NEW CONSTRUCTIONS IN TRANSCENDENTAL DYNAMICS

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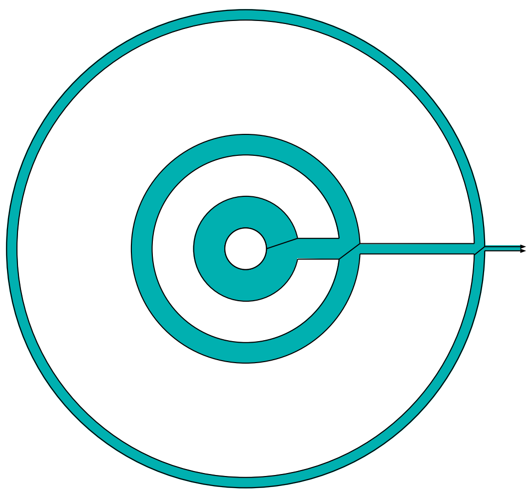
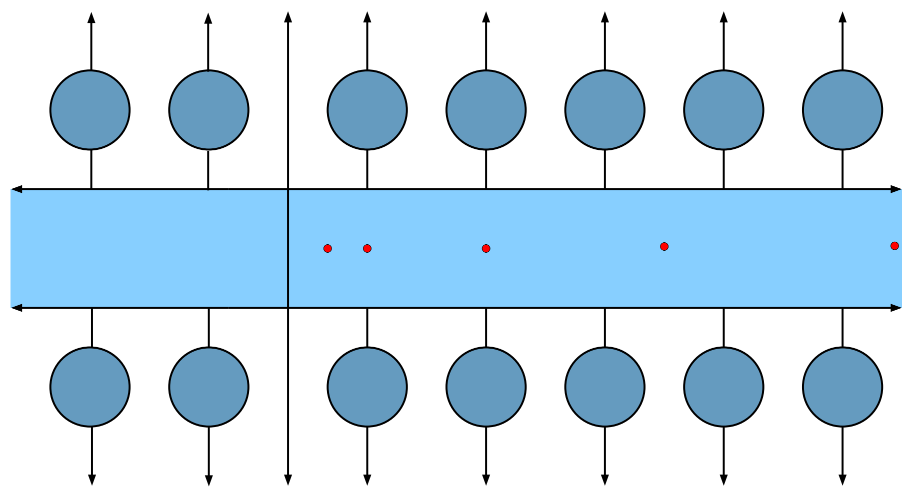
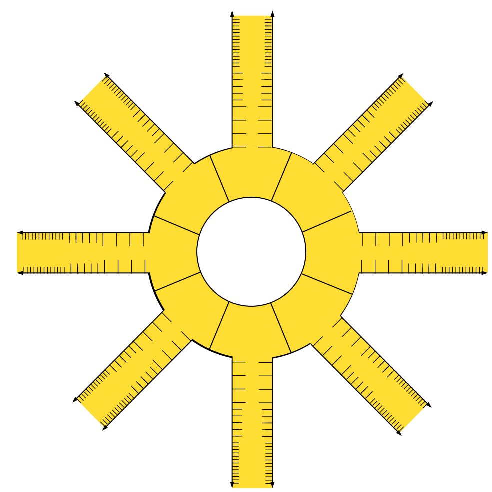
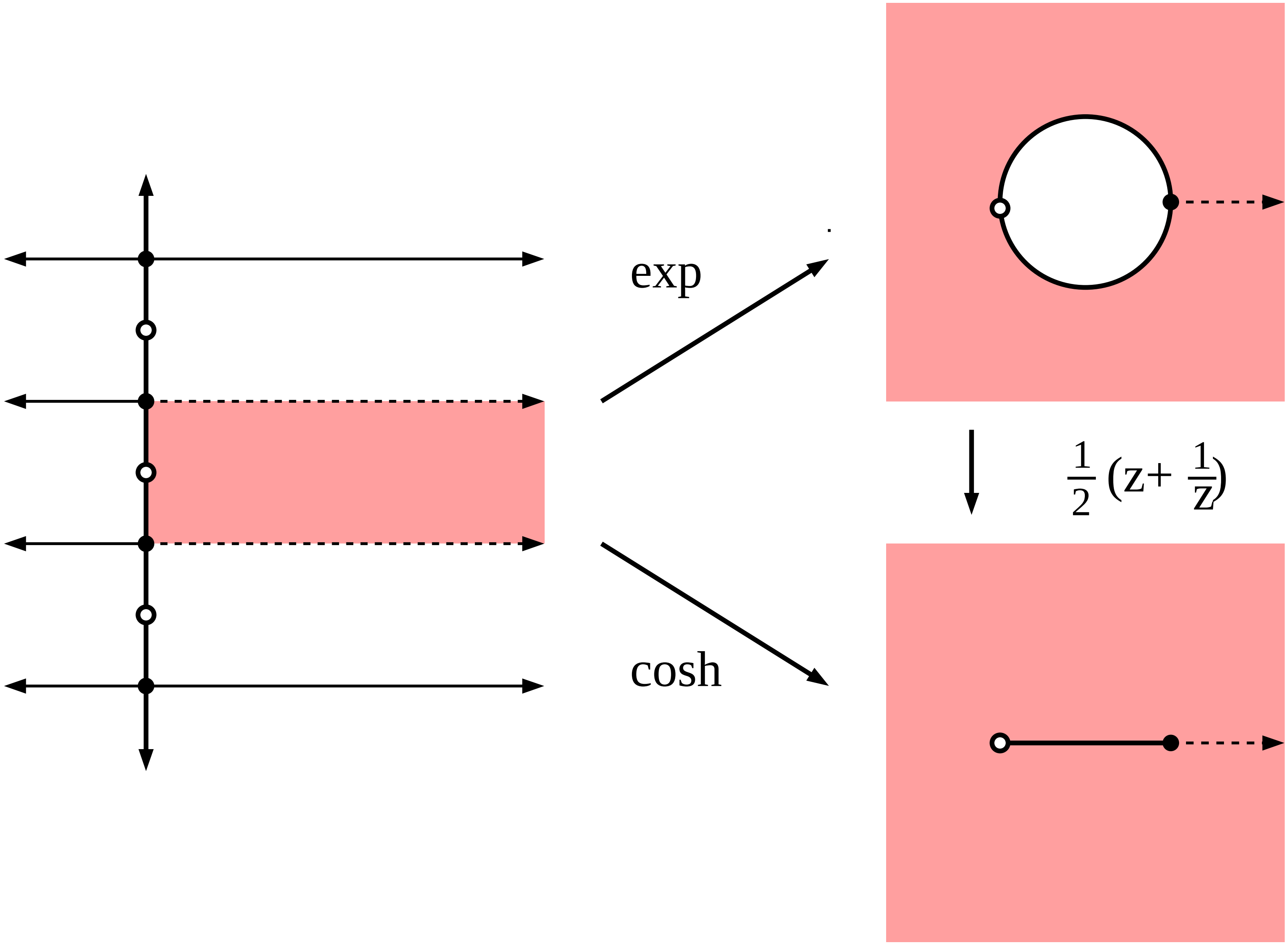
Analytic Low-Dimensional Dynamics:
a celebration of Misha Lyubich's 60th birthday

June 7, 2019 Fields Institute, Toronto

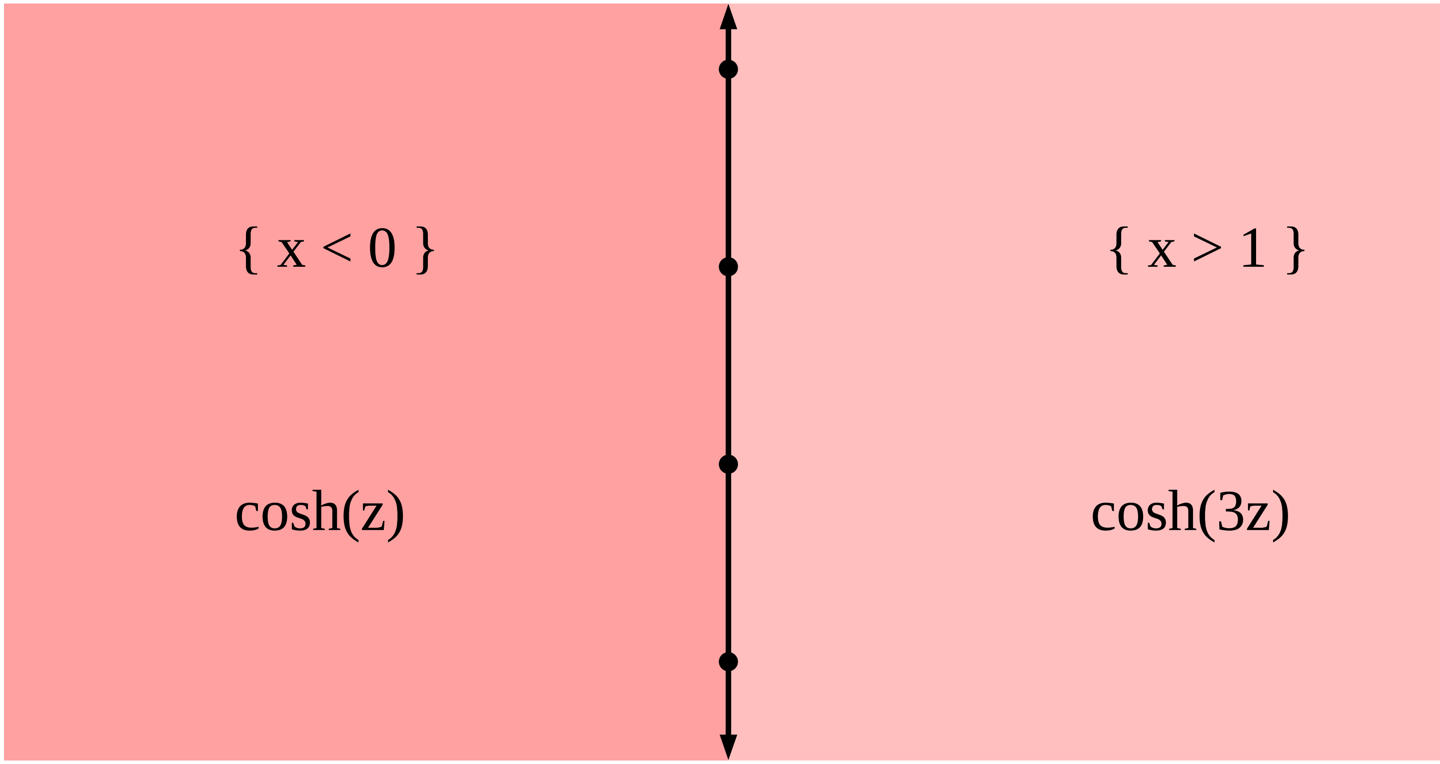








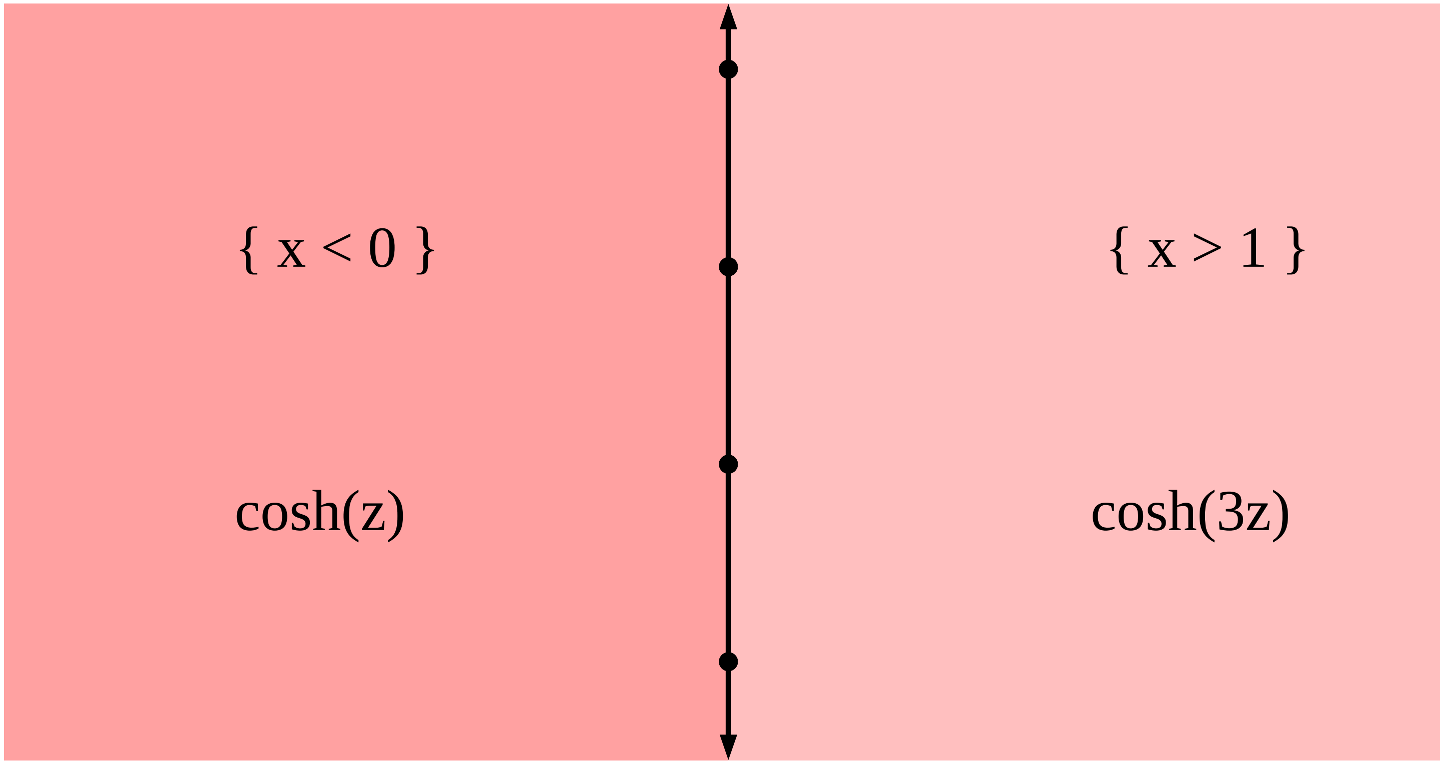
Definitions of exp and cosh.



Construct an entire function F so that

$$F(z) = \cosh(z) \text{ on left half-plane } = \mathbb{H}_l = \{z = x + iy : x < 0\}$$

$$F(z) = \cosh(3z) \text{ on right half-plane } = \mathbb{H}_r = \{z = x + iy : x > 0\}$$



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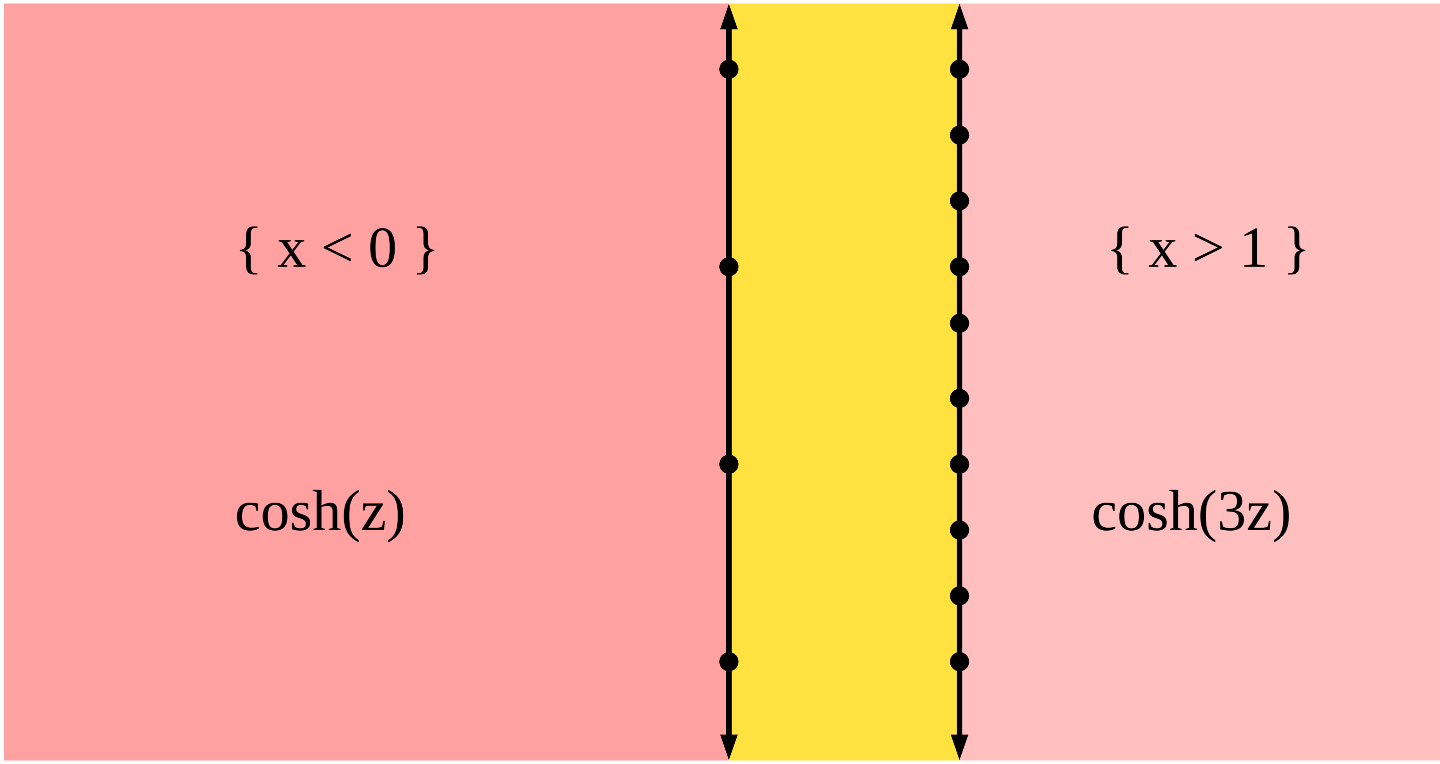
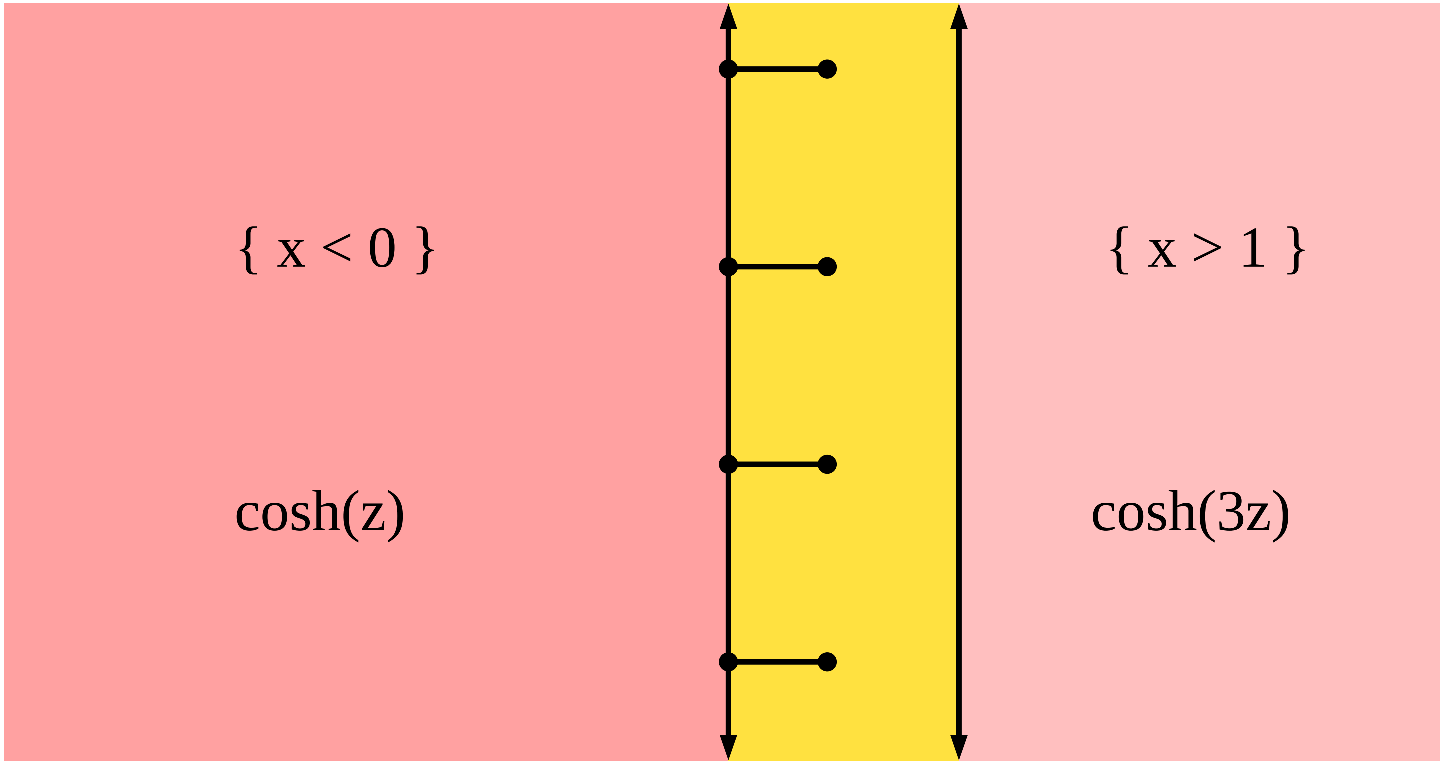
Impossible: violates unique continuation



Construct an **quasiregular** function F so that

$$F(z) = \cosh(z) \text{ on } \{x < 0\}$$

$$F(z) = \cosh(3z) \text{ on } \{x > 1\}$$



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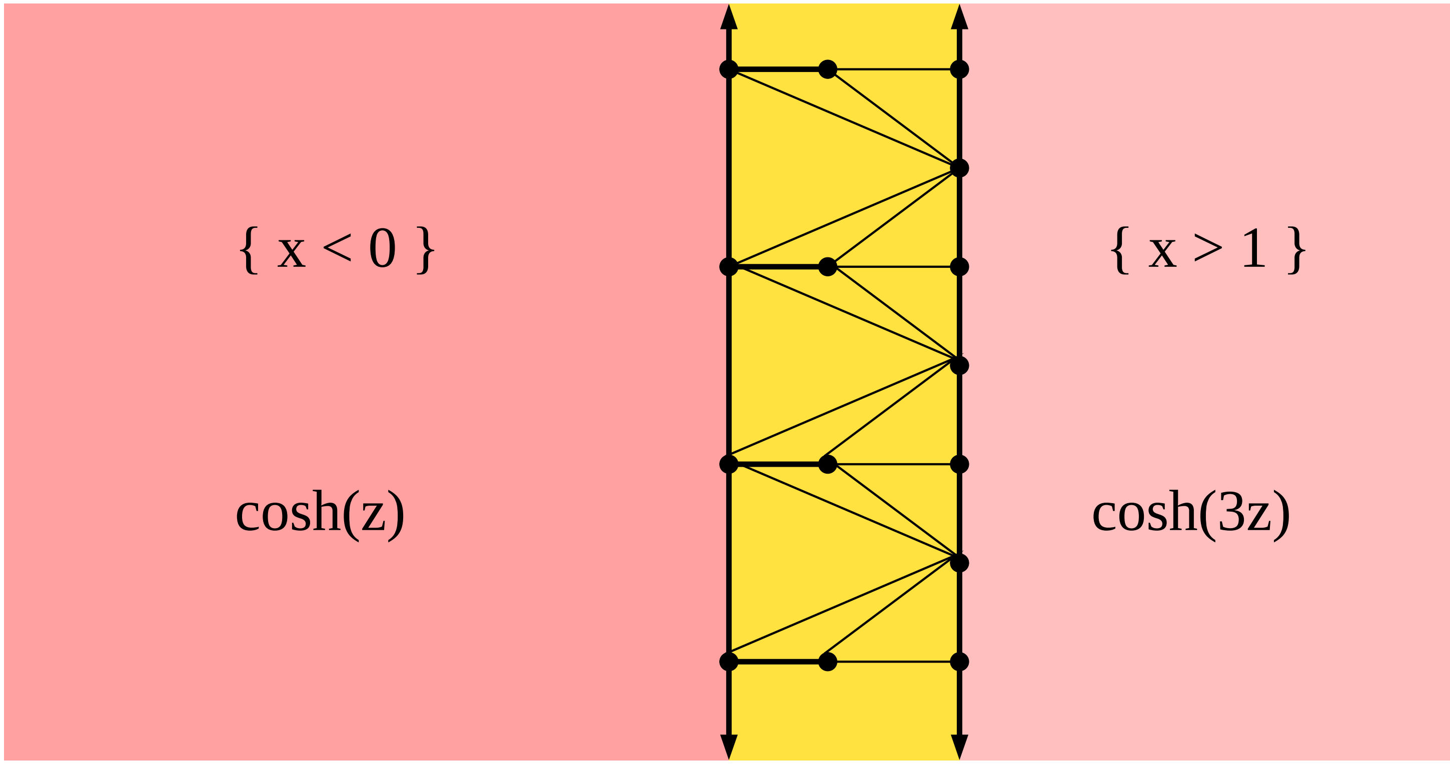
This is easy. First add some spikes in the strip.



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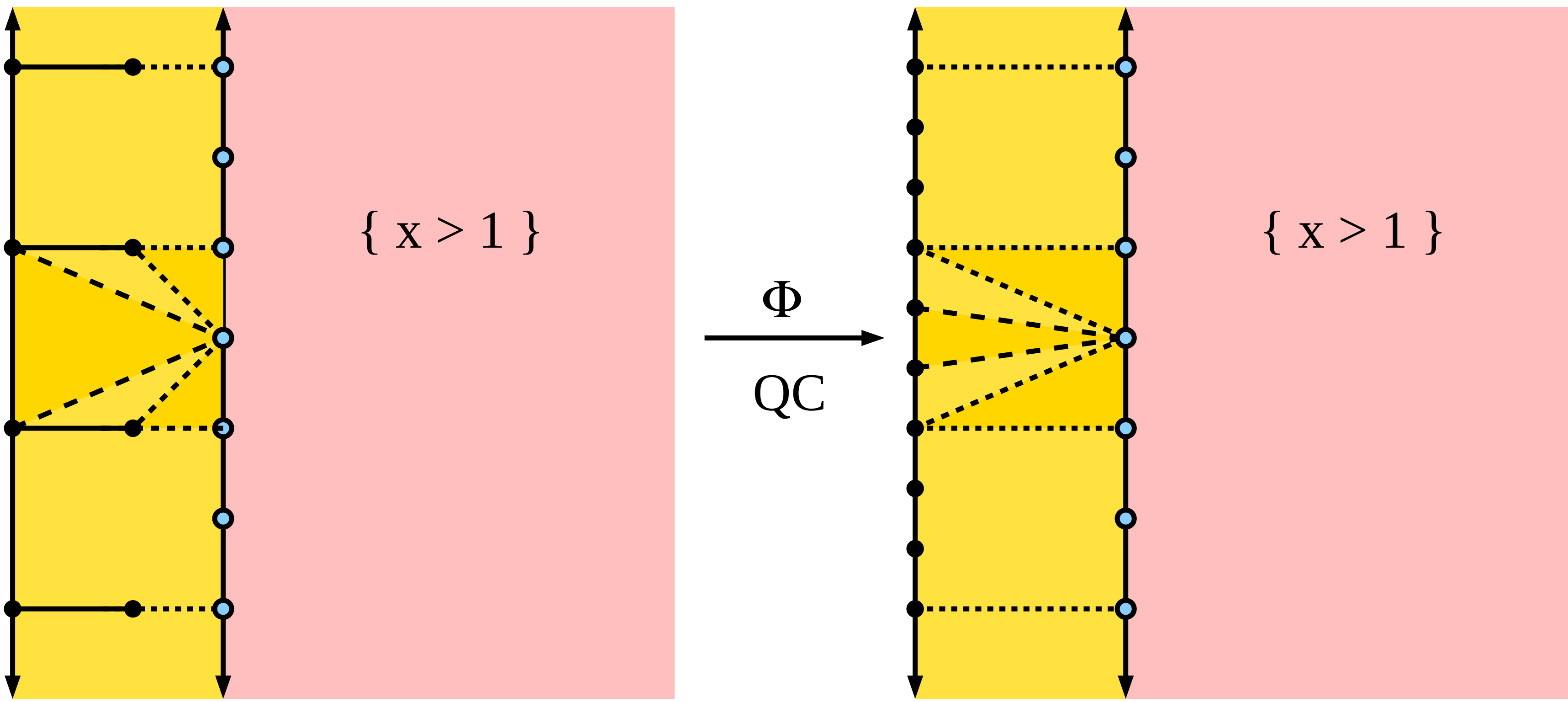
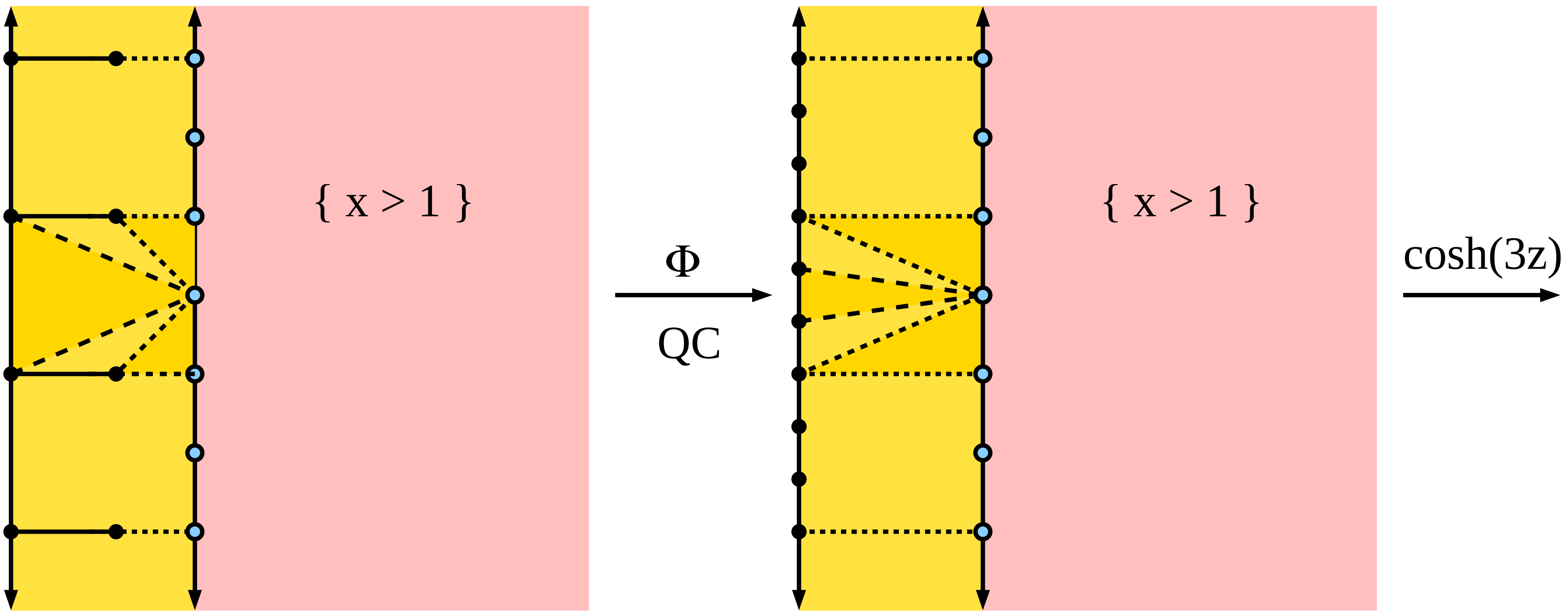
Triangulate the folded strip.



