TRUE TREES AND TRANSCENDENTAL TRACTS

Christopher Bishop, Stony Brook

Low Dimensional and Complex Dynamics December 2-6, 2019
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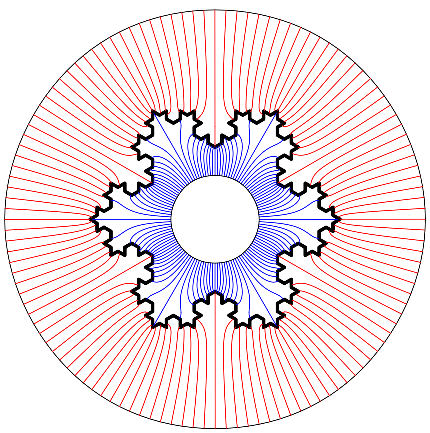
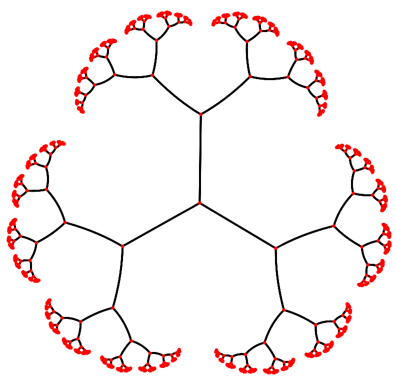
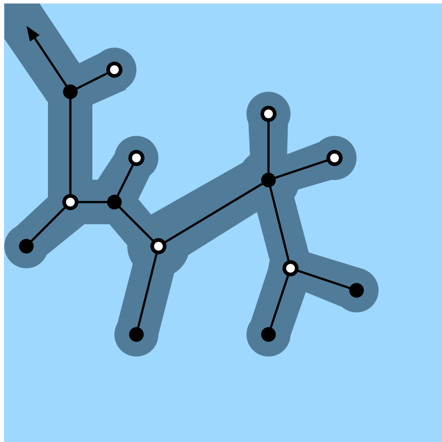
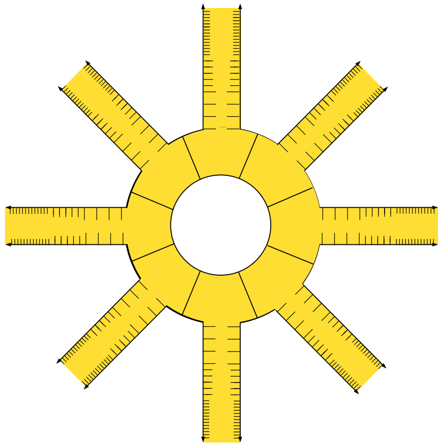
`www.math.sunysb.edu/~bishop/lectures`







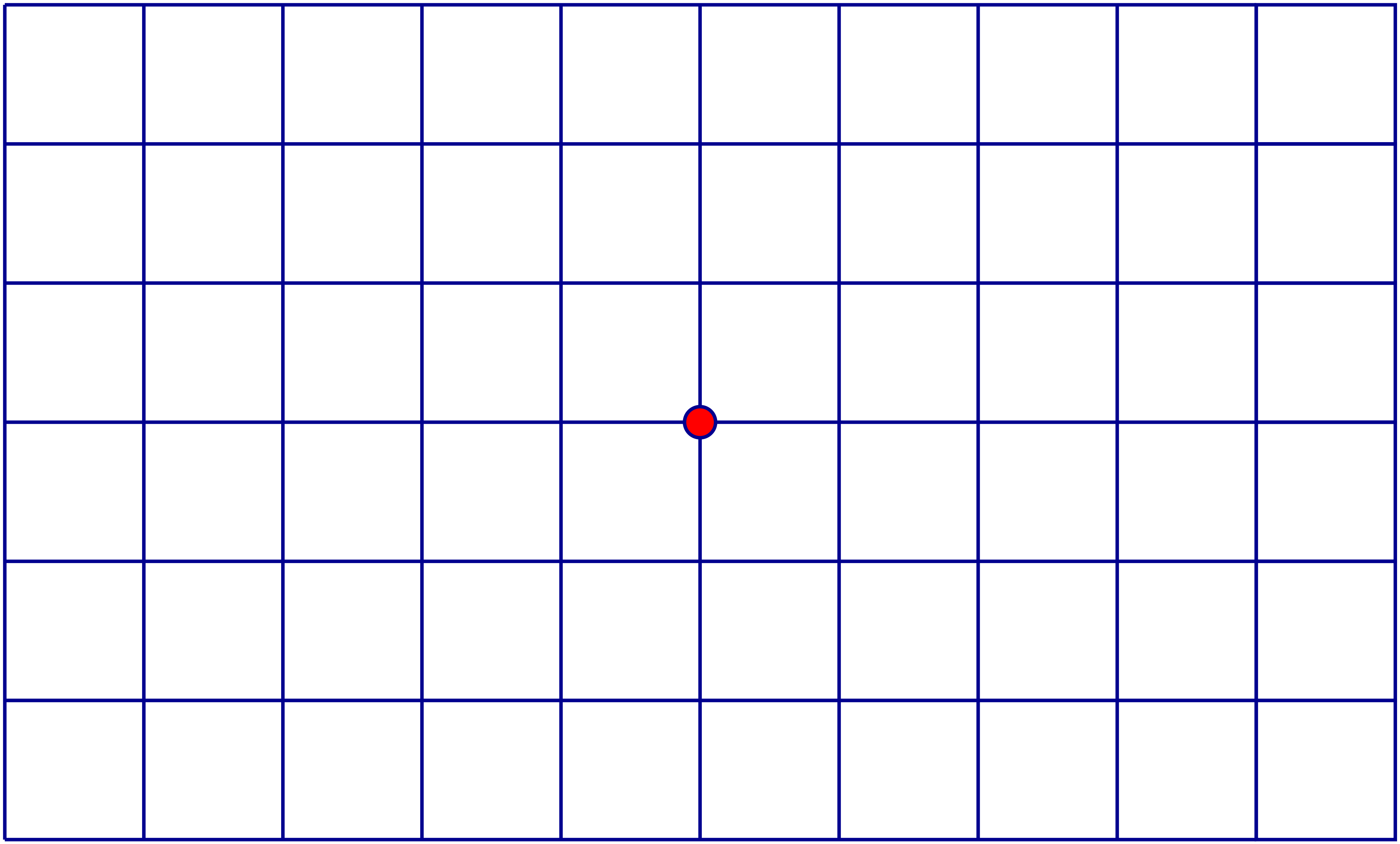


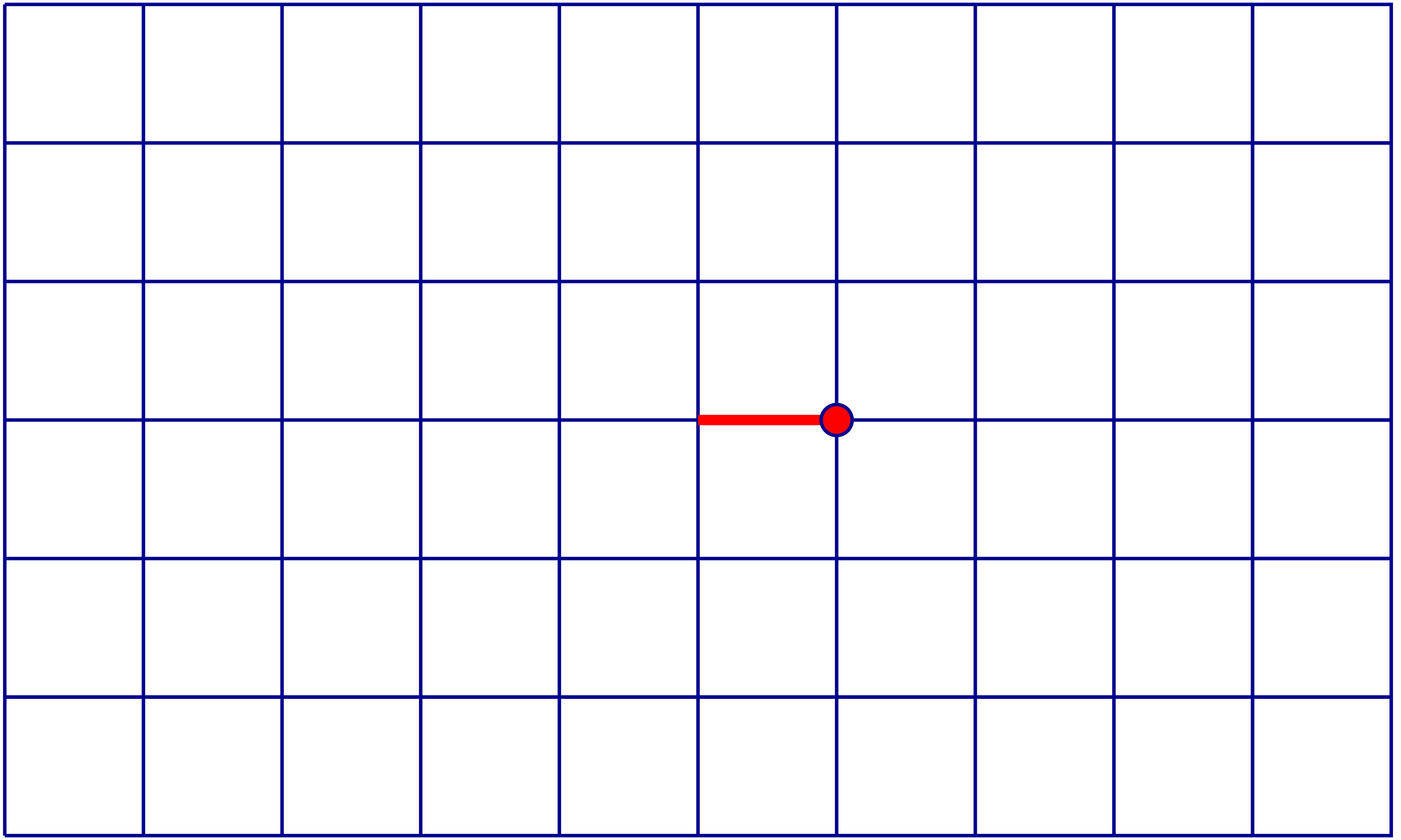
"The best friend on earth of man is the tree. When we use the tree respectfully and economically, we have one of the greatest resources on the earth."

