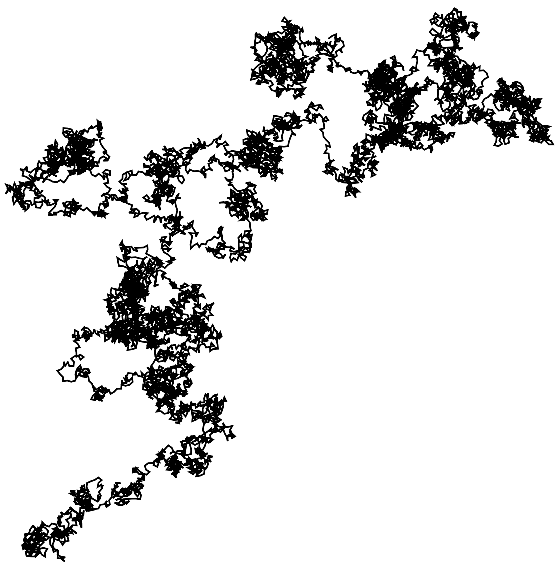
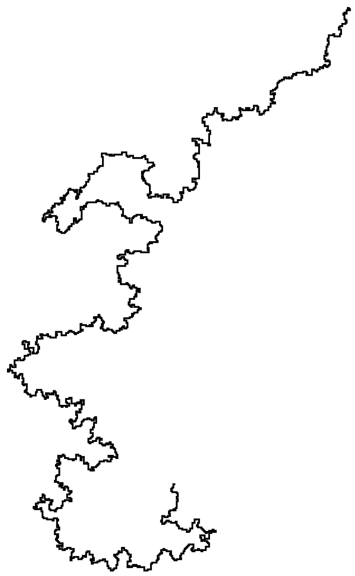
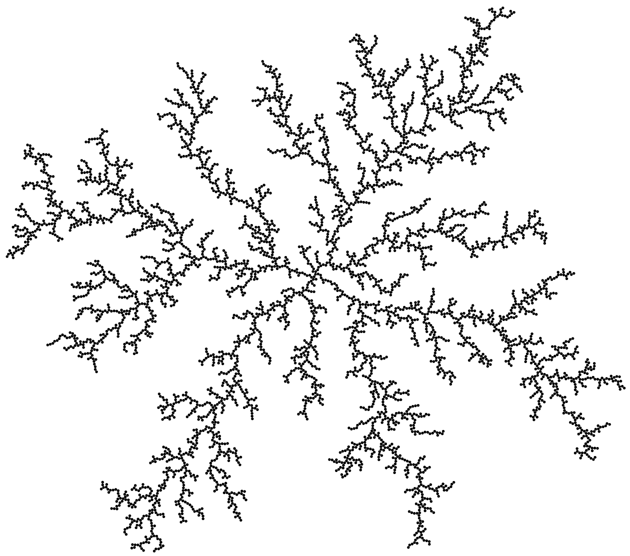
Some Random Geometry Problems