Let Φ be piecewise affine map: folded strip $\rightarrow$ un-folded strip.

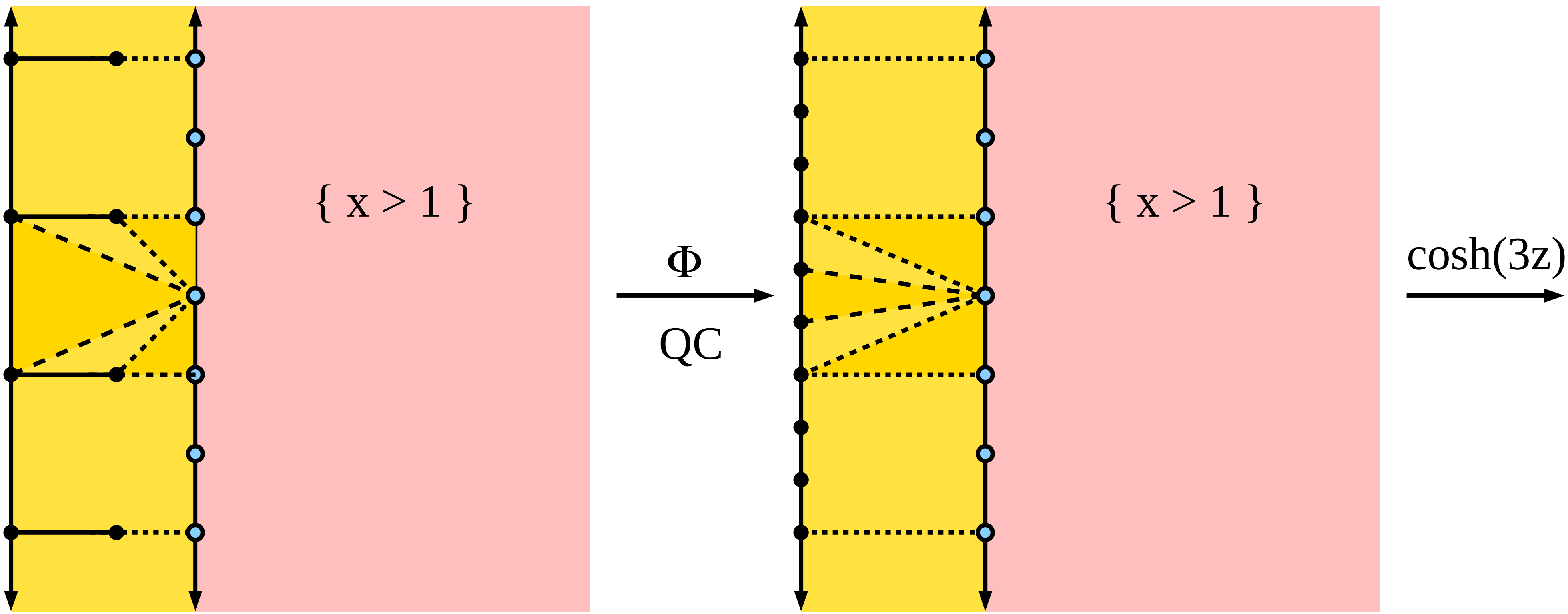
Take $\Phi =$ identity map off the strip.



$G(z) = \cosh(3 \cdot \Phi(z))$ satisfies:

- equals $\cosh 3z$ on $\{x \geq 1\}$
- equals $\cosh z$ on $\{x \leq 0\}$
- Is quasiregular (composition of QC and holomorphic)

Measurable Riemann mapping theorem gives entire $F = G \circ \varphi$.



$G(z) = \cosh(3 \cdot \Phi(z))$ satisfies:

- equals $\cosh 3z$ on $\{x > 1\}$
- equals $\cosh z$ on $\{x = 0\}$
- Is quasiregular (composition of QC and holomorphic)

Can replace 3, by any n and keep QC constant bounded.

There are many, many entire functions, so it is convenient to focus on special classes, e.g., exponential families.

Eremenko and Lyubich introduced a class that has nice structure, contains many common examples, but also allows a wide variety of new, geometrically defined examples.

• A. Eremenko and M. Lyubich, *Dynamical properties of some classes of entire functions*, Ann. Inst. Fourier 42 (1992), no. 4, 989-1020

• D.J. Sixsmith, *Dynamics in the Eremenko-Lyubich class*, Conform. Geom. Dyn. 22 (2018), 185-224.

$$
\begin{aligned}
\textbf{Singular set} \quad &= \quad \text{closure of critical values and finite asymptotic values} \\
&= \quad \text{smallest set so that } f \text{ is a covering map onto } \mathbb{C} \setminus S
\end{aligned}
$$

Singular set = closure of critical values and finite asymptotic values
= smallest set so that f is a covering map onto $\mathbb{C} \setminus S$

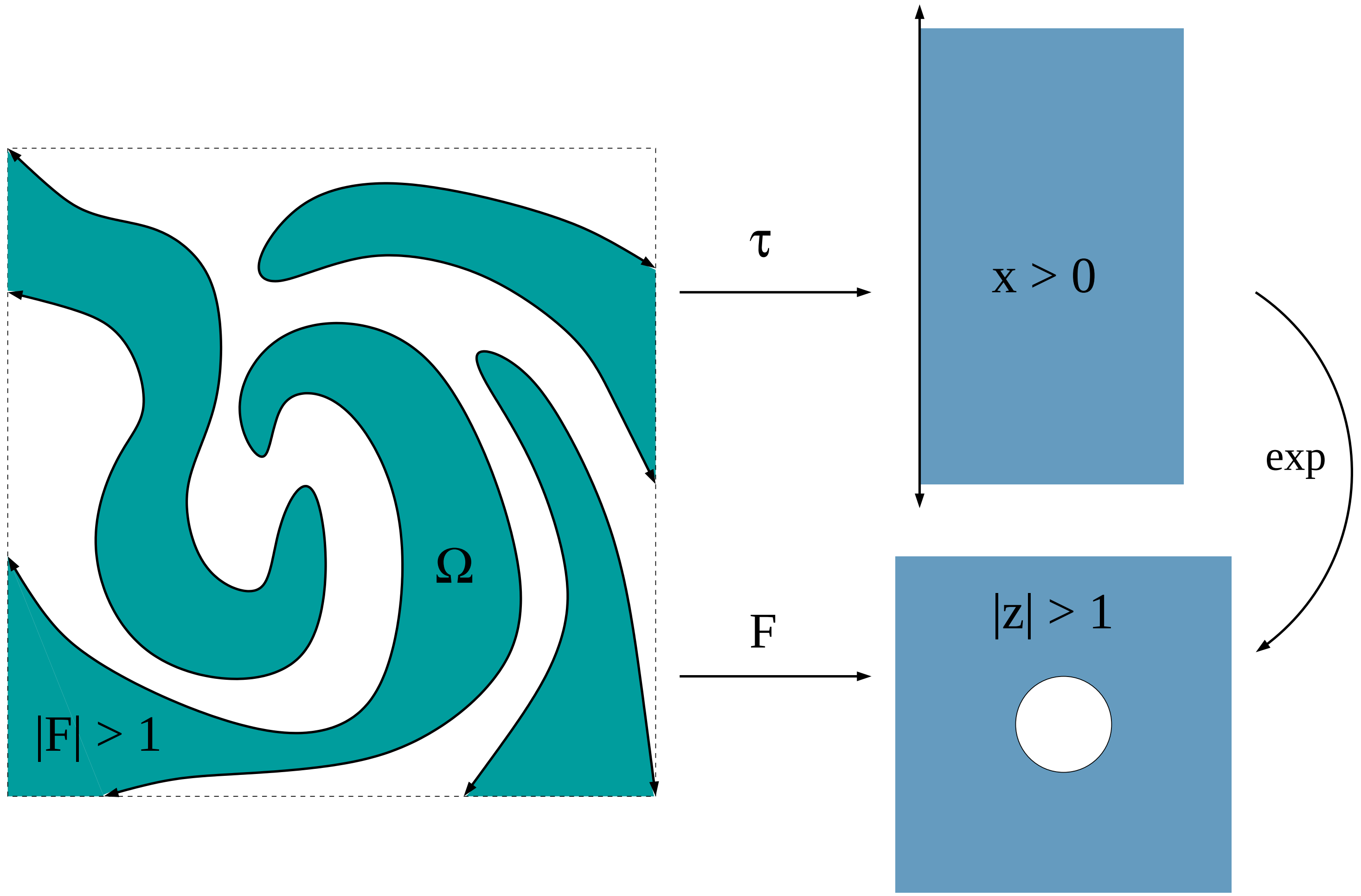
Eremenko-Lyubich class = bounded singular set = $\mathcal{B}$

Speiser class = finite singular set = $\mathcal{S} \subset \mathcal{B}$

Suppose $F \in \mathcal{B}$ and $S(F) \subset \mathbb{D} = \{|z| < 1\}$.

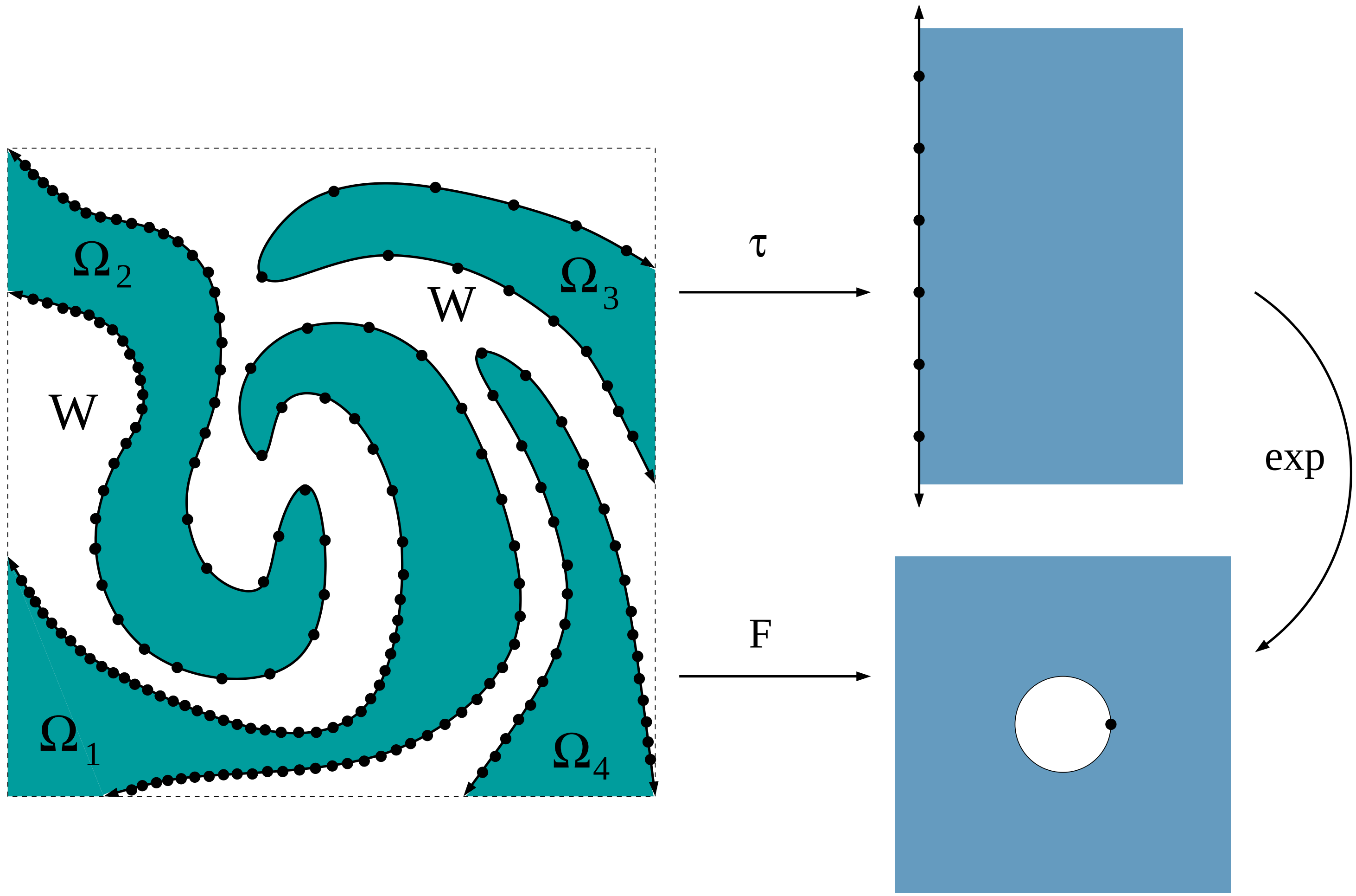
$\Omega = \{|F| > 1\}$ has simply connected components, called **tracts**

$W = \mathbb{C} \setminus \overline{\Omega} = \{|F| < 1\}$ is connected, simply connected



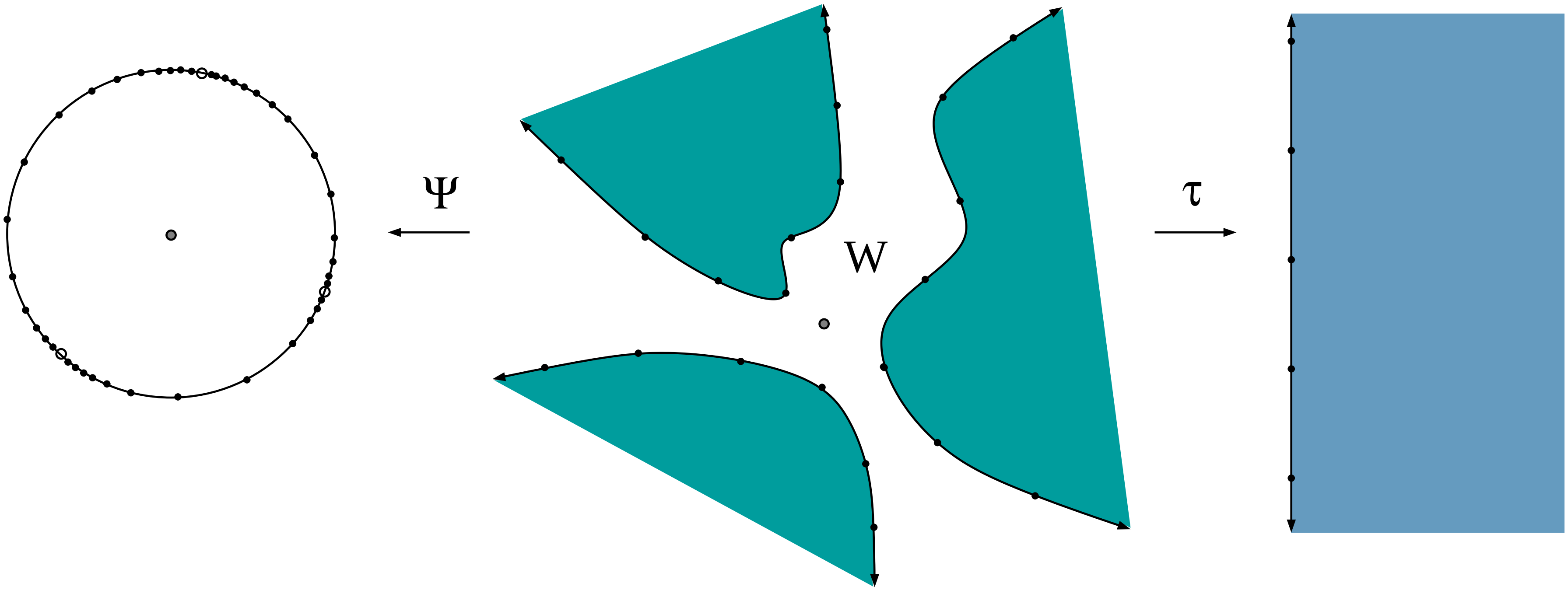
F is a covering map $\Omega \to \mathbb{D}^* = \{|z| > 1\}$, $F = \exp \circ \tau$.

τ is conformal from each tract to $\mathbb{H}_r$ = right half-plane



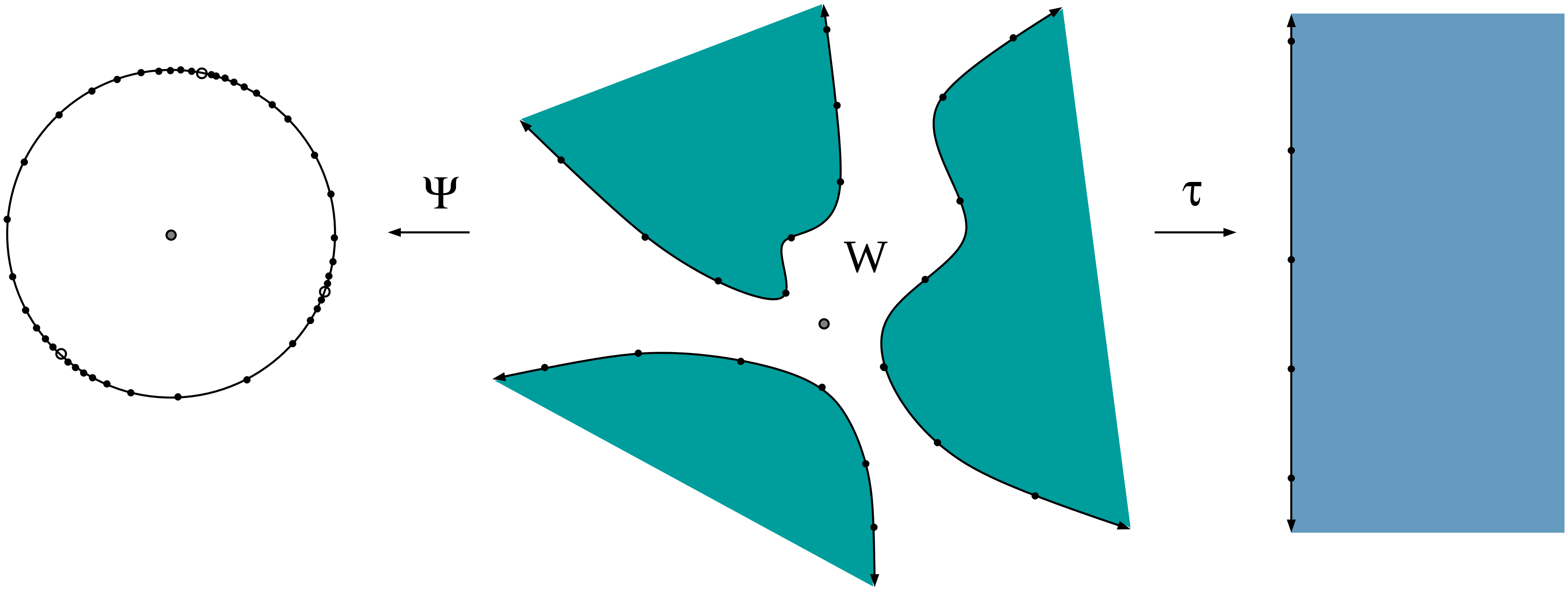
Dots $=$ F pre-images of $1 = \tau^{-1}(2\pi i\mathbb{Z})$

Tract boundaries have natural partition into arc via τ.



Let $\Psi : W \to \mathbb{D}$ be conformal map

Then $G = F \circ \Psi^{-1} : \mathbb{D} \to \mathbb{D}$ is an inner function.



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Every Eremenko-Lyubich function is a built from conformal maps on the tracts "glued together" by an inner function between the tracts.

Can we reverse this? Build entire function given only Ω and τ?

A **model** is a pair (Ω, τ) where

- $\Omega = \cup \Omega_j$ is a disjoint union of unbounded Jordan domains
- τ is conformal from each Ω_j to $\mathbb{H}_r$ $(\infty \to \infty)$.

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Obviously impossible:

(1) violates unique continuation.
(2) $\partial\Omega$ must be analytic if F is.

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Obviously impossible:

(1) violates unique continuation. **Use QR function.**
(2) $\partial\Omega$ must be analytic if F is. **Use conformal approximation.**

Theorem: Suppose (Ω, τ) is a model and $\rho > 0$. Define

$$\Omega(\rho) = \tau^{-1}(\{x + iy : x > \rho\}) \subset \Omega.$$

Then there is a quasiregular g so that

(1) $g = e^{\tau}$ on $\Omega(\rho)$,

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(1) $g = e^{\tau}$ on $\Omega(\rho)$,

(2) $|g| \leq e^{\rho}$ off Ω.

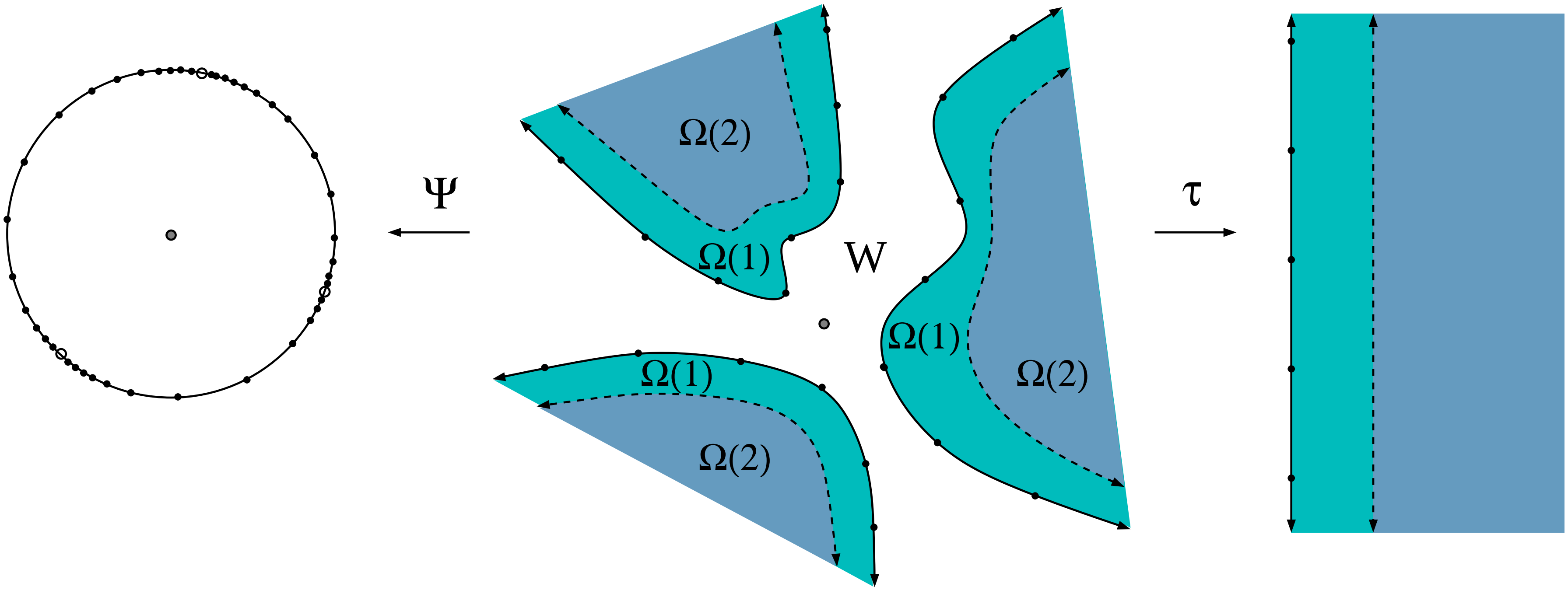
The QR constant depends on ρ, but not on Ω.

There is a quasiconformal φ so that $f = g \circ \varphi \in \mathcal{B}$.

g is holomorphic except on $\Omega(\rho) \setminus \Omega(\rho/2)$ (often has finite area).

Tracts of f correspond to components of Ω. Very similar shapes.

Can take $f \in \mathcal{S}$ if we drop (2); twice as many tracts may be needed.

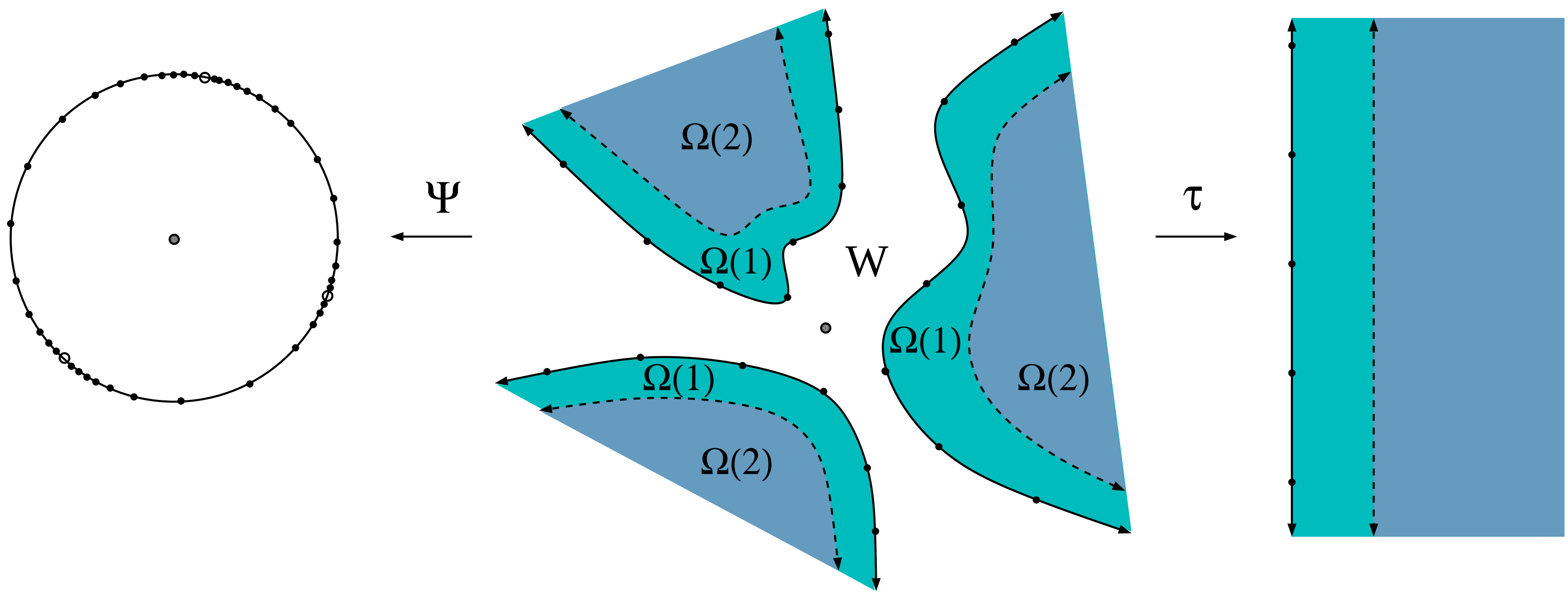


Construction of EL functions from models (Ω, τ):

- Set $F = e^{\tau}$ on Ω.

- Consider $\rho = 1$. Partition $\Gamma = \partial W = \partial\Omega(1)$ by $F^{-1}(1)$.