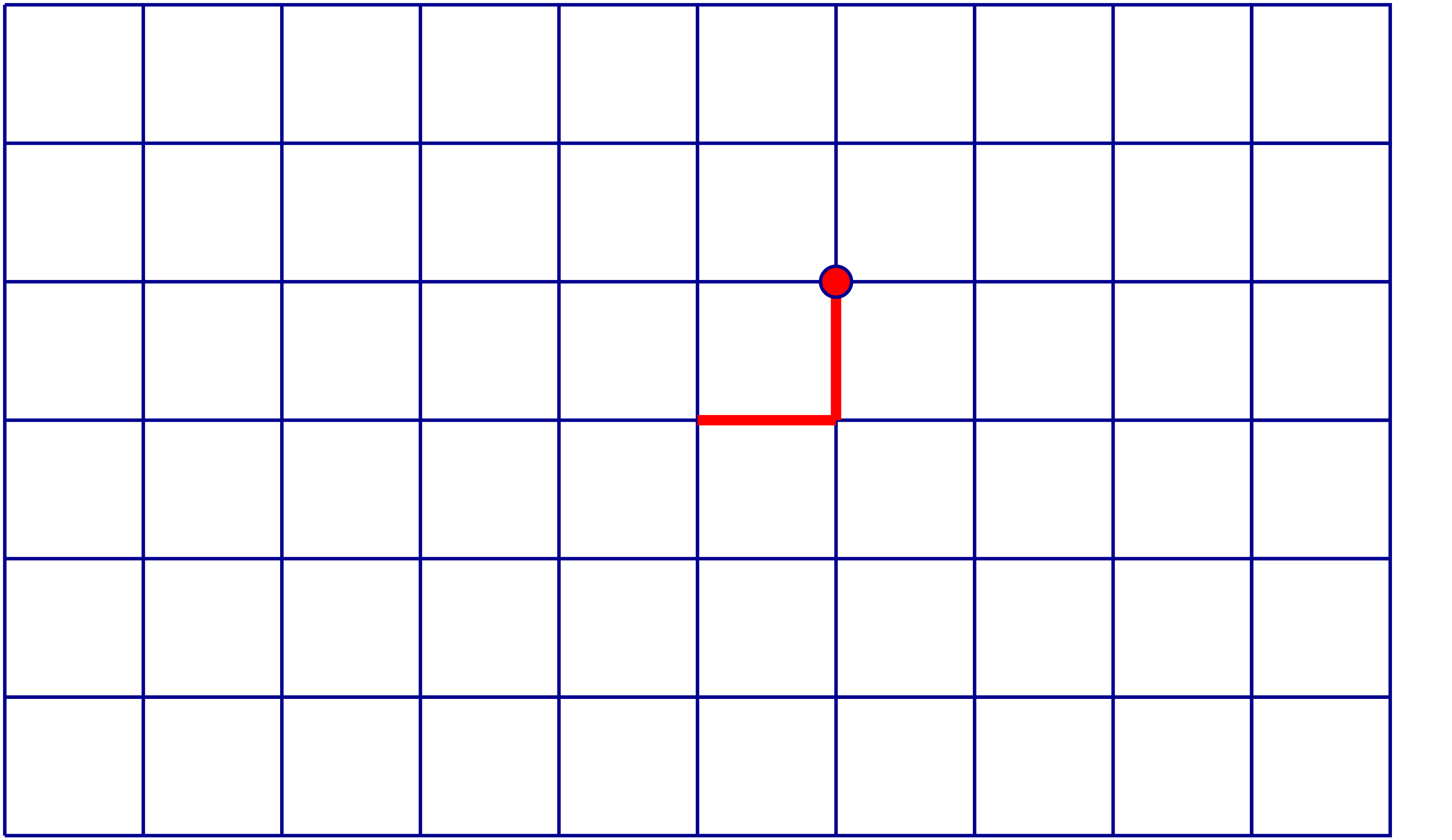
– Frank Lloyd Wright

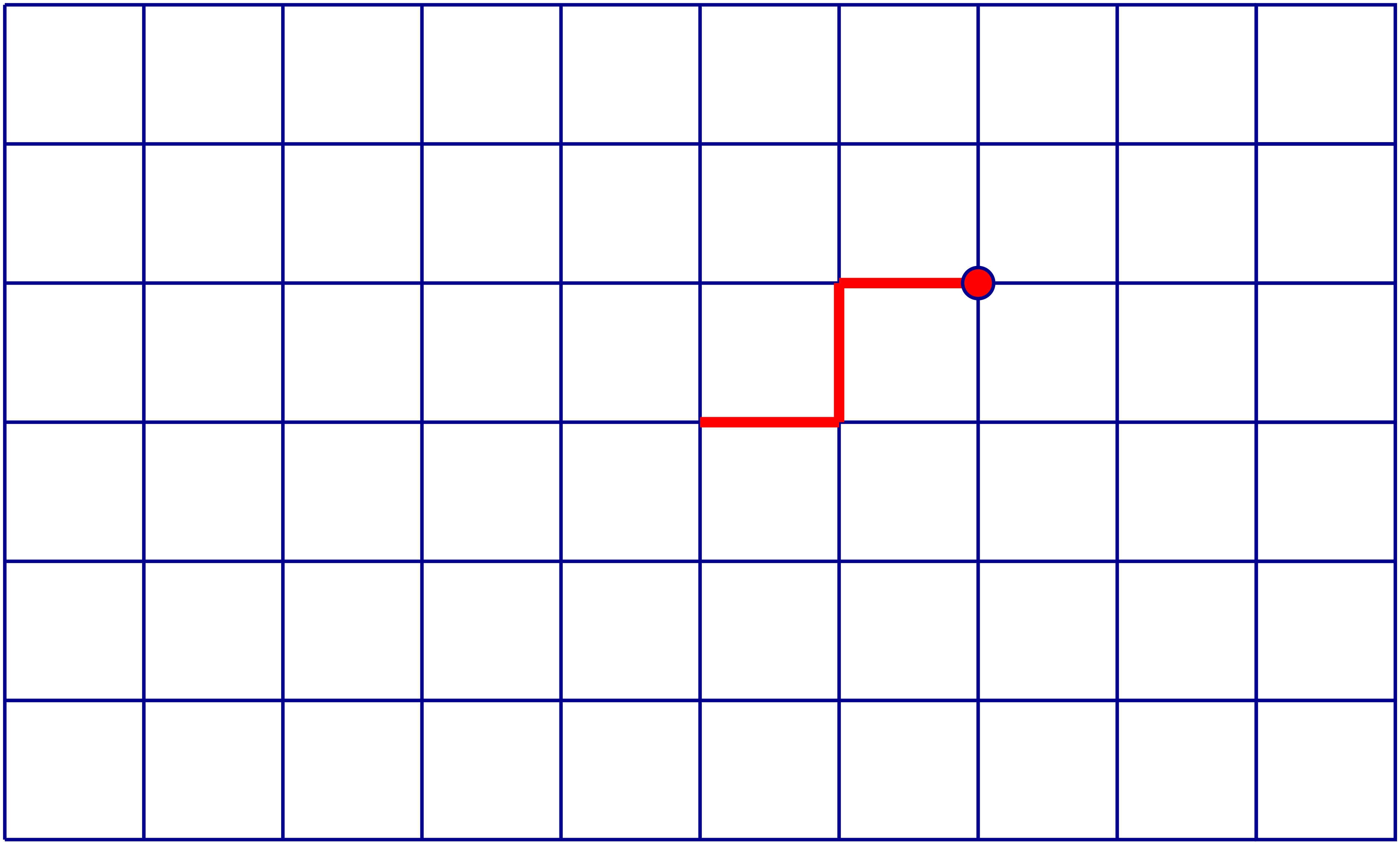
"Give me six hours to chop down a tree and I will spend the first four sharpening the axe."

