“The best friend on earth of man is the tree. When we use the tree respectfully and economically, we have one of the greatest resources on the earth.”
– Frank Lloyd Wright

“Give me six hours to chop down a tree and I will spend the first four sharpening the axe.”
– Abraham Lincoln
200 step random walk.
1000 step random walk.
10000 step random walk.
Harmonic measure $= \text{hitting distribution of Brownian motion}$

Suppose $\Omega$ is a planar Jordan domain.
Harmonic measure = hitting distribution of Brownian motion

Let $E$ be a subset of the boundary, $\partial \Omega$. 
Harmonic measure = hitting distribution of Brownian motion

Choose an interior point $z \in \Omega$. 
Harmonic measure $= \text{hitting distribution of Brownian motion}$

$\omega(z, E, \Omega) = \text{probability a particle started at } z \text{ first hits } \partial \Omega \text{ in } E.$
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**Harmonic measure** = hitting distribution of Brownian motion

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$\omega(z, E, \Omega) =$ probability a particle started at $z$ first hits $\partial \Omega$ in $E$. 
Harmonic measure = hitting distribution of Brownian motion

$$\omega(z, E, \Omega) \approx \frac{35}{100}.$$ What if we move starting point $z$?
Harmonic measure $= \text{hitting distribution of Brownian motion}$

Different $z$ gives $\omega(z, E, \Omega) \approx 10/100$.

$\omega$ is harmonic in $z$ with boundary values $\omega = 1$ on $E$, $\omega = 0$ off $E$. 
Riemann Mapping Theorem: If $\Omega \subset \mathbb{R}^2$ is simply connected, then there is a conformal map $f : \mathbb{D} \to \Omega$.

Conformal = angle preserving
Brownian motion is conformally invariant, so normalized length measure maps to harmonic measure. Fastest way to compute harmonic measure.
Thm (F & M Riesz 1916):
For rectifiable boundaries, $\omega(E) = 0$ iff $E$ has zero length.

“Inside” and “outside” harmonic measures have same null sets. Measures are mutually absolutely continuous. Same measure class.
Thm (Makarov 1985):
For fractal domains, $\omega$ gives full measure to a set of zero length.

First such examples due to Lavrentiev (1936).
Thm (Makarov 1985):
For fractal domains, $\omega$ gives full measure to a set of zero length.

Outside is also fractal. Same set of length zero?
Theorem (B. 1987):
\( \omega_1 \perp \omega_2 \) iff tangents points have zero length.

Inside and outside harmonic measures are singular
Images of radial lines for conformal maps to inside and outside.
For which curves is $\omega_1 = \omega_2$?
For which curves is $\omega_1 = \omega_2$? True for lines:
For which curves is $\omega_1 = \omega_2$? True for lines:

Also for circles ($= \text{lines conformally}$).

Converse? If $\omega_1 = \omega_2$ must $\Gamma$ be a circle/line?
For which curves is $\omega_1 = \omega_2$? True for lines:

Also for circles ( = lines conformally).

Converse? If $\omega_1 = \omega_2$ must $\Gamma$ be a circle/line? Yes
Suppose $\omega_1 = \omega_2$ for a curve $\gamma$.

Conformally map two sides of circle to two sides of $\gamma$ so $f(1) = g(1)$.

$\omega_1 = \omega_2$ implies maps agree on whole boundary.

So $f, g$ define homeomorphism $h$ of plane holomorphic off circle.

Then $h$ is entire by Morera’s theorem.

Entire and 1-1 implies $h$ is linear (Liouville’s thm), so $\gamma$ is a circle.
A planar graph is a finite set of points connected by non-crossing edges. It is a tree if there are no closed loops.
A planar tree is **conformally balanced** if

- every edge has equal harmonic measure from $\infty$
- edge subsets have same measure from both sides
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This is also called a “**true tree**”.
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A planar tree is \textbf{conformally balanced} if

- every edge has equal harmonic measure from $\infty$
- edge subsets have same measure from both sides

This is also called a \textit{“true tree”}. A line segment is an example.
Trivially true by symmetry
Non-obvious true tree
Definition of critical value: if $p$ = polynomial, then

$$CV(p) = \{p(z) : p'(z) = 0\} = \text{critical values}$$

If $CV(p) = \pm 1$, $p$ is called generalized Chebyshev or Shabat.