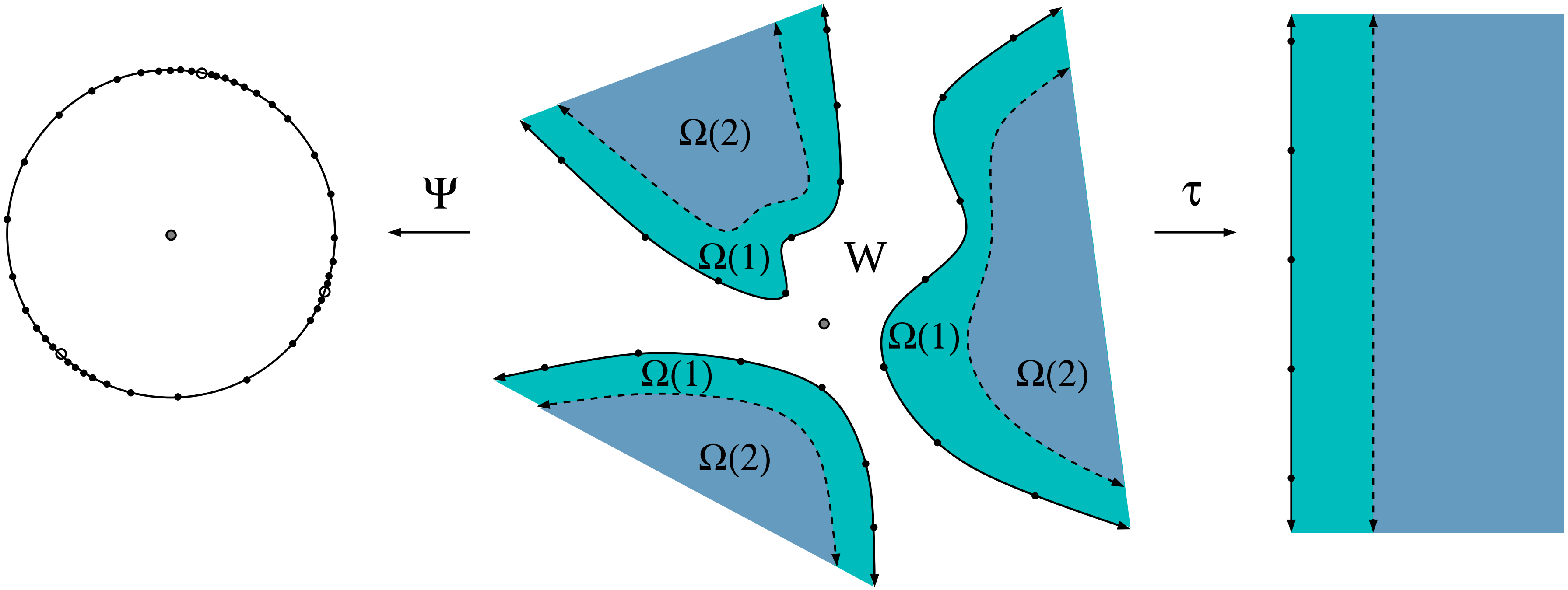
- Adjacent arcs have comparable sizes.



- Recall $\Psi : W \to \mathbb{D}$ is conformal. Set $E = \Psi(\infty)$

- Partition of Γ maps to partition of $\mathbb{T} \setminus E$ via Ψ. Set $Z_\tau = \Psi(F^1(1))$.

- For g inner with singular set E, $Z_g = g^{-1}(1)$ also partitions $\mathbb{T} \setminus E$.



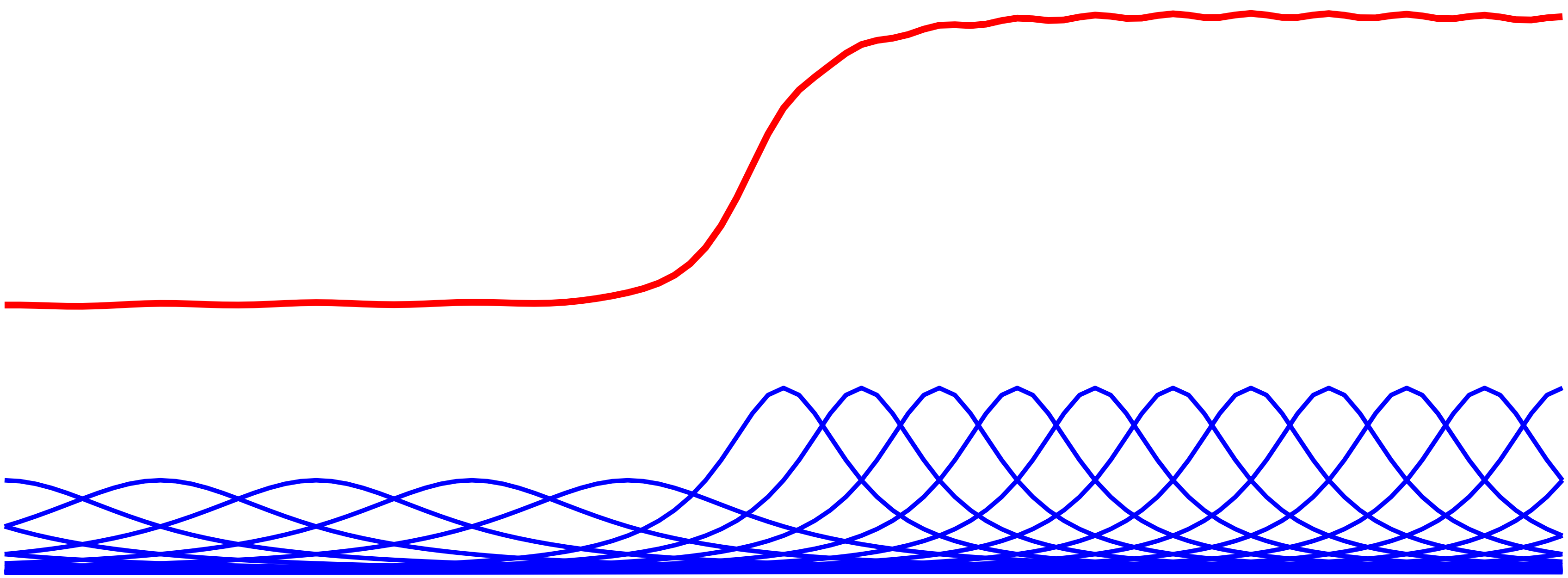
- Ideally, we want the two partitions to be the same.
- **Main Step:** construct g so $Z_g \simeq Z_\tau$ (but want Z_τ "denser").
- Each pair in $Z\tau$ is separated at most 1 point in Z_g
- Each pair in Z_g is separated by a bounded number of points in Z_τ.

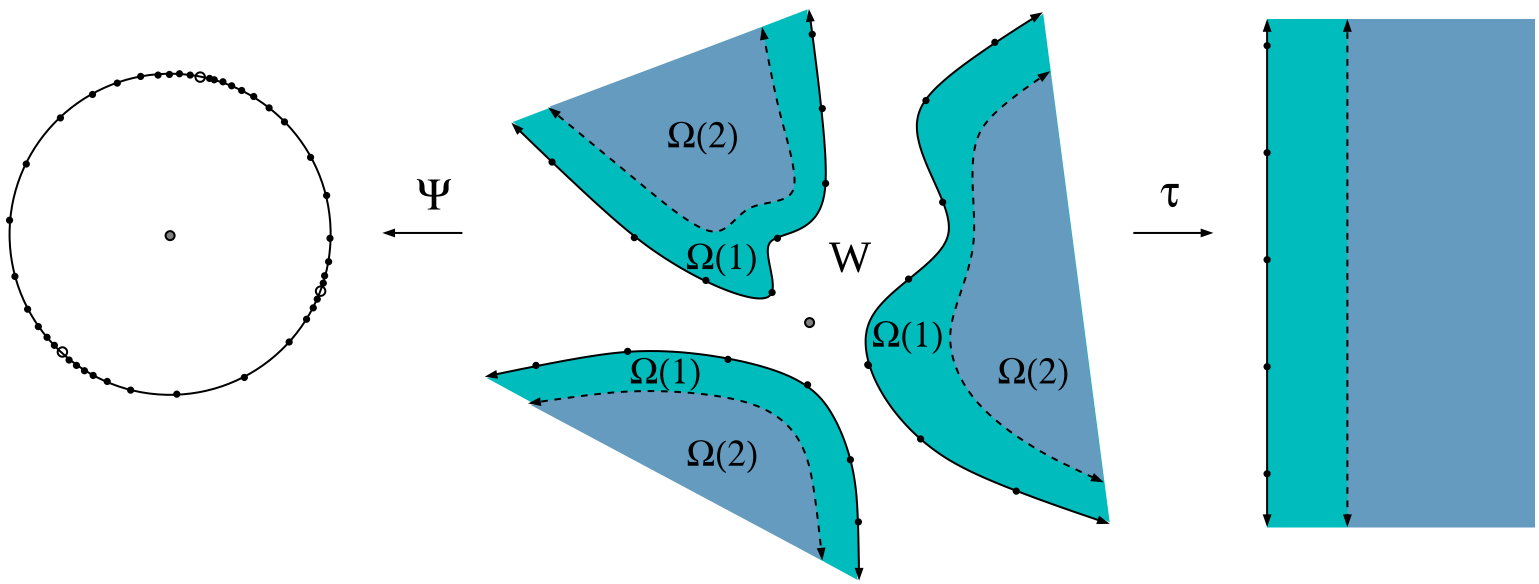
- We can do this using an infinite Blaschke product

$$B(z) = \prod \frac{z - z_n}{1 - \overline{z_n} z}.$$

Place one zero near center of each desired partition element.

Idea: $|B'|$ is sum of Poisson kernels.





Now we have two holomorphic functions:

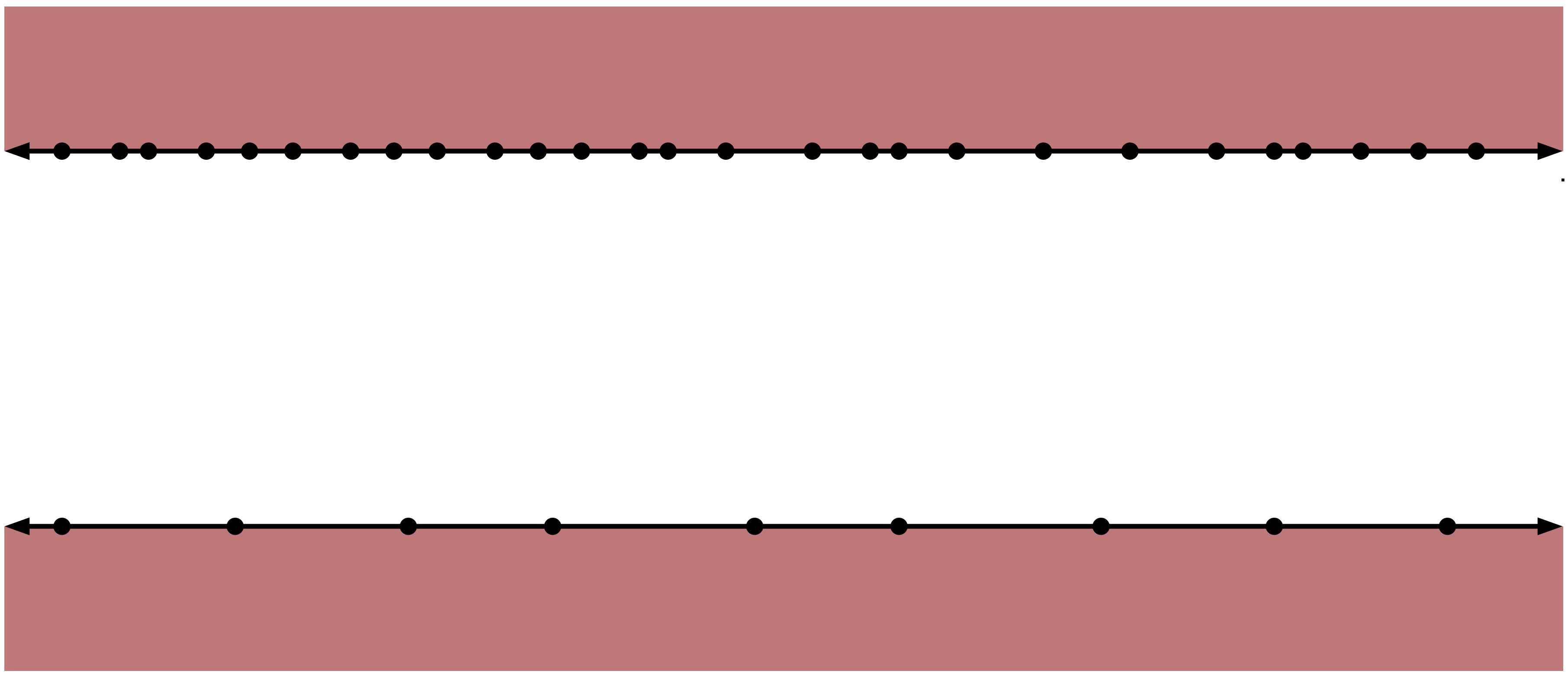
(1) $\exp \circ \tau$ on $\Omega(\rho)$

(2) $g \circ \Psi$ on $W(\rho/2)$.

Both have boundary values in unit circle.

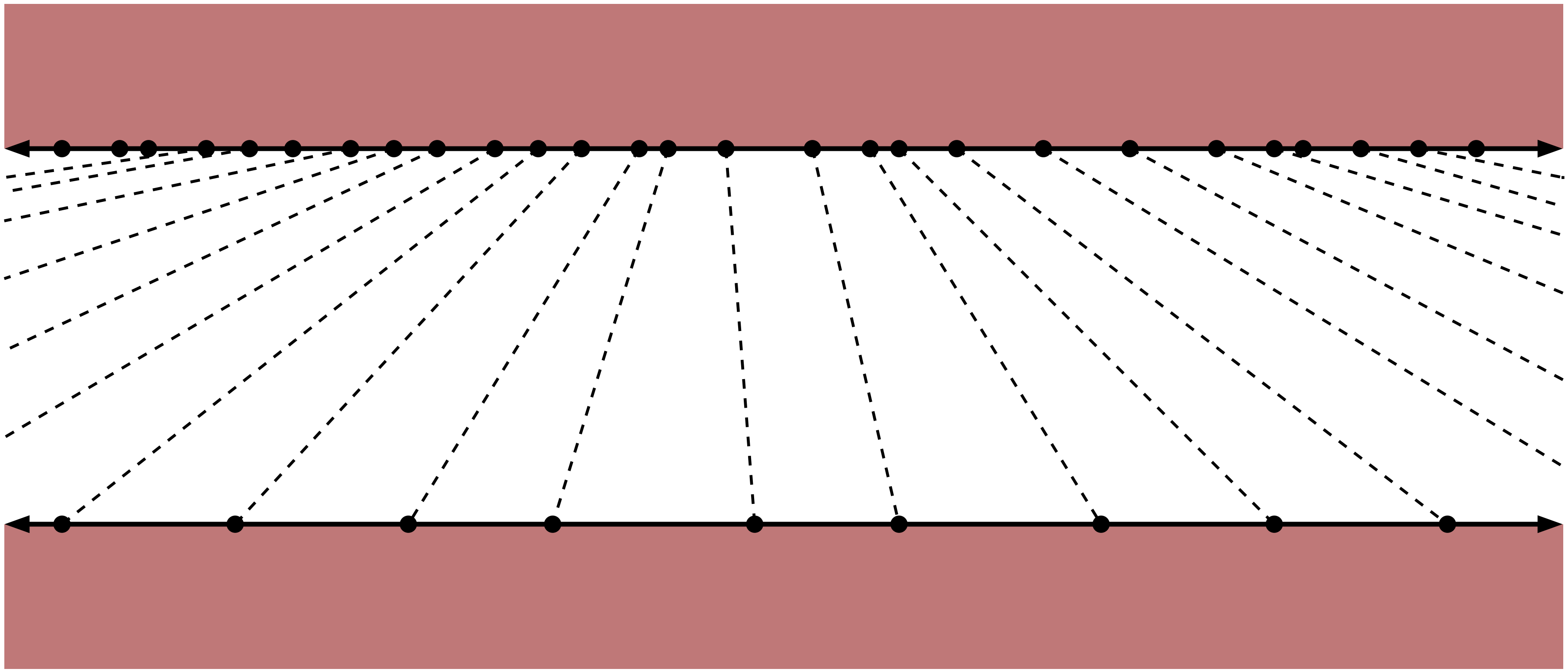
Interpolate these two functions in $\Omega(\rho) \setminus \Omega(\rho/2)$.

This is a "strip-like" neighborhood of Γ.

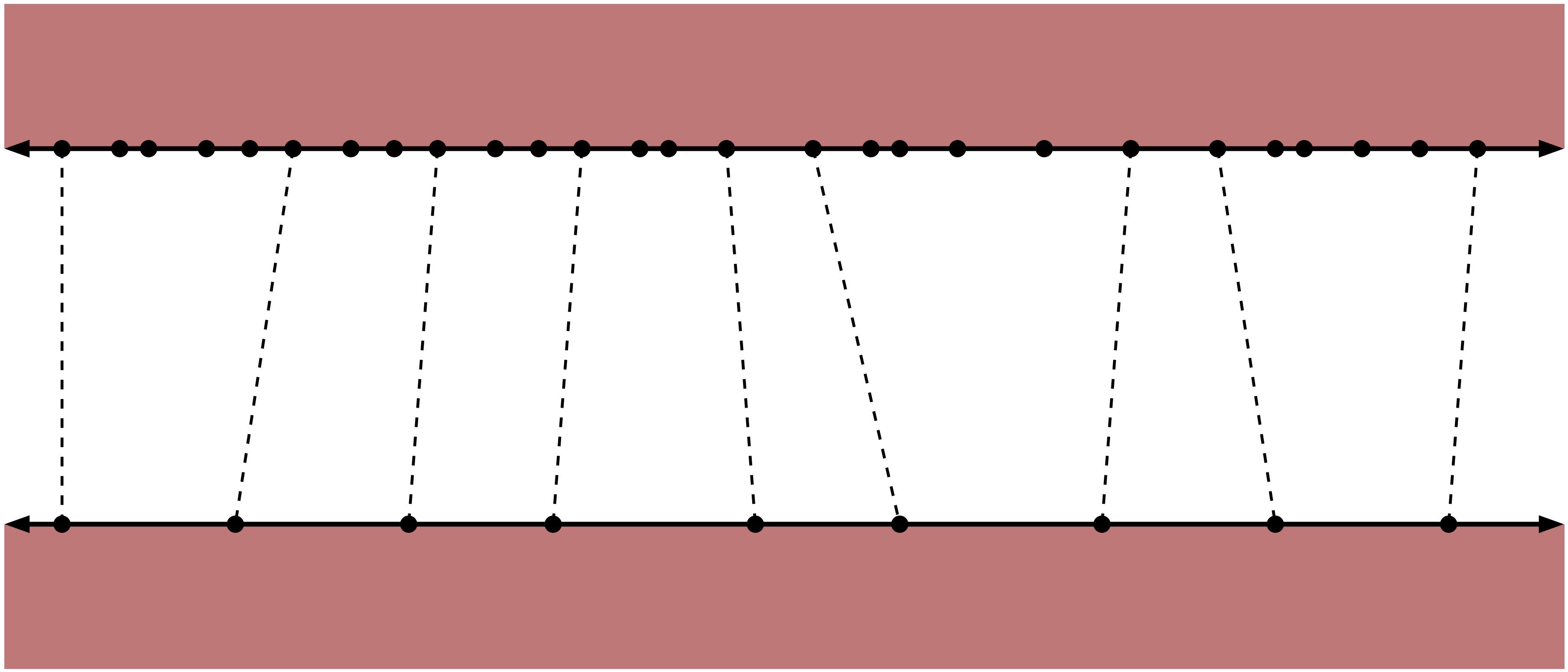


We want to glue together roughly similar partitions.

By construction, Z_τ more dense than Z_g, but boundedly so.

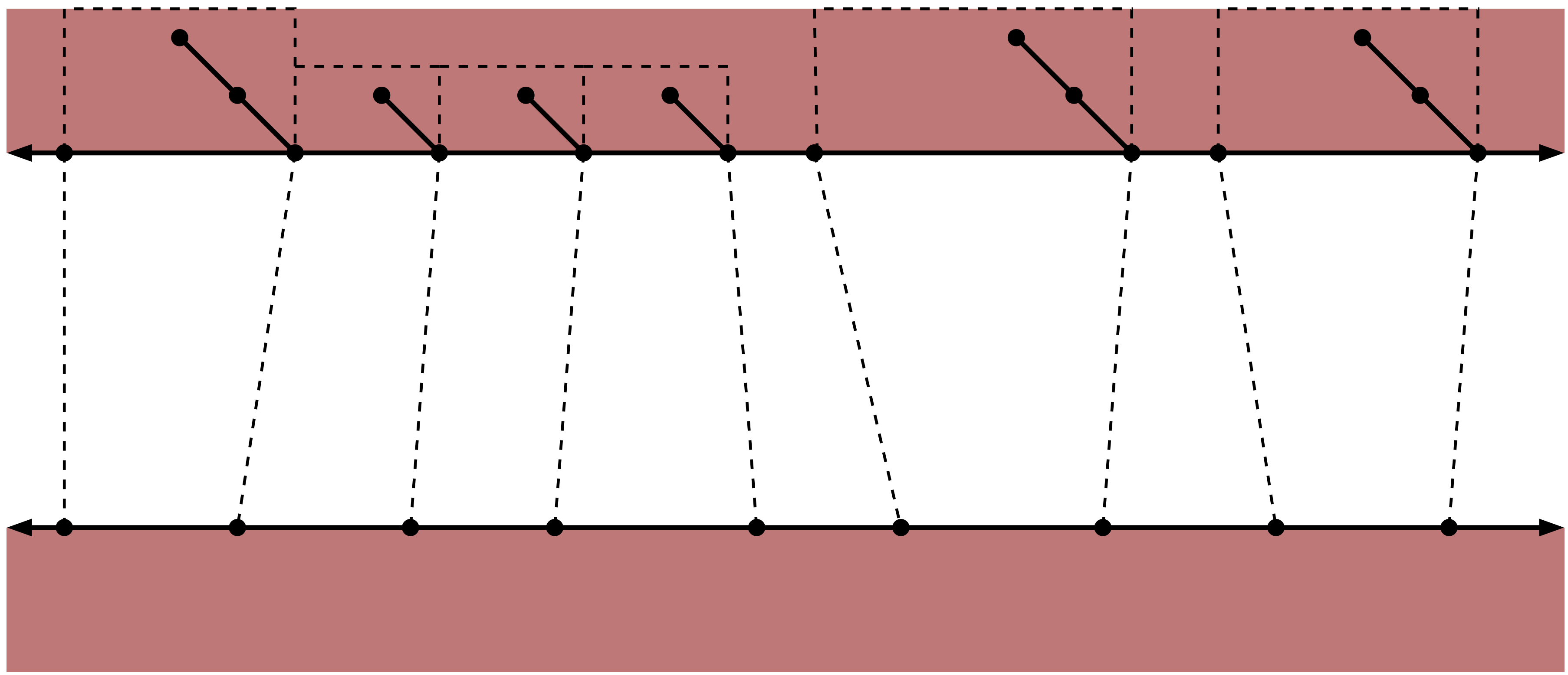


Identifying in obvious way, matched points might become far apart.



So identity all inner points with some outer points that are nearby.

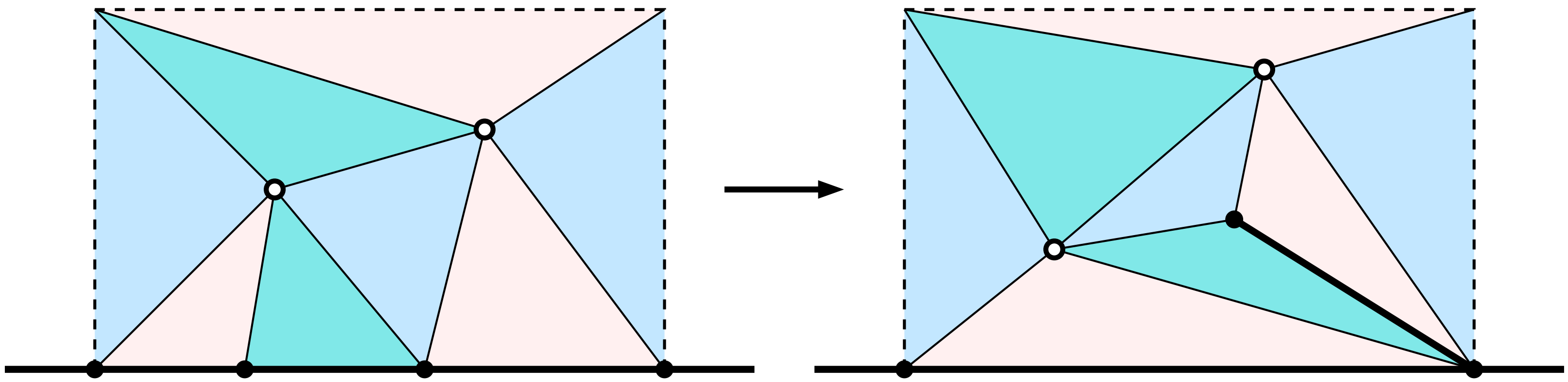
What to do with unmatched points?



Match leftover points with each other by folding boundary.

Folding map is a QC map of region above strip to itself.

Equals the identity outside dashed boxes.



The folding can be done in a QC way by piecewise linear maps

The maximum dilatation depends on the number of edges being folded.

This number was chosen to be uniformly bounded.

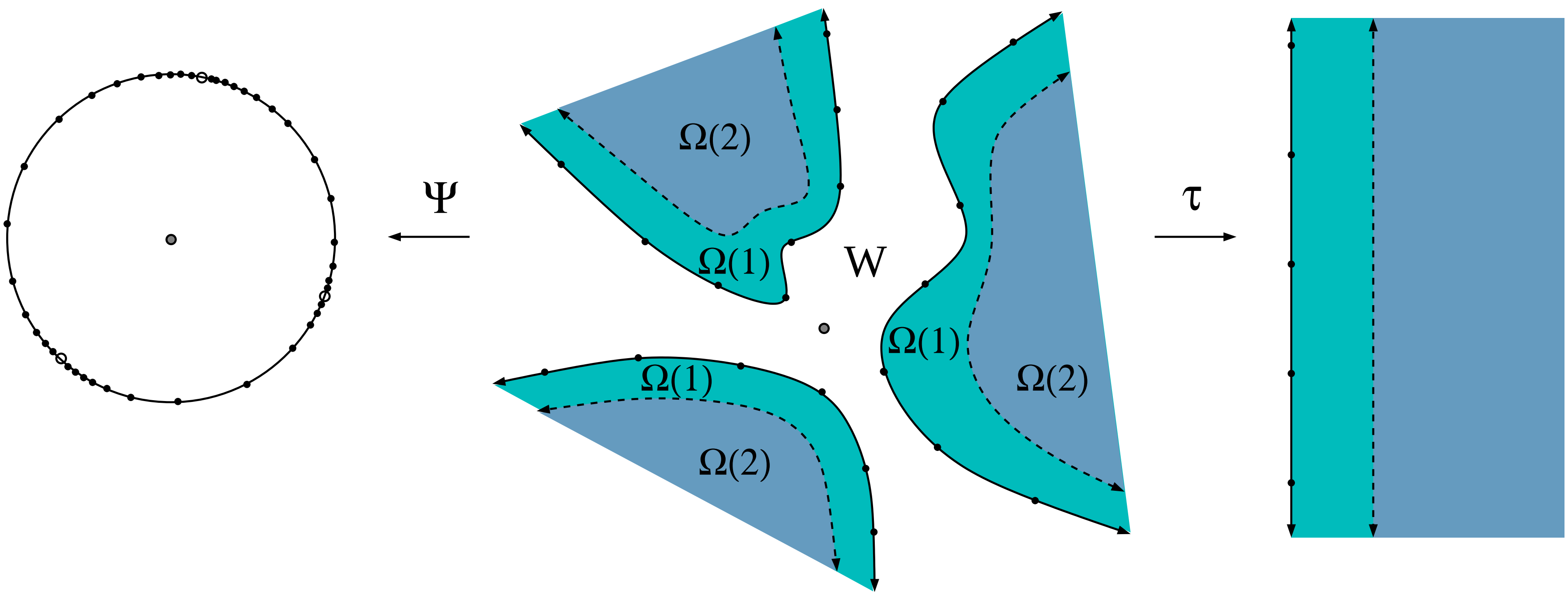
This gives a QR function with all desired properties.

Holomorphic except near $\Gamma = \partial\Omega$. QC correction often close to identity.

Only critical points are due to the inner function or the folding.

- Inner function CV's are in $\mathbb{D}$. Possibly infinitely many.
- Folding CV's are ± 1.

Result map is in Eremenko-Lyubich class.



How do we improve from bounded singular set to finite set?