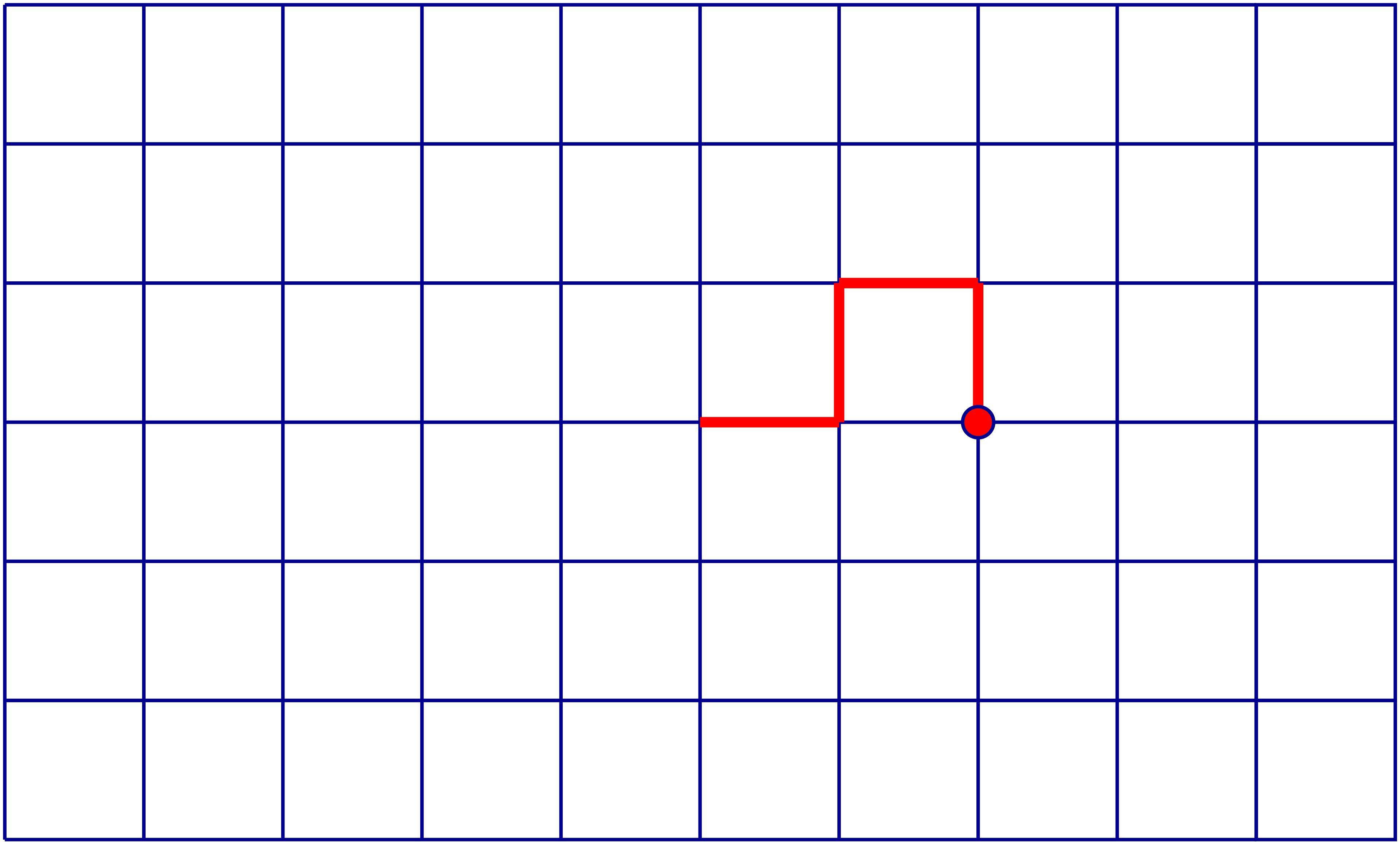
– Abraham Lincoln

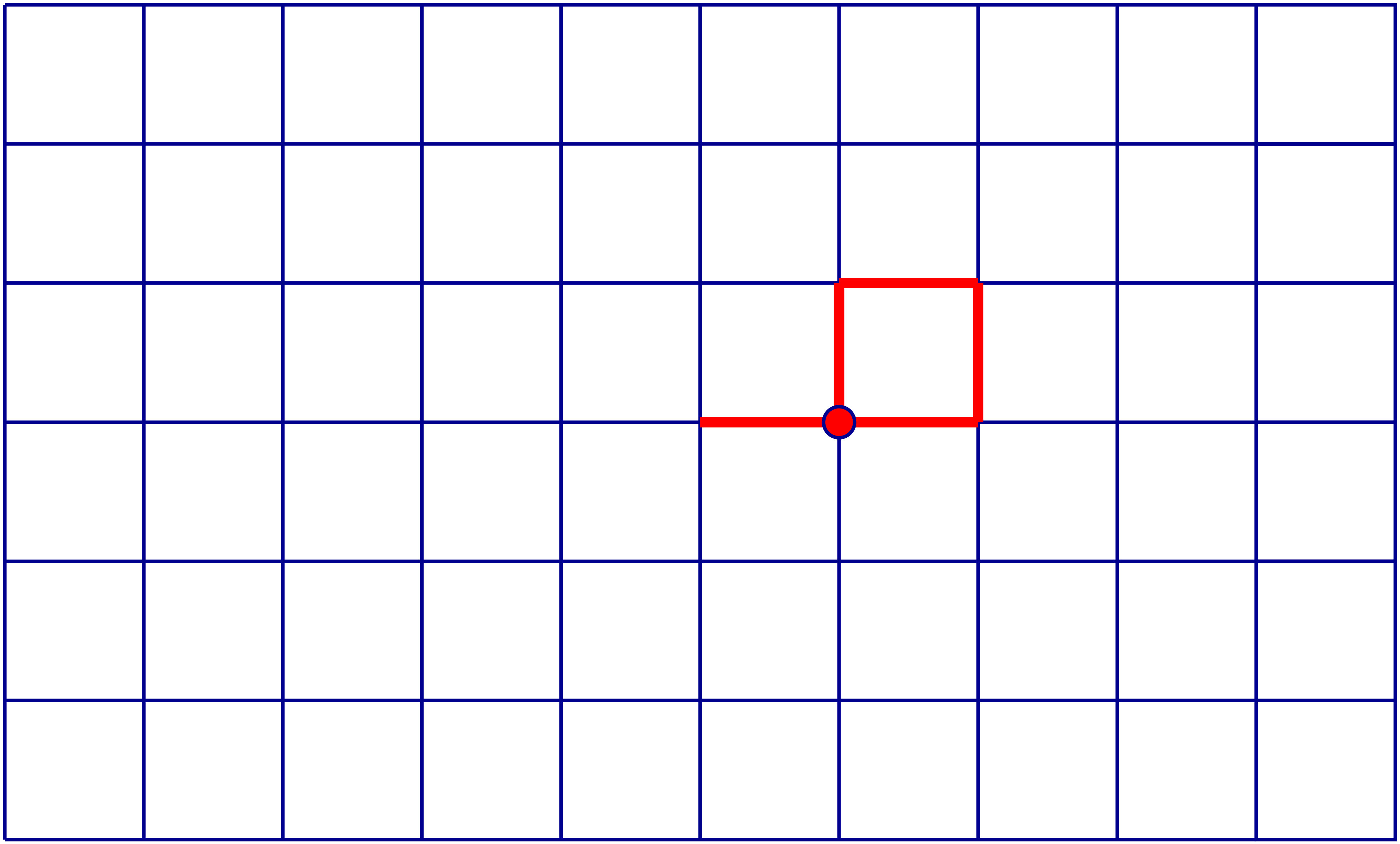


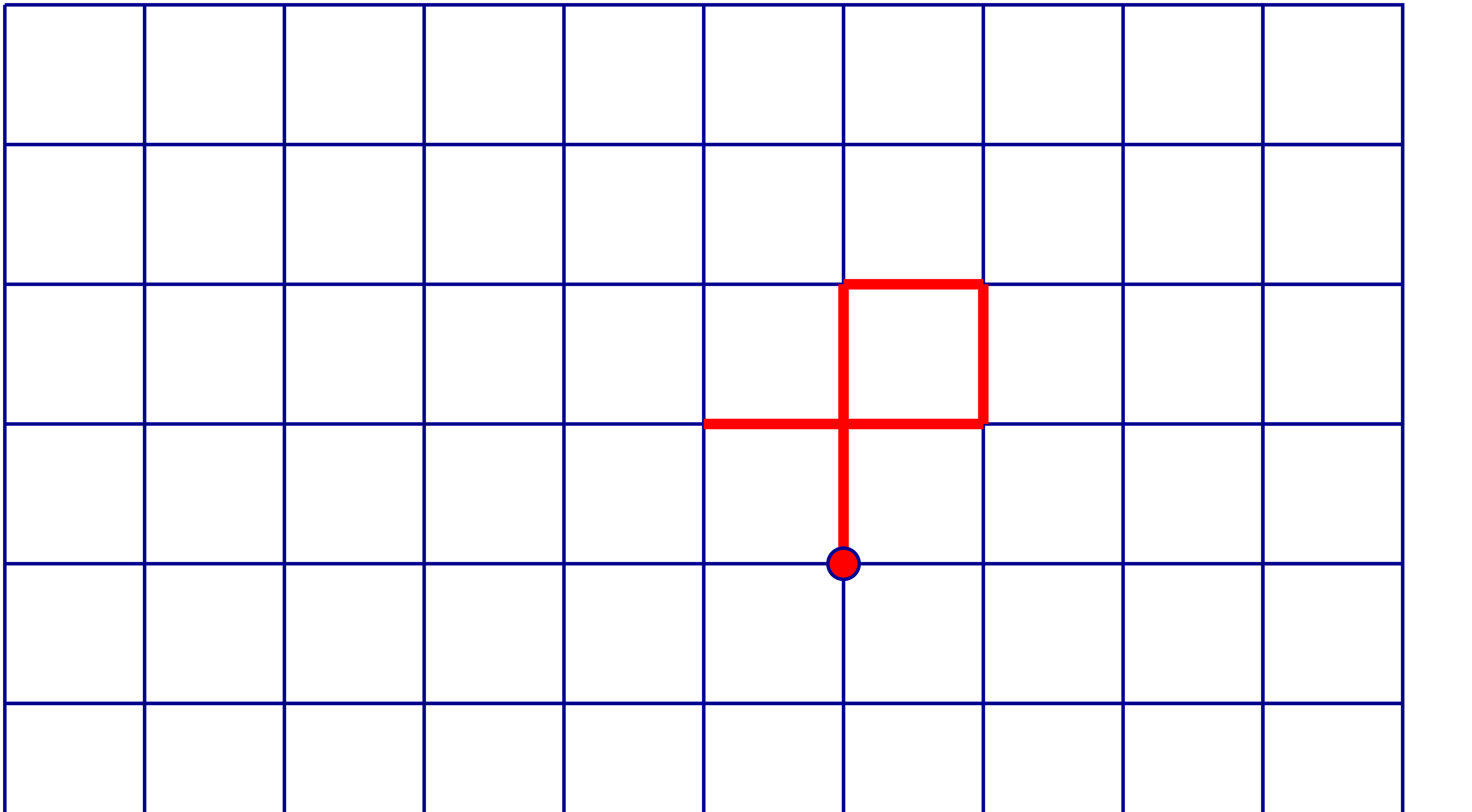