10 classical Chebyshev polynomials
Balanced trees $\leftrightarrow$ Shabat polynomials

**Fact:** $T$ is balanced iff $T = p^{-1}([-1, 1])$, $p = \text{Shabat}$. 

\[ \Omega = \mathbb{C} \setminus T \quad \text{and} \quad U = \mathbb{C} \setminus [-1, 1] \]
$T$ balanced $\iff p$ Shabat.

$p$ is entire and $n$-to-$1 \iff p = \text{polynomial}.$

$CV(p) \not\in U \iff p : \Omega \to U$ is covering map.
Theorem: Every finite tree has a true form.

Standard proof uses the uniformization theorem.
Standard proof:

- start with a finite tree.
Standard proof:

- connect vertices of $T$ to infinity; gives finite triangulation of sphere.
- Defines adjacencies between triangles.
Standard proof:

- Glue equilateral triangles using adjacencies: get a conformal 2-sphere.
- By uniformization theorem, conformal maps to Riemann sphere.
- Can check that tree maps to balanced tree.
Note: conformal 2-sphere has many “equilateral triangulations”.
Which other Riemann surfaces can be obtained in this way?
This is subject of Lasse Rempe-Gillen’s talk later.
**Algebraic aside:**

True trees are examples of Grothendieck’s *dessins d’enfants* on sphere.

Normalized polynomials are algebraic, so planar trees correspond to number fields. Absolute Galois group acts on trees, but orbits unknown.

Six graphs of type 5 1 1 1 1 1 - 3 3 2 1 1, two orbits.
Even computing number field from tree is difficult.

For example, the polynomial for this 9-edge tree is

\[ p(z) = z^4(z^2 + az + b)^2(z - 1), \]

where \( a \) is a root of ...
\[ 0 = 126105021875 \, a^{15} + 873367351500 \, a^{14} \\
+ 2340460381665 \, a^{13} + 2877817869766 \, a^{12} \\
+ 3181427453757 \, a^{11} - 68622755391456 \, a^{10} \\
- 680918281137097 \, a^9 - 2851406436711330 \, a^8 \\
- 7139130404618520 \, a^7 - 12051656256571792 \, a^6 \\
- 14350515598839120 \, a^5 - 12058311779508768 \, a^4 \\
- 6916678783373312 \, a^3 - 2556853615656960 \, a^2 \\
- 561846360735744 \, a - 65703906377728 \]

This is not the most complicated formula in Kochetkov’s paper.

However, true form can be drawn without knowing the polynomial.
Don Marshall’s **ZIPPER** uses conformal mapping to draw true trees.
Don Marshall’s **ZIPPER** uses conformal mapping to draw true trees.

Marshall and Rohde approximated all true trees with $\leq 14$ edges.

They can compute vertices to 1000’s of digits of accuracy.

Test if $\alpha \in \mathbb{C}$ is algebraic by seeking integer relationships between $1, \alpha, \alpha^2, \ldots$ using lattice reduction or Helaman Ferguson’s PSLQ algorithm.
Some true trees, courtesy of Marshall and Rohde
Alternate proof that all trees have true forms using QC mappings.
Quasiconformal (QC) maps send infinitesimal ellipses to circles.

Eccentricity = ratio of major to minor axis of ellipse.

For $K$-QC maps, ellipses have eccentricity $\leq K$
Quasiconformal (QC) maps send infinitesimal ellipses to circles.

Eccentricity = ratio of major to minor axis of ellipse.

For $K$-QC maps, ellipses have eccentricity $\leq K$

Ellipses determined a.e. by measurable dilatation $\mu = f_{\overline{z}}/f_z$ with

$$|\mu| \leq \frac{K - 1}{K + 1} < 1.$$

Conversely, . . .
Quasiconformal (QC) maps send infinitesimal ellipses to circles.

Mapping theorem: any such $\mu$ comes from some QC map $f$. 
Quasiconformal (QC) maps send infinitesimal ellipses to circles.

Mapping theorem: any such $\mu$ comes from some QC map $f$.

**Cor:** If $f$ is holomorphic and $\psi$ is QC, then there is a QC map $\varphi$ so that $g = \psi \circ f \circ \varphi$ is also holomorphic.
Quasiconformal (QC) maps send infinitesimal ellipses to circles.

**Mapping theorem:** any such $\mu$ comes from some QC map $f$.

**Cor:** If $f$ is holomorphic and $\psi$ is QC, then there is a QC map $\varphi$ so that $g = \psi \circ f \circ \varphi$ is also holomorphic.

Such $f, g$ are called **QC-equivalent** (weaker than being conjugate).