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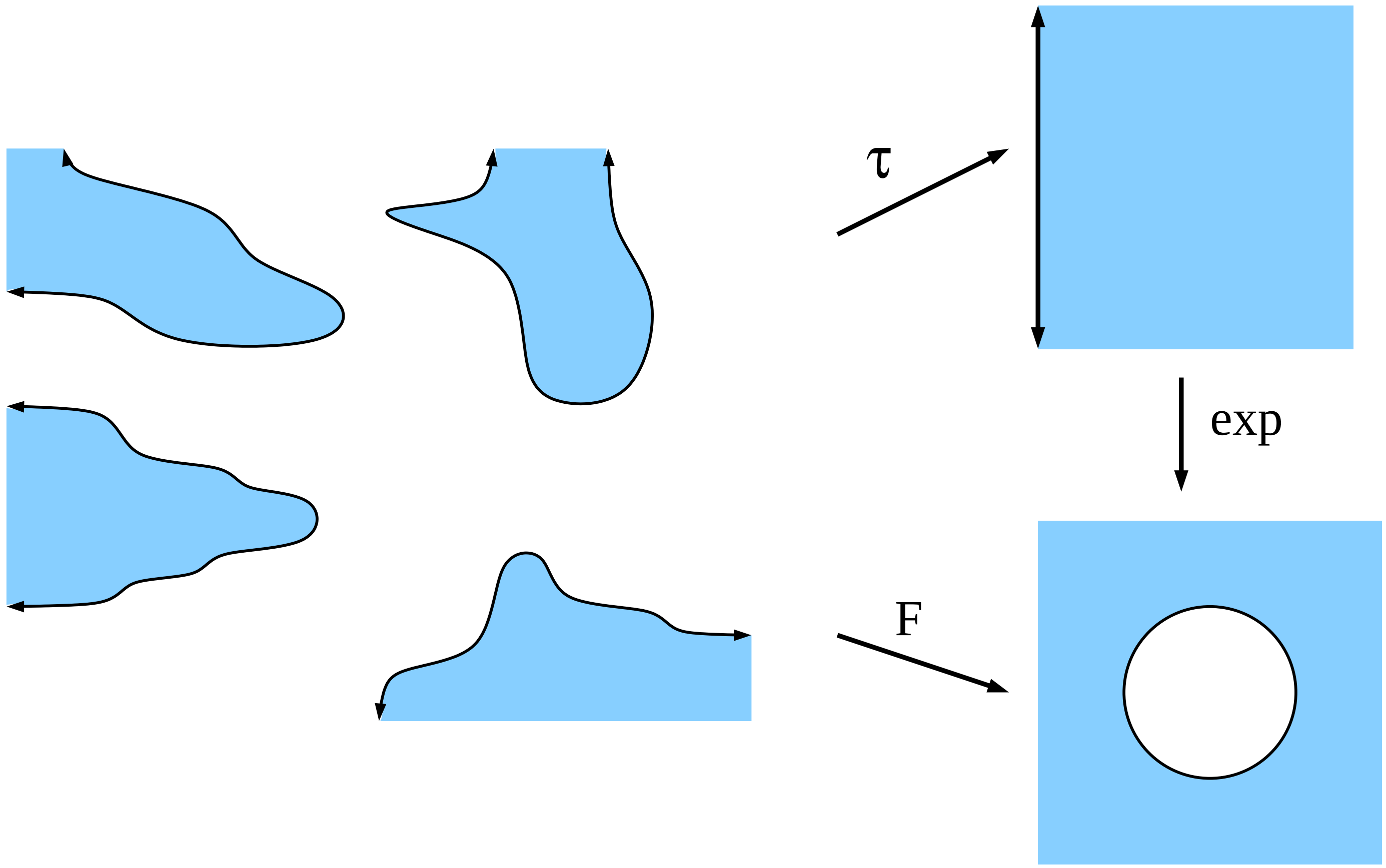
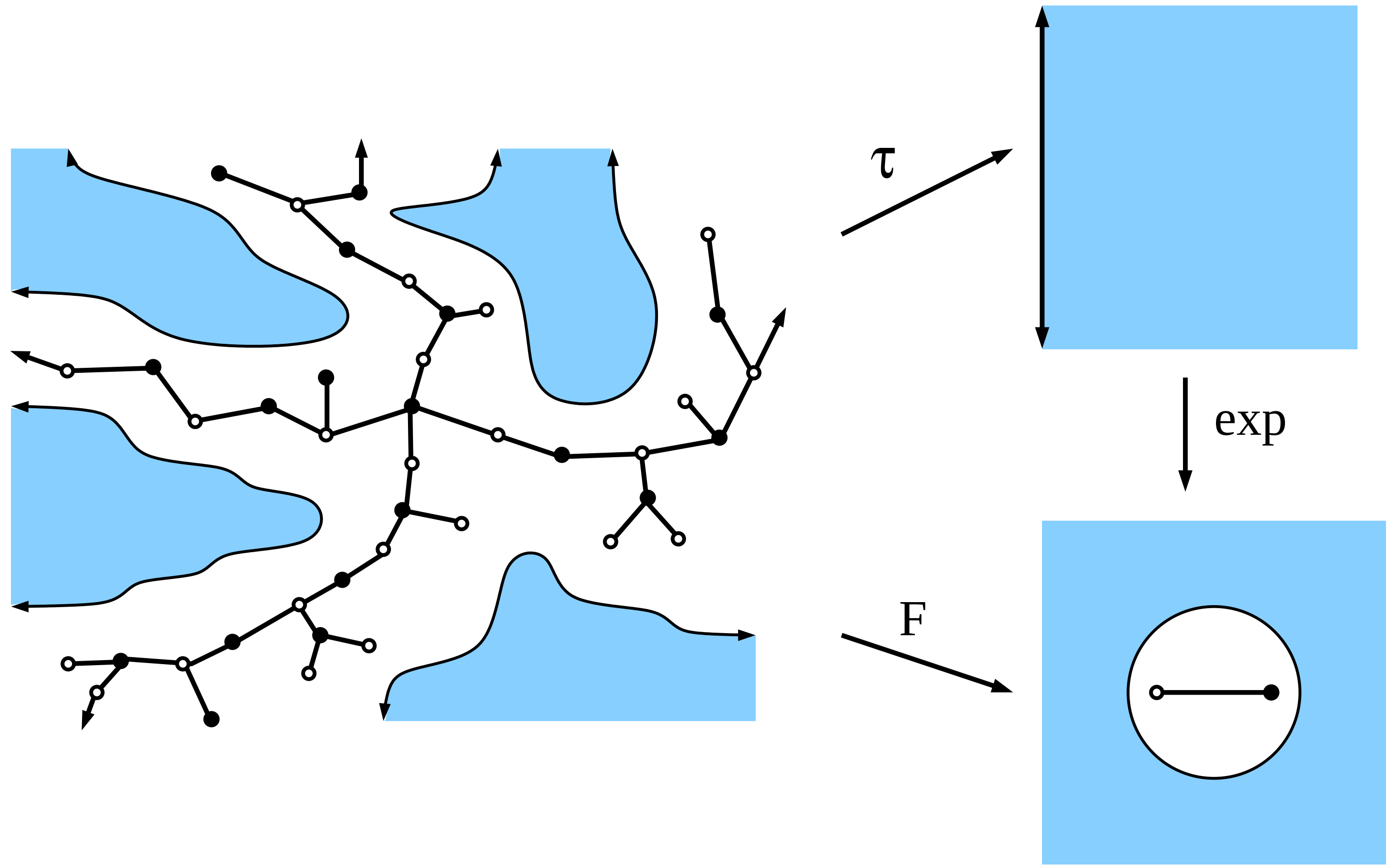


Eremenko-Lyubich function maps tracts to $\{|z| > 1\}$.

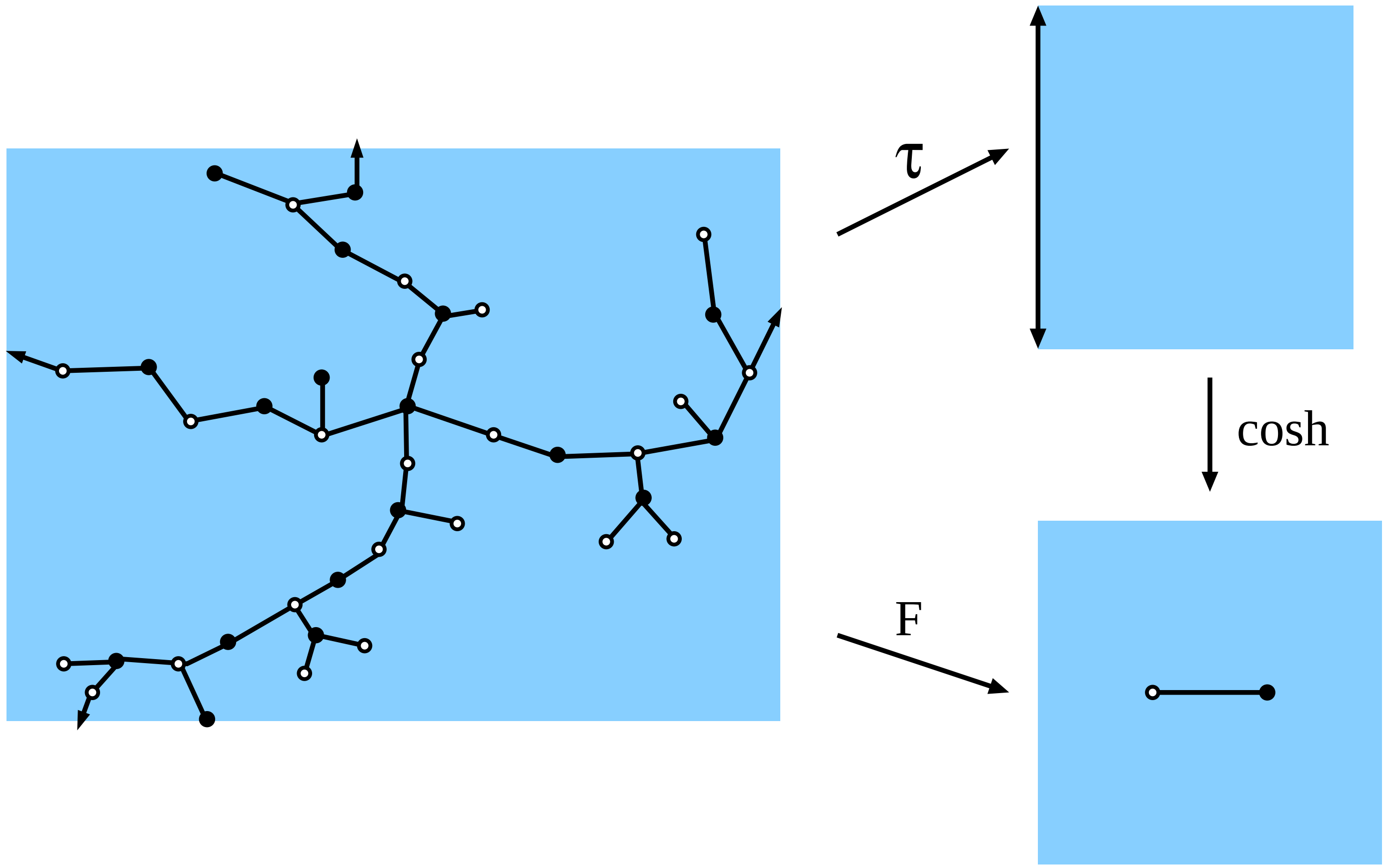
How do we improve from bounded singular set to finite set?



Suppose there are only two critical values, ± 1.

Connect by segment. Segment lifts to a tree.

How do we improve from bounded singular set to finite set?

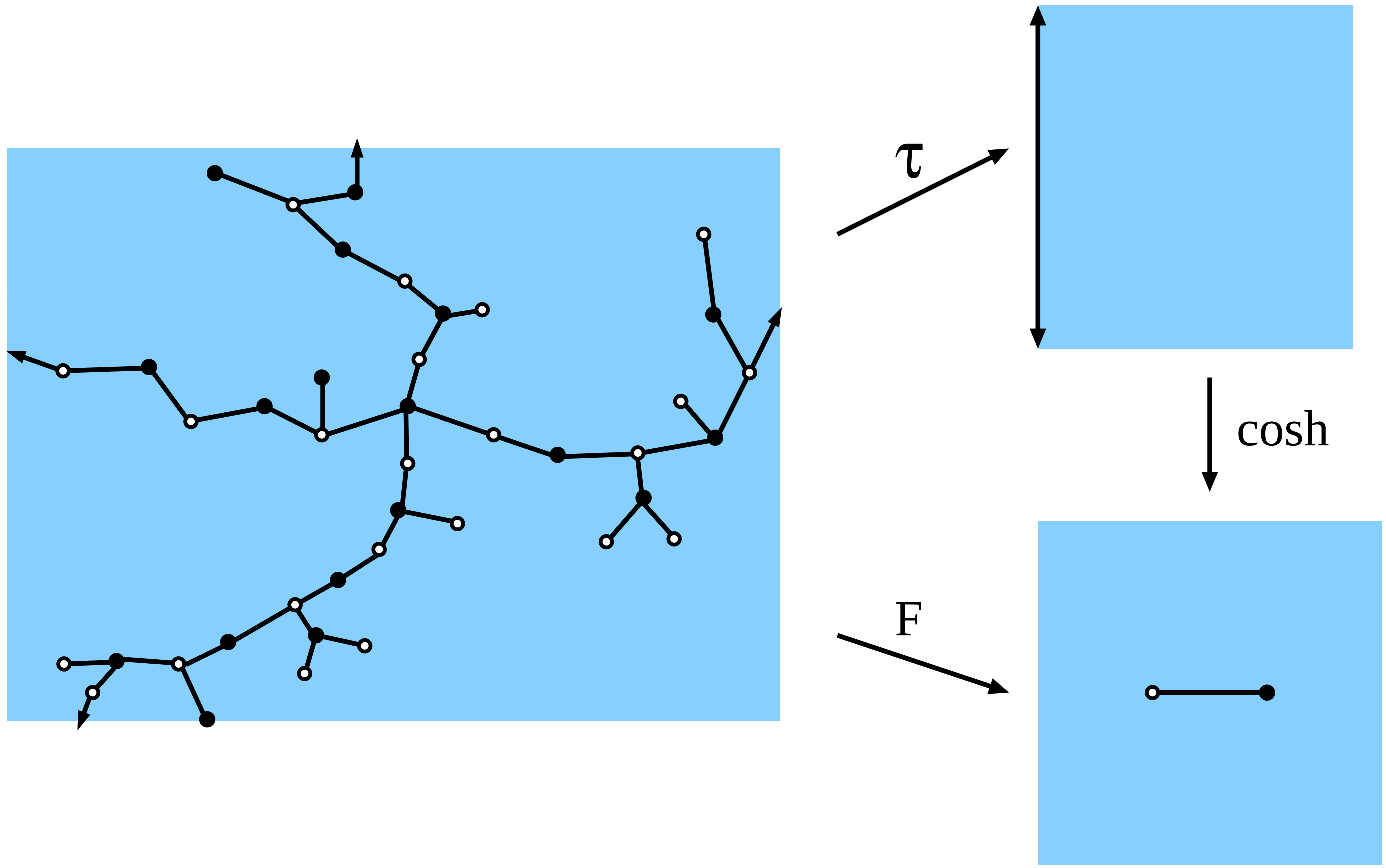


Let complementary components of tree be tracts.

$F(z) = \cosh(\tau(z))$; maps to complement of $[-1, 1]$.

All Speiser functions give trees. Do all trees give functions?

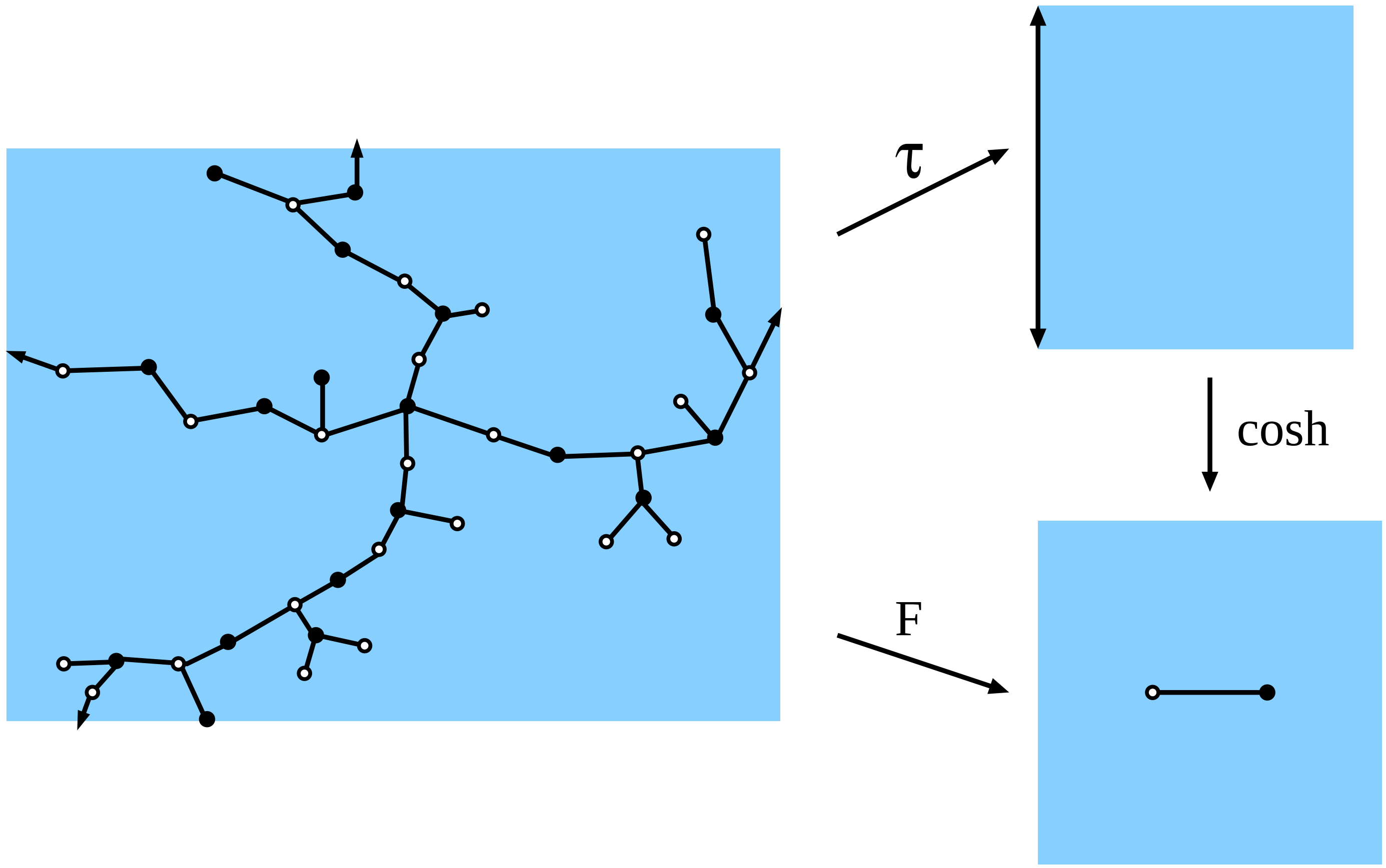
How do we improve from bounded singular set to finite set?



Given tree T, e^{τ} need not be continuous across T.

Can we replace e^{τ} by QR function in a neighborhood of tree?

How do we improve from bounded singular set to finite set?



Yes. Need some definitions:

- Appropriate neighborhood of a tree (adapted Hausdorff nbhd)
- Some edge smoothness.
- Want tree edges to be unions of τ-intervals.

If e is an edge of T and $r > 0$ let

$$e(r) = \{z : \mathrm{dist}(z, e) \leq r \cdot \mathrm{diam}(e)\}$$

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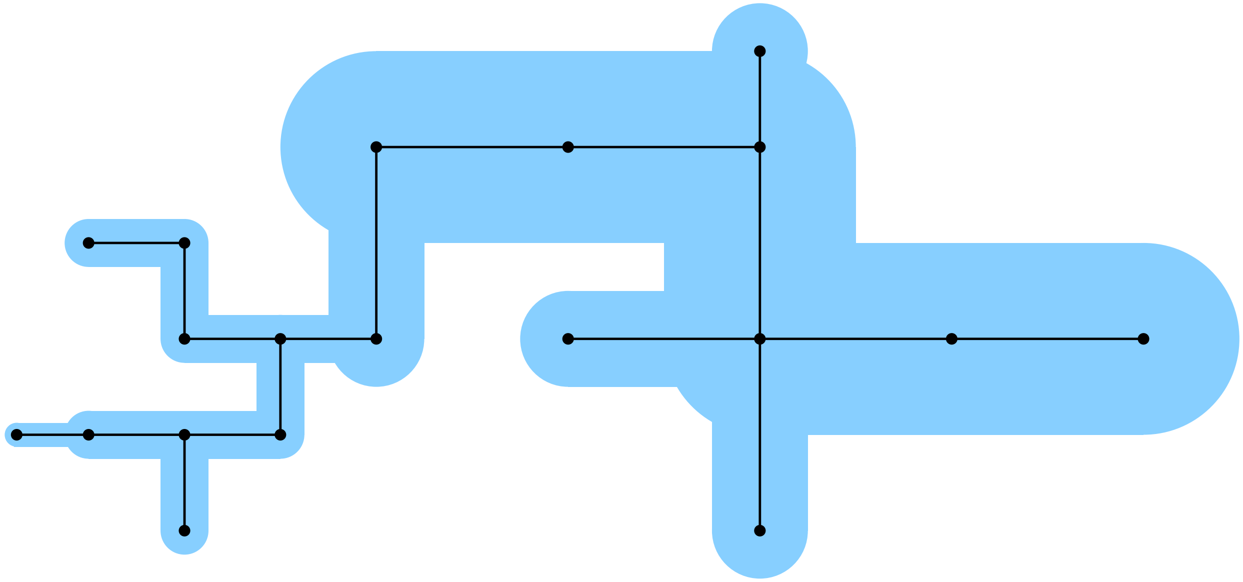
Define neighborhood of T: $T(r) = \cup\{e(r) : e \in T\}$.



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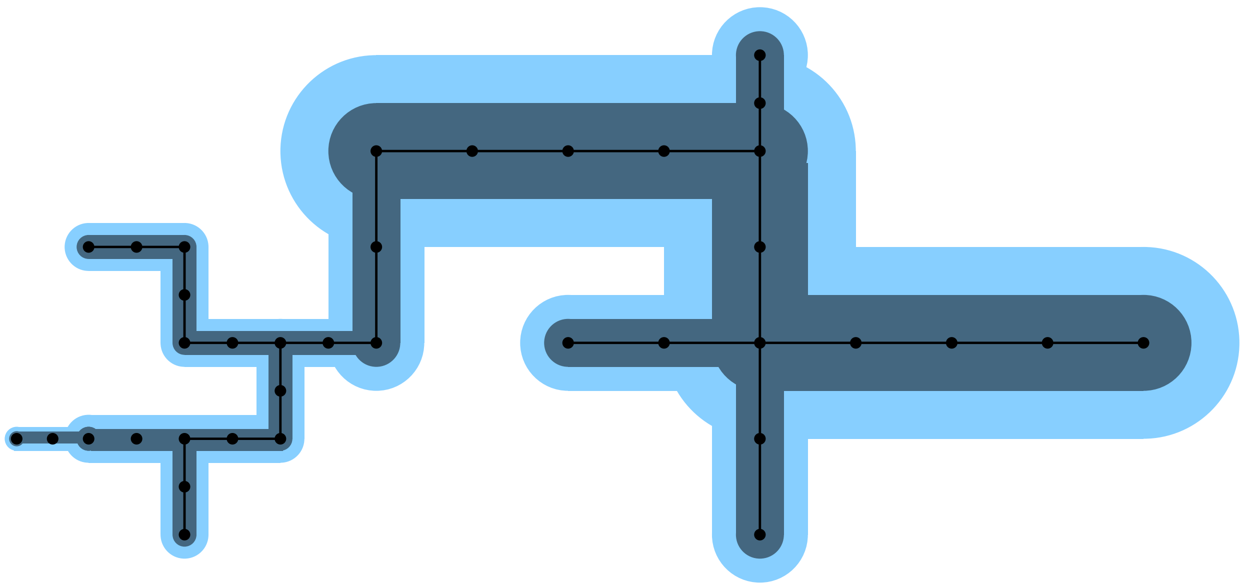
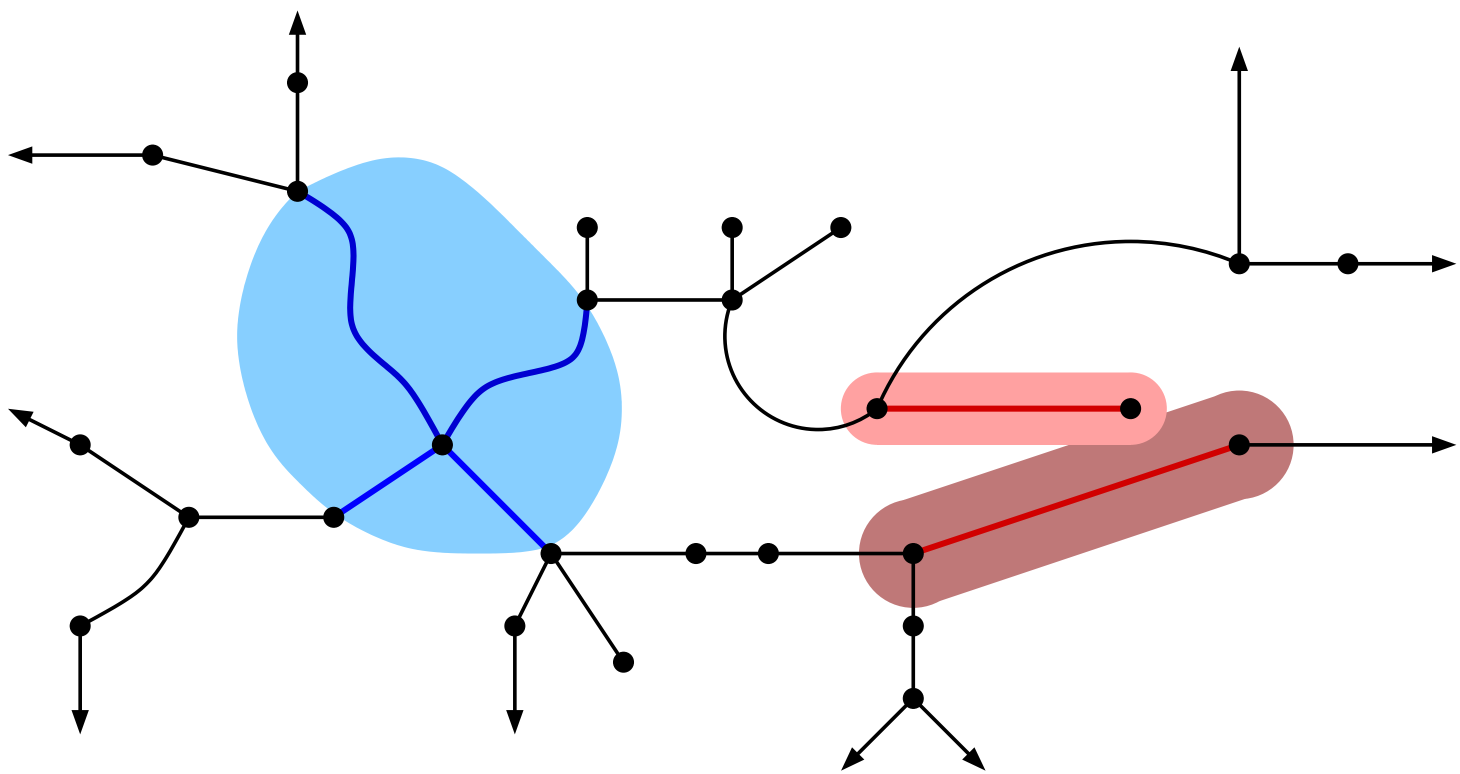


Adding vertices reduces $T(r)$. Useful scaling property.

Bounded Geometry (local condition; easy to verify):

- edges are uniformly smooth.
- adjacent edges form bi-Lipschitz image of a star $= \{z^n \in [0, r]\}$
- non-adjacent edges are well separated,

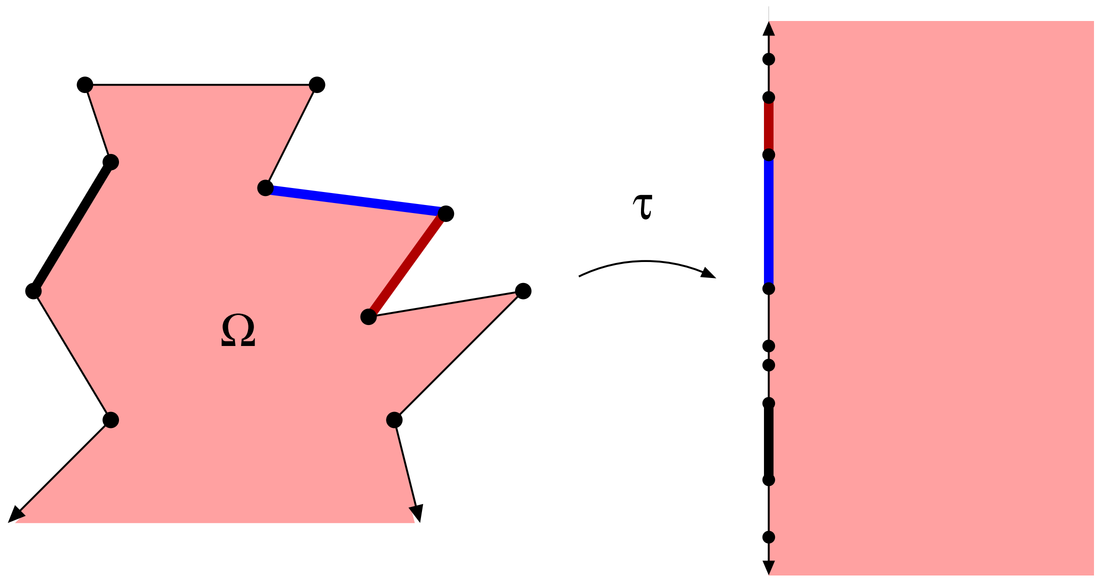
$$\mathrm{dist}(e, f) \geq \epsilon \cdot \max(\mathrm{diam}(e), \mathrm{diam}(f)).$$



τ-Lower Bound (global condition; harder to check):

Complementary components of tree are simply connected.

Each can be conformally mapped to right half-plane. Call map τ.



Assume all images have length $\geq \pi$.

