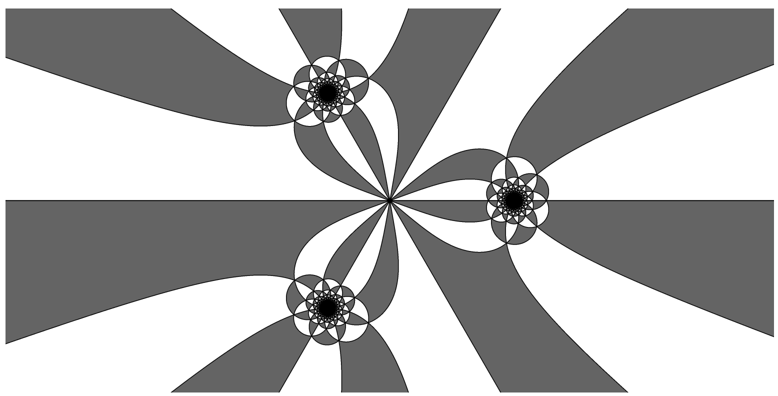
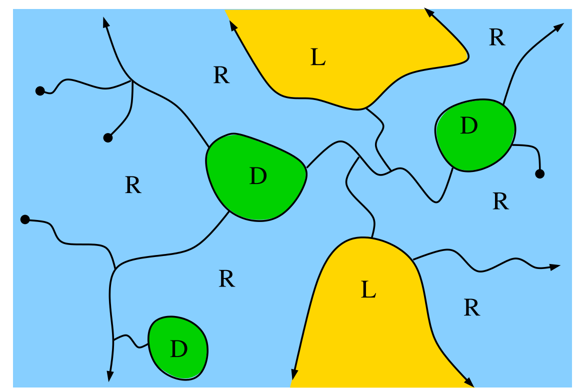
TRANSCENDENTAL FUNCTIONS WITH SMALL SINGULAR SETS

Christopher Bishop, Stony Brook

Advancing Bridges in Complex Dynamics
Luminy, September 20-24, 2021

`www.math.sunysb.edu/~bishop/lectures`





THE PLAN

- Speiser and Eremenko-Lyubich classes
- Quasiconformal maps
- All models occur in the EL-class
- Quasiconformal folding
- Meromorphic functions with prescribed singular orbits
- Equilateral triangulations of Riemann surfaces

Singular set $=$ closure of critical values and finite asymptotic values

$=$ smallest set so that f is a covering map onto $\mathbb{C} \setminus S$

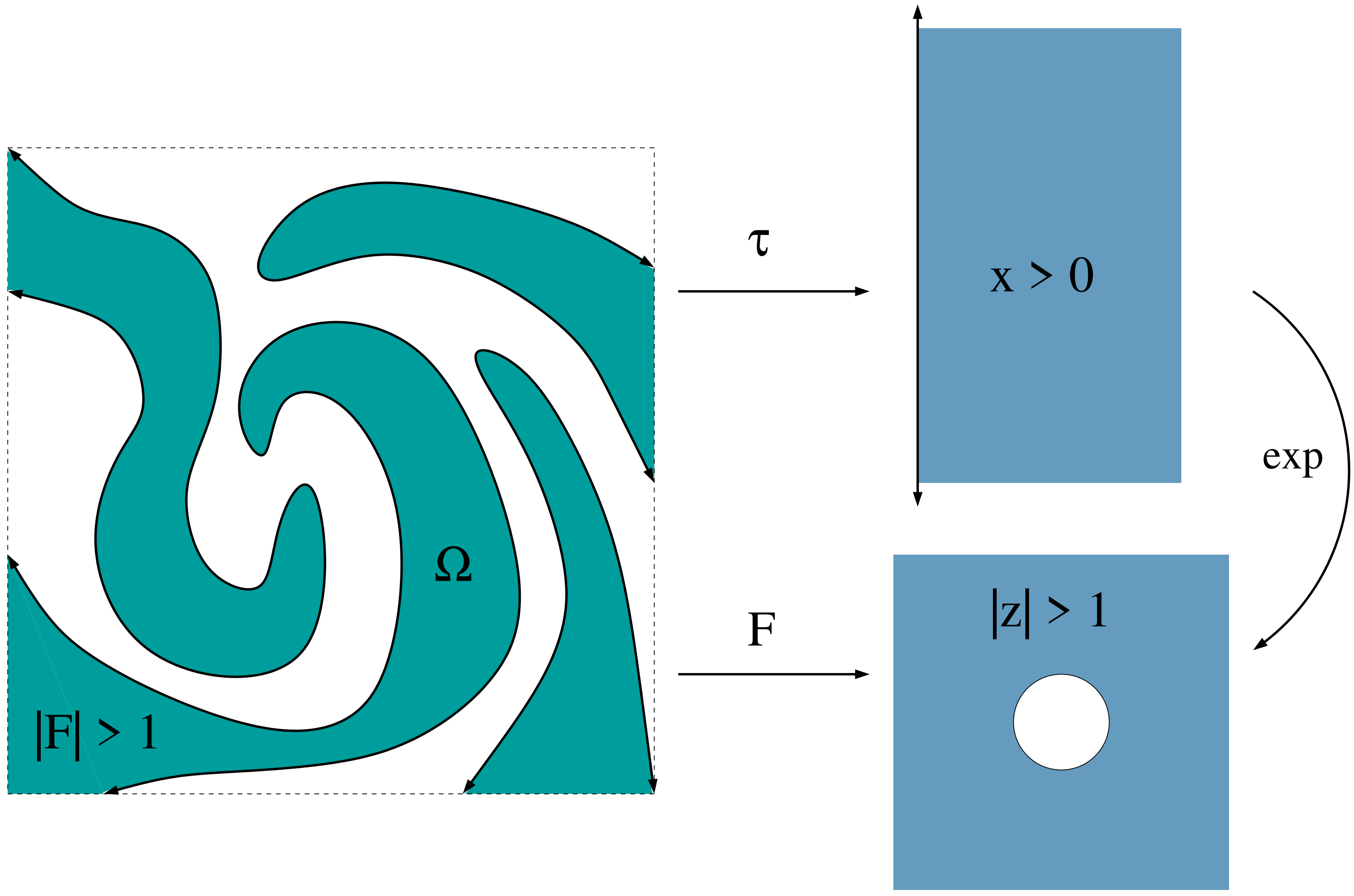
Eremenko-Lyubich class $=$ bounded singular set $= \mathcal{B}$

Speiser class $=$ finite singular set $= \mathcal{S} \subset \mathcal{B}$

Suppose $F \in$ EL-class and $S(F) \subset \mathbb{D} = \{|z| < 1\}$.

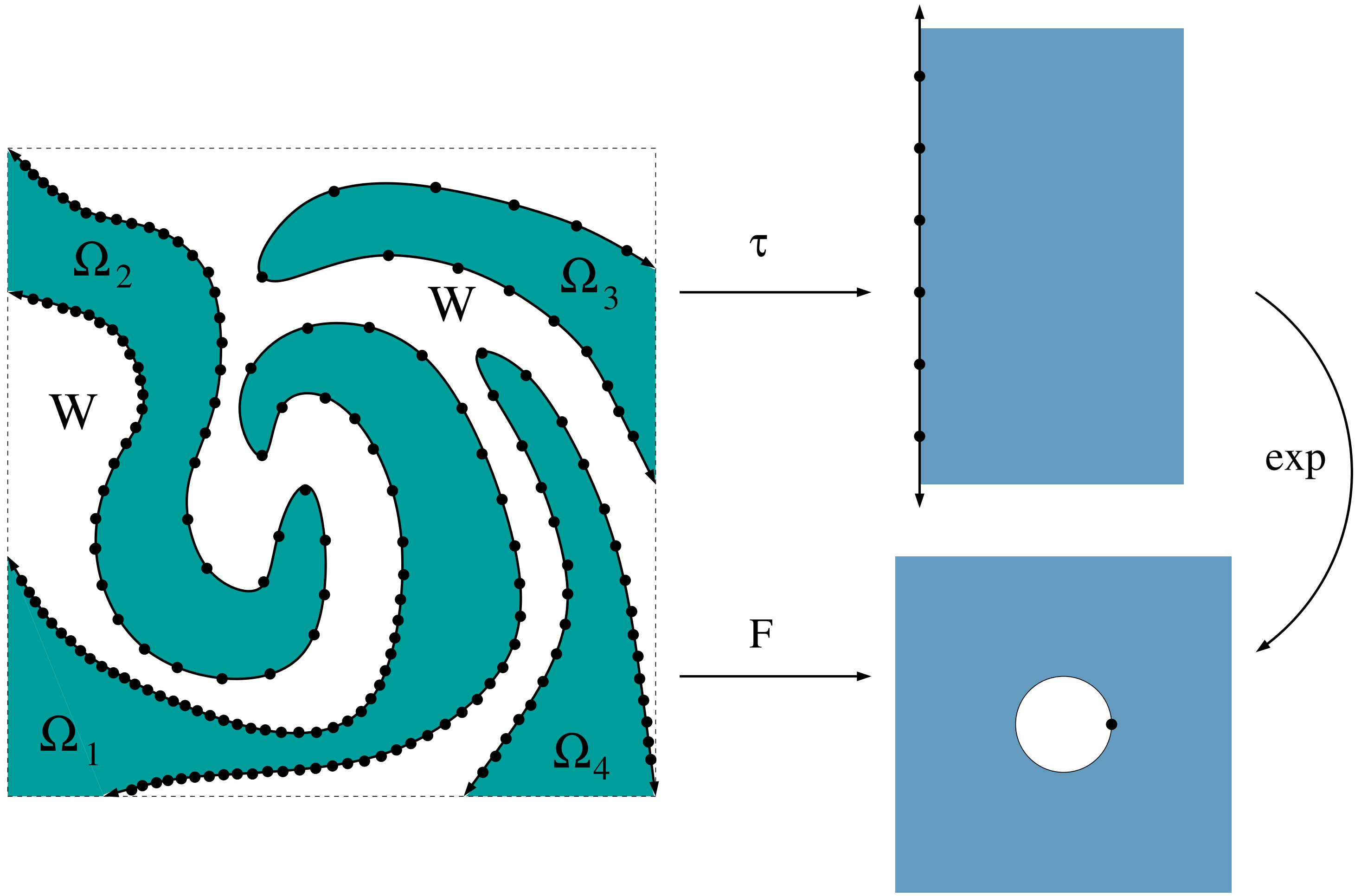
$\Omega = \{|F| > 1\}$ has simply connected components, called **tracts**

$W = \mathbb{C} \setminus \overline{\Omega} = \{|F| < 1\}$ is connected, simply connected



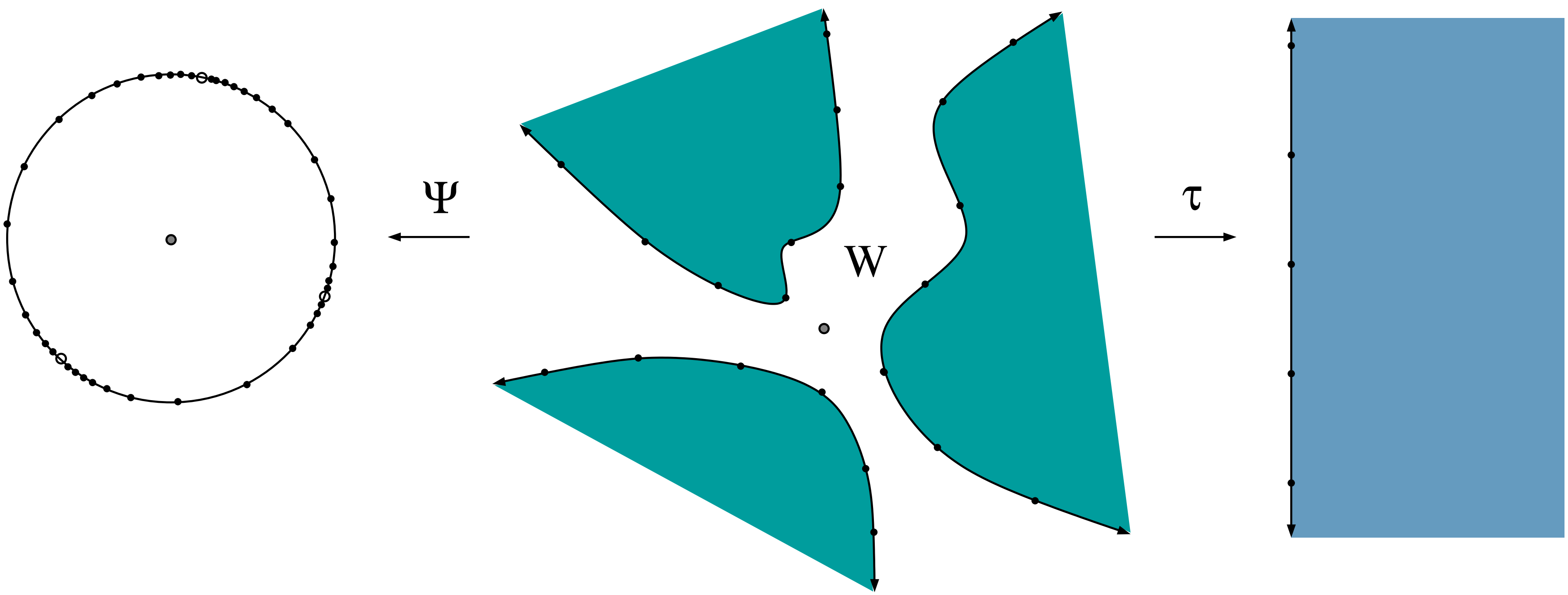
F is a covering map $\Omega \to \mathbb{D}^* = \{|z| > 1\}$, $F = \exp \circ \tau$.

τ is conformal from each tract to $\mathbb{H}_r$ = right half-plane



Dots = F pre-images of $1 = \tau^{-1}(2\pi i\mathbb{Z})$

Tract boundaries have natural partition into arc via τ.

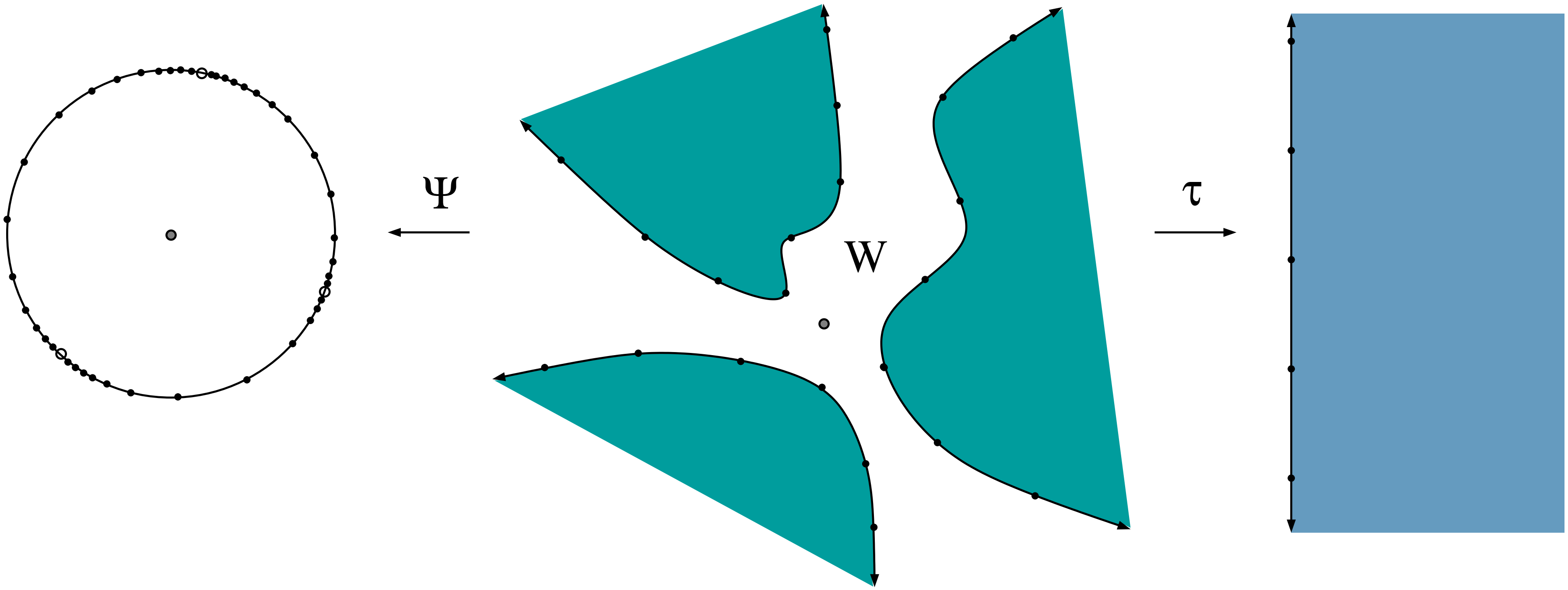


Let $\Psi : W \to \mathbb{D}$ be conformal map

Then $G = F \circ \Psi^{-1} : \mathbb{D} \to \mathbb{D}$ is an inner function.

inner function = holomorphic $\mathbb{D} \to \mathbb{D}$, $\mathbb{T} \to \mathbb{T}$ almost everywhere.

Examples: Blaschke products, $B(z) = \prod_n \frac{z-a_n}{1-\overline{a_n}z} \cdot \frac{|a_n|}{a_n}$.



Let $\Psi : W \to \mathbb{D}$ be conformal map

Then $G = F \circ \Psi^{-1} : \mathbb{D} \to \mathbb{D}$ is an inner function.

Every Eremenko-Lyubich function is a built from conformal maps on the tracts "glued together" by an inner function between the tracts.

Can we reverse this? Build entire function given only Ω and τ?

A **model** is a pair (Ω, τ) where

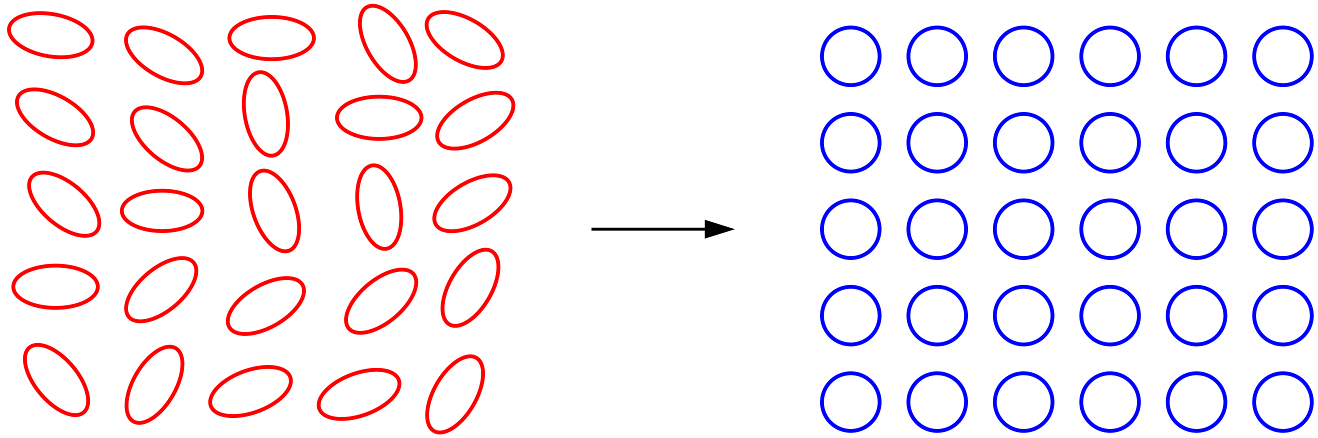
- $\Omega = \cup \Omega_j$ is a disjoint union of unbounded Jordan domains
- τ is conformal from each Ω_j to $\mathbb{H}_r$ ($\infty \to \infty$).

Every Eremenko-Lyubich function F gives a model with $\Omega = \{|F| > R\}$.

Does every model give an Eremenko-Lyubich function?

We need to review a little about quasiconformal and quasiregular maps.

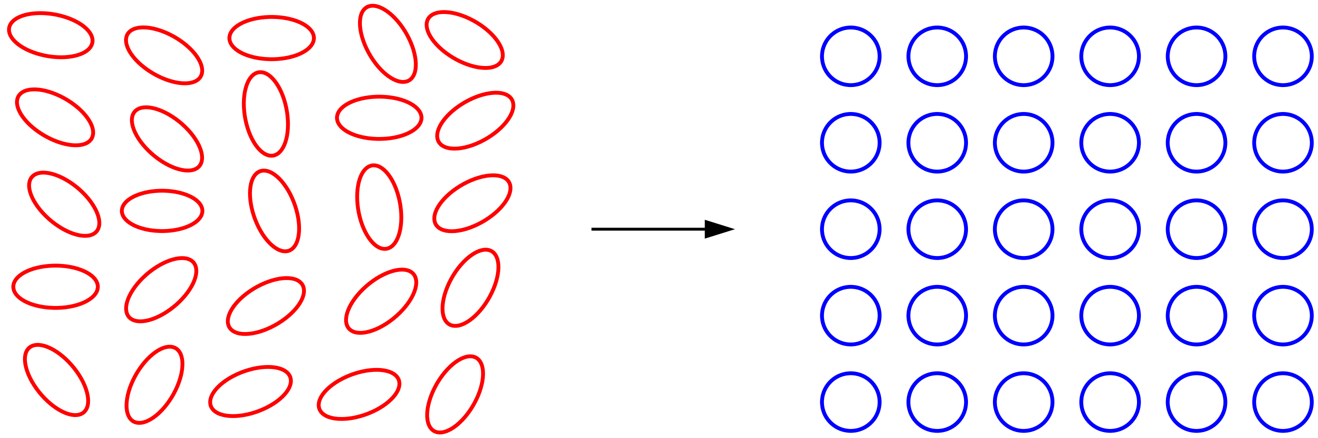
Quasiconformal (QC) maps are homeomorphisms that are differentiable a.e. and send infinitesimal ellipses to circles.



Eccentricity = ratio of major to minor axis of ellipse.

For K-QC maps, ellipses have eccentricity $\leq K$

Quasiconformal (QC) maps are homeomorphisms that are differentiable a.e. and send infinitesimal ellipses to circles.



Eccentricity = ratio of major to minor axis of ellipse.

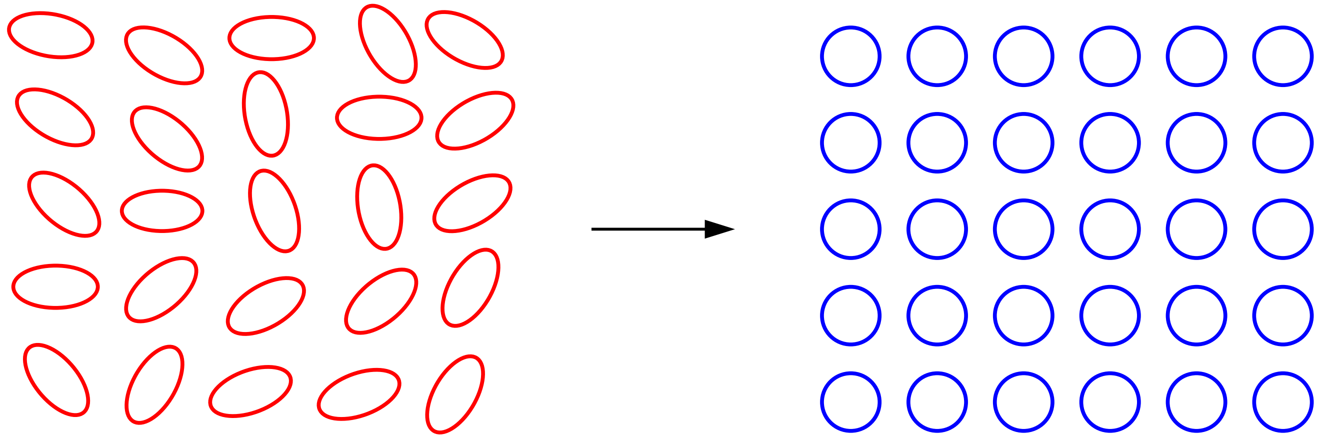
For K-QC maps, ellipses have eccentricity $\leq K$

Ellipses determined a.e. by measurable dilatation

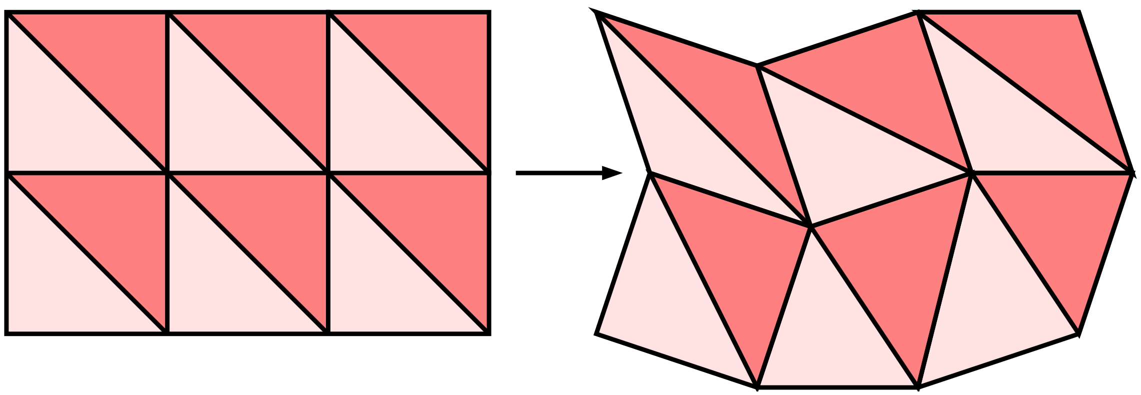
$$\mu = f_{\overline{z}}/f_z, \qquad f_{\overline{z}} = \mu \cdot f_z, \quad \text{with } |\mu| \leq \frac{K-1}{K+1} < 1.$$

Here $f_z = f_x - i f_y$ and $f_{\overline{z}} = f_x + i f_y$.