In general, $\psi \circ f \circ \varphi$ is not holomorphic, but is **quasiregular** (QR).
QC proof that every finite tree has a true form:

Map $\Omega = \mathbb{C} \setminus T$ to $\{|z| > 1\}$ conformally.

“Equalize intervals” by diffeomorphism. Composition is quasiconformal.
QC proof that every finite tree has a true form:

Map $\Omega = \mathbb{C} \setminus T$ to $\{|z| > 1\}$ conformally.

“Equalize intervals” by diffeomorphism. Composition is quasiconformal.
Mapping theorem implies there is a QC $\varphi$ so $p = q \circ \varphi$ is a polynomial.

Only possible critical points are vertices of tree; these map to $\pm 1$.

Thus every planar tree has a true form.
Do true trees approximate all possible shapes?
Do true trees approximate all possible shapes?

YES
Do true trees approximate all possible shapes?

**Thm:** (B. 2013) Every planar continuum is Hausdorff limit of true trees.
Suffices to approximate subtrees of a grid.
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Suffices to approximate subtrees of a grid.
**Theorem:** Every planar continuum is a limit of true trees.

**Idea of Proof:** reduce harmonic measure ratio by adding edges.

Vertical side has much larger harmonic measure from left.
Theorem: Every planar continuum is a limit of true trees.

Idea of Proof: reduce harmonic measure ratio by adding edges.

“Left” harmonic measure is reduced (roughly 3-to-1).

New edges are approximately balanced (universal constant).
**Theorem:** Every planar continuum is a limit of true trees.