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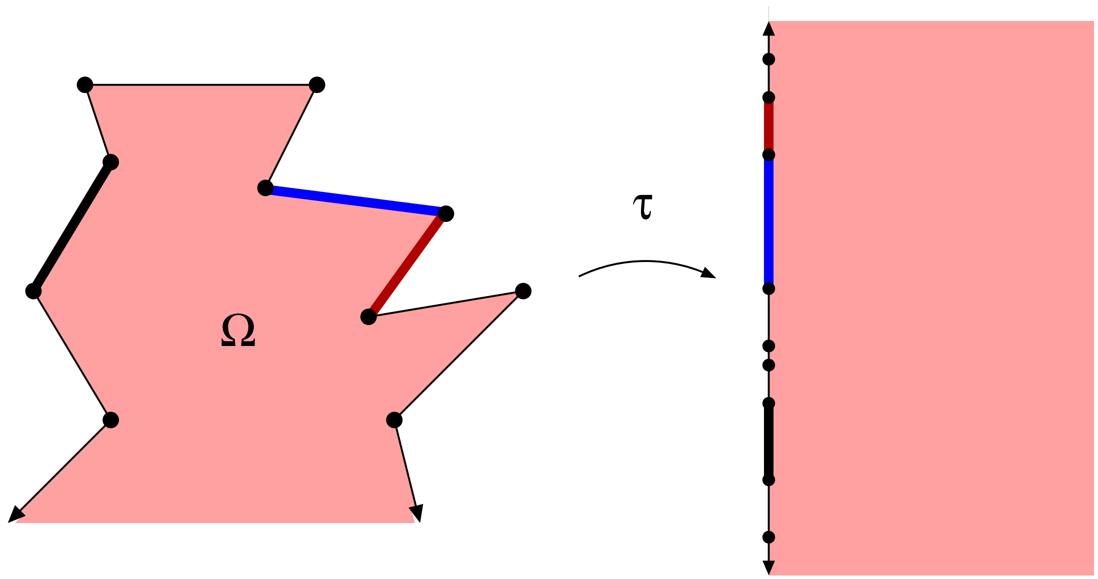


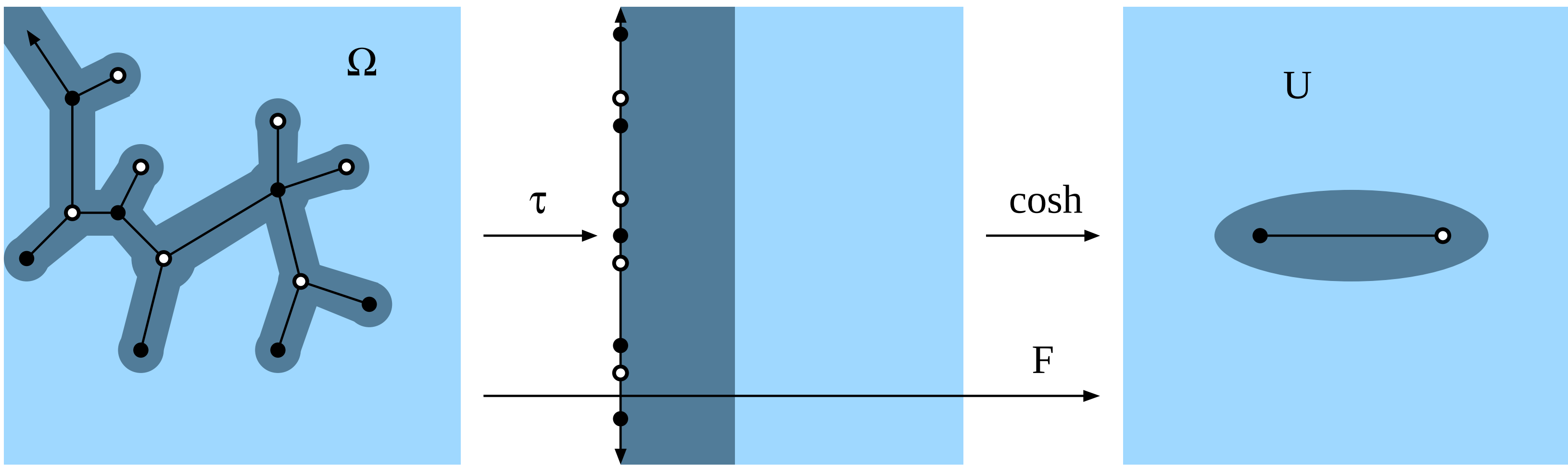
Assume all images have length $\geq \pi$.

We want τ-vertices ($\tau^{-1}(i\pi\mathbb{Z})$) to be "denser" then tree vertices.

Ideally, tree edges should be unions of τ-edges.

Don't need comparable density; just a lower bound.

QC-Folding Theorem: If T has bounded geometry and the τ-lower bound, then there is a K-quasiregular g and $r > 0$ such that $g = \cosh \circ \tau$ off $T(r)$ (shaded) and $\mathrm{CV}(g) = \pm 1$.



K and r only depend on the bounded geometry constants.

$g = F$ on light blue.

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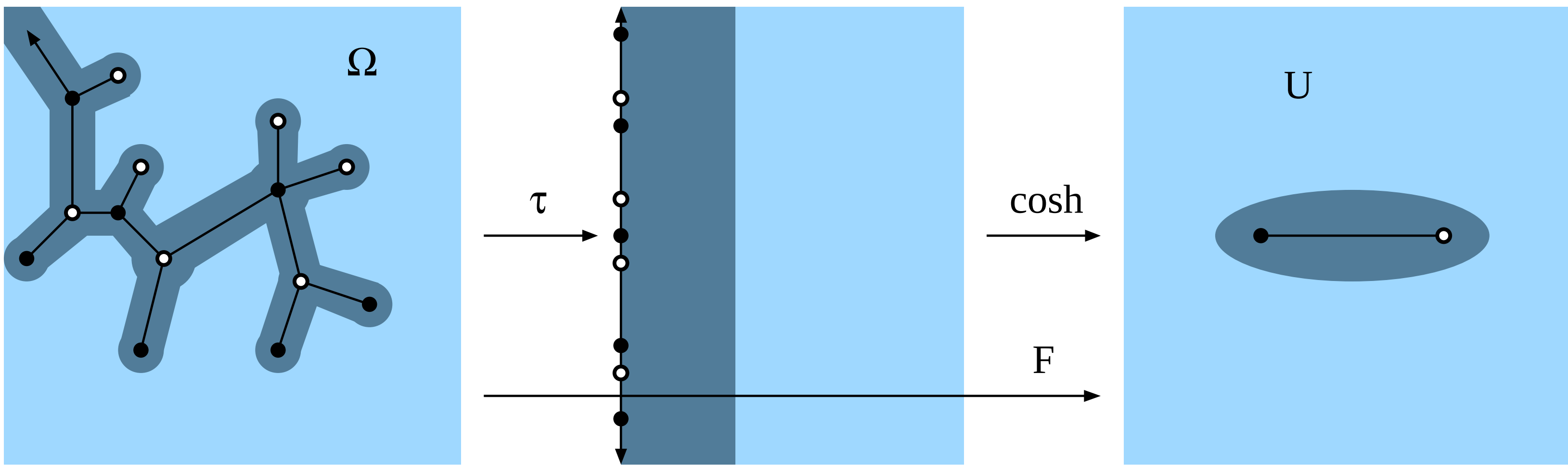

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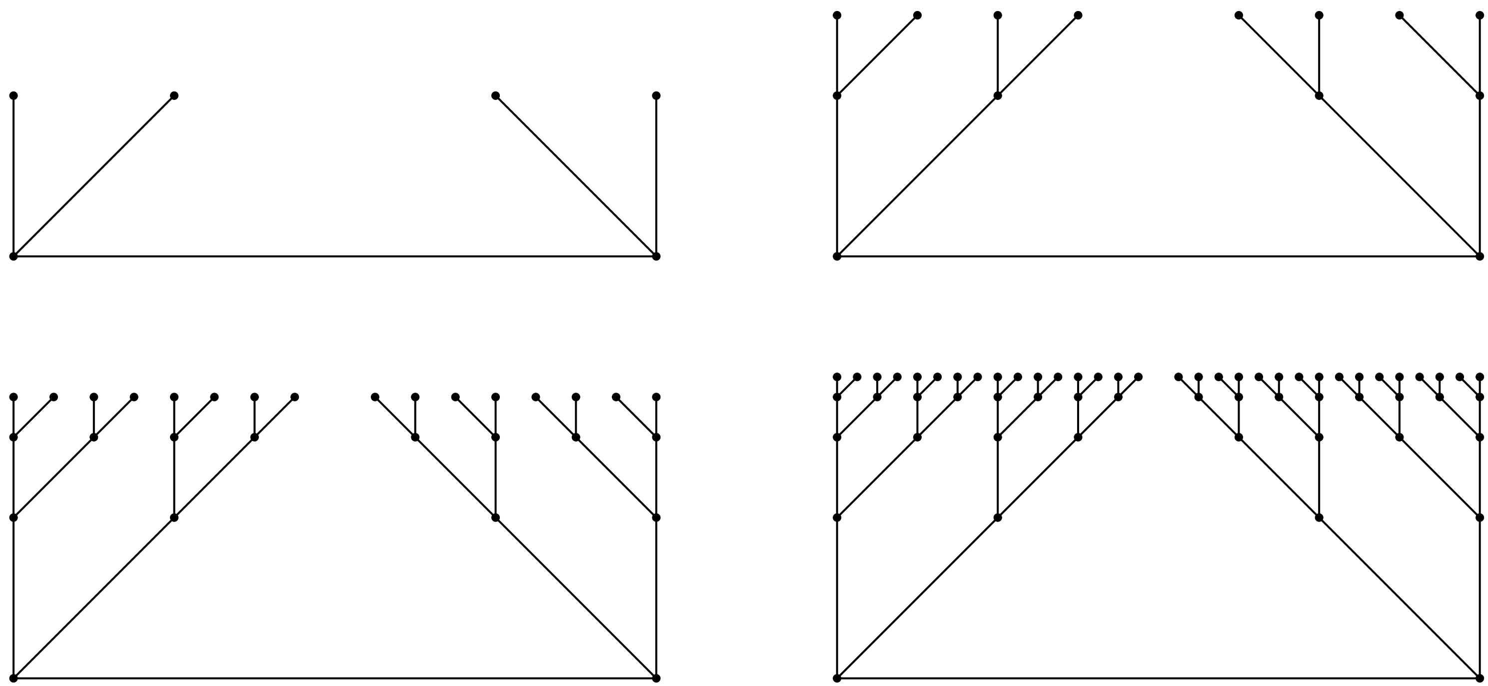
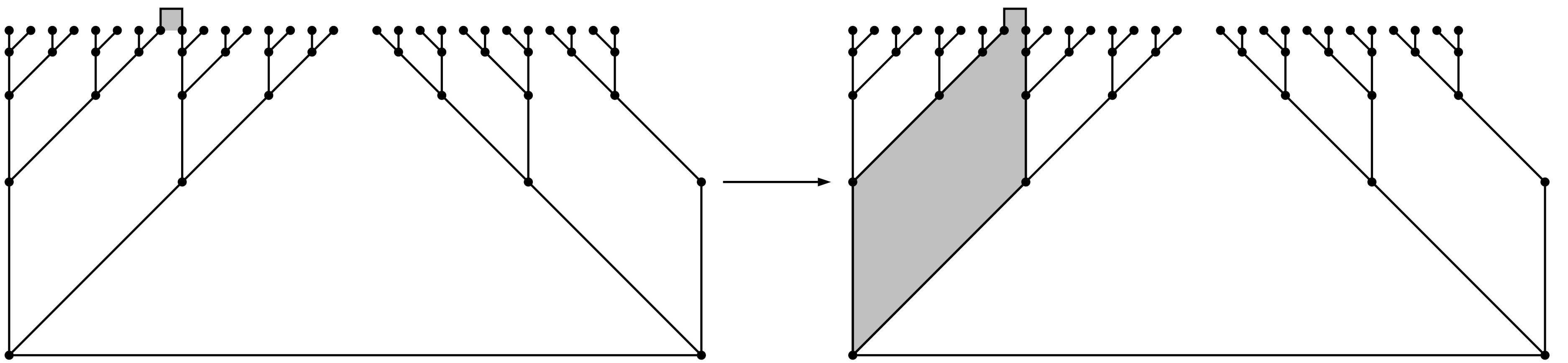
Cor: Any T as above is approximated by $\varphi(T')$ where $T' = f^{-1}([-1, 1])$, f is entire with $\mathrm{CV}(f) = \pm 1$, φ is QC and conformal off $T(r)$.

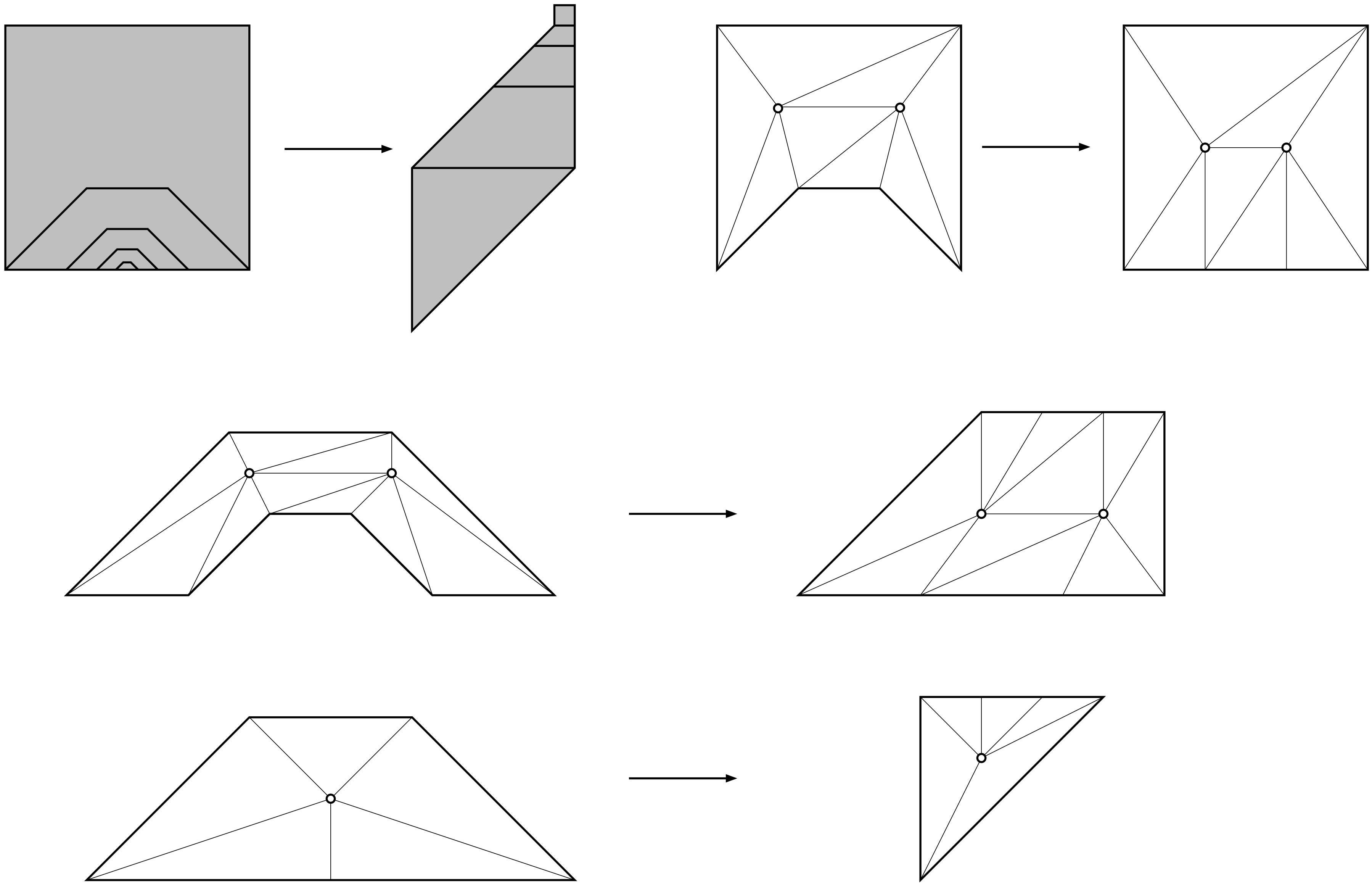
In many applications, φ is close to identity, so $T \approx T'$.

Proof of Folding Theorem: similar to finite case: add finite trees (instead of spikes) to sides of T, so one edge is glued to tree edge; other are folded onto each other.

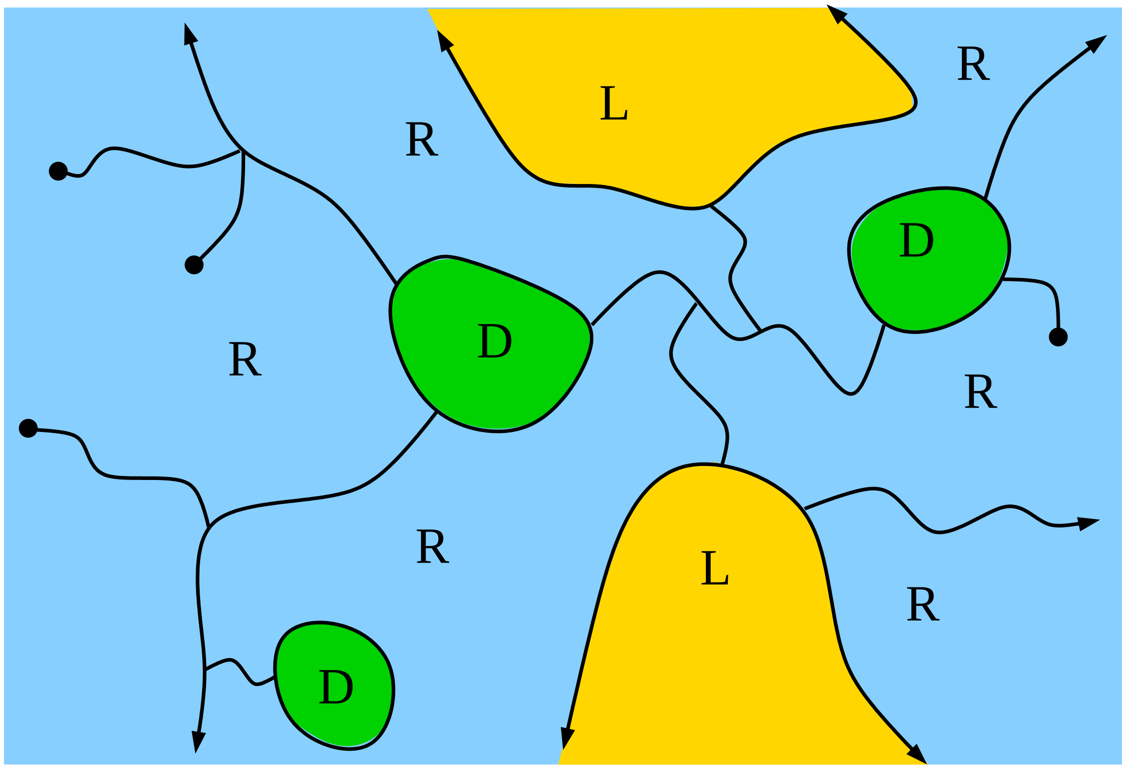


This allows arbitrarily large foldings with uniform QC bounds.





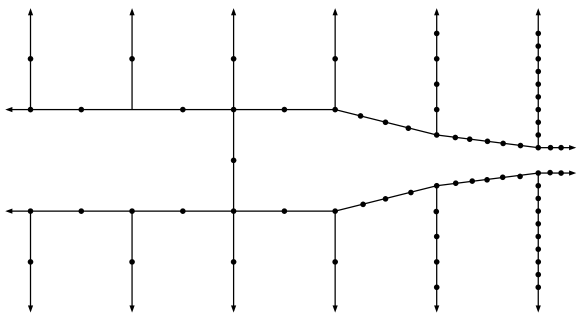
More general folding theorem: replace tree by graph, faces labeled D,L,R.



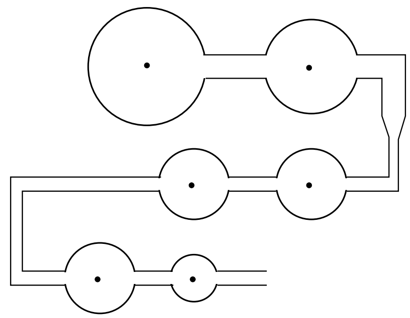
D = bounded Jordan domains (z^n, high degree critical points)

L = unbounded Jordan domains ($e^{-\tau(z)}$, finite asymptotic values)

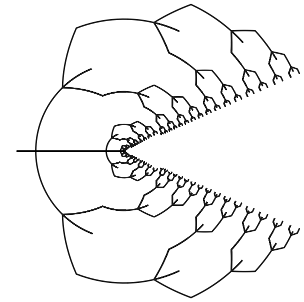
RR-edges map to $[-1, 1]$, other edges map to $\mathbb{T}$.



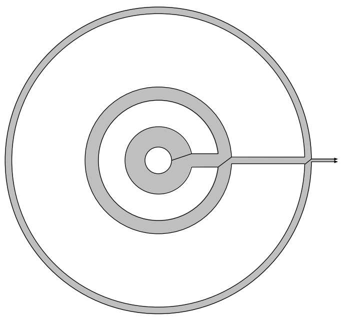
Rapid increase



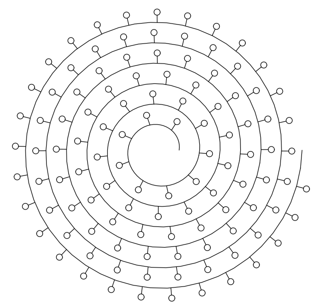
Post-singular dynamics



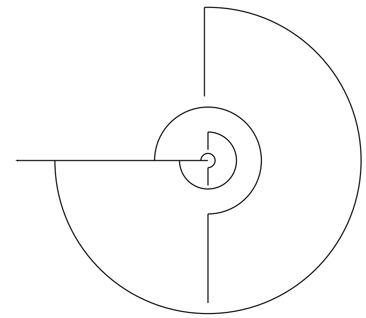
Order Conjecture



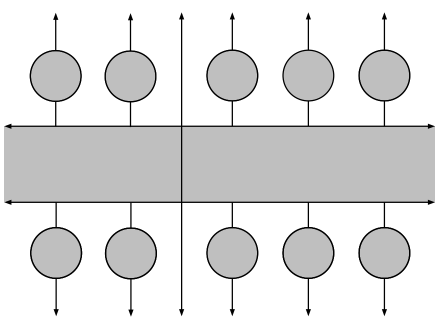
Area Conjecture



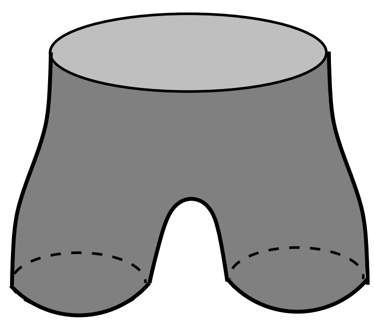
Wiman's Conjecture



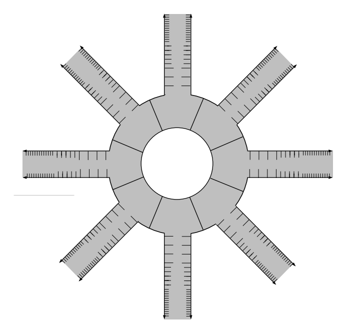
Eremenko Conjecture



Wandering domain

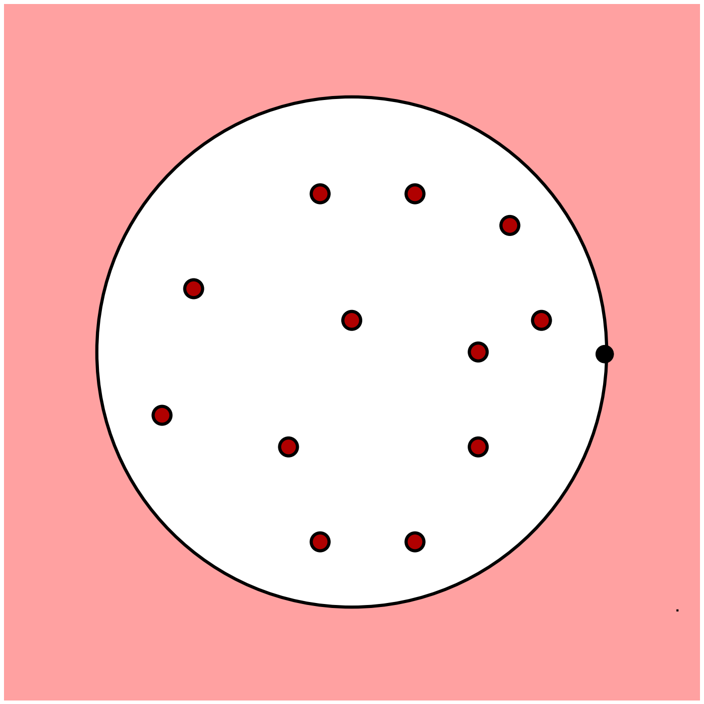


Folding a pair-of-pants



Dimension near 1

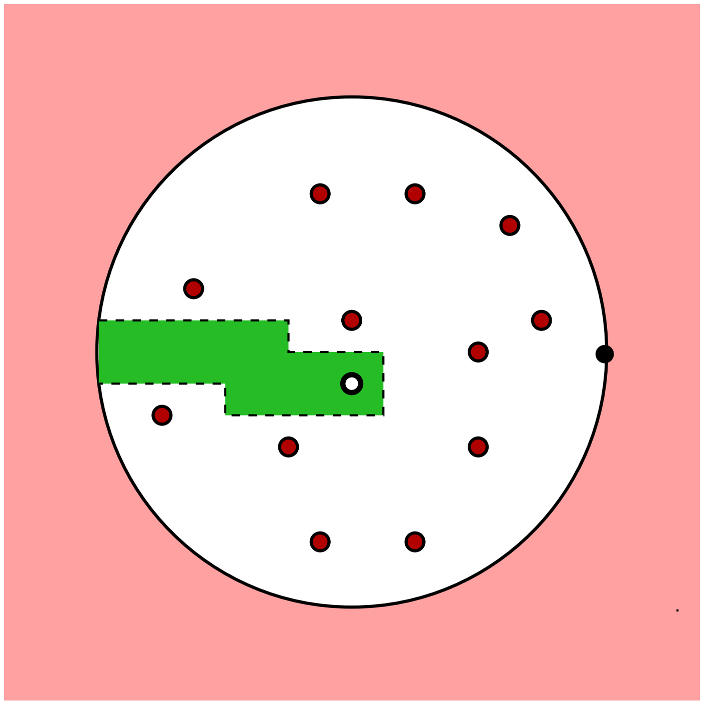
Tracts of Speiser class functions must satisfy certain geometric conditions.



Suppose we have a finite set of singular values in $\mathbb{D}$.

These values have minimal separation $\epsilon > 0$.

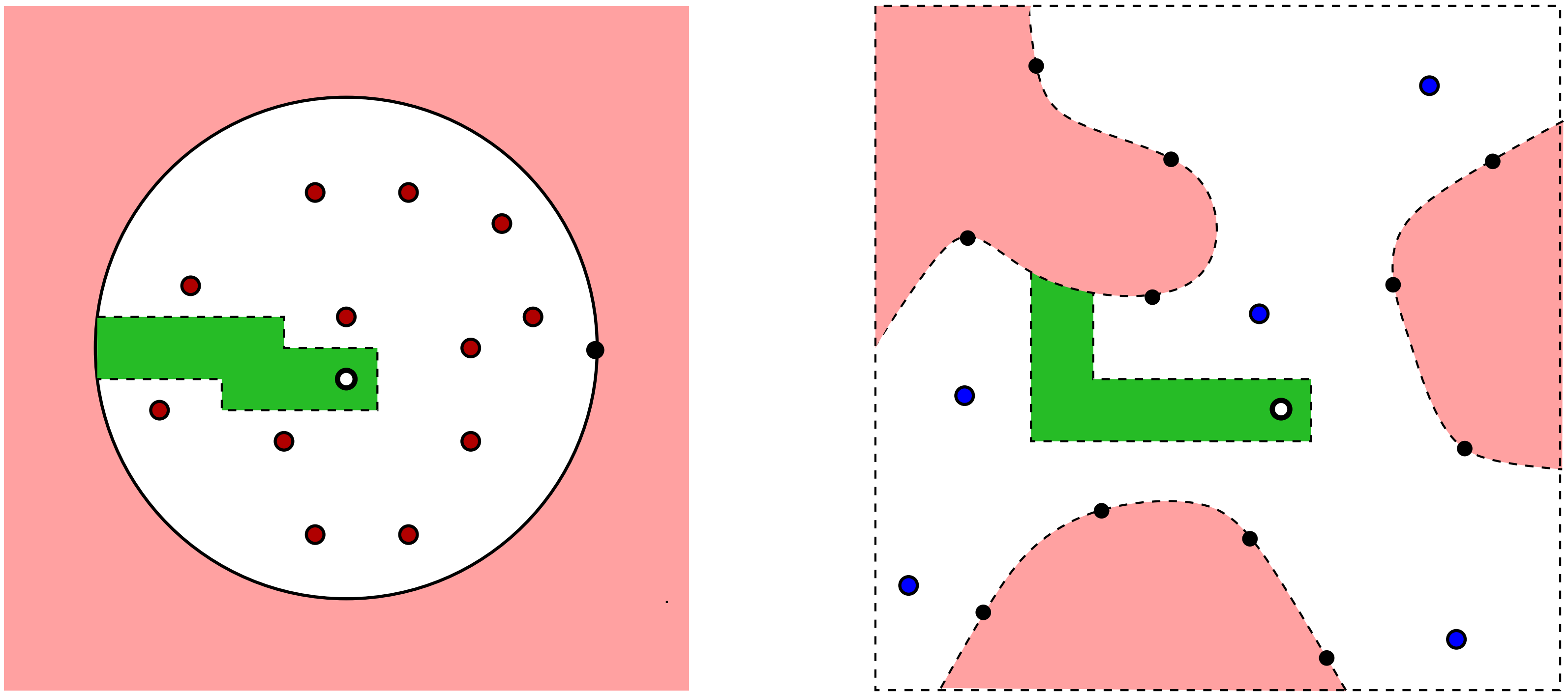
Tracts of Speiser class functions must satisfy certain geometric conditions.



Suppose z is ϵ-far from a singular value. It can be connected to $\mathbb{T}$ by a ϵ-thick tube of bounded length.

Harmonic measure of $\mathbb{T}$ from z in tube is $\delta = \delta(\epsilon) > 0$.

Tracts of Speiser class functions must satisfy certain geometric conditions.



Lifted point z has associated tract edge I with $\omega(z, I, W) \geq \delta > 0$.