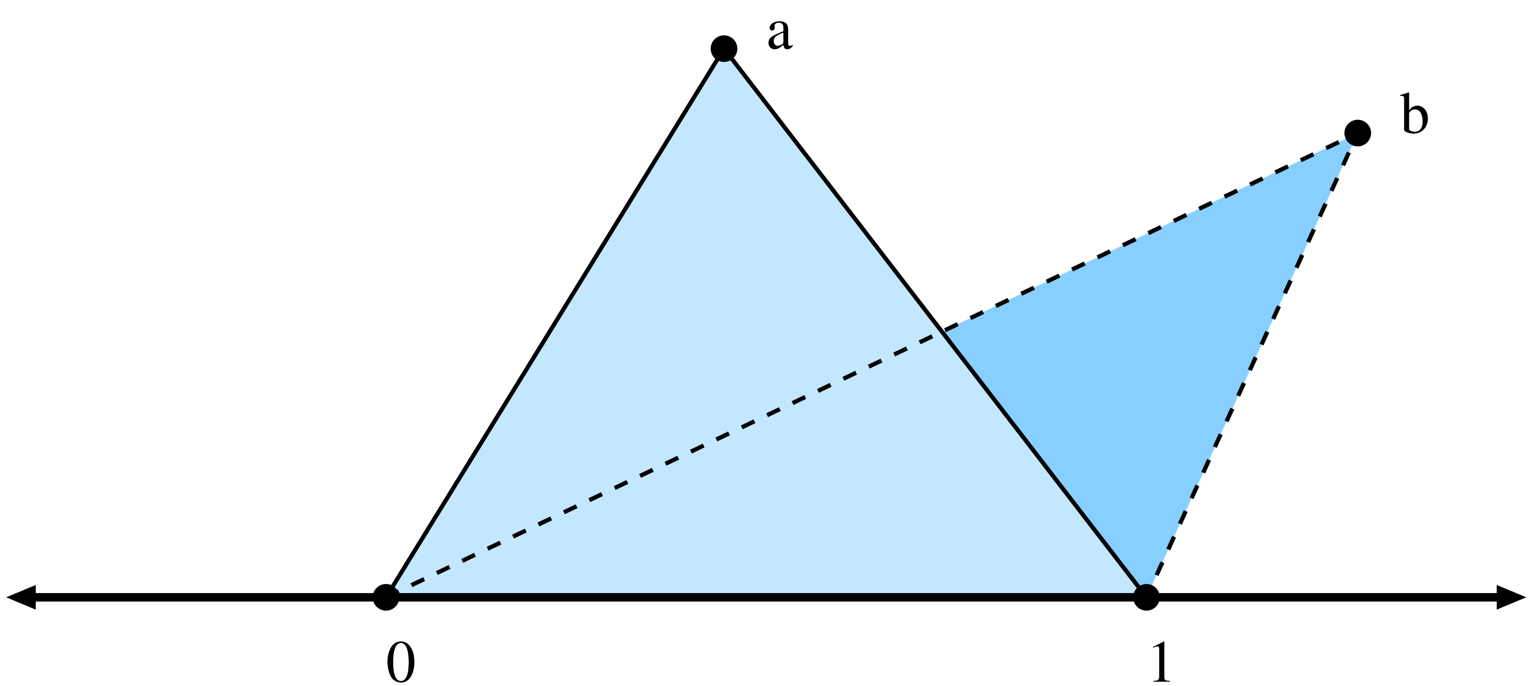
Quasiconformal (QC) maps are homeomorphisms that are differentiable a.e. and send infinitesimal ellipses to circles.



Example: piecewise affine maps between triangulations.



Map is QC if all angles bounded above and below.

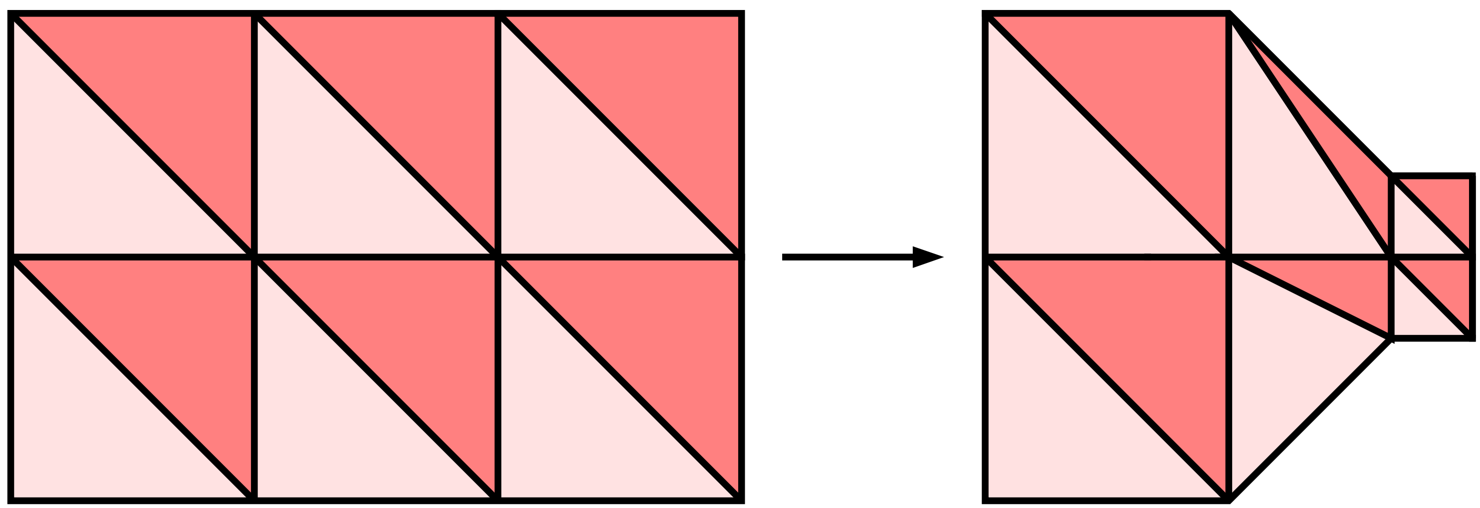


Affine map between triangles $\{0, 1, a\}$ and $\{0, 1, b\}$ has constant dilatation

$$\mu = \frac{b-a}{b-\overline{a}}$$

(For experts, this is pseudo-hyperbolic distance in upper half-plane.)

Example: piecewise affine maps between triangulations.



Recall that the dilatation of a QC map is

$$\mu = \frac{f_{\overline{z}}}{f_z} \quad \text{or} \quad f_{\overline{z}} = \mu \cdot f_z$$

Measurable Riemann Mapping Theorem:

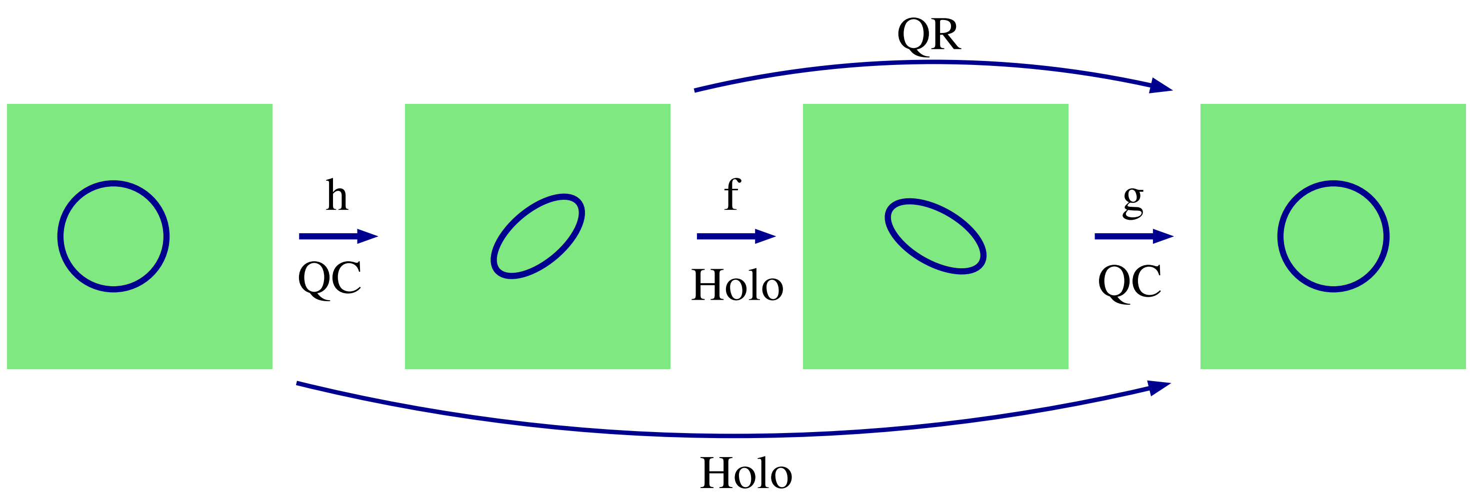
Given a measurable dilatation μ on the unit disk with $\|\mu\|_\infty < 1$, there is a quasiconformal $f : \mathbb{D} \to \mathbb{D}$ with this dilatation.

Recall that the dilatation of a QC map is

$$\mu = \frac{f_{\overline{z}}}{f_z} \quad \text{or} \quad f_{\overline{z}} = \mu \cdot f_z$$

Measurable Riemann Mapping Theorem:
Given a measurable dilatation μ on the unit disk with $\|\mu\|_\infty < 1$, there is a quasiconformal $f : \mathbb{D} \to \mathbb{D}$ with this dilatation.

Cor: If f is holomorphic and g is QC, then there is a QC map h so that $F = g \circ f \circ h$ is also holomorphic. ($g \circ f$ is quasiregular.)



Now, back to models for the Eremenko-Lyubich class.

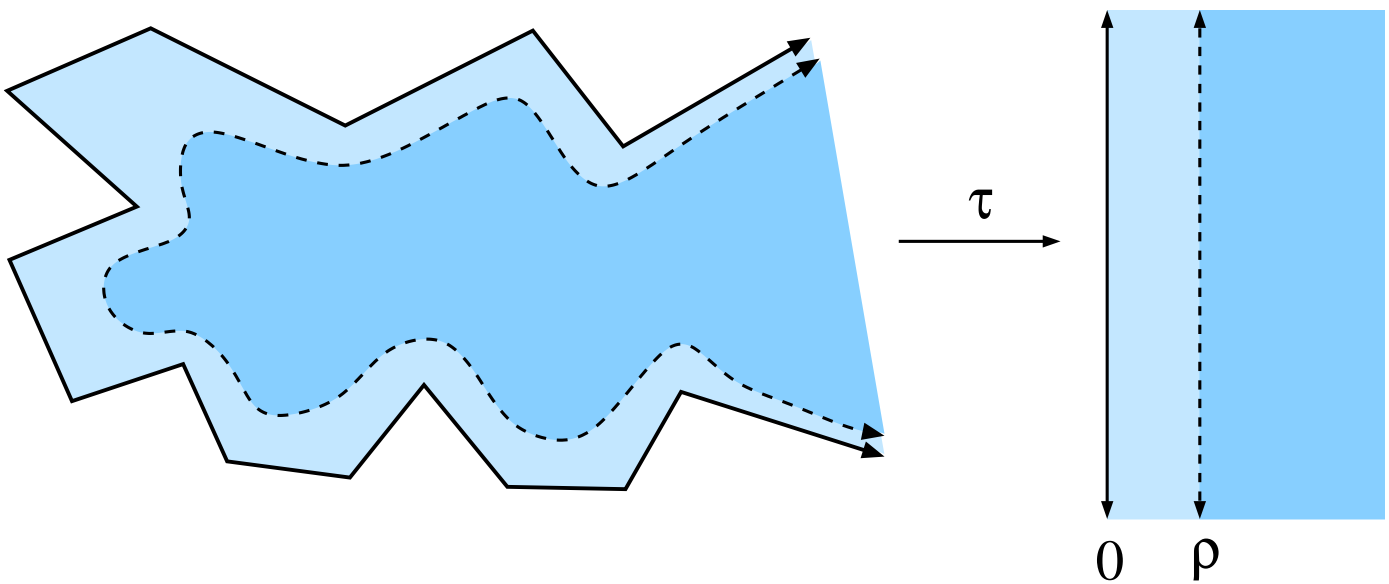
Theorem: Suppose (Ω, τ) is a model and $\rho > 0$. Define

$$\Omega(\rho) = \tau^{-1}(\{x + iy : x > \rho\}) \subset \Omega.$$

Then there is a quasiregular g so that

(1) $g = e^{\tau}$ on $\Omega(\rho)$,

(2) $|g| \leq e^{\rho}$ off Ω.



Theorem: Suppose (Ω, τ) is a model and $\rho > 0$. Define

$$\Omega(\rho) = \tau^{-1}(\{x + iy : x > \rho\}) \subset \Omega.$$

Then there is a quasiregular g so that

(1) $g = e^{\tau}$ on $\Omega(\rho)$,

(2) $|g| \leq e^{\rho}$ off Ω.

The QR constant depends on ρ, but not on Ω.

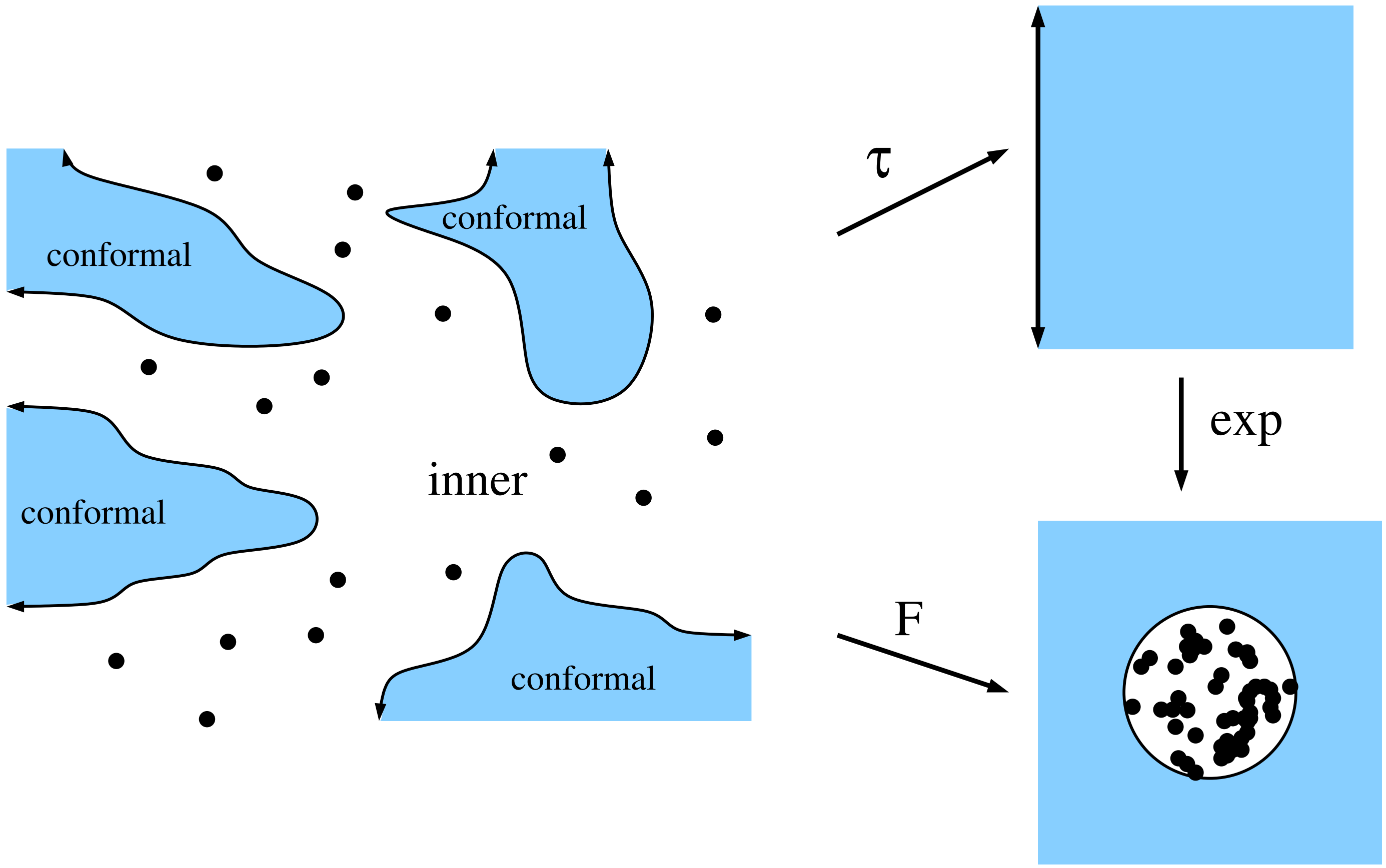
There is a quasiconformal φ so that $f = g \circ \varphi \in \mathcal{B}$.

g is holomorphic except on $\Omega(\rho) \setminus \Omega(\rho/2)$ (often has finite area).

Tracts of f correspond to components of Ω. Very similar shapes.

How do we improve from bounded singular set to finite set?

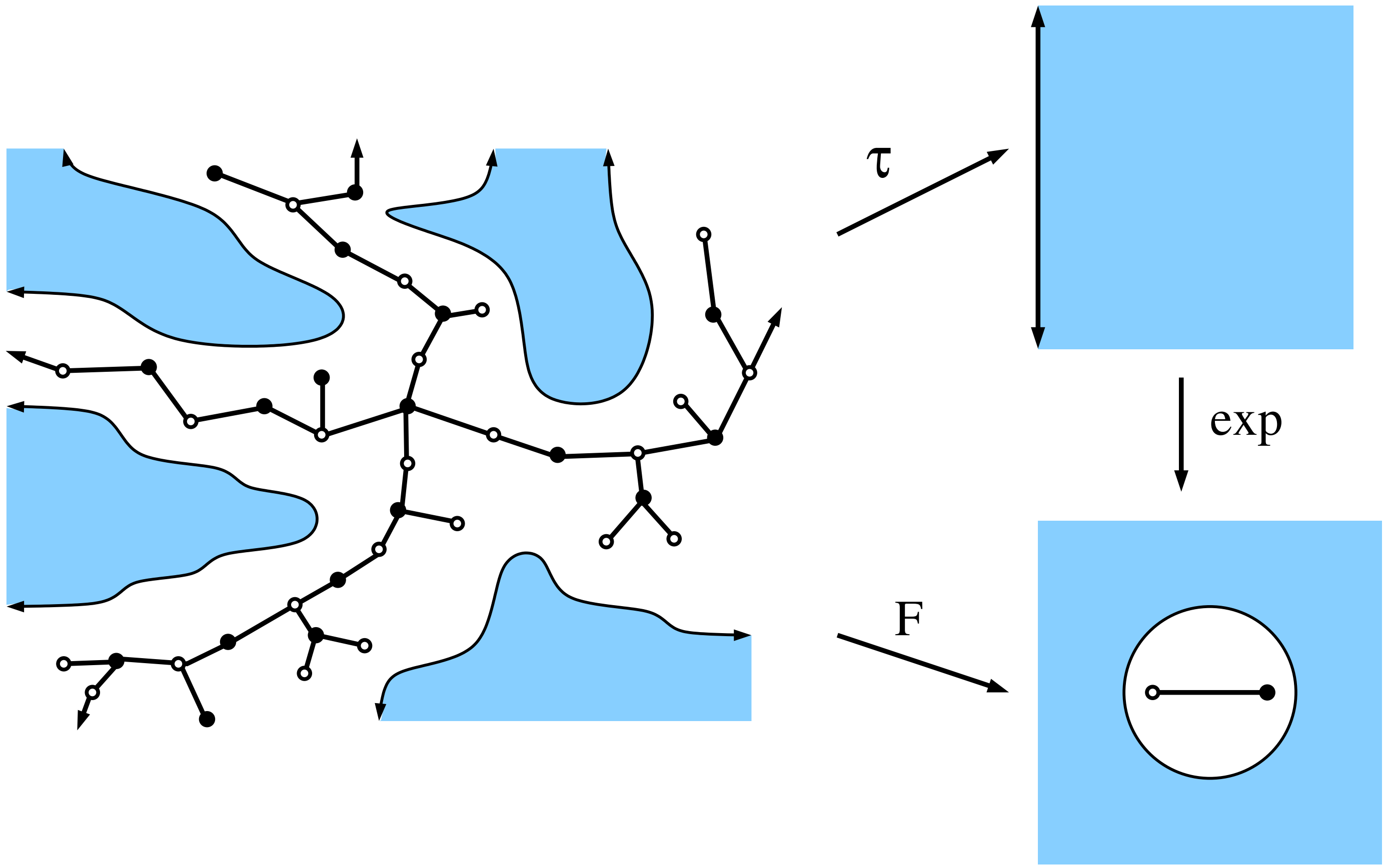
How do we improve from bounded singular set to finite set?



Eremenko-Lyubich function maps tracts to $\{|z| > 1\}$.

Inner function has many critical points. Critical values anywhere in disk.

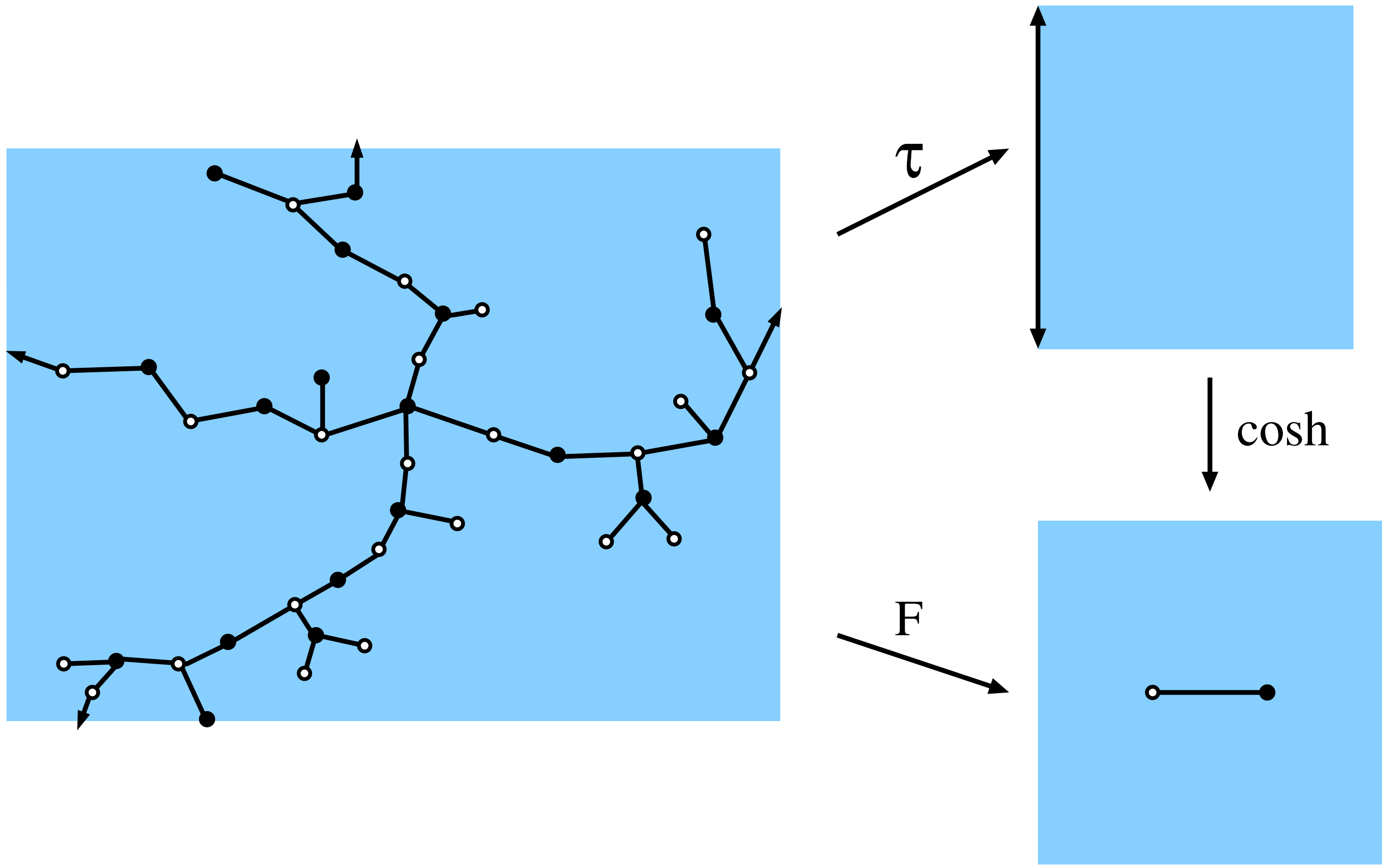
How do we improve from bounded singular set to finite set?



Suppose there are only two critical values, ± 1.

Connect by segment. Segment lifts to a tree.

How do we improve from bounded singular set to finite set?