**Idea of Proof:** reduce harmonic measure ratio by adding edges.

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  [Diagram]
```

“Left” harmonic measure is reduced (roughly 3-to-1).

New edges are approximately balanced (universal constant).

Mapping theorem gives exactly balanced.

QC correction map is near identity if “spikes” are short.

New tree approximates shape of old tree; different combinatorics.
Corollary (B., Pilgrim): Every continuum is a limit of Julia sets of polynomials with two post-critical points.

Also see work of Ivrii, Lindsey, Younsi,...
Can replace trees by meshes. Can replace plane by other surfaces. Get rational or meromorphic functions.

Lasse’s talk later today will explore folding on surfaces.

The rest of my lectures will concentrate on infinite trees in the plane.
Do infinite trees correspond to entire functions with 2 critical values?
Main difference:
\( \mathbb{C} \setminus \text{finite tree} = \text{one topological annulus} \)
\( \mathbb{C} \setminus \text{infinite tree} = \text{many simply connected components} \)
Finite case

$T$ is true tree $\iff p = \frac{1}{2}(\tau^n + 1/\tau^n)$ is continuous across $T$. 
Infinite case

\[
\Omega \xrightarrow{\text{conformal}} \tau \xrightarrow{\cosh} \exp \xrightarrow{\text{continuously}} T
\]

Infinite balanced tree \(\Leftrightarrow f = \cosh \circ \tau\) is continuous across \(T\).
Definitions of $\exp$ and $\cosh$. 

\[ \frac{1}{2} \left( z + \frac{1}{z} \right) \]
Do all infinite planar trees have true forms?

Do infinite true trees approximate any shape?
Do all infinite planar trees have true forms? **No.**

Do infinite true trees approximate any shape? **Yes (sort of).**
The infinite 3-regular tree has no true form in plane.

Reason 1 (W. Cui, 2018): Corresponding $f$ would contradict a theorem of Nevanlinna: $\sum_{i=1}^{q}(1 - \frac{1}{m_i}) \leq 2$, $q = 3$, $m_1 = m_2 = 3$, $m_3 = \infty$. Cui also gives sufficient conditions for an infinite tree to have a true form.
The infinite 3-regular tree has no true form in plane.

**Reason 2:** Planar homeomorphisms mapping tree to itself permute the complementary components. Corresponding conformal maps are continuous across tree, hence linear, hence isometries. But this group has exponential growth: “too large” to fit inside isometries of plane.
The infinite 3-regular tree has no true form in plane.

“Reason” 3: Computation of true form of truncated regular tree indicate convergence to bounded set. (Image due to Marshall and Rohde.)
Coincidence of shapes noted by Rohde and Werness. Deltoid fractal studied by Lee, Lyubich, Makarov, Mukherjee. Arises in anti-holomorphic dynamics.
Now, the second question.

Which infinite planar trees can be approximated by true trees?

We need two assumptions that substitute for finiteness.

First we define a neighborhood $T(r)$ of an infinite tree.

(Replaces Hausdorff $\epsilon$-neighborhood in finite case.)
If $e$ is an edge of $T$ and $r > 0$ let

$$e(r) = \{ z : \text{dist}(z, e) \leq r \cdot \text{diam}(e) \}$$
If $e$ is an edge of $T$ and $r > 0$ let

$$e(r) = \{ z : \text{dist}(z, e) \leq r \cdot \text{diam}(e) \}$$

Define neighborhood of $T$: $T(r) = \bigcup \{ e(r) : e \in T \}$. 
If $e$ is an edge of $T$ and $r > 0$ let

$$e(r) = \{ z : \text{dist}(z, e) \leq r \cdot \text{diam}(e) \}$$

Define neighborhood of $T$: $T(r) = \bigcup \{ e(r) : e \in T \}$.

Adding vertices reduces $T(r)$. Useful scaling property.
(1) **Bounded Geometry** (local condition; easy to verify):
- edges are uniformly smooth.
- adjacent edges form bi-Lipschitz image of a star = \( \{ z^n \in [0, r] \} \)
- non-adjacent edges are well separated,

\[
\text{dist}(e, f) \geq \epsilon \cdot \min(\text{diam}(e), \text{diam}(f)).
\]
(2) $\tau$-Lower Bound (global condition; harder to check):
Complementary components of tree are simply connected.
Each can be conformally mapped to right half-plane. Call map $\tau$.

We assume all images have length $\geq \pi$.
Need positive lower bound; actual value usually not important.
Components are “thinner” than half-plane near $\infty$. 
QC-Folding Theorem (B 2015): If $T$ has bounded geometry and the $\tau$-lower bound, then $T$ can be approximated by a true tree in the following sense. Let $F = \cosh \circ \tau$. Then there is a $K$-quasiregular $g$ and $r > 0$ such that $g = F$ off $T(r)$ (shaded) and $\text{CV}(g) = \pm 1$.