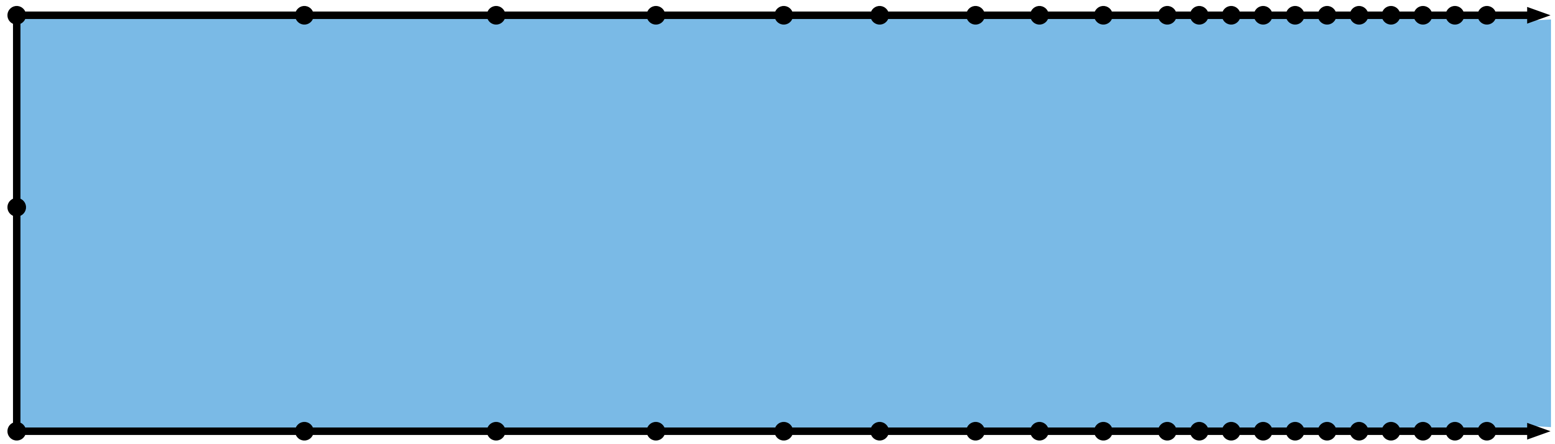
Beurling's estimate implies: $\mathrm{dist}(z, I) \leq C(\delta) \cdot \mathrm{diam}(I)$.

Thus we must have "dot spacing $\simeq$ gap between tracts".

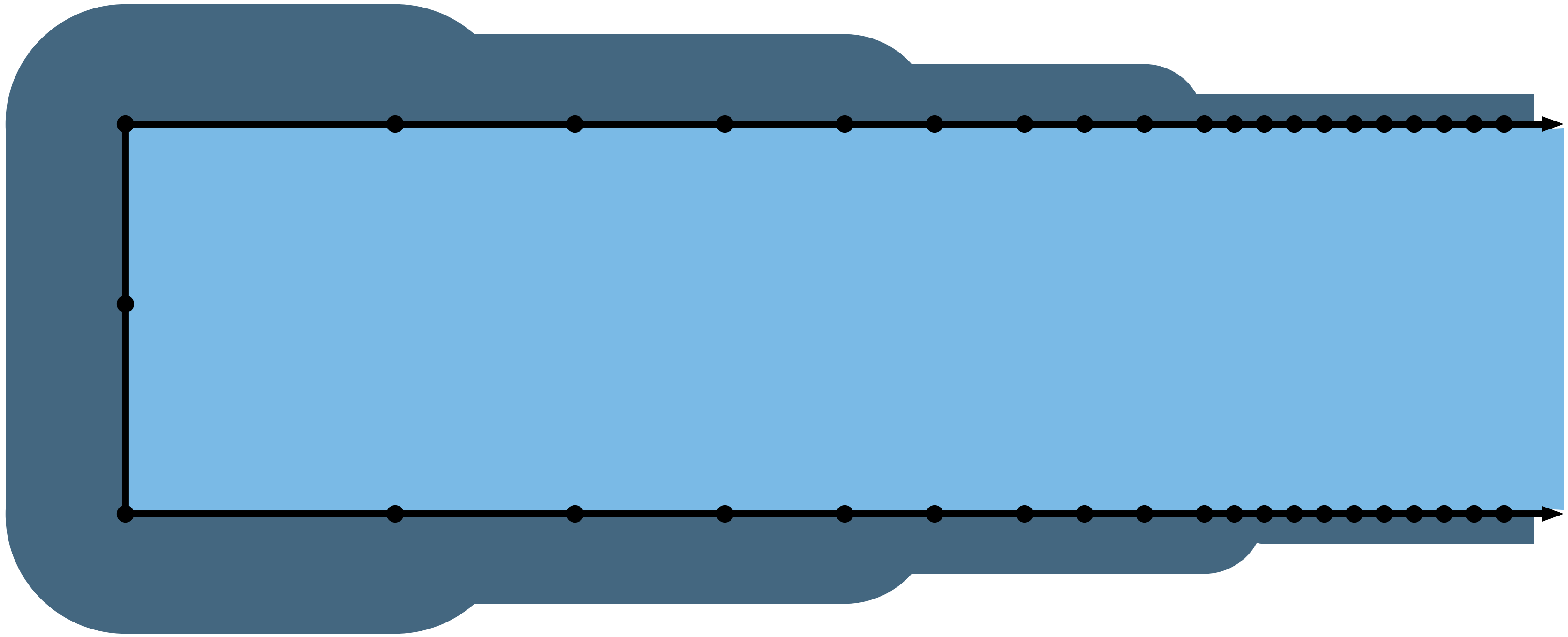
Theorem: Assume f is Speiser class f with no finite asymptotic values and all critical points have degree $\leq D < \infty$. Then

$$\Omega \cup T(r) = \mathbb{C}$$

for some $r < \infty$; $T = \partial\Omega$, vertices $= \tau$-partition.



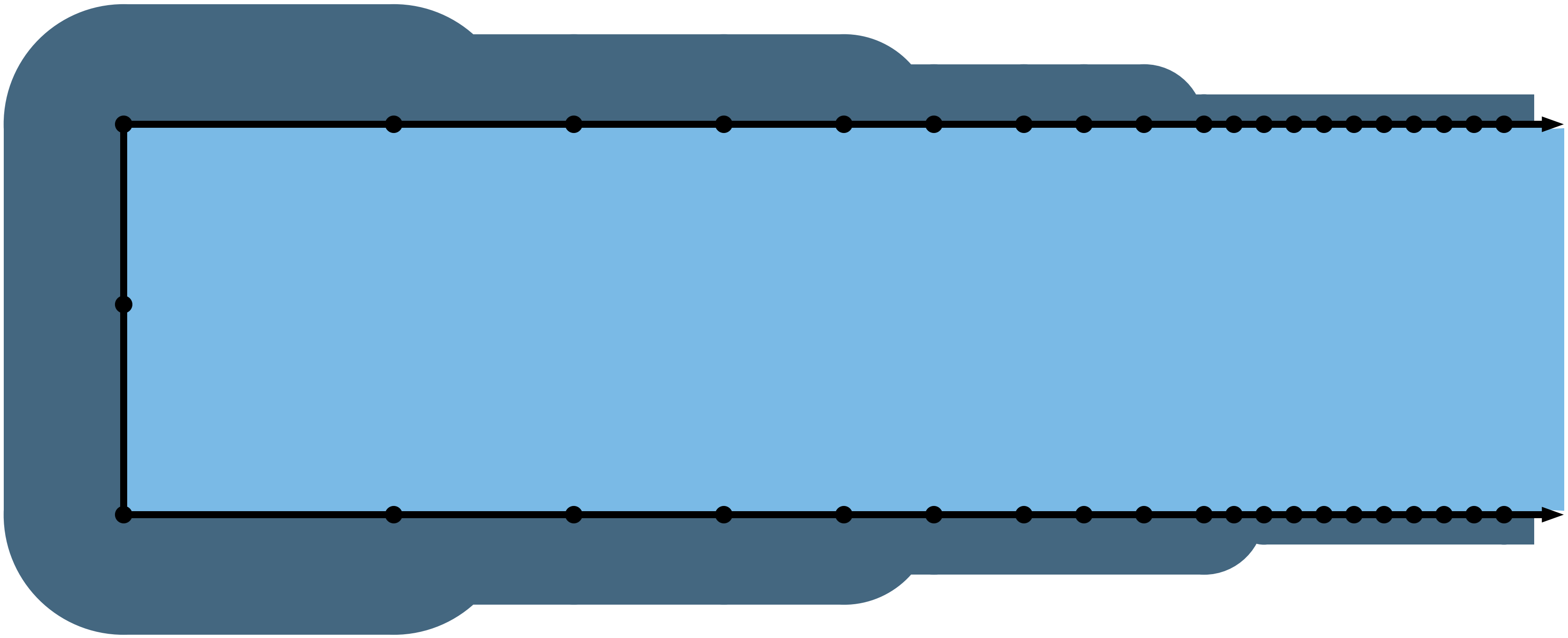
If Ω = half-strip, the τ-partition decays exponentially.



If $\Omega =$ half-strip, the τ-partition decays exponentially.

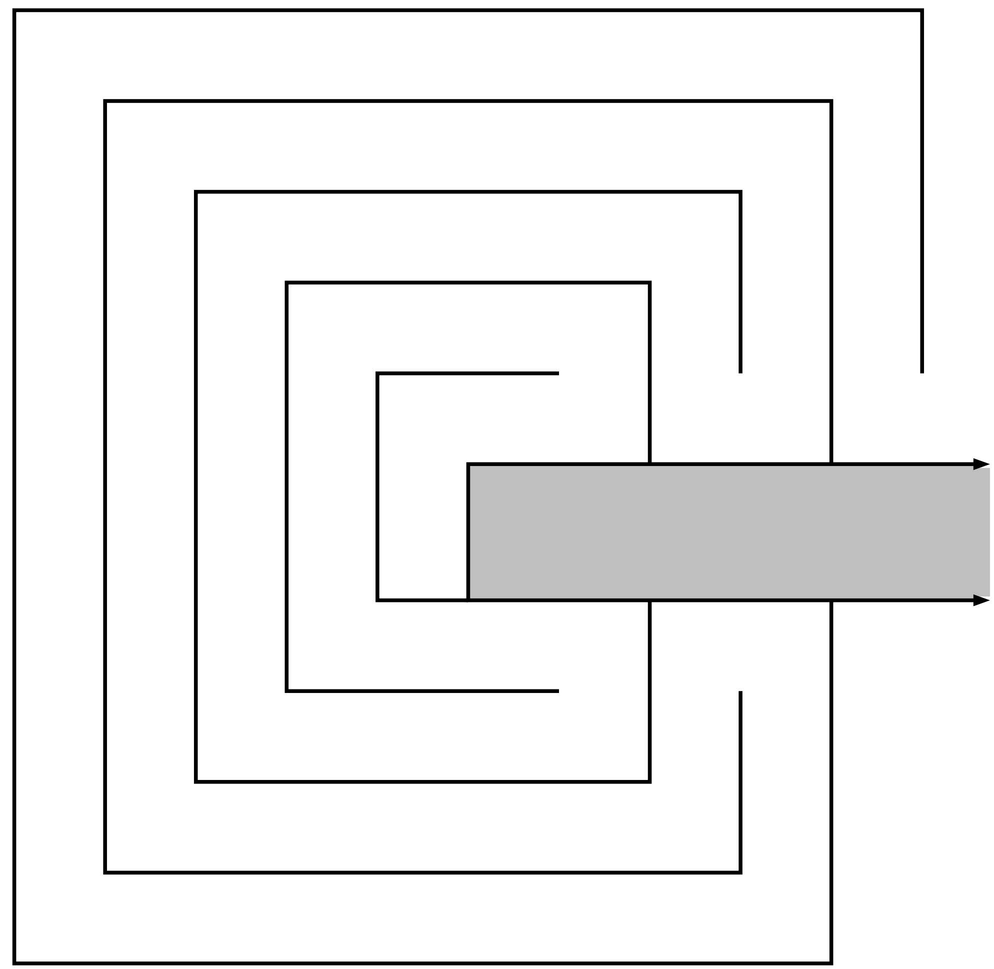
So $\Omega \cup T(r)$ is not the whole plane.

Theorem: If $f \in \mathcal{S}$ has a single tract Ω, then Ω is not the image of a half-strip under any QC mapping of the plane.

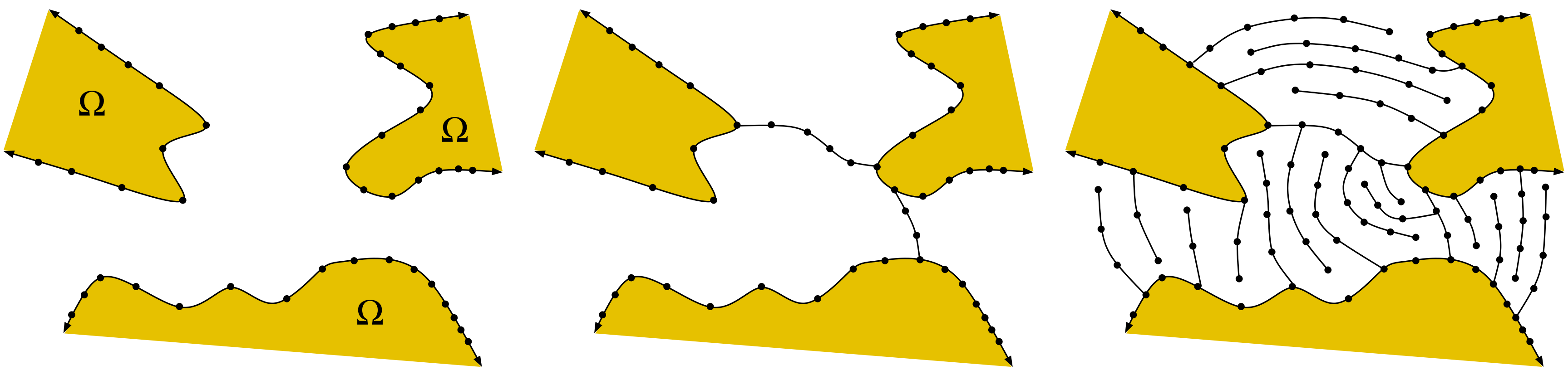


• There are Eremenko-Lyubich functions with a single tract that is as close to a half-strip as we wish.

• There are Speiser functions with multiple tracts, one of which is a close to a half-strip as we wish.



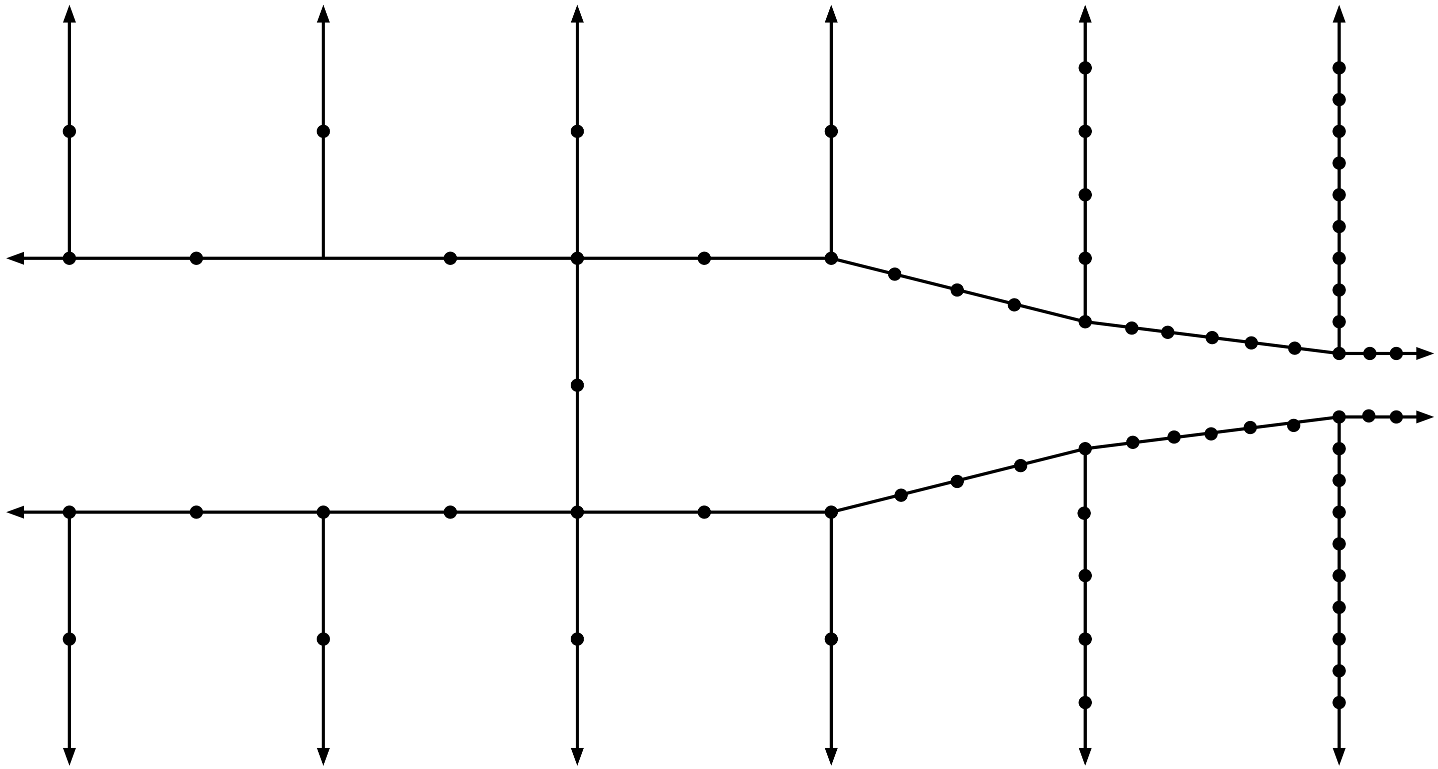
A Speiser class function with two tracts can have one of them very close to a half-strip.



In general, only need to double the number of tracts

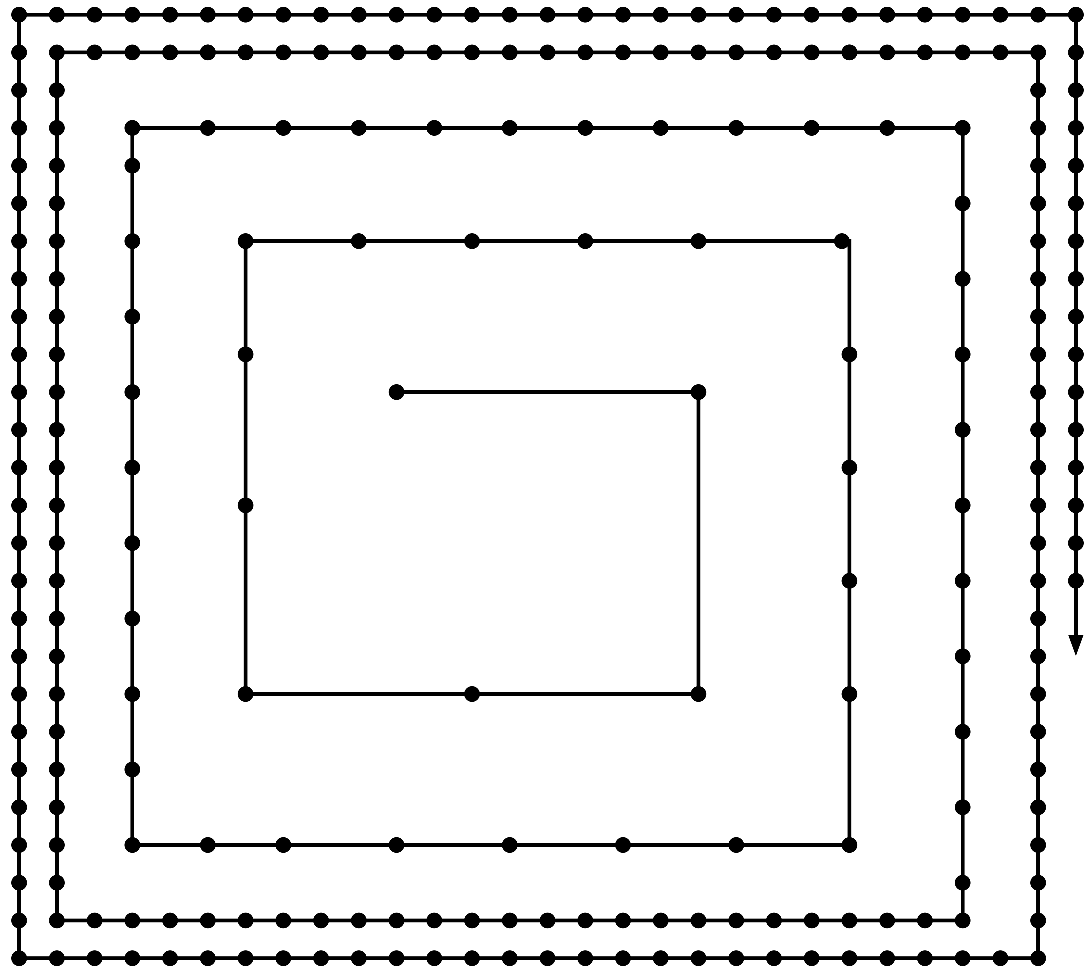
THANKS

Rapid increase in Speiser class



f has 2 singular values, $f(x) \nearrow \infty$ as fast as we wish.

Similar examples due to Sergei Merenkov 2008 (3 singular values).



Speiser example with single tract; rapid growth and spiraling.

DeMarco, Koch and McMullen proved that a post-critically finite rational map can have **any** given dynamics.

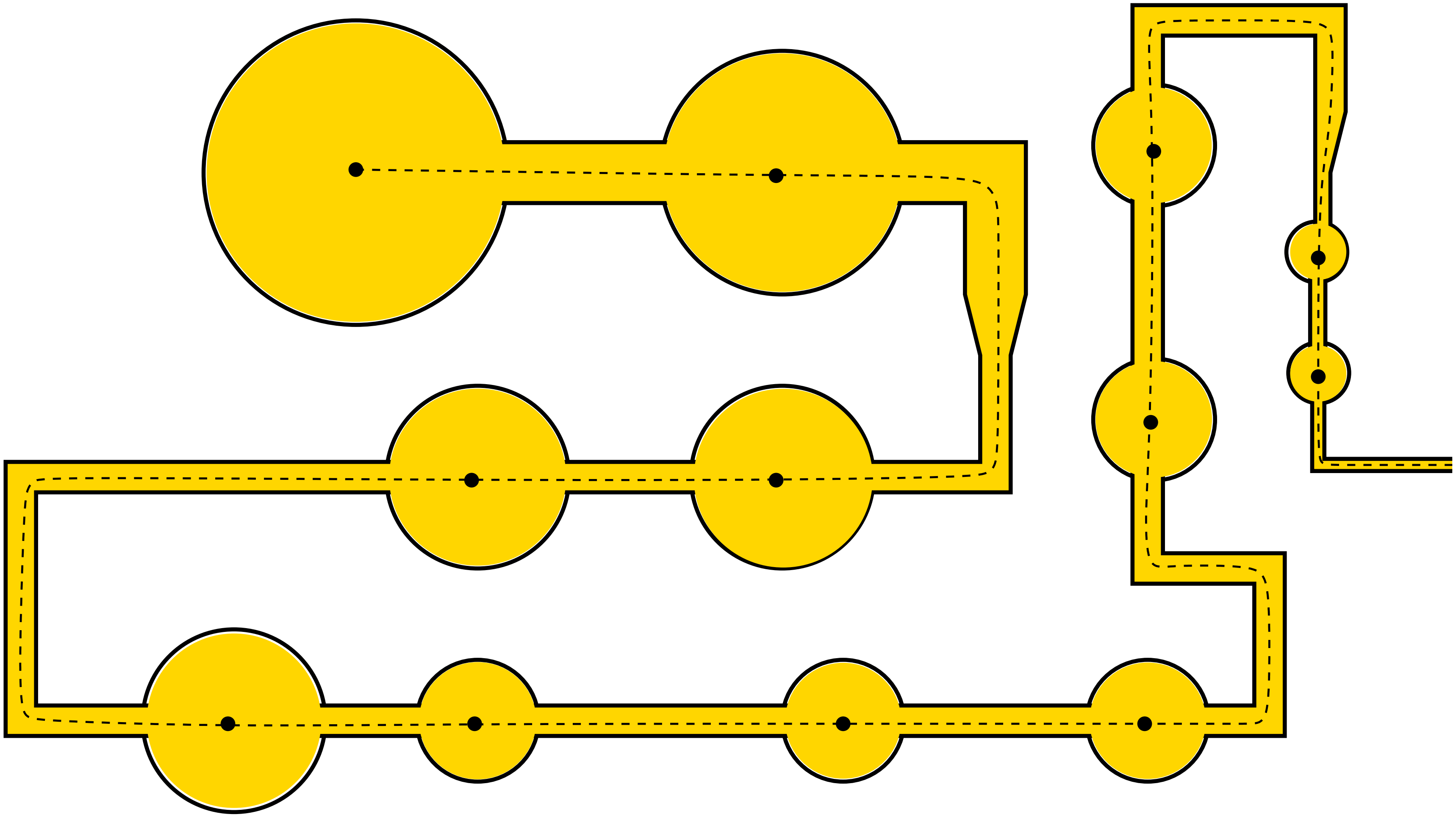
More precisely, given any $\epsilon > 0$, any finite set X and any map $h : X \to X$ there is a rational r whose post-singular set $P(r)$ is the same size as X, $P(r)$ ϵ-approximates X in Hausdorff metric, and $|r - h| < \epsilon$ on P.

Theorem (Lazebnik, B.): Let X be discrete (≥ 4 points), let $h : X \to X$ be any map, and let $\epsilon > 0$. Then there is a transcendental meromorphic function f and a bijection $\psi : X \to P(f)$ so that

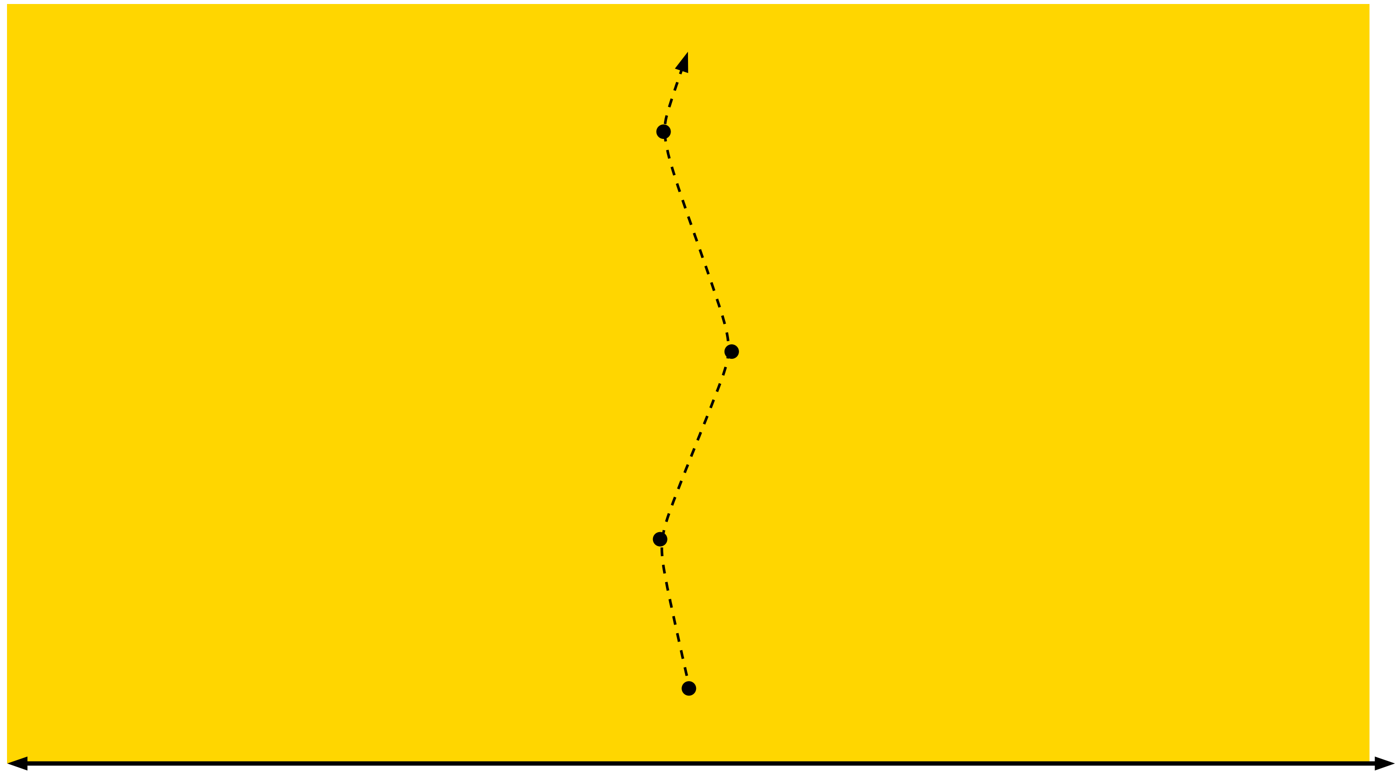
$$|\psi(z) - z| \leq \max(\epsilon, o(1)) \qquad \text{and} \qquad f = \psi \circ h \circ \psi^{-1} \text{ on } P(f).$$

Main argument (folding + fixed point thm) due to Kirill.