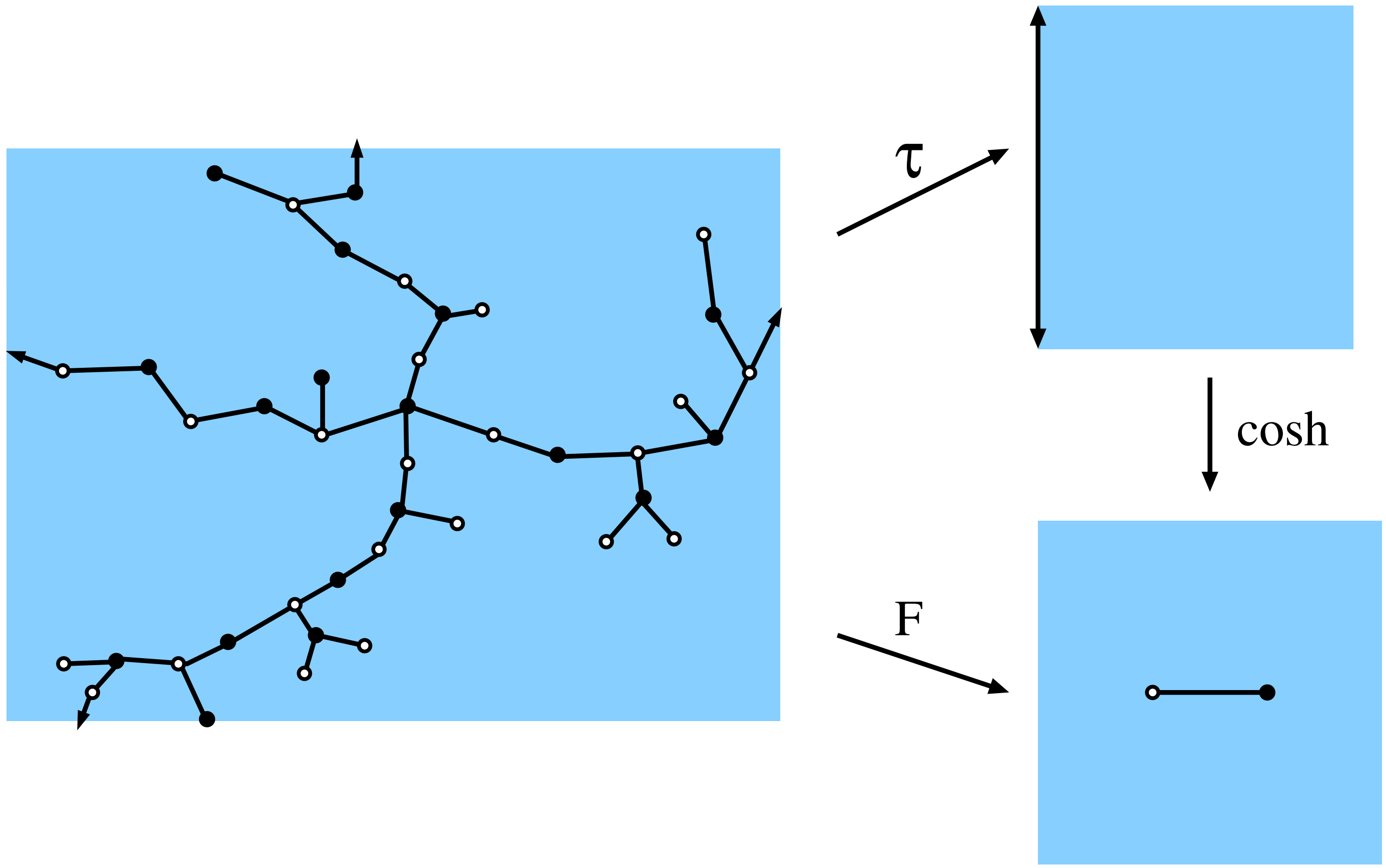


Let complementary components of tree be tracts.

$F(z) = \cosh(\tau(z))$; maps to complement of $[-1, 1]$.

All Speiser functions give trees. Do all trees give functions?

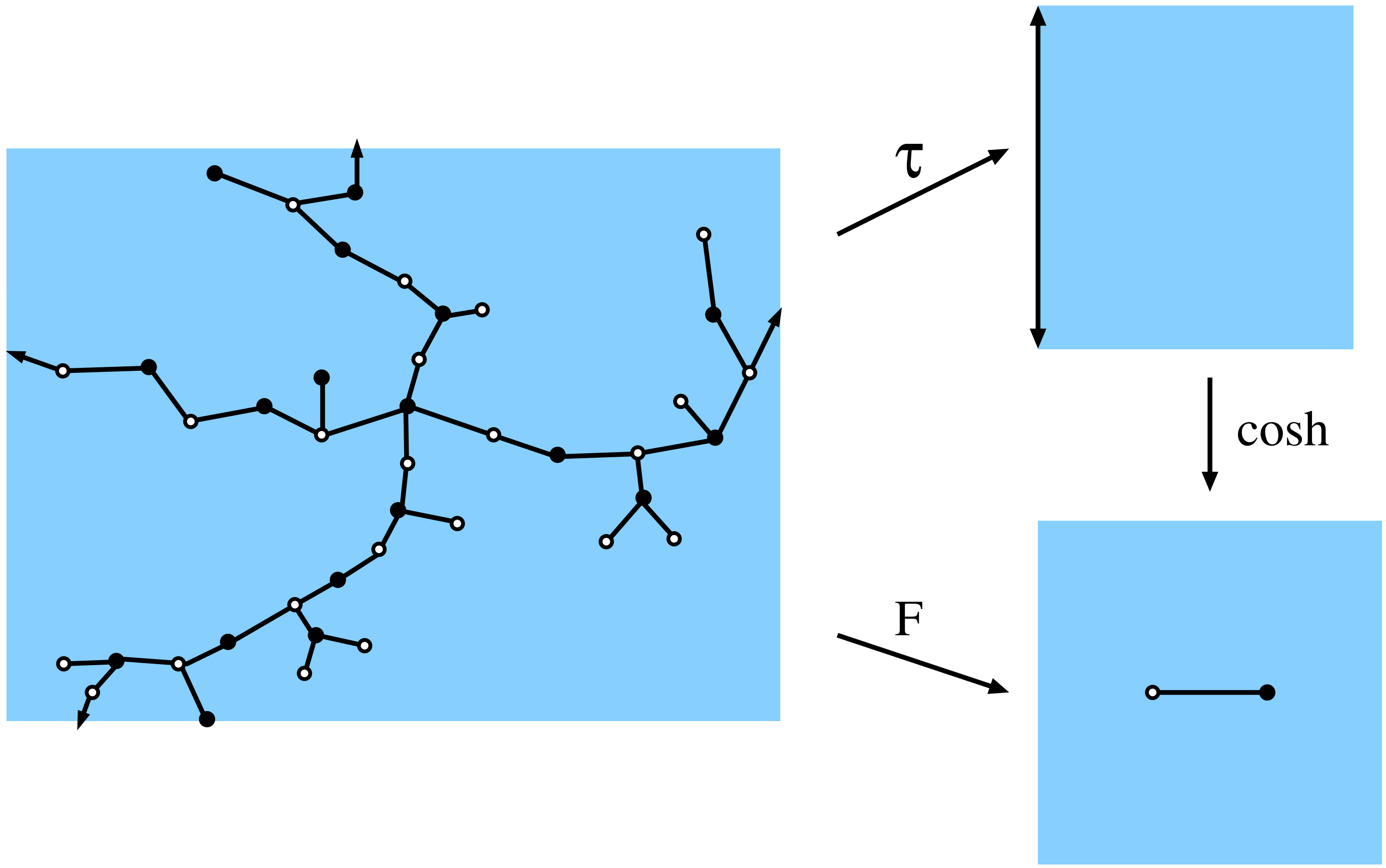
How do we improve from bounded singular set to finite set?



Given tree T, $\cosh \tau$ need not be continuous across T.

Can we replace $\cosh \tau$ by QR function in a neighborhood of tree?

How do we improve from bounded singular set to finite set?



Yes. Need some definitions:

- Localized Hausdorff neigbourhood of tree
- Good local behavior: edges smooth and separated
- Good global behavior: τ-images of edges are not too short

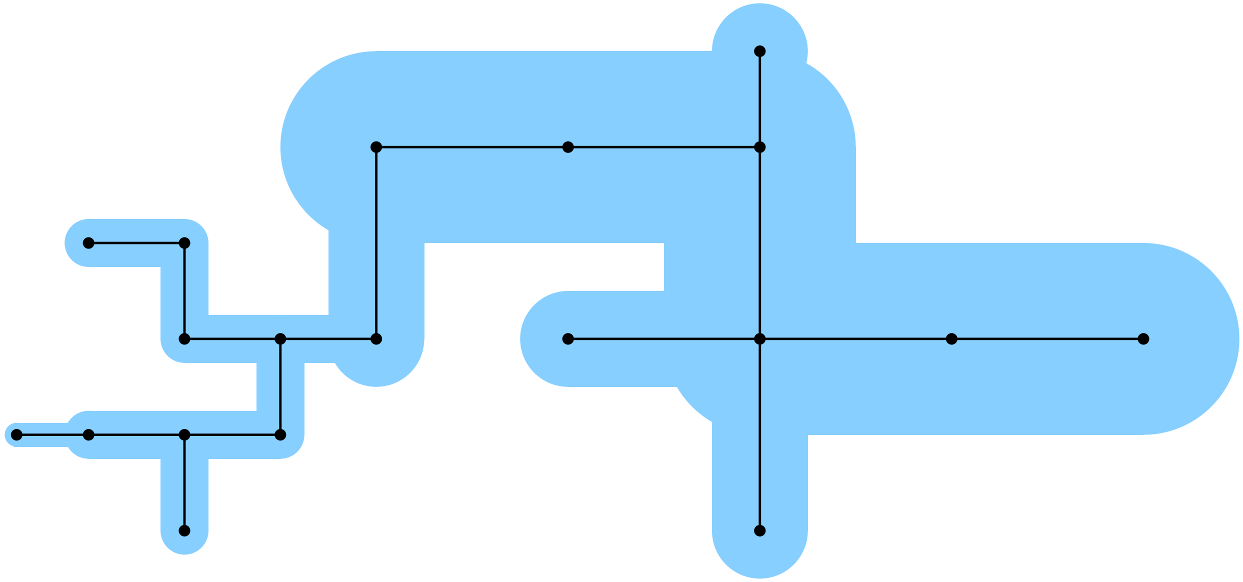
If e is an edge of T and $r > 0$ let

$$e(r) = \{z : \mathrm{dist}(z, e) \leq r \cdot \mathrm{diam}(e)\}$$

If e is an edge of T and $r > 0$ let

$$e(r) = \{z : \text{dist}(z, e) \leq r \cdot \text{diam}(e)\}$$

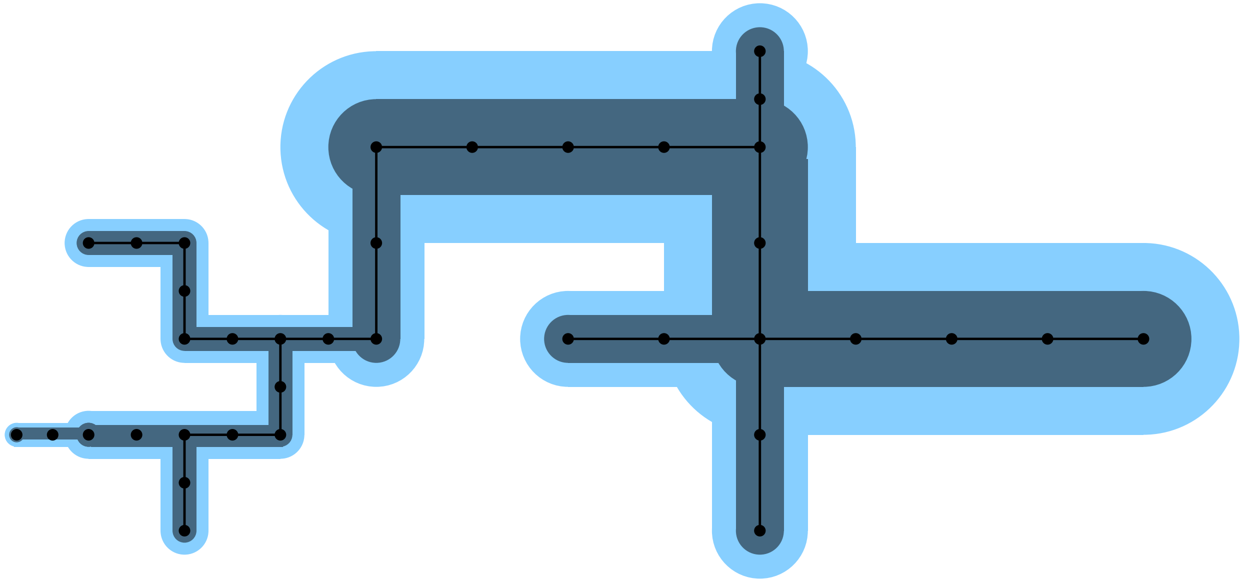
Define neighborhood of T: $T(r) = \cup\{e(r) : e \in T\}$.



If e is an edge of T and $r > 0$ let

$$e(r) = \{z : \text{dist}(z, e) \leq r \cdot \text{diam}(e)\}$$

Define neighborhood of T: $T(r) = \cup\{e(r) : e \in T\}$.

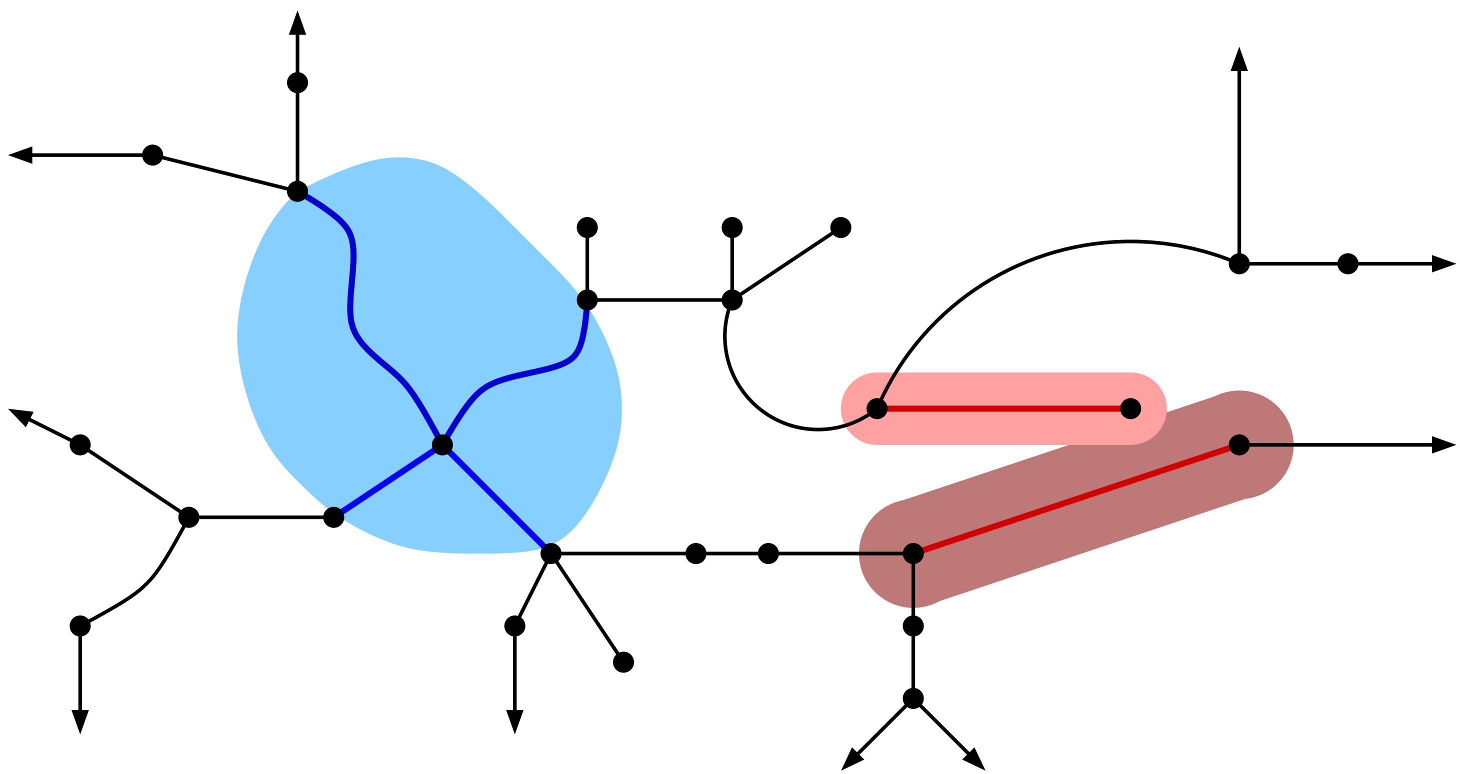


Adding vertices reduces $T(r)$. Useful scaling property.

Bounded Geometry (local condition; easy to verify):

- edges are uniformly smooth.
- adjacent edges form bi-Lipschitz image of a star $= \{z^n \in [0, r]\}$
- non-adjacent edges are well separated,

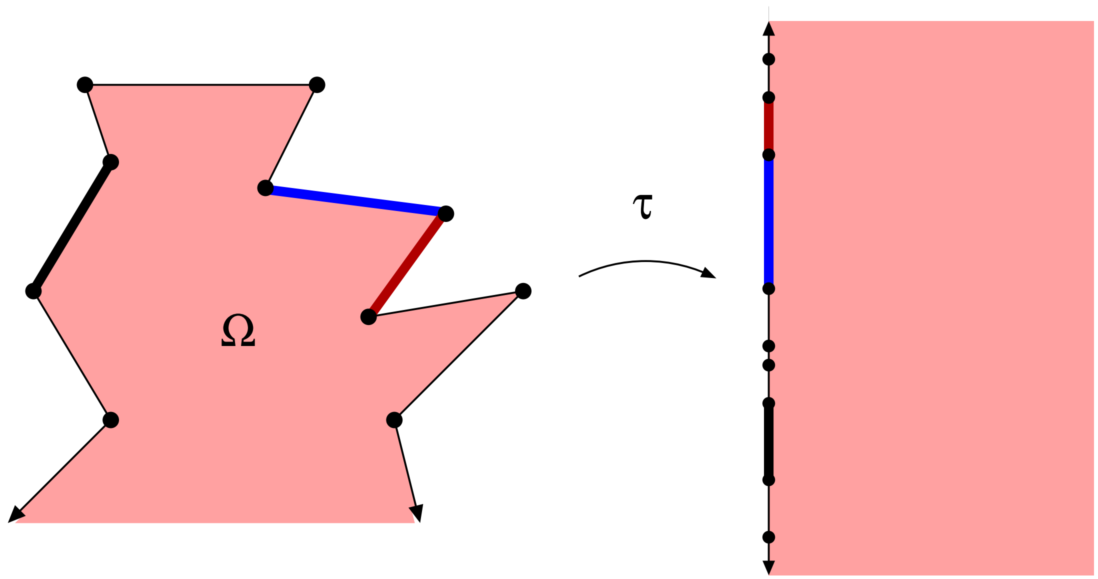
$$\mathrm{dist}(e, f) \geq \epsilon \cdot \max(\mathrm{diam}(e), \mathrm{diam}(f)).$$



τ-Lower Bound (global condition; harder to check):

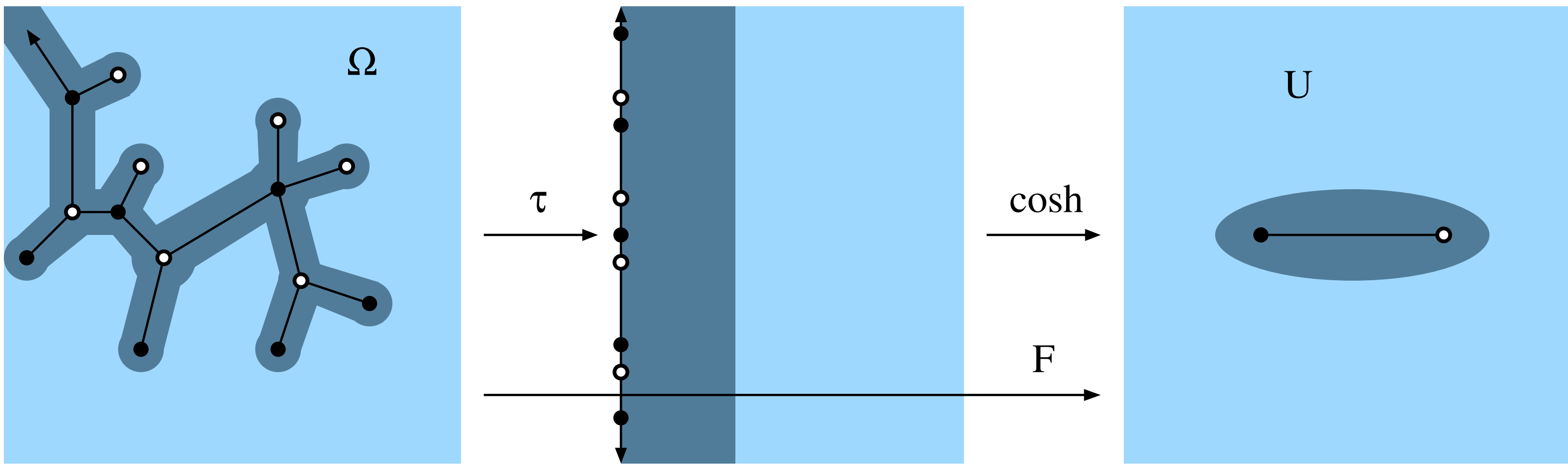
Complementary components of tree are simply connected.

Each can be conformally mapped to right half-plane. Call map τ.



Assume all images have length $\geq \pi$.

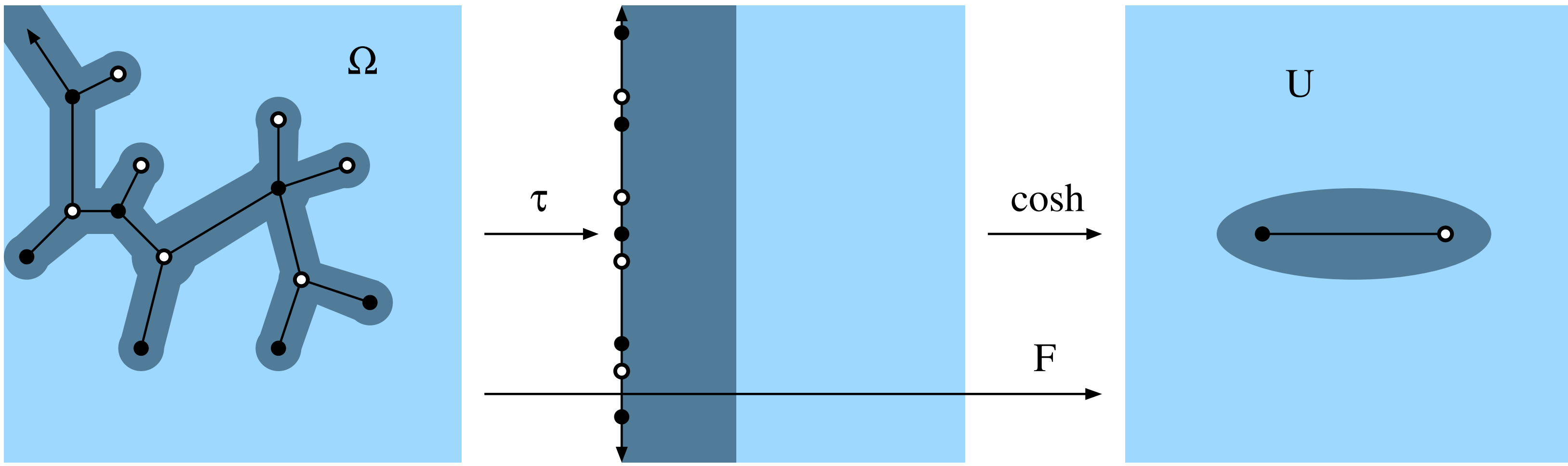
QC-Folding Theorem: If T has bounded geometry and the τ-lower bound, then there is a K-quasiregular g and $r > 0$ such that $g = \cosh \circ \tau$ off $T(r)$ (shaded) and $\mathrm{CV}(g) = \pm 1$.



K and r only depend on the bounded geometry constants.

$g = F$ on light blue.

QC-Folding Theorem: If T has bounded geometry and the τ-lower bound, then there is a K-quasiregular g and $r > 0$ such that $g = \cosh \circ \tau$ off $T(r)$ (shaded) and $\mathrm{CV}(g) = \pm 1$.

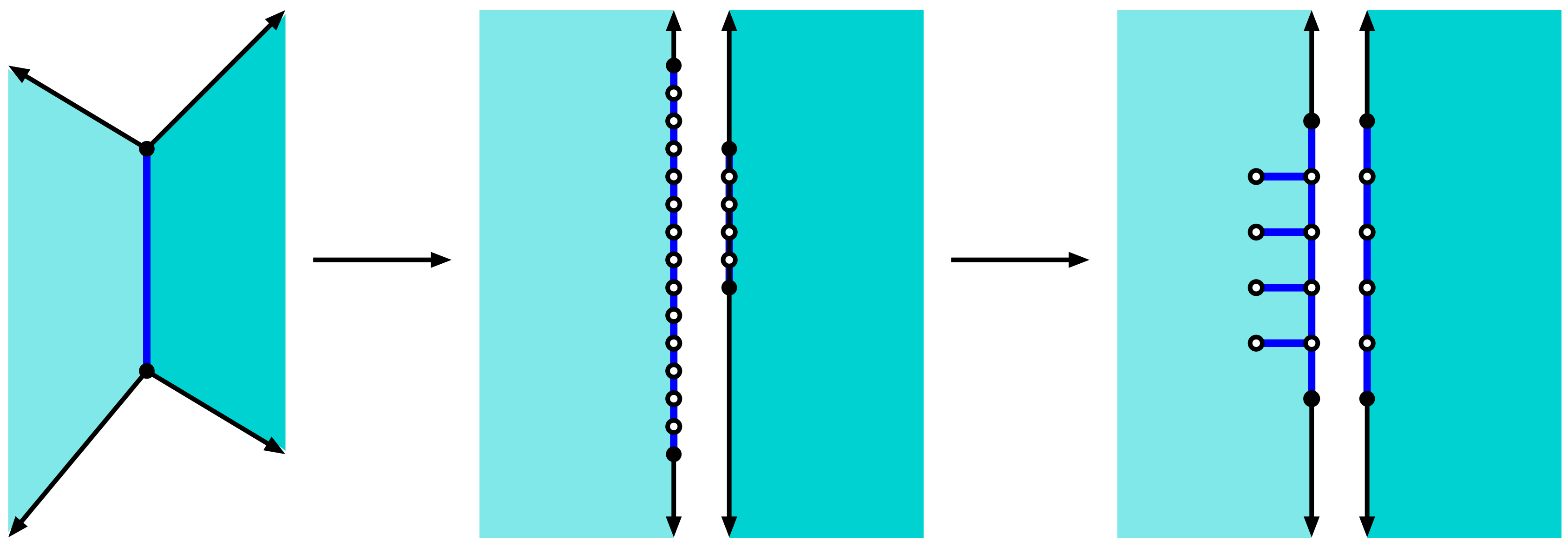


K and r only depend on the bounded geometry constants.

Cor: Any T as above is approximated by $\varphi(T')$ where $T' = f^{-1}([-1, 1])$, f is entire with $\mathrm{CV}(f) = \pm 1$, φ is QC and conformal off $T(r)$.

In many applications, φ is close to identity, so $T \approx T'$.