$K$ and $r$ only depend on the bounded geometry constants.

$g = F$ on light blue.

$F$ may be discontinuous across $T$. $g$ is continuous everywhere.
**QC-Folding Theorem (B 2015):** If $T$ has bounded geometry and the $\tau$-lower bound, then $T$ can be approximated by a true tree in the following sense. Let $F = \cosh \circ \tau$. Then there is a $K$-quasiregular $g$ and $r > 0$ such that $g = F$ off $T(r)$ (shaded) and $\text{CV}(g) = \pm 1$.

$K$ and $r$ only depend on the bounded geometry constants.

**Cor:** Any $T$ as above is approximated by $\varphi(T')$ where $T' = f^{-1}([-1, 1])$, $f$ is entire with $\text{CV}(f) = \pm 1$, $\varphi$ is QC and conformal off $T(r)$. (In many applications, $\varphi$ is close to identity, so $T \approx T'$.)
First, a sample application.
Rapid increase in Speiser class

\[ f \text{ has } 2 \text{ singular values, } f(x) \to \infty \text{ as fast as we wish.} \]

Similar examples due to Sergei Merenkov 2008 (3 singular values).
Speiser example with single tract; rapid growth and spiraling.
Previous examples had logarithmic spirals.
Next, the idea of the proof of folding in a simple case.

The tree is a vertical line with even spaced vertices.

The $\tau$ map are different linear maps on each side.
Construct an entire function $F$ so that

\[ F(z) = \cosh(z) \text{ on left half-plane } = \mathbb{H}_l = \{z = x + iy : x < 0\} \]
\[ F(z) = \cosh(3z) \text{ on right half-plane } = \mathbb{H}_r = \{z = x + iy : x > 0\} \]
Construct an entire function $F$ so that

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$$F(z) = \cosh(3z) \text{ on right half-plane } = \mathbb{H}_r = \{ z = x + iy : x > 0 \}$$

**Impossible:** violates unique continuation
Construct an **quasiregular** function $F$ so that

\[ F(z) = \cosh(z) \text{ on } \{x < 0\} \]
\[ F(z) = \cosh(3z) \text{ on } \{x > 1\} \]
Construct an **quasiregular** function $F$ so that

$$F(z) = \cosh(z) \text{ on } \{x < 0\}$$

$$F(z) = \cosh(3z) \text{ on } \{x > 1\}$$

This is easy. First add some spikes in the strip.
Construct an **quasiregular** function $F$ so that

\[ F(z) = \cosh(z) \quad \text{on} \quad \{ x < 0 \} \]
\[ F(z) = \cosh(3z) \quad \text{on} \quad \{ x > 1 \} \]

Triangulate the folded strip.
Let $\Phi$ be piecewise affine map: folded strip $\rightarrow$ un-folded strip.

Take $\Phi = \text{identity map off the strip}$. 
\( G(z) = \cosh(3 \cdot \Phi(z)) \) satisfies:
- equals \( \cosh 3z \) on \( \{ x \geq 1 \} \)
- equals \( \cosh z \) on \( \{ x = 0 \} \)
- Is quasiregular (composition of QC and holomorphic)

Measurable Riemann mapping theorem gives entire \( F = G \circ \varphi \).
\[ G(z) = \cosh(3 \cdot \Phi(z)) \] satisfies:

- equals \( \cosh 3z \) on \( \{ x \geq 1 \} \)
- equals \( \cosh z \) on \( \{ x = 0 \} \)
- Is quasiregular (composition of QC and holomorphic)

Can replace 3, by any \( n \) and keep QC constant bounded.
**QC-Folding Theorem:** If $T$ has bounded geometry and the $\tau$-lower bound, then there is a $K$-quasiregular $g$ and $r > 0$ such that $g = \cosh \circ \tau$ off $T(r)$ (shaded) and $\text{CV}(g) = \pm 1$.

$K$ and $r$ only depend on the bounded geometry constants.
Proof of Folding Theorem: add finite trees (instead of spikes) to sides of $T$, so one edge is glued to tree edge; other are folded onto each other.

This allows arbitrarily large foldings with uniform QC bounds.
More general folding theorem: replace tree by graph, faces labeled D,L,R.

\[ D = \text{bounded Jordan domains } (z^n, \text{high degree critical points}) \]

\[ L = \text{unbounded Jordan domains } (e^{-\tau(z)}, \text{finite asymptotic values}) \]

RR-edges map to \([-1, 1]\), other edges map to \(\mathbb{T}\).

Another variation inverts the D-components, introduce poles.
APPLICATIONS OF QC-FOLDING

EL Models

Order Conjecture

Post-singular dynamics

Eremenko Conjecture

Area Conjecture

Wiman’s Conjecture

Dimension near 1

Folding in disk

Wandering domain
THANKS
**Singular set**  =  closure of critical values and finite asymptotic values
= smallest set so that $f$ is a covering map onto $\mathbb{C} \setminus S$
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= smallest set so that $f$ is a covering map onto $\mathbb{C} \setminus S$

Eremenko-Lyubich class = bounded singular set = $\mathcal{B}$

Speiser class = finite singular set = $\mathcal{S} \subset \mathcal{B}$