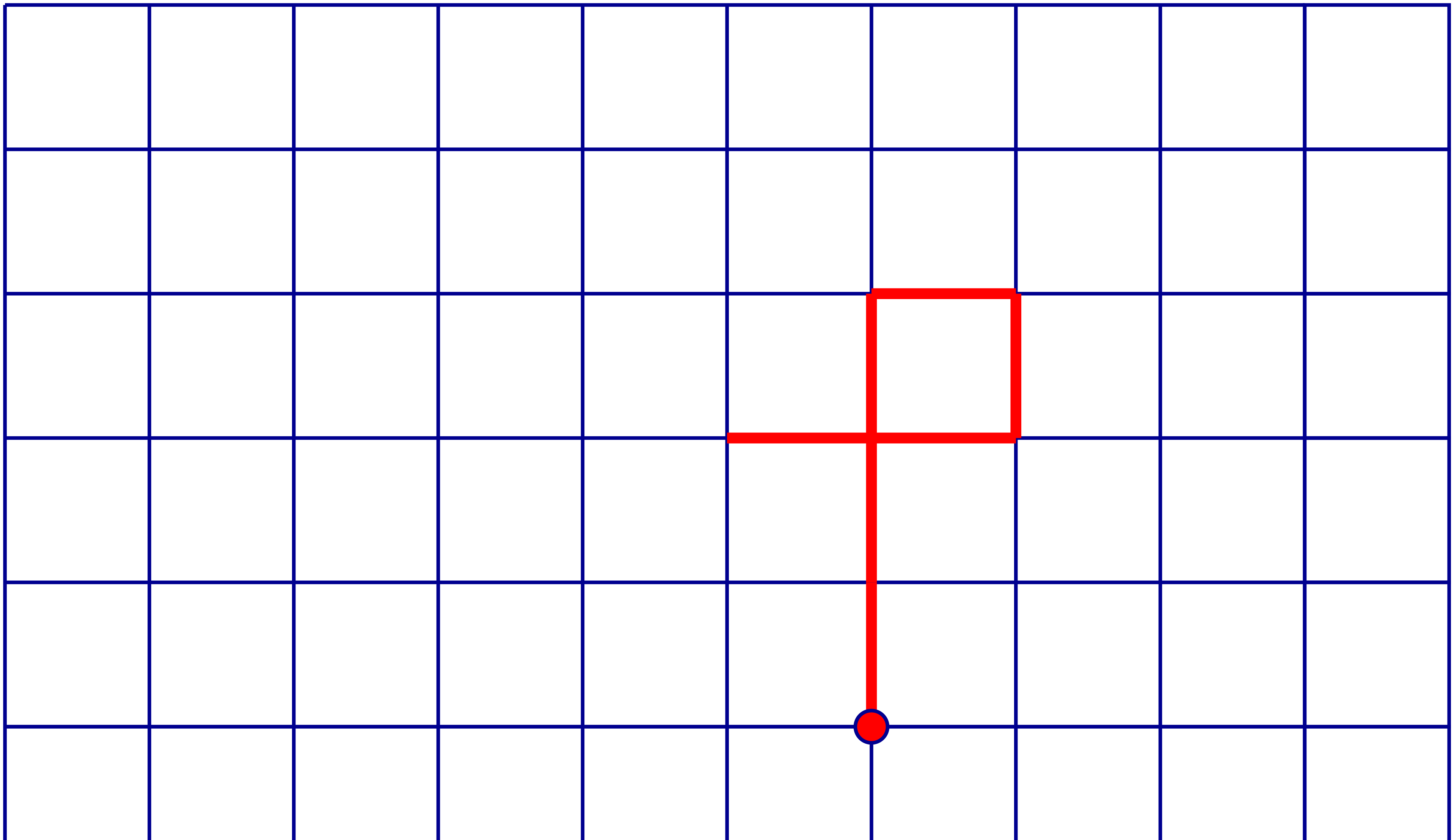


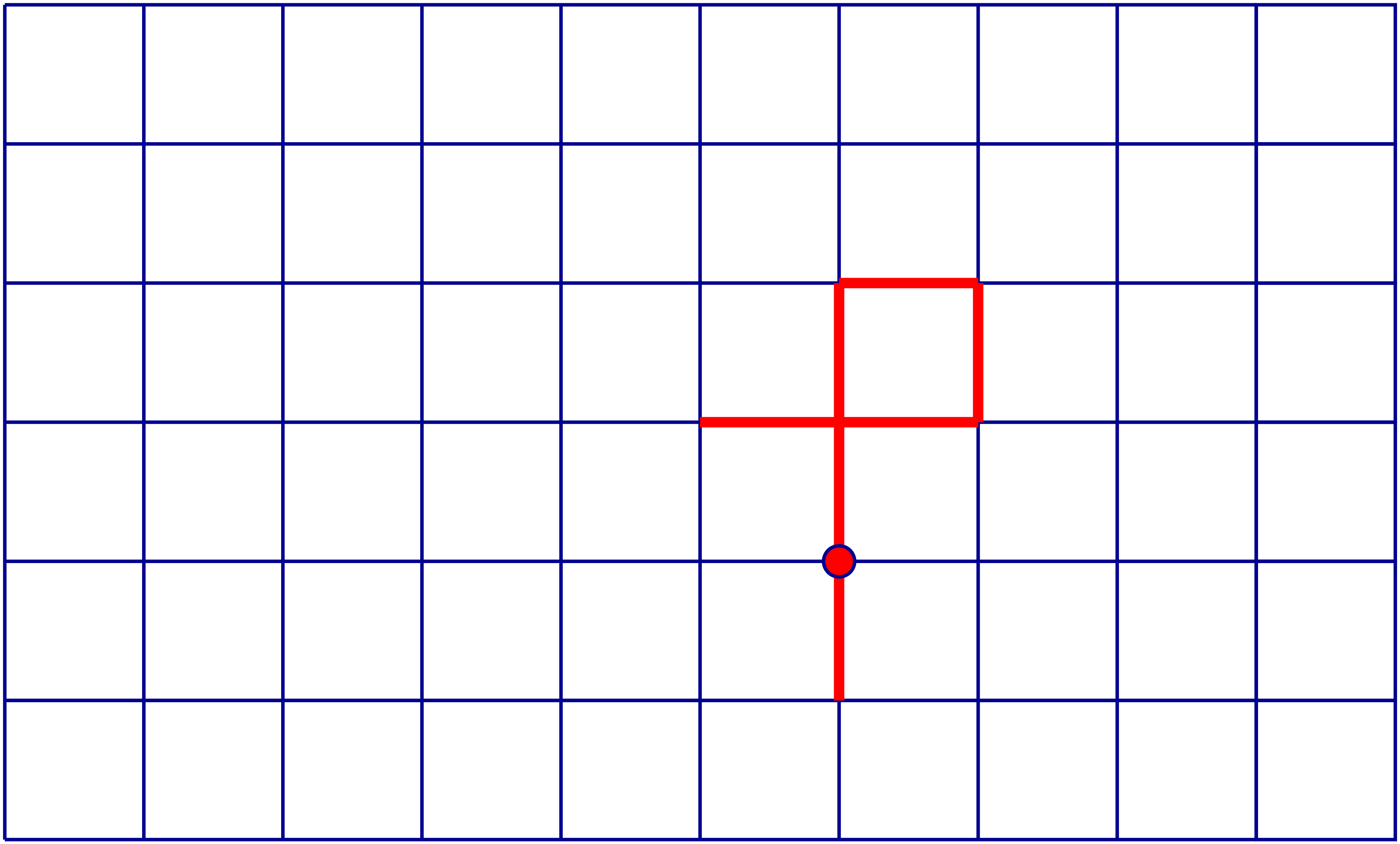