Where the "folding" occurs:



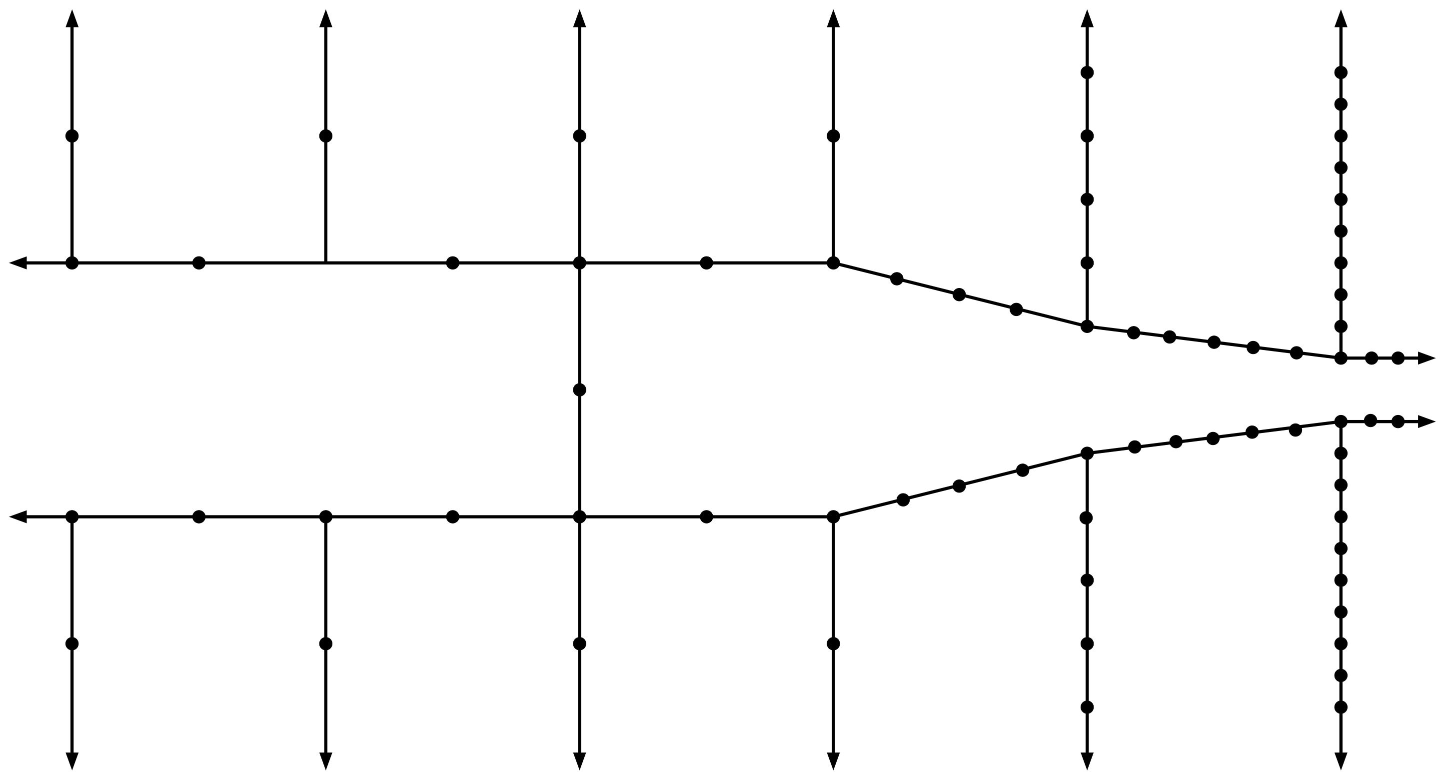
One edge of tree has two images under τ, possibly different lengths.

QC-folding makes length the same, by folding one side into itself.

Hard part is to keep QC constant bounded, independent of length ratio.

Application 1: Speiser functions that grow quickly

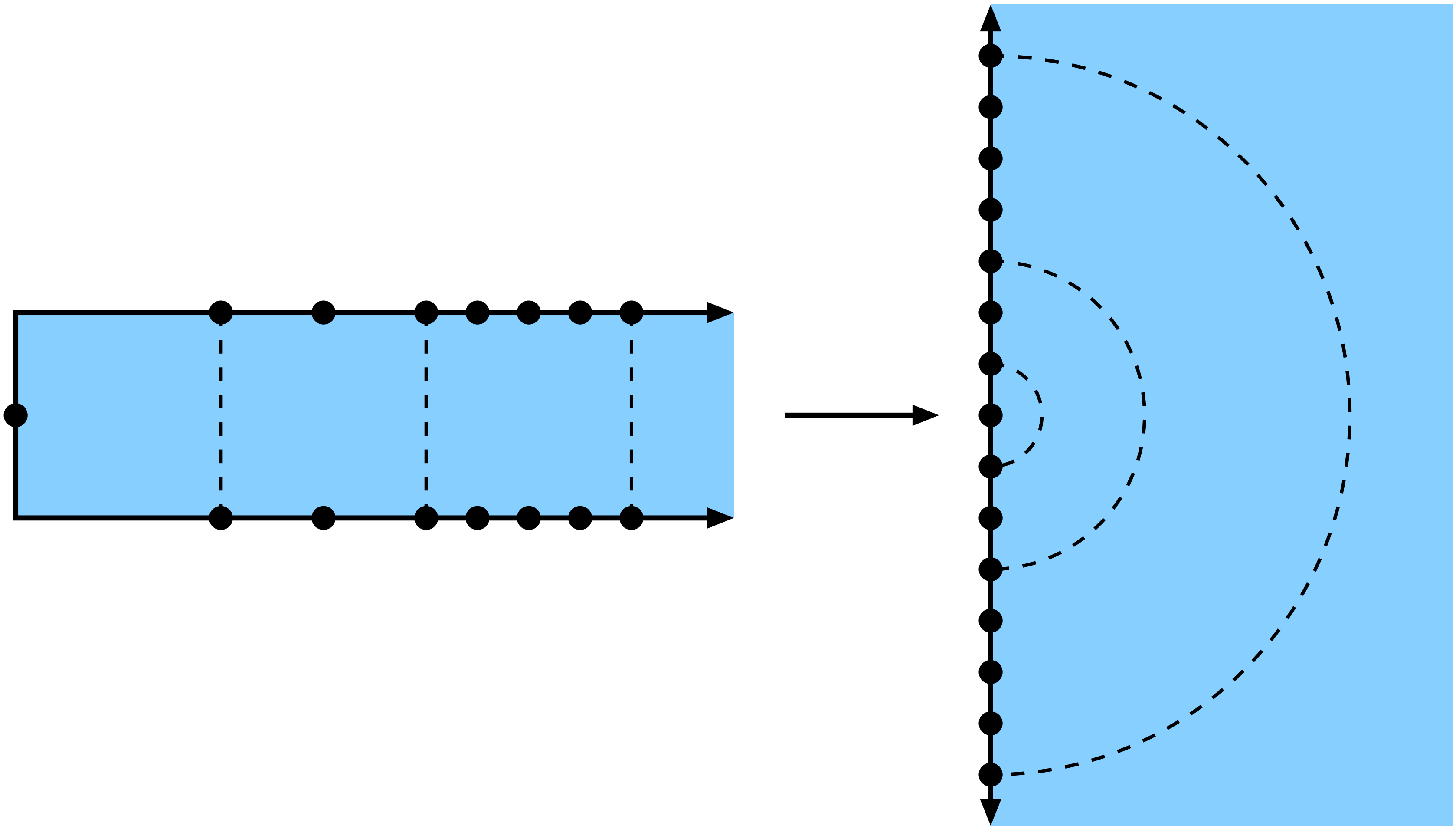
Rapid increase in Speiser class



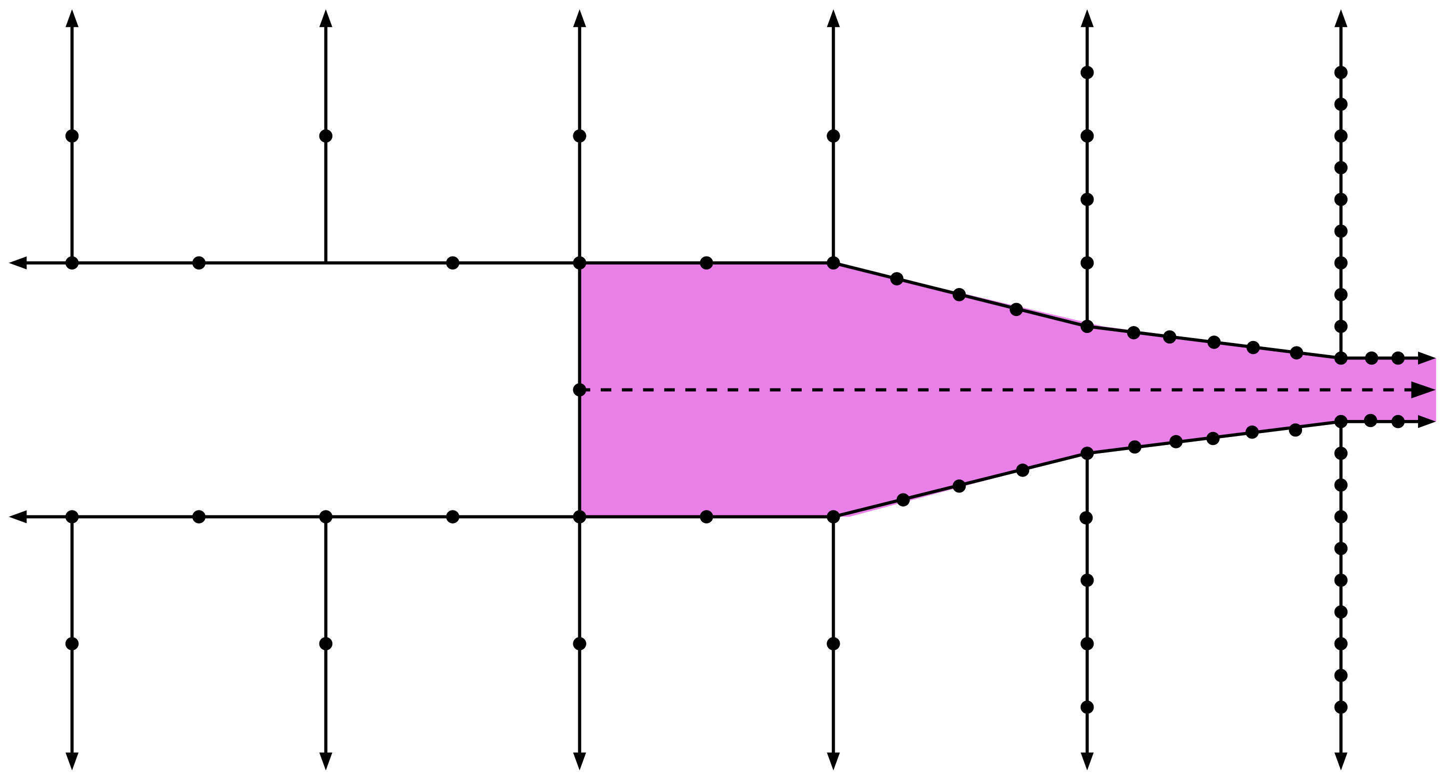
Check that this tree has:

(1) bounded geometry,

(2) the τ-lower bound.



Rapid increase in Speiser class

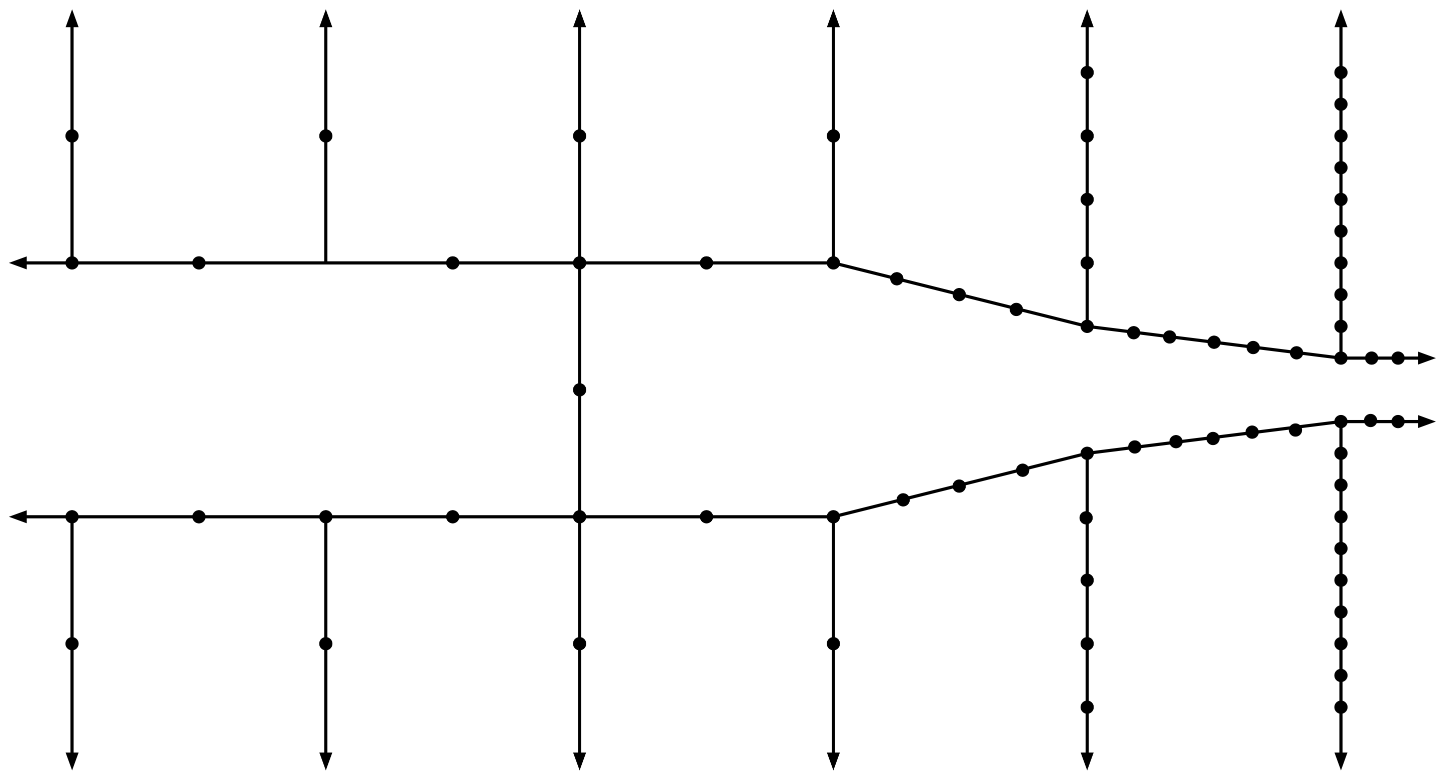


Conformal map from pinched strip to half-plane grows as fast as we want.

Folding gives quasiregular g with radpid growth.

What about holomorphic $f = g \circ \varphi$?

Rapid increase in Speiser class

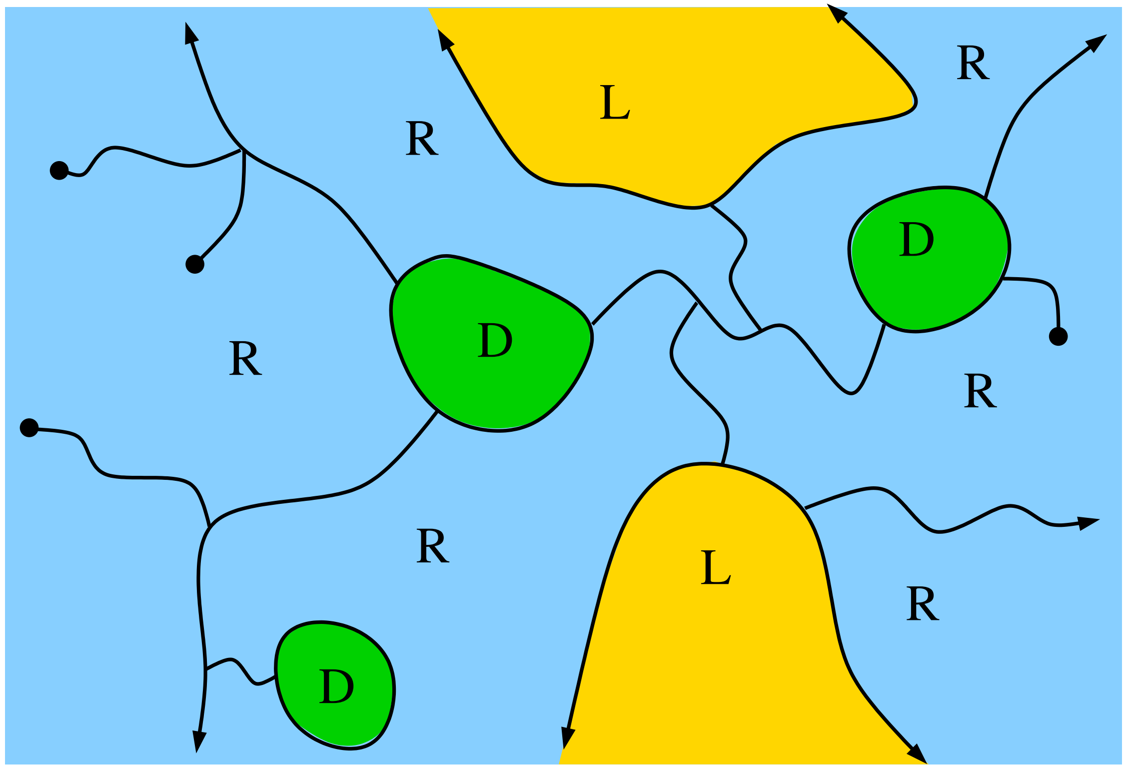


Correction map φ is Hölder, only slows growth a little.

We get f with 2 singular values, $f(x) \nearrow \infty$ as fast as we wish.

Similar examples due to Sergei Merenkov (2008) (3 singular values).

Other applications need a more general folding theorem: replace tree by graph, faces labeled D,L,R.

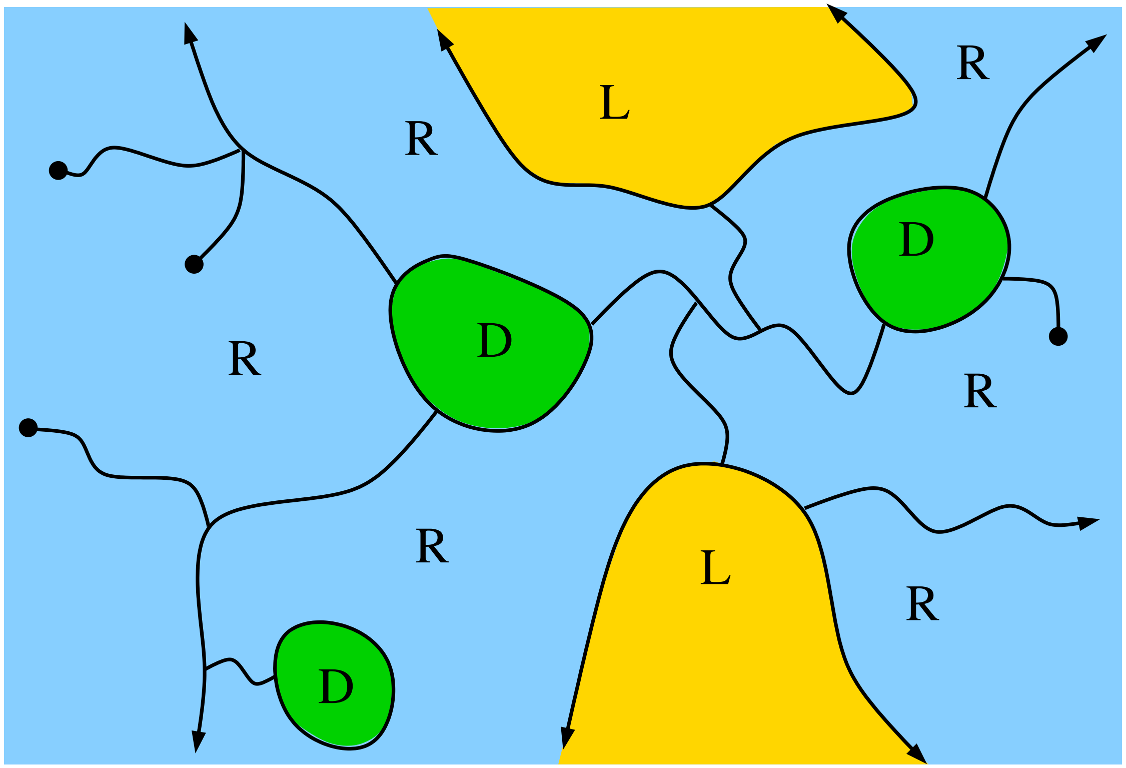


D = bounded Jordan domains (z^n, high degree critical points)

L = unbounded Jordan domains ($e^{-\tau(z)}$, finite asymptotic values)

RR-edges map to $[-1, 1]$, other edges map to $\mathbb{T}$.

Other applications need a more general folding theorem: replace tree by graph, faces labeled D,L,R.



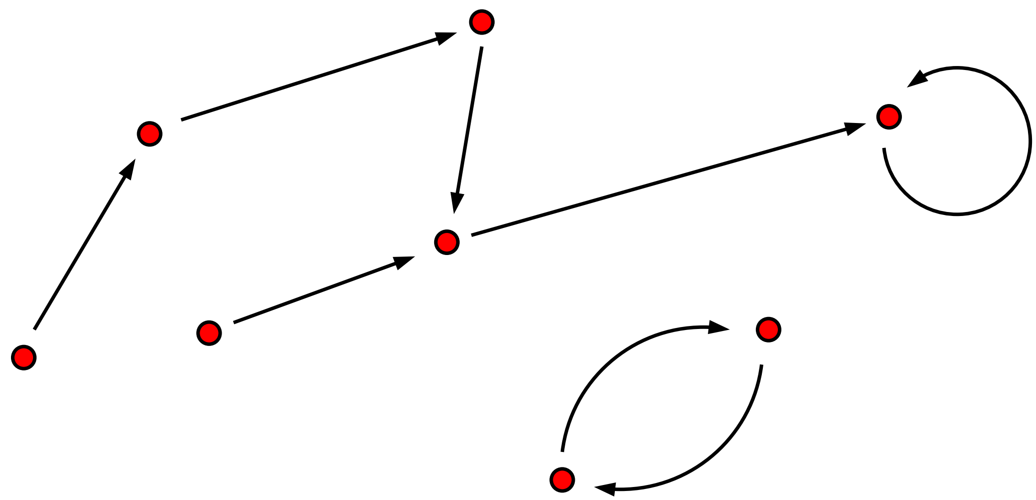
Further generaliztions allow for poles: “inverted disk components”.

Creates meromorphic function using folding.

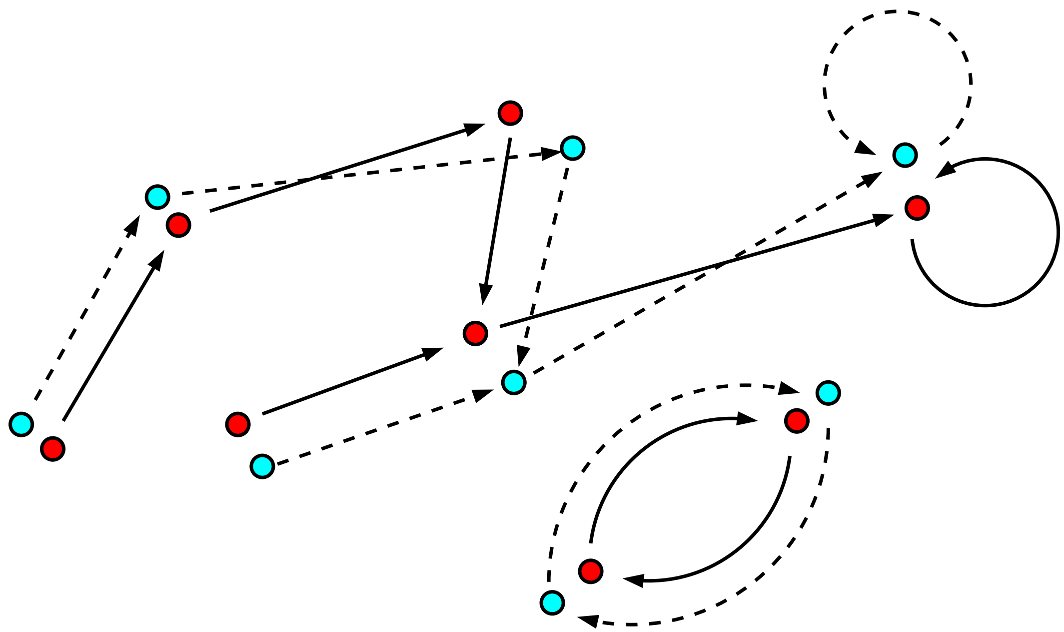
Needed in following applications.

Application 2: Prescribing the postsingular orbit

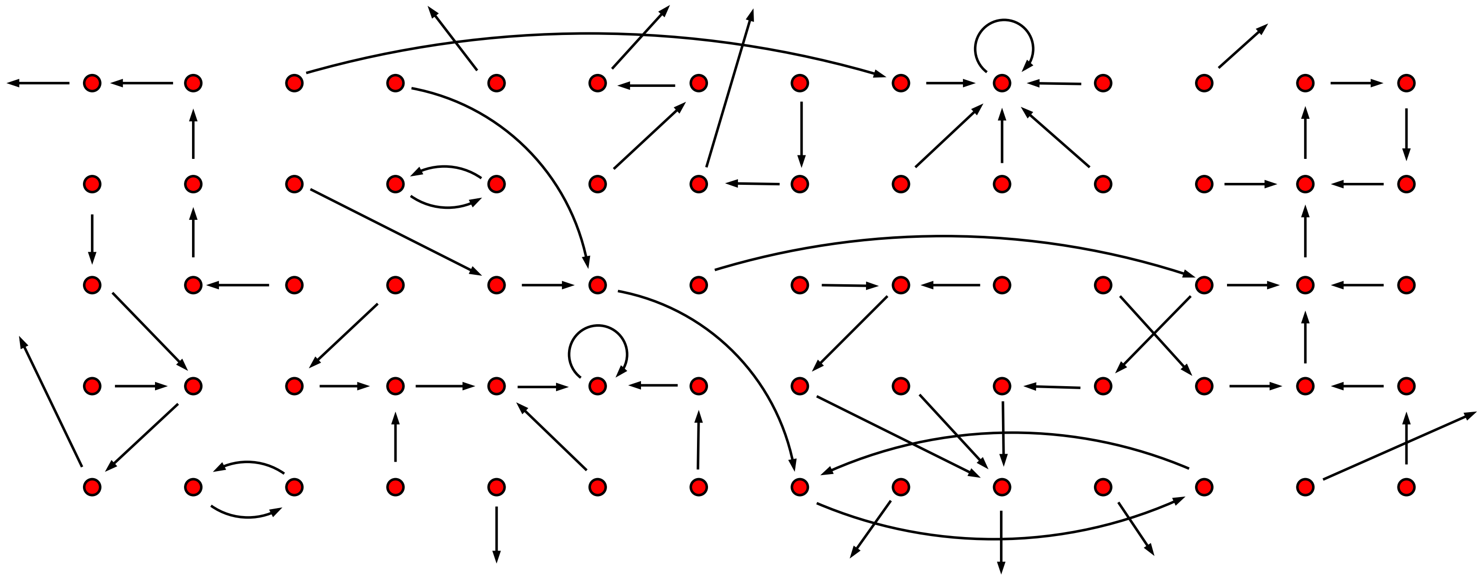
DeMarco, Koch and McMullen proved that a post-critically finite rational map can have **any** given dynamics.