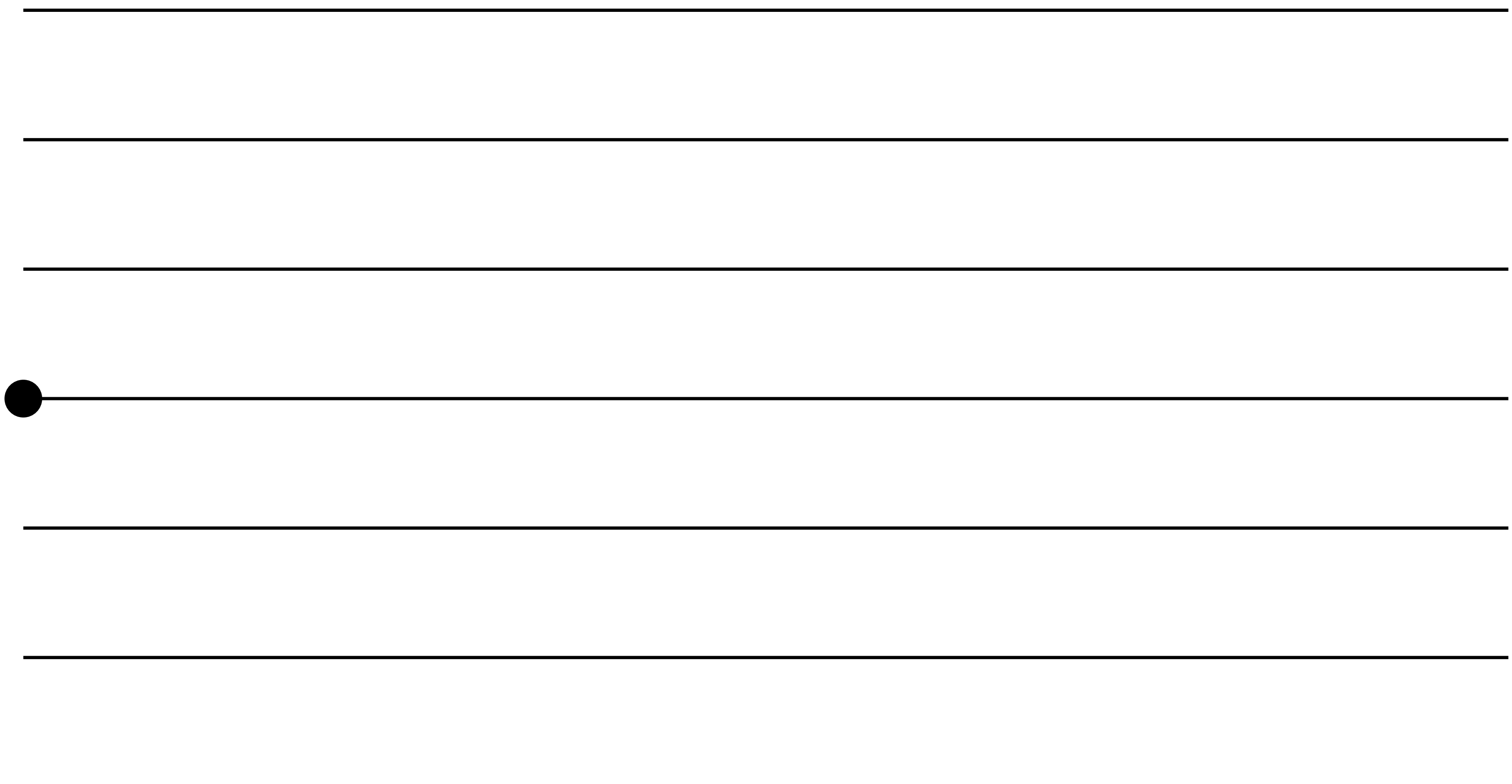
Christopher Bishop, Stony Brook

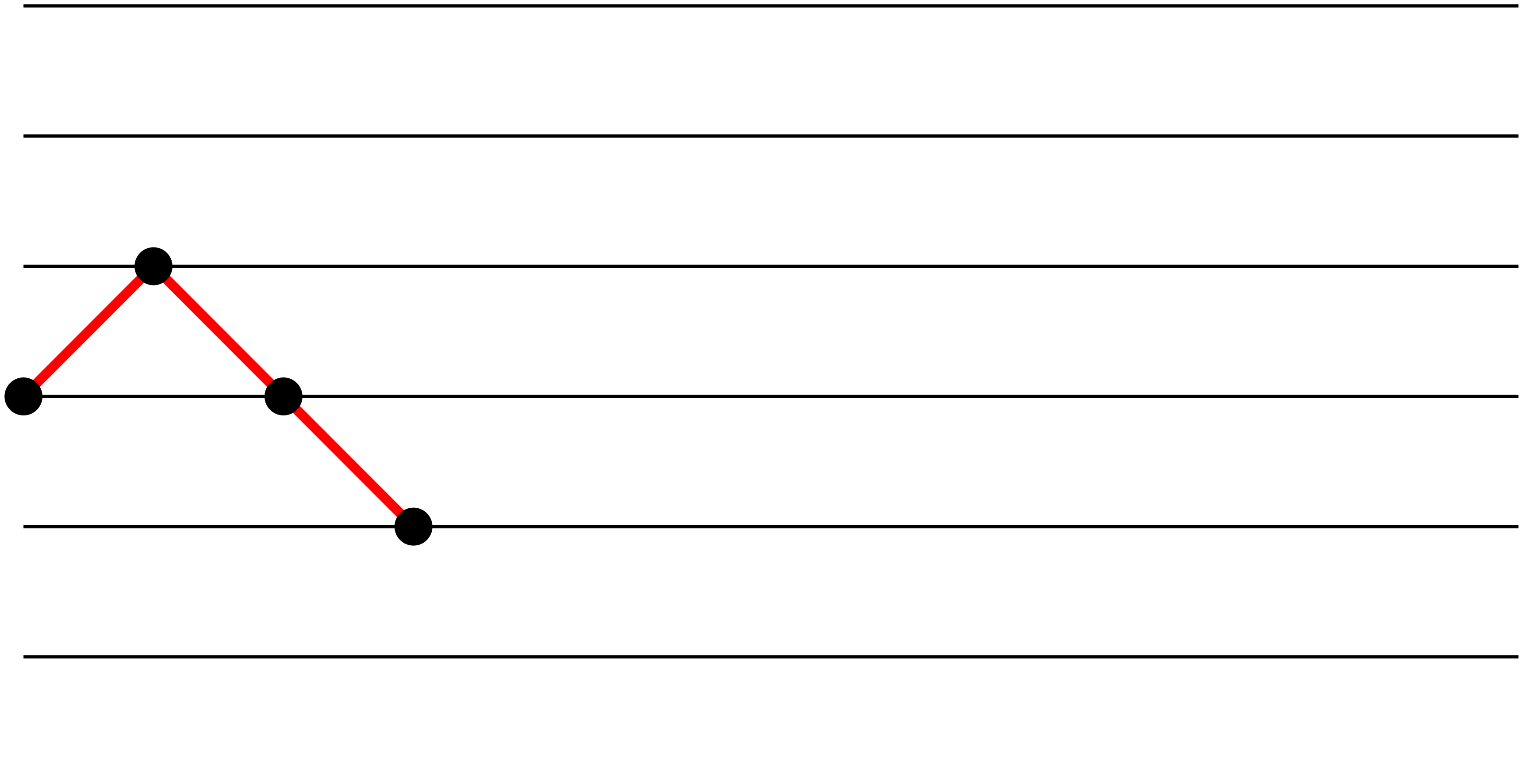
www.math.sunysb.edu/~bishop/lectures

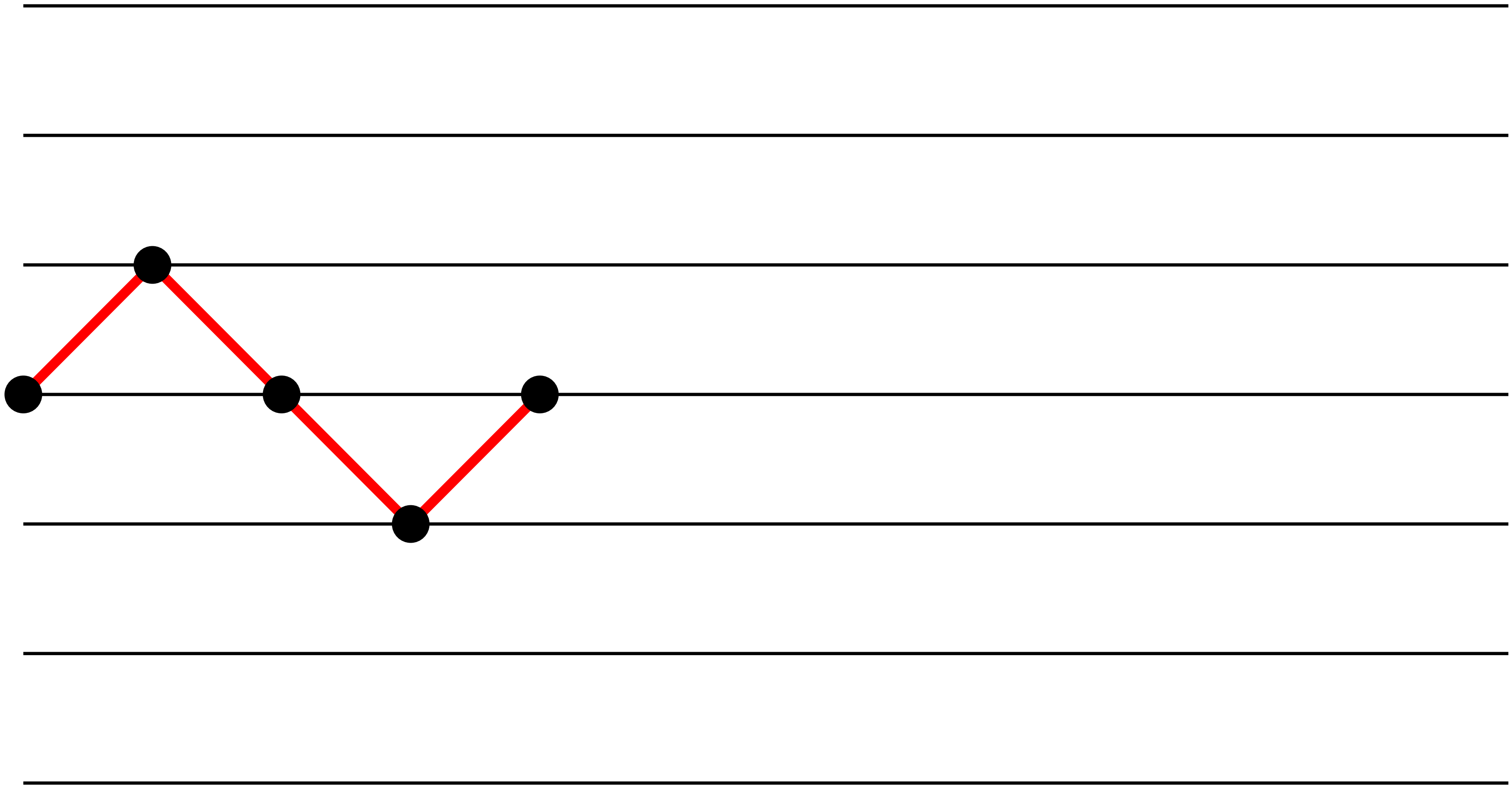


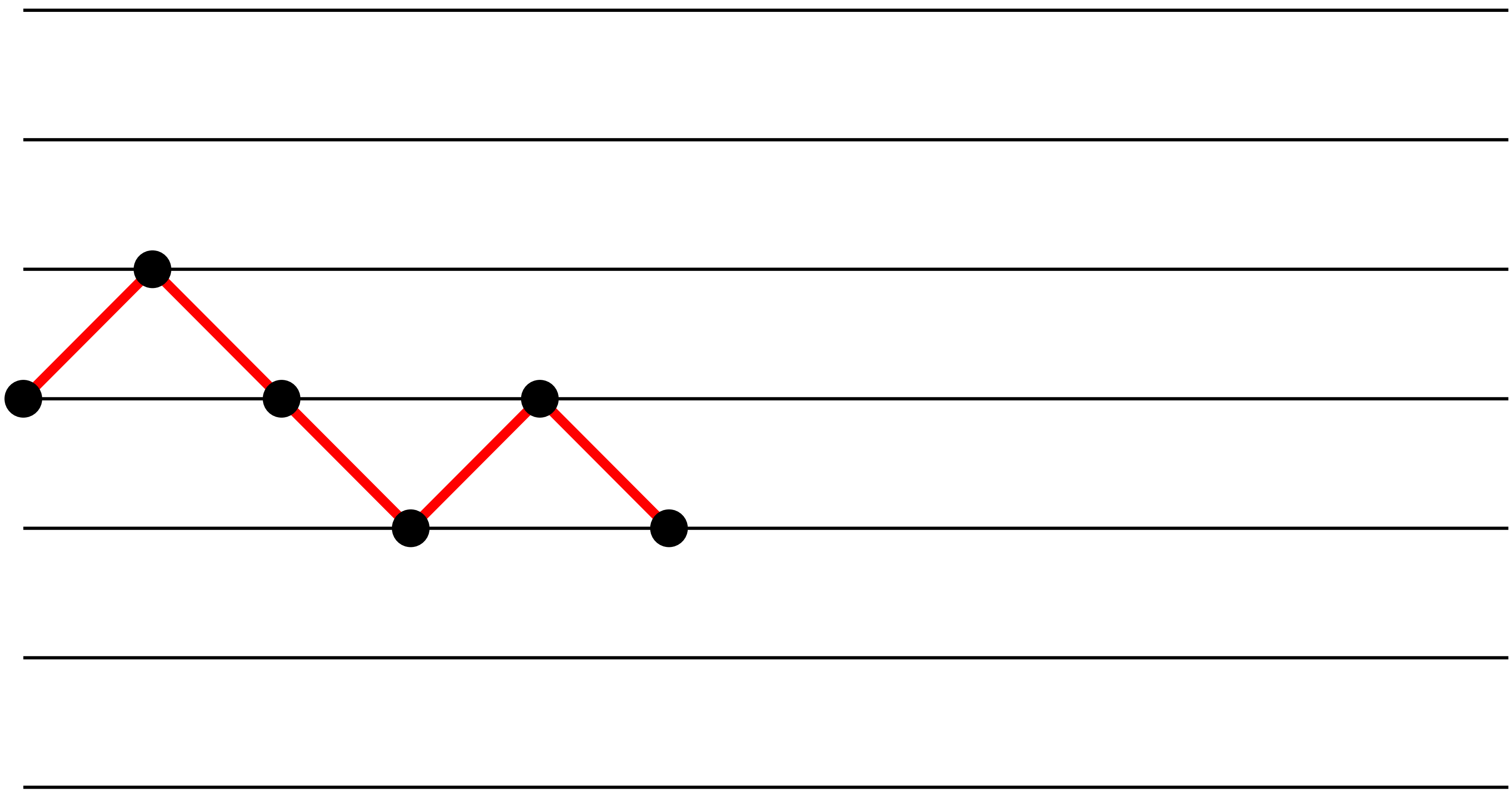


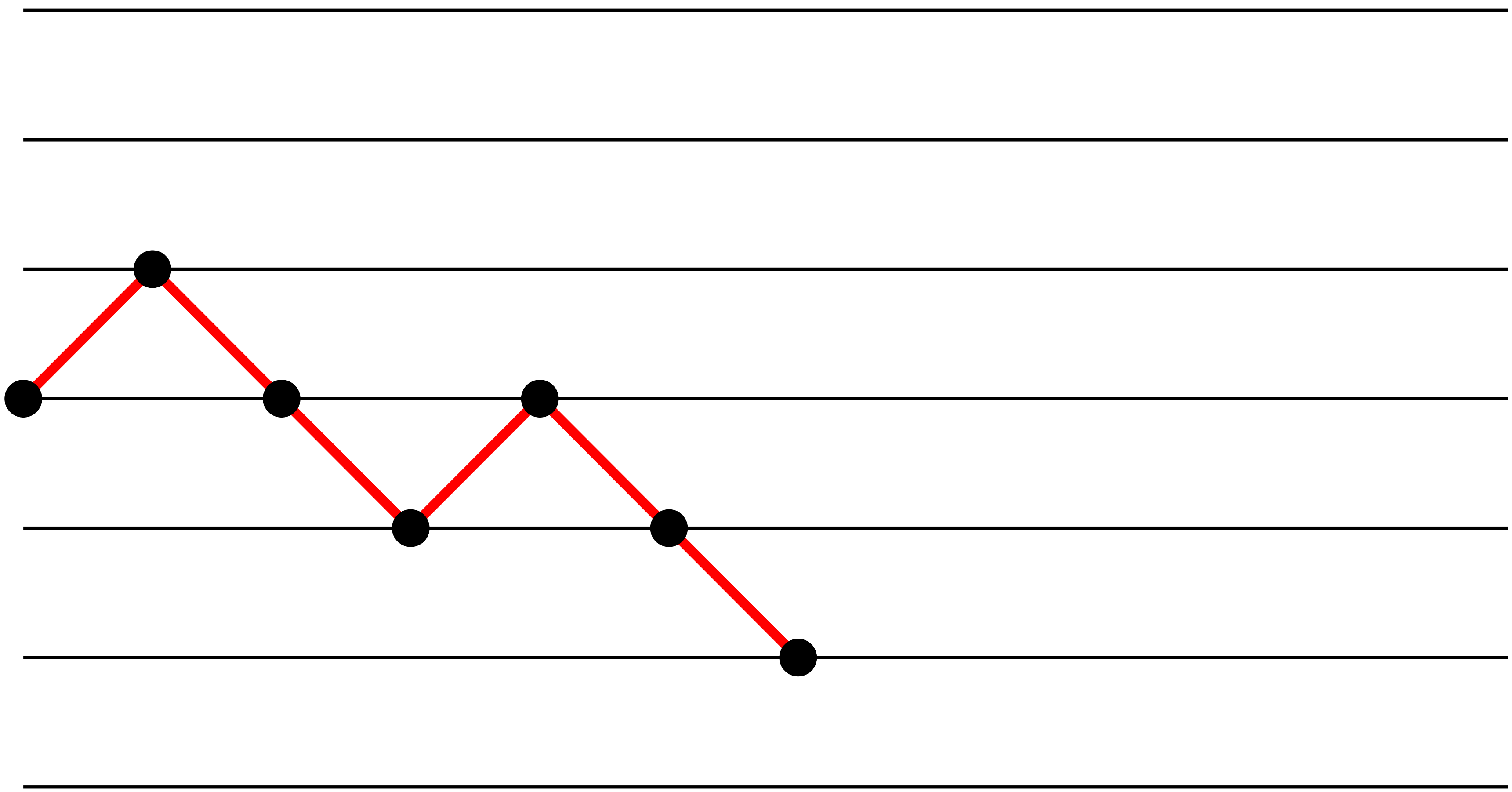


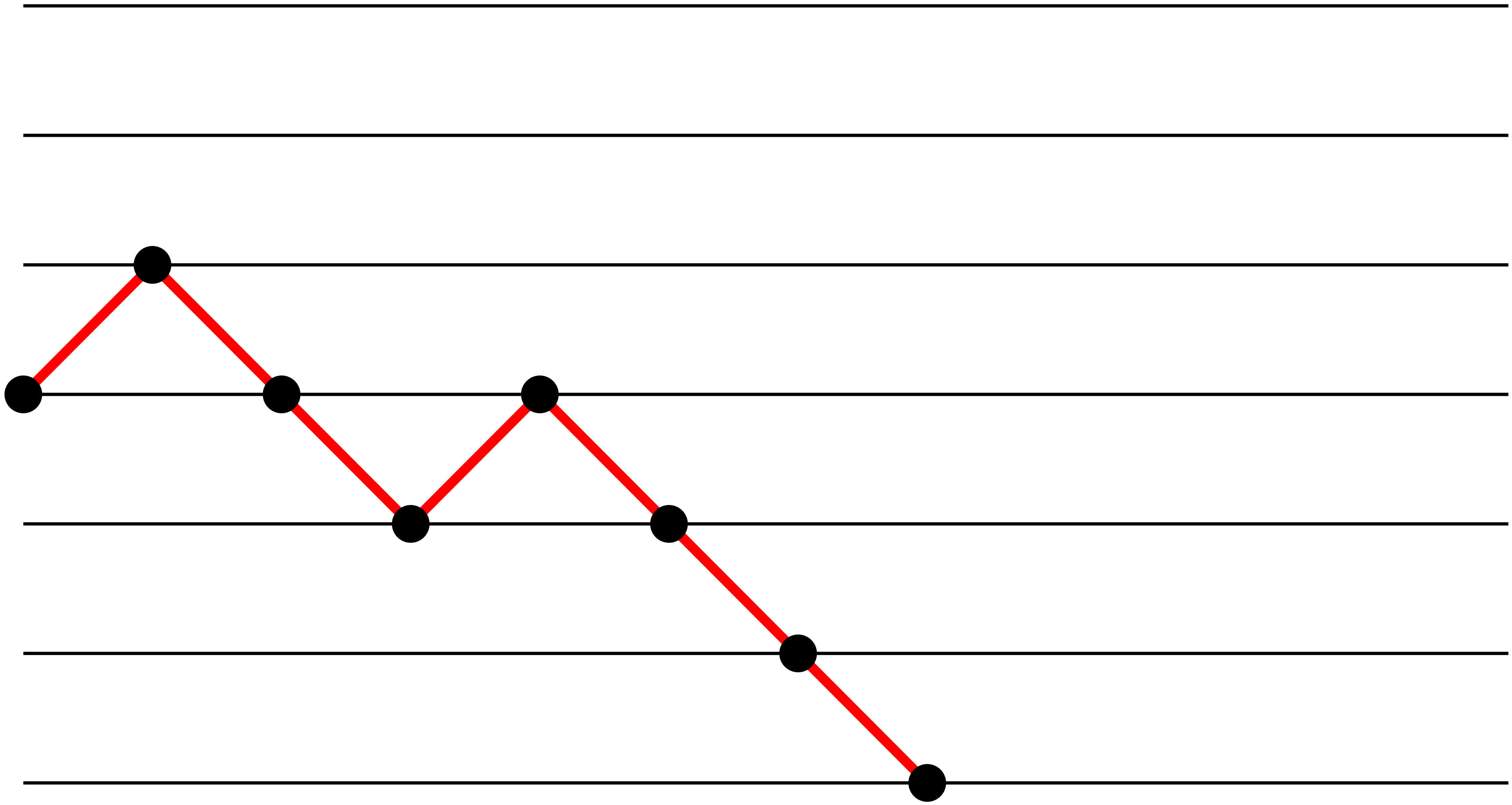


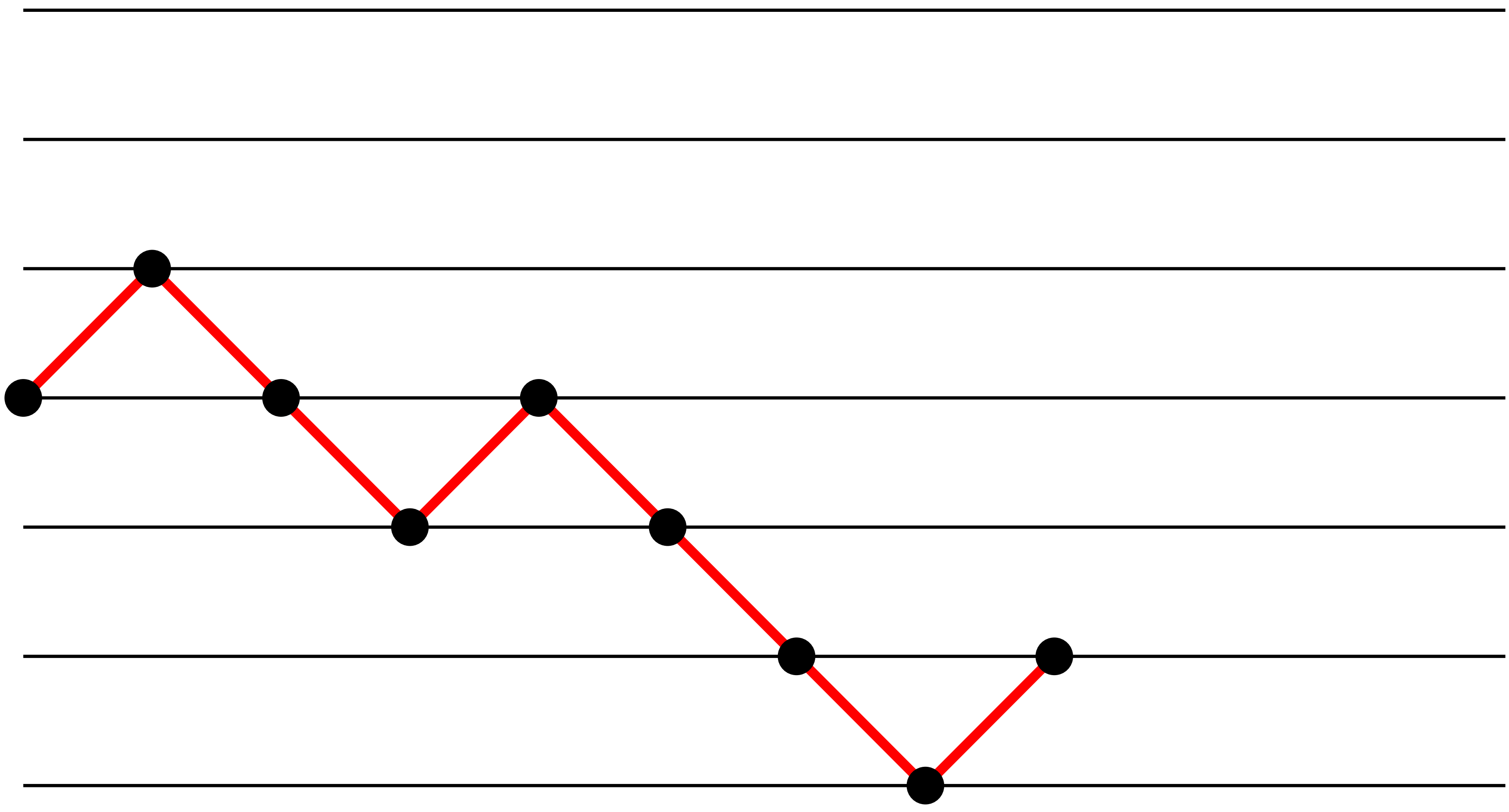


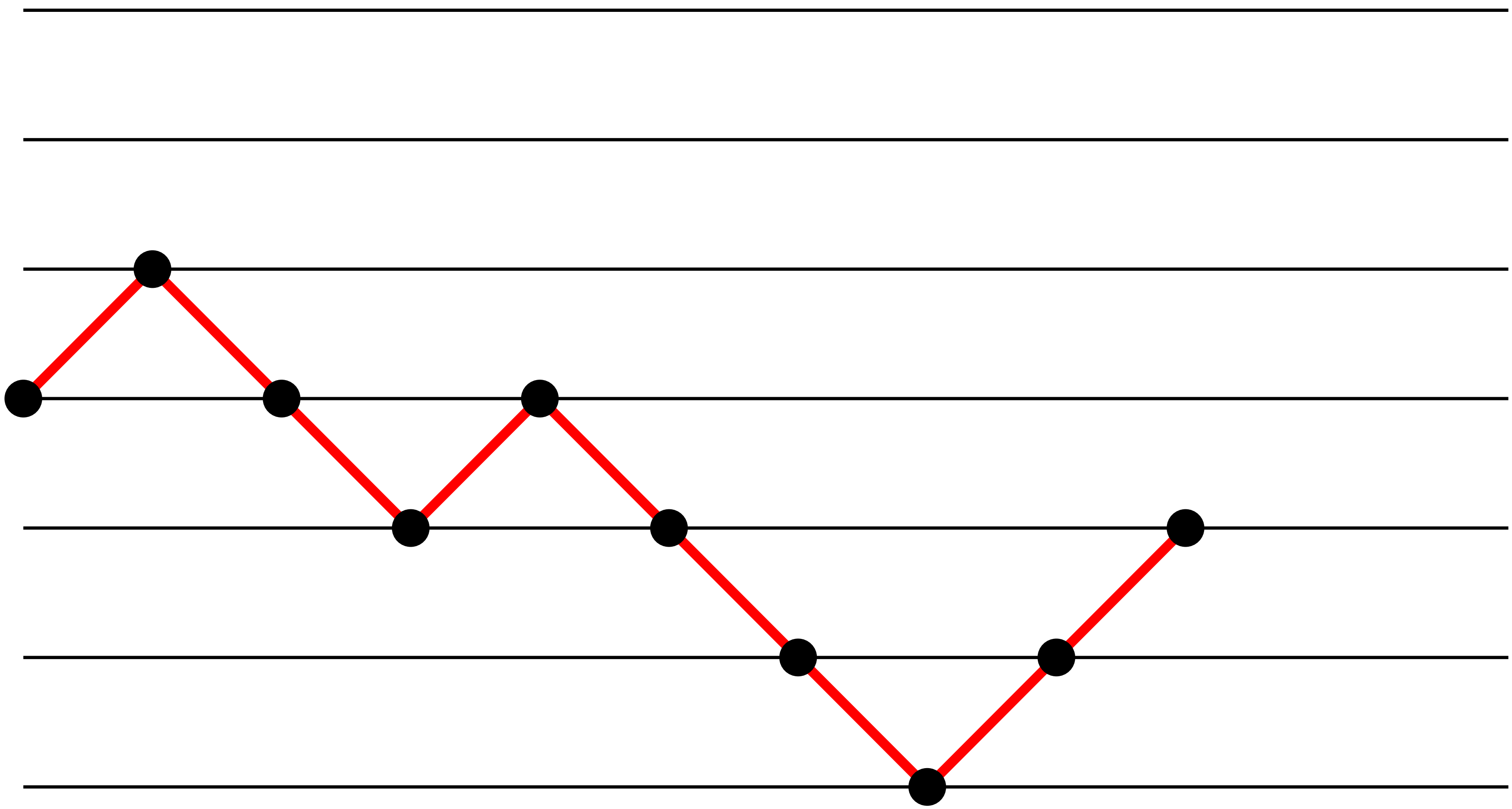


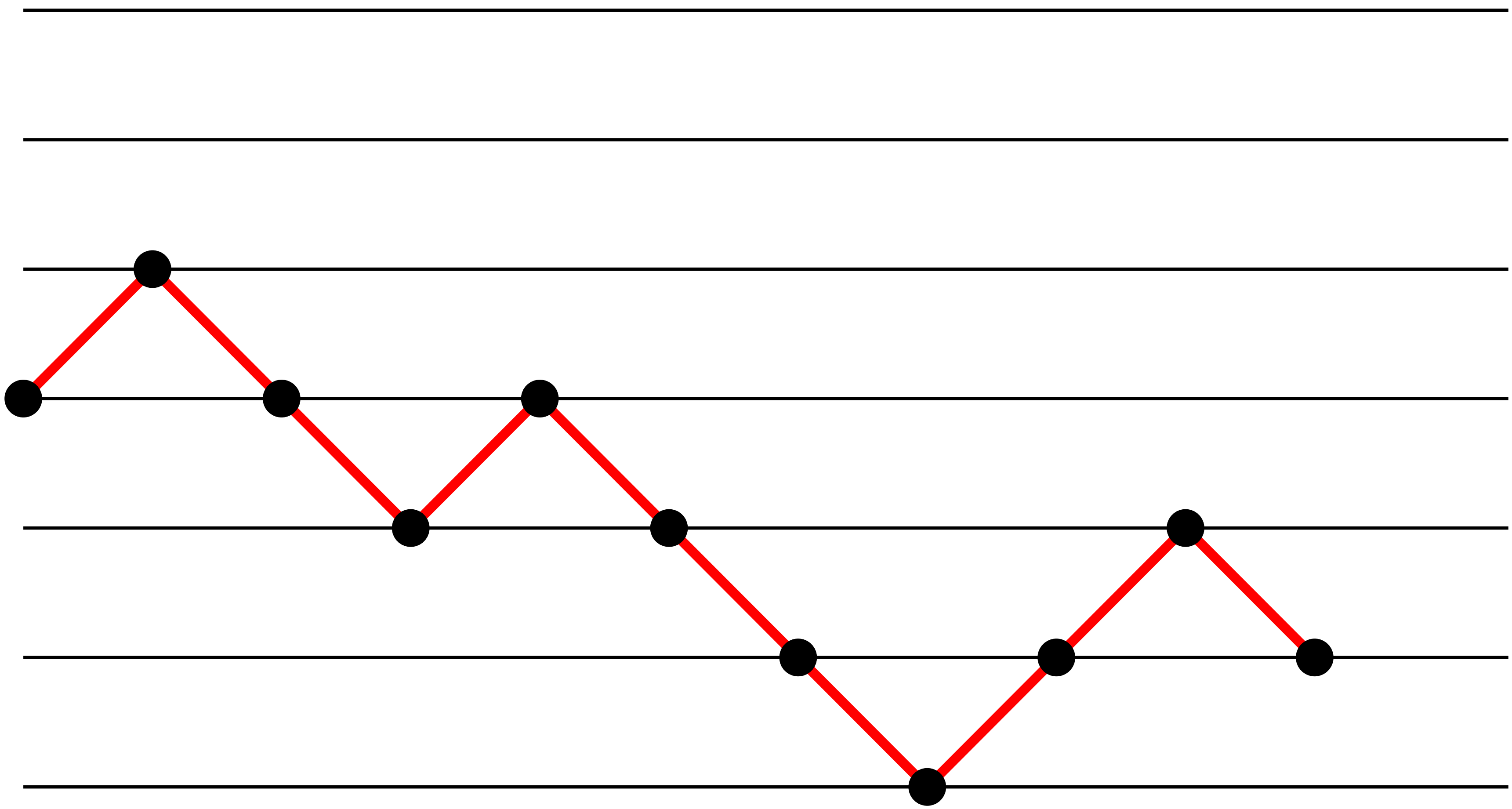


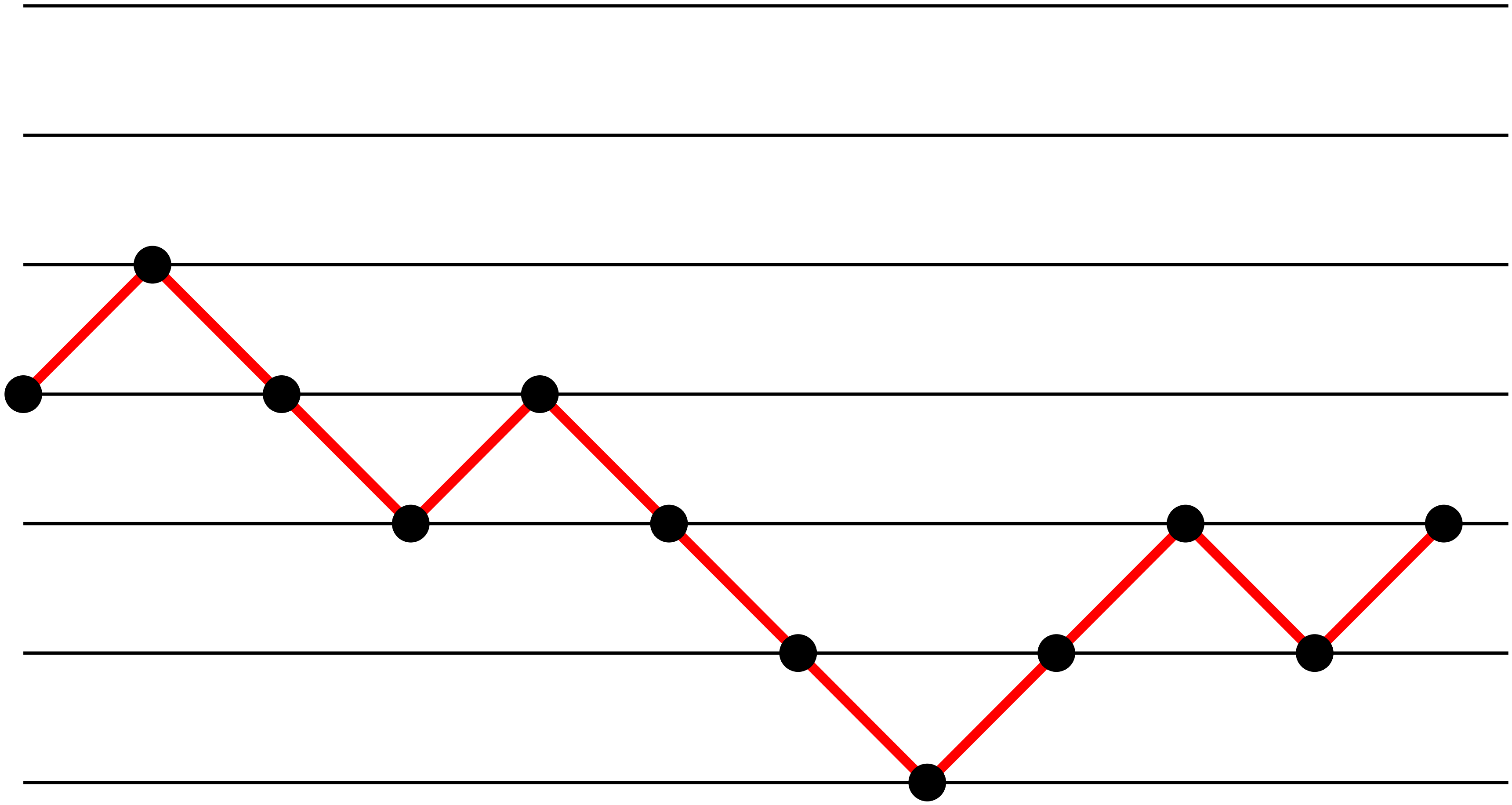


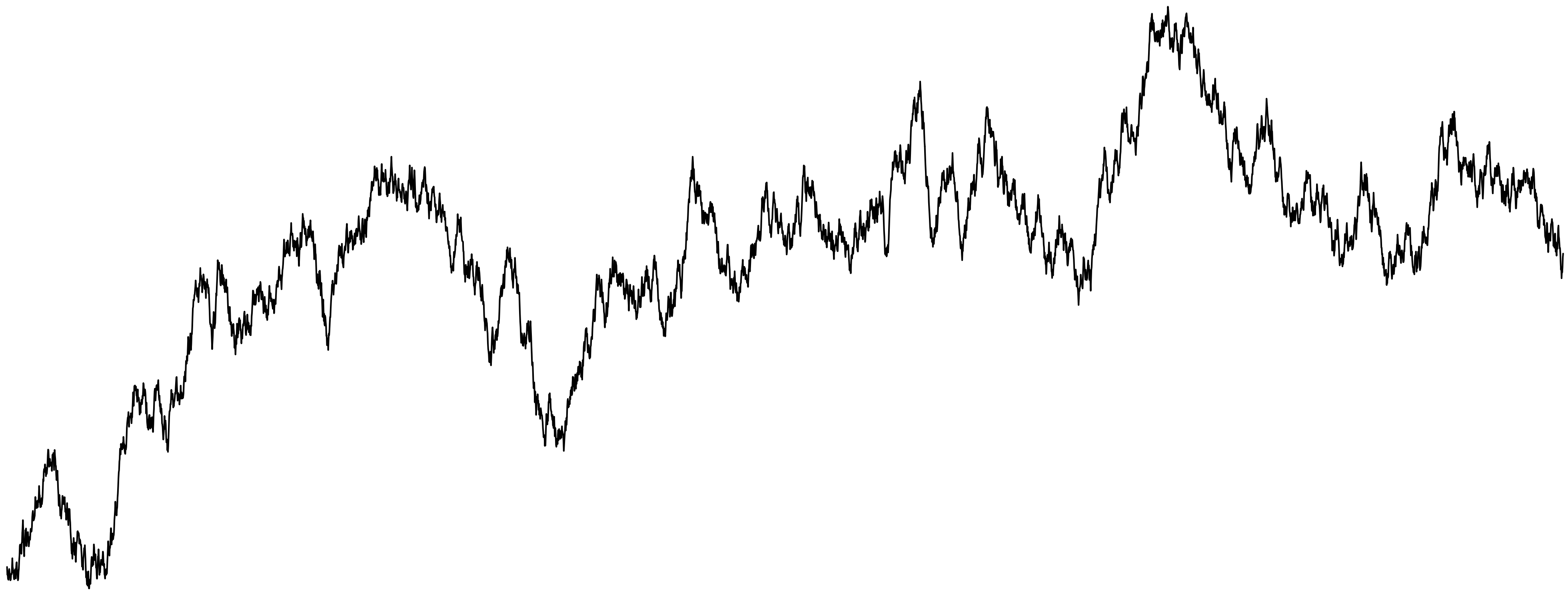




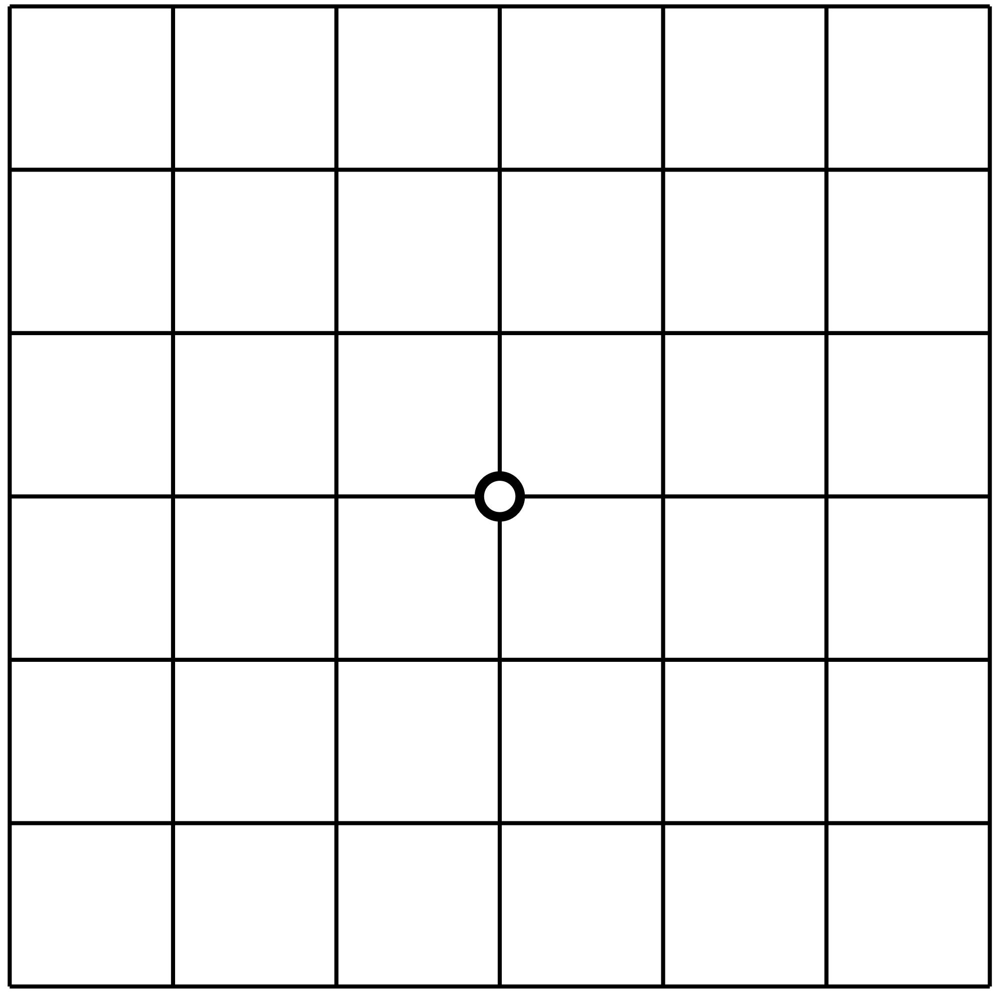


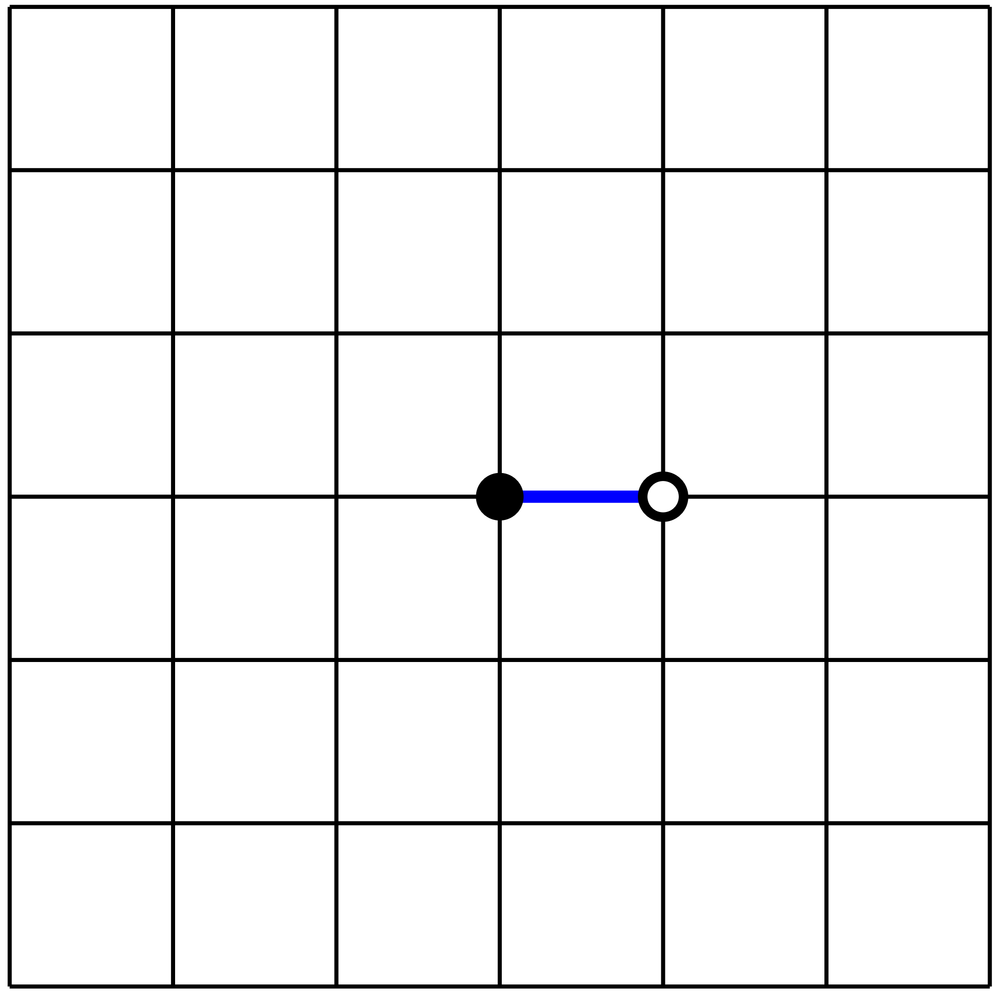


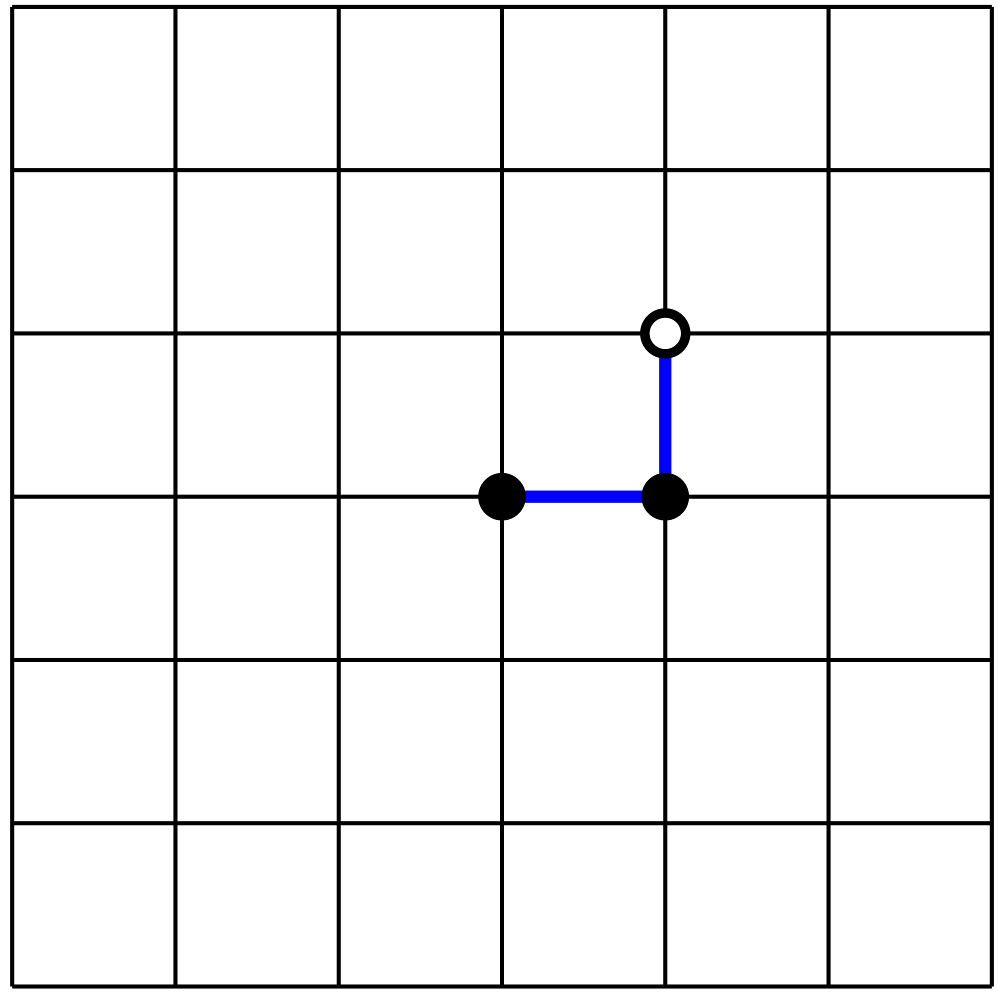


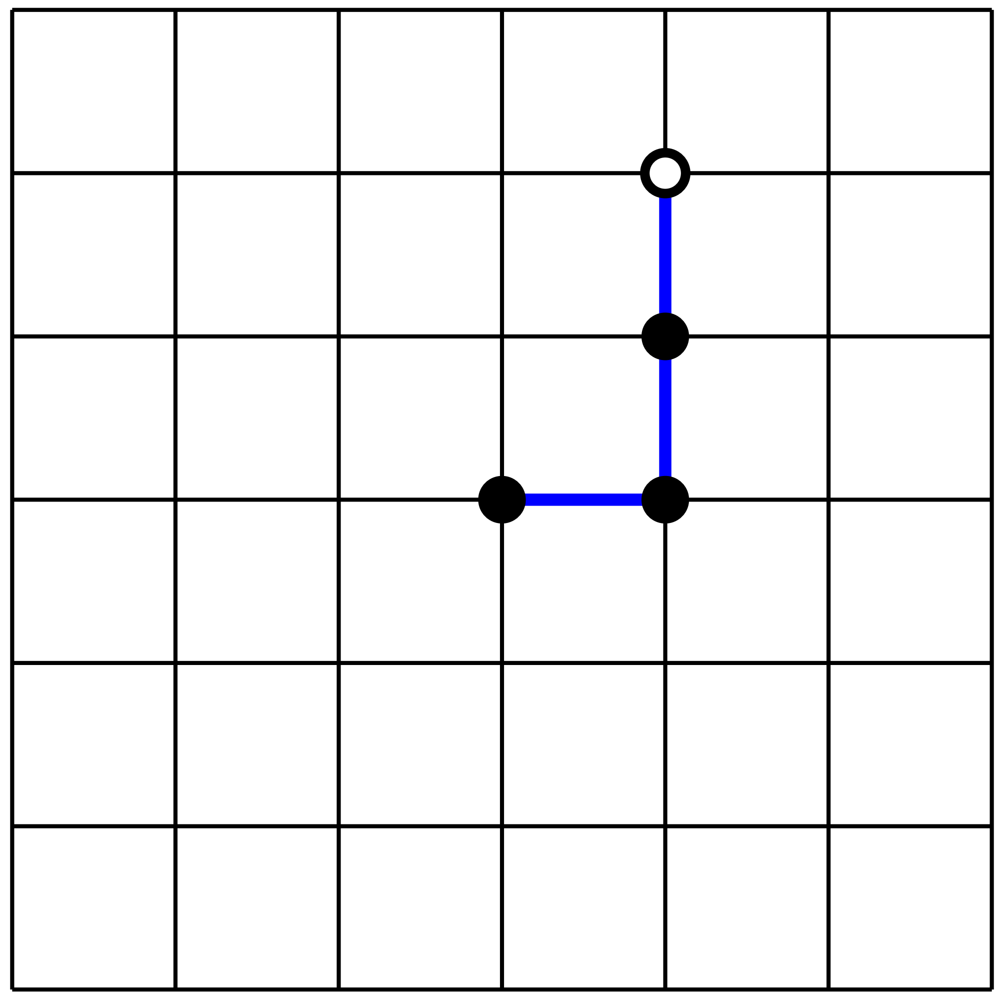


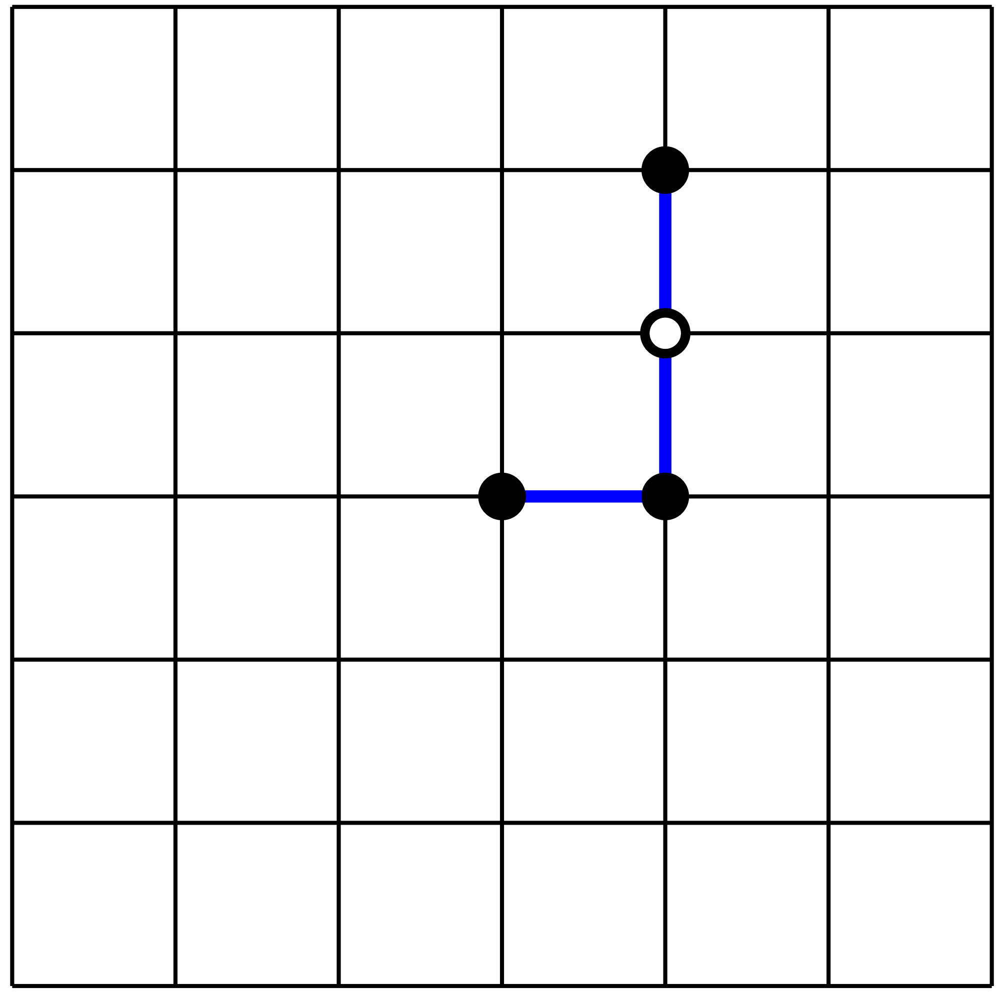
1-dimensional Brownian motion

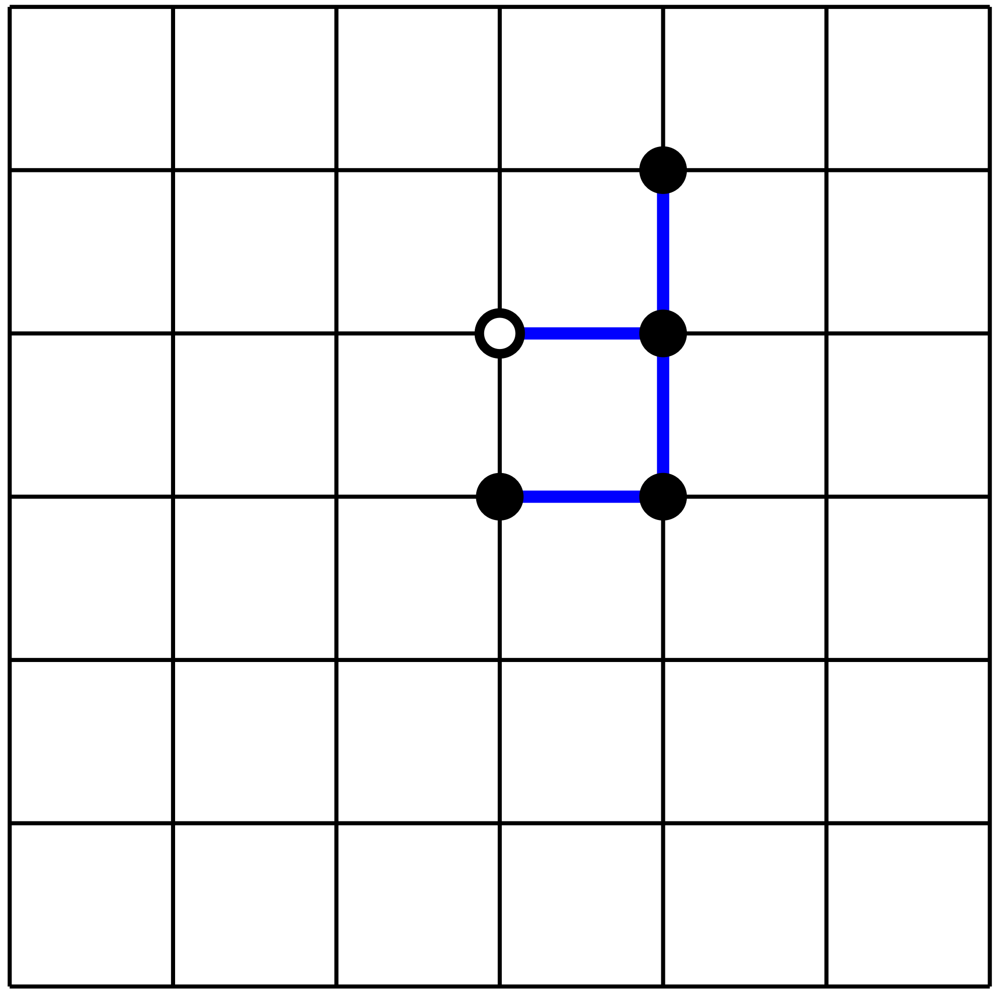


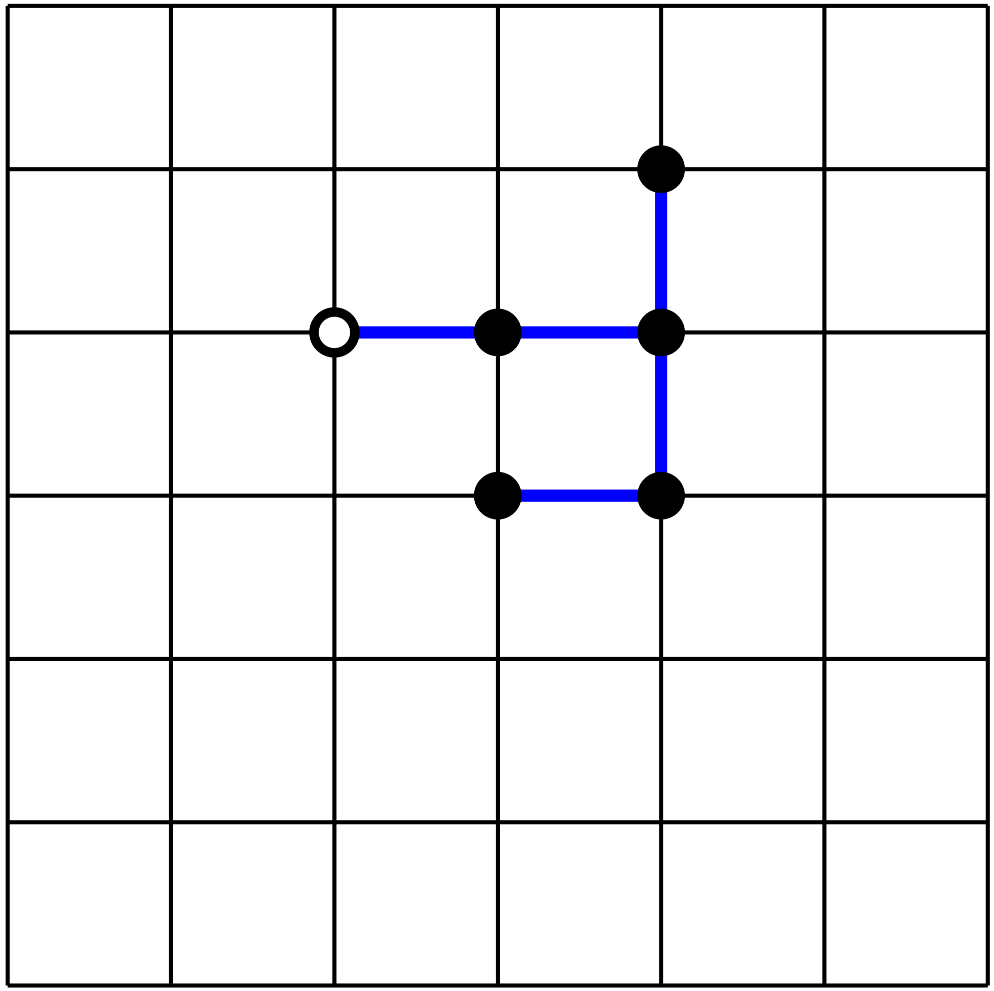


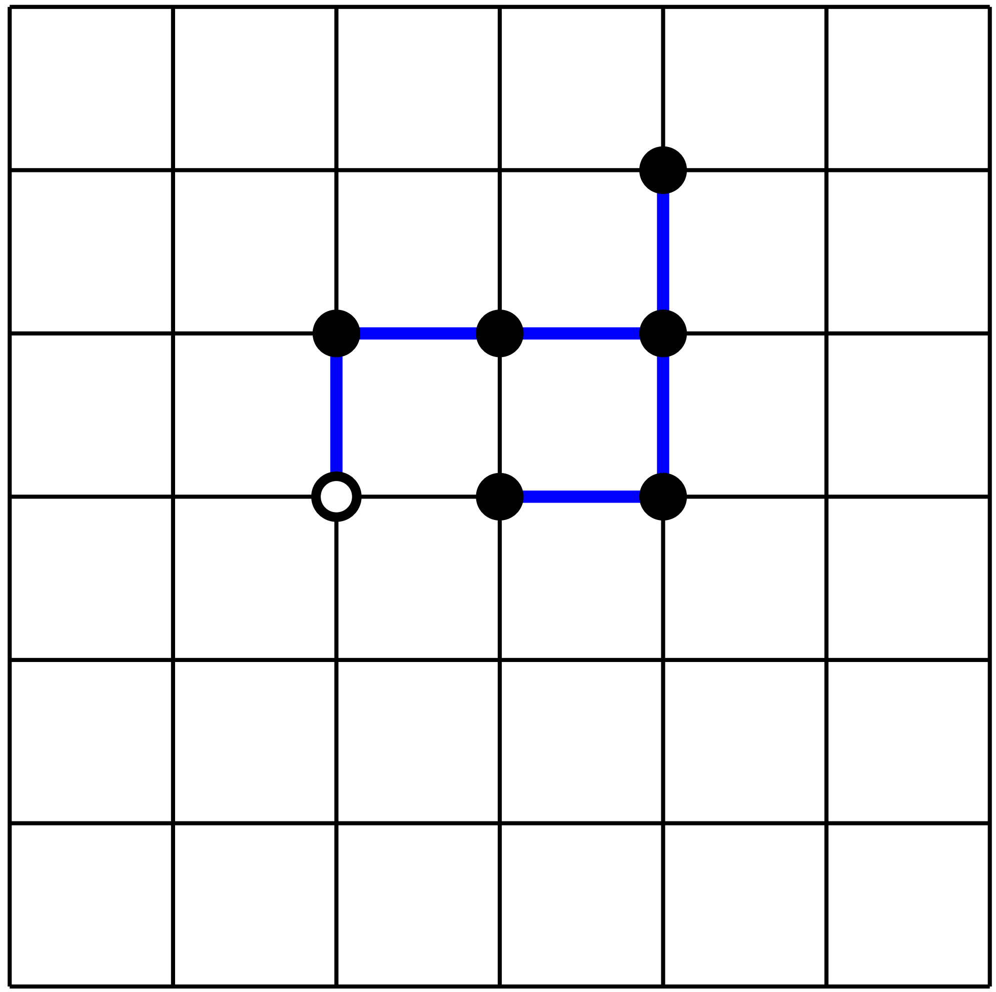


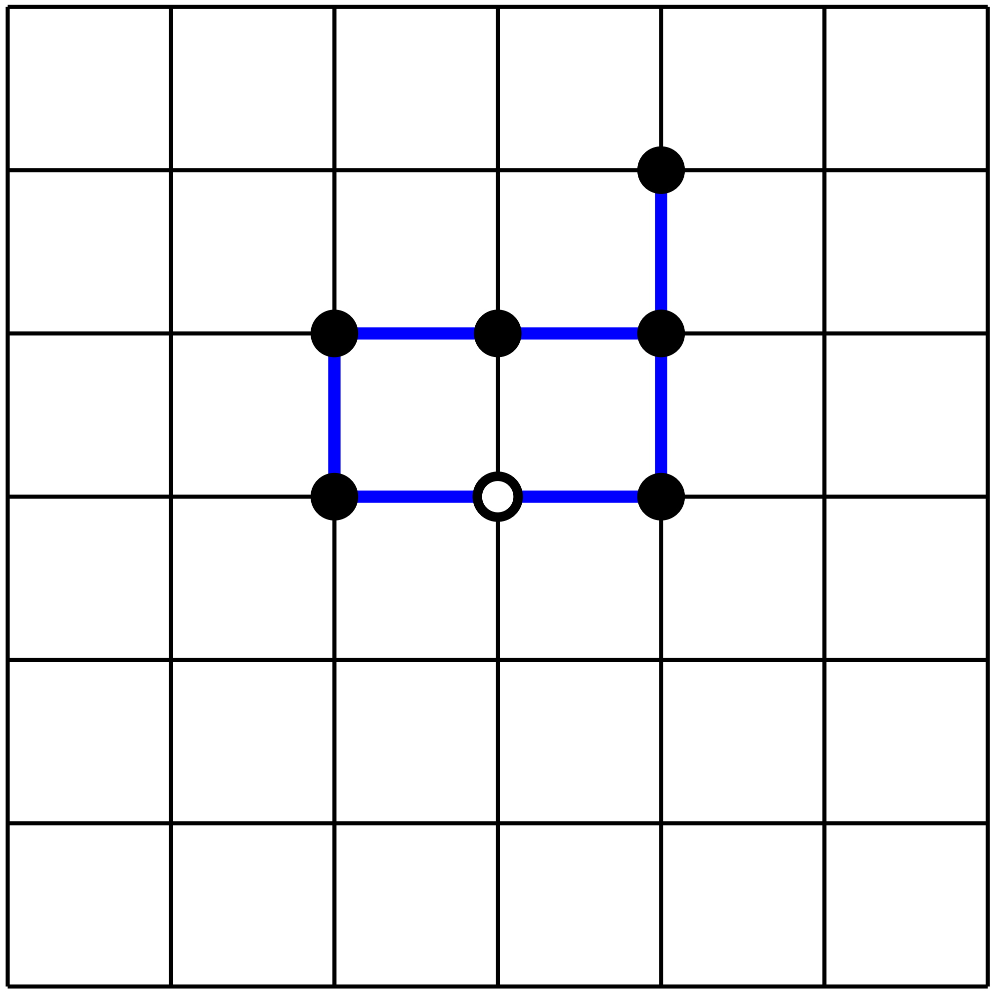