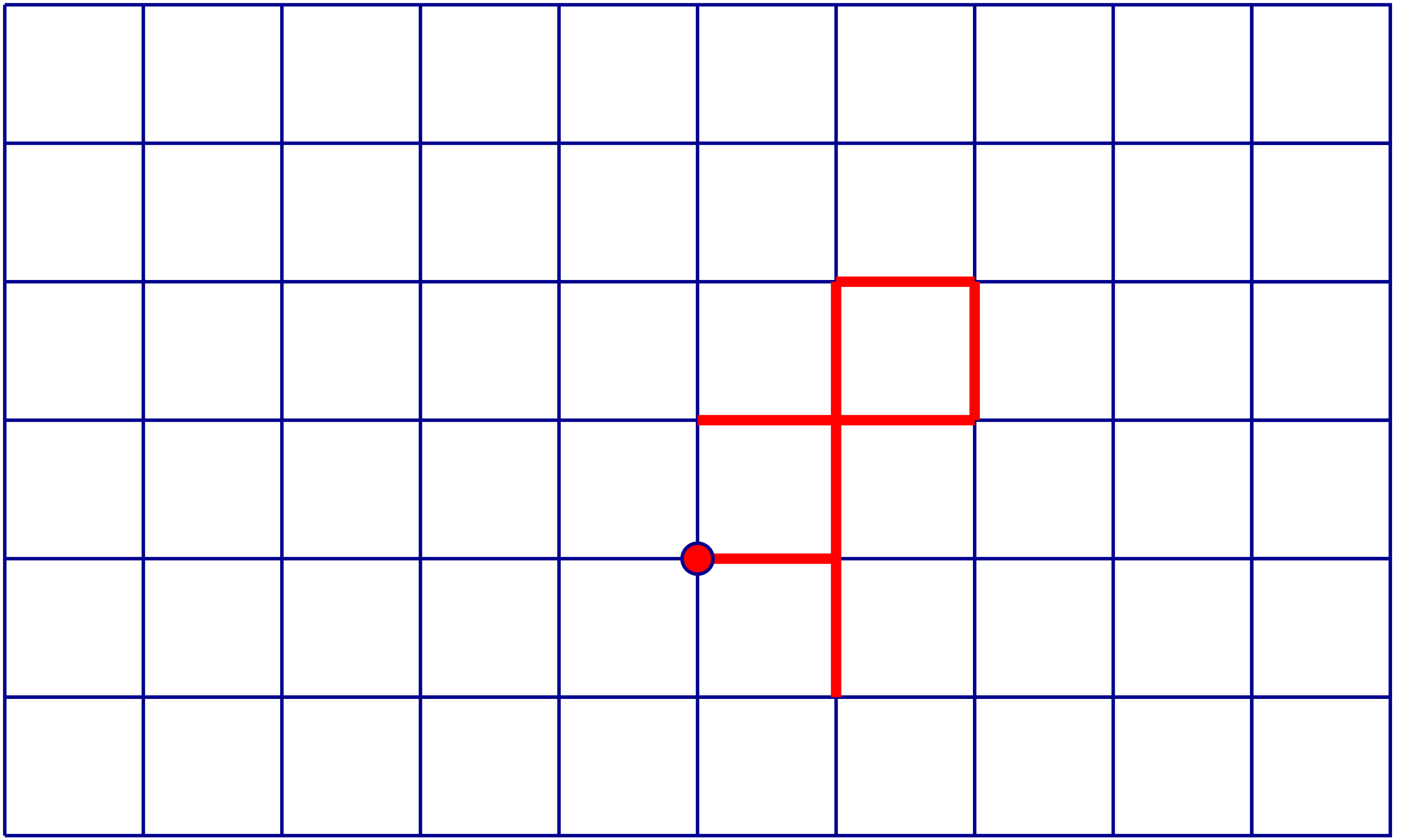


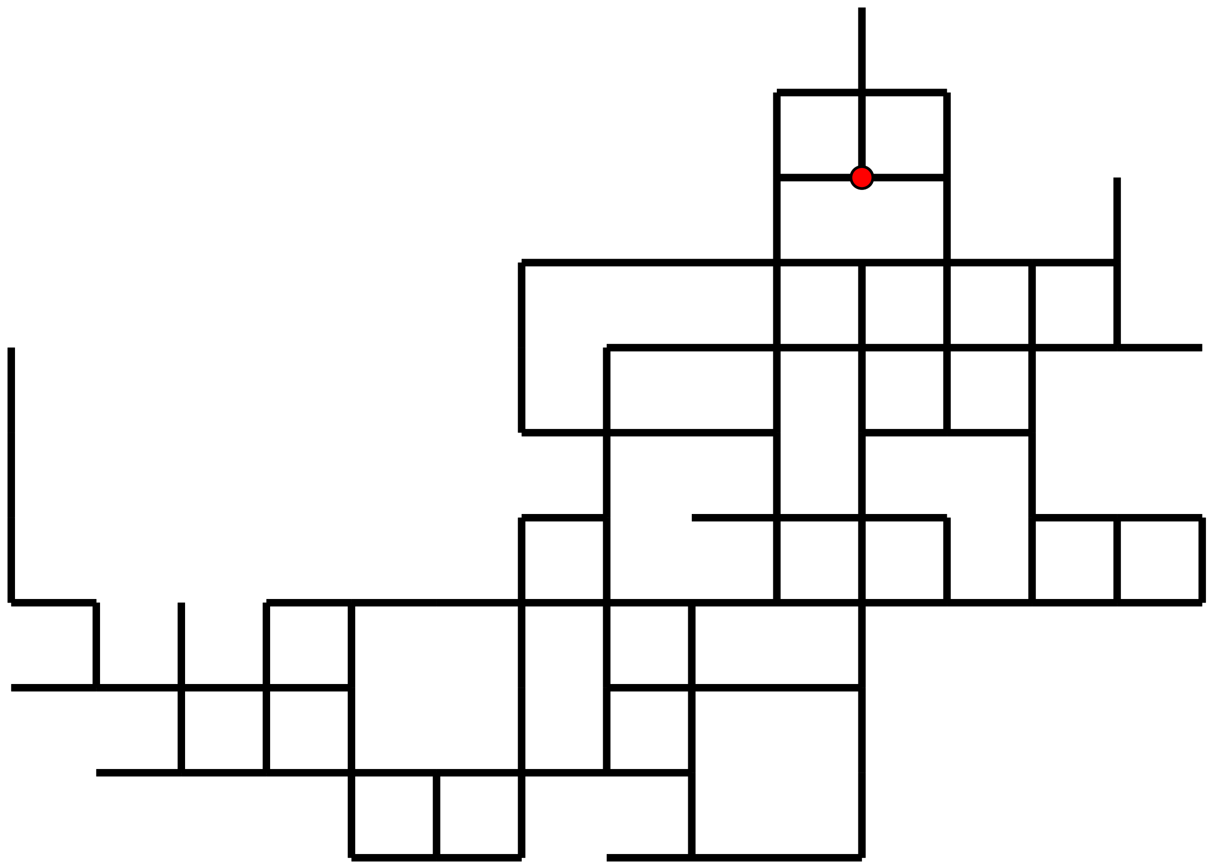




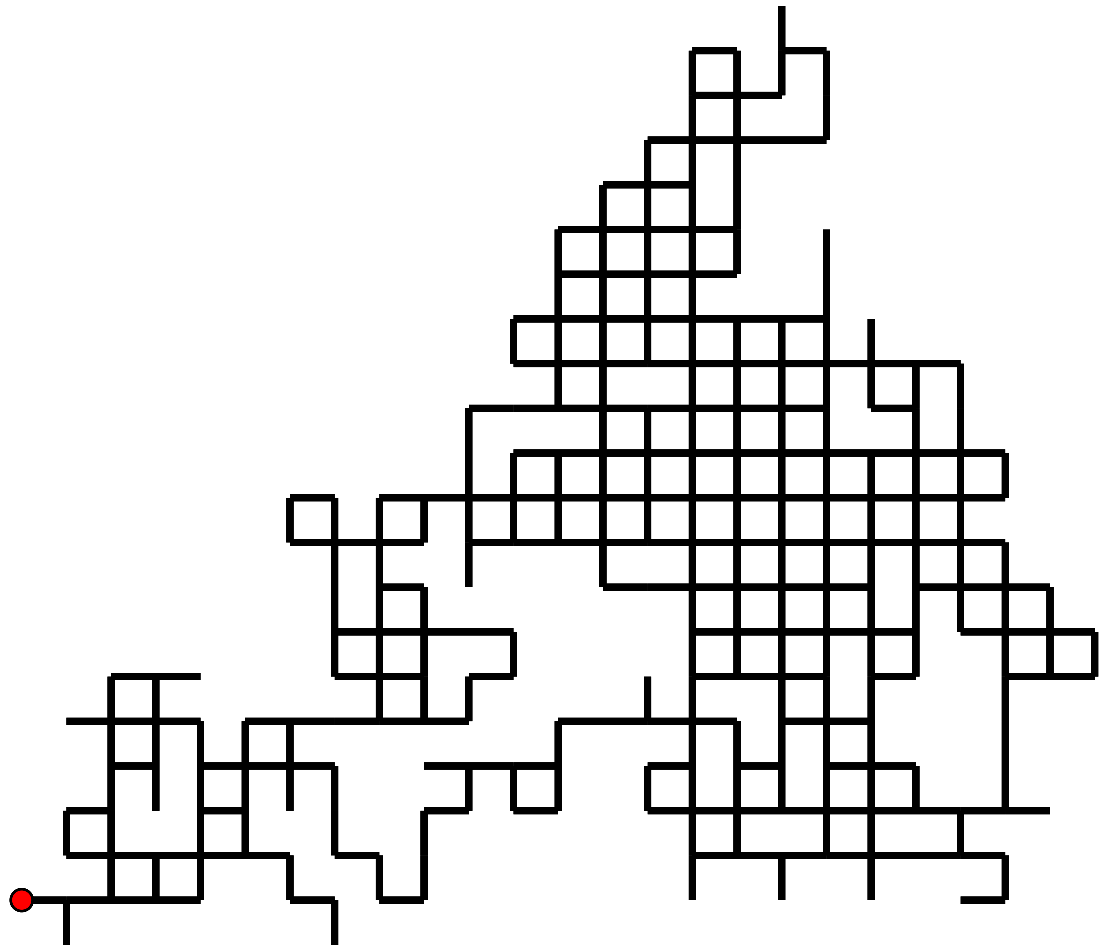