DeMarco, Koch and McMullen proved that a post-critically finite rational map can have **any** given dynamics.



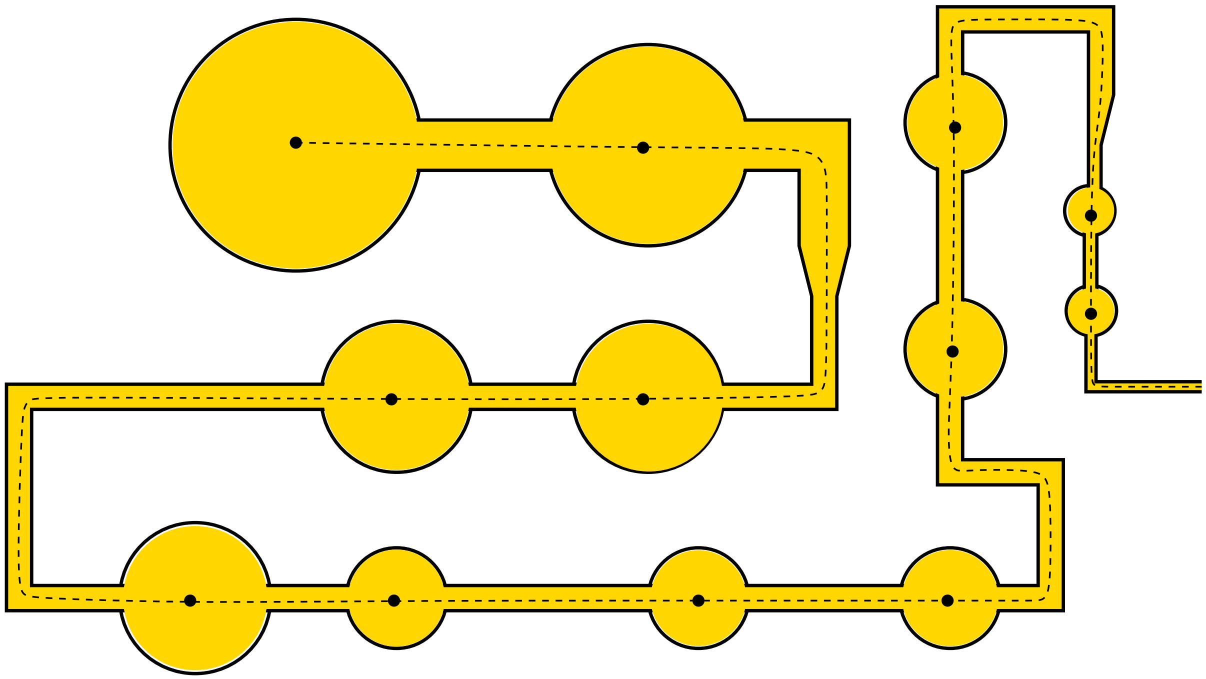
More precisely, given any $\epsilon > 0$, any finite set X and any map $h : X \to X$ there is a rational r whose post-singular set $P(r)$ is the same size as X, $P(r)$ ϵ-approximates X in Hausdorff metric, and $|r - h| < \epsilon$ on P.



Theorem (Lazebnik, B.): Let X be discrete ($\geq$ 4 points), let $h : X \to X$ be any map, and let $\epsilon > 0$. Then there is a transcendental meromorphic function f and a bijection $\psi : X \to P(f)$ so that

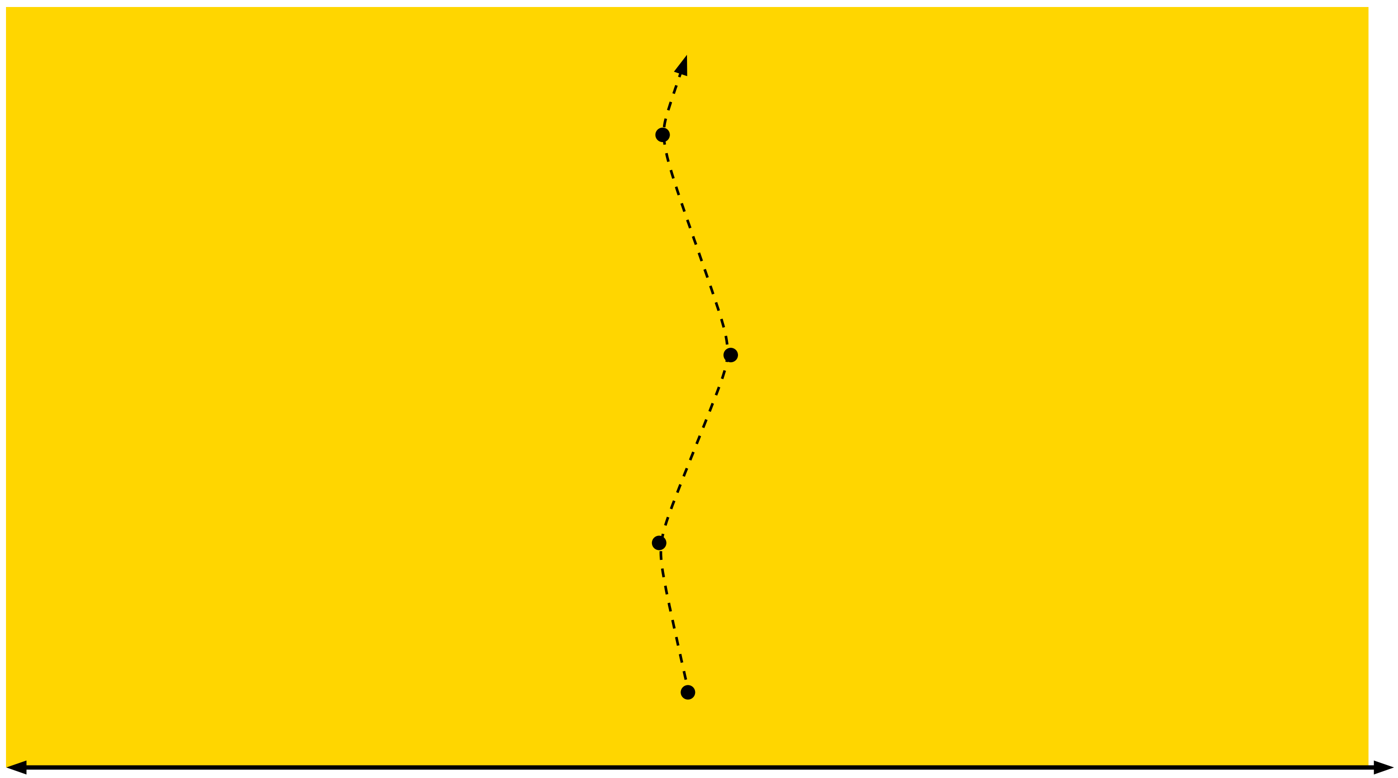
$$|\psi(z) - z| \leq \max(\epsilon, o(1)) \quad \text{and} \quad f = \psi \circ h \circ \psi^{-1} \text{ on } P(f).$$

Main argument (folding + fixed point thm) due to Kirill.

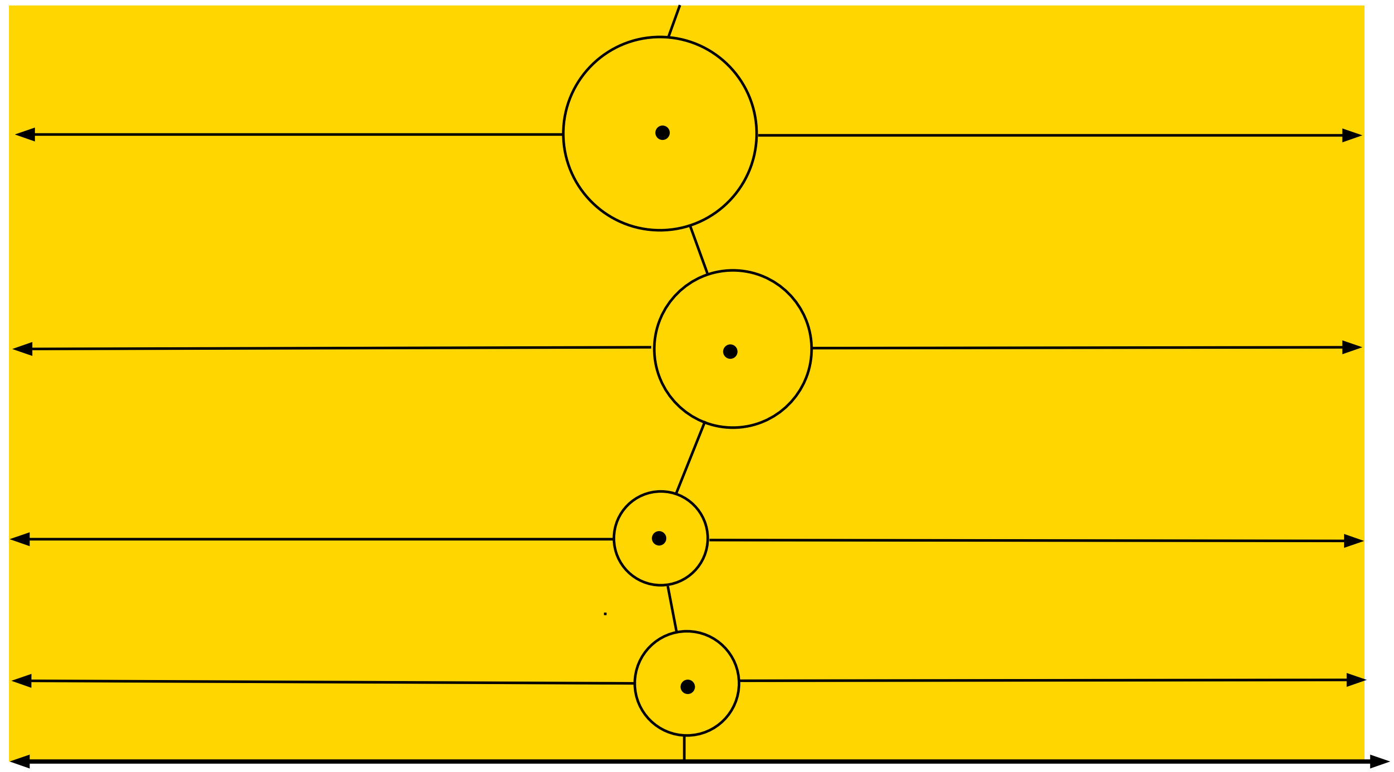


Reduce to case where ± 1 are in post-singular set and all other post-singular points are off the unit circle.

Construct a domain W that contains the given set, and so that these points lie very close to hyperbolic geodesic to ∞.



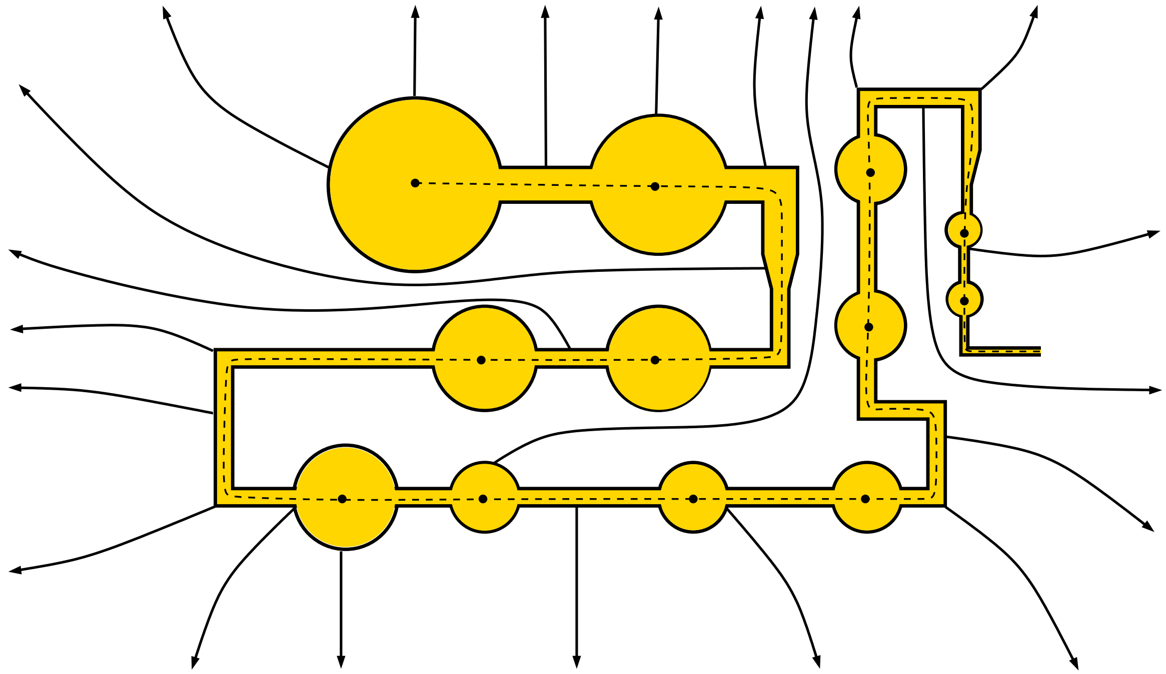
Conformally map W to upper half-plane; points almost on vertical ray.



Divide upper half-plane into R and D components.

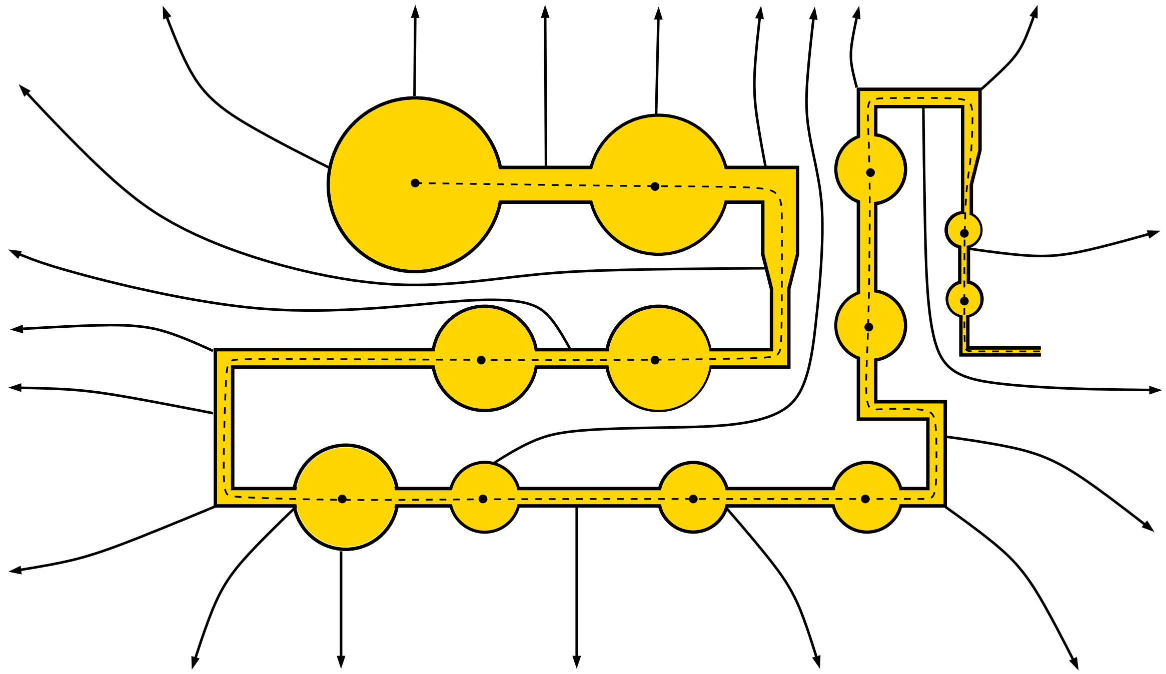
Folding map sends centers anywhere we want in $\mathbb{D}$.

Analogous construction maps centers outside $\mathbb{D}$; introduces poles.



Transfer graph back to W. Decompose complement of W into R-components (not hard) and define map on whole plane by folding.

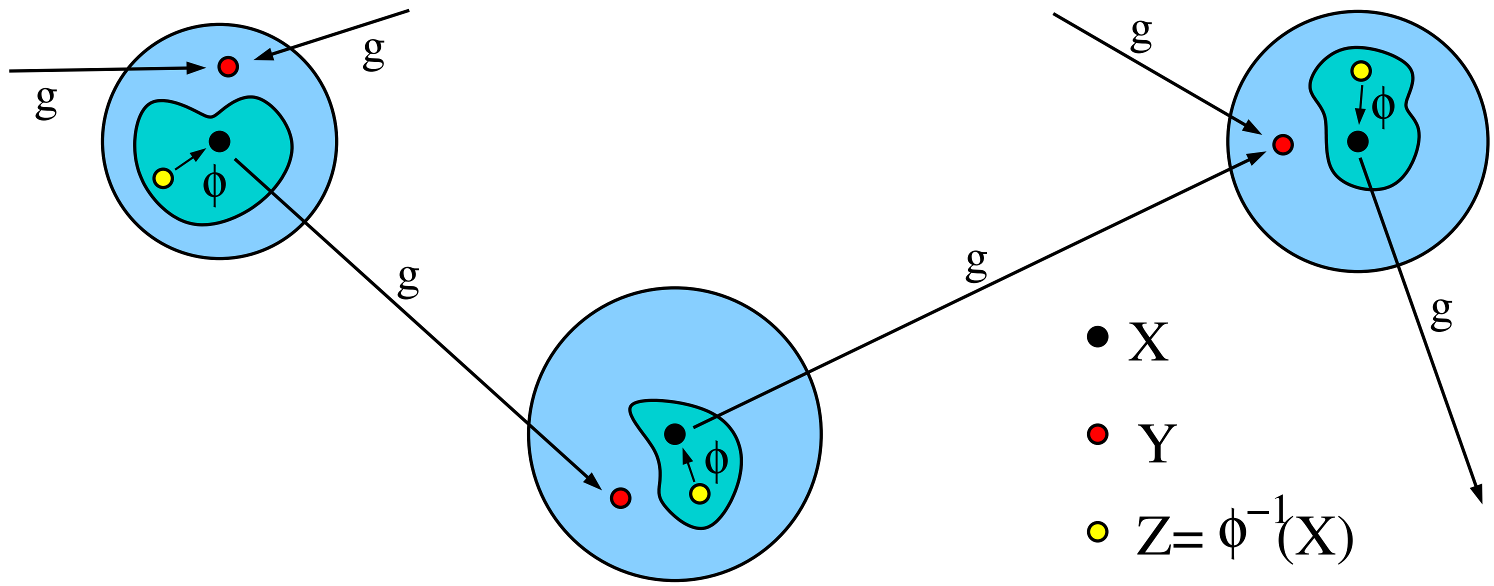
Gives quasi-meromorphic map g with desired post-singular behavior.



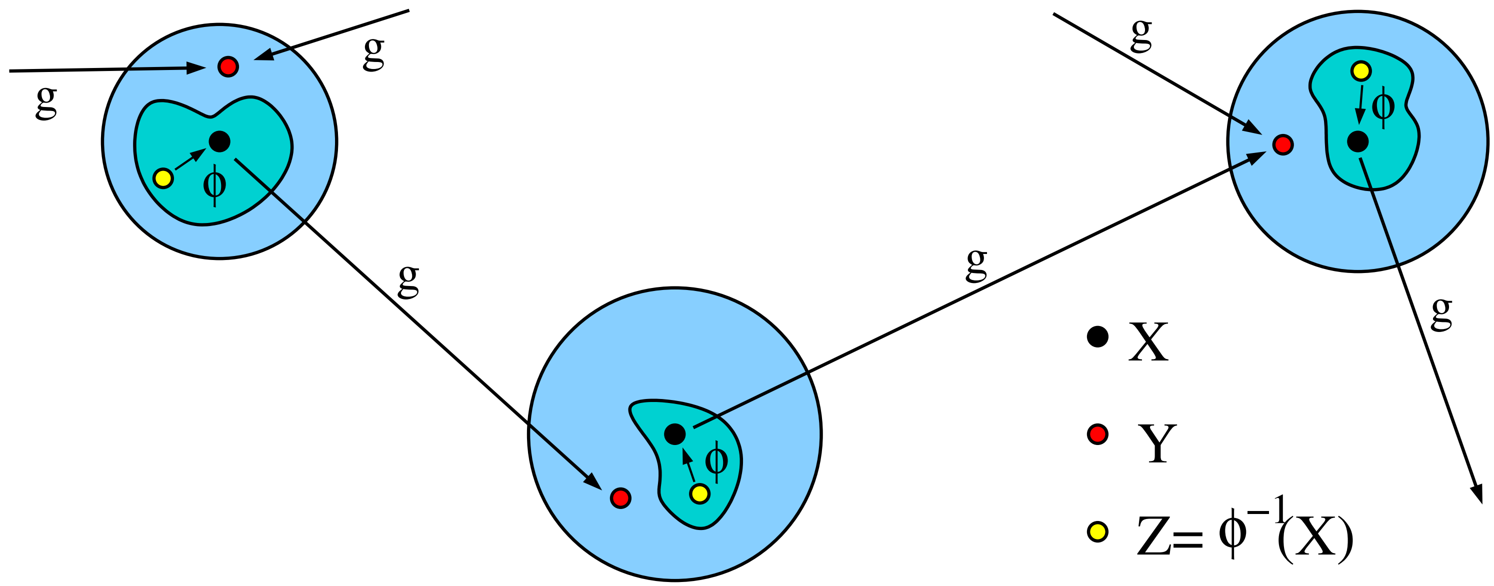
QC corection gives a meromorphic $f = g \circ \phi$.

Because of ϕ, the post-singular set might not be invariant under f.

How to fix this? Perturb X and used fixed point theorem.

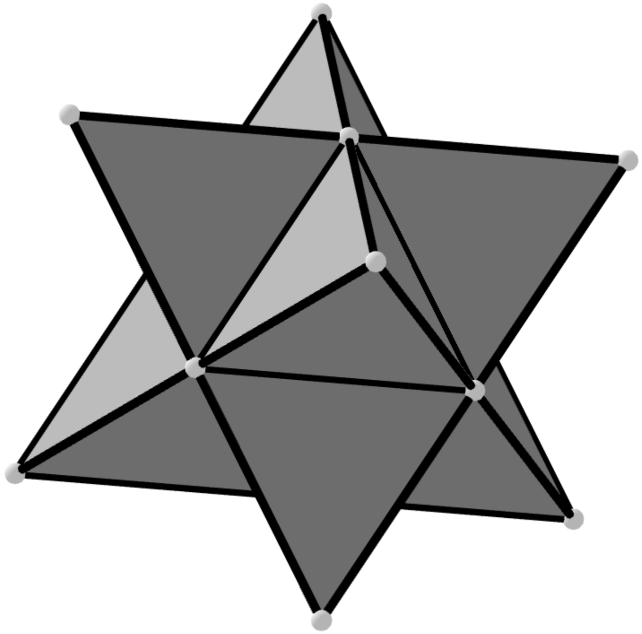


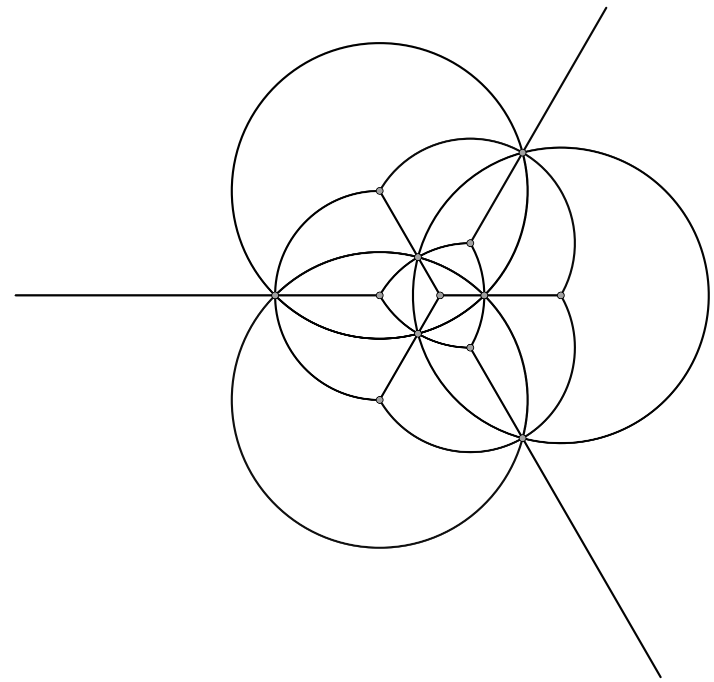
- X = given set (black points). Choose diosjoint disks (blue) around them.
- Choose $Y \approx X$ (red points), taking one from each disk.
- By folding construct g so $X \to Y$ (black $\to$ red).
- Meromorphic $f = g \circ \phi$ maps $Z = \phi^{-1}(X)$ to Y (yellow $\to$ red).
- We want $Z = Y$ (yellow points = red points).
- Then $f : Y \to Y$ (red = yellow $\to$ black $\to$ red) and $Y \approx X$.



- $Z = Z(Y)$ (yellow) is continuous in Y (red).
- $\phi \approx$ ID, so $Y \to Z(Y)$ maps light blue disks into themselves.
- Infinite-dimensional fixed point theorem says $Z(Y) = Y$, for some Y.
- Gives meromorphic f with singular orbit $Y \approx X$.