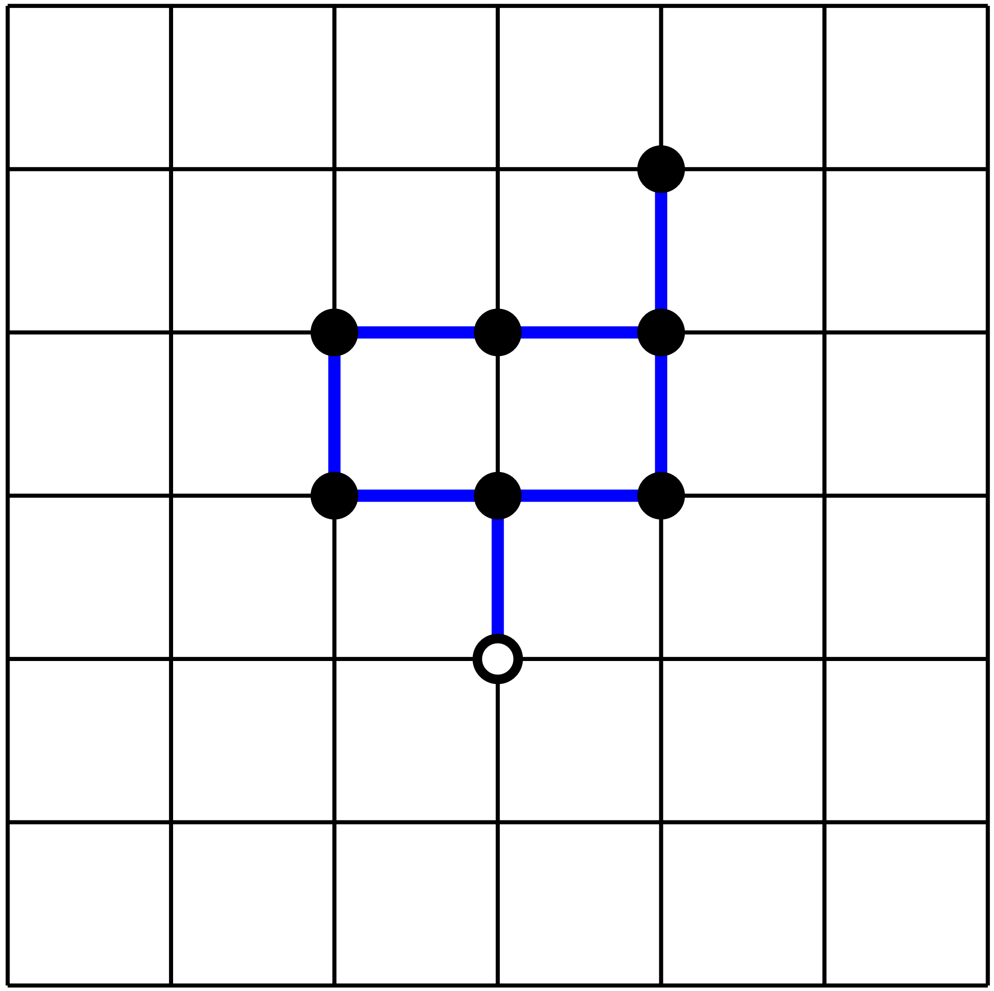


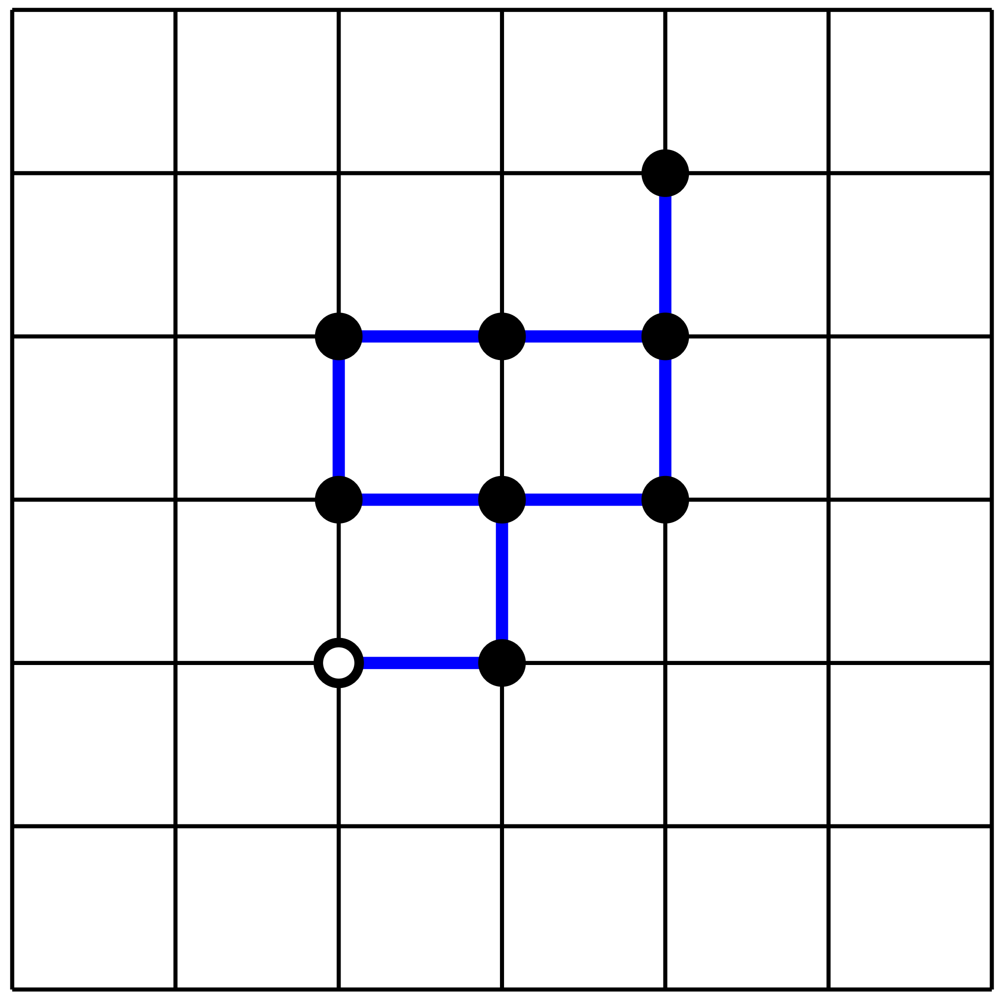


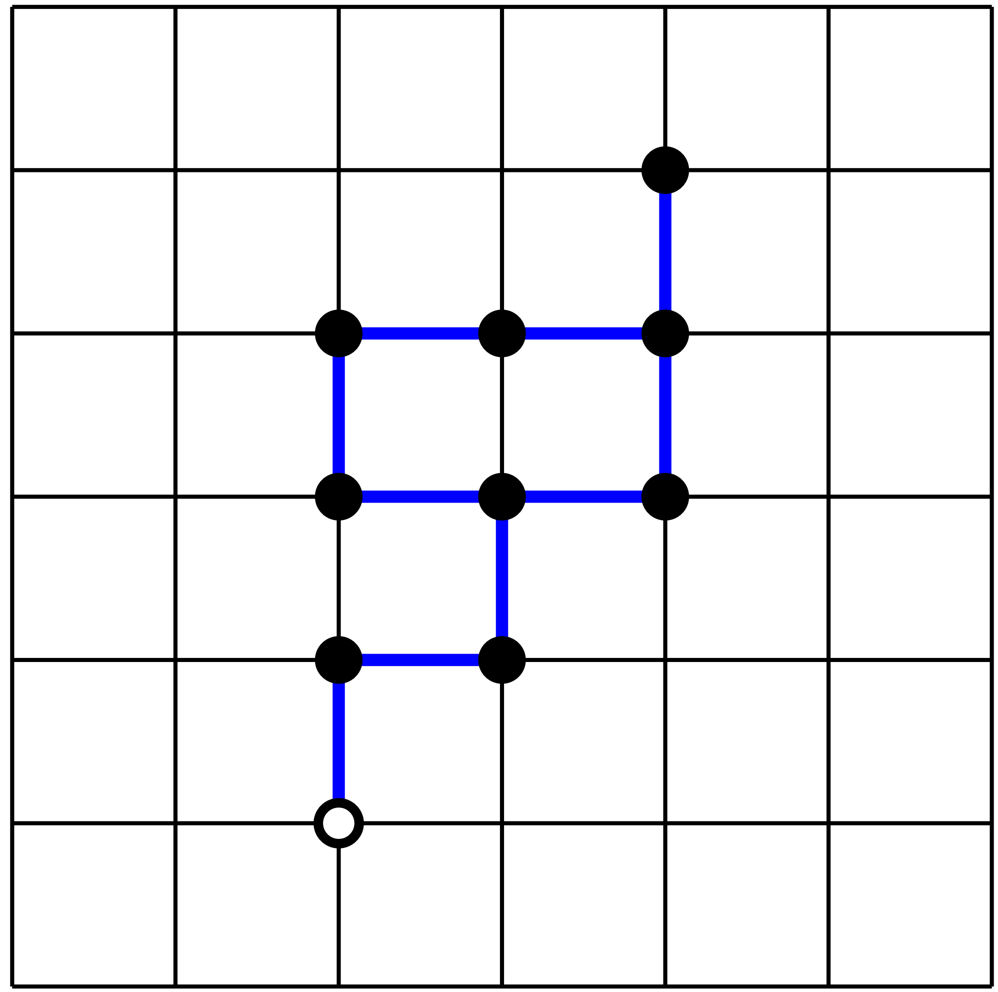


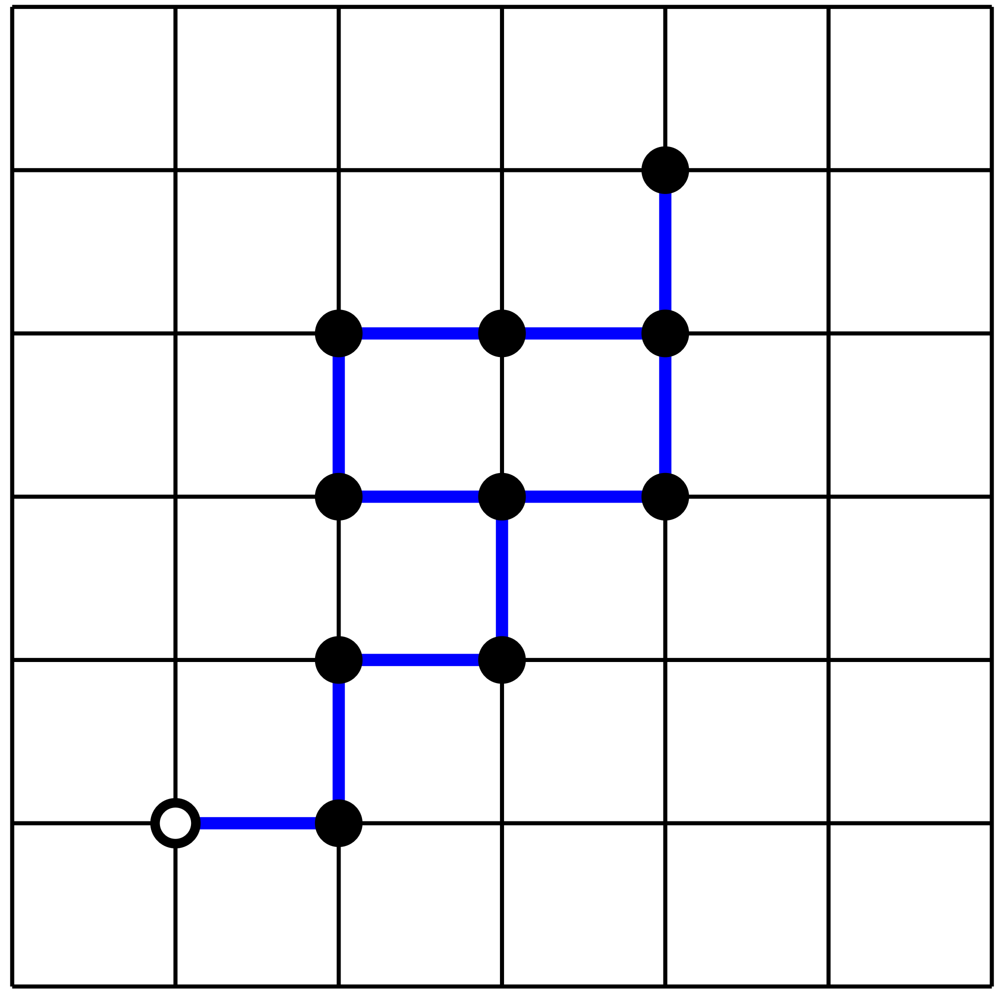


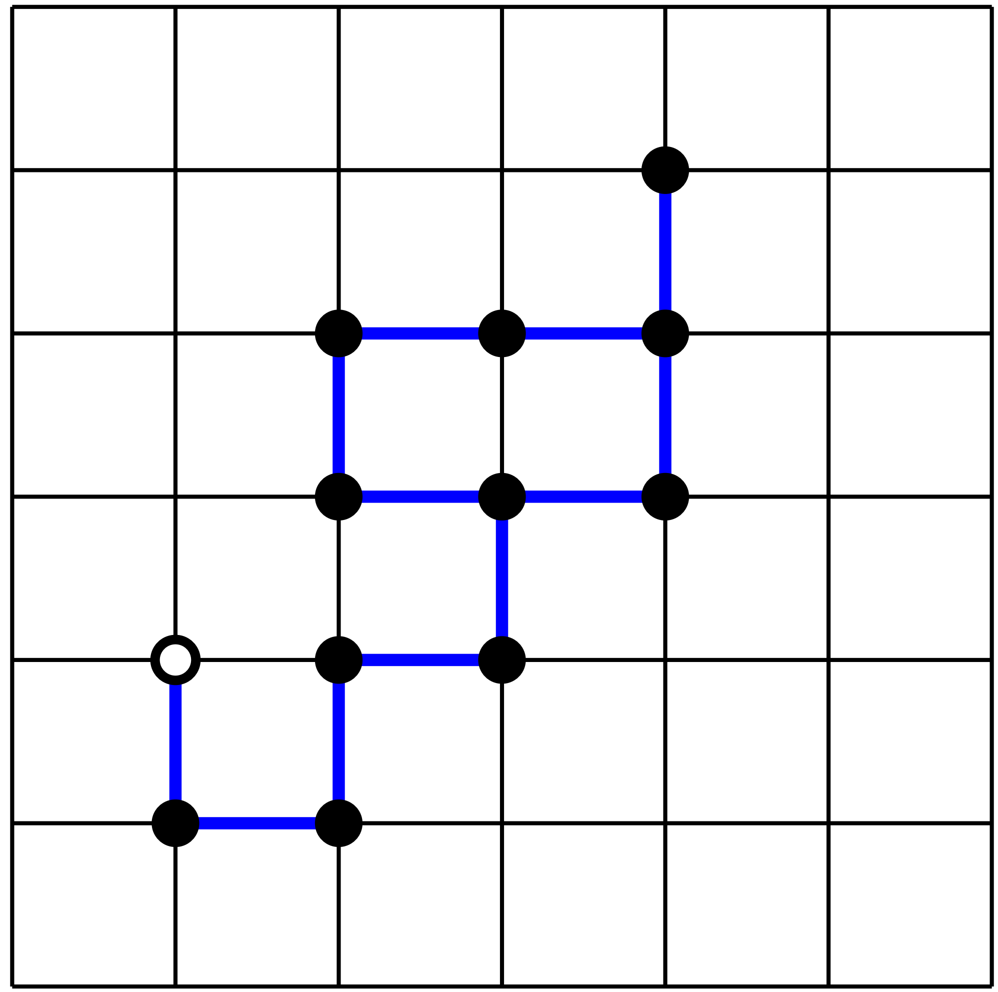


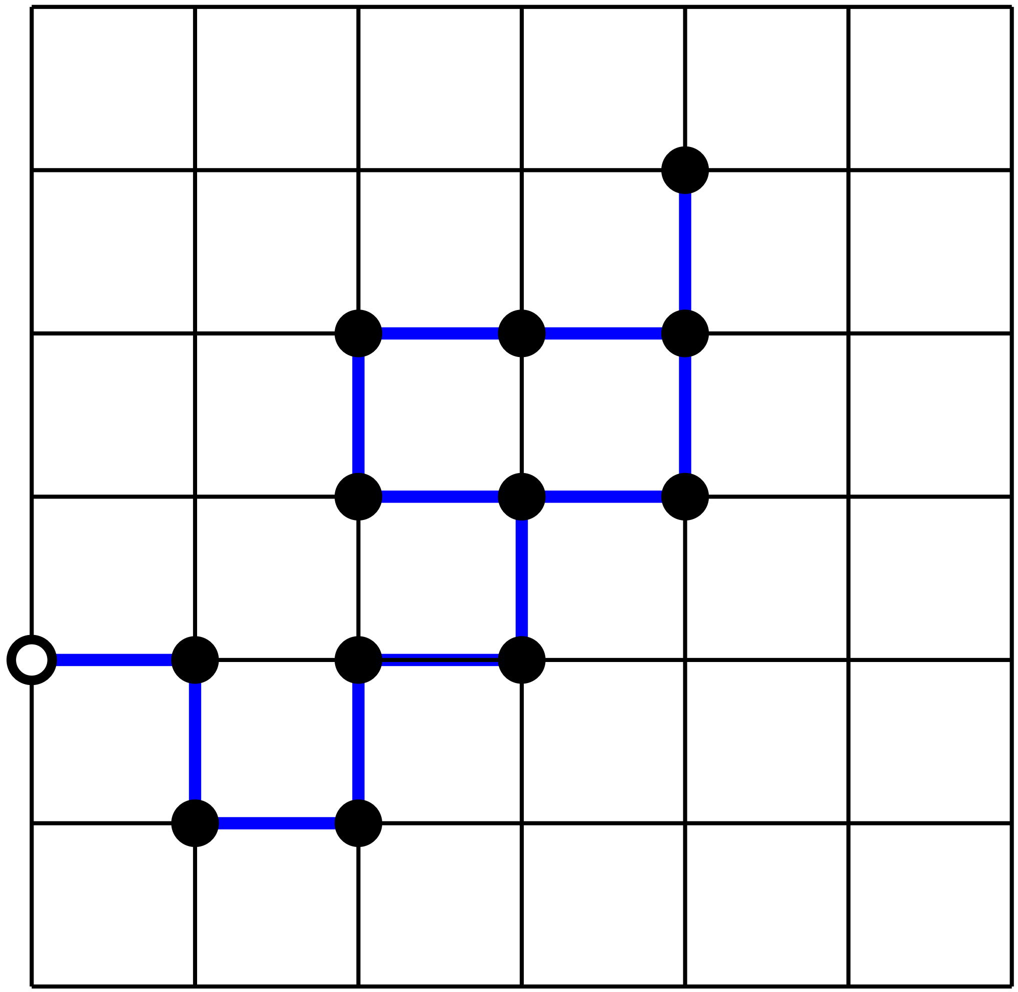


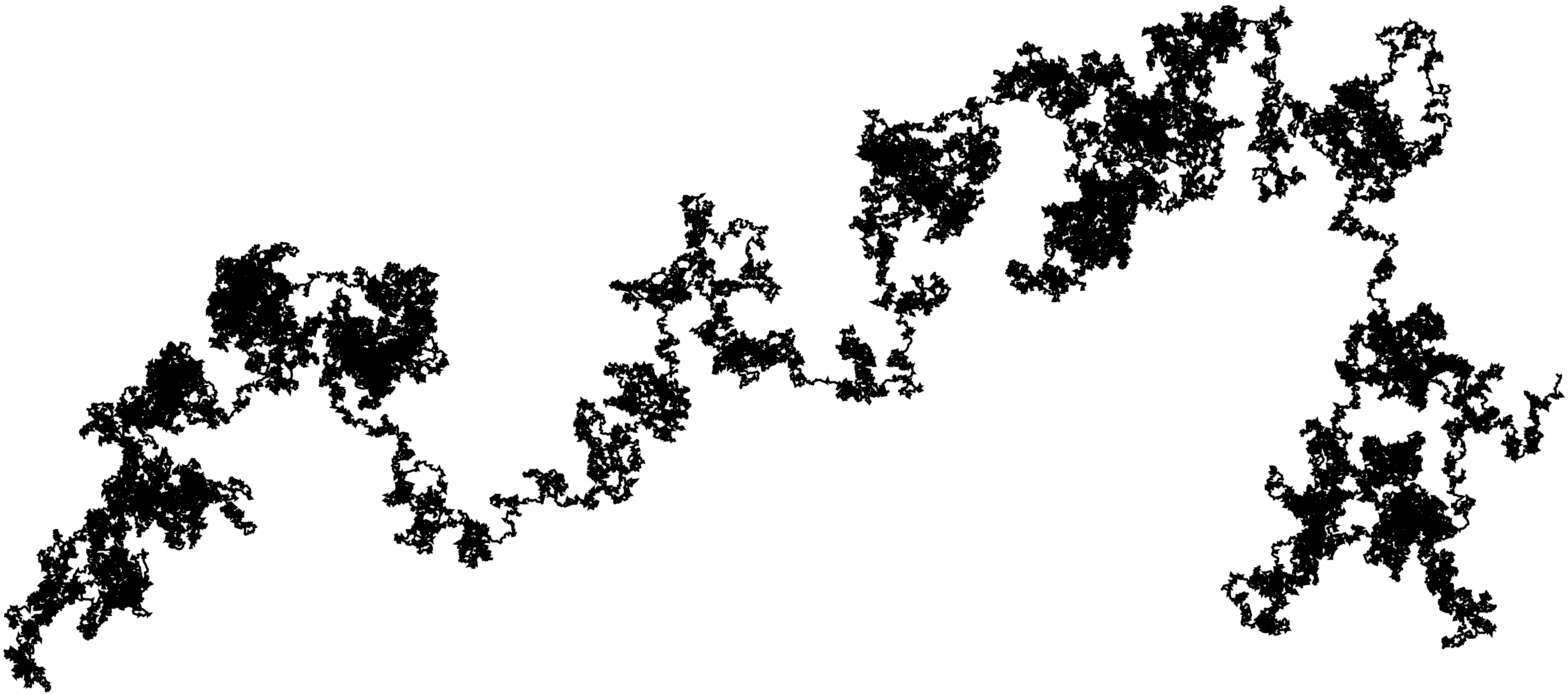




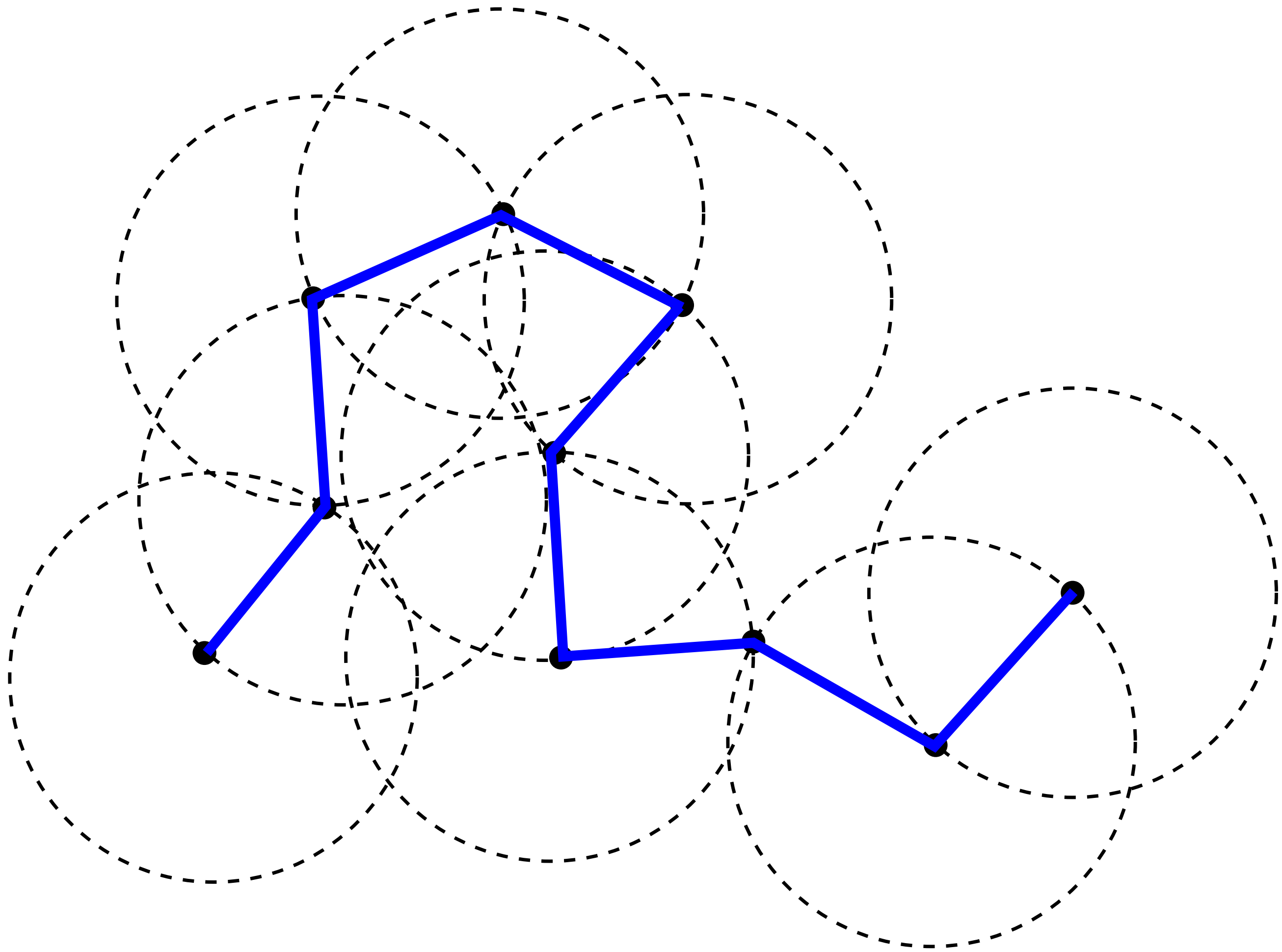




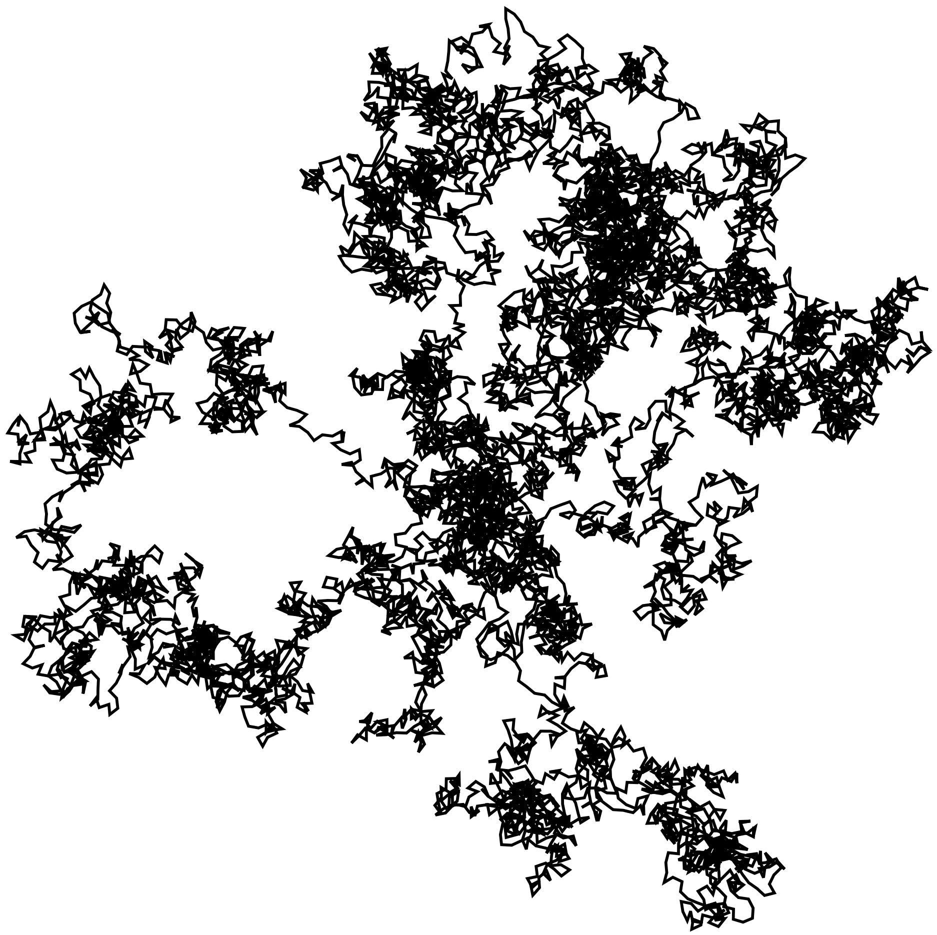




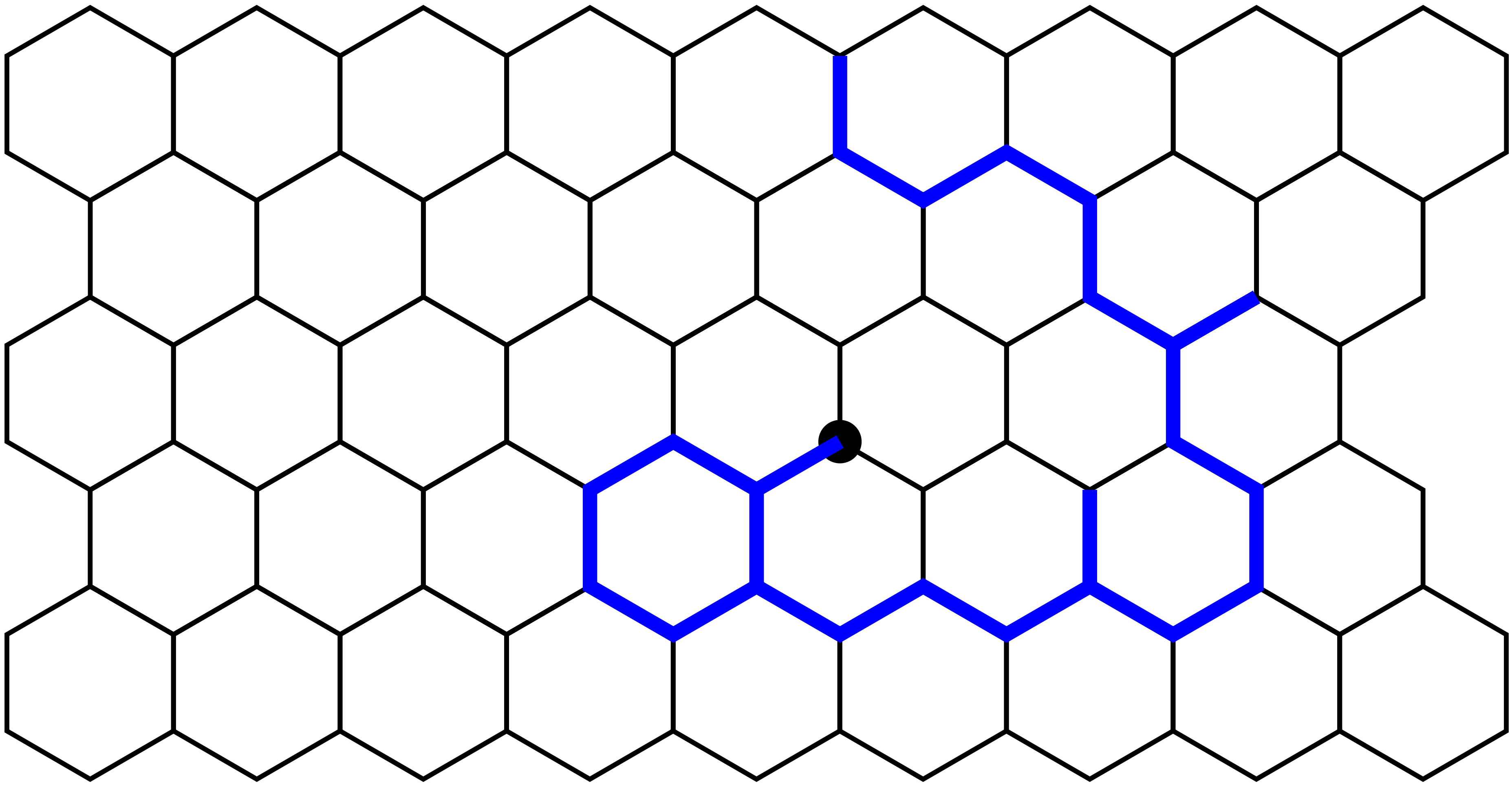
2-dimensional Brownian motion



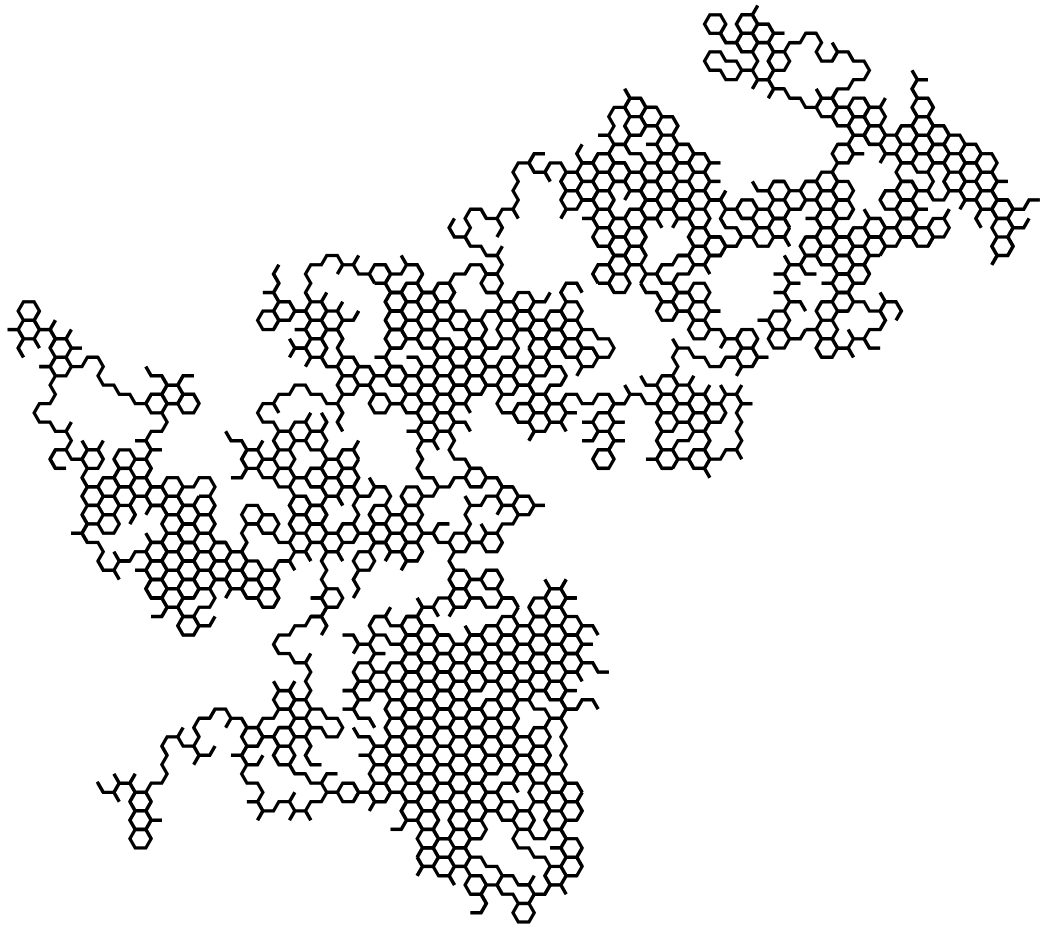
Take unit step in randomly chosen direction.



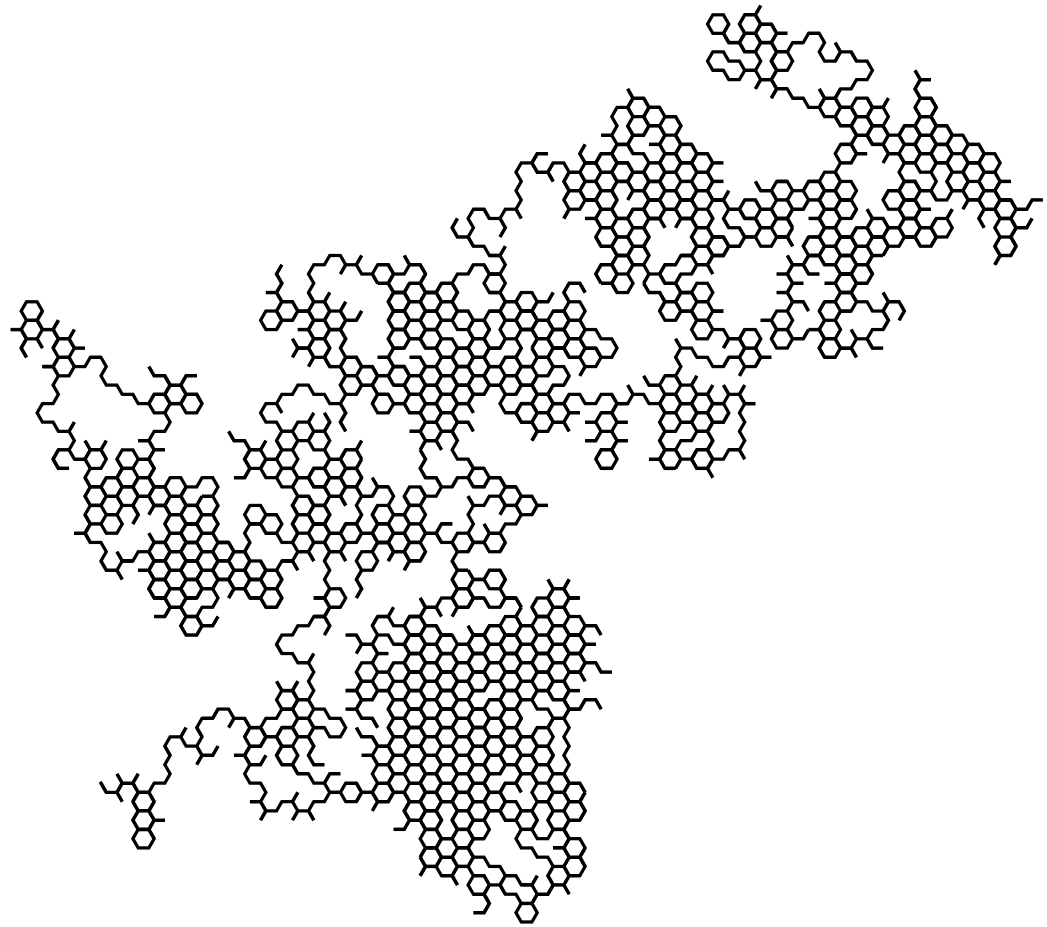
10,000 unit steps in randomly chosen direction.



Random walk on the hexagonal grid.

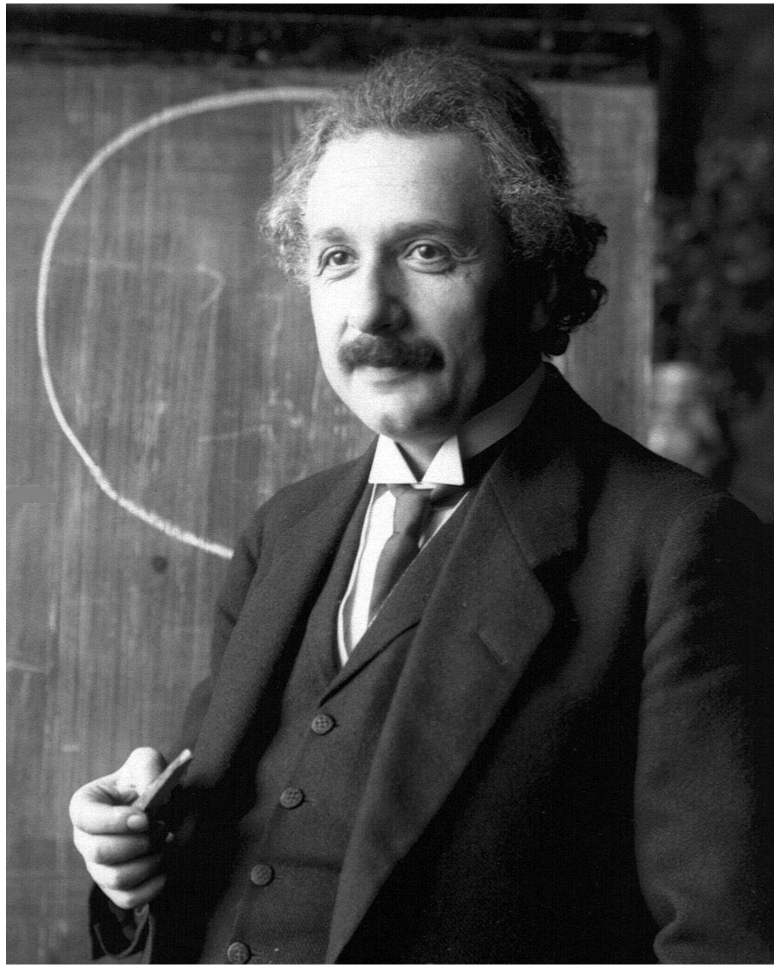


10,000 steps on the hexagonal grid.

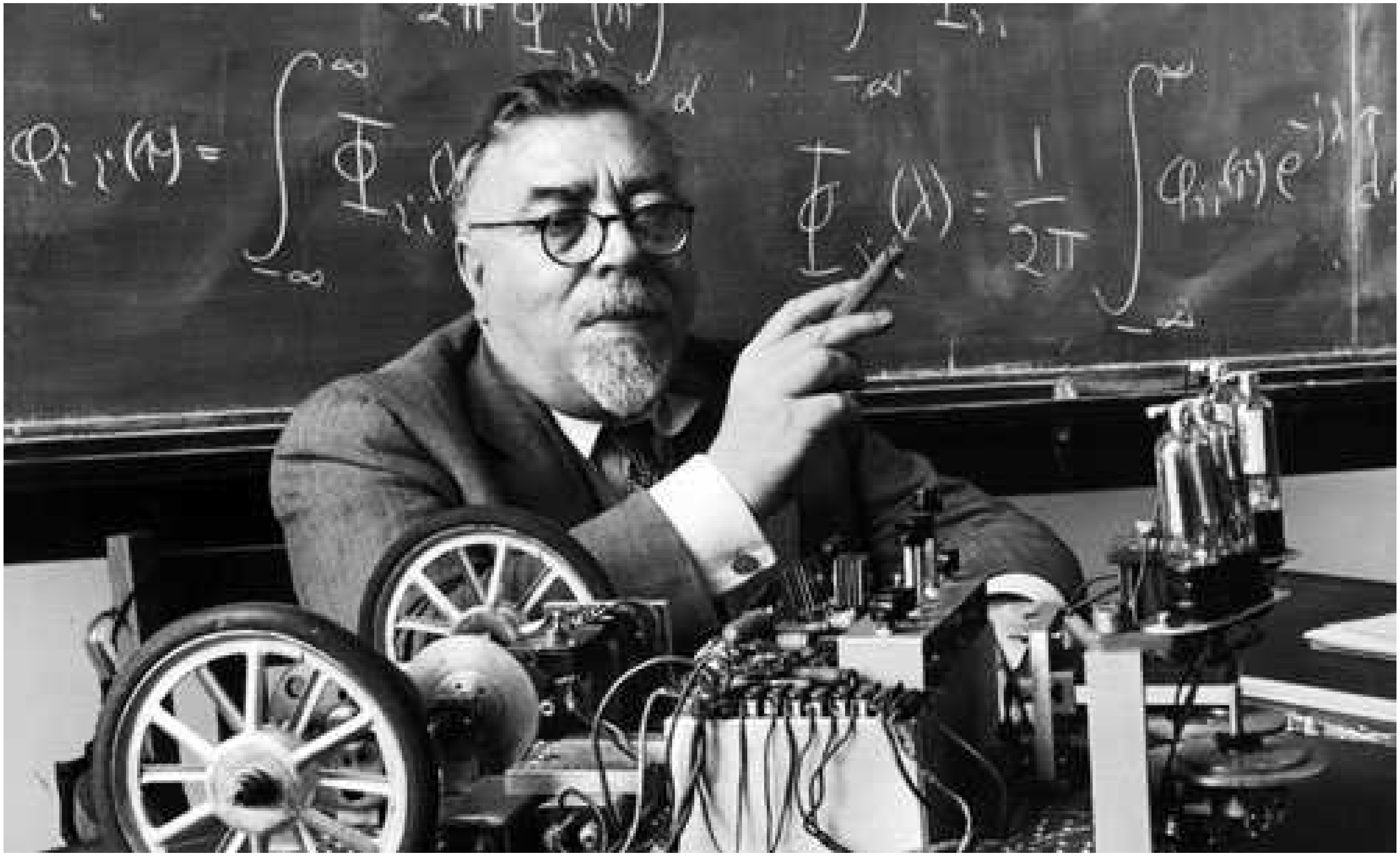


Every "reasonable" discrete random walk converges to Brownian motion in the limit (Donsker's invariance principle).

