P)$.

Let $F(z) = H(z) - z_0$.

Find solution of $F(z) = 0$ by Newton’s method.

Then $H(z) = z_0 \Rightarrow G \circ S(z) = G(P) \Rightarrow S(z) = P$ (if $G$ is 1-1).

We need:
  - $G$ to be 1-1.
  - $G$ to be computable.
How to solve $F(z) = 0$?

Define an iteration by

$$z_{n+1} = z_n - D_F^{-1}(F(z_n)).$$

where $D_F$ is the derivative matrix of $F = (F_1, \ldots, F_d)$

$$(D_F)_{jk} = \frac{dF_k}{dx_j}.$$

In one dimension, this is

$$z_{n+1} = z_n - \frac{F(z_n)}{F'(z_n)}.$$
A couple of problems with this:

- What is a good $G$ to choose?
- How do we compute $D_F$?
- $z_n, F(z_n) \in \mathbb{T}$. How do we do linear algebra?

We will deal with these in reverse order.
How to make \( n \)-tuples on circle into a vector space?

Triangulate the points.
How to make $n$-tuples on circle into a vector space?

Choose a root triangle.
How to make $n$-tuples on circle into a vector space?

For each triangle adjacent to root, form quadrilateral by union of the two triangles
How to make $n$-tuples on circle into a vector space?

Record the cross ratio of the four points

\[ \text{cr}(a, b, c, d) = \frac{(d - a)(b - c)}{(c - d)(a - b)}. \]

Invariant under Möbius transformations.

Points on circle $\Rightarrow$ cross ratio real valued.
How to make $n$-tuples on circle into a vector space?

The $n-2$ numbers $\log |\rho|$ determine $n$-tuple on circle up to a Möbius transformation of disk (free to place vertices of root triangle where we please).

If two $n$-tuples differ by a Möbius transformation, Schwarz-Christoffel gives similar polygons.
\( T^*_n = \mathbb{R}^{n-3} = \) equivalence classes of ordered \( n \)-tuples on circle identified via Möbius transformations.

\( \mathbb{C}^*_n = n \)-tuples on complex numbers modulo similarities.

We can think of
\[
G : \mathbb{C}^*_n \rightarrow T^*_n, \quad S : T^*_n \rightarrow \mathbb{C}^*_n,
\]
and
\[
F : T^*_n \rightarrow T^*_n,
\]
or
\[
F : \mathbb{R}^{n-3} \rightarrow \mathbb{R}^{n-3}.
\]

So now we can do linear algebra.
How do we compute derivative of $F = G \circ S$?

**1)** Use a discrete approximation

$$\partial_j F_k(x_1, \ldots, x_m) = \frac{1}{h} [(F_k(x_1, \ldots, x_j + he_j, \ldots, x_m) - F_k(x_1, \ldots, x_j, \ldots x_m)].$$

Gives good result but slow ($m + 1$ evaluations of $F$).

**2)** Assume $DF = Id$. Easy, fast, often works.

**3)** Broyden updates. Start by assuming $DF = Id$, but update $DF$ after each evaluation of $F$. Often best compromise between speed and accuracy.
What is a good choice for \( G \), the guessing function?

**Davis’s method:** based on edge lengths.

**CRDT:** based on triangulation and cross ratios.

**Iota:** based on hyperbolic geometry.
CRDT

Cross Ratios and Delaunay Triangulations

Toby Driscoll and Stephen Vavasis, 1998

Triangulate polygon

Choose root triangle

For non-roots form quadrilateral of triangle and parent

Compute cross ratio $\rho$ of 4 vertices (complex number).

Record $\log |\rho|$.

Identify with $n$-tuple (modulo Möbius) as before.

Full CRFT: $DF = \text{discrete approximation}$

Simple CRDT: $DF = \text{Identity}$

Shortcut CRDT: $DF$ using Broyden updates
What is Delaunay Triangulation?

A triangulation is Delaunay if whenever triangles share and edge, the opposite angles sum to $\leq \pi$.

A DT always exists and minimizes the maximum angle. Not needed to define CRDT, but makes it work better.
QC error for CRDT
Comparison of Davis and CRDT
A 98-gon and its Delaunay triangulation.
10 iterations of shortcut CRDT applied to a 98-gon.
QC error of shortcut CRDT applied to the 98-gon.
For a pentagon, the iteration is on $\mathbb{R}^2$.

We can draw a picture: connect $z$ to $F(z)$ by a segment.
The map $F = G \circ S$. We want to solve $F(z) = z_0$. 
Iteration $\mathbf{z} \rightarrow \mathbf{z} - (F(\mathbf{z}) - \mathbf{z}_0)$. We want fixed point.
Another guessing map: iota

Consider interior disks with $\geq 2$ contacts on boundary.
Another guessing map: iota

Consider interior disks with $\geq 2$ contacts on boundary.
Another guessing map: iota

Consider interior disks with $\geq 2$ contacts on boundary.
Another guessing map: iota

Centers of all such disks define **medial axis**.
Another guessing map: iota

Centers of all such disks define **medial axis**.
Another guessing map: iota

Take a finite set of medial axis disks. Choose a root.
Another guessing map: iota

Foliate crescents by orthogonal arcs.
Another guessing map: \( \iota \)

Follow arcs to define map of boundary to circle.
Similar flow for any simply connected domain.
**Thm:** Iota is 8-QC close to conformal.

**Thm:** Iota is computable in $O(n)$ time.
How does Iota compare to CRDT?