Singular set \(=\) closure of critical values and finite asymptotic values
\(=\) smallest set so that \(f\) is a covering map onto \(\mathbb{C} \setminus S\)

Eremenko-Lyubich class \(=\) bounded singular set \(=\) \(\mathcal{B}\)

Speiser class \(=\) finite singular set \(=\) \(S \subset \mathcal{B}\)

Folding produces functions in the smaller Speiser class. There is an “easier” version that gives functions in the larger Eremenko-Lyubich class. Uses fewer assumptions and the proof is simpler.
For Speiser class, the tree is preimage of a segment.
The preimage of a surrounding circle is a union of infinite Jordan domains.
This preimage makes sense as long as all singular values are strictly inside the circle, i.e., for Eremenko-Lyubich functions.
A **model** is a pair \((\Omega, \tau)\) where

- \(\Omega = \bigcup \Omega_j\) is a disjoint union of tracts (unbounded Jordan domains).
- \(\tau\) is conformal from each \(\Omega_j\) to \(\mathbb{H}_r\) \((\infty \rightarrow \infty)\).
Every Eremenko-Lyubich function $F$ gives a model with $\Omega = \{|F| > R\}$.

Does every model give an Eremenko-Lyubich function?
Given any model \((\Omega, \tau)\) can we extend \(F = e^{\tau}\) holomorphically to \(\mathbb{C}\)?
Given any model $(\Omega, \tau)$ can we extend $F = e^\tau$ holomorphically to $\mathbb{C}$?

**Obviously impossible:**

(1) violates unique continuation.

(2) $\partial \Omega$ must be analytic if $F$ is.
Given any model \((\Omega, \tau)\) can we extend \(F = e^\tau\) holomorphically to \(\mathbb{C}\)?

Obviously impossible:

1. violates unique continuation. **Use QR function.**
2. \(\partial\Omega\) must be analytic if \(F\) is. **Use conformal approximation.**
**Theorem:** Suppose $(\Omega, \tau)$ is a model and $\rho > 0$. Define

$$\Omega(\rho) = \tau^{-1}(\{x + iy : x > \rho\}) \subset \Omega.$$ 

Then there is a quasiregular $g$ so that

1. $g = e^\tau$ on $\Omega(2\rho)$,
2. $|g| \leq e^\rho$ off $\Omega(2\rho)$.
3. $g$ is holomorphic off $\Omega(\rho)$.
**Theorem:** Suppose $(\Omega, \tau)$ is a model and $\rho > 0$. Define

$$\Omega(\rho) = \tau^{-1}(\{x + iy : x > \rho\}) \subset \Omega.$$ 

Then there is a quasiregular $g$ so that

1. $g = e^\tau$ on $\Omega(2\rho)$,
2. $|g| \leq e^\rho$ off $\Omega(2\rho)$.
3. $g$ is holomorphic off $\Omega(\rho)$.

The QR constant depends on $\rho$, but not on $\Omega$.

There is a quasiconformal $\varphi$ so that $f = g \circ \varphi \in B$.

$g$ is holomorphic except on $\Omega(\rho) \setminus \Omega(\rho/2)$ (often has finite area).

Formulation inspired by Lasse Rempe-Gillen.
If $\Omega =$ half-strip, the $\tau$-partition for inside decays exponentially.

But $\tau$-partition for outside grows like $\sqrt{n}$.

Can’t choose vertices with both bounded geometry and $\tau$ lower bound.

So QC-folding theorem does not apply here.
Indeed, no Speiser class function with a single tract can have that tract approximate a half-strip.

However, the models theorem does apply: there are Eremenko-Lyubich functions with tracts approximating the half-strip.
In fact, such examples give a theorem of Stallard:

**Theorem:** There are Eremenko-Lyubich functions whose Julia sets have Hausdorff dimension close to 1.
We will define an entire function $f$ that has an attracting fixed point at 0 and the basin contains a large disk $D(0, R)$. Julia set is outside this disk.

$$\mathcal{J}(f) \subset \bigcap X_n,$$

where

$$X_n = \{ z : |f^k(z)| \geq R, k = 1, \ldots, n \}.$$
We will define an entire function $f$ that has an attracting fixed point at 0 and the basin contains a large disk $D(0, R)$. Julia set is outside this disk.

$$\mathcal{J}(f) \subset \bigcap X_n,$$

where

$$X_n = \{ z : |f^k(z)| \geq R, k = 1, \ldots, n \}.$$  

Suffices to cover $X_1$ by disks $\{D_j\}$ so that

$$\sum_j \text{diam}(D_j)^{1+\delta} < \infty,$$

and prove that if $D$ is a disk that hits $\mathcal{J}$ and is contained in $\{|z| > R\}$, then the preimages satisfy

$$\sum_{W_j \in f^{-1}(D)} \text{diam}(W_j)^{1+\delta} \leq \epsilon \cdot \text{diam}(D)^{1+\delta}.$$
The models theorem applies to this tract.

Implies there is a QR extension of $\exp(\exp(z - K))$ from tract to plane, that is bounded off the tract.

Region outside tract maps compactly into itself, hence iterates to an attractive fixed point, so it is in Fatou set. Hence Julia set is contained in the (blue) tract.
Preimage of gold disk $D = D(w, r)$ defined in two steps:

- stack of regions of diameter $O(r/|w|)$ on line $\{x = \log |w|\}$.
- region at height $2\pi k$ in stack has single preimage $U_k$ of diameter

$$O \left( \frac{r}{|w| (\log |w| + 2\pi |k|)} \right).$$

These estimates only use $(\log z)' = 1/z$. 

\[\begin{align*}
\end{align*}\]
If $\delta > 0$ is fixed and $R$ is large enough, then

$$\sum_k \text{diam}(U_k)^{1+\delta} \lesssim \left(\frac{r}{|w|}\right)^{1+\delta} \sum_k \frac{1}{(\log |w| + 2\pi |k|)^{1+\delta}}$$

$$\lesssim \left|\frac{r}{w}\right|^{1+\delta} \frac{1}{\delta \log^{1+\delta} |w|} \ll \left|\frac{r}{w}\right|^{1+\delta} \ll r^{1+\delta}$$

This proves \( \dim(\mathcal{J}) \leq 1 + \delta \).

(Ignore the QC correction; it’s bi-Lipschitz on a nbhd of the Julia set.)
DeMarco, Koch and McMullen proved that a post-critically finite rational map can have any given dynamics.

More precisely, given any $\epsilon > 0$, any finite set $X$ and any map $h : X \to X$ there is a rational $r$ whose post-singular set $P(r)$ is the same size as $X$, $P(r)$ $\epsilon$-approximates $X$ in Hausdorff metric, and $|r - h| < \epsilon$ on $P$. 
DeMarco, Koch and McMullen proved that a post-critically finite rational map can have any given dynamics.

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**Theorem (Lazebnik, B.):** Let $X$ be discrete ($\geq 4$ points), let $h : X \to X$ be any map, and let $\epsilon > 0$. Then there is a transcendental meromorphic function $f$ and a bijection $\psi : X \to P(f)$ so that

$$|\psi(z) - z| \leq \max(\epsilon, o(1)) \quad \text{and} \quad f = \psi \circ h \circ \psi^{-1} \text{ on } P(f).$$

Main argument (folding + fixed point thm) due to Kirill.

Unknown if we can take $\epsilon = 0$ in either case
Construct a domain $W$ that contains the given points, and so that points lie very close to hyperbolic geodesic to $\infty$. 
Conformally map $W$ to upper half-plane; points almost on vertical ray.
Divide upper half-plane into R and D components.

Folding map sends centers anywhere we want in \( \mathbb{D} \).

Analogous construction maps centers outside \( \mathbb{D} \); introduces poles.
Transfer map back to $W$. Decompose complement of $W$ into R-components and define map on whole plane by folding.

Gives quasi-meromorphc map $g$ with desired post-singular behavior. In particular post-singular set is mapped into itself by $g$.

Mapping theorem gives a meromorphic $f = g \circ \phi$. However, because of $\phi$, the post-singular set no longer invariant under $f$. How to fix this?
Replace $X$ by a union of disjoint disks, and let $Y \approx X$ consist of one point from each disk (red points). Repeat construction to get $g : X \to Y$ and meromorphic $f = g \circ \phi$ that has singular points at $\phi^{-1}(Y)$ (yellow).

Each yellow point is a continuous function of all the red ones and we want the yellow and red sets to coincide. Because $\phi$ is close to the identity the yellow points remain inside the disks. An infinite-dimensional fixed point theorem gives choice of $Y$ so that $f(Y) = Y$ (yellow = red), as desired.
Order of growth of entire function:

\[ \rho(f) = \limsup_{|z| \to \infty} \frac{\log \log |f(z)|}{\log |z|}, \quad \rho(e^{zd}) = d. \]

**QC-equivalent:** \( f \sim g \) if \( g = \psi \circ f \circ \phi \) for quasiconformal \( \psi, \phi \).

Speiser class QC-equivalence classes \( M_f \) are finite dimensional.

**Order conjecture:** Is \( \rho \) constant on \( M_f \)?

Adam Epstein observed this holds in many cases where order is determined by “combinatorial data”. He and Rempe-Gillen gave EL counterexample.
Speiser class counterexample with 3 singular values.
Same tree in logarithmic coordinates.
Speiser function that is “statistically zero”: for all $\epsilon > 0$,
\[
\text{area}(\{|f| > \epsilon\}) < \infty.
\]
Counterexample to Eremenko-Lyubich area conjecture.
Wiman’s conjecture:

Define $m(r) = \min_{|z|=r} |f(z)|$, $M(r) = \max_{|z|=r} |f(z)|$.

Wiman (1916) conjectured that for all entire functions

$$\limsup_{r \to \infty} \frac{\log m(r, f)}{\log M(r, f)} \geq -1.$$ 

Sharp for $f(z) = \exp(z)$.

True in special cases, e.g., $|f(r)| = m(r)$ (Beurling, 1949).

False in general (Hayman, 1952).
Speiser class counterexample to Wiman’s conjecture

\[
\frac{\log m(r,f)}{\log M(r,f)} \leq -C \cdot \log \log \log \log M(r,f)
\]
Given an entire function $f$,