Robert Brown
observed particles in microscope, 1827



Albert Einstein
explained observations, confirmed atomic theory, 1905



Norbert Wiener

mathematical existence, “Wiener measure”

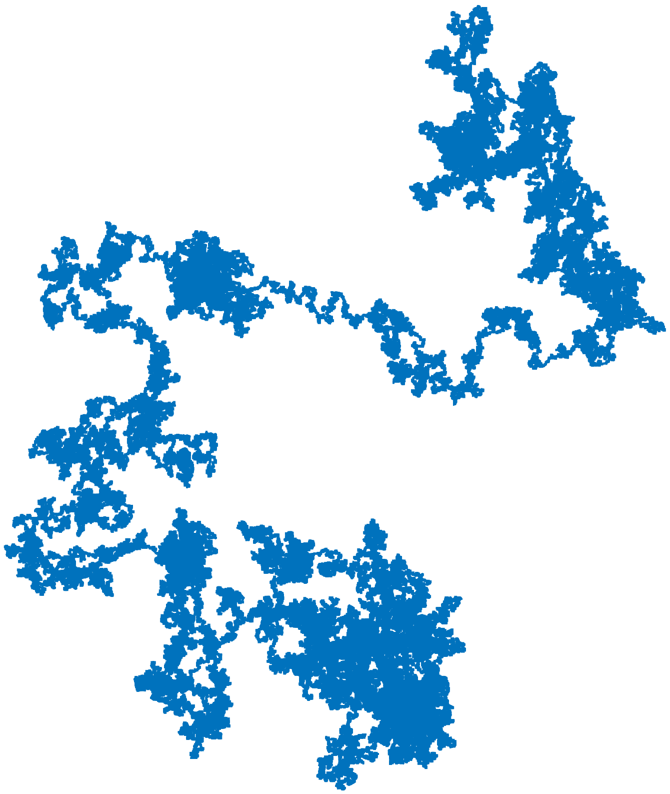
Louis Bachelier
Studied finance using random walks, 1900

Robert Merton

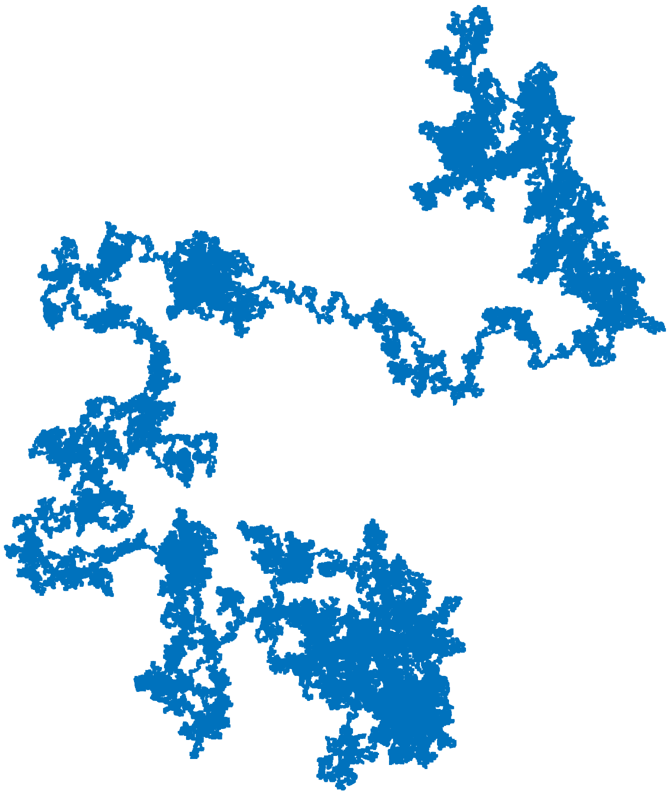
Myron Scholes

1997 Nobel prize, options pricing assuming prices are random walk

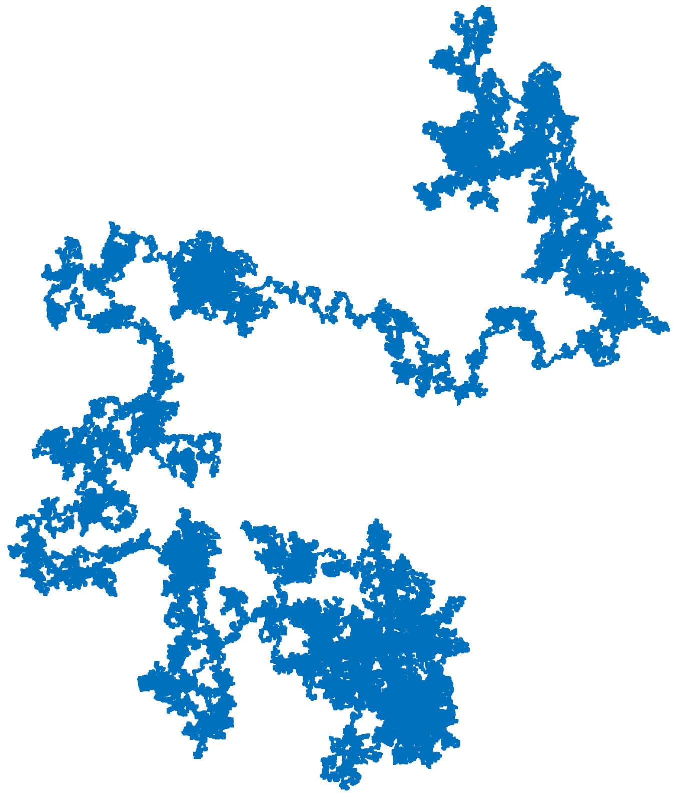
Proof that prices are not a random walk



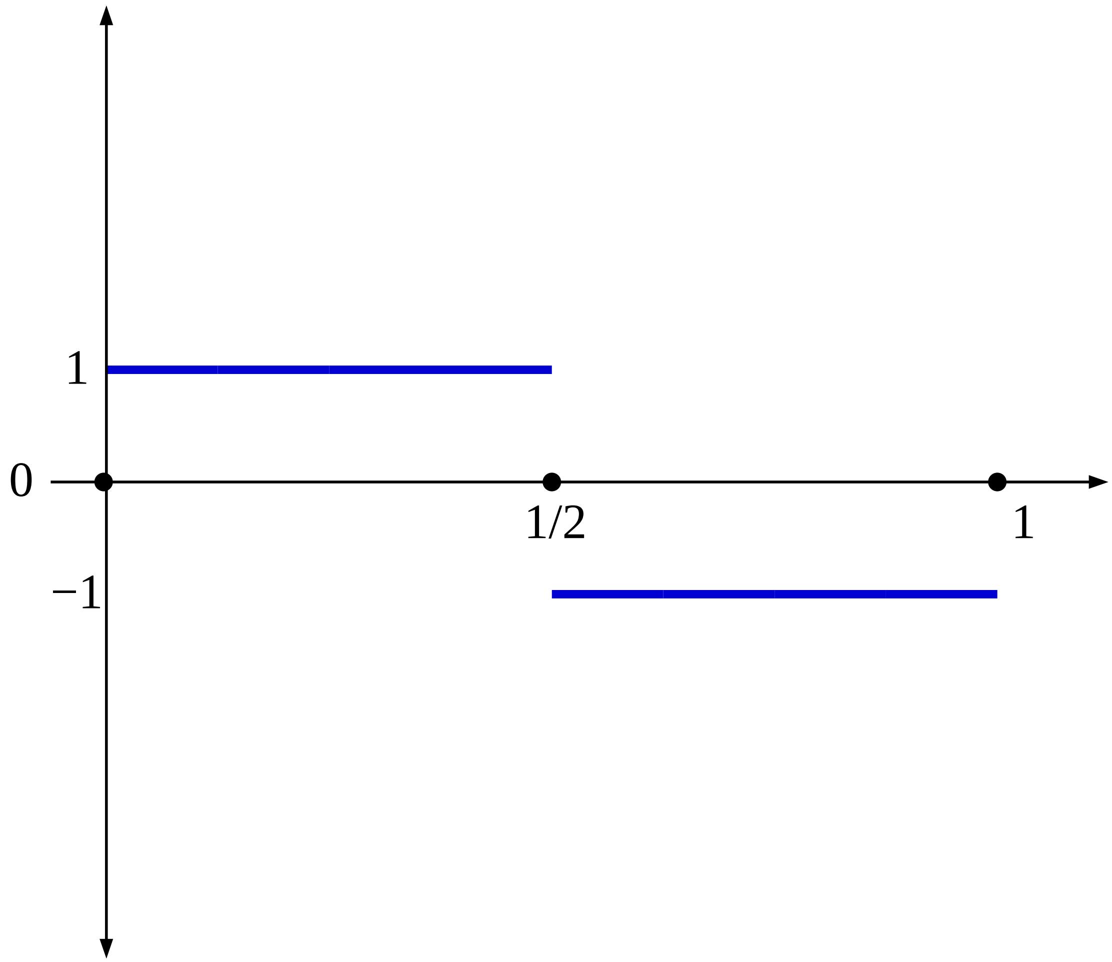
How big is a random walk after n steps?



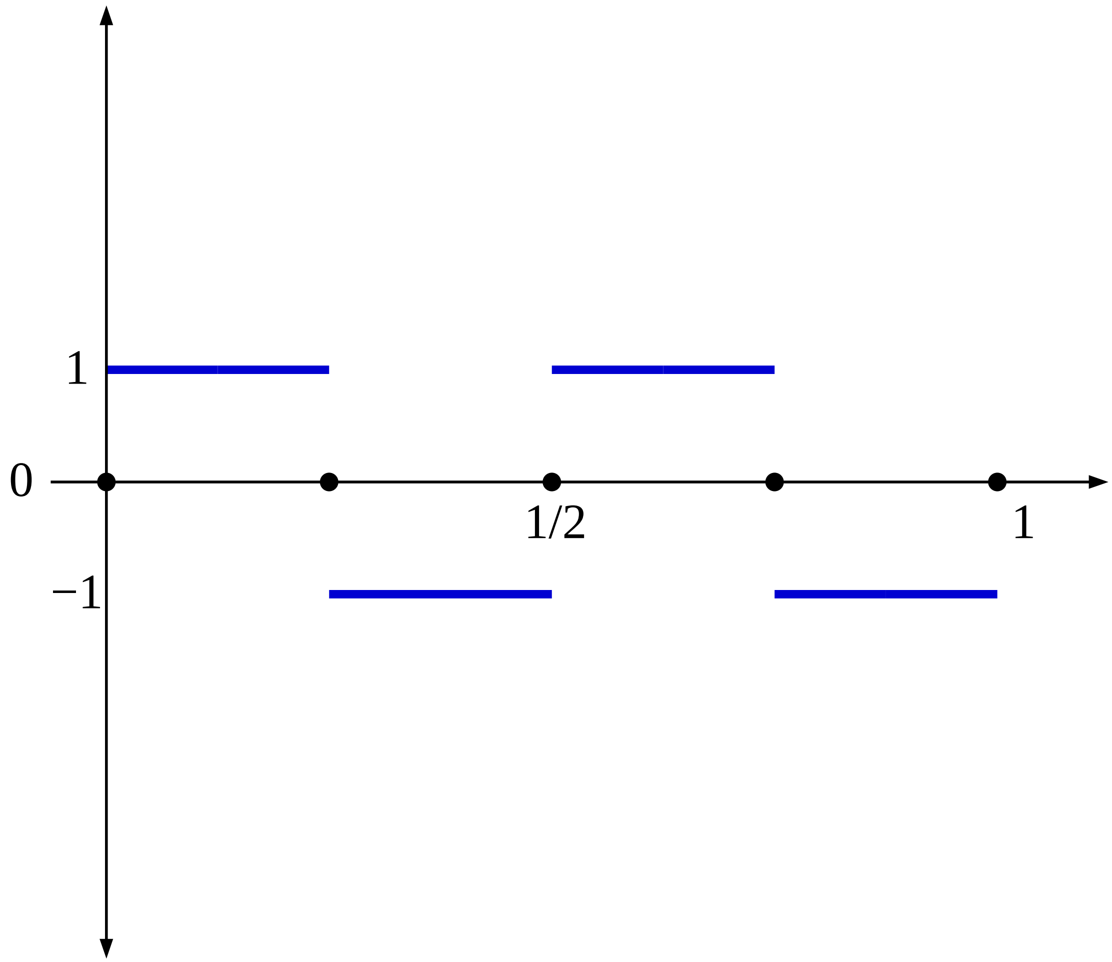
$$\text{diameter} \approx \sqrt{n}$$



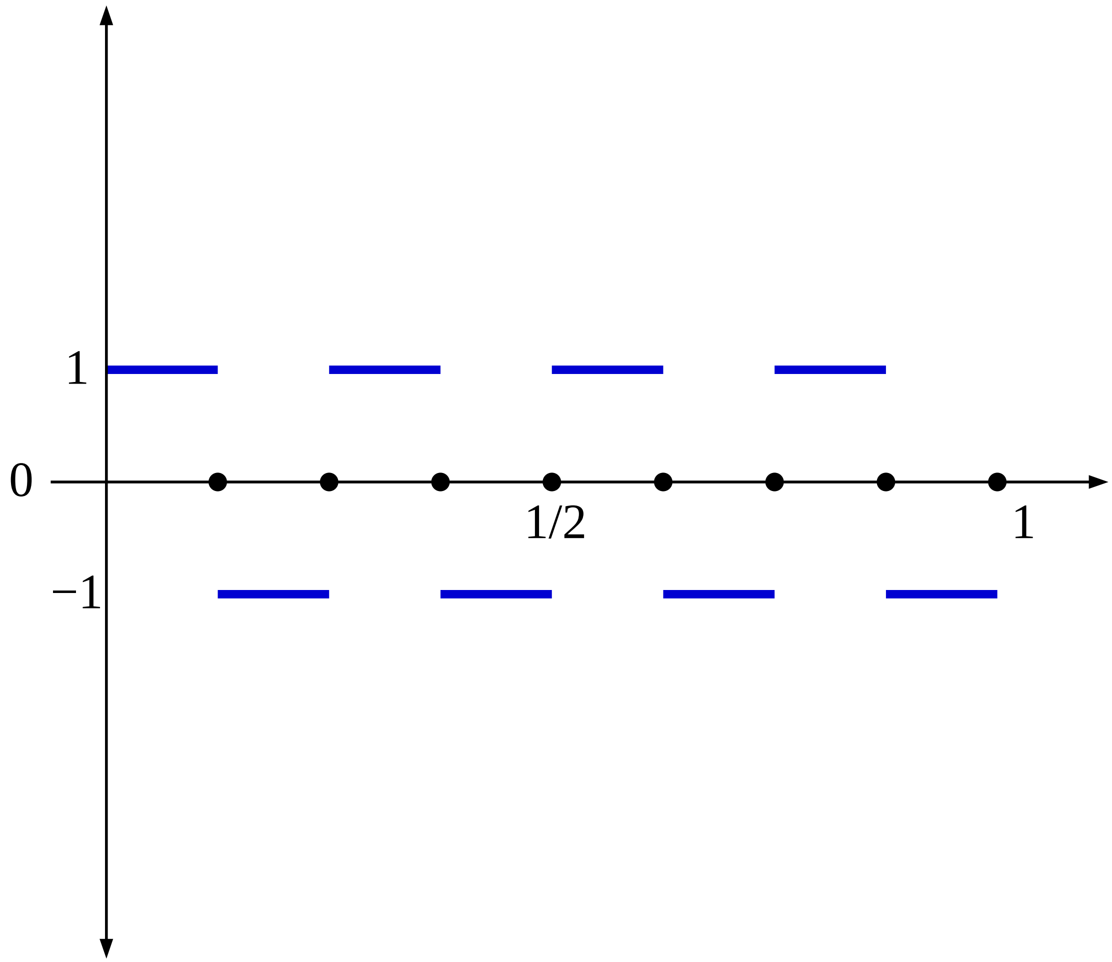
Look at all possible random paths at once using Haar functions.



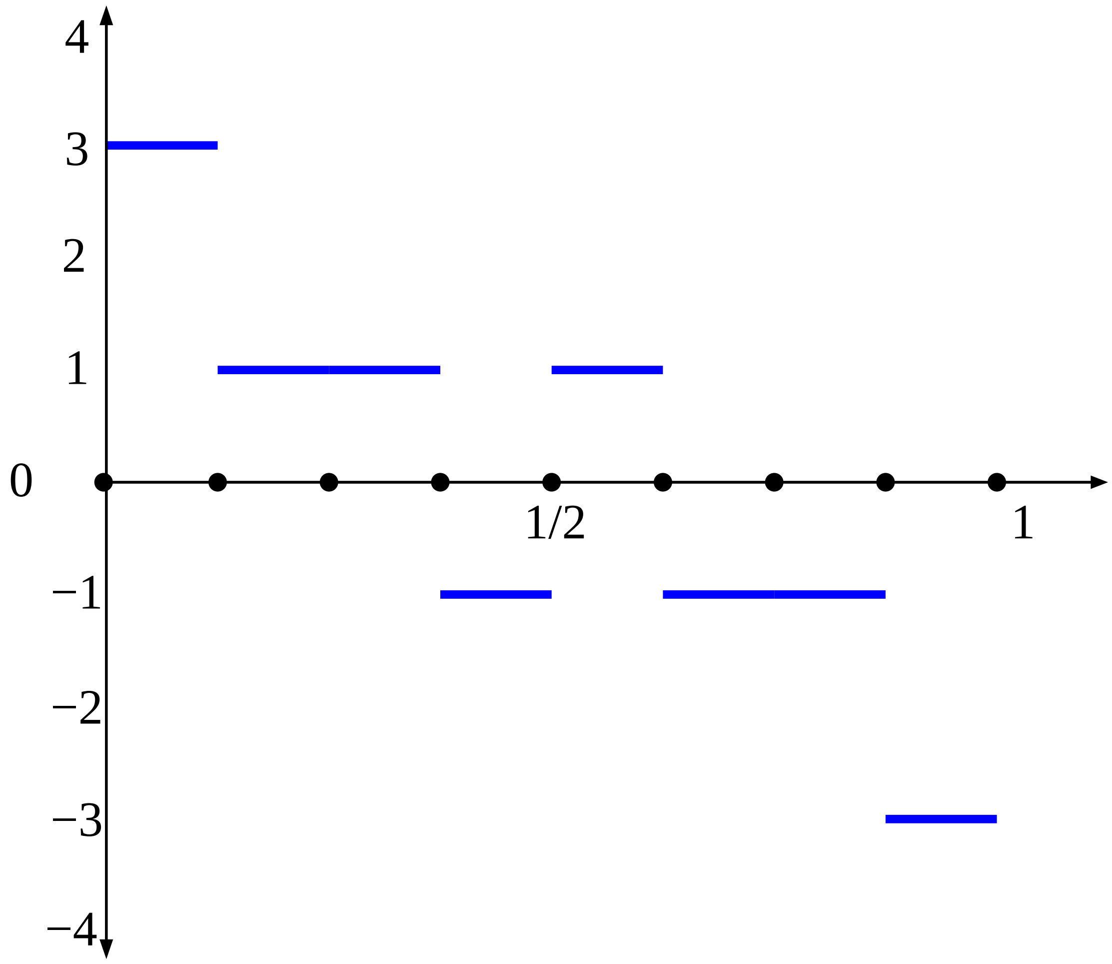
First Haar function, H_1.



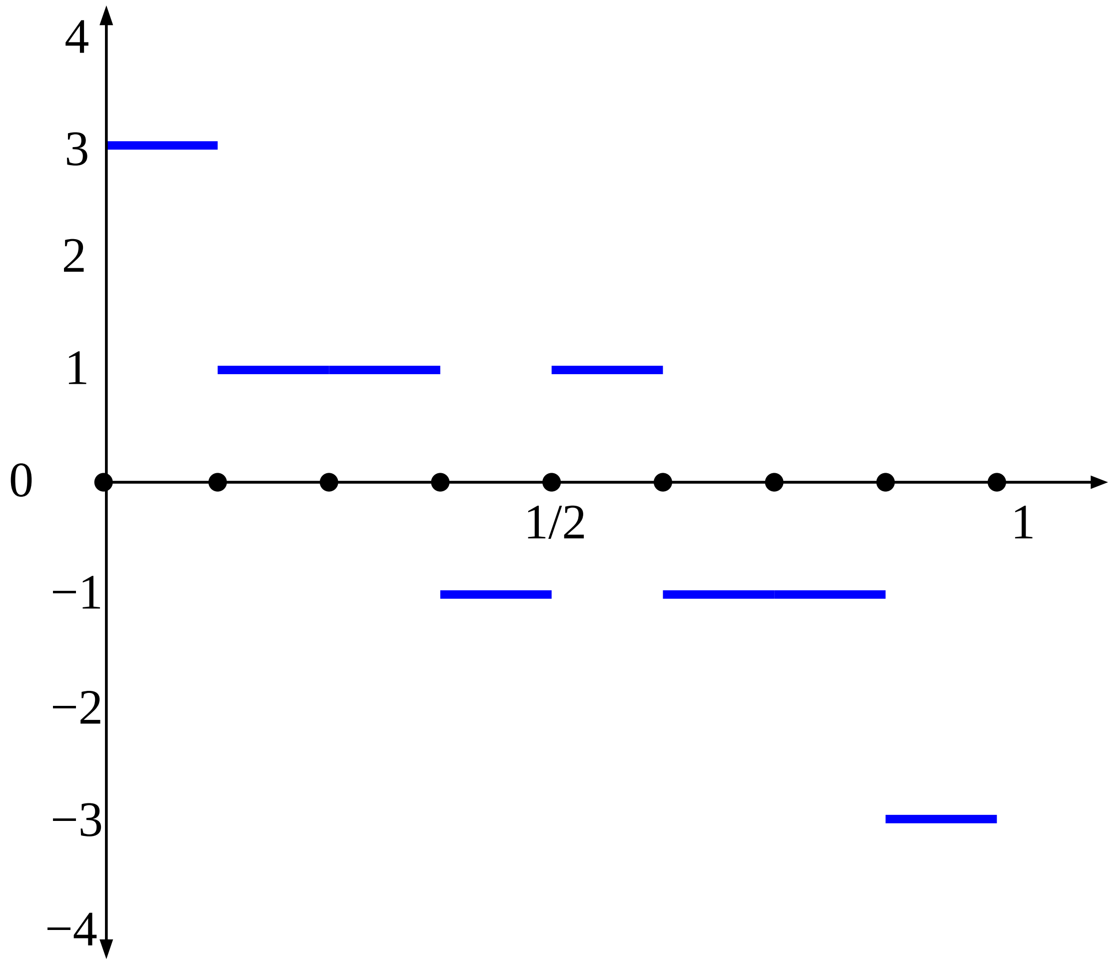
Second Haar function, H_2.



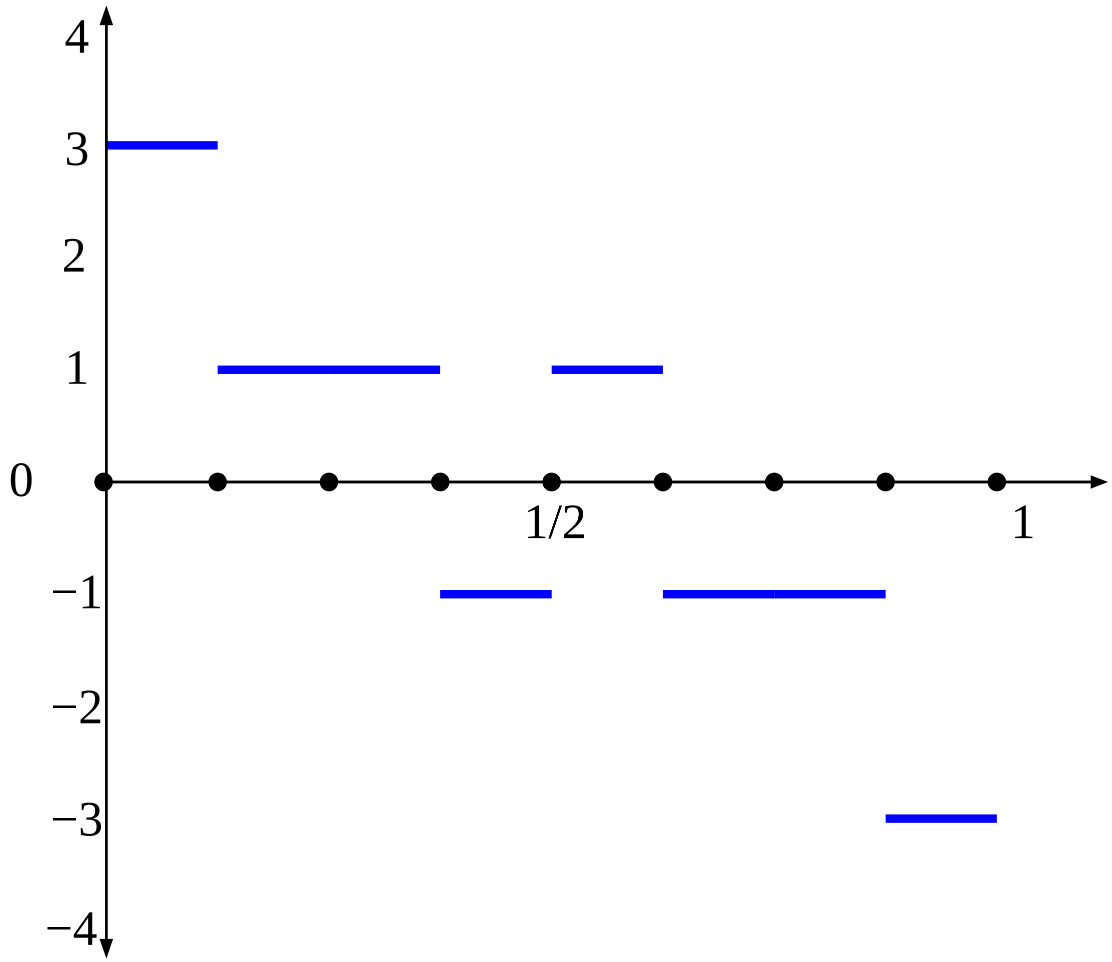
Thrid Haar function, H_3.



Sum of first three Haar functions, $S_3 = H_1 + H_2 + H_3$



Evaluating S_n at random point in [0,1] is the same as running a random walk for n steps



Average position of walk after n steps is $\int_0^1 |\sum_{k=1}^n H_k| dx$.

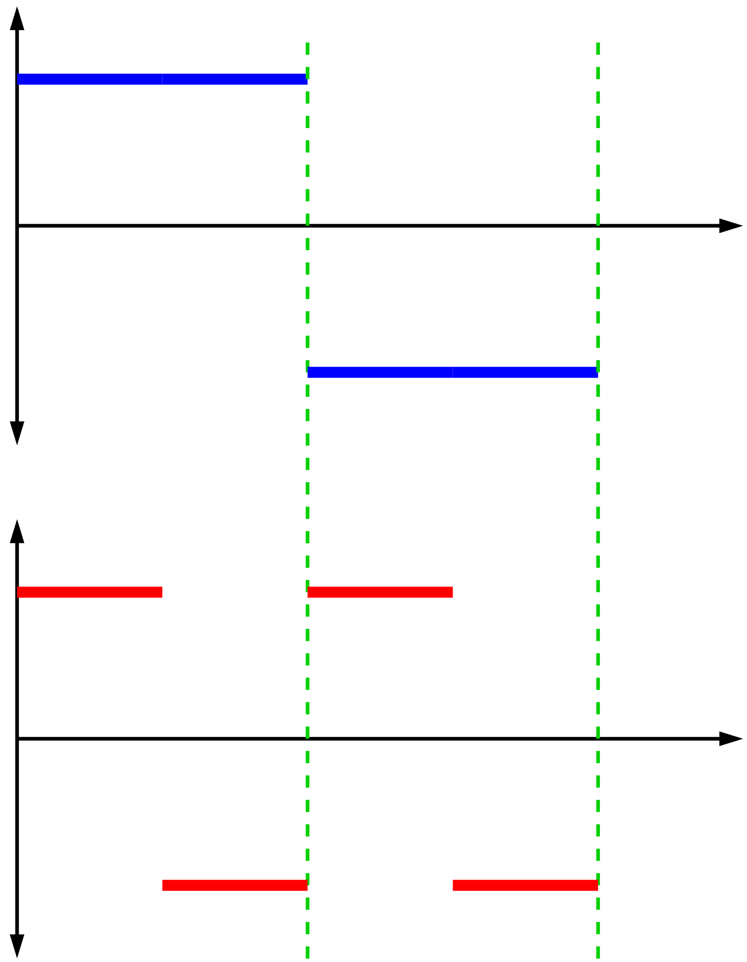
$$\int_0^1 |\sum_{k=1}^{n} H_n(x)|dx \le \left[\int_0^1 (\sum_{k=1}^{n} H_k(x))^2 dx\right]^{1/2}$$

Cauchy-Schwarz inequality

$$
\begin{aligned}
\int_0^1 |\sum_{k=1}^{n} H_n(x)|dx &\leq \left[\int_0^1 (\sum_{k=1}^{n} H_k(x))^2 dx\right]^{1/2} \\
&\leq \left[\int_0^1 \sum_{k=1}^{n} H_k(x) \cdot \sum_{j=1}^{n} H_j(x) dx\right]^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
\int_0^1 |\sum_{k=1}^{n} H_n(x)|dx &\leq \left[\int_0^1 (\sum_{k=1}^{n} H_k(x))^2 dx\right]^{1/2} \\
&\leq \left[\int_0^1 \sum_{k=1}^{n} H_k(x) \cdot \sum_{j=1}^{n} H_j(x) dx\right]^{1/2} \\
&\leq \left[\sum_{k=1}^{n} \sum_{j=1}^{n} \int_0^1 H_k(x) H_j(x) dx\right]^{1/2}
\end{aligned}
$$

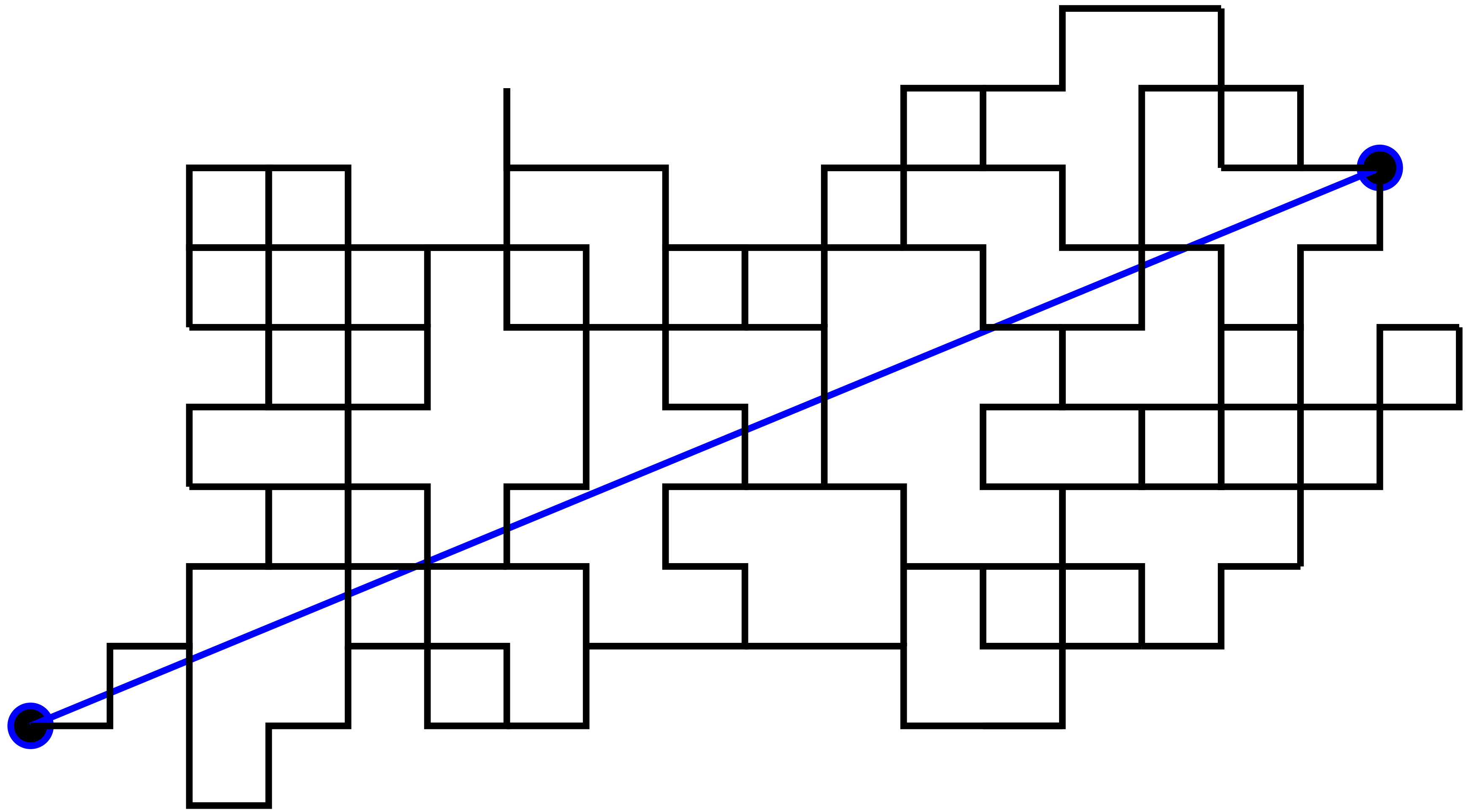
$$
\begin{aligned}
\int_0^1 |\sum_{k=1}^{n} H_n(x)|dx &\leq \left[\int_0^1 (\sum_{k=1}^{n} H_k(x))^2 dx\right]^{1/2} \\
&\leq \left[\int_0^1 \sum_{k=1}^{n} H_k(x) \cdot \sum_{j=1}^{n} H_j(x) dx\right]^{1/2} \\
&\leq \left[\sum_{k=1}^{n} \sum_{j=1}^{n} \int_0^1 H_k(x) H_j(x) dx\right]^{1/2} \\
&\leq \left[\sum_{k=1}^{n} \int_0^1 H_k^2(x) dx\right]^{1/2}
\end{aligned}
$$



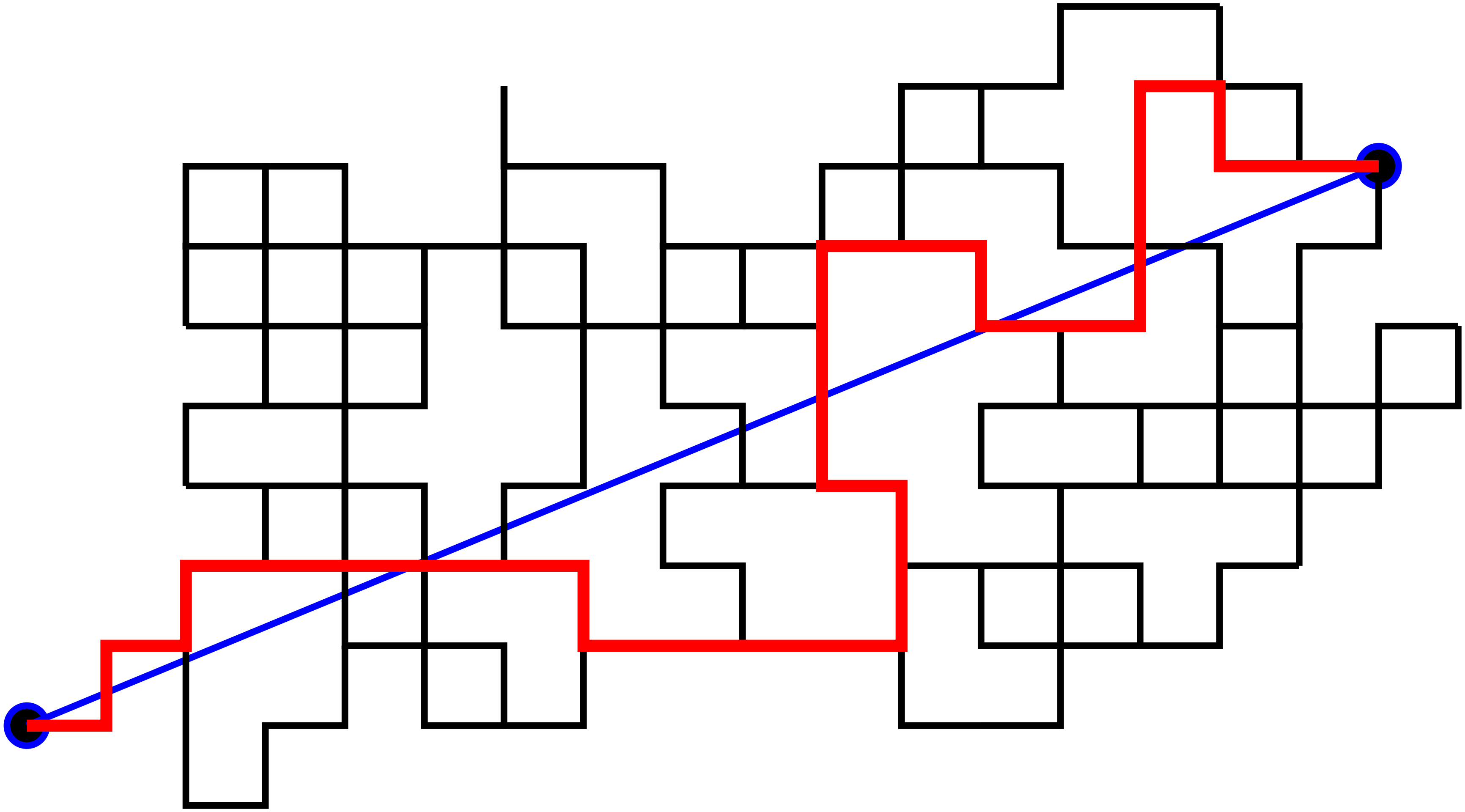
$$\int_0^1 H_j(x)H_k(x)dx = 0, \text{ if } j \neq k, \qquad \int_0^1 H_k^2(x)dx = 1$$

$$
\begin{aligned}
\int_0^1 \left|\sum_{k=1}^n H_n(x)\right| dx &\leq \left[\int_0^1 \Big(\sum_{k=1}^n H_k(x)\Big)^2 dx\right]^{1/2} \\
&\leq \left[\int_0^1 \sum_{k=1}^n H_k(x) \cdot \sum_{j=1}^n H_j(x) dx\right]^{1/2} \\
&\leq \left[\sum_{k=1}^n \int_0^1 H_k^2(x) dx\right]^{1/2} \\
&\leq \left[\sum_{k=1}^n 1\right]^{1/2} \\
&= \sqrt{n}
\end{aligned}
$$

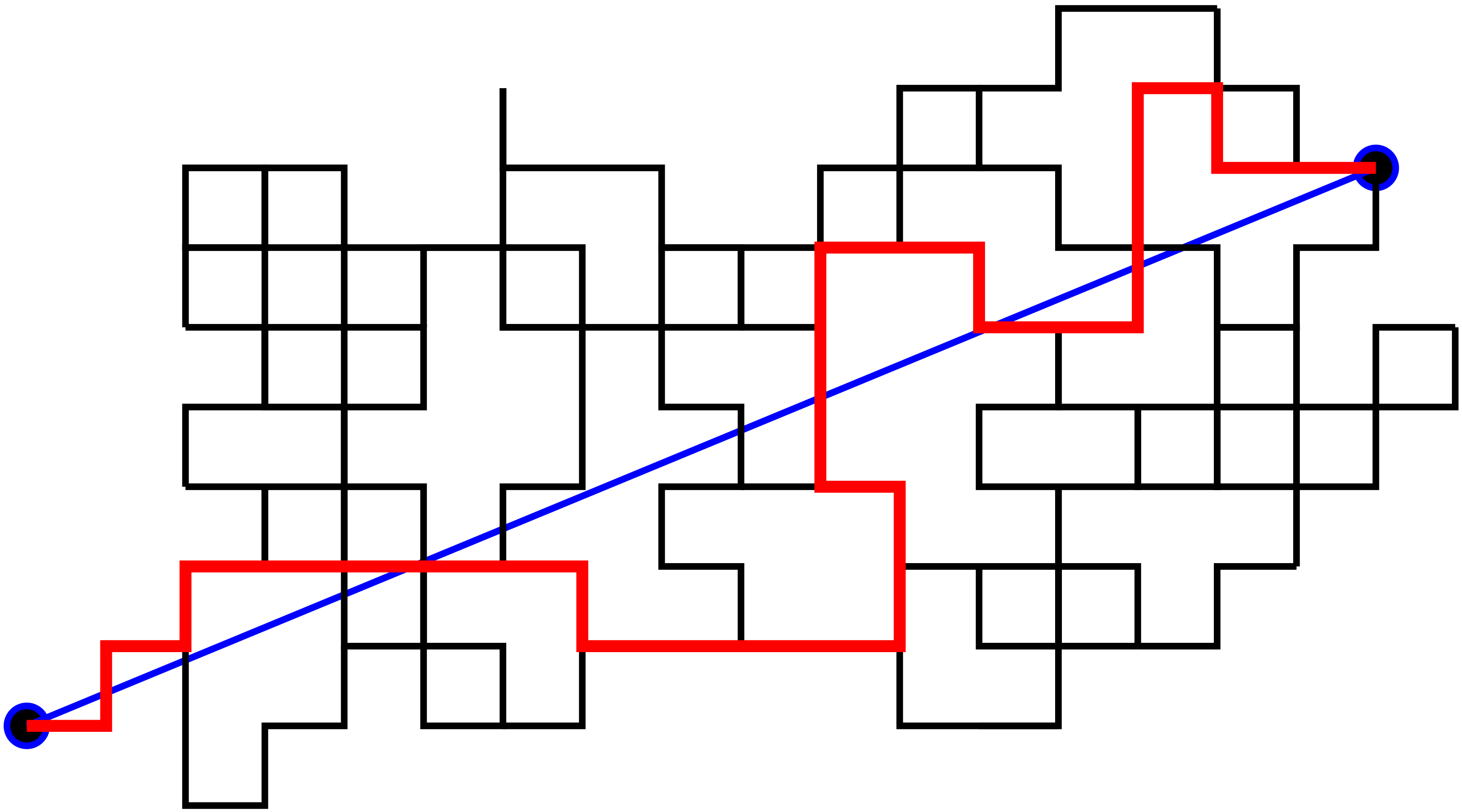
Diameter of n-step random walk is $\simeq \sqrt{n}$



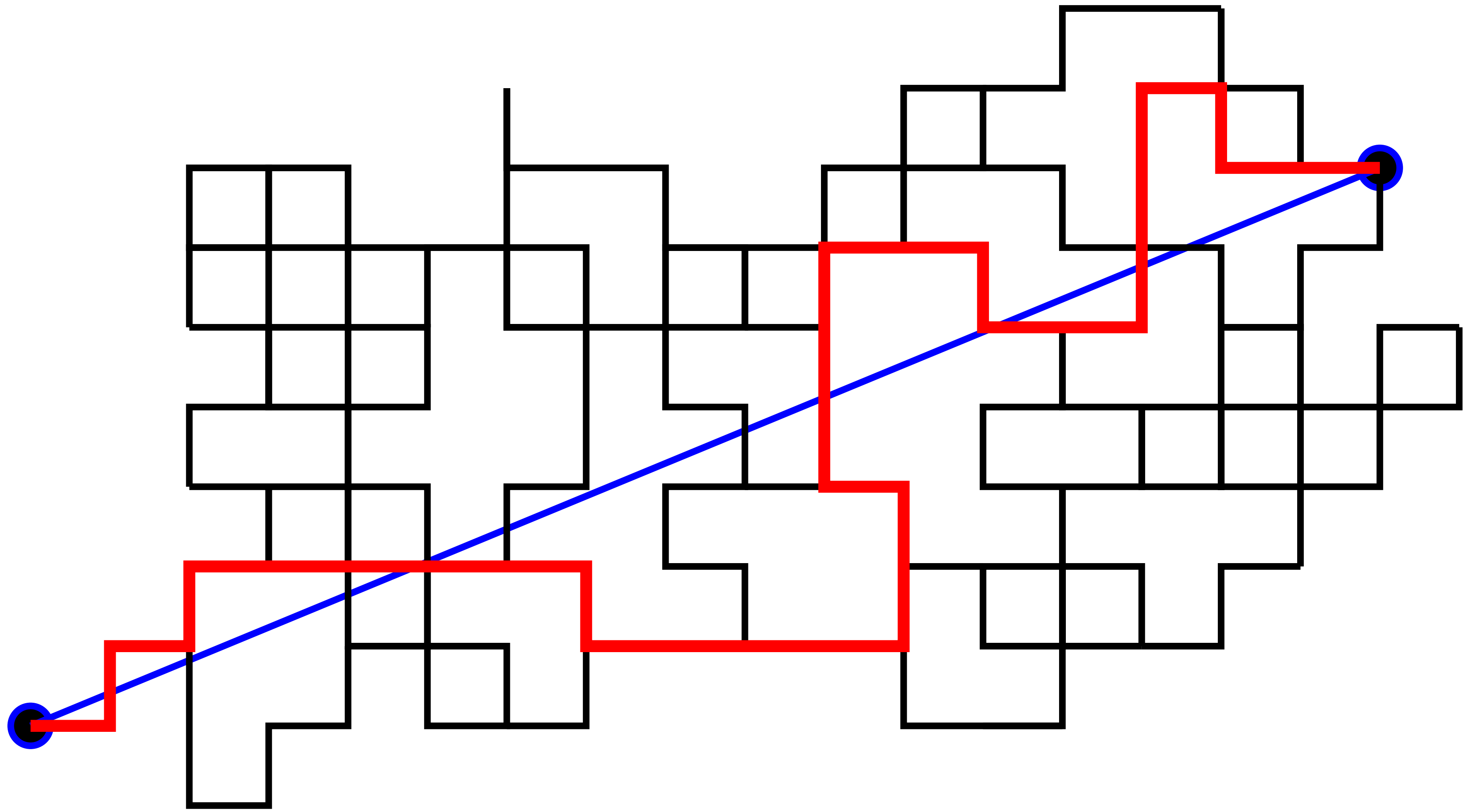
Euclidean diameter of n-step walk is $\simeq \sqrt{n}$



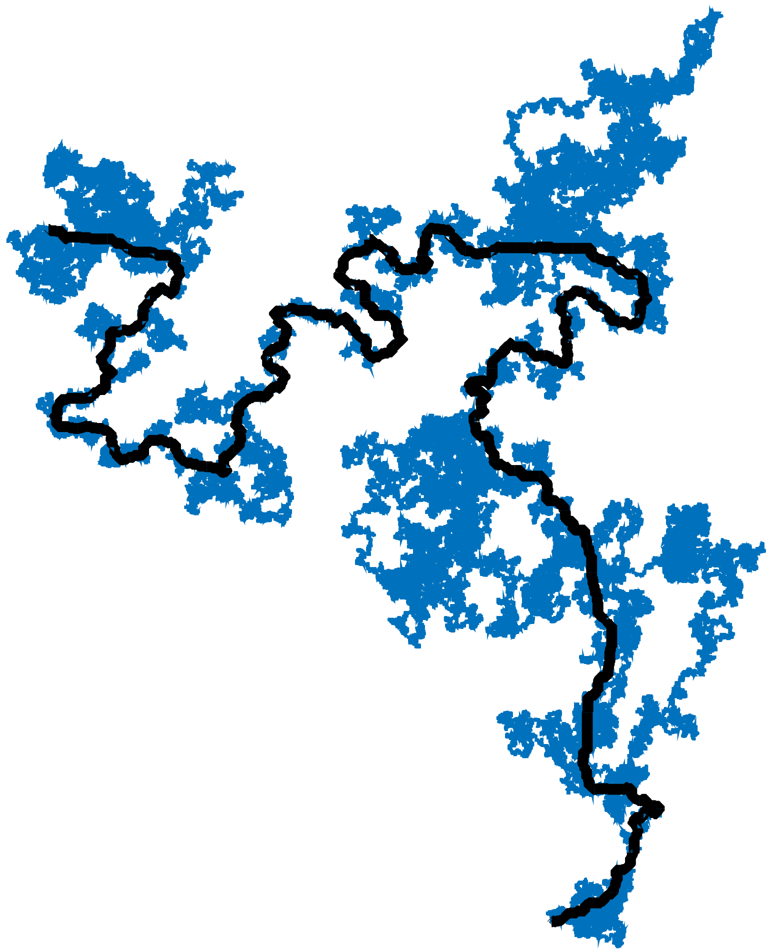
What is path diameter of n-step walk?