Application 3: Belyi functions and triangulations





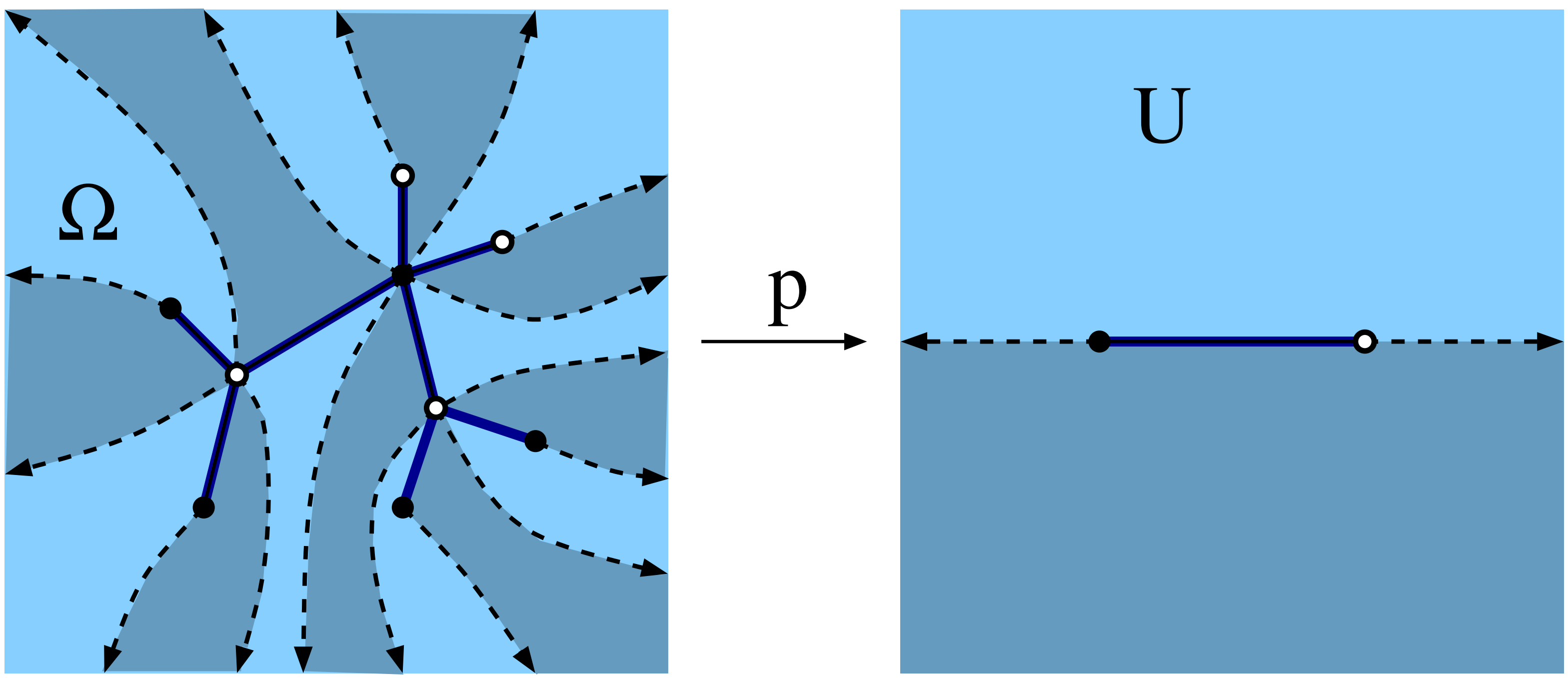
Attaching equilateral triangles edge-to-edge gives conformal structure.

This a topological sphere, so edges embed in plane.

Which Riemann surfaces can occur this way?

Belyi function = holomorphic map from Riemann surface with 3 critical values (usually $0, 1, \infty$).

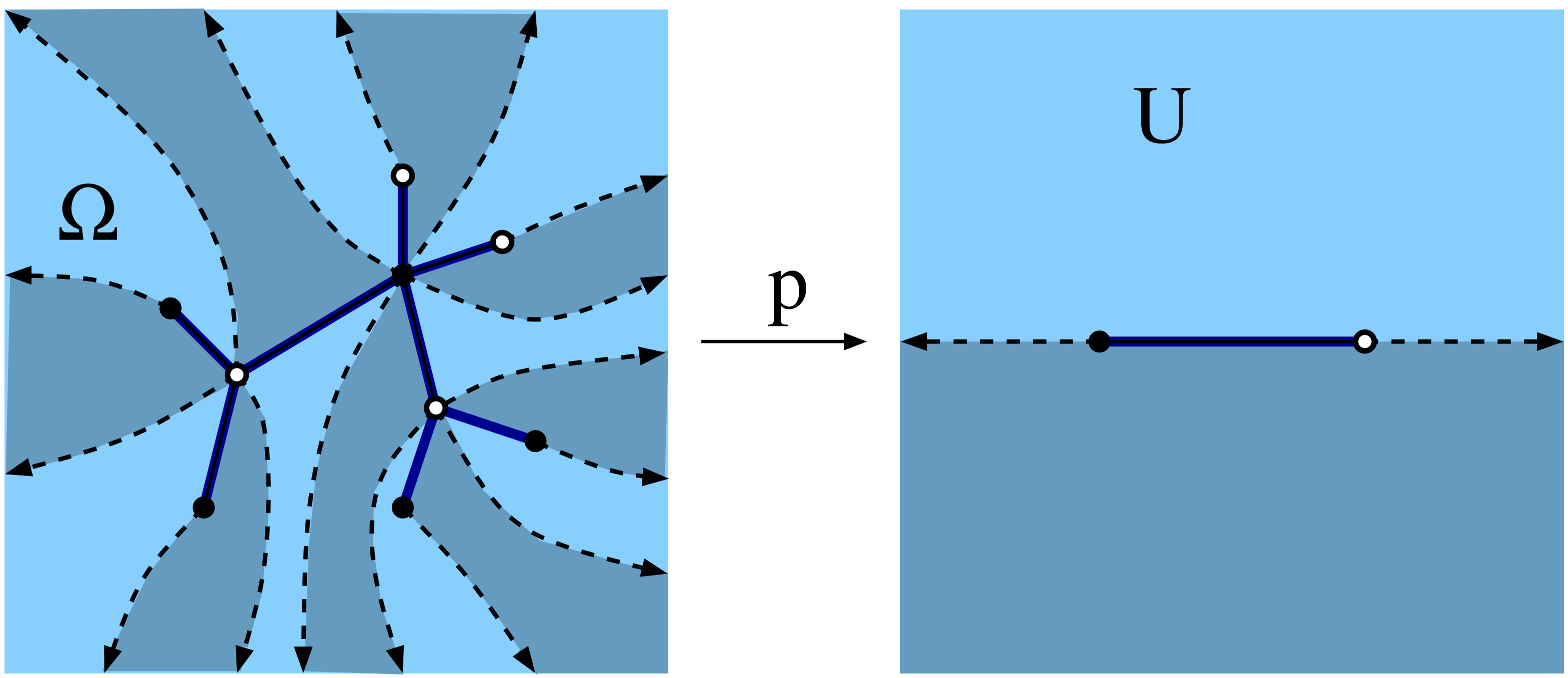
Theorem: A Riemann surface can be constructed from equilateral triangles iff it has a Belyi function.



Triangles are inverse images of upper and lower half-planes.

Belyi function = holomorphic map from Riemann surface with 3 critical values (usually $0, 1, \infty$).

Theorem: A Riemann surface can be constructed from equilateral triangles iff it has a Belyi function.



A triangulation is equilateral iff every edge has an anti-holomorphic reflection fixing it pointwise and swapping the two adjacent triangles.

Belyi function = holomorphic map from Riemann surface with 3 critical values (usually $0, 1, \infty$).

Theorem: A Riemann surface can be constructed from equilateral triangles iff it has a Belyi function.

There are only finitely many ways to glue together n triangles.

$\Rightarrow$ only countably many compact surfaces have a Belyi function.

Belyi function = holomorphic map from Riemann surface with 3 critical values (usually $0, 1, \infty$).

Theorem: A Riemann surface can be constructed from equilateral triangles iff it has a Belyi function.

There are only finitely many ways to glue together n triangles.

$\Rightarrow$ only countably many compact surfaces have a Belyi function.

Belyi's Thm: A compact surface has a Belyi function iff it is algebraic.

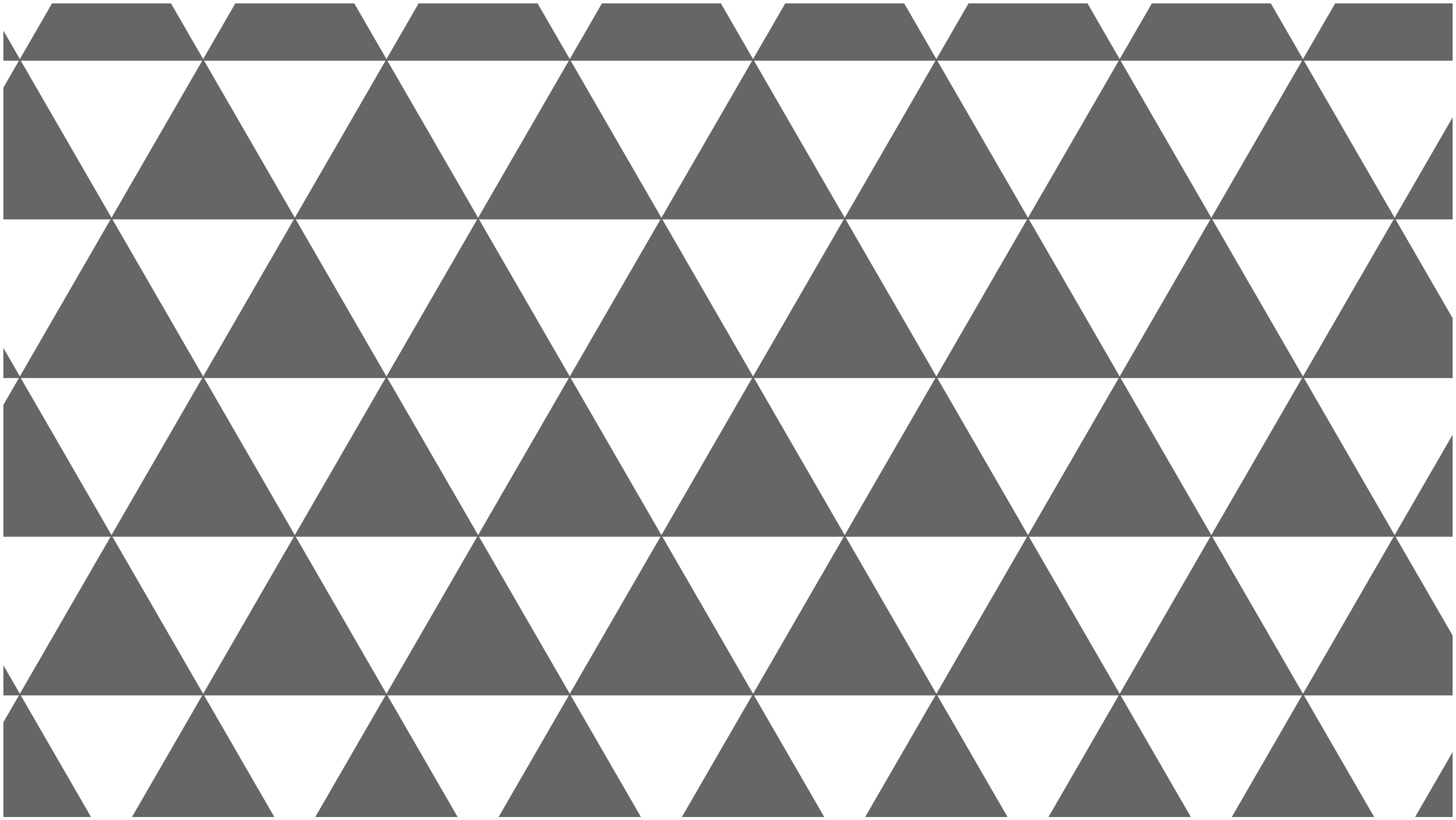
(Zero set of $P(z, w)$ with algebraic integers as coefficients).

Motivated Grothendieck's theory of *dessins d'enfant*.

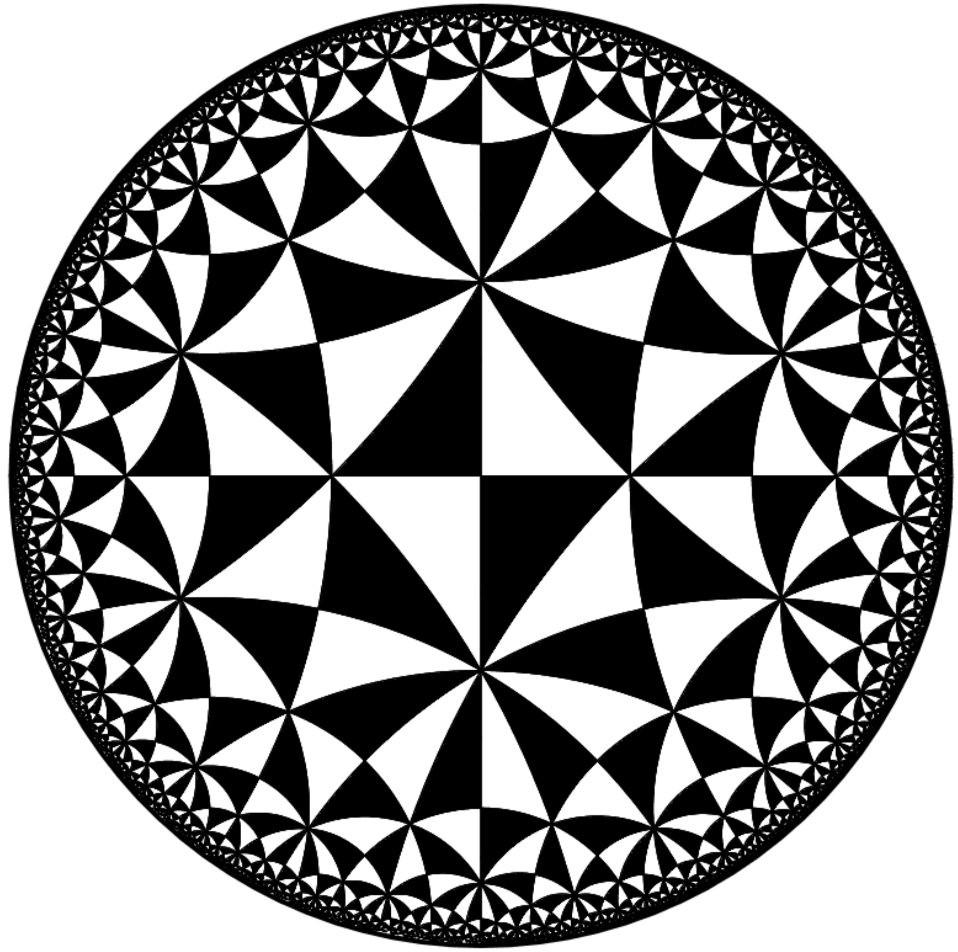
Which non-compact Riemann surfaces are Belyi?

= has a Belyi function

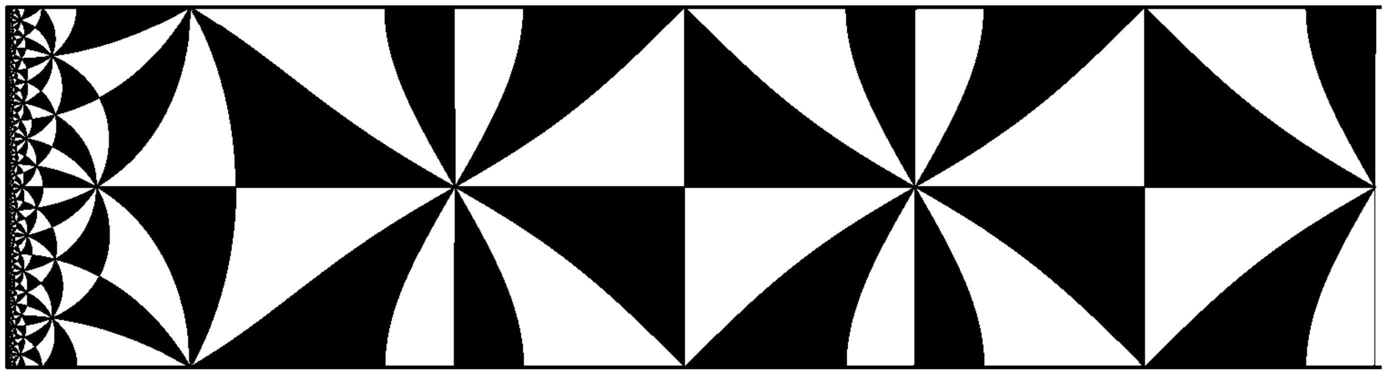
= has an equilateral triangulation



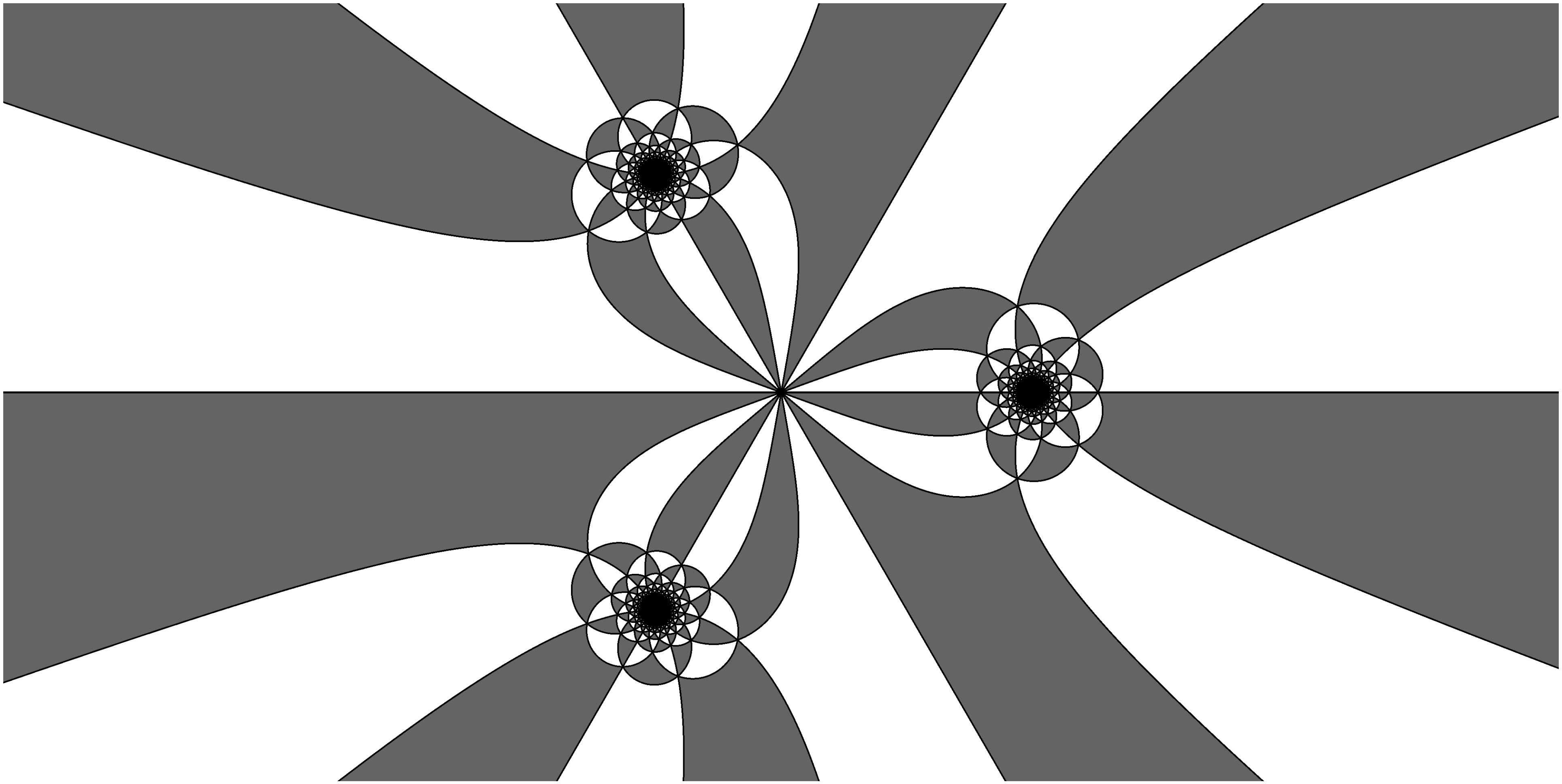
The plane



The disk



The punctured disk (identity top and bottom sides)

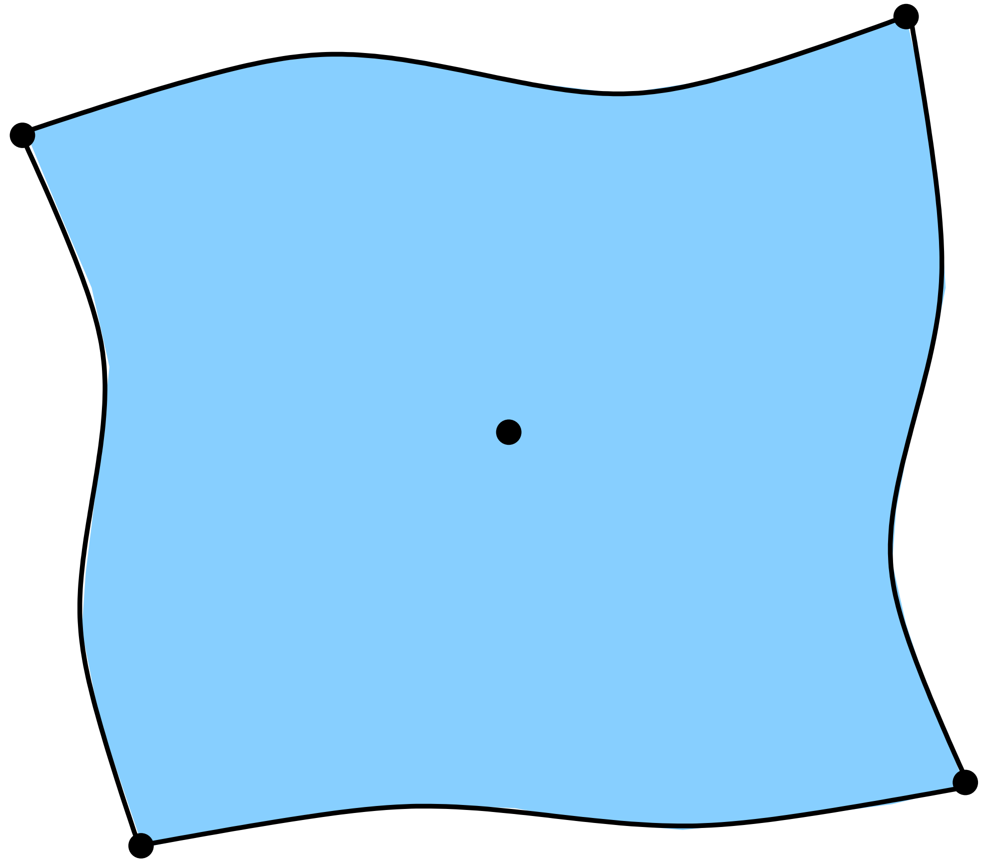


The trice punctured sphere

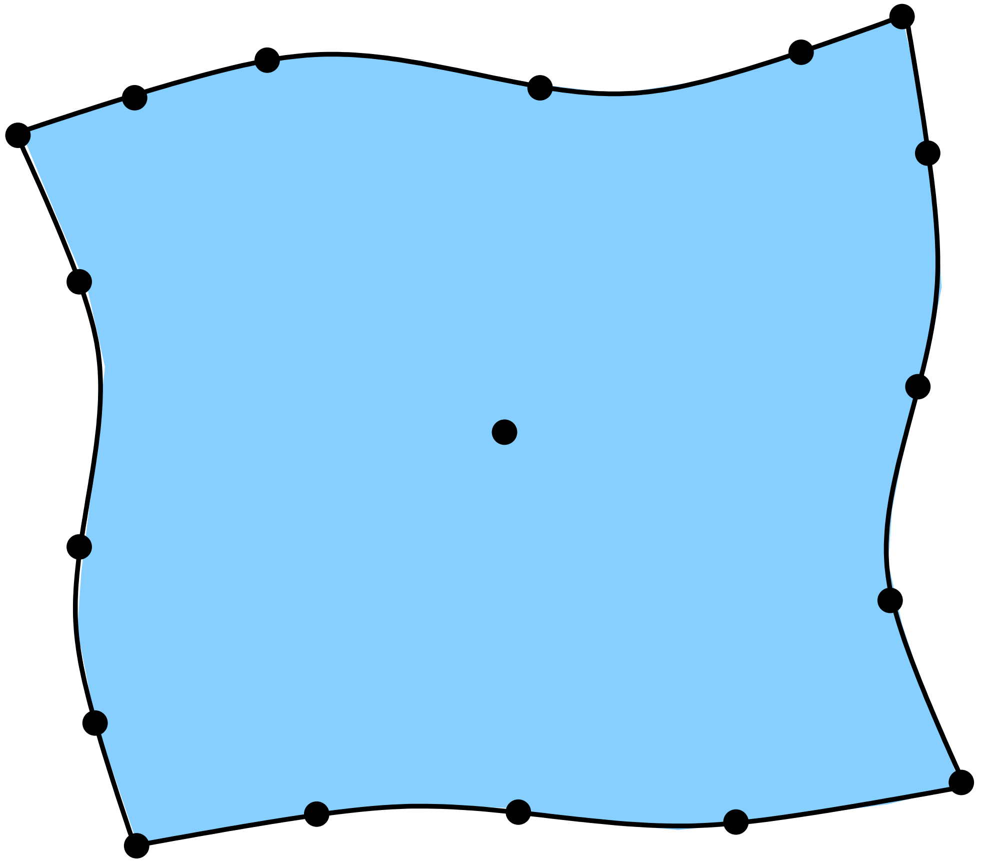
In general we want to:

- Decompose surface into pieces
- Build QR Belyi function on each piece.
- Pieces attach to give QR Belyi function on whole surface.
- Use MRMT to get holomorphic Belyi function.