**Fatou set** $= \mathcal{F}(f) =$ open set where iterates are normal family.

**Julia set** $= \mathcal{J}(f) =$ complement of Fatou set.

Julia set is usually fractal. What is its (Hausdorff) dimension?

$\mathcal{J}((e^z - 1)/2)$, courtesy of Arnaud Ch´eritat
Given an entire function $f$,

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- Baker (1975): $f$ transcendental $\Rightarrow \mathcal{J}$ contains a continuum $\Rightarrow \dim \geq 1$.
- Misiurewicz (1981), McMullen (1987) $\dim = 2$ occurs (is common)
- Stallard (1997, 2000): $\{\dim(\mathcal{J}(f)) : f \in \mathcal{B}\} = (1, 2]$.
- $1 < \text{H-dim} < 2$ can also occur in Speiser class (Albrecht-B.)
- Rippon-Stallard (2005) Packing dim $= 2$ for $f \in \mathcal{B}$.
- B (2018) H-dim $= \text{P-dim} = 1$ can occur (example outside $\mathcal{B}$).
- Burkart (2019) $1 < \text{packing dim} < 2$ can occur (outside $\mathcal{B}$).
\[ \inf \{ \dim(J(f)) : f \in \mathcal{S} \} = 1 \quad \text{(B.-Albrecht, 2018).} \]
Uses upper and lower \( \tau \)-bounds to control dimension.
Radial edges have exponential imbalance. “Spikes” are chosen to give bounded imbalance. Must do this explicitly to control number and sizes of preimages of a disk.
Inspiration for $\dim = 1$ example. Actual proof uses infinite product with zeros on concentric circles.

$$f(z) = f_0(z) \cdot \prod_{k=1}^{\infty} \left( 1 - \frac{1}{2} (z/R_k)^{n_k} \right)$$
Fatou component in “Dim = 1” example: $C^1$ boundaries
First multiply connected Fatou component where dynamics is understood.
See also thesis of Markus Baumgartner under Bergweiler.
Given an entire function $f$,

**Fatou set** = $\mathcal{F}(f) =$ open set where iterates are normal family.

**Julia set** = $\mathcal{J}(f) =$ complement of Fatou set.

$f$ permutes components of its Fatou set.

**Wandering domain** = Fatou component with infinite orbit.

- Entire functions can have wandering domains (Baker 1975).
- No wandering domains for rational functions (Sullivan 1985).
- Also none in Speiser class (Eremenko-Lyubich, Goldberg-Keen).

Are there wandering domains in Eremenko-Lyubich class?
Graph giving wandering domain in Eremenko-Lyubich class.

Original proof corrected by Marti-Pete and Shishikura, who also give alternate construction.
Graph giving wandering domain in Eremenko-Lyubich class.

Variations by Lazebnik, Fagella-Godillon-Jarque, Osborne-Sixsmith.
Graph giving wandering domain in Eremenko-Lyubich class.
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Graph giving wandering domain in Eremenko-Lyubich class.

Variations by Lazebnik, Fagella-Godillon-Jarque, Osborne-Sixsmith.
A curve spiraling out the boundary of the disk. We can add vertices to make this bounded geometry and satisfy the $\tau$-lower bound.

The folding construction gives a function on the unit disk with two critical values so that $\{|f| > 2\}$ spirals out to disk (first constructed by Valiron).
A holomorphic function on the disk with two critical values so that the set \( \{ z : |f(z)| > \epsilon \} \) has finite hyperbolic area for any \( \epsilon > 0 \).

Decompose \( \mathbb{D} \) into Whitney boxes. Thicken the edges to form disjoint R-components, each tending to boundary. The remaining boxes are D-components, where \( f \) is close to zero (except very near the boundary), if we place a high degree critical point in each.
For $f$ entire, **escaping set** is $I(f) = \{z : f^n(z) \to \infty\}$.

Known that $\mathcal{J}(f) = \partial I(f)$.

Fatou observed $I(f)$ often consists of curves to $\infty$.

Curve in escaping set, courtesy of Lasse Rempe-Gillen
Eremenko Conj: components of $I(f)$ are unbounded (still open).

Strong Eremenko Conj (SEC): path components are unbounded.

Rottenfusser, Rüchert, Rempe-Gillen and Schleicher (2011) proved:

- SEC true for EL functions with finite order of growth.
- SEC false for some EL functions with infinite order.
- Examples with trivial path components.

QC-Folding gives examples in Speiser class.
Speiser class counterexample to SEC.
Path components of escaping set can be points.
Other exotic examples by Rempe-Gillen.
Speiser class counterexample to SEC. 
Path components of escaping set can be points. 
Other exotic examples by Rempe-Gillen.
SEC counterexample in logarithmic coordinates
Ceci n’est pas un ensemble de Julia.
True tree based on combinatorics of Julia set of $z^2 + i$. Example of “rigidity”: combinatorics determines geometry.
True DLA