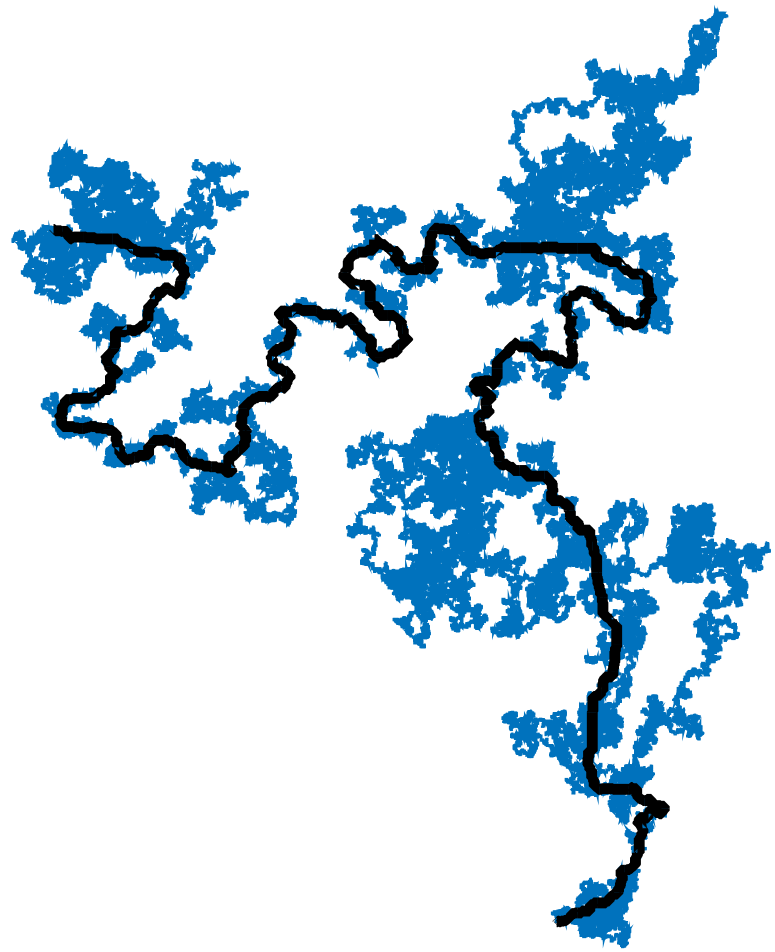
$$\frac{\text{path diameter}}{\text{Euclidean diameter}} \to \infty \quad ?$$



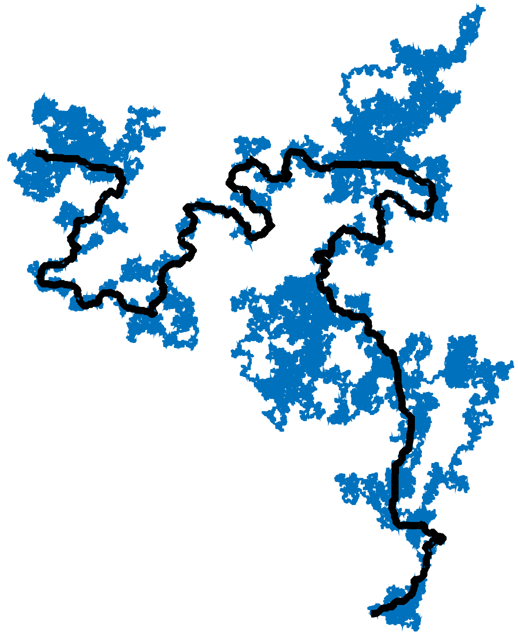
$$\text{path diameter} \leq C \cdot (\text{Euclidean diameter})^{1+\epsilon}, \qquad \text{any } \epsilon > 0 \qquad ?$$



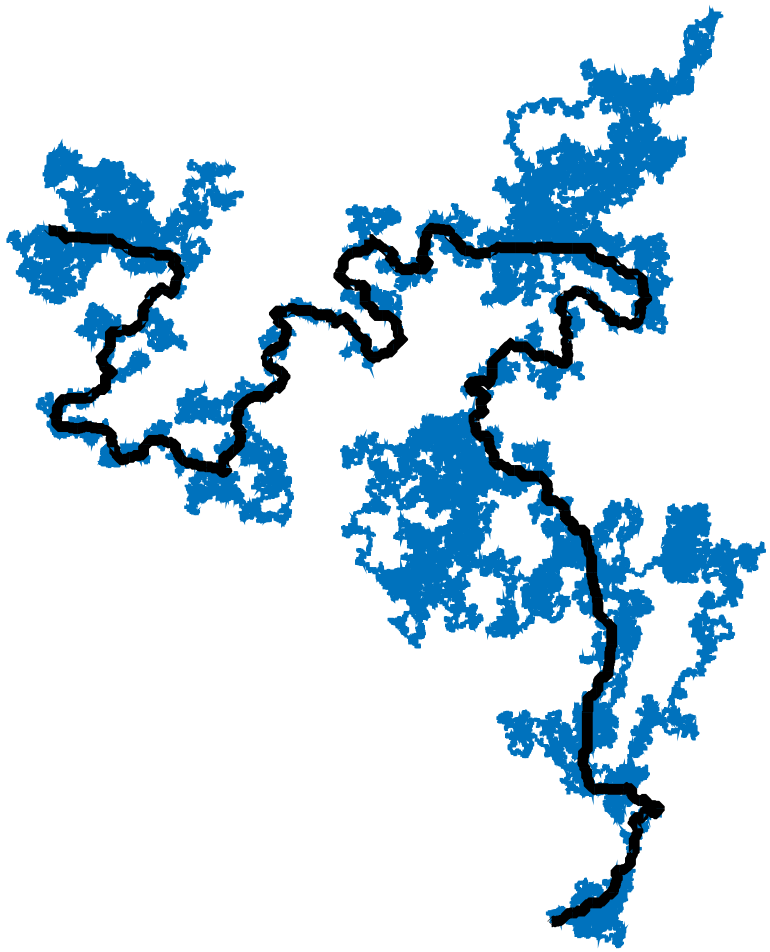
A random walk (blue) and shortest path between two points (black)



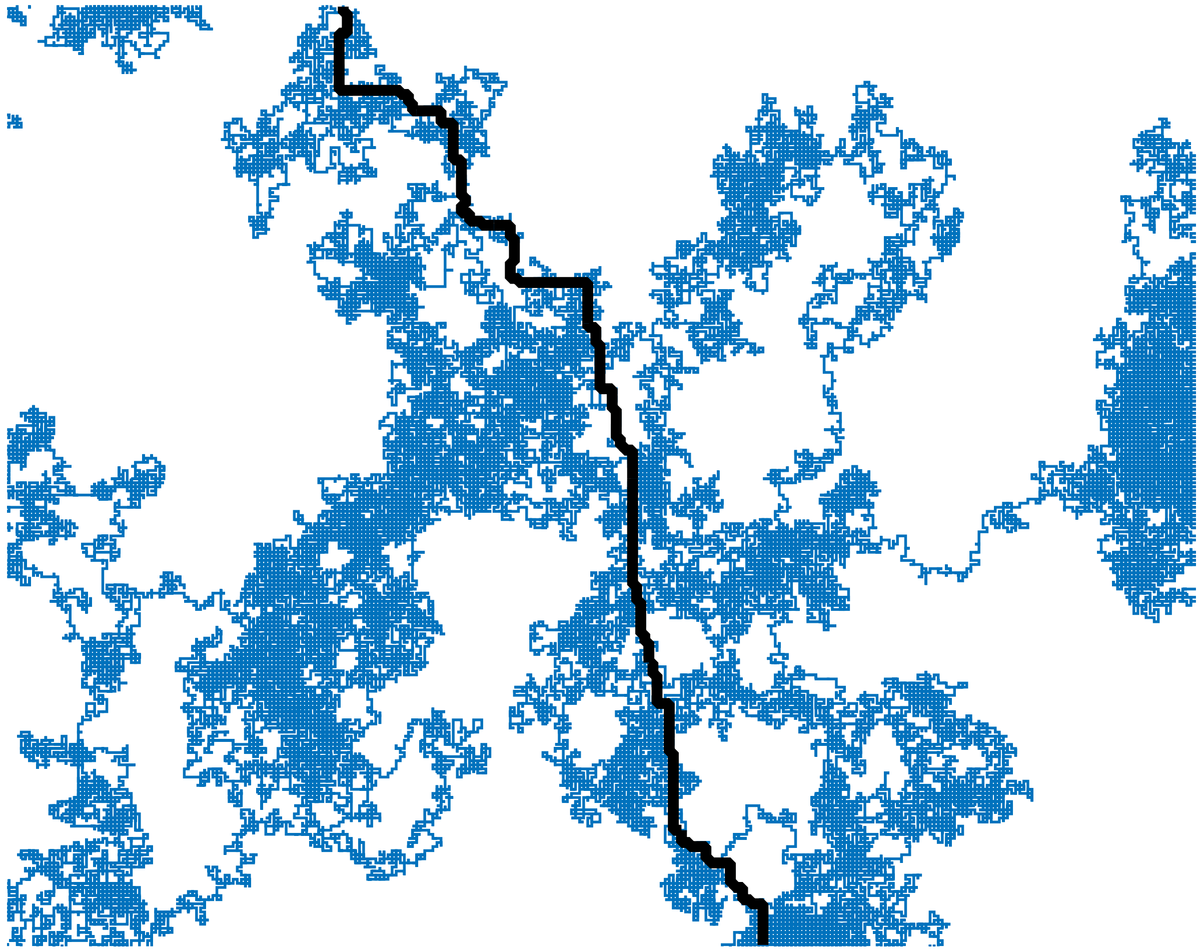
In the limit, does Brownian motion contain a finite length arc?

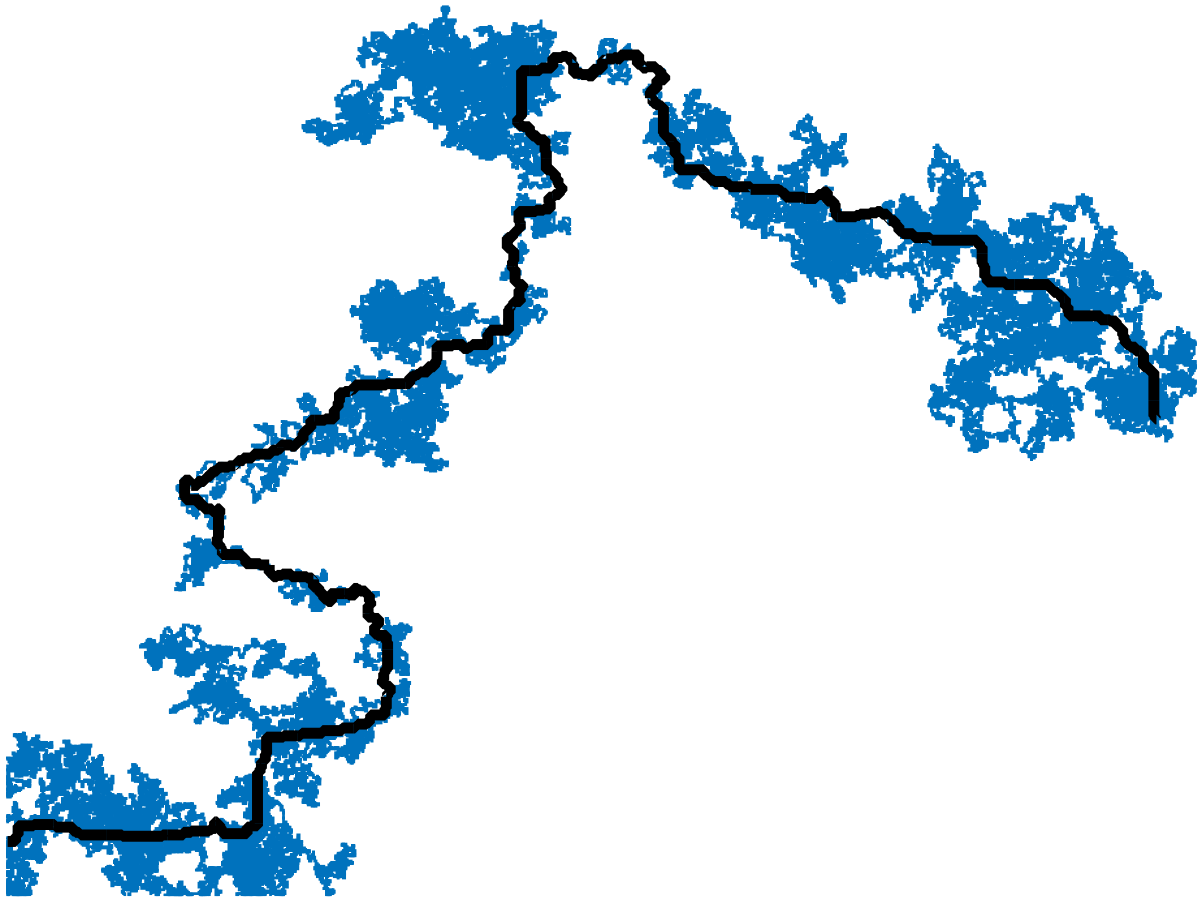
Brownian motion contains no line segments. Robin Pemantle, 1997.



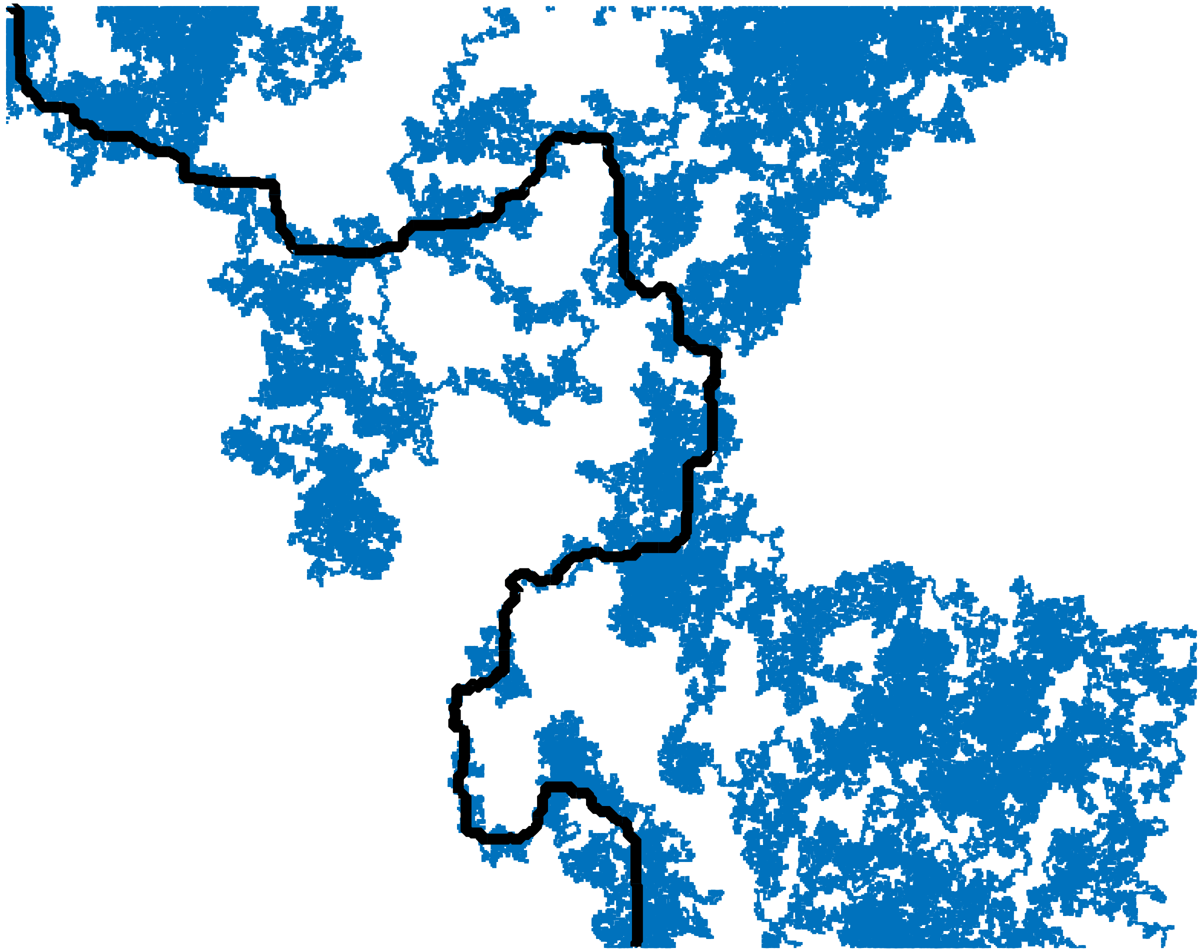
Random walk and a shortest path between points



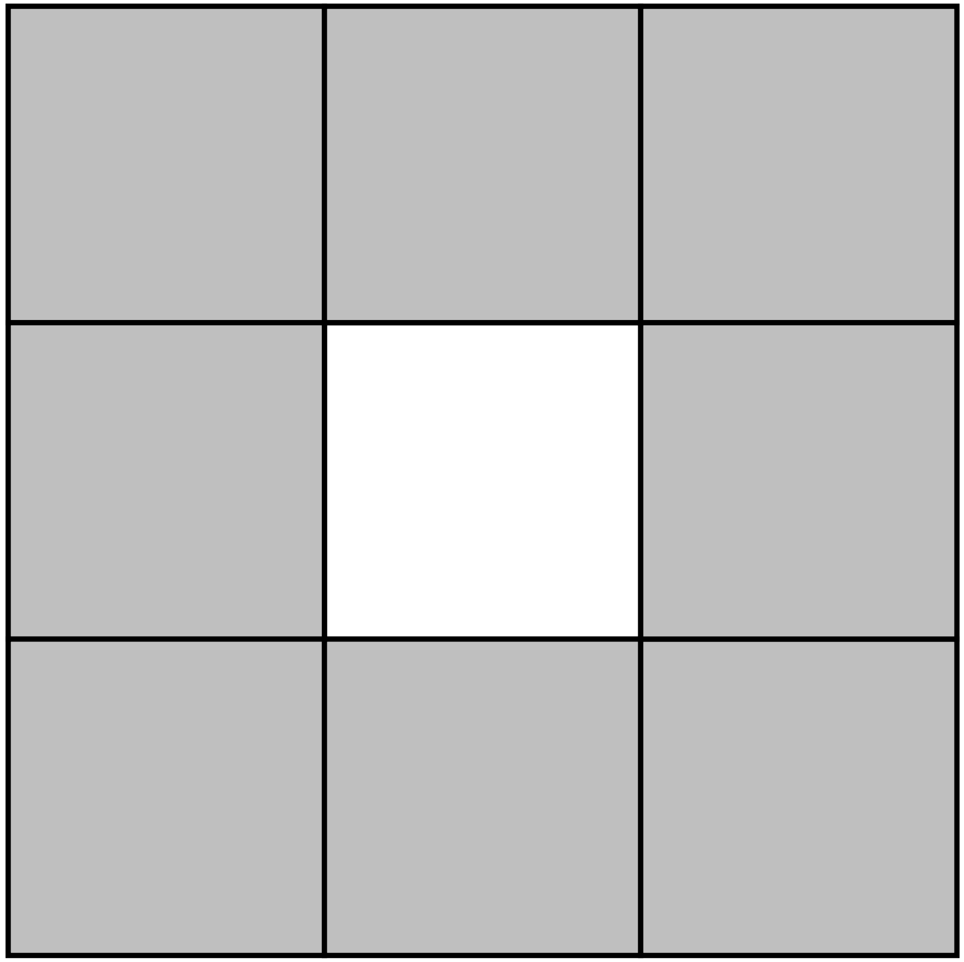
Blowup of shortest path: path is “short” where trace is “thick”



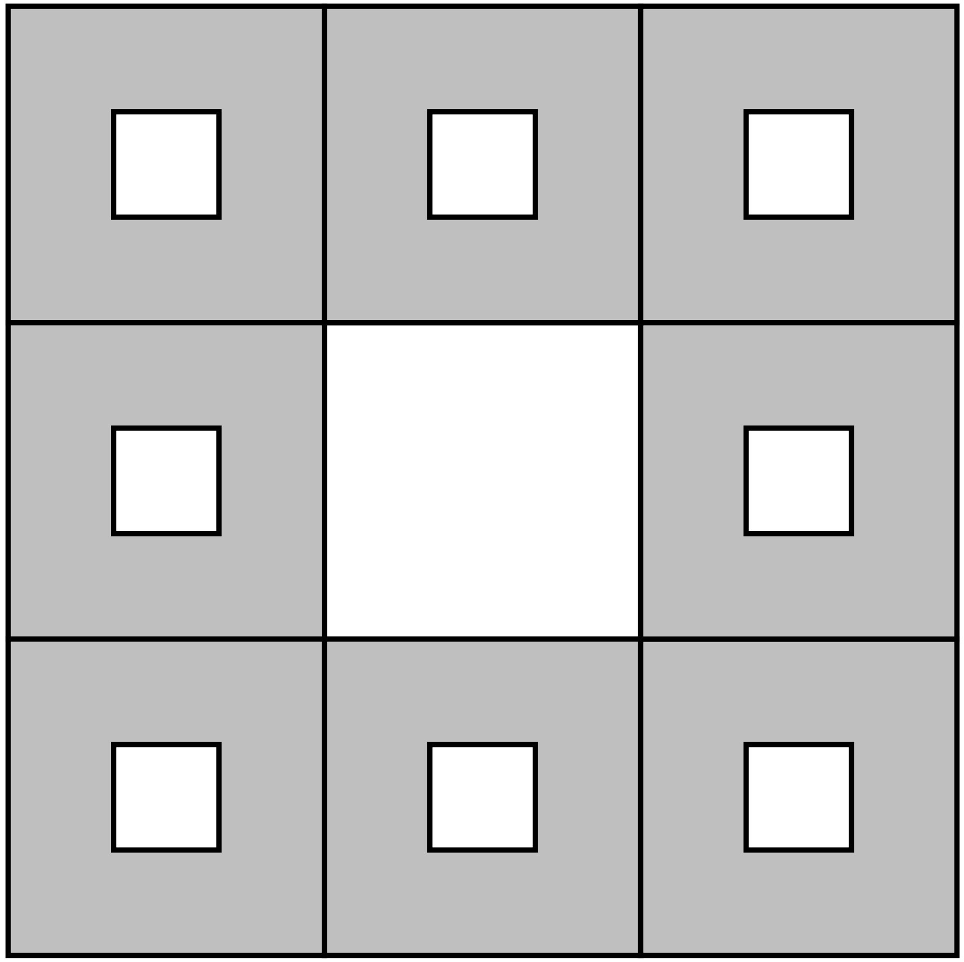
Blowup of shortest path: path is “long” where trace is thin



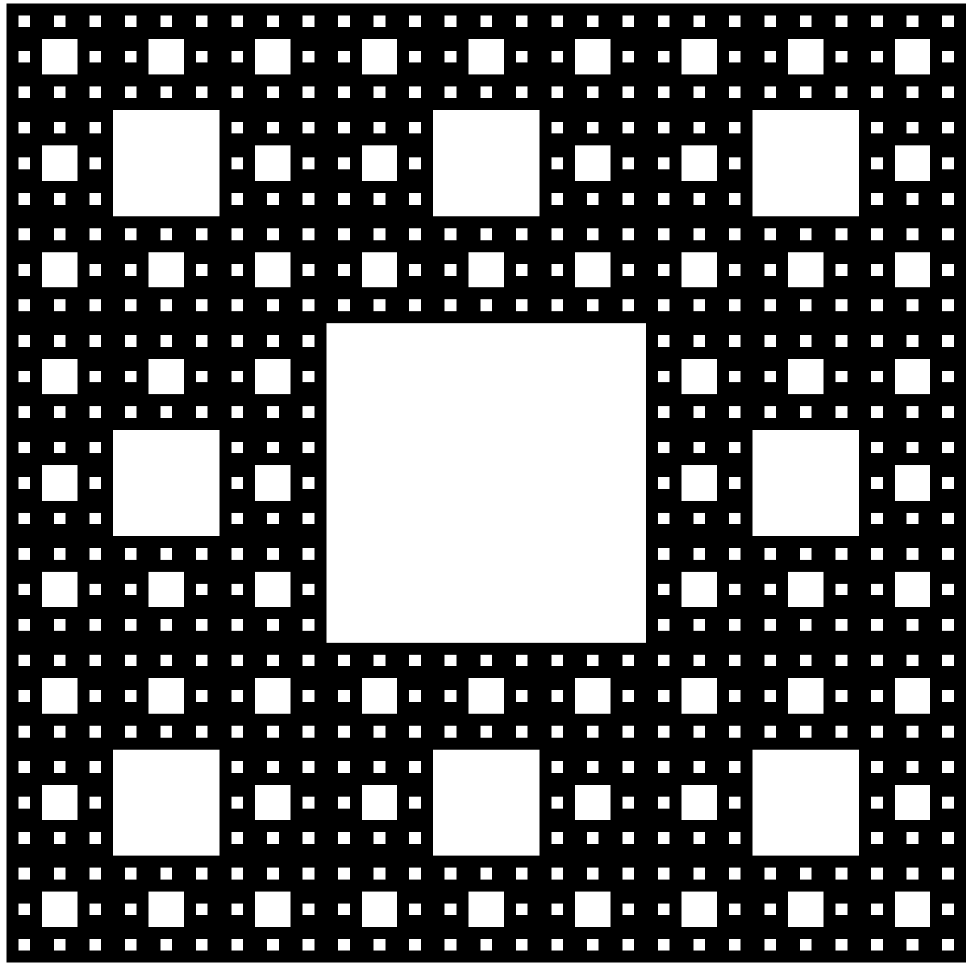
Do the complementary components of Brownian motion touch?



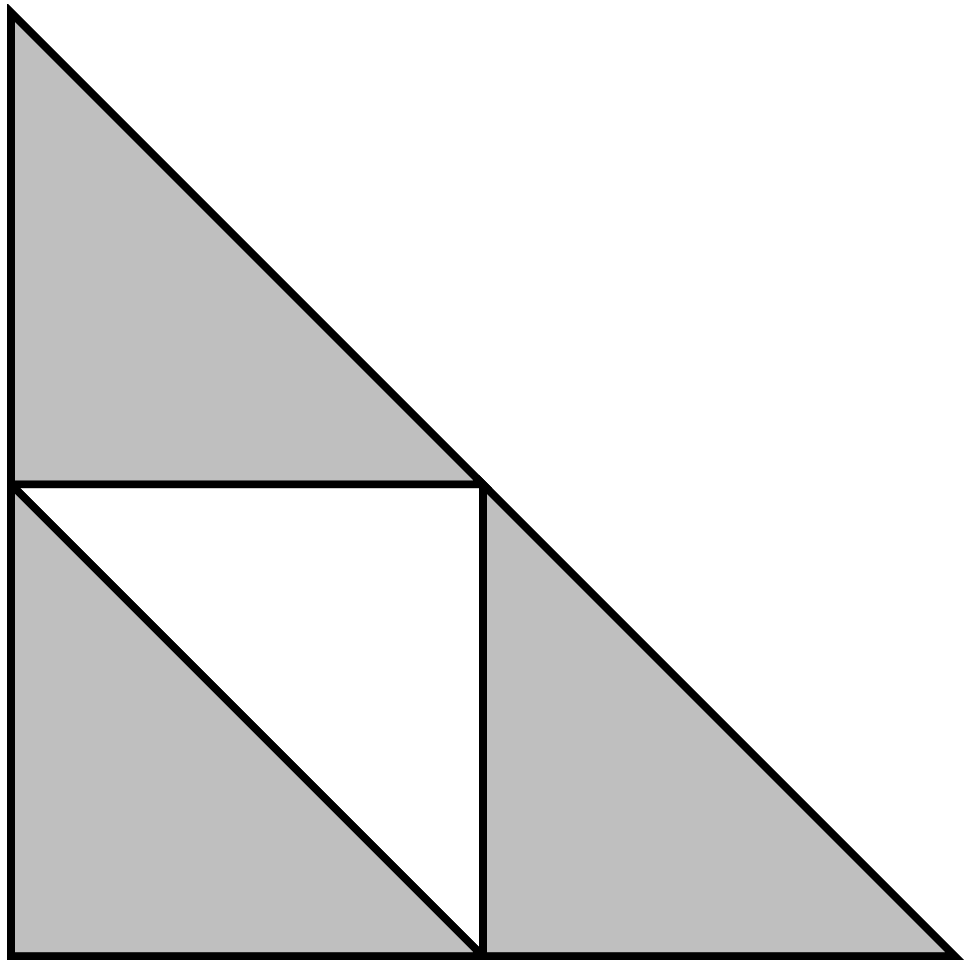
Sierpinski Carpet



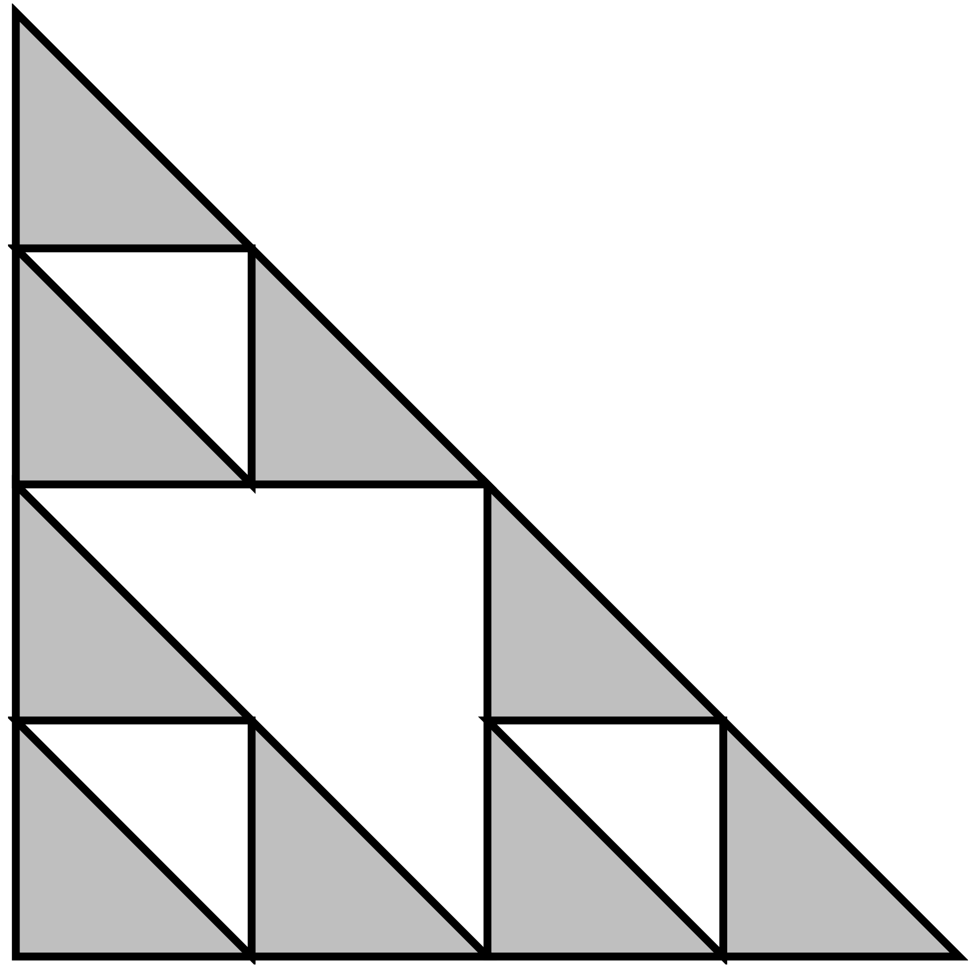
Sierpinski Carpet



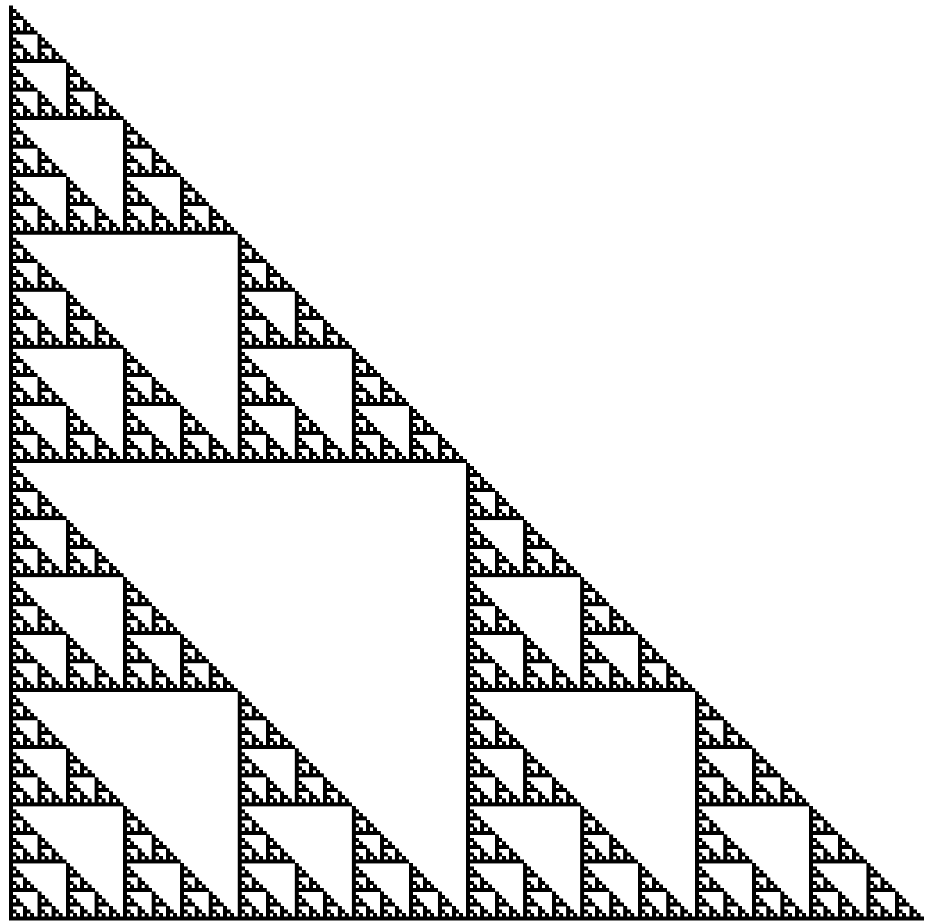
Sierpinski Carpet



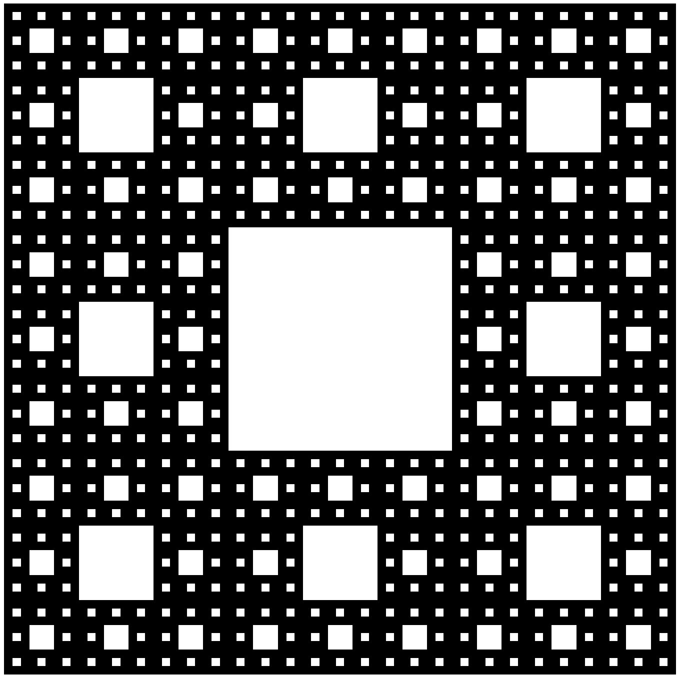
Sierpinski Gasket



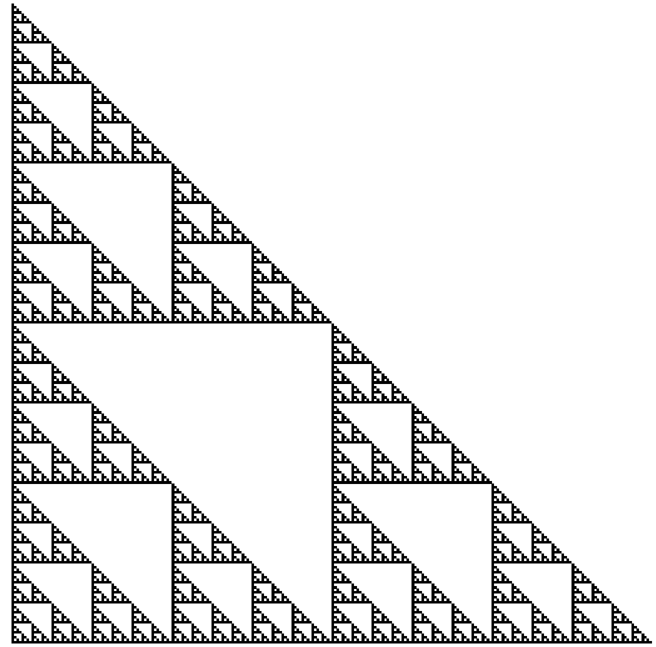
Sierpinski Gasket



Sierpinski Gasket



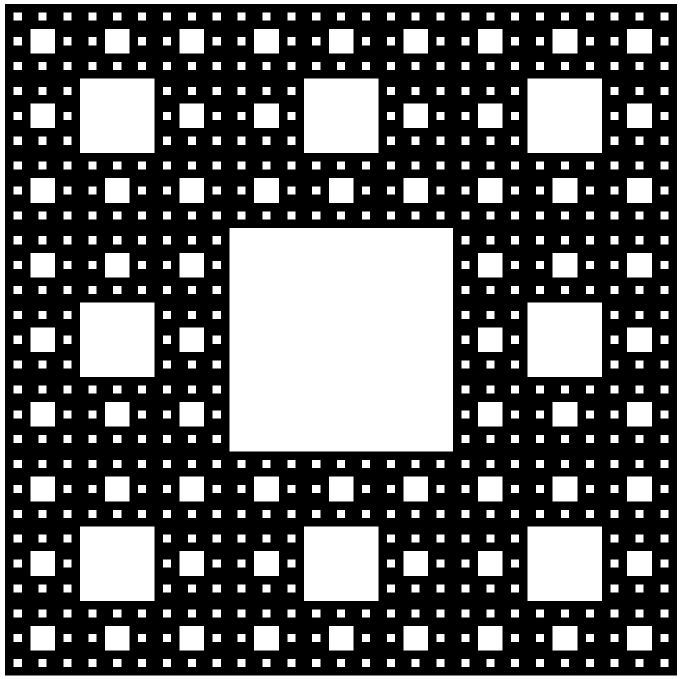
Sierpinski carpet



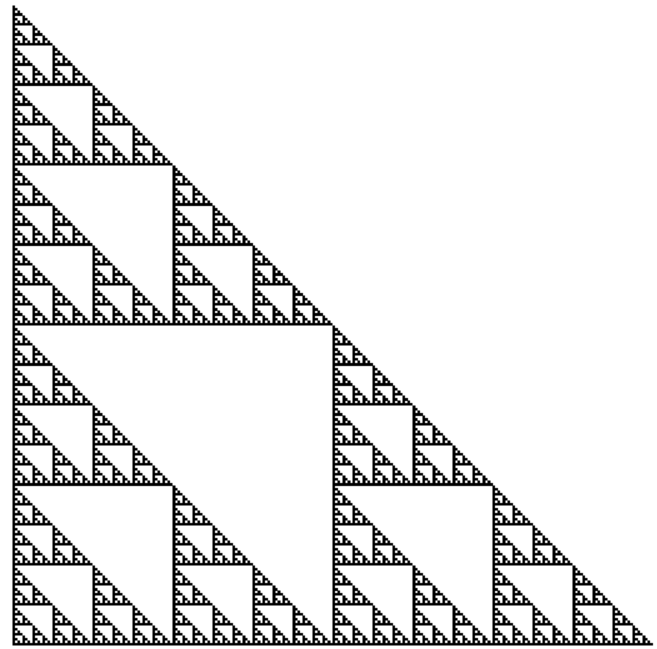
Sierpinski gasket

Holes of carpet don't touch each other.