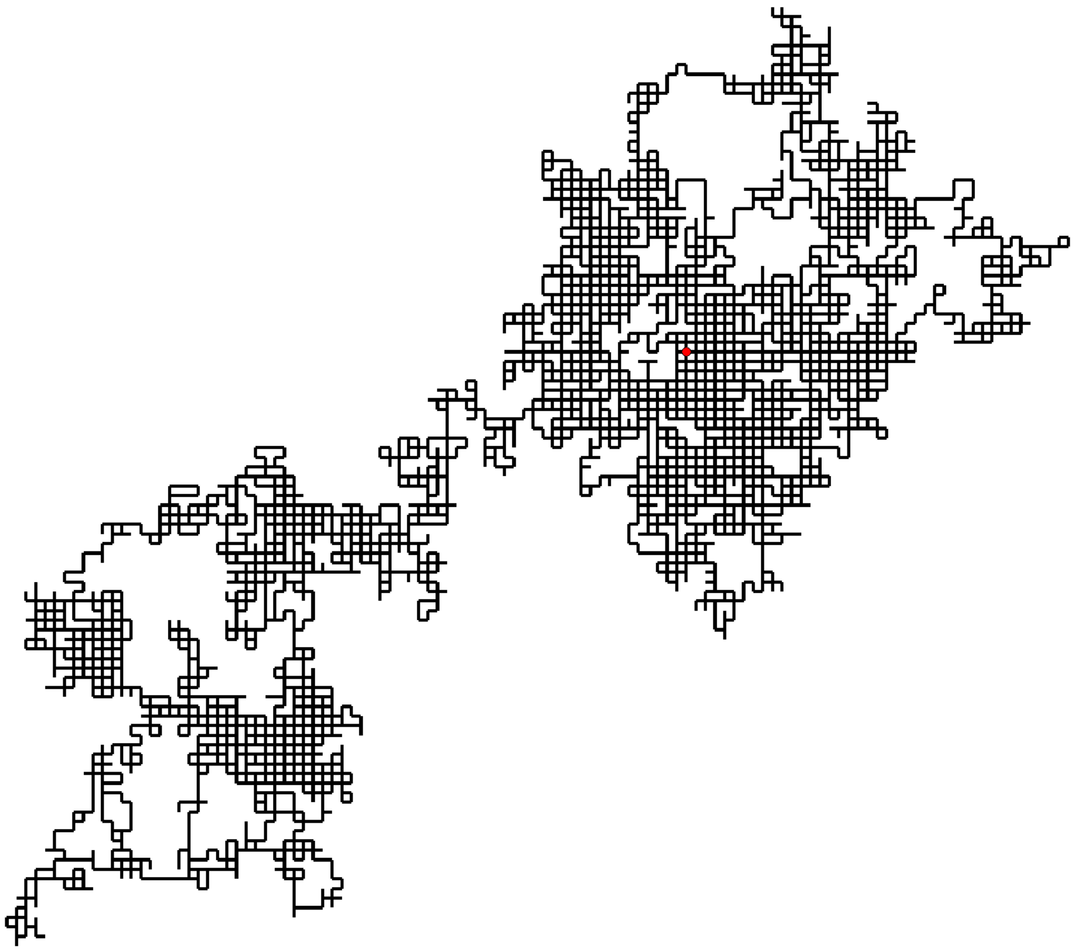




200 step random walk.

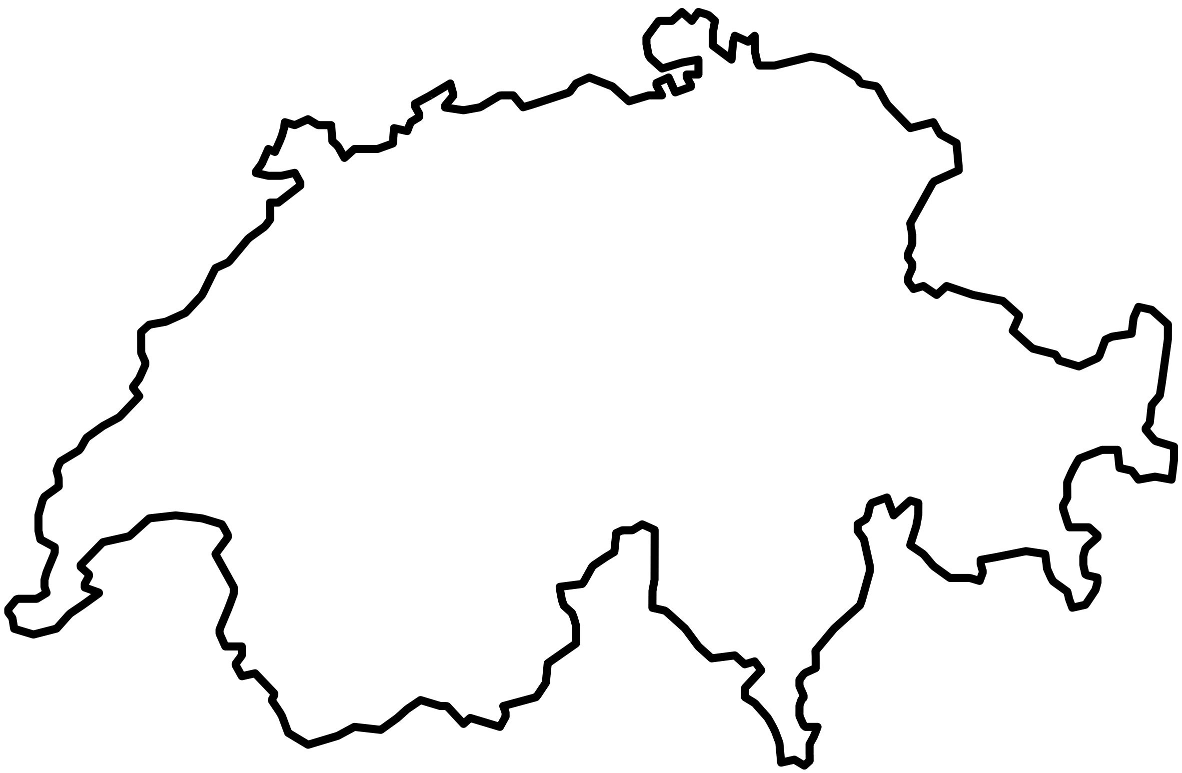


1000 step random walk.