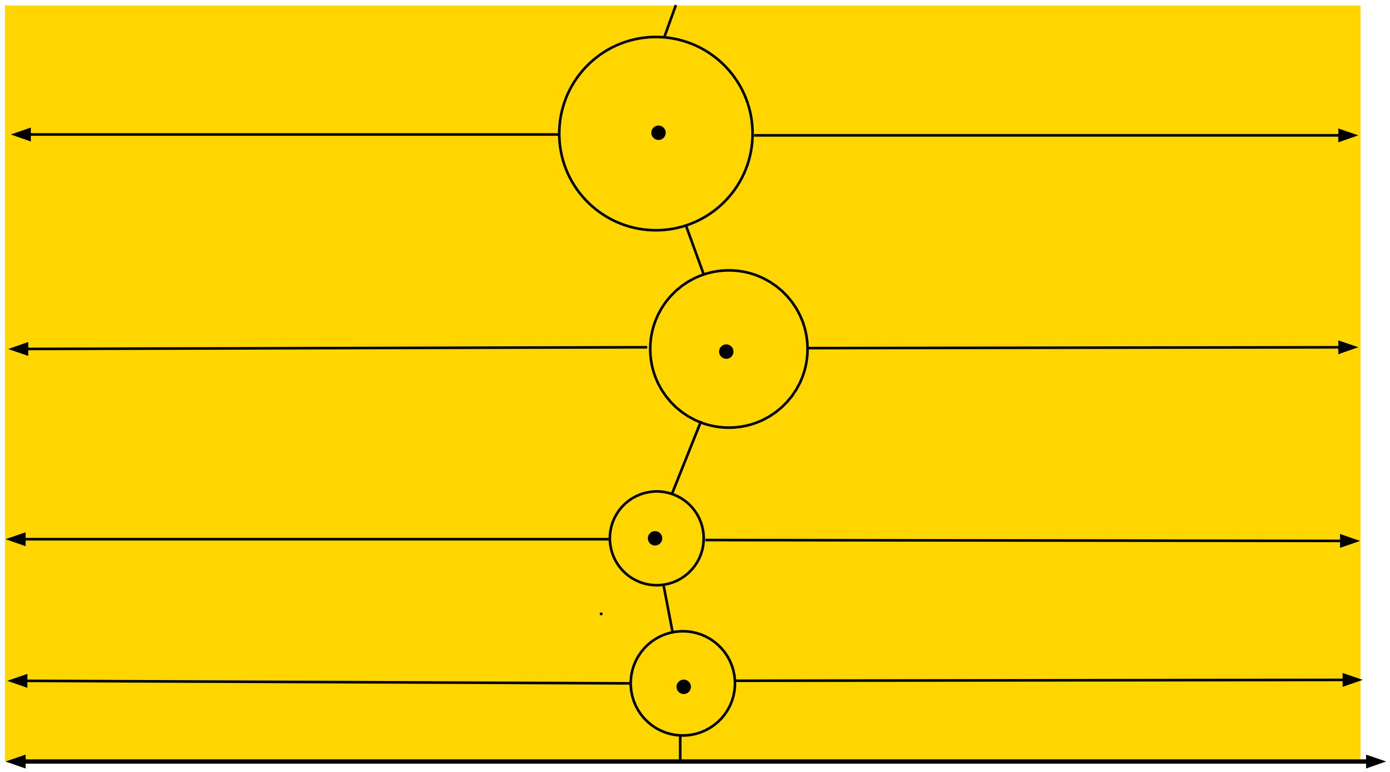
Unknown if we can take $\epsilon = 0$ in either case



Construct a domain W that contains the given points, and so that points lie very close to hyperbolic geodesic to ∞.



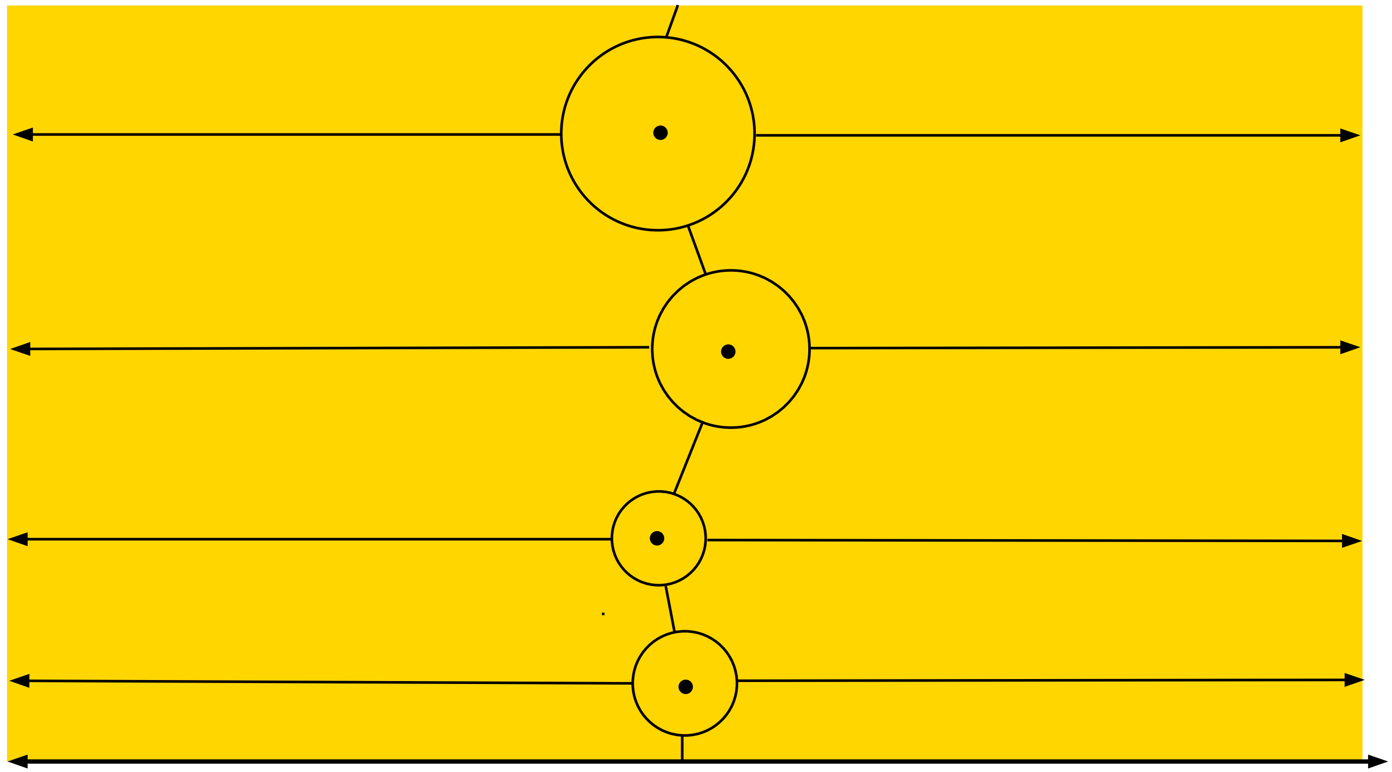
Conformally map W to upper half-plane; points almost on vertical ray.



Divide upper half-plane into R and D components.

Folding map sends centers anywhere we want in $\mathbb{D}$.

Analogous construction maps centers outside $\mathbb{D}$; introduces poles.



Transfer map back to W. Decompose complement of W into R components and define map on whole plane by folding.

Gives QM map with exact desired post-singular behavior.

MRMT and fixed point theorem give approximating meromorphic map. Post-singular set is moves slightly, but with correct dynamics.

Order of growth of entire function:

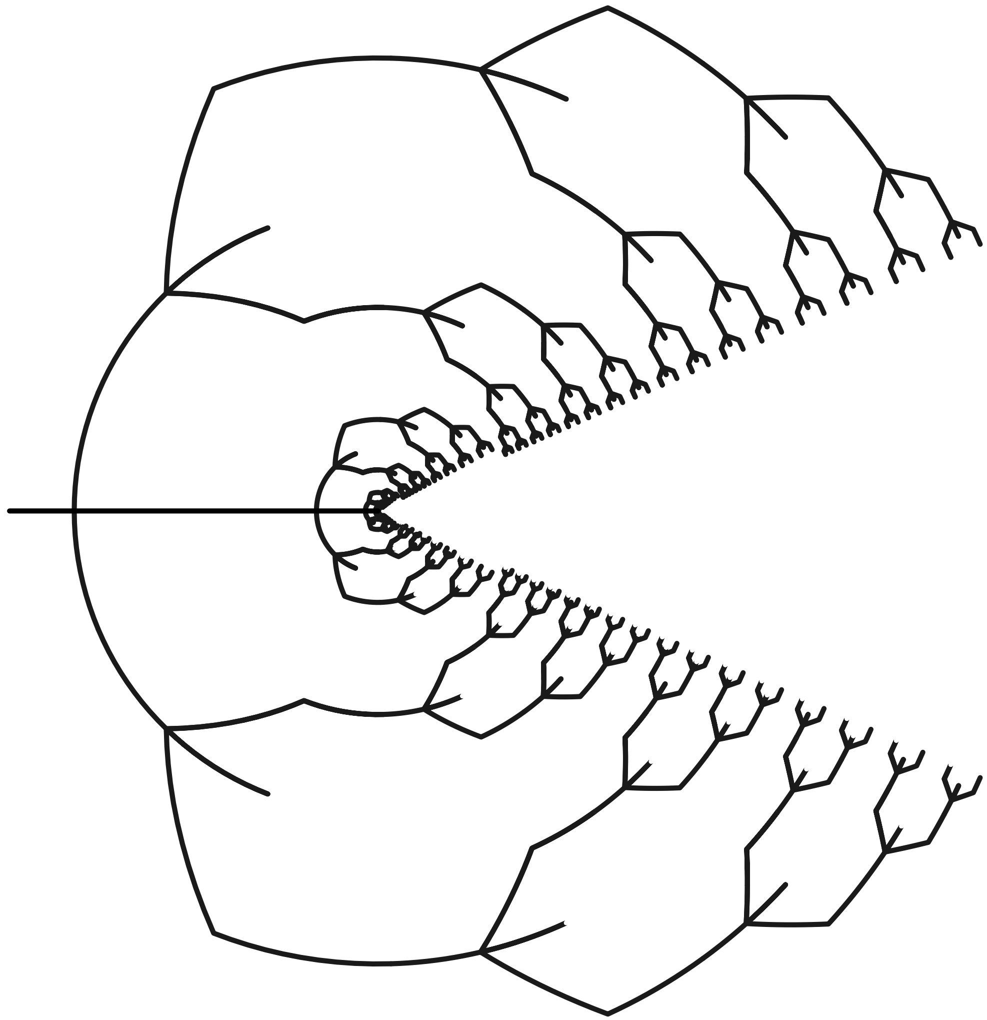
$$\rho(f) = \limsup_{|z|\to\infty} \frac{\log\log|f(z)|}{\log|z|}, \qquad \rho(e^{z^d}) = d.$$

QC-equivalent: $f \sim g$ if $g = \psi \circ f \circ \phi$ for quasiconformal ψ, ϕ.

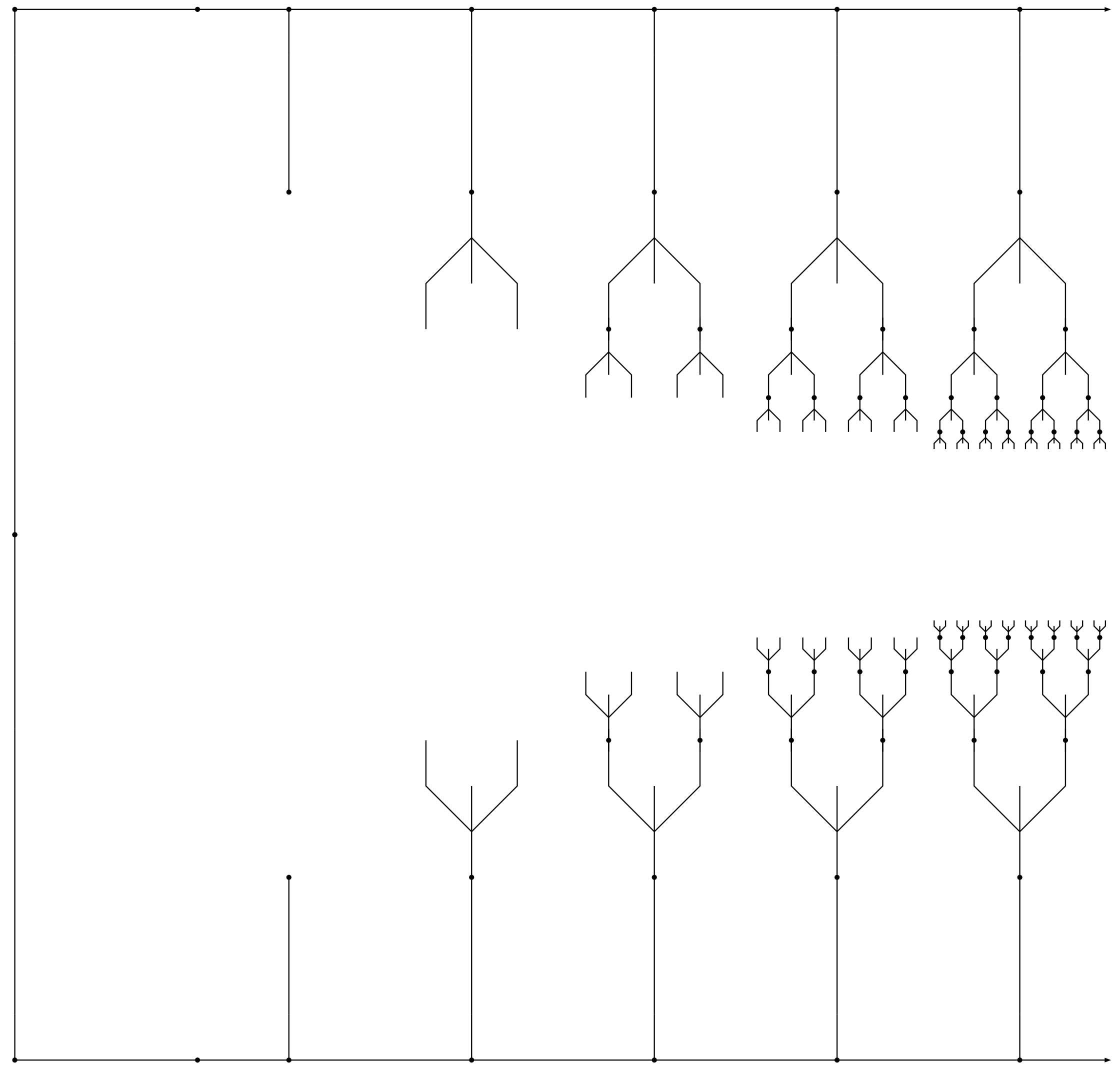
Speiser class QC-equivalence classes M_f are finite dimensional.

Order conjecture: Is ρ constant on M_f?

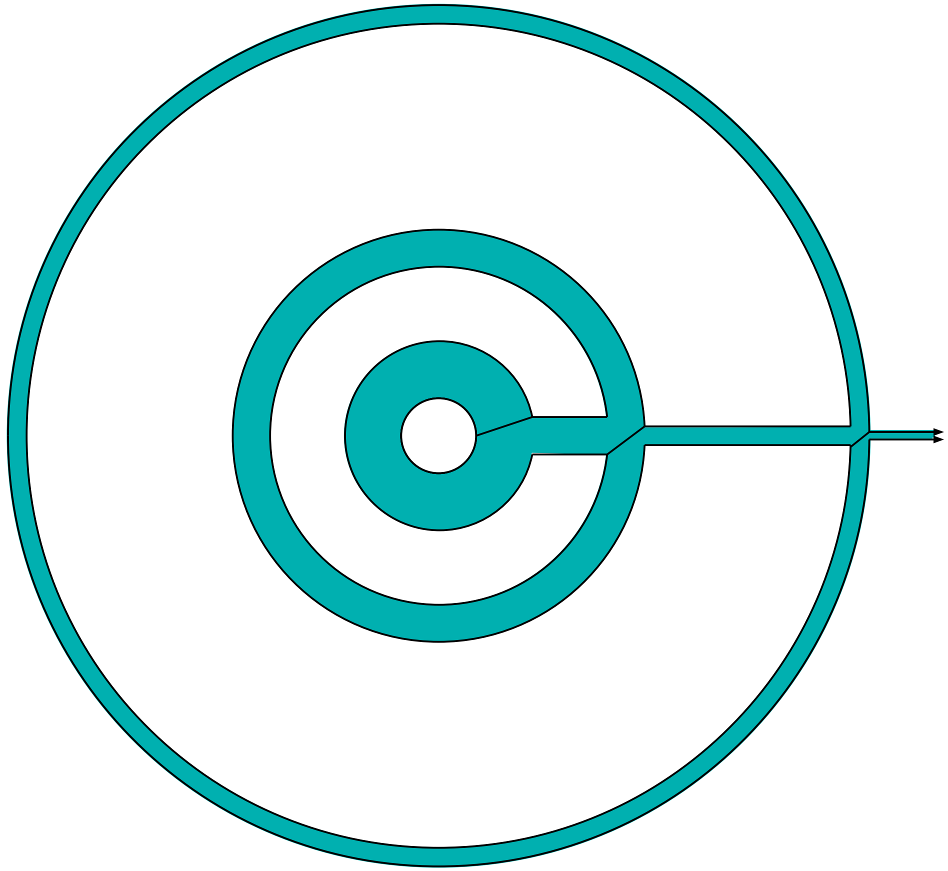
Adam Epstein observed this holds in many cases where order is determined by "combinatorial data". He and Rempe-Gillen gave EL counterexample.



Speiser class counterexample with 3 singular values.



Same tree in logarithmic coordinates.



Speiser function that is "statistically zero": for all $\epsilon > 0$,

$$\text{area}(\{|f| > \epsilon\}) < \infty.$$

Counterexample to Eremenko-Lyubich area conjecture.

Wiman's conjecture:

Define $m(r) = \min_{|z|=r} |f(z)|, \qquad M(r) = \max_{|z|=r} |f(z)|.$

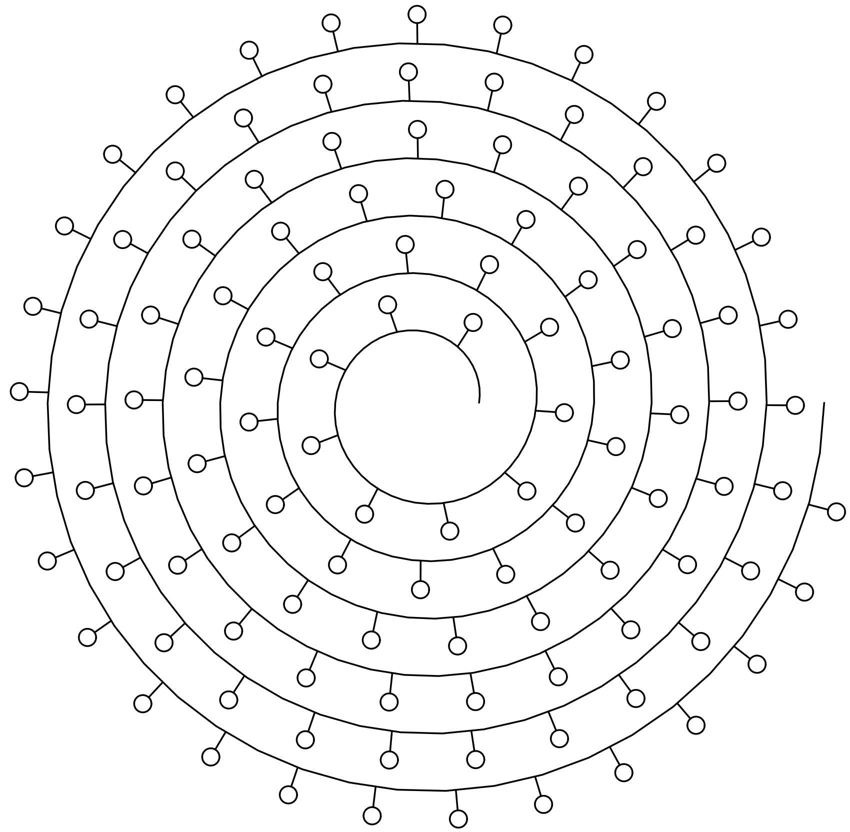
Wiman (1916) conjectured that for all entire functions

$$\limsup_{r \nearrow \infty} \frac{\log m(r,f)}{\log M(r,f)} \geq -1.$$

Sharp for $f(z) = \exp(z)$.

True in special cases, e.g., $|f(r)| = m(r)$ (Beurling, 1949).

False in general (Hayman, 1952).



Speiser class counterexample to Wiman's conjecture

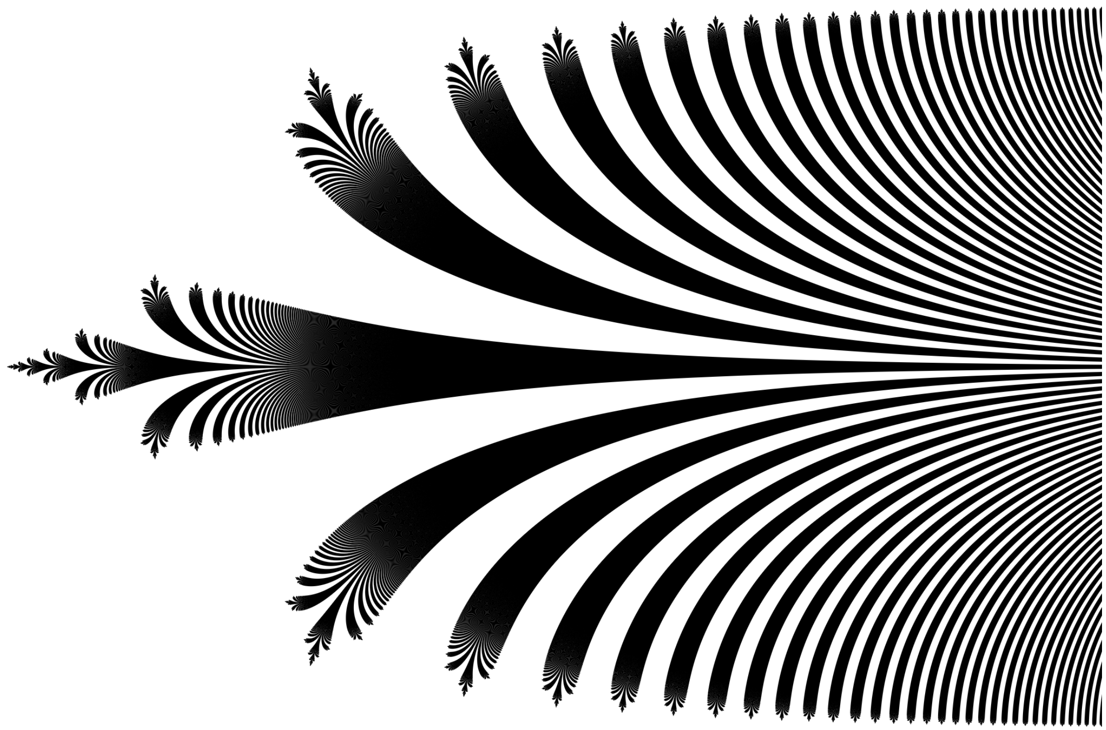
$$\frac{\log m(r,f)}{\log M(r,f)} \leq -C \cdot \log\log\log M(r,f)$$

Given an entire function f,

Fatou set $= \mathcal{F}(f) =$ open set where iterates are normal family.

Julia set $= \mathcal{J}(f) =$ complement of Fatou set.

Julia set is usually fractal. What is its (Hausdorff) dimension?



$\mathcal{J}((e^z - 1)/2)$, courtesy of Arnaud Chéritat

Given an entire function f,

Fatou set $= \mathcal{F}(f) =$ open set where iterates are normal family.

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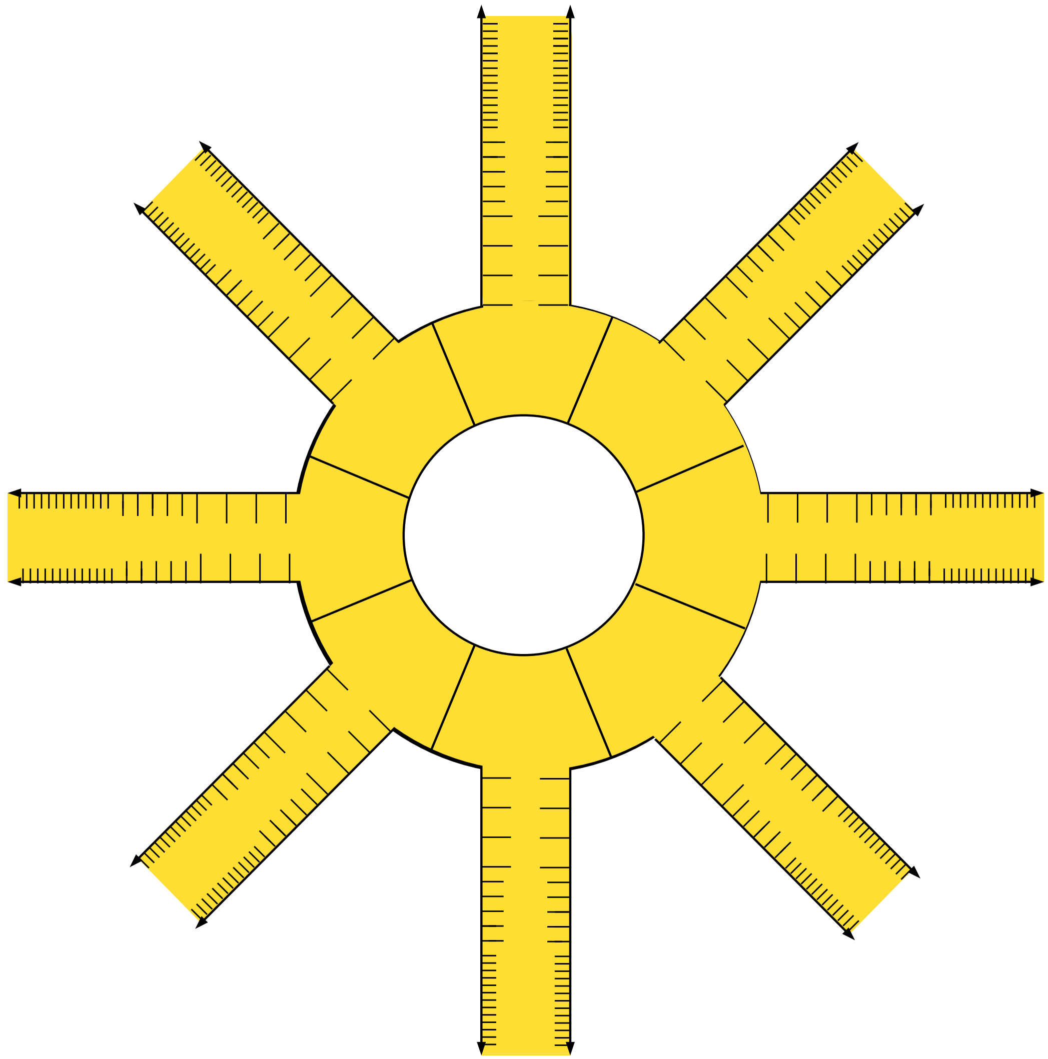
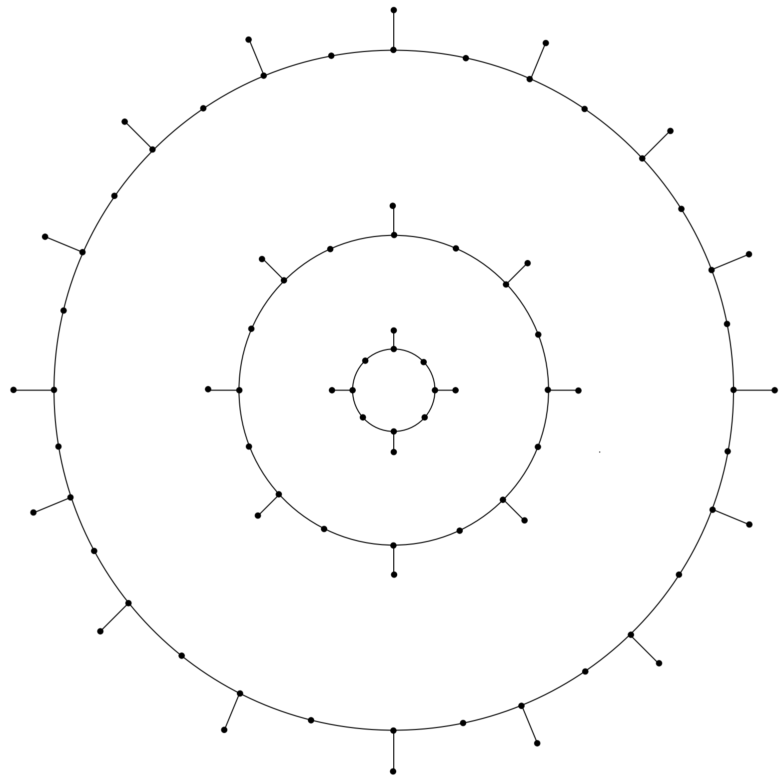
Julia set is usually fractal. What is its (Hausdorff) dimension?

- Baker (1975): f transcendental $\Rightarrow \mathcal{J}$ contains a continuum $\Rightarrow \dim \geq 1$.
- Misiurewicz (1981), McMullen (1987) dim =2 occurs (is common)
- Stallard (1997, 2000): $\{\dim(\mathcal{J}(f)) : f \in \mathcal{B}\} = (1, 2]$.
- $1 <$ H-dim < 2 can also occur in Speiser class (Albrecht-B.)
- Rippon-Stallard (2005) Packing dim $= 2$ for $f \in \mathcal{B}$.
- B (2018) H-dim $=$ P-dim $= 1$ can occur (example outside $\mathcal{B}$).
- Burkart (2019) $1 <$ packing dim < 2 can occur (outside $\mathcal{B}$).



$\inf\{\dim(\mathcal{J}(f)) : f \in \mathcal{S}\} = 1$ (B.-Albrecht, 2018).