Holes of gasket do touch.

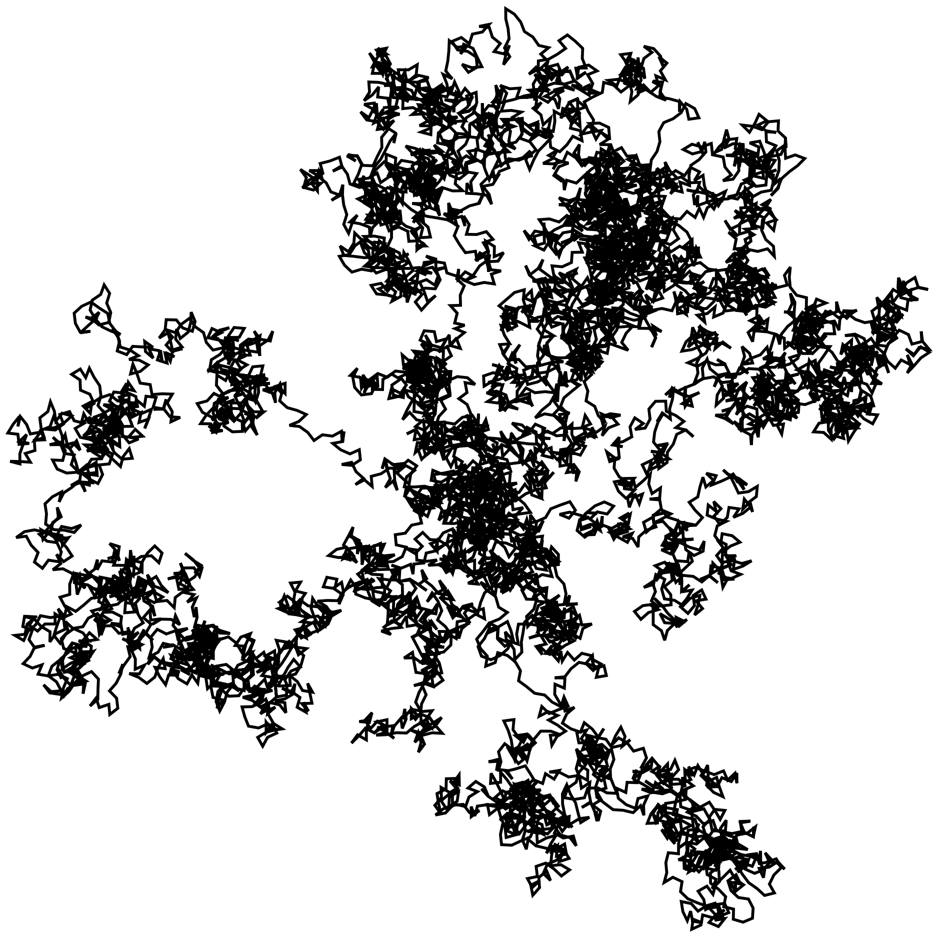


Sierpinski carpet

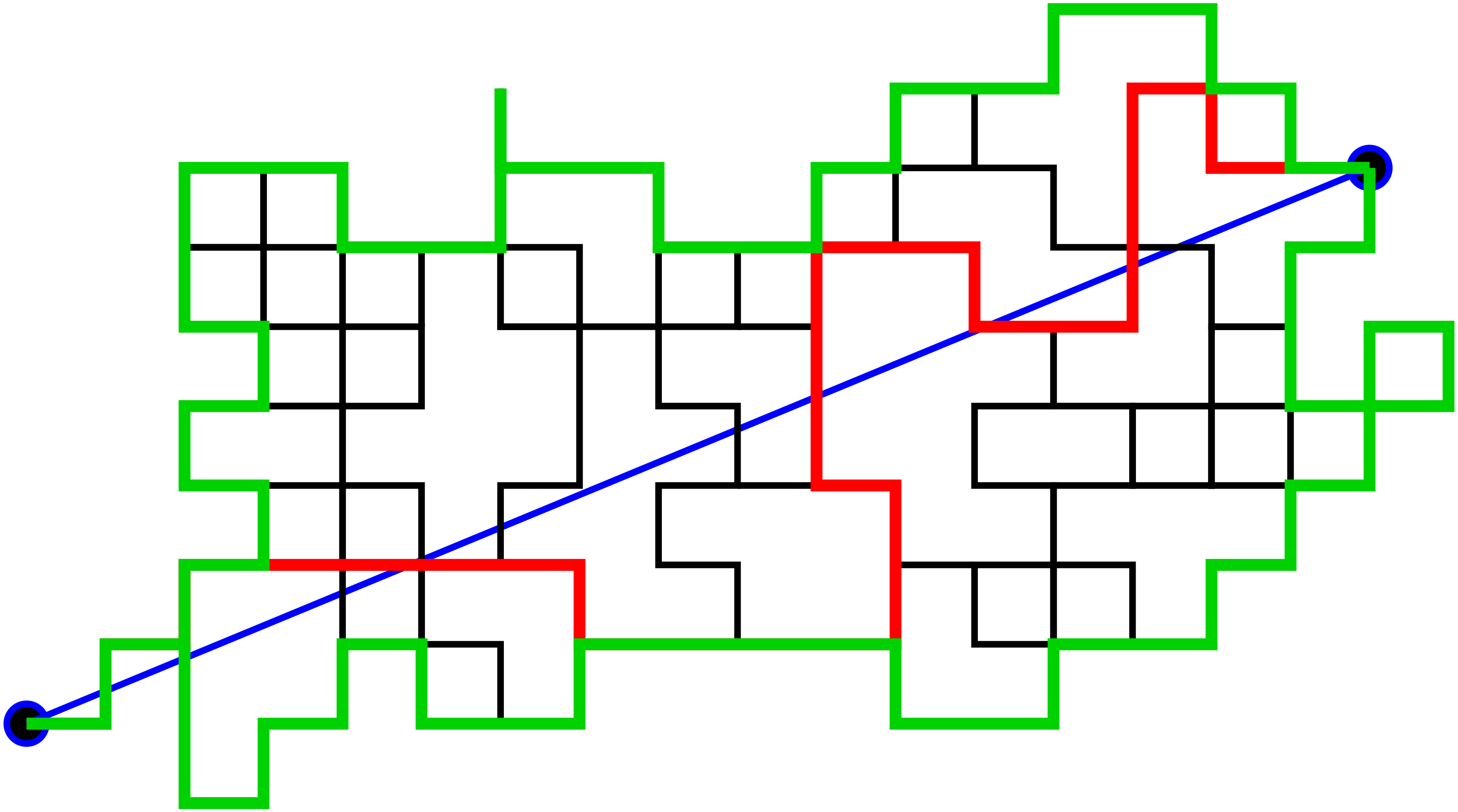


Sierpinski gasket

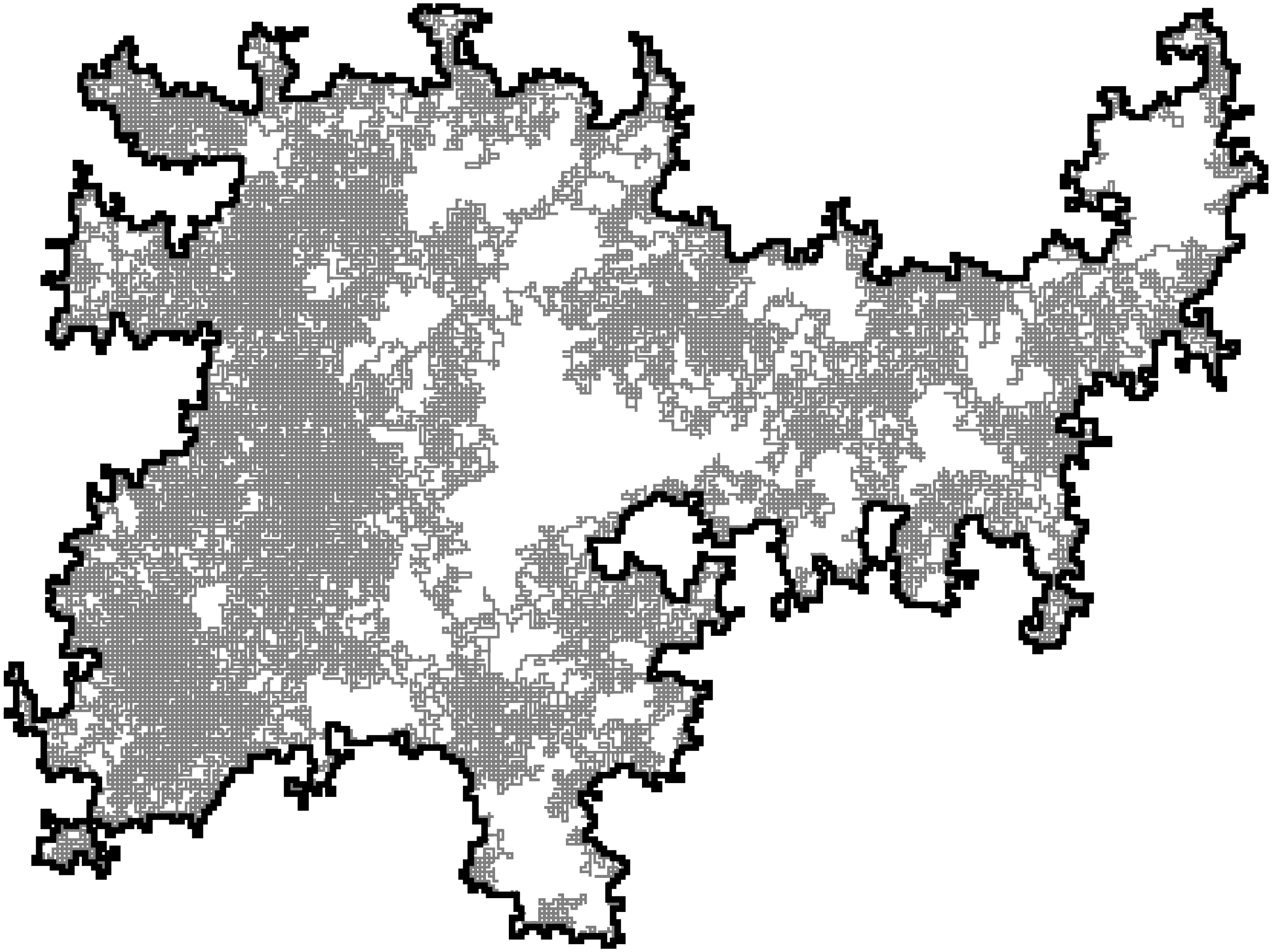
Any two points in complement of gasket can be connected by a path that hits the gasket only finite often.

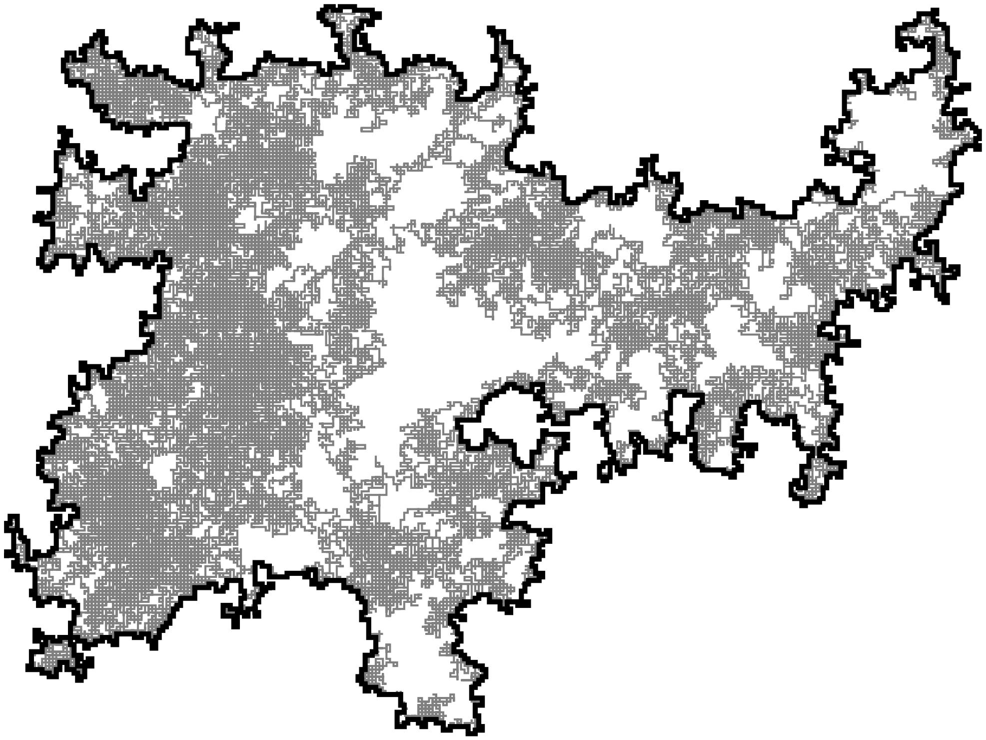


Conjecture: any two points in the complement of a Brownian path can be connected by a path that hits the Browian path only finitely often.



How long is the perimeter of n-step random walk?



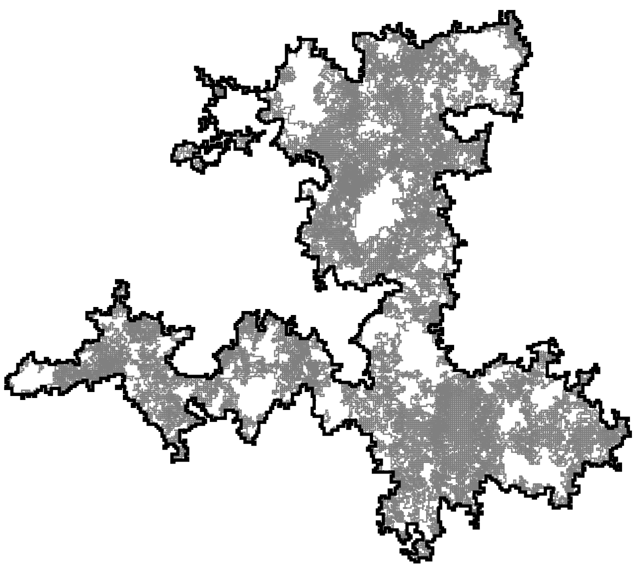


Benoit Mandelbrot conjectured #(perimeter) $\simeq$ diameter$^{4/3}$, 1982.

Outer boundary of Brownian motion is called its "frontier".

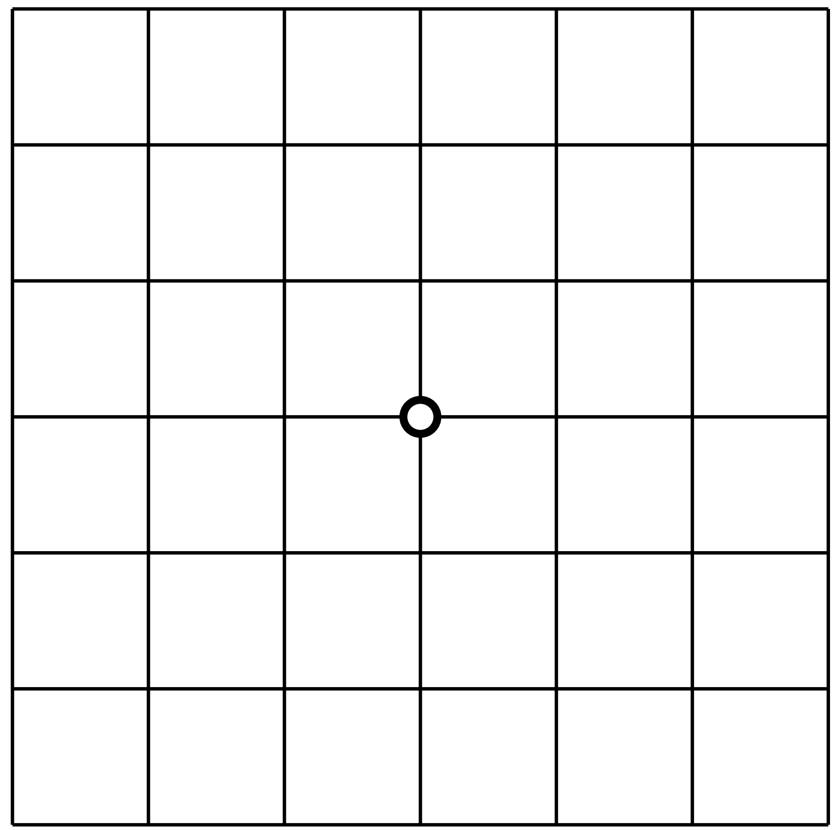
  

Proven by Greg Lawler, Oded Schramm, Wendelin Werner, 2002.

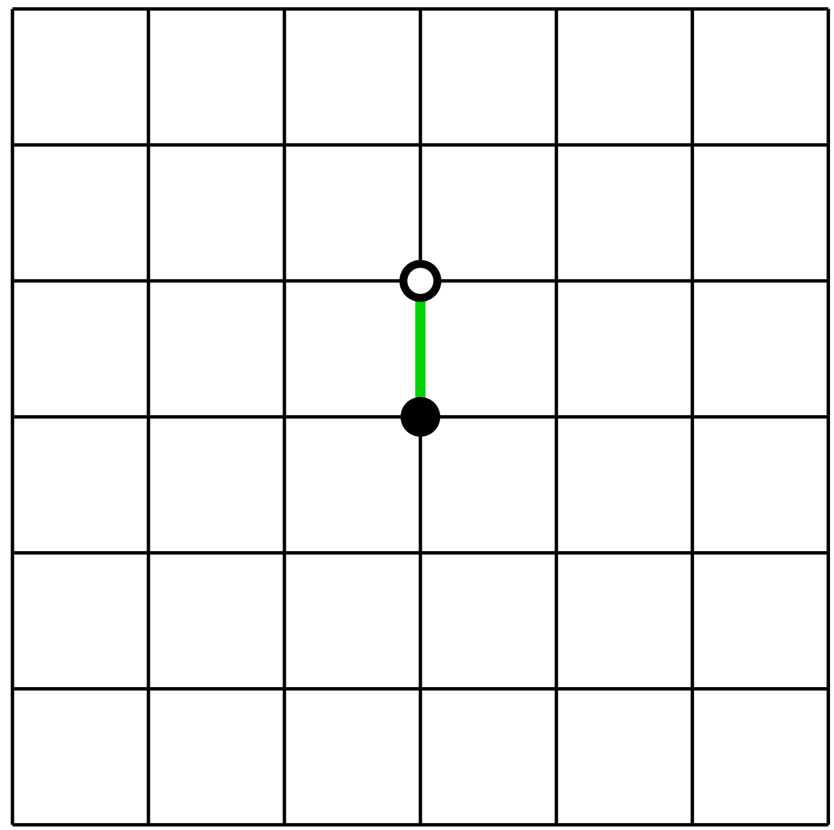


In limit, Brownian frontiers have “dimension” 4/3.

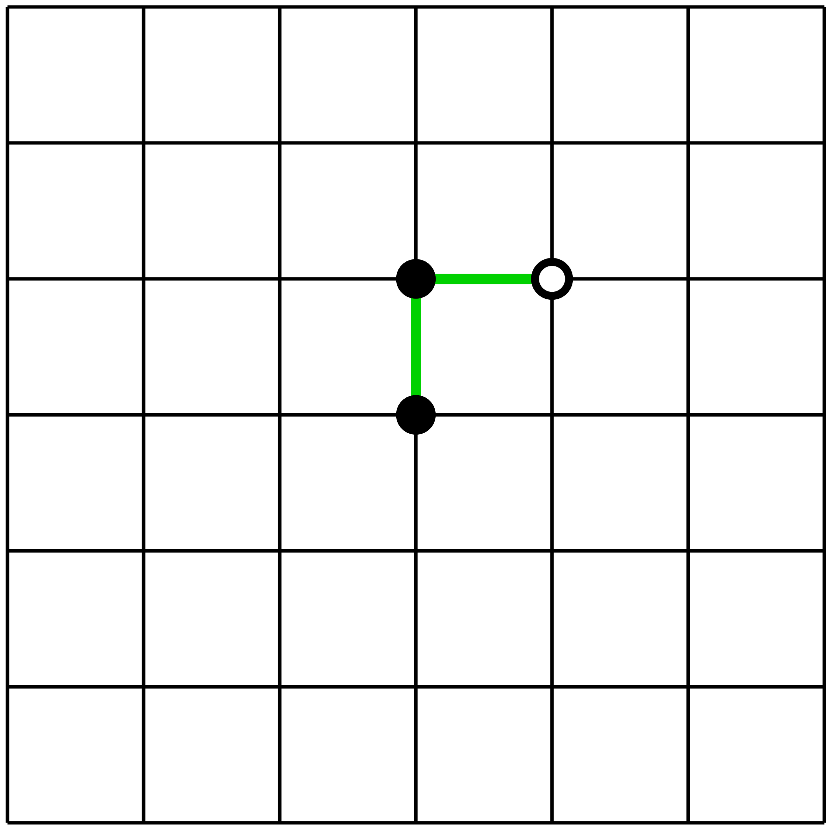
Frontiers are a type of random Jordan curve.
(no self-intersections)



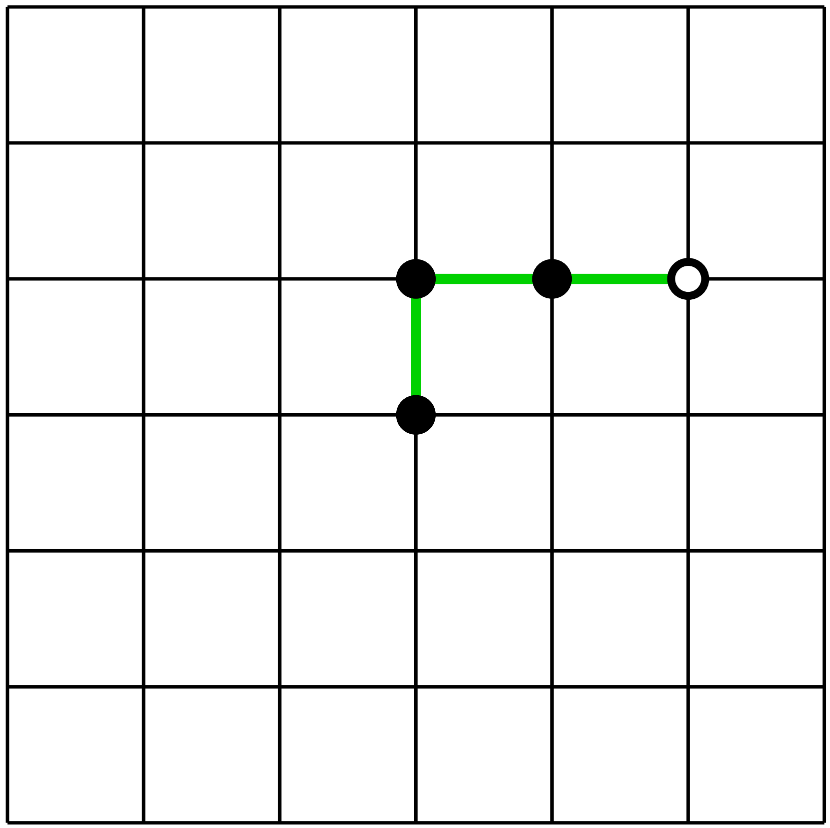
LERW = **L**oop **E**rased **R**andom **W**alk