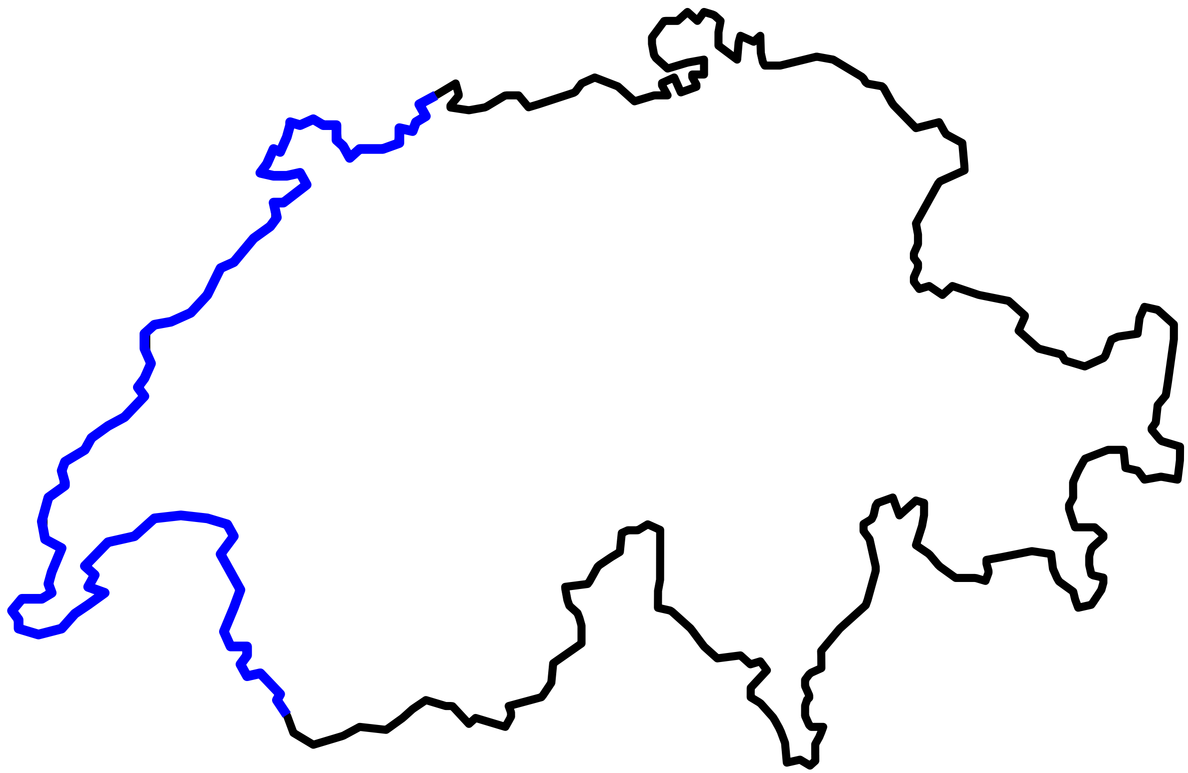
10000 step random walk.

Harmonic measure = hitting distribution of Brownian motion



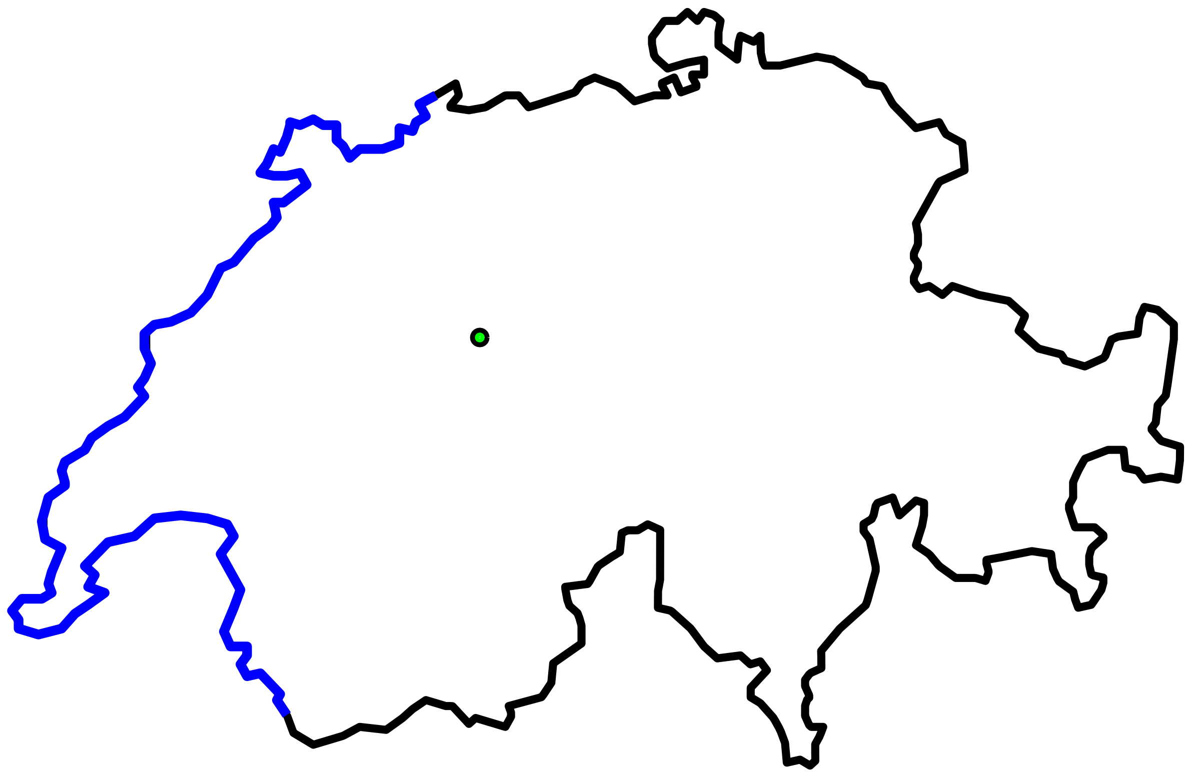
Suppose Ω is a planar Jordan domain.

Harmonic measure = hitting distribution of Brownian motion



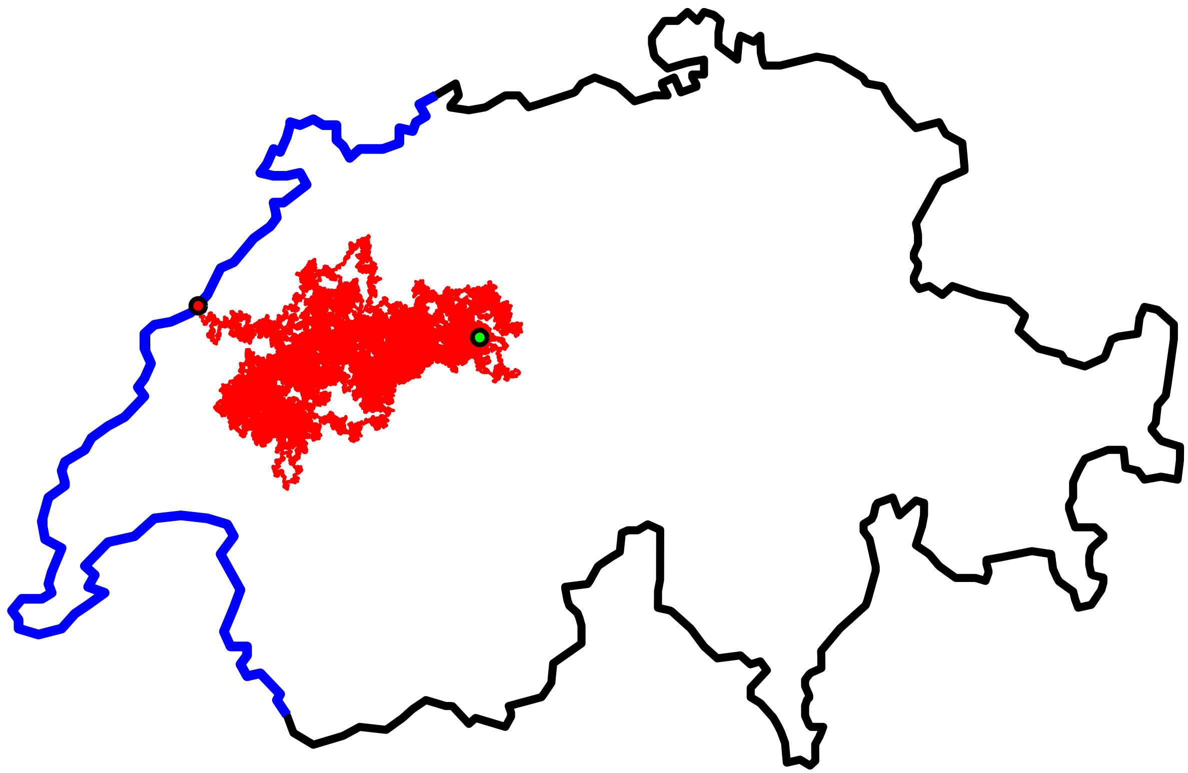
Let E be a subset of the boundary, $\partial\Omega$.