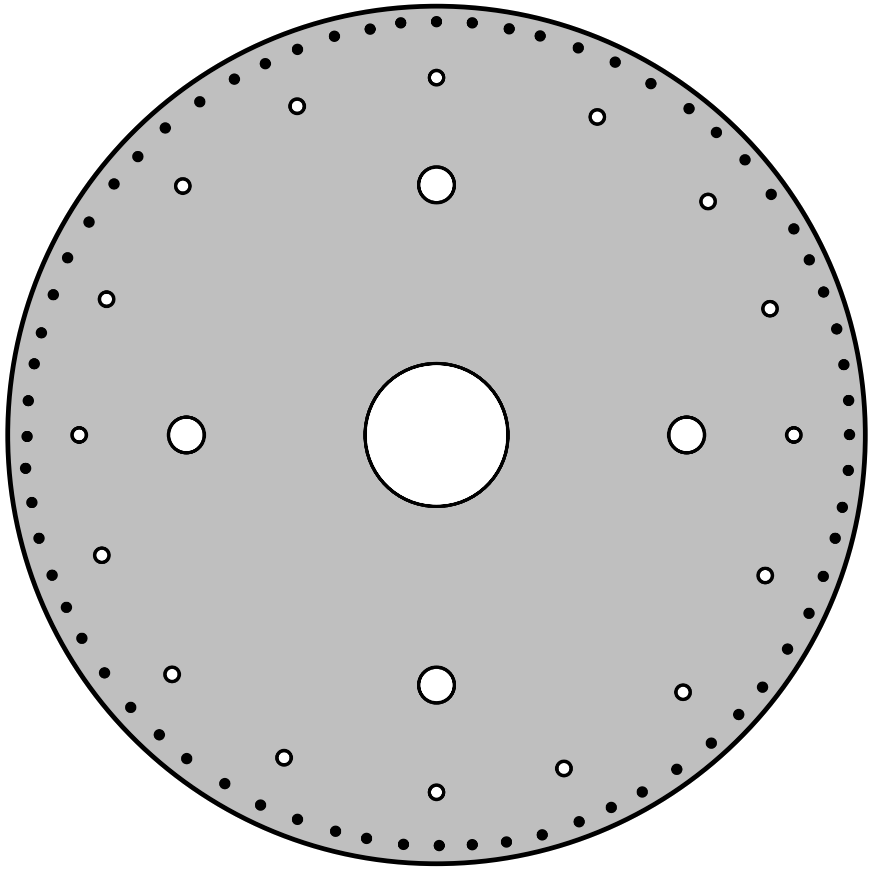
Uses upper and lower τ-bounds to control dimension.



Inspiration for dim = 1 example.

Actual proof uses infinite product with zeros on concentric circles.

$$f(z) = f_0(z) \cdot \prod_{k=1}^{\infty} \left(1 - \frac{1}{2}\,(z/R_k)^{n_k}\right)$$



Fatou component in "Dim = 1" example: C^1 boundaries
Burkart's example also has components with fractal boundaries.

Given an entire function f,

Fatou set $= \mathcal{F}(f) =$ open set where iterates are normal family.

Julia set $= \mathcal{J}(f) =$ complement of Fatou set.

f permutes components of its Fatou set.

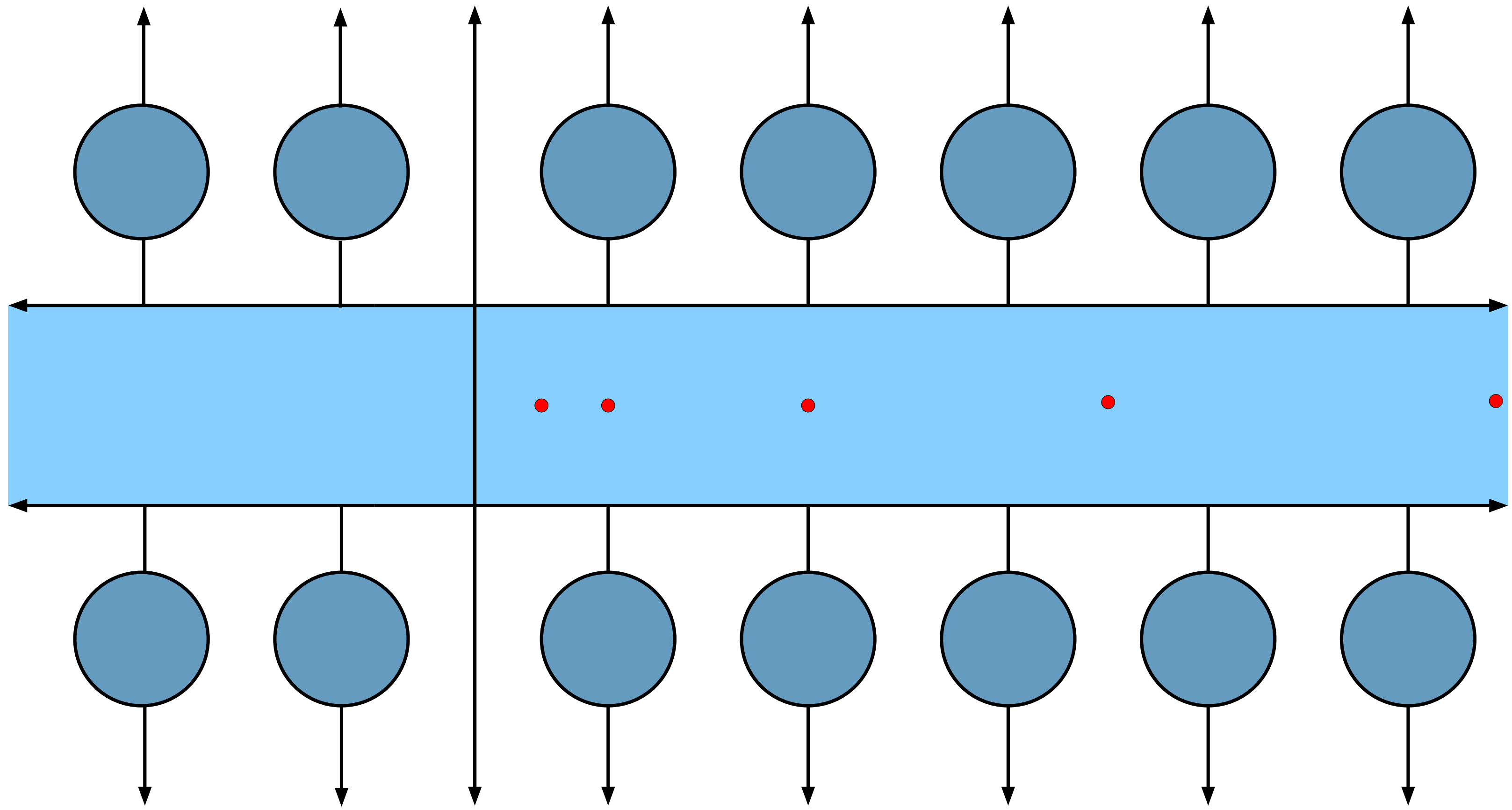
Wandering domain = Fatou component with infinite orbit.

- Entire functions can have wandering domains (Baker 1975).

- No wandering domains for rational functions (Sullivan 1985).

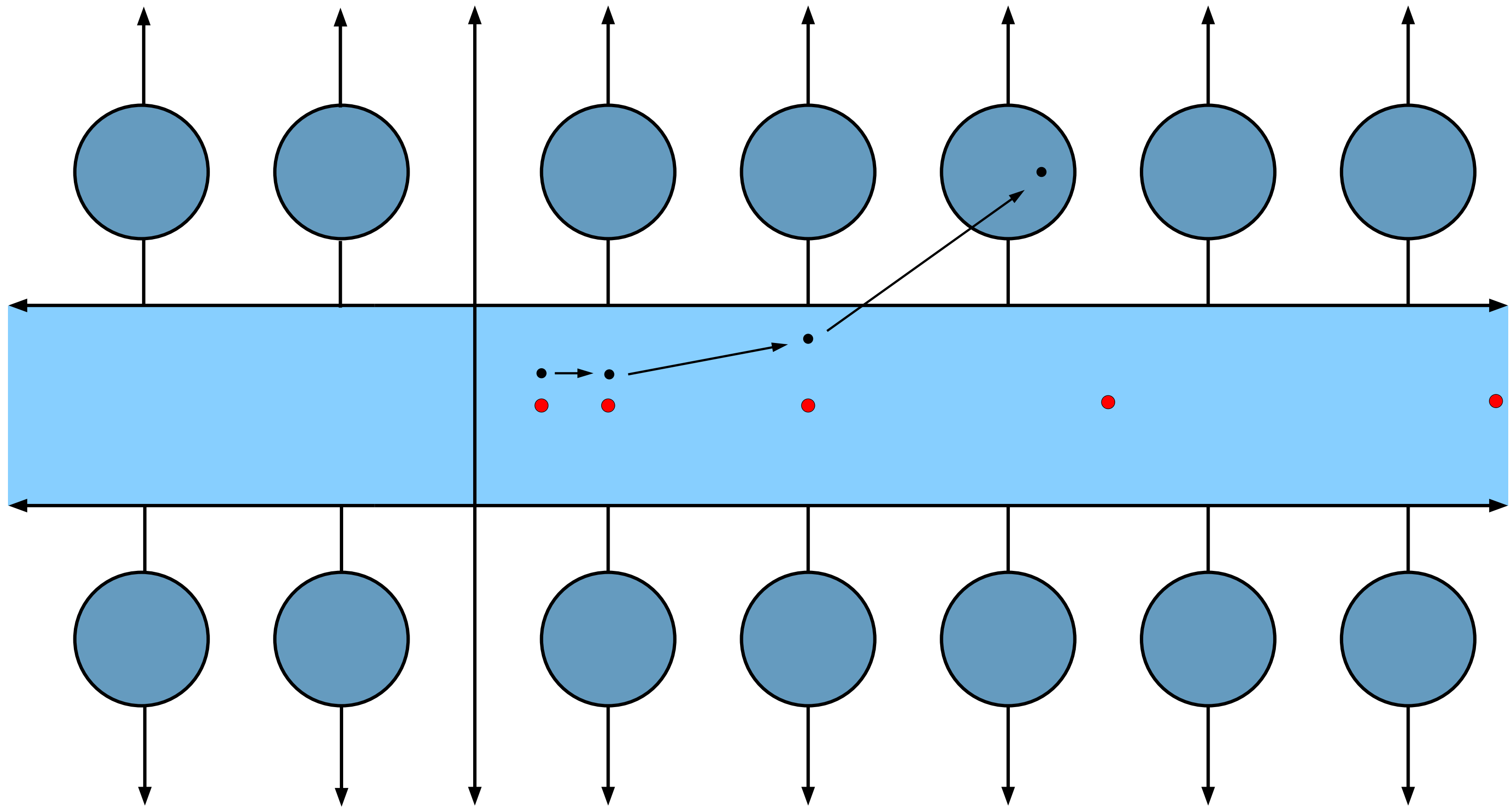
- Also none in Speiser class (Eremenko-Lyubich, Goldberg-Keen).

Are there wandering domains in Eremenko-Lyubich class?



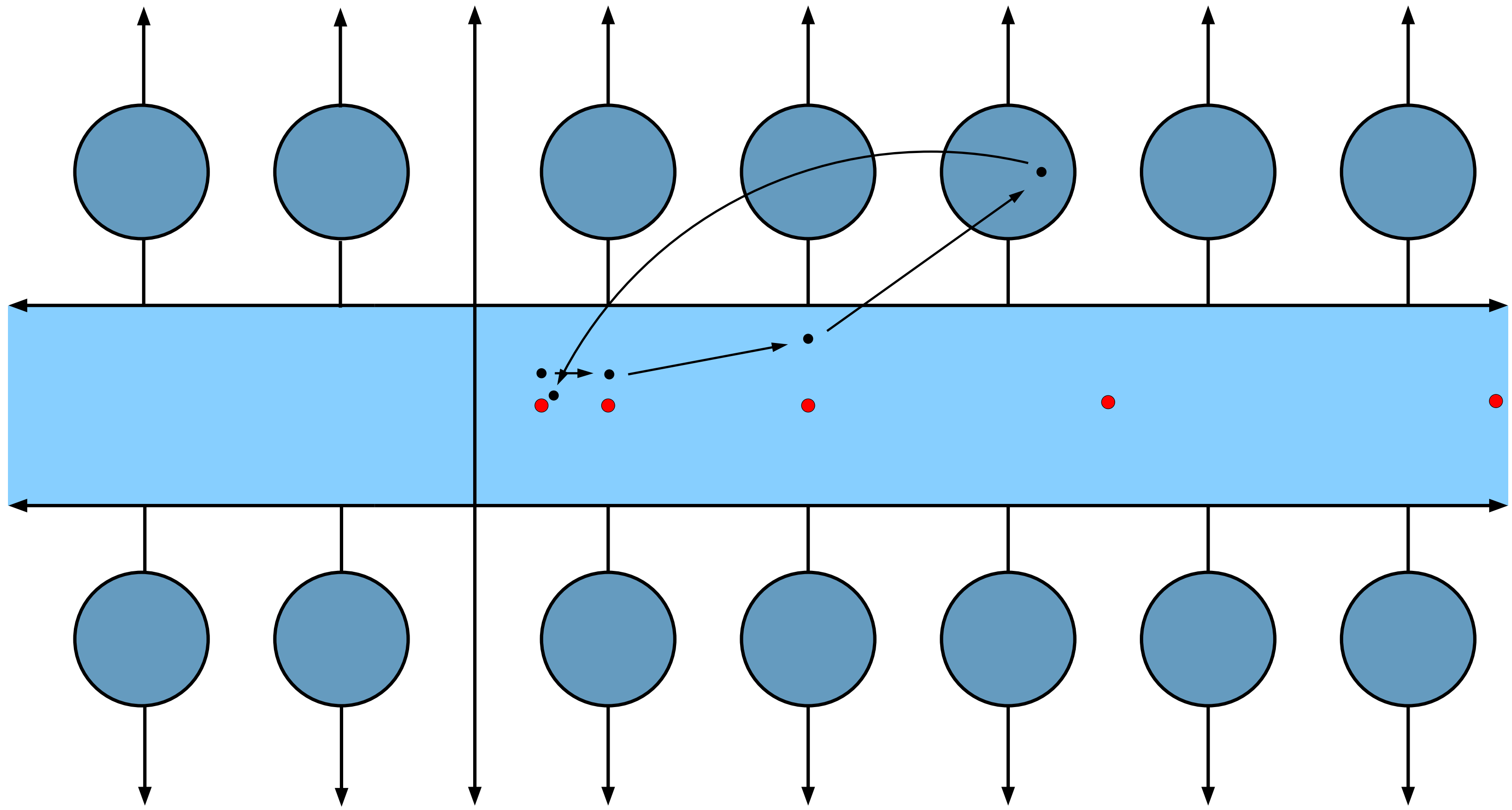
Graph giving wandering domain in Eremenko-Lyubich class.

Original proof corrected by Marti-Pete and Shishikura.



Graph giving wandering domain in Eremenko-Lyubich class.

Variations by Lazebnik, Fagella-Godillon-Jarque, Osborne-Sixsmith.



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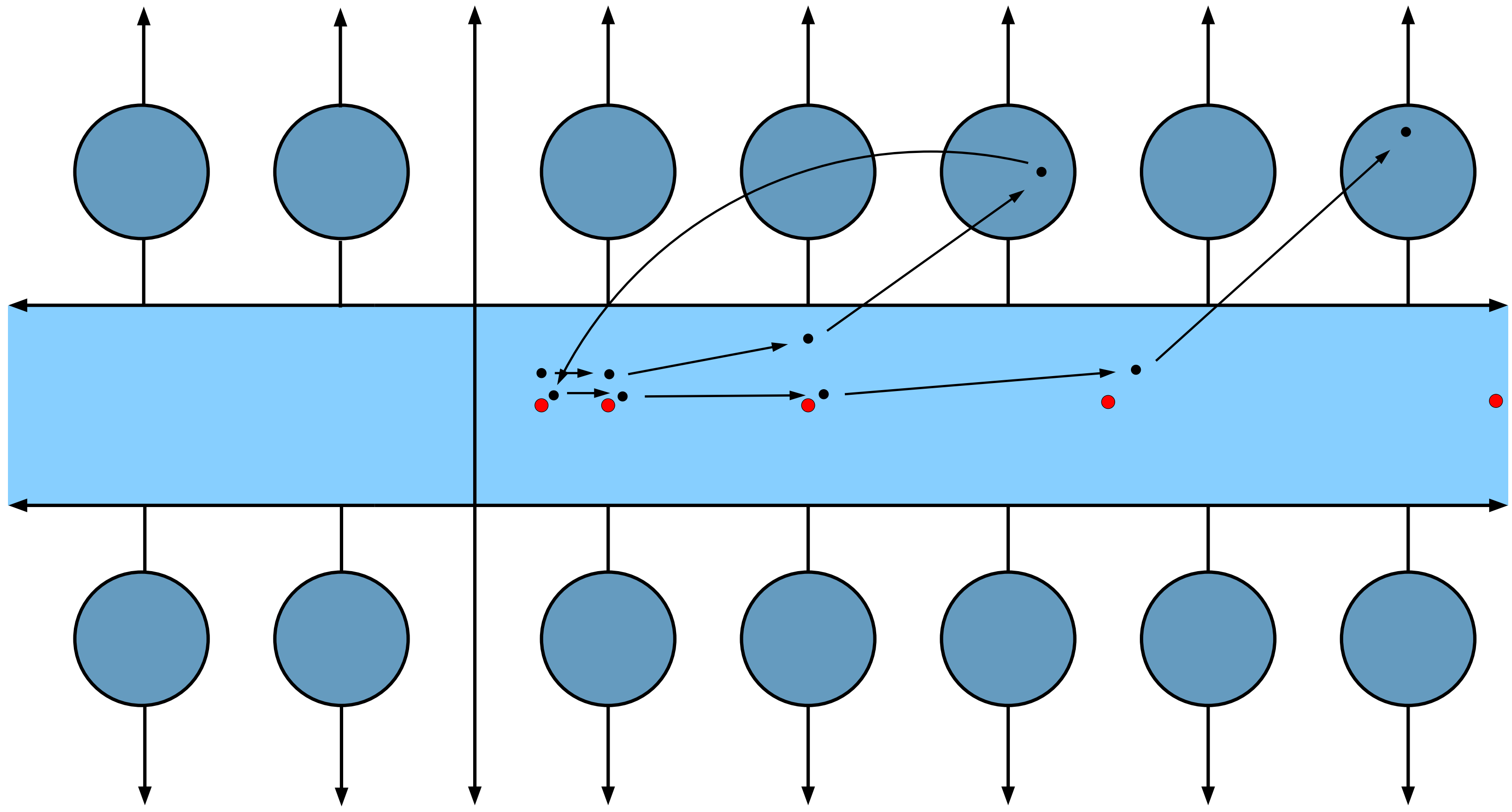


Graph giving wandering domain in Eremenko-Lyubich class.

Variations by Lazebnik, Fagella-Godillon-Jarque, Osborne-Sixsmith.

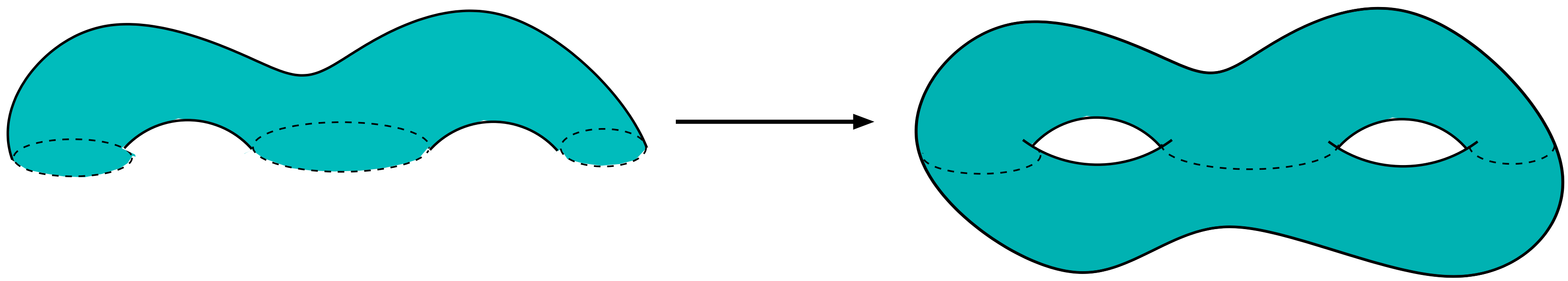
Folding a pair of pants

For Riemann surfaces, A. Epstein defines a non-constant $f : Y \to X$ to be **finite type** if set of singular values is finite. If $Y \subset X$, f can be iterated until an orbit leaves Y.

- Rational maps $S^2 \to S^2$,
- Speiser class $\mathbb{C} \to \mathbb{C}$,
- Covering map $\mathbb{D} \to X$.

Are there examples where domain is not simply connected?

Folding creates non-trivial examples, where $Y \subset X$ is a "pair of pants".



Pair of pants = sphere minus three disks.

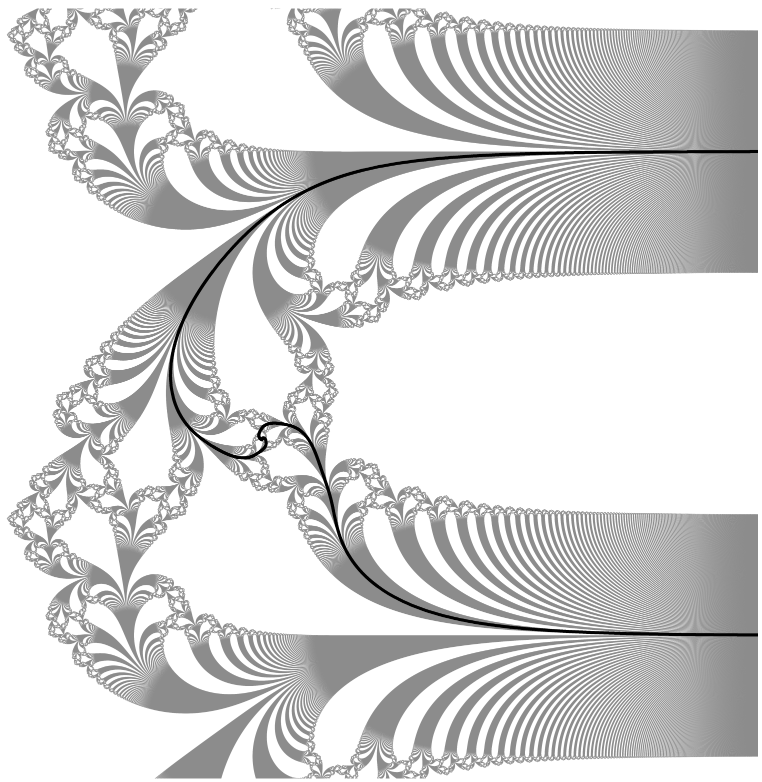
Every surface is union of these.

Does every compact surface have such self-maps? If not, which do?

For f entire, **escaping set** is $I(f) = \{z : f^n(z) \to \infty\}$.

Known that $\mathcal{J}(f) = \partial I(f)$.

Fatou observed $I(f)$ often consists of curves to ∞.



Curve in escaping set, courtesy of Lasse Rempe-Gillen

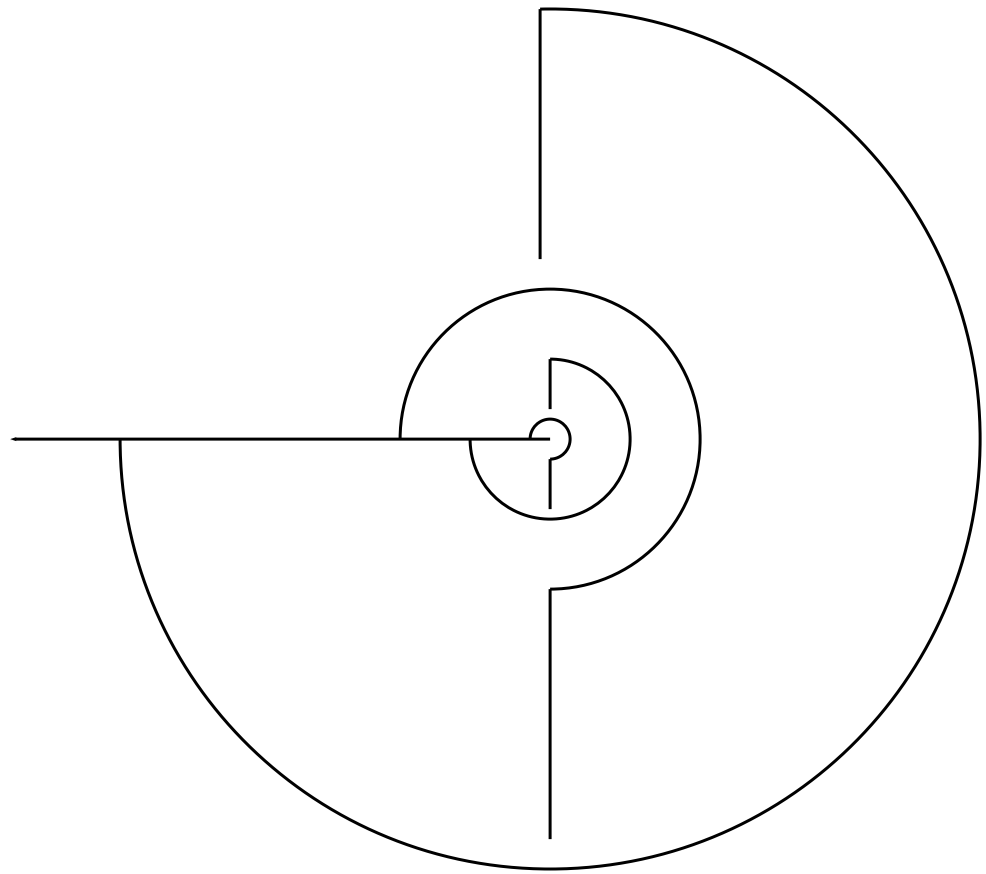
Eremenko Conj: components of $I(f)$ are unbounded (still open).

Strong Eremenko Conj (SEC): path components are unbounded.

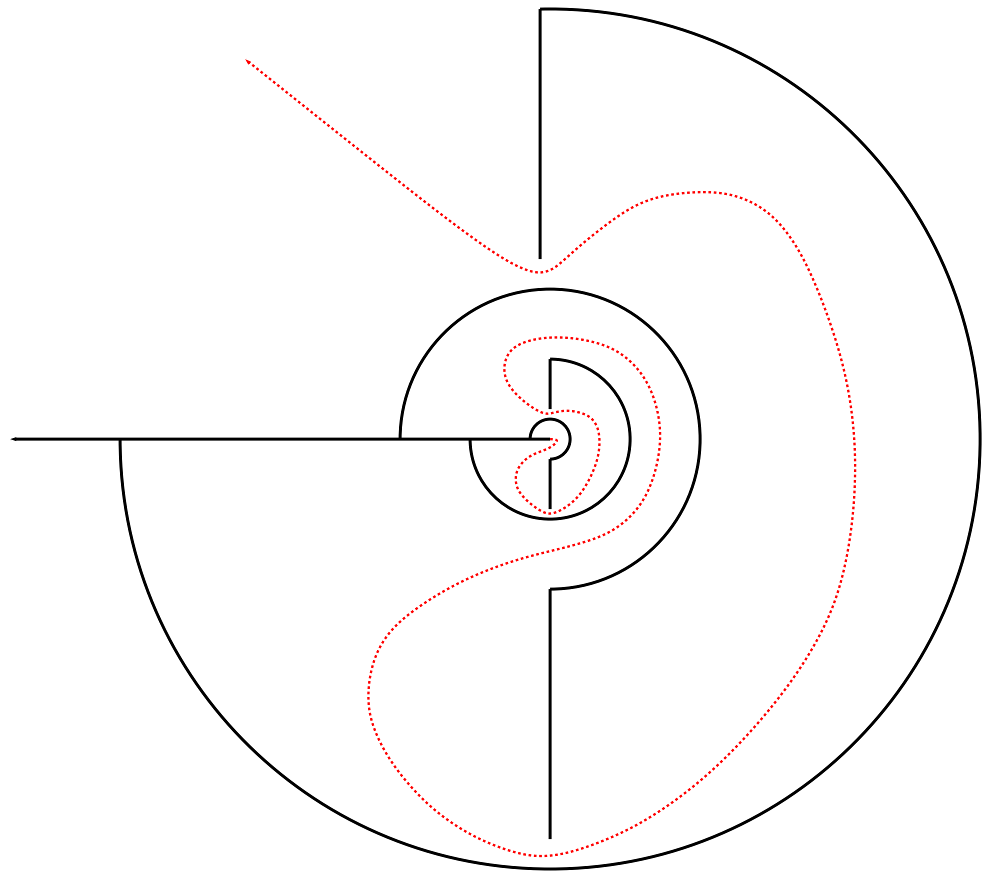
Rottenfusser, Rüchert, Rempe-Gillen and Schleicher (2011) proved:

- SEC true for EL functions with finite order of growth.
- SEC false for some EL functions with infinite order.
- Examples with trivial path components.

QC-Folding gives examples in Speiser class.



Speiser class counterexample to SEC.
Path components of escaping set can be points.
Other exotic examples by Rempe-Gillen.



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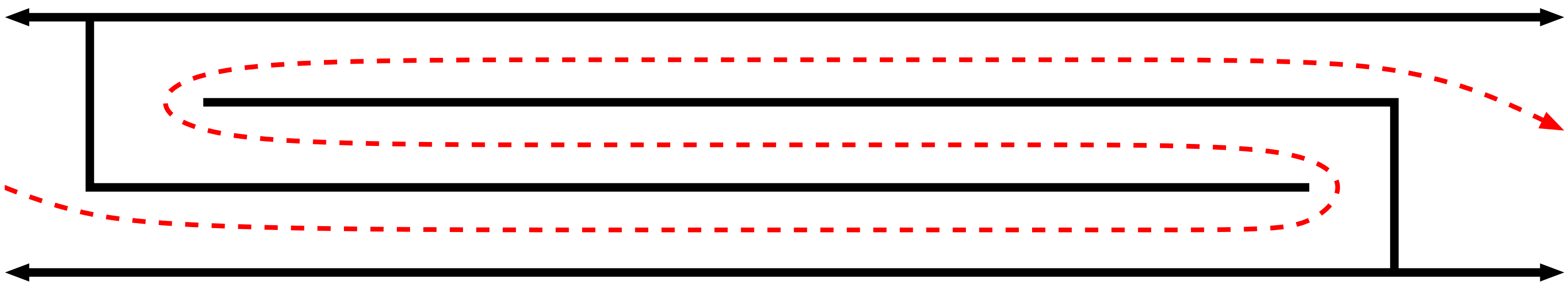

SEC counterexample in logarithmic coordinates