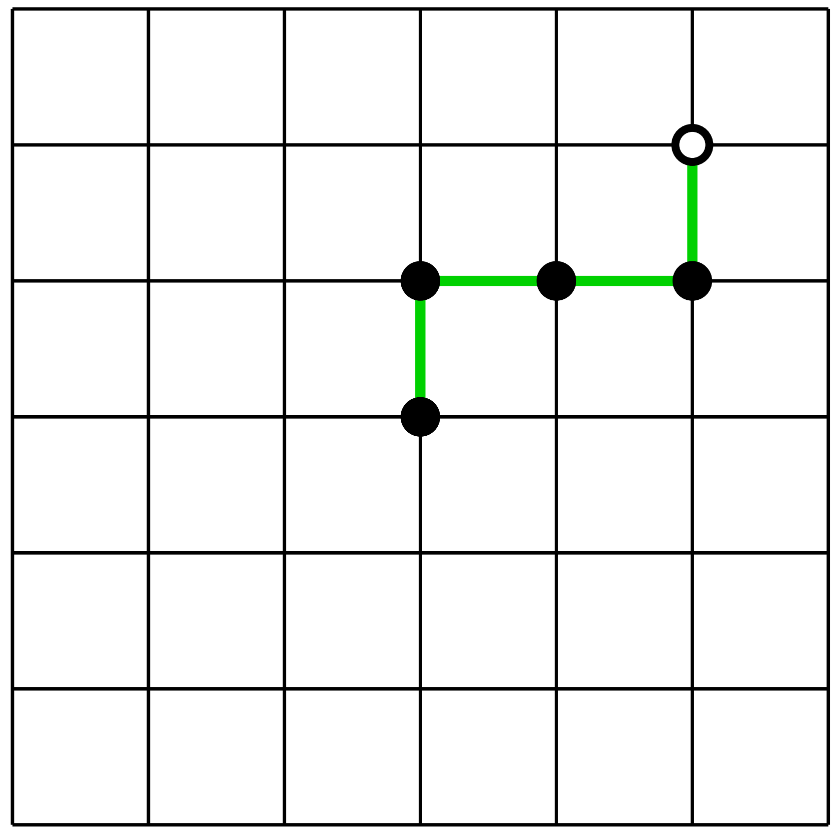
LERW = Loop Erased Random Walk



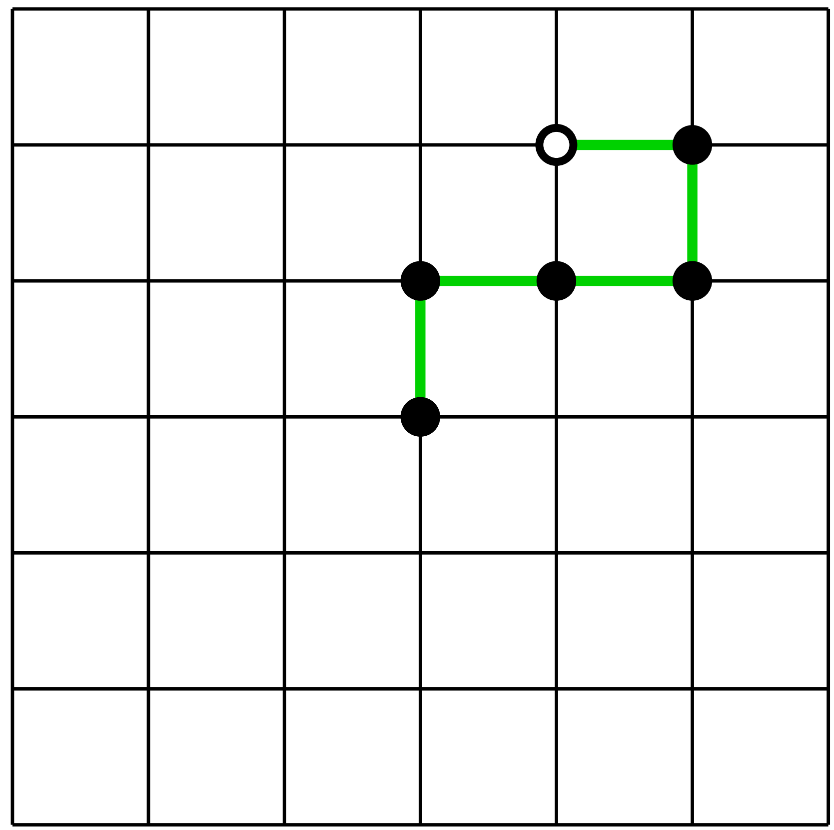
LERW = Loop Erased Random Walk



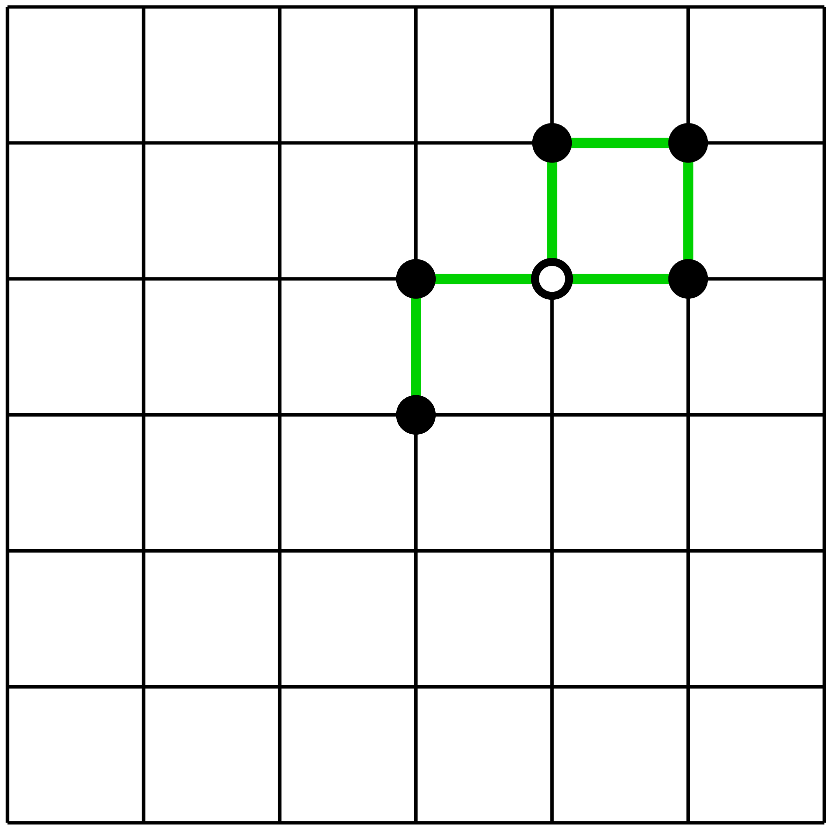
LERW = Loop Erased Random Walk



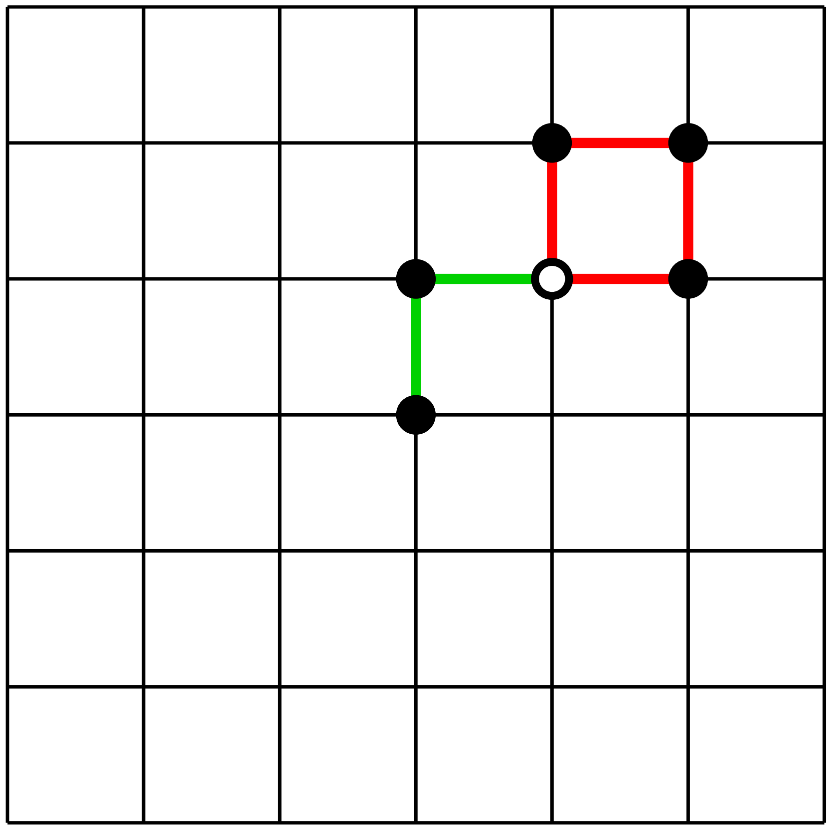
LERW = Loop Erased Random Walk



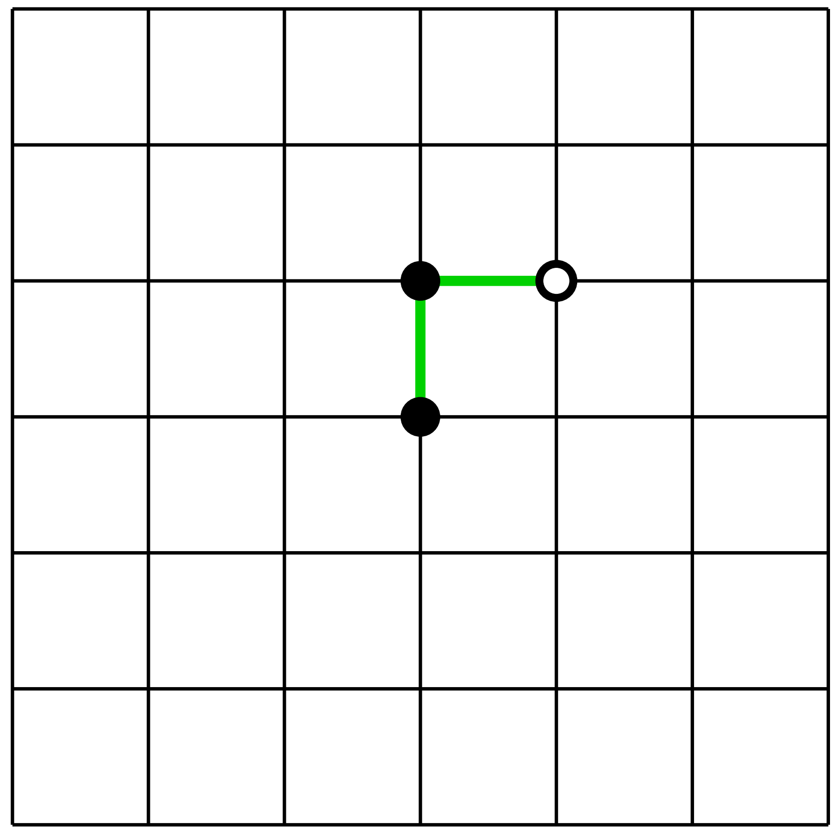
LERW = Loop Erased Random Walk



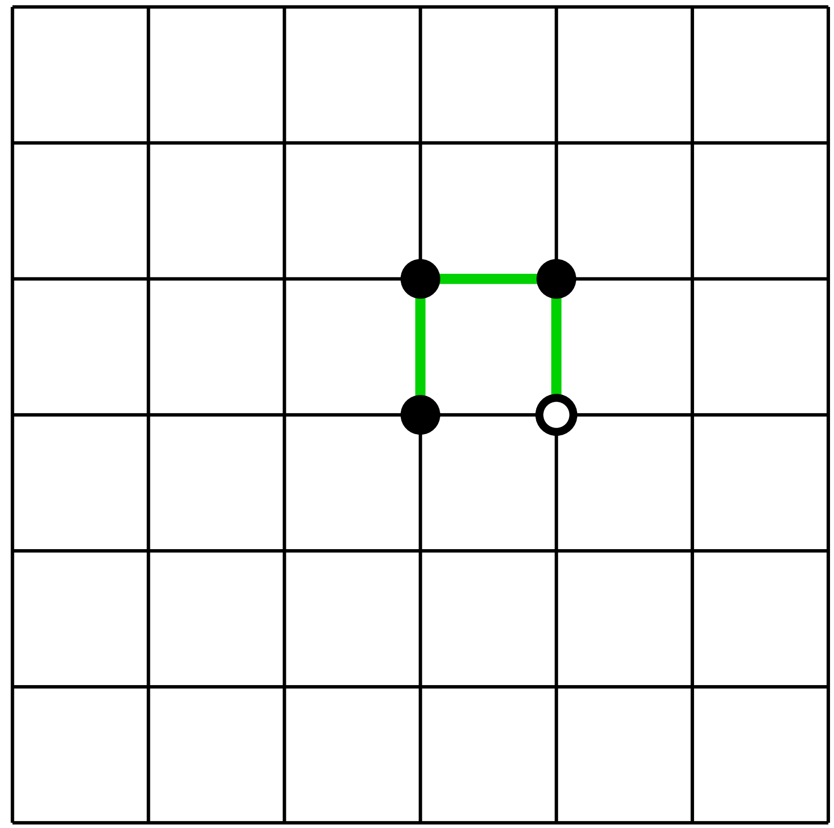
LERW = Loop Erased Random Walk



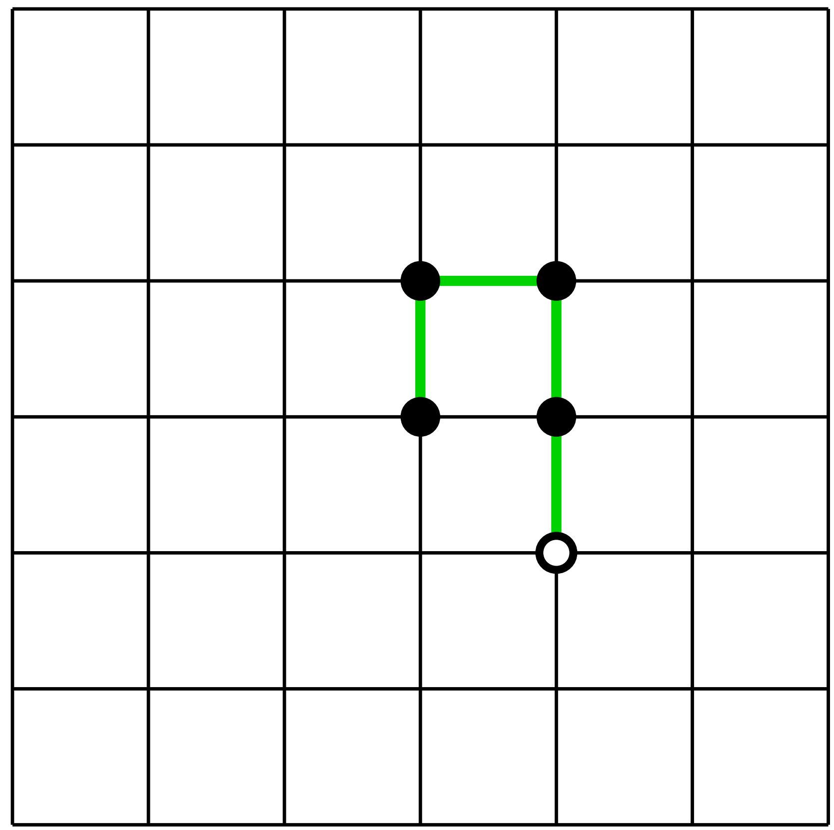
LERW = Loop Erased Random Walk



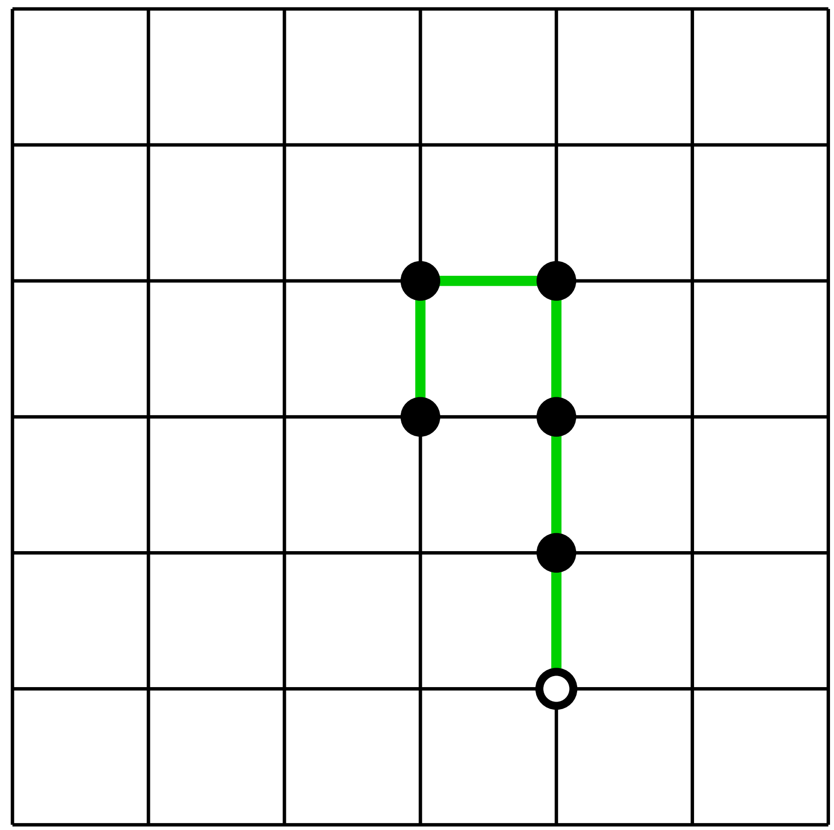
LERW = Loop Erased Random Walk



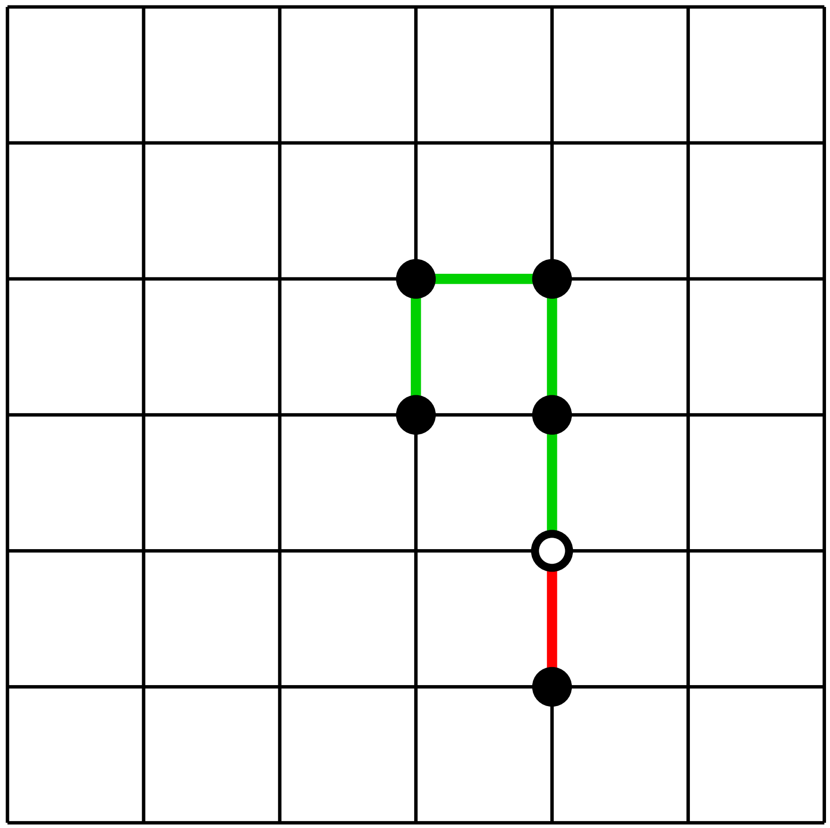
LERW = Loop Erased Random Walk



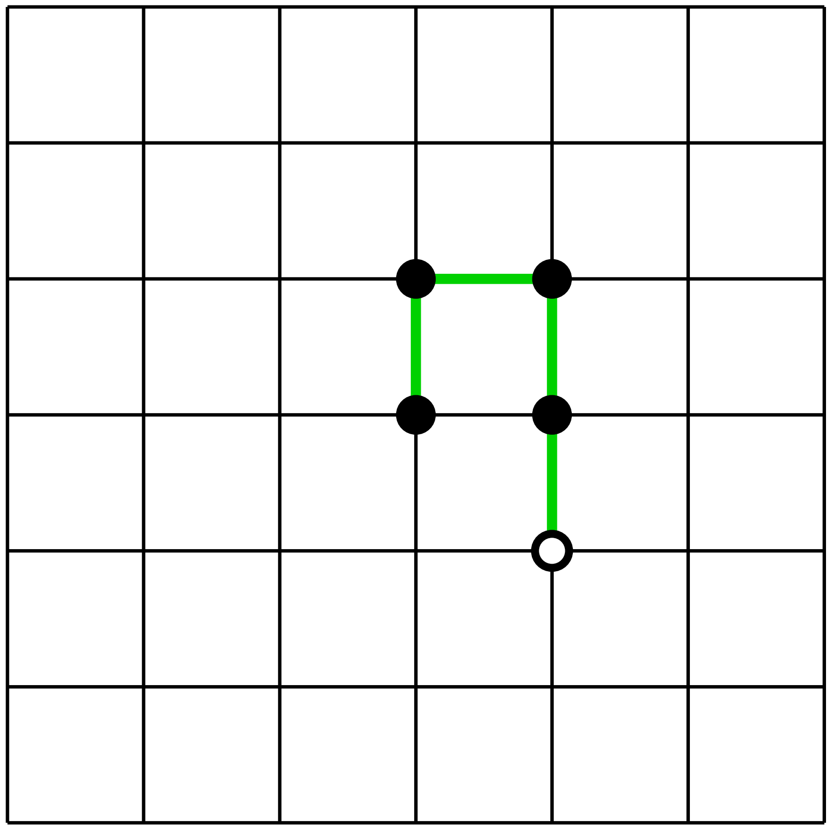
LERW = Loop Erased Random Walk



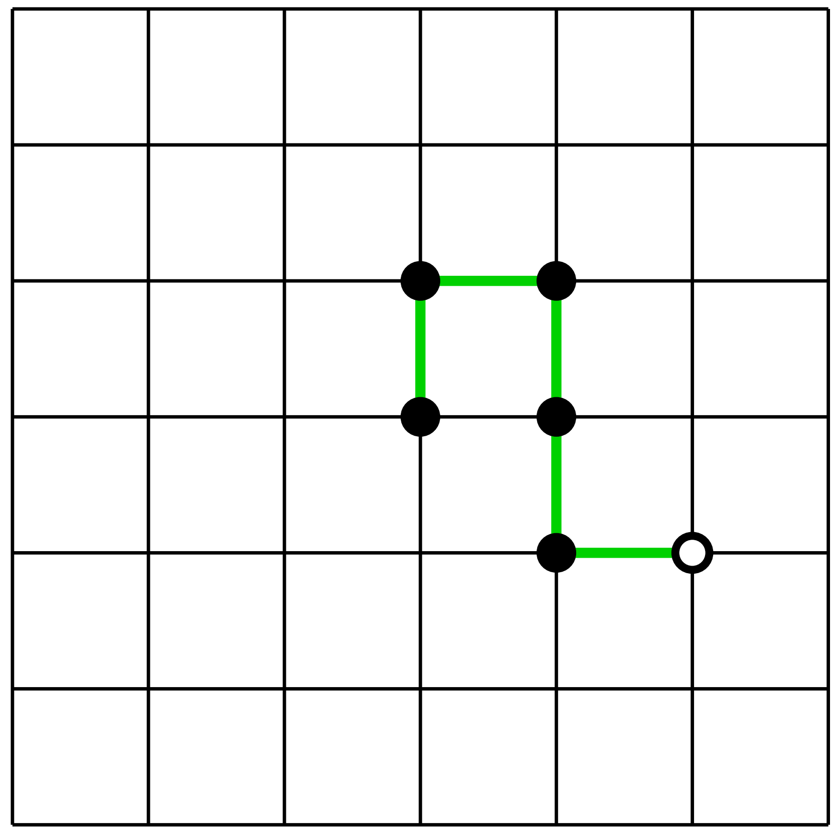
LERW = Loop Erased Random Walk



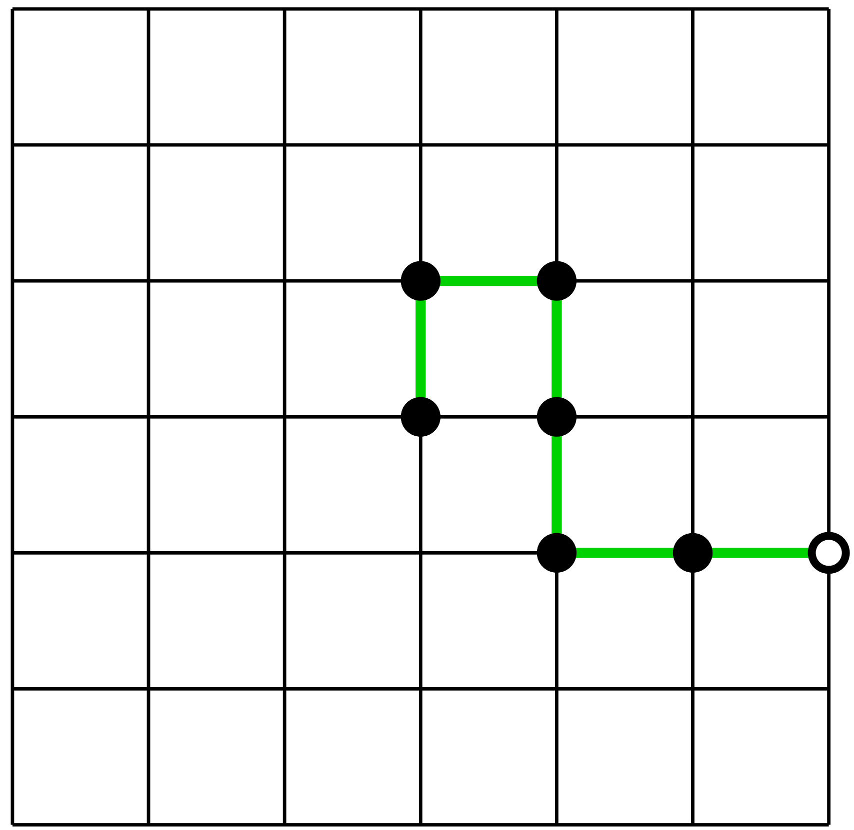
LERW = Loop Erased Random Walk



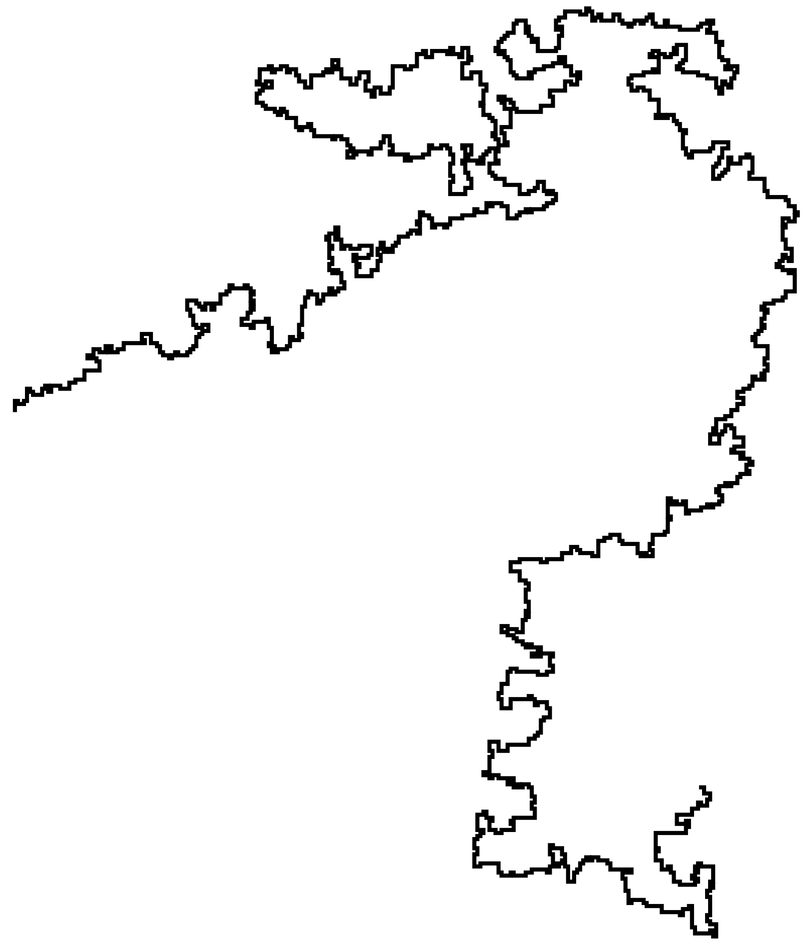
LERW = Loop Erased Random Walk



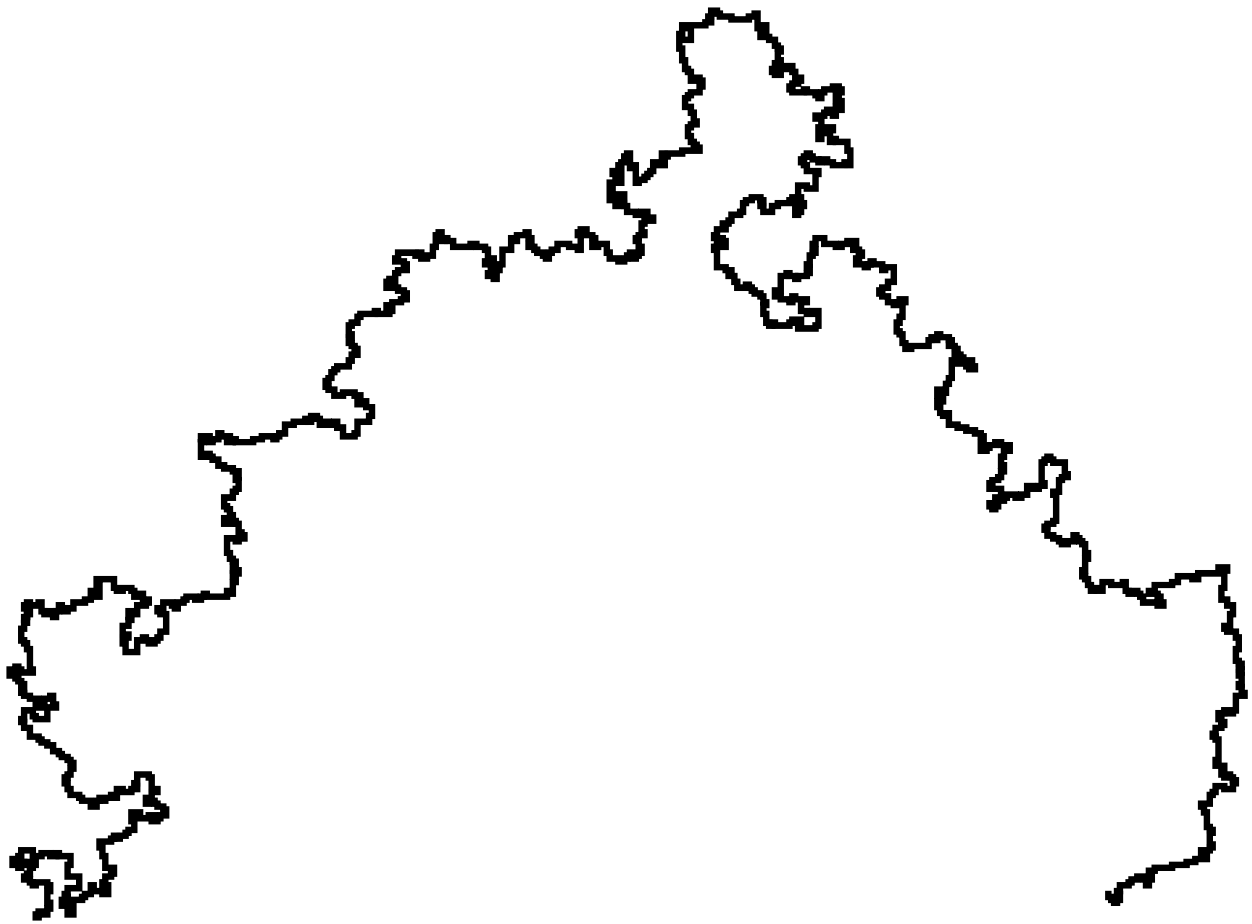
LERW = Loop Erased Random Walk



LERW = Loop Erased Random Walk

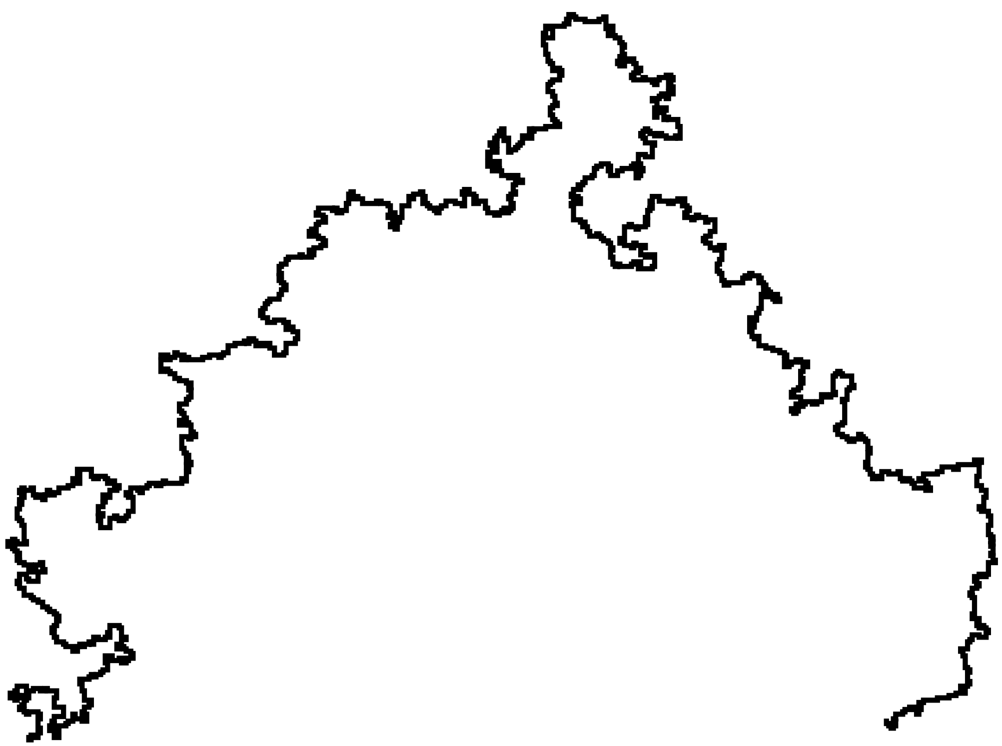


LERW = Loop Erased Random Walk



As $n \to \infty$, $\#(\text{LERW}) \approx (\text{diameter})^{5/4}$,

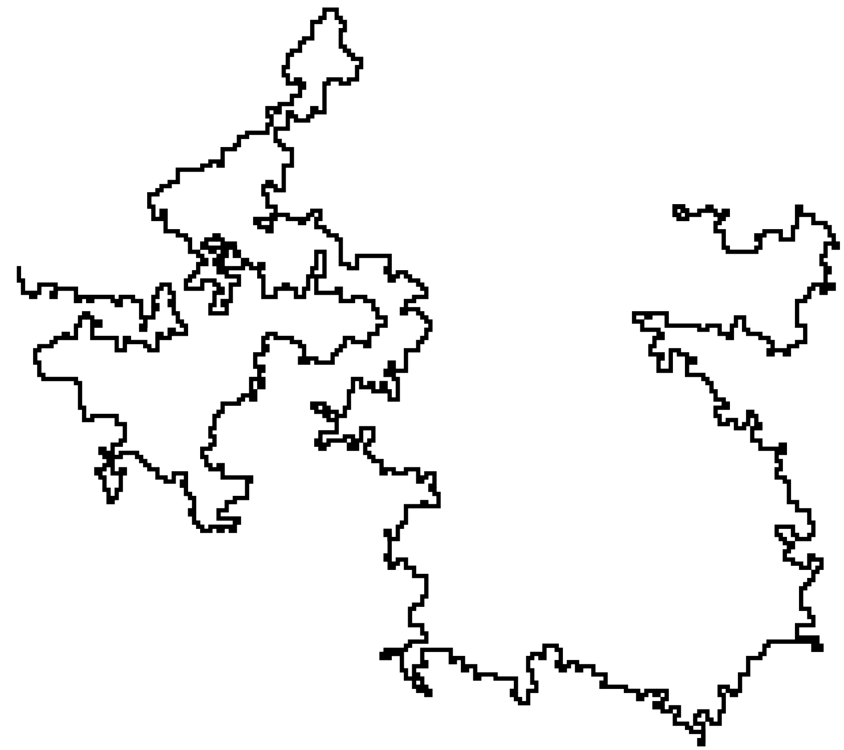
LERW converges to curve of "dimension" 5/4.



Loop Erased Random Walk

dimension $\frac{5}{4}$

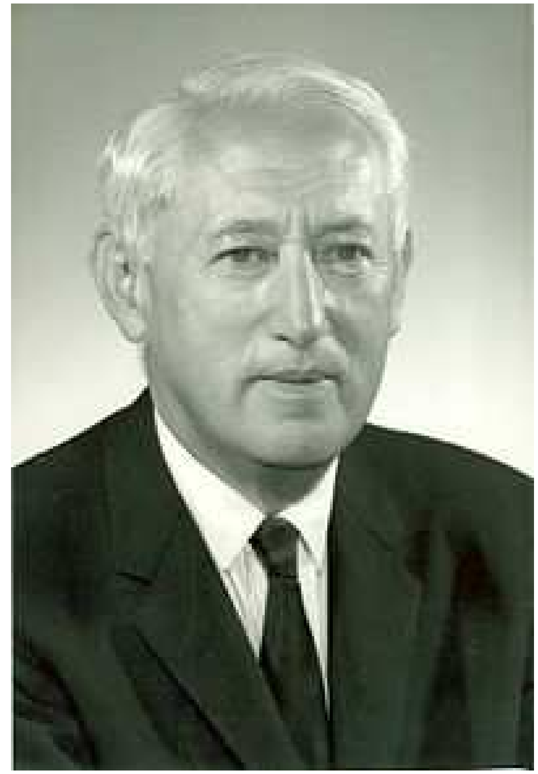
$\neq$



Brownian Frontier

dimension $\frac{4}{3}$

Oded Schramm

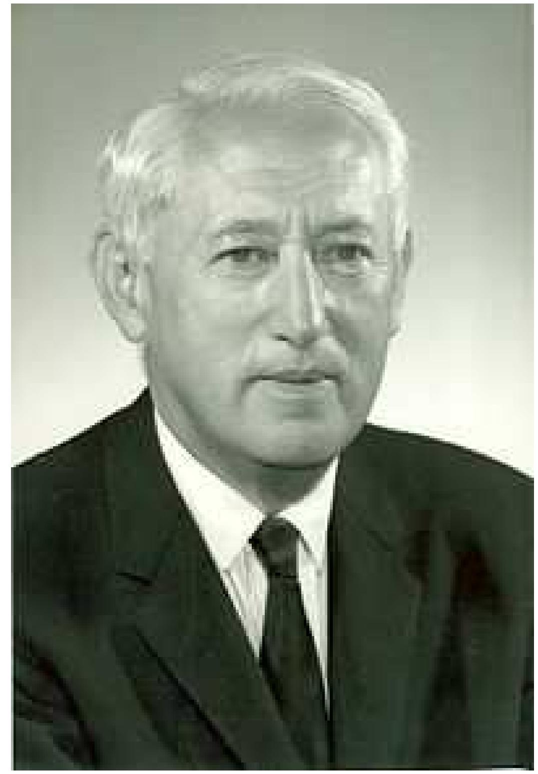


Charles Loewner

Schramm invented 1-parameter families of random Jordan curves, SLE(κ).

Schramm used classical differential equation for conformal maps, due to Charles Lowener, with multiples of 1-dim Brownian motion as input data. These are the only “nice” families of random Jordan curves.

Oded Schramm

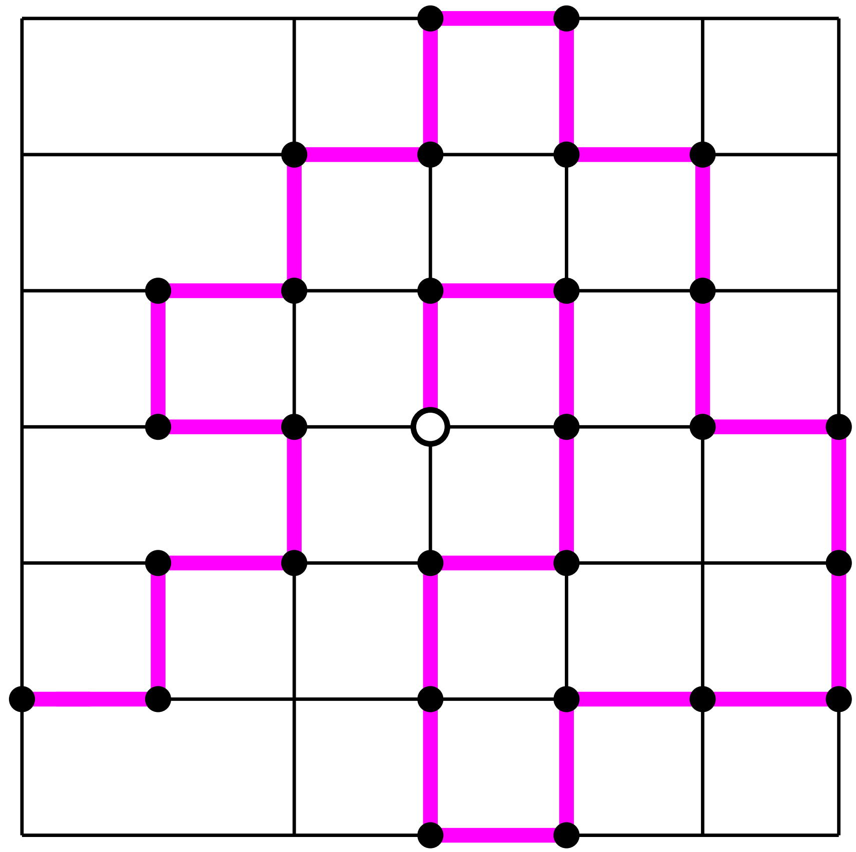


Charles Loewner

SLE = Stochastic Loewner Equation = Schramm-Loewner Evolutions

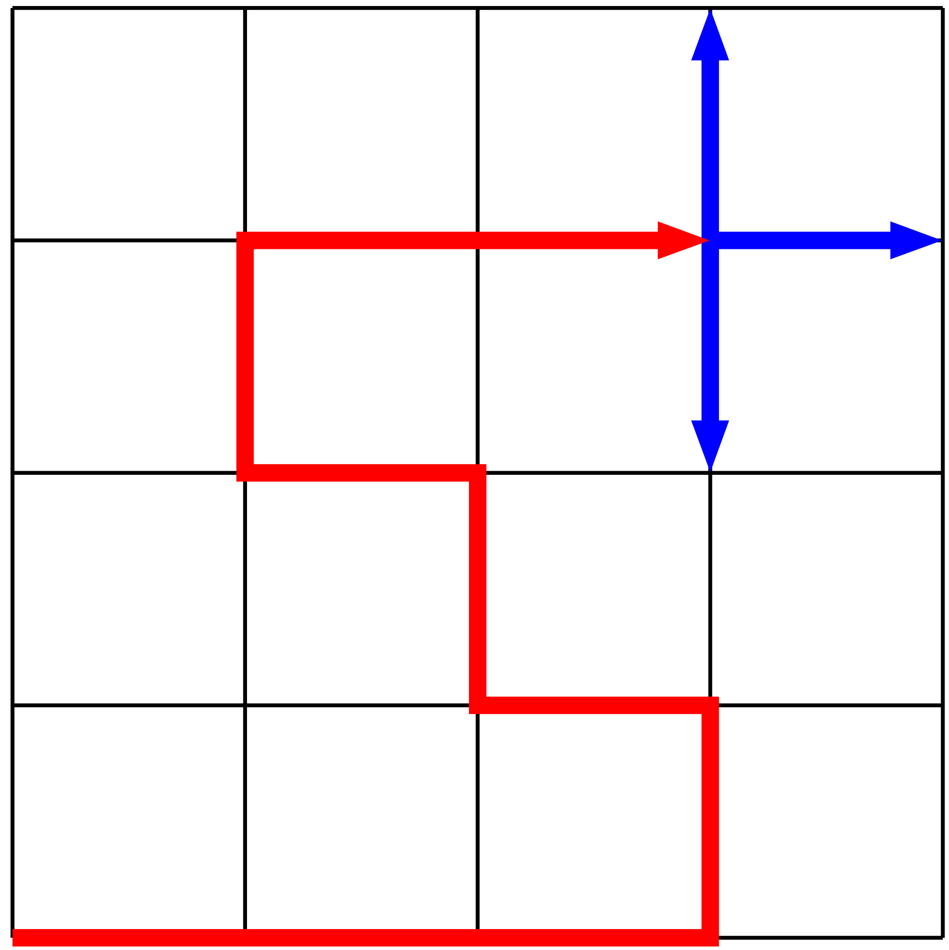
Brownian Frontiers = SLE(8/3) (Lawler, Schramm, Werner, 2002)

LERW $\rightarrow$ SLE(2) (Lawler, Schramm, Werner, 2004).