Harmonic measure = hitting distribution of Brownian motion



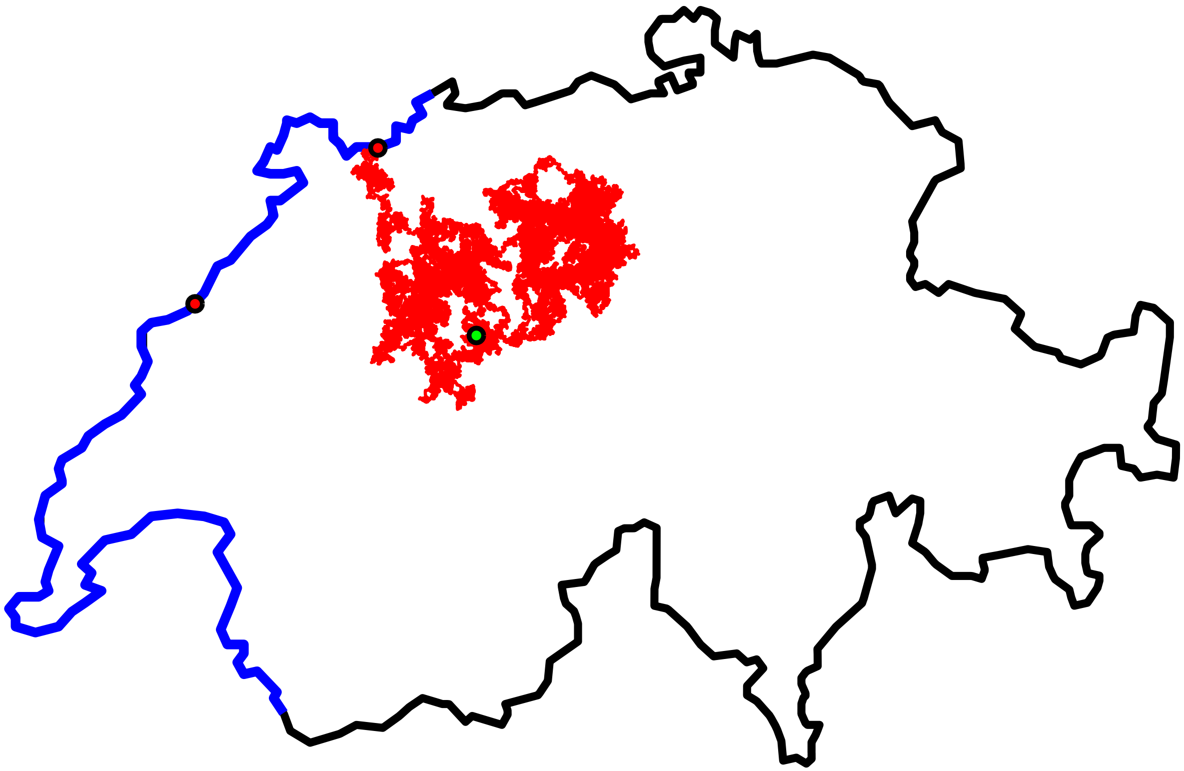
Choose an interior point $z \in \Omega$.

Harmonic measure = hitting distribution of Brownian motion



$\omega(z, E, \Omega)$ = probability a particle started at z first hits $\partial\Omega$ in E.

Harmonic measure = hitting distribution of Brownian motion



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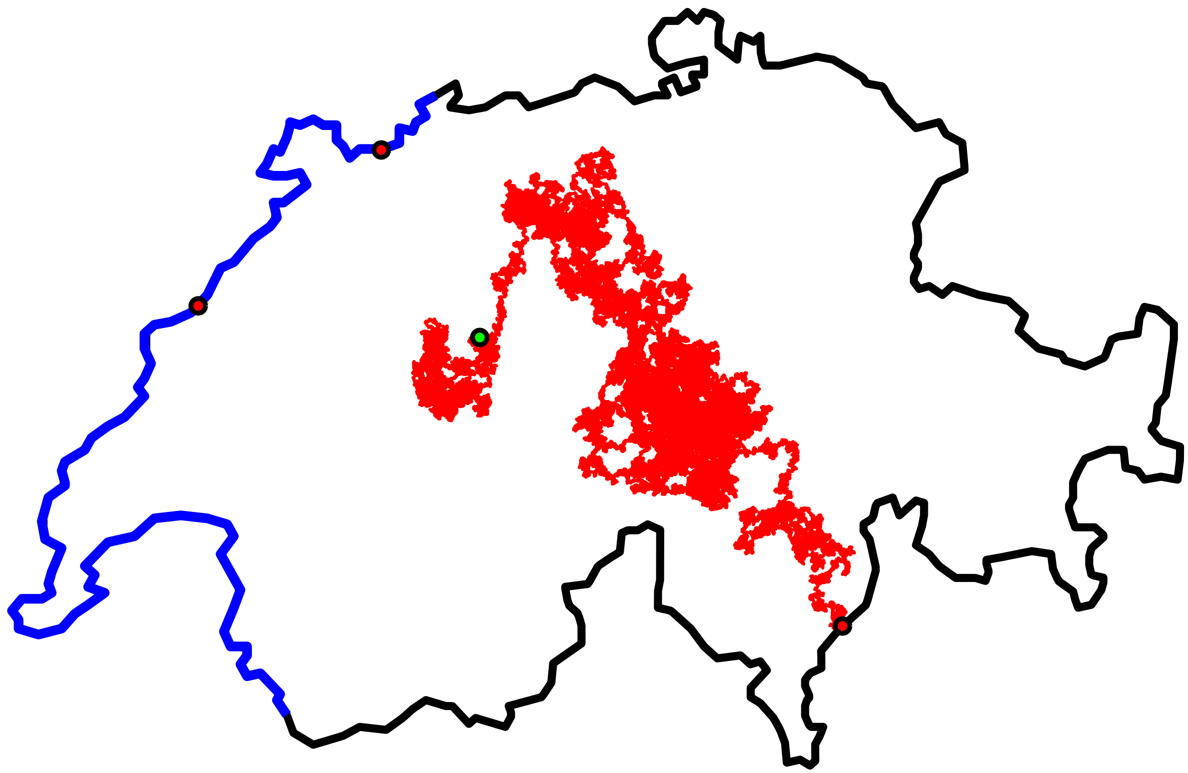

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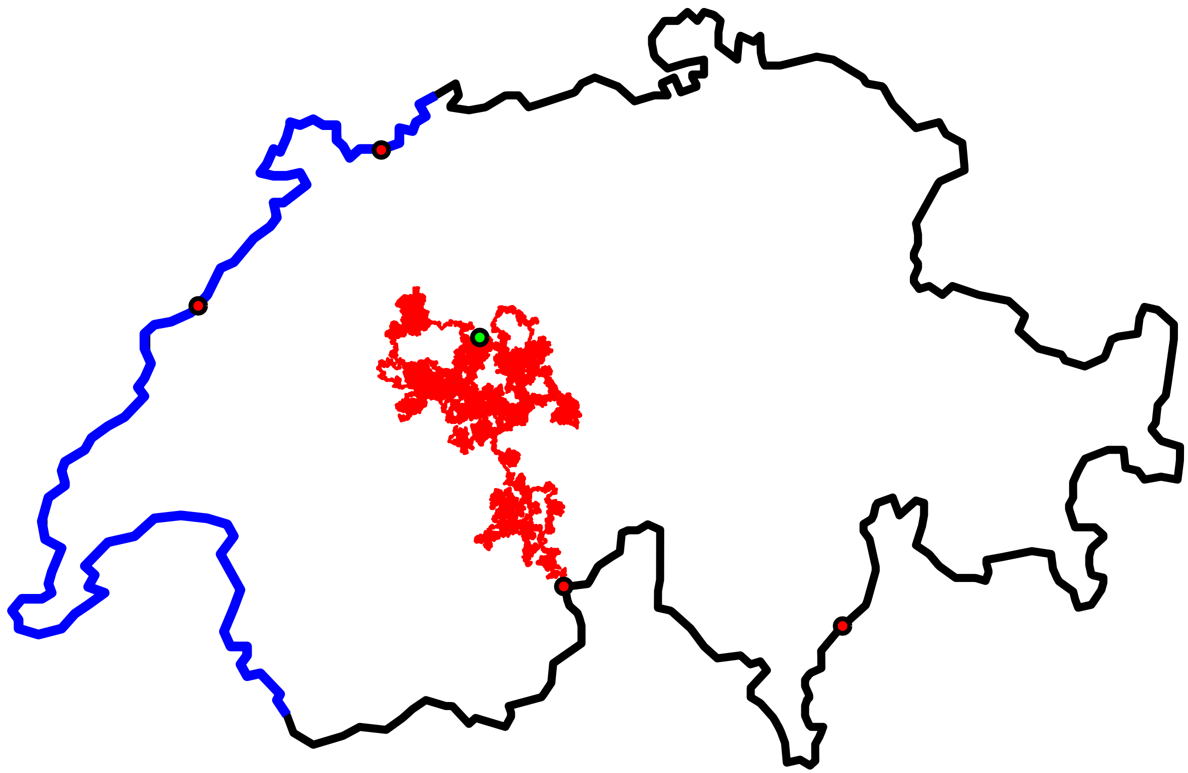

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