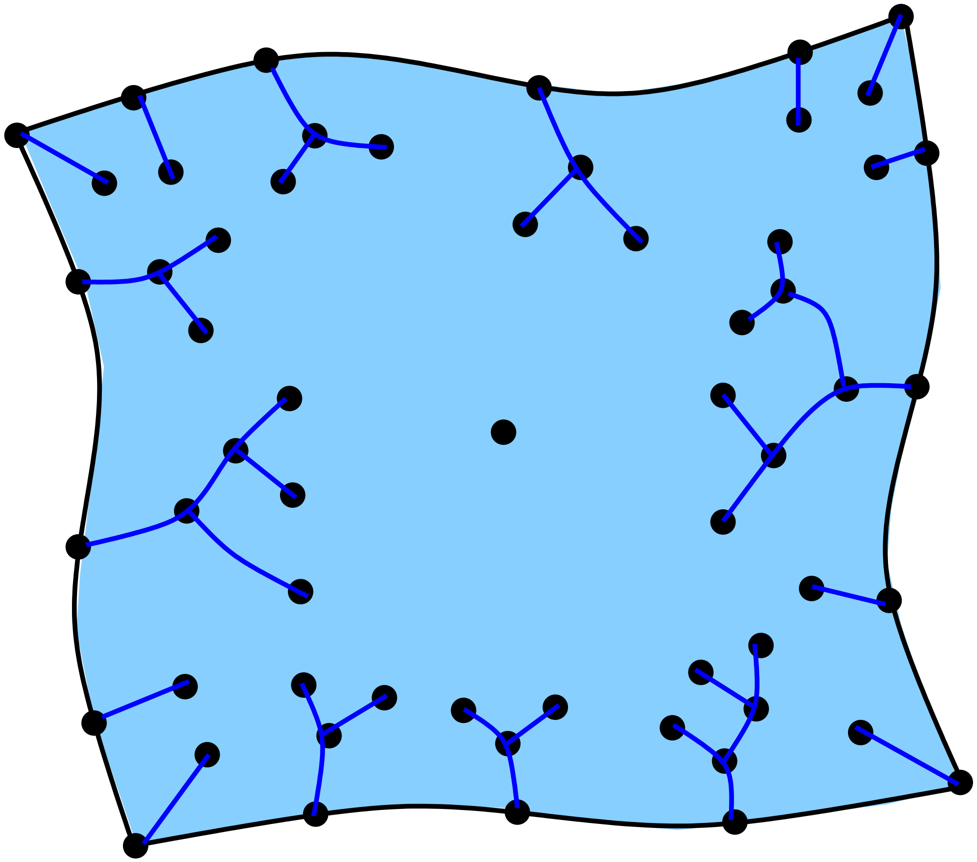
Last step is problematic: using MRMT can change conformal structure.



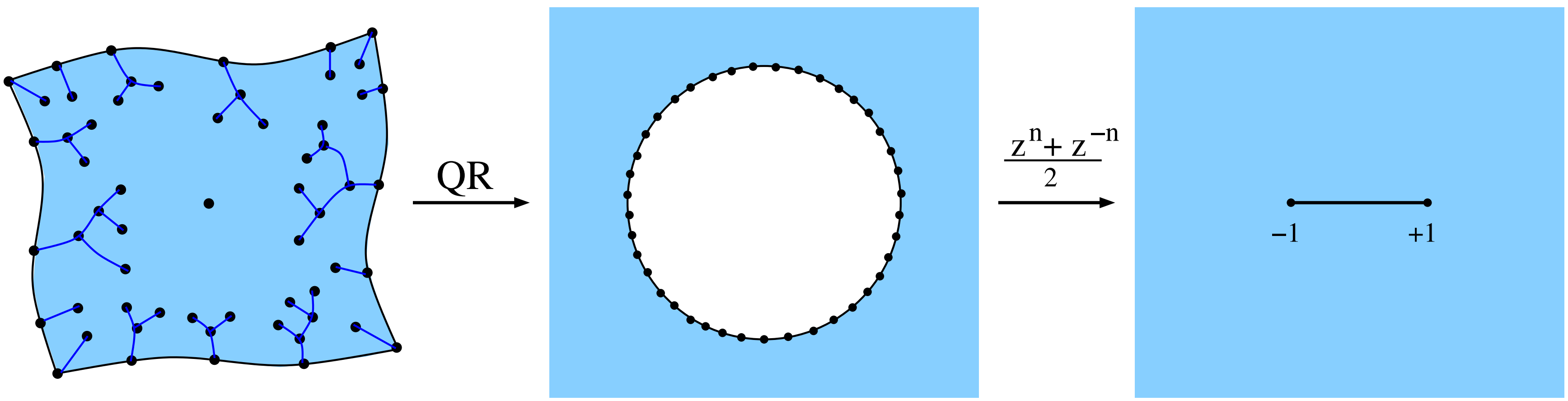
Start with a piecewise smooth domain.



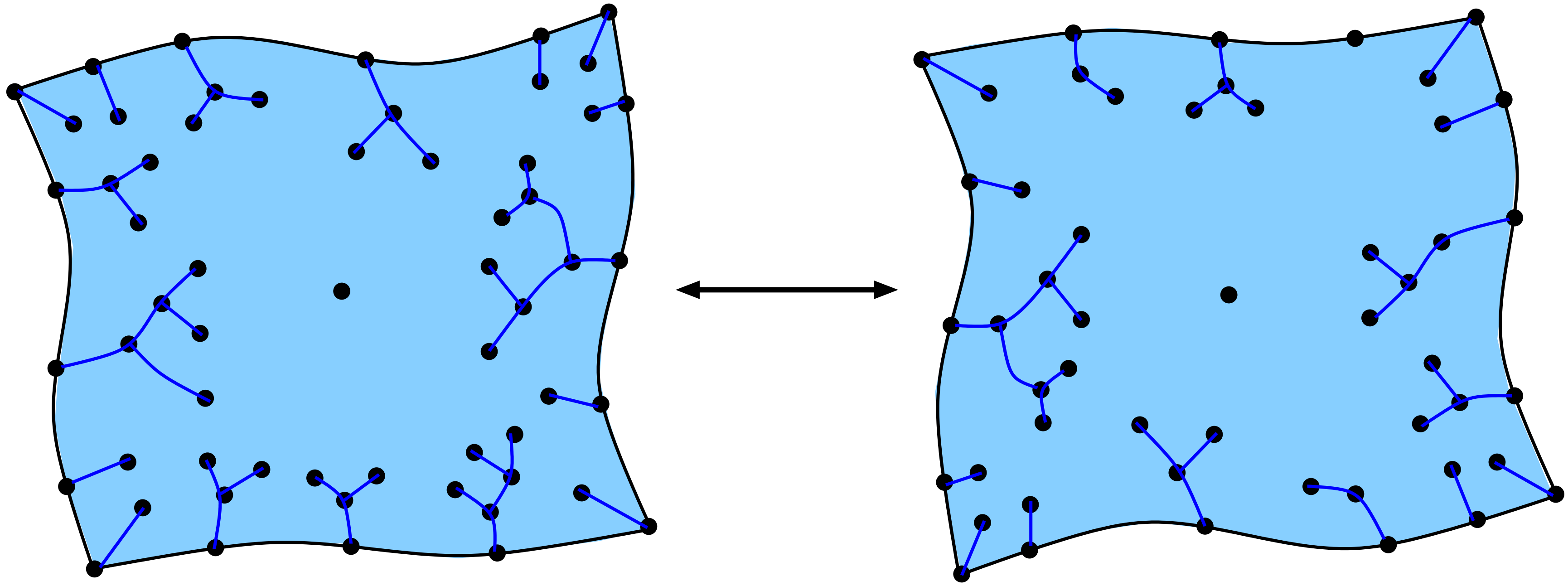
Add boundary points. Roughly evenly spaced.
Add interior point. Harmonic measures not all the same.



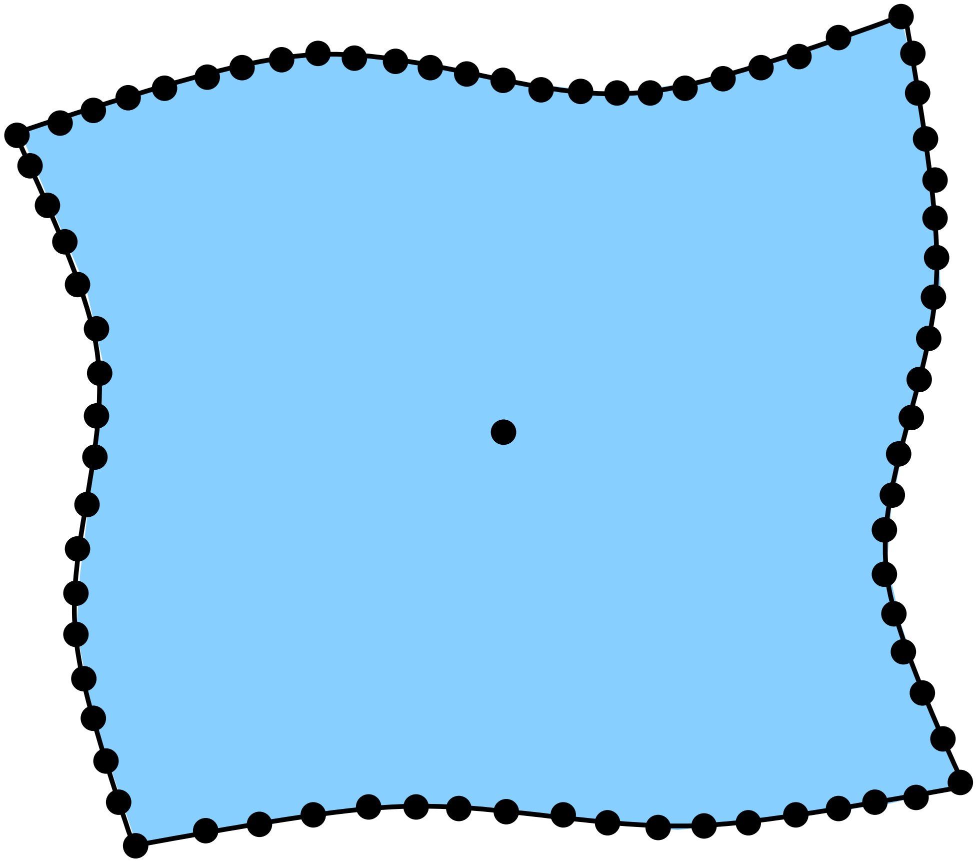
Add slits and trees so that all boundary arcs have roughly the same harmonic measure. This mimics the QC folding construction.



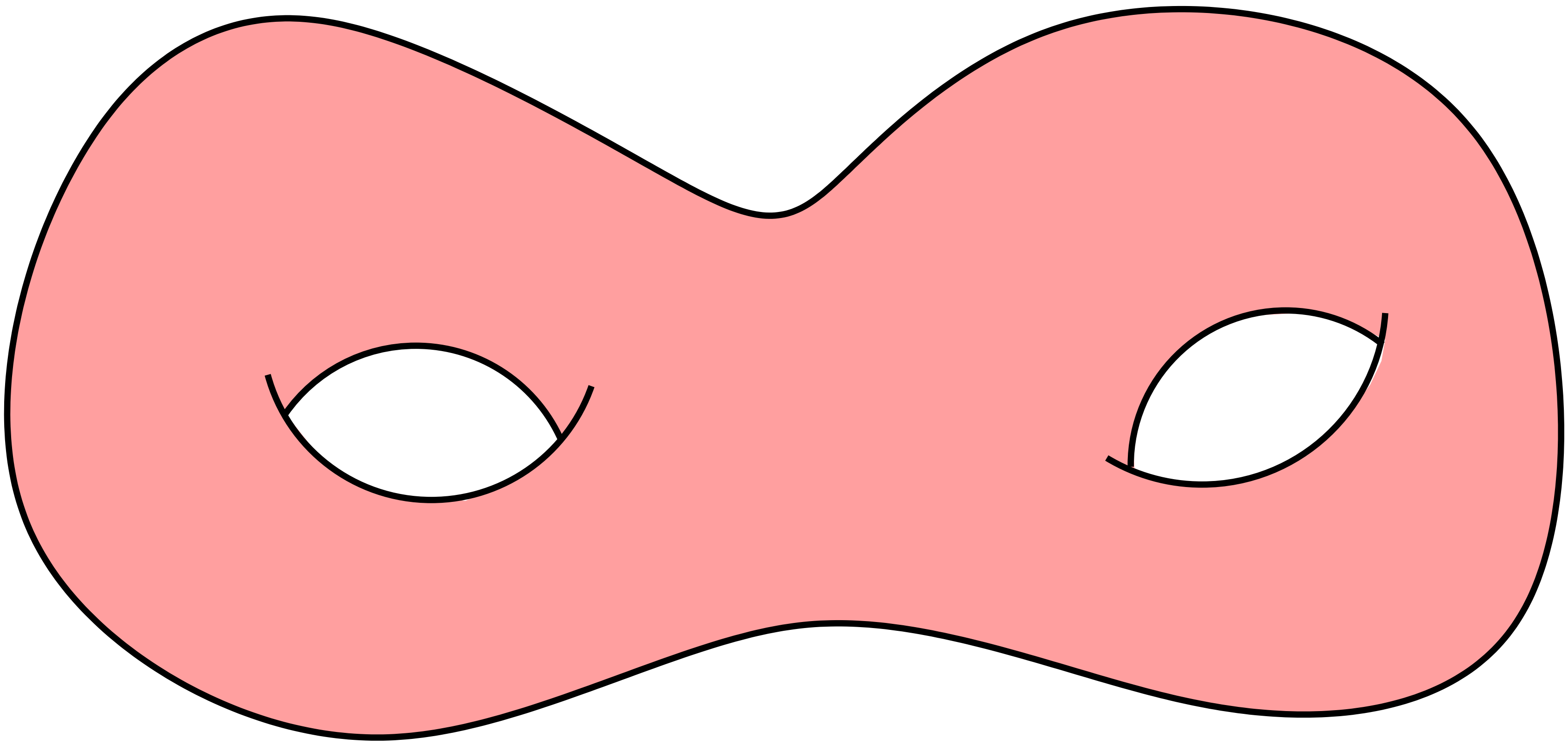
Find quasiregular map taking base point to ∞, sending boundary arcs to equal intervals on circle. Then each arc is mapped to $[-1, 1]$ via arclength.



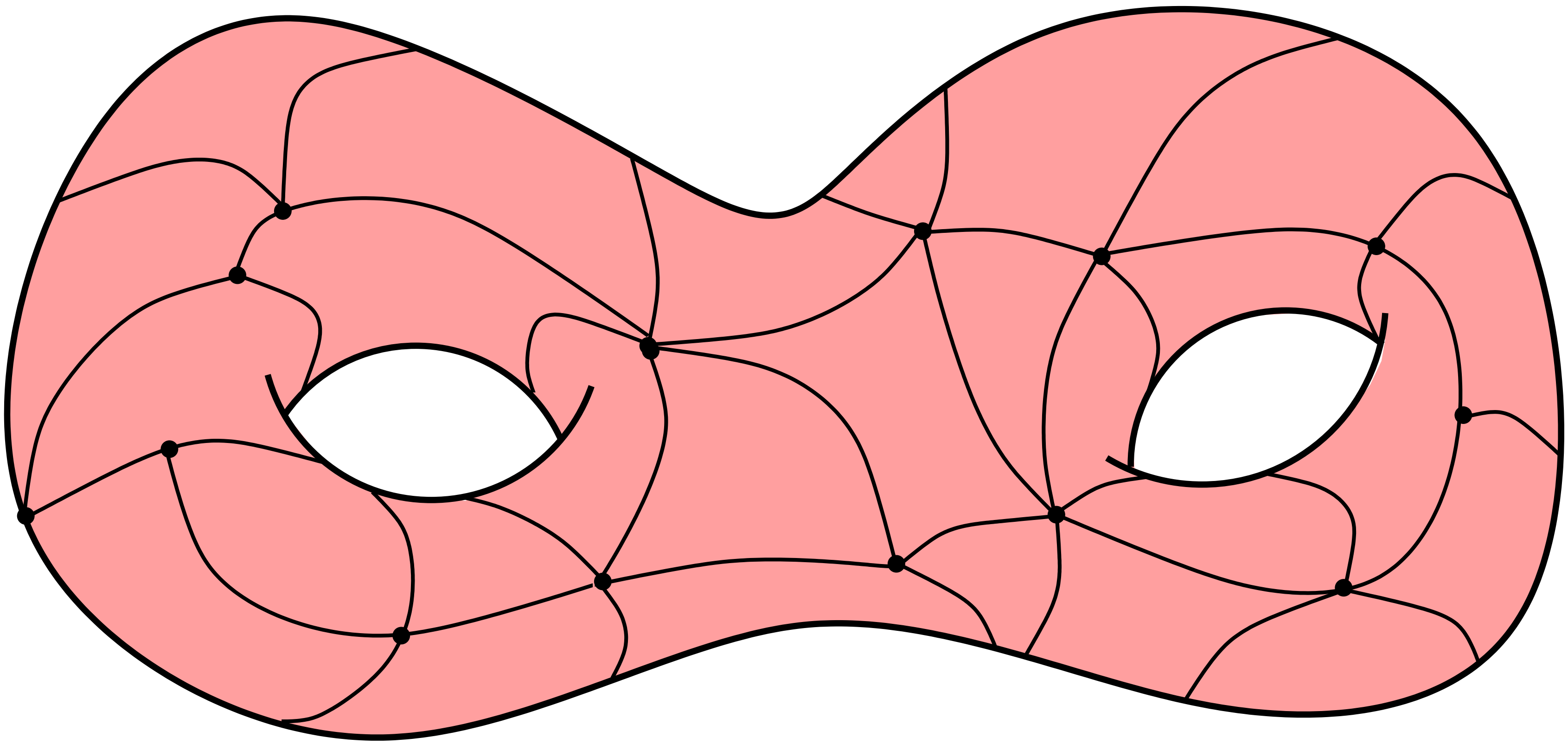
Arcs on both boundaries map to $[0, 1]$ via arclength.
We get a well defined, continuous function along boundary.
Uniformly quasiregular. Holomorphic except on small area.



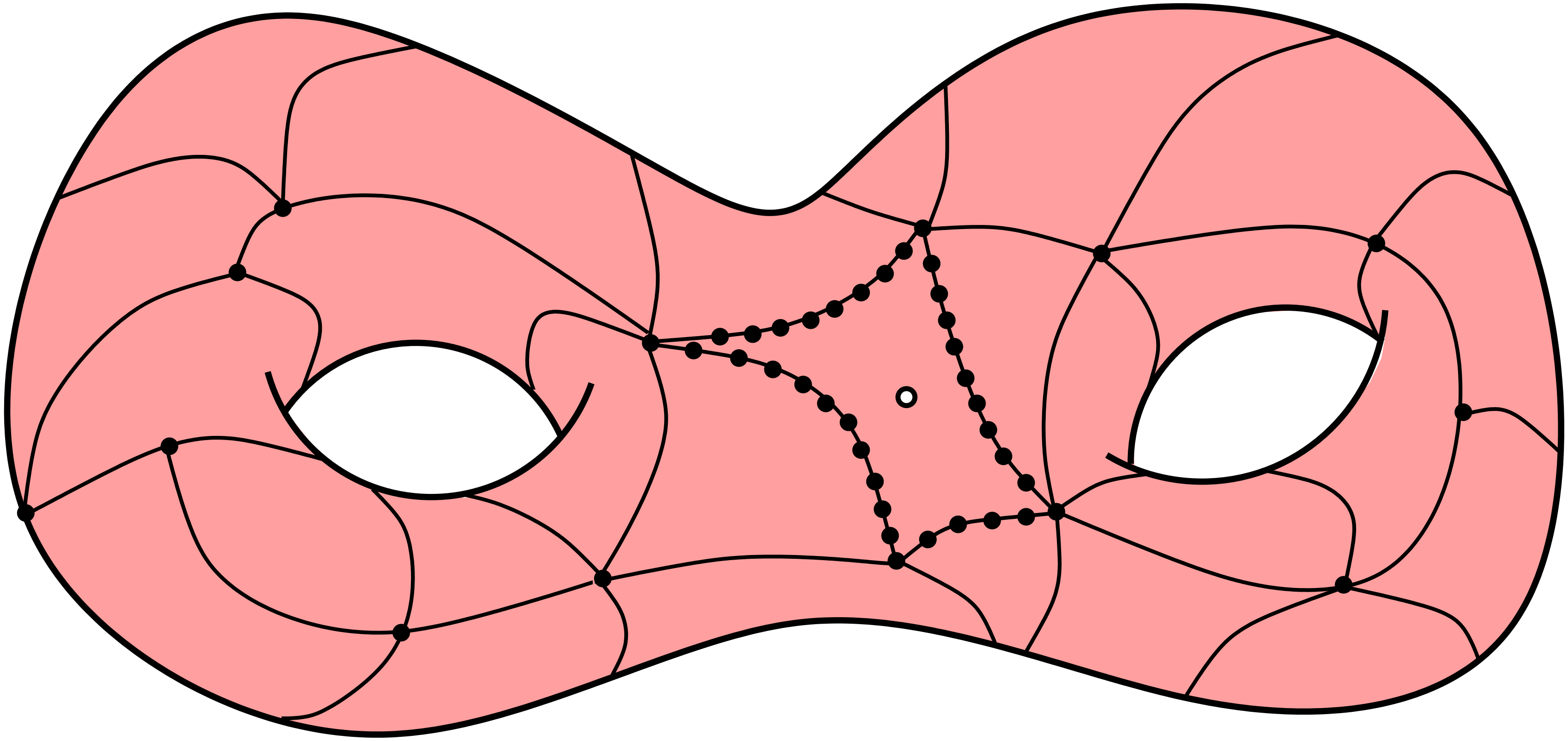
Starting with more closely spaced points gives QR map that is holomorphic on thinner neighborhood of boundary.



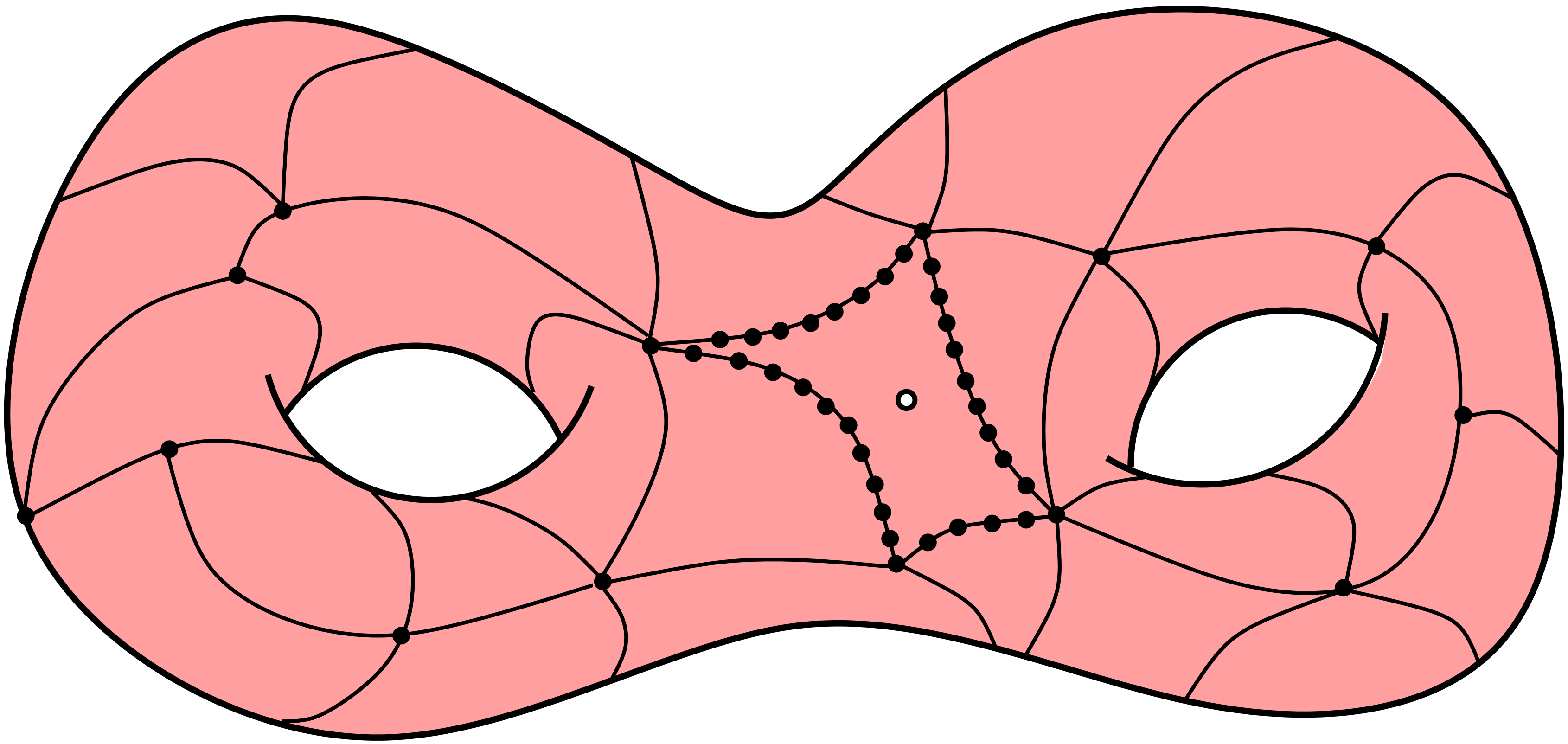
Now do this on a Riemann surface.