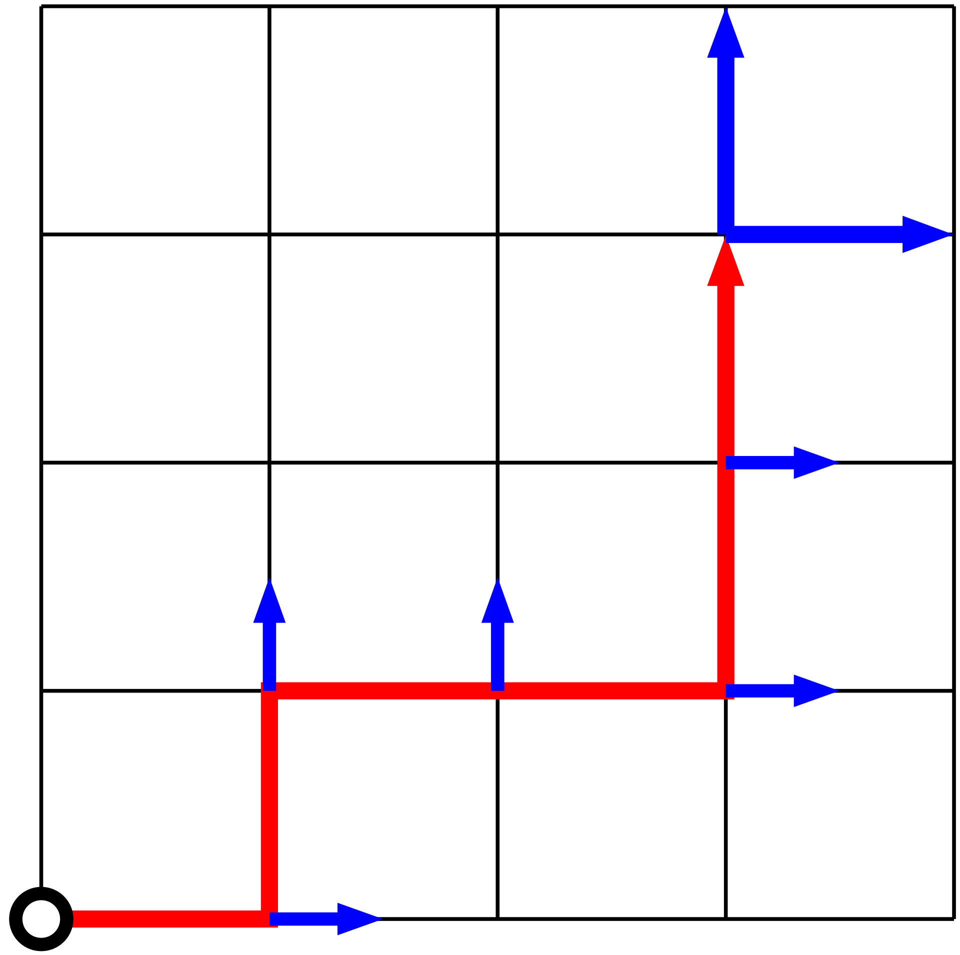


SAW = **S**elf **A**voiding **W**alk = no repeated vertices

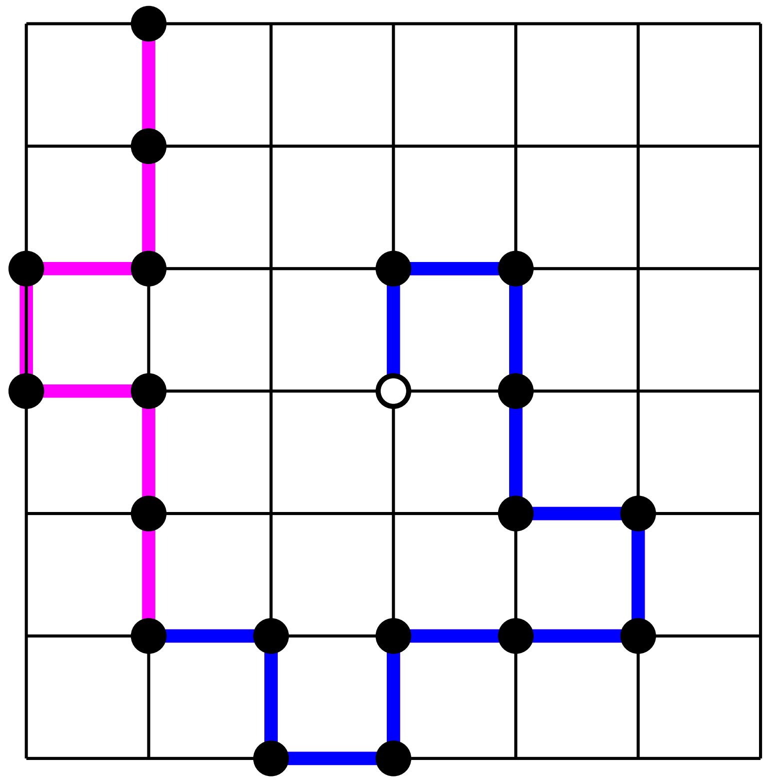
Estimate SAW(n) = number of n step SAWs.



$$\mathrm{SAW}(n) \leq 4 \cdot 3^n.$$



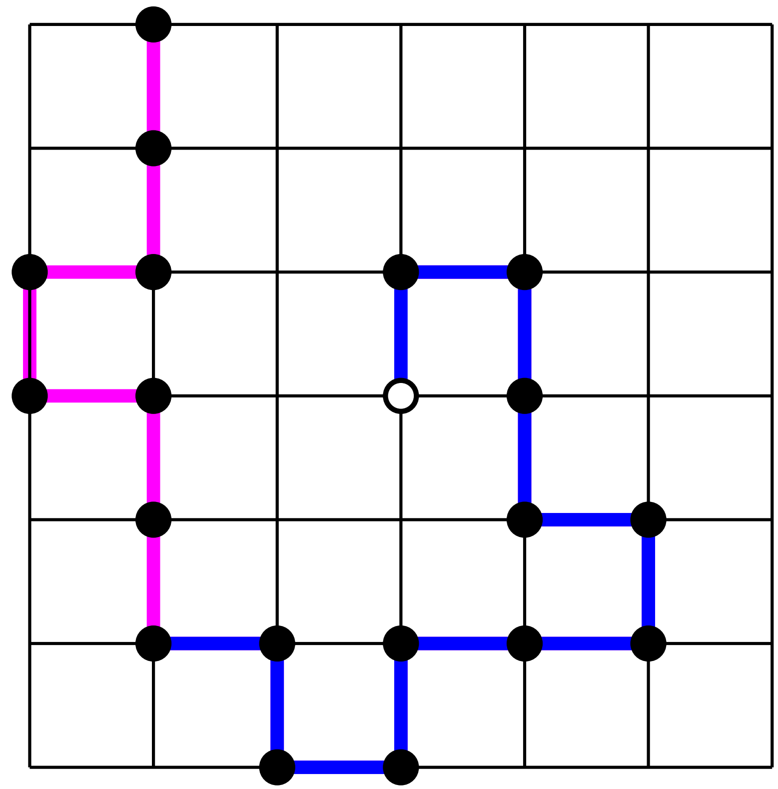
$$\mathrm{SAW}(n) \geq 2^n.$$



$$\mathrm{SAW}(n+m) \leq \mathrm{SAW}(n) \cdot \mathrm{SAW}(m)$$

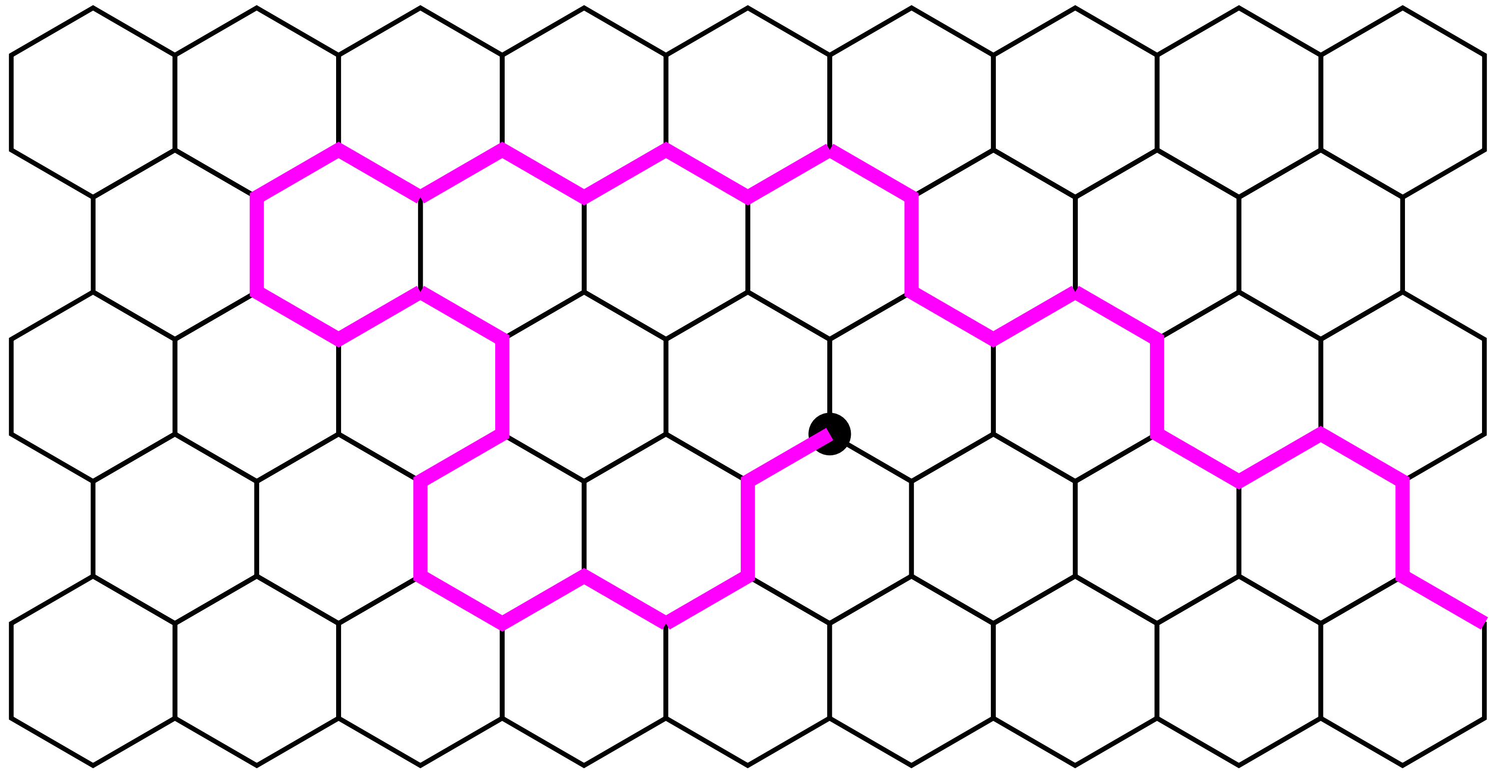
Fekete's lemma $\Rightarrow \alpha = \lim \mathrm{SAW}(n)^{1/n}$ exists.

$$\Rightarrow \quad \mathrm{SAW}(n) \approx \alpha^n \text{ for some } \alpha$$



Numerically, $\quad \mathrm{SAW}(n) \approx (2.64)^n$

Exact value unknown. Algebraic?



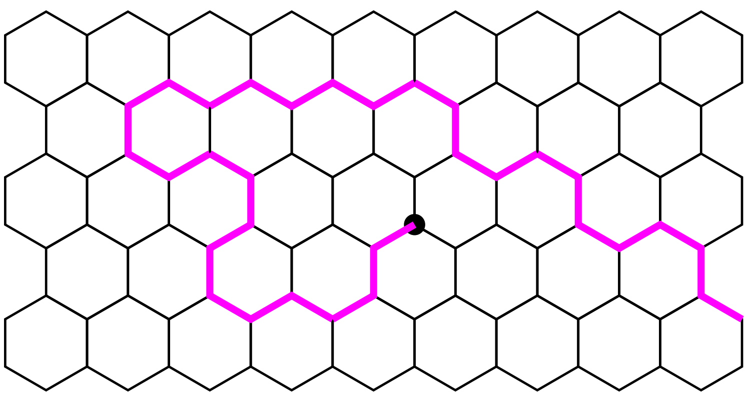
If we replace square grid by hex grid,

$$\#\mathrm{SAWs} \simeq \left(\sqrt{2+\sqrt{2}}\right)^n.$$

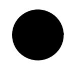
 

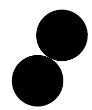
Proven by Stas Smirnov and Hugo Duminil-Copin in 2012.

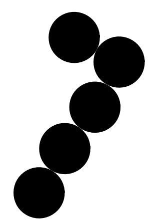
(non-rigorously derived in theoretical physics by Nienhuis, 1982)

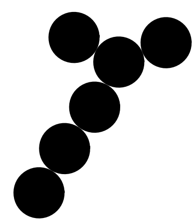


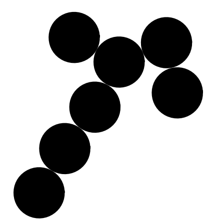
$$\#\text{SAWs} \simeq \left(\sqrt{2+\sqrt{2}}\right)^n$$

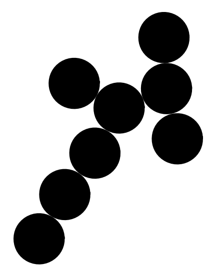


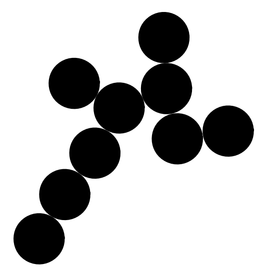


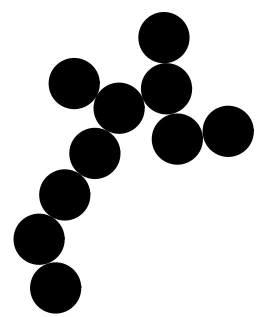


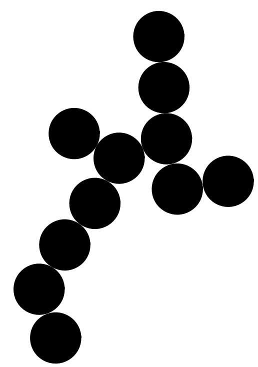


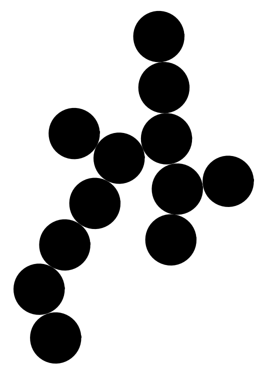


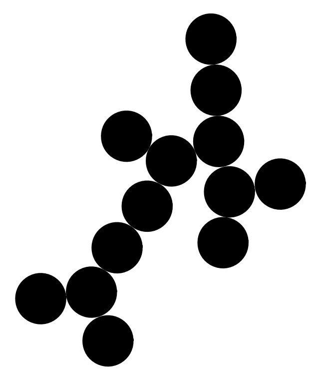


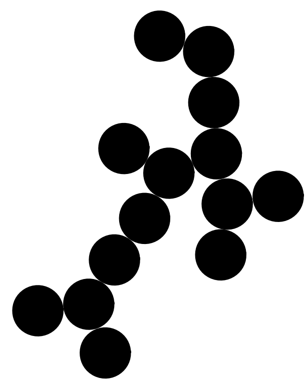


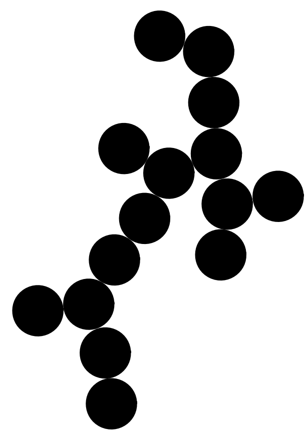


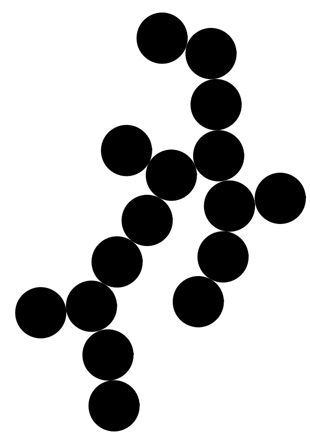


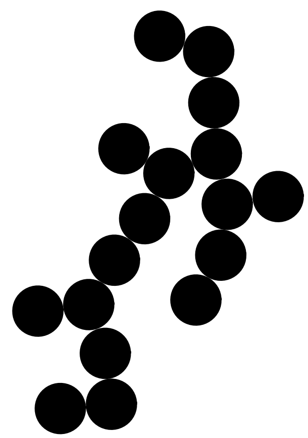


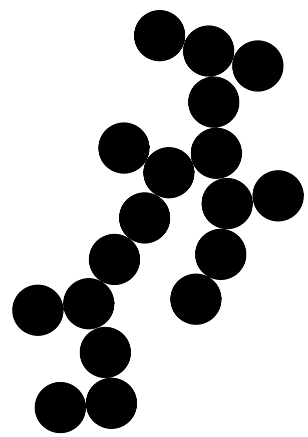


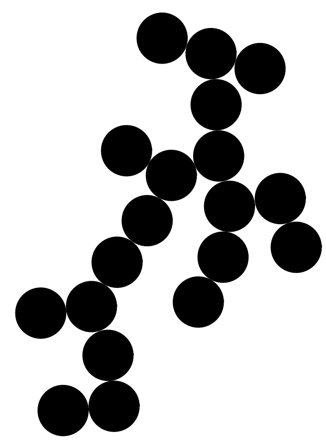


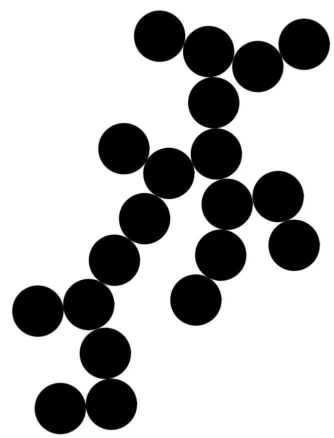


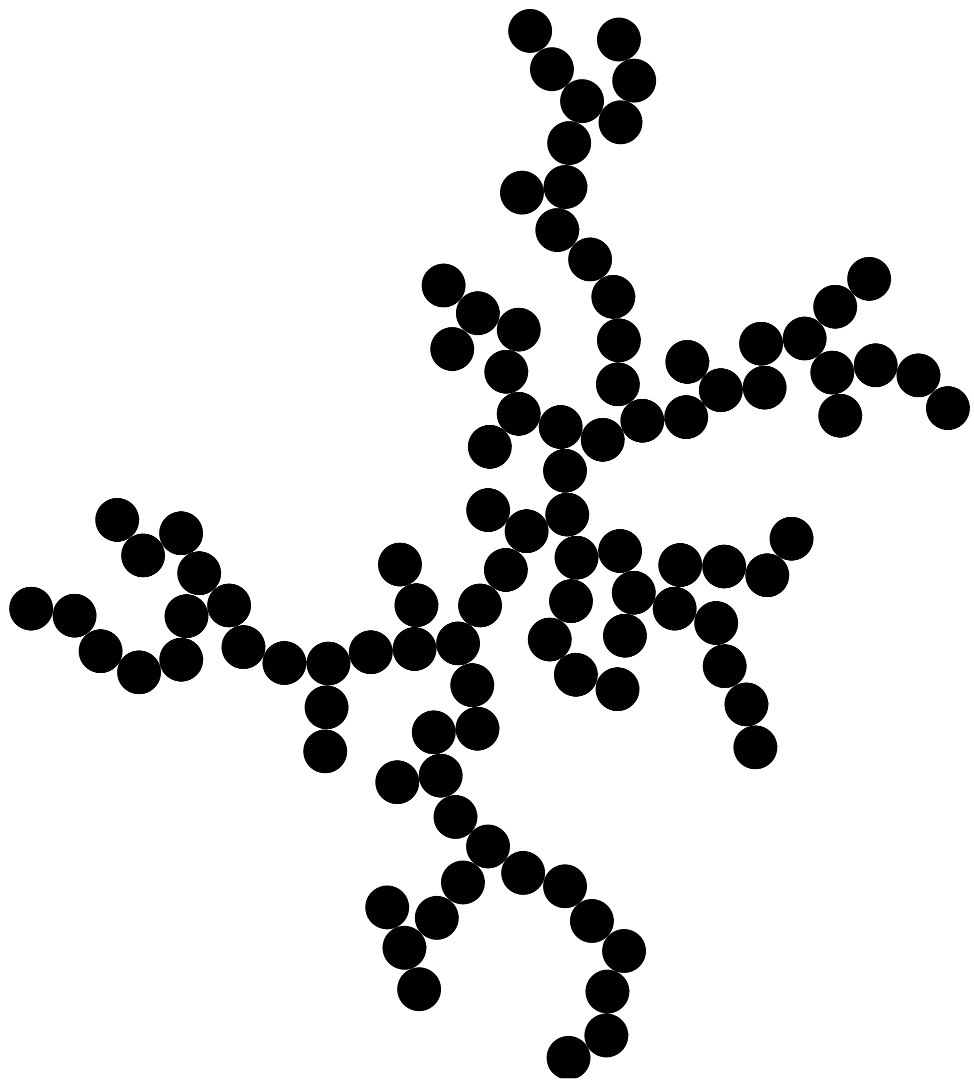




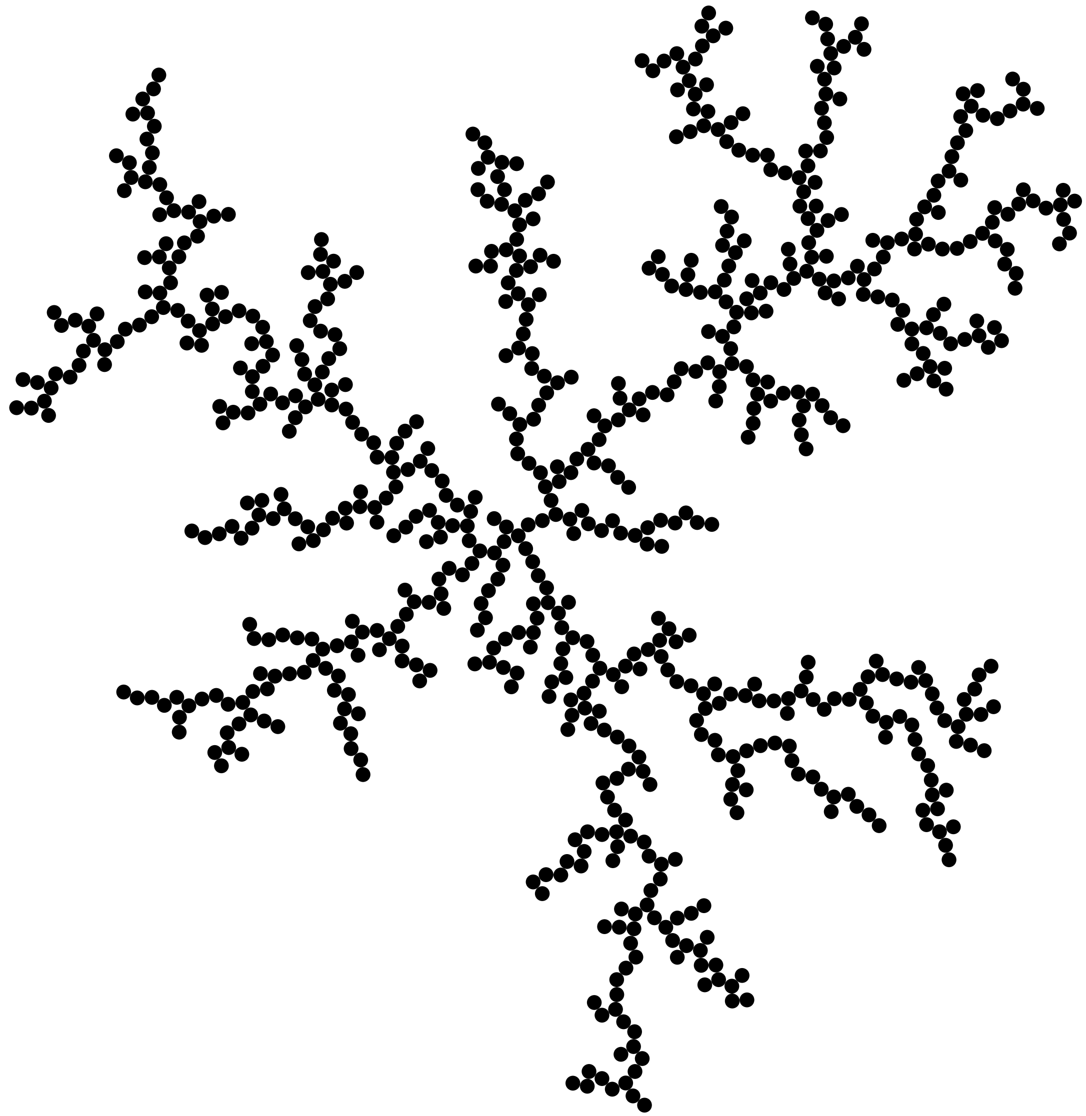




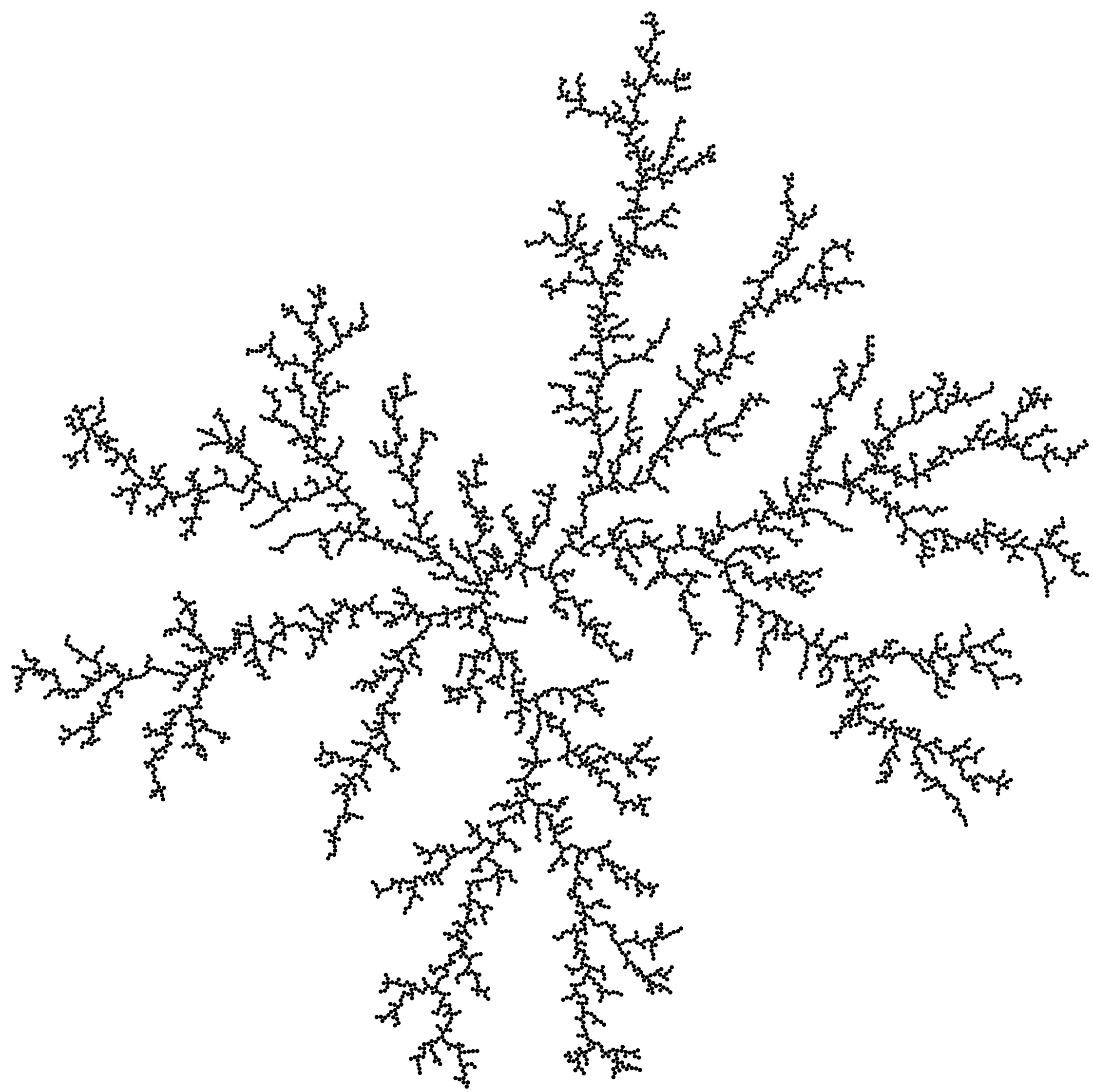




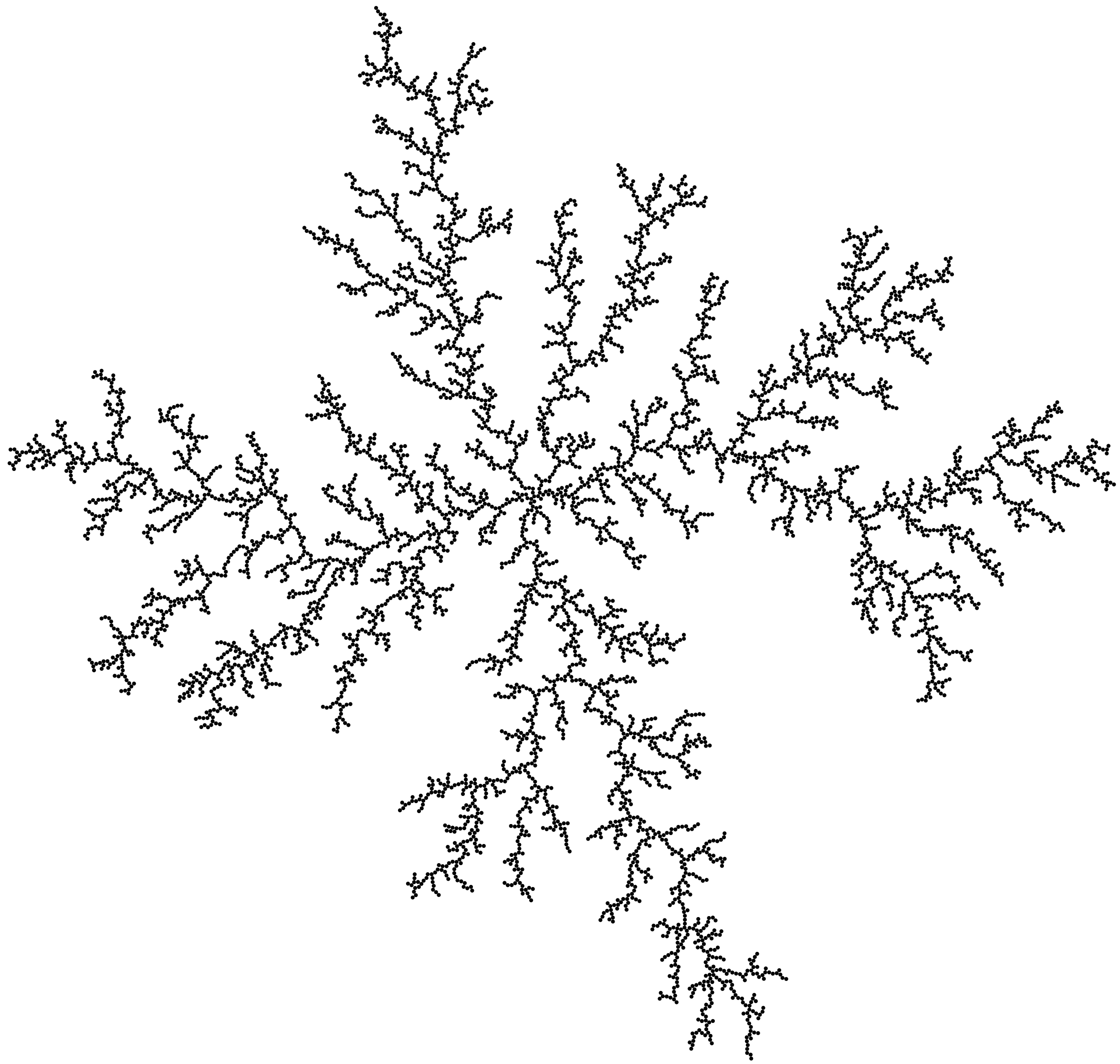
Diffusion Limited Aggregation (DLA), n=100



Diffusion Limited Aggregation (DLA), n=1000

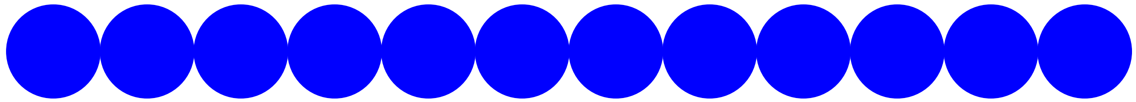


Diffusion Limited Aggregation (DLA), n=10000



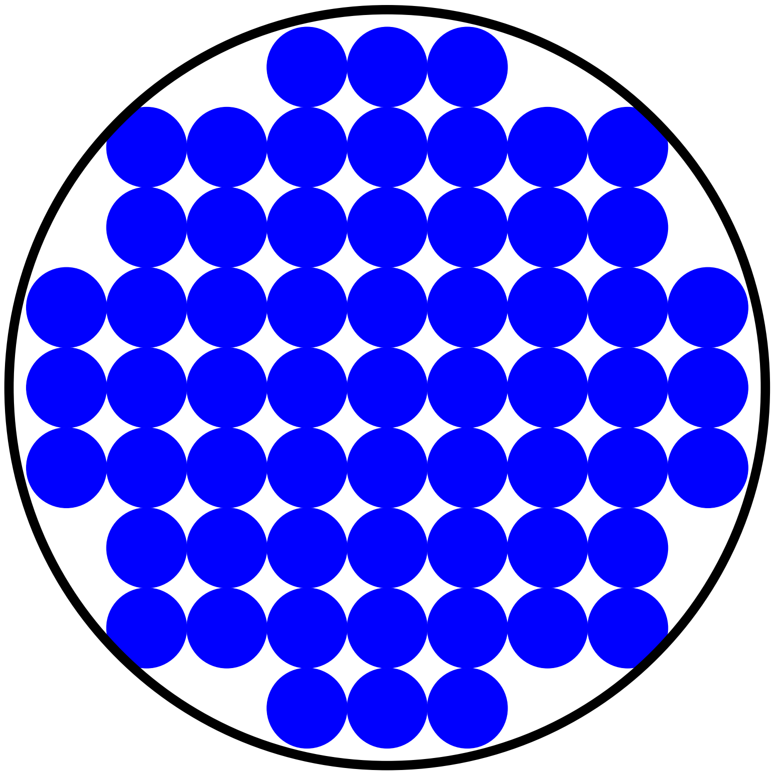
How does diameter grow with n? $\approx n^{\alpha}$?

Diffusion Limited Aggregation (DLA)



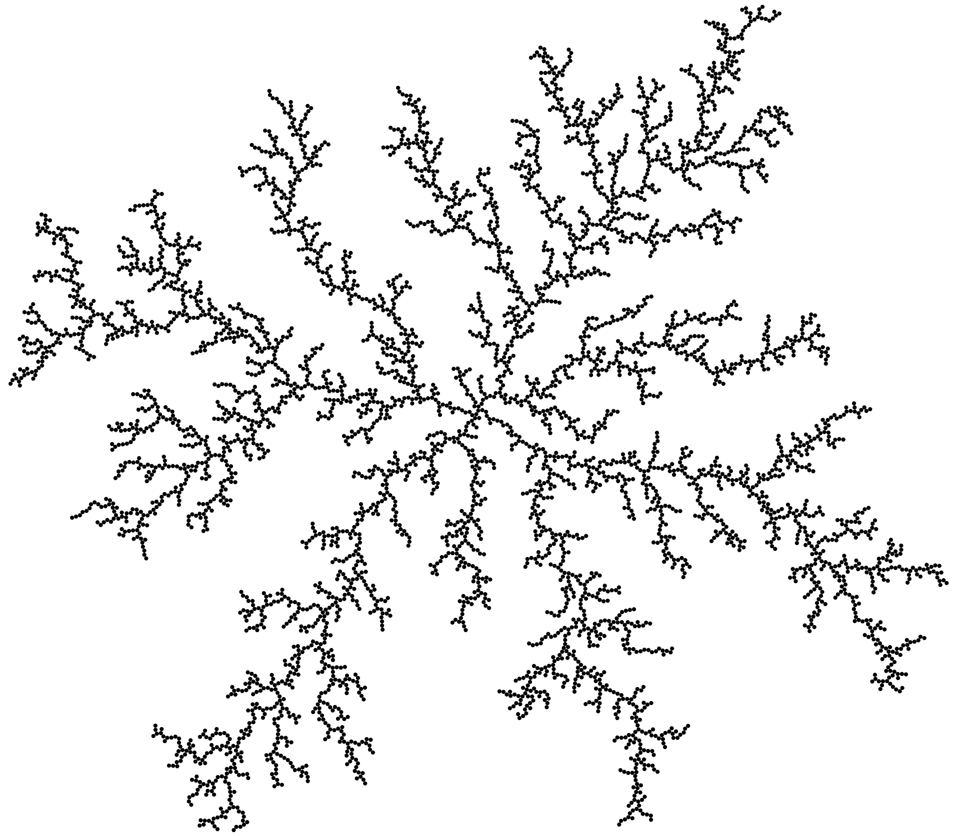
Trival: diameter $\leq n$

Diffusion Limited Aggregation (DLA)



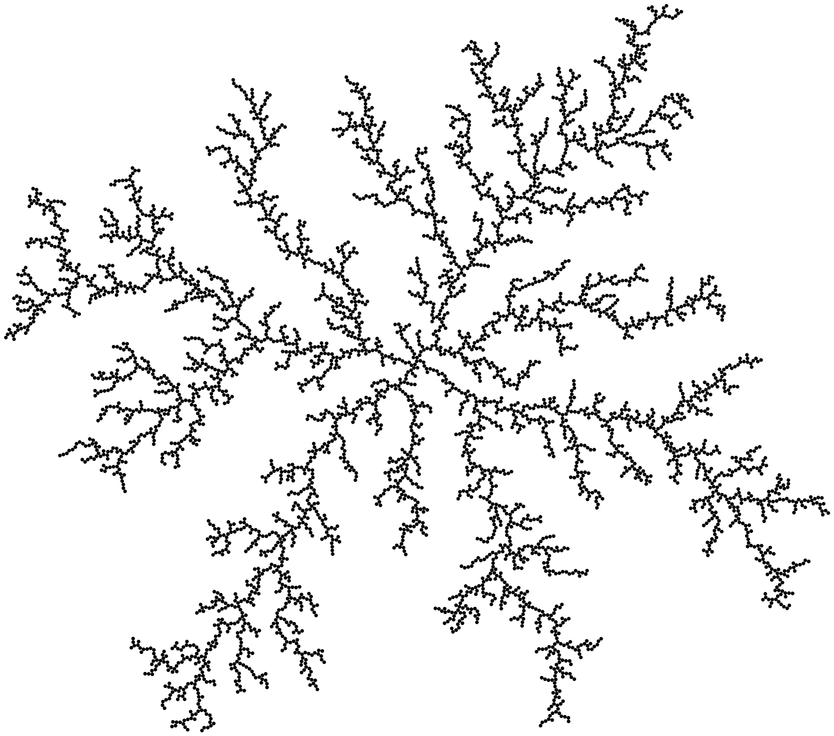
Trival: diameter $\geq \sqrt{n}$

Diffusion Limited Aggregation (DLA)



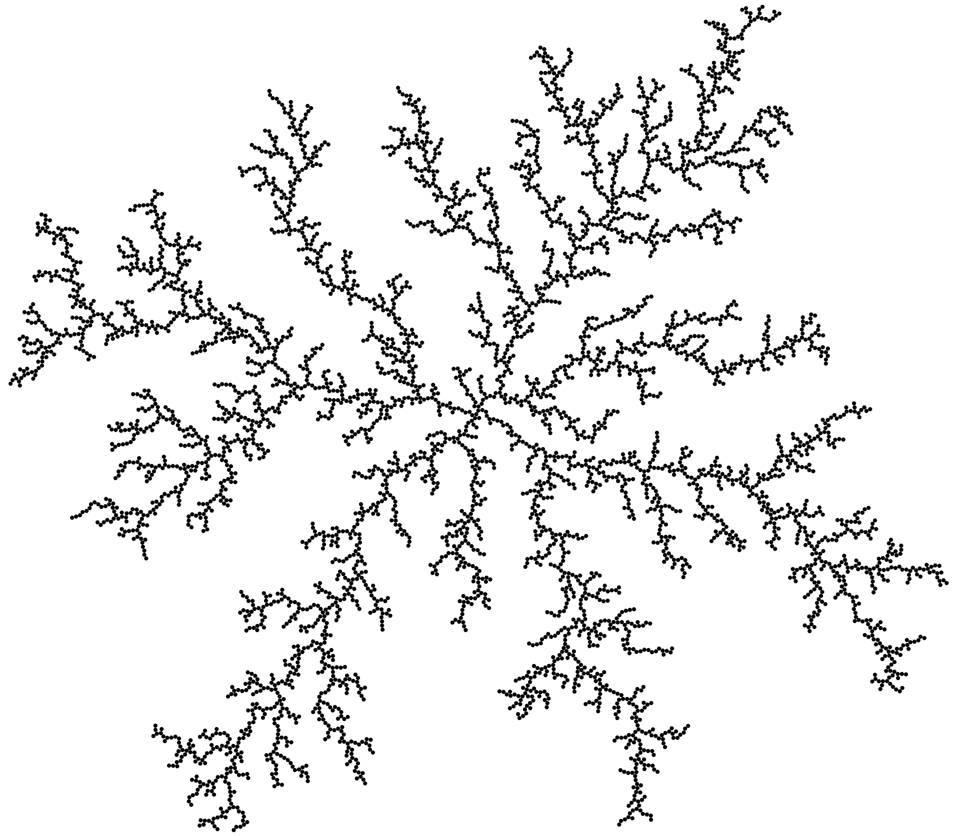
Experiments $\simeq N^{.585}$.

Diffusion Limited Aggregation (DLA)

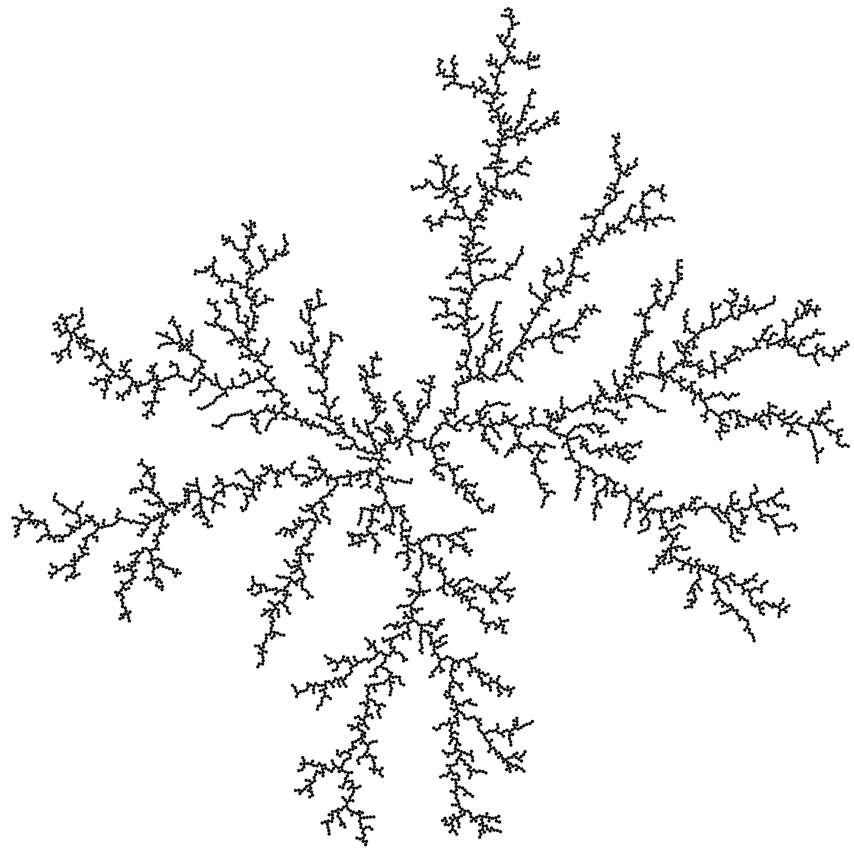
Harry Kesten proved diameter $\leq N^{2/3}$, 1987.

Diffusion Limited Aggregation (DLA)



Improve the trival lower bound: $\frac{\text{Diameter}}{\sqrt{n}} \to \infty$.

Diffusion Limited Aggregation (DLA)



Improve the trival lower bound: Diameter $\geq n^{\frac{1}{2}+\epsilon}$.

Thanks for listening. Answers?