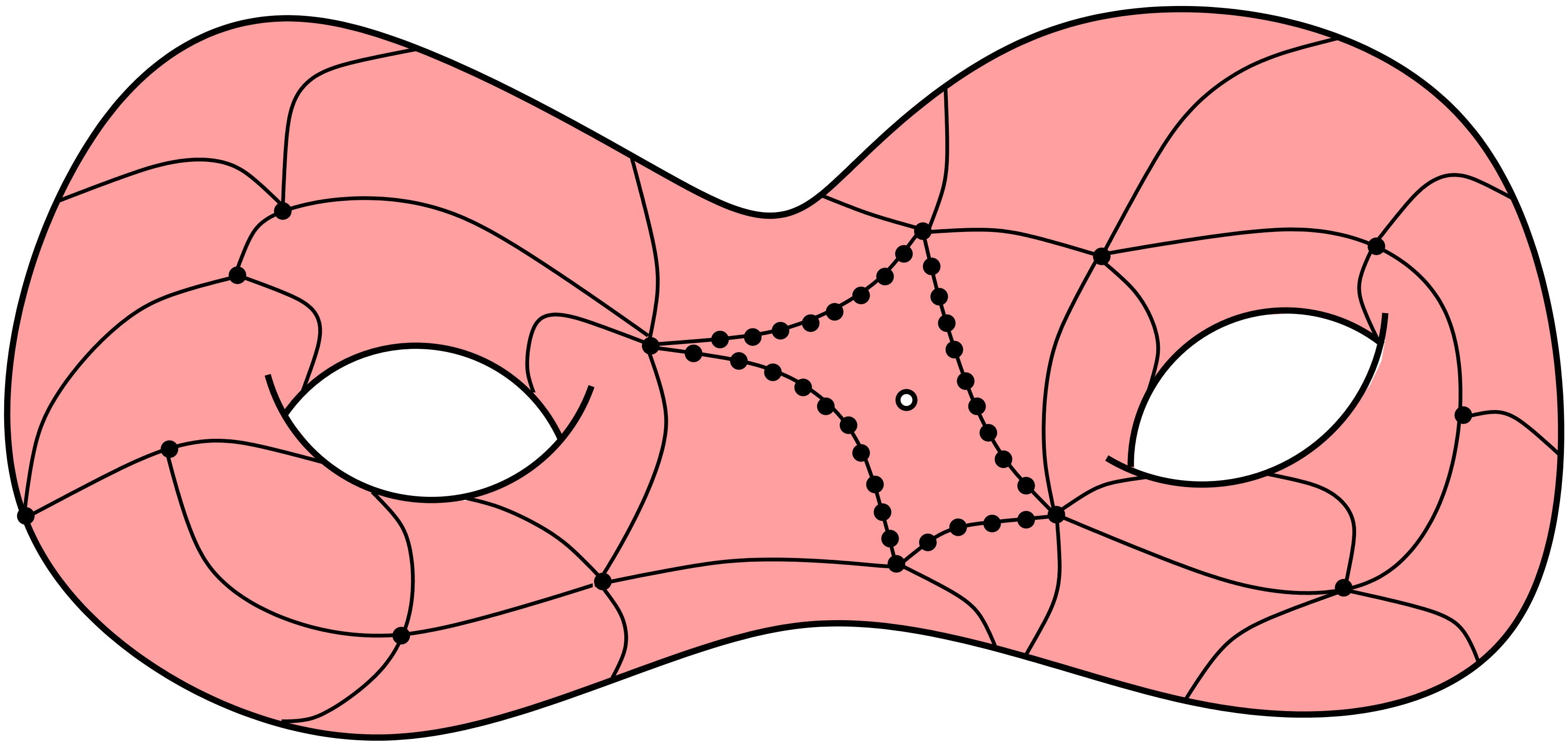
Cut the surface into nice pieces.



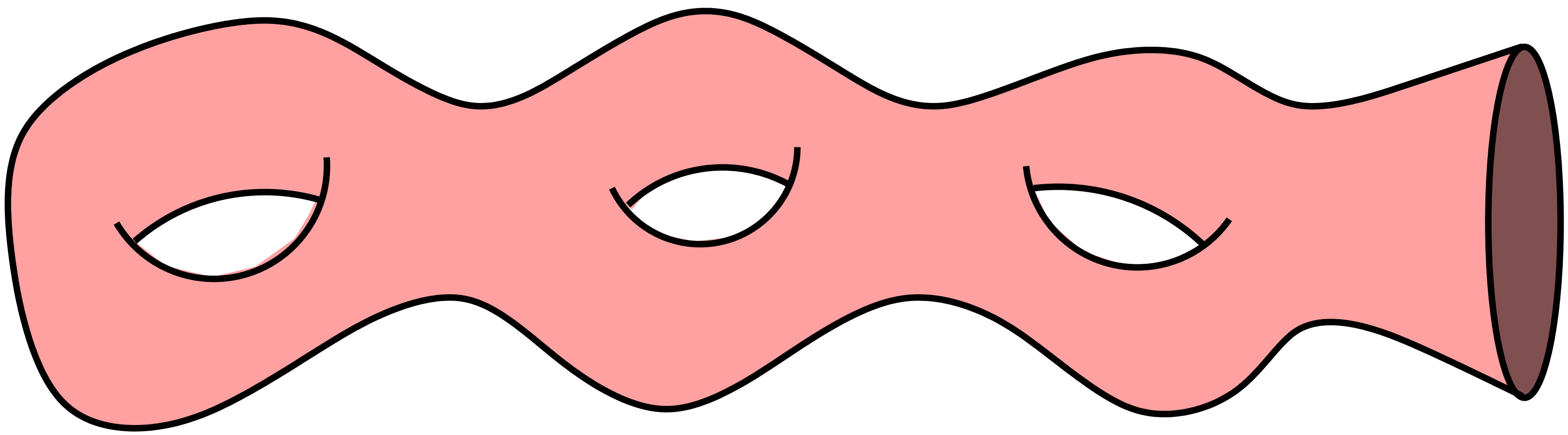
Subdivide edges into many small arcs.
Our construction gives QR maps on each piece.
Continuous across boundaries. QR on whole surface.



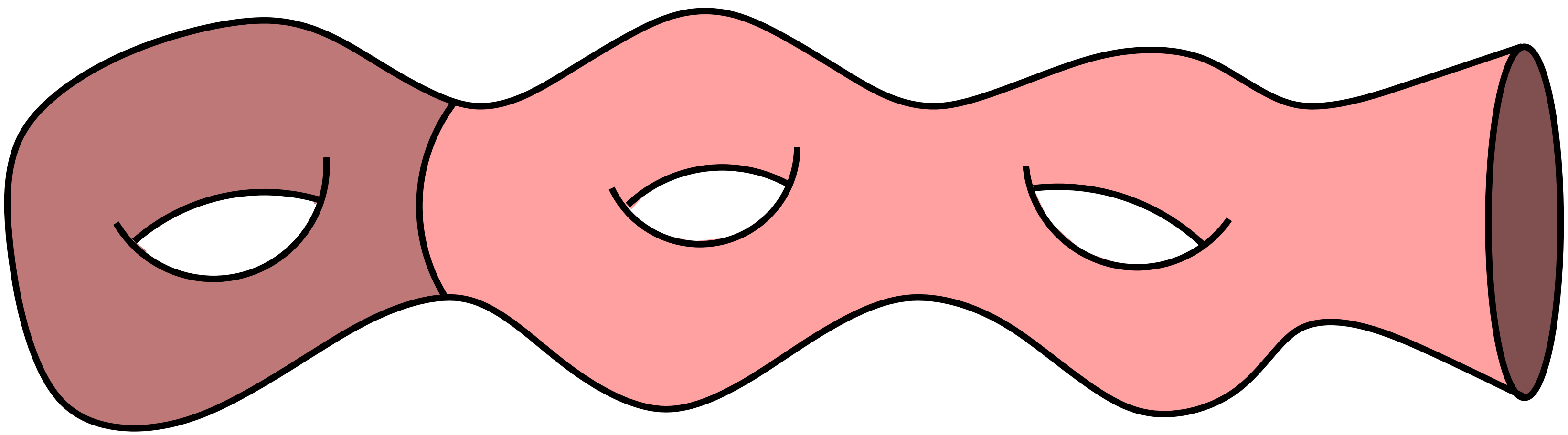
Change conformal structure to make QR map holomorphic.
Closely spaced dots ⇒ small conformal change.
⇒ Belyi surfaces are dense in all surfaces.



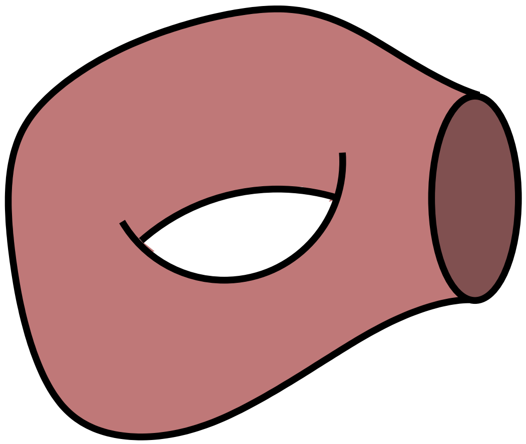
Every Riemann surface has a "QR Belyi" function.
But not every compact suface has a holomoprhic one.
What about non-compact surfaces? .



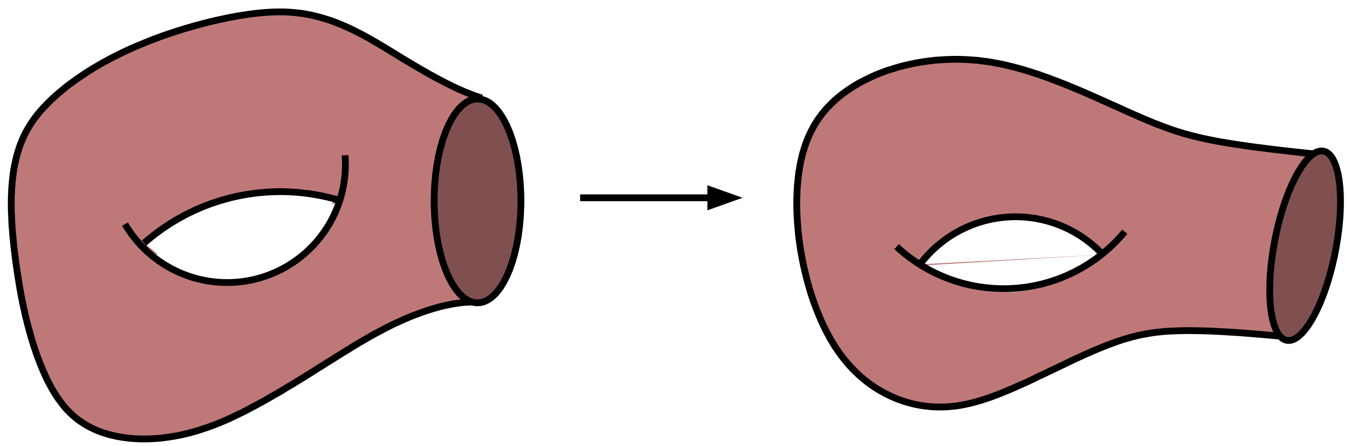
Consider a non-compact surface R.



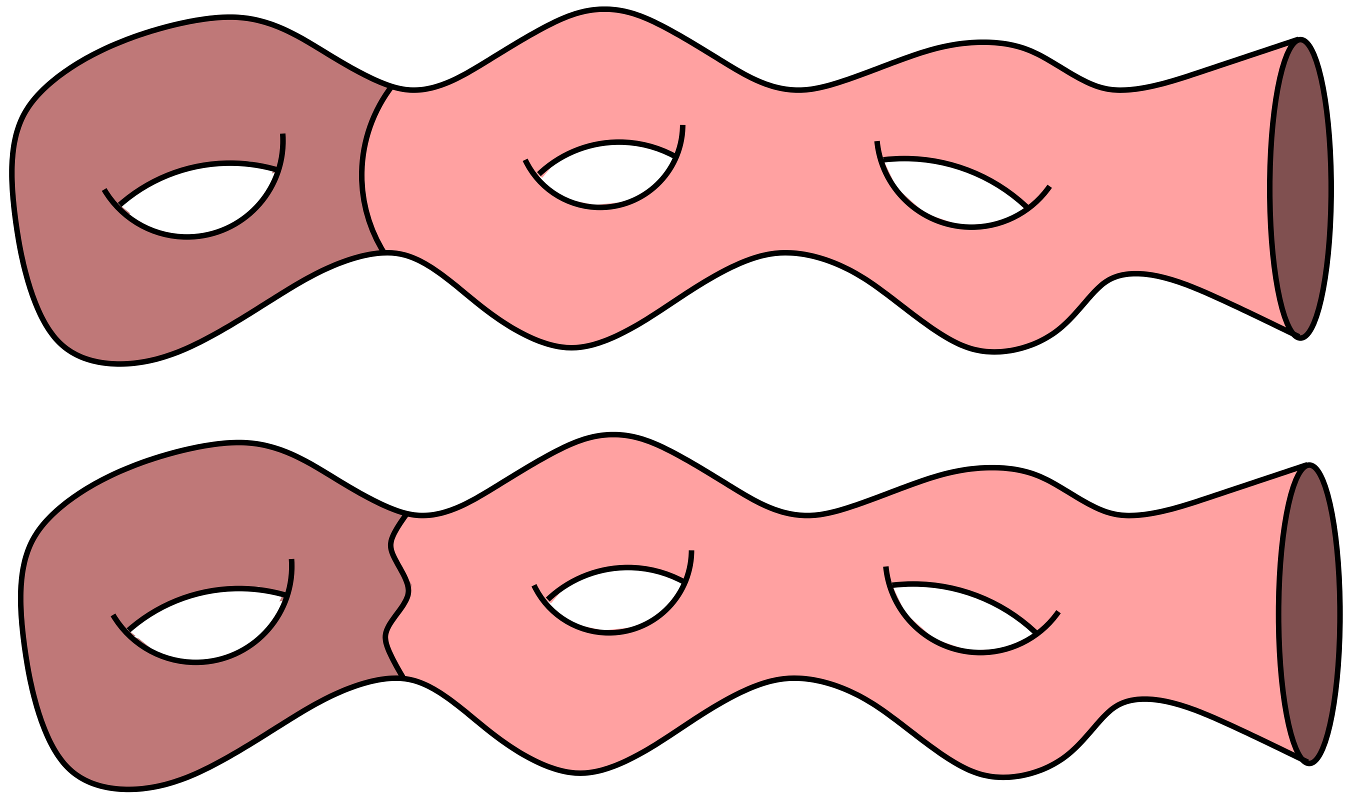
Take a compact, smoothly bordered piece Y_1.



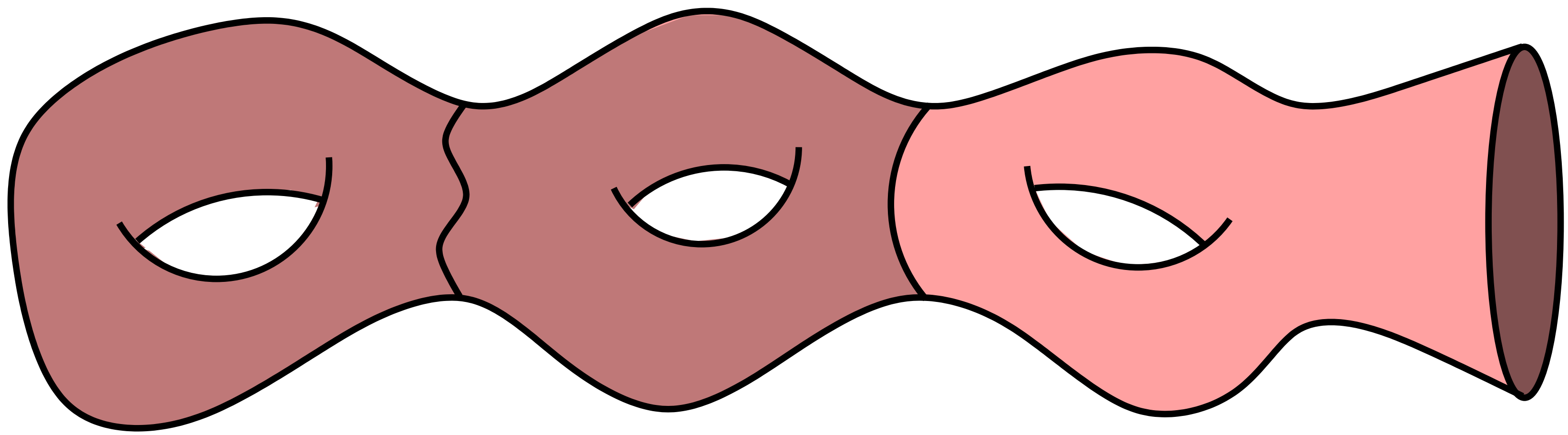
Take a compact, smoothly bordered piece Y_1.
Build a QR covering map on this piece as before.



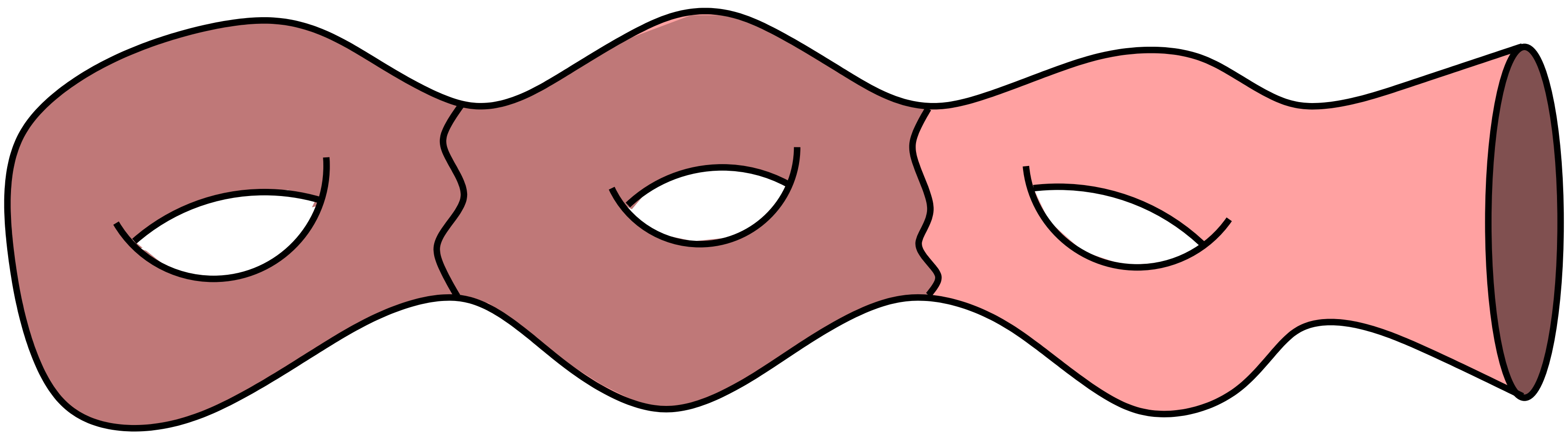
Take a compact, smoothly bordered piece Y_1.
Build a QR covering map on this piece as before.
Solve Beltrami on the compact piece. Get new surface $\widetilde{Y}_1 \neq Y_1$.



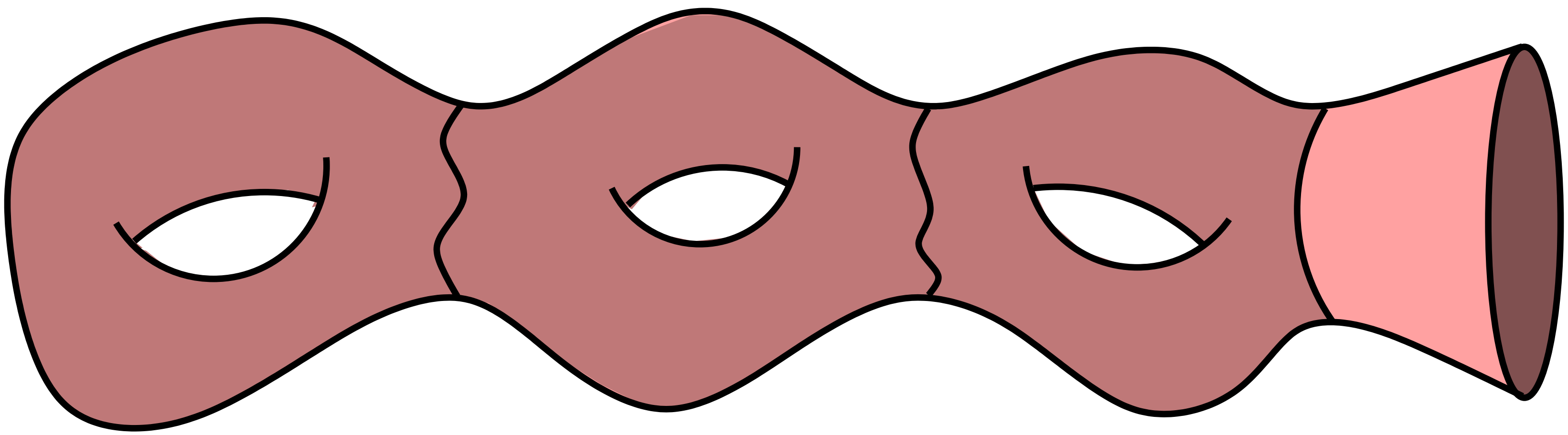
Key idea: small dilatation implies $\widetilde{Y}_1$ embeds in R.



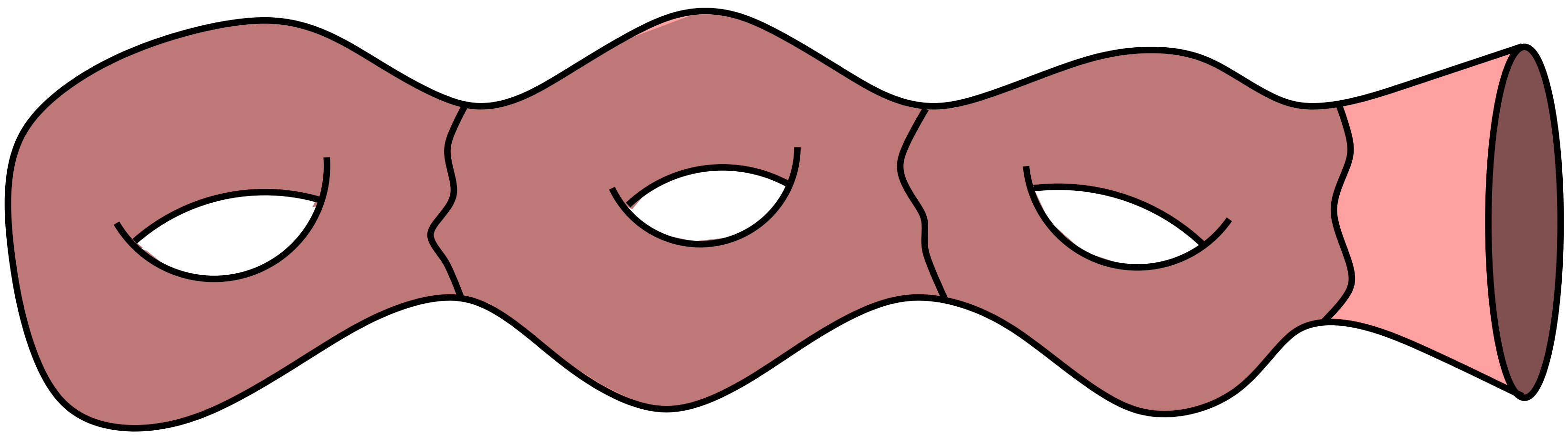
Take a larger compact, bordered piece Y_2.



Construct QR, solve Beltrami, embed $\widetilde{Y}_2$ in R.



Induction: Construct, Beltrami, re-embed in R,...



Induction: Construct, Beltrami, re-embed in R,...
Eventually build Belyi function on R.

Thm (B-Rempe): Every non-compact surface has a Belyi function.

Thm (B-Rempe): Every non-compact surface has a Belyi function.

Cor: Every non-compact surface has a equilateral triangulation.

Thm (B-Rempe): Every non-compact surface has a Belyi function.

Cor: Every non-compact surface has a equilateral triangulation.

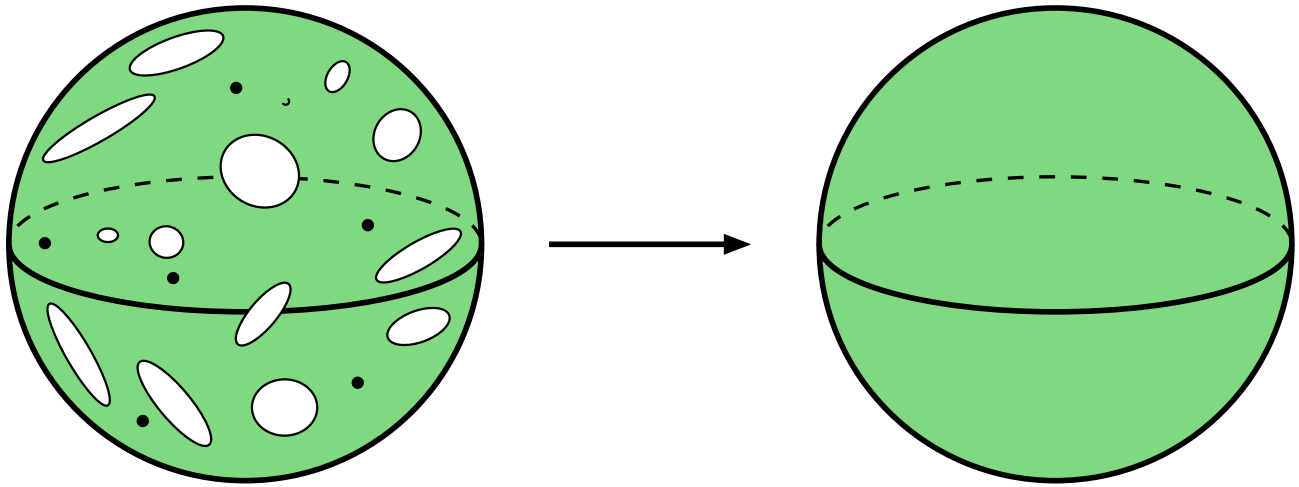
Cor: Every Riemann surface is a branched cover of the sphere, branched over finitely many points.

For compact surfaces, this is Riemann-Roch.

Compact, genus g sometimes needs $3g$ branch points.

3 branch points suffice for all non-compact surfaces.

Cor: Any open $U \subset \mathbb{C}$ is 3-branched cover of the sphere.



Adam Epstein: a holomorphic map $X \to Y$ is finite type if

- Y is compact,
- f is open,
- f has no isolated removable singularities,
- the set of singular values is finite.

Gives many new dynamical systems of finite type.

E.g., escaping points have more than one point they can escape to.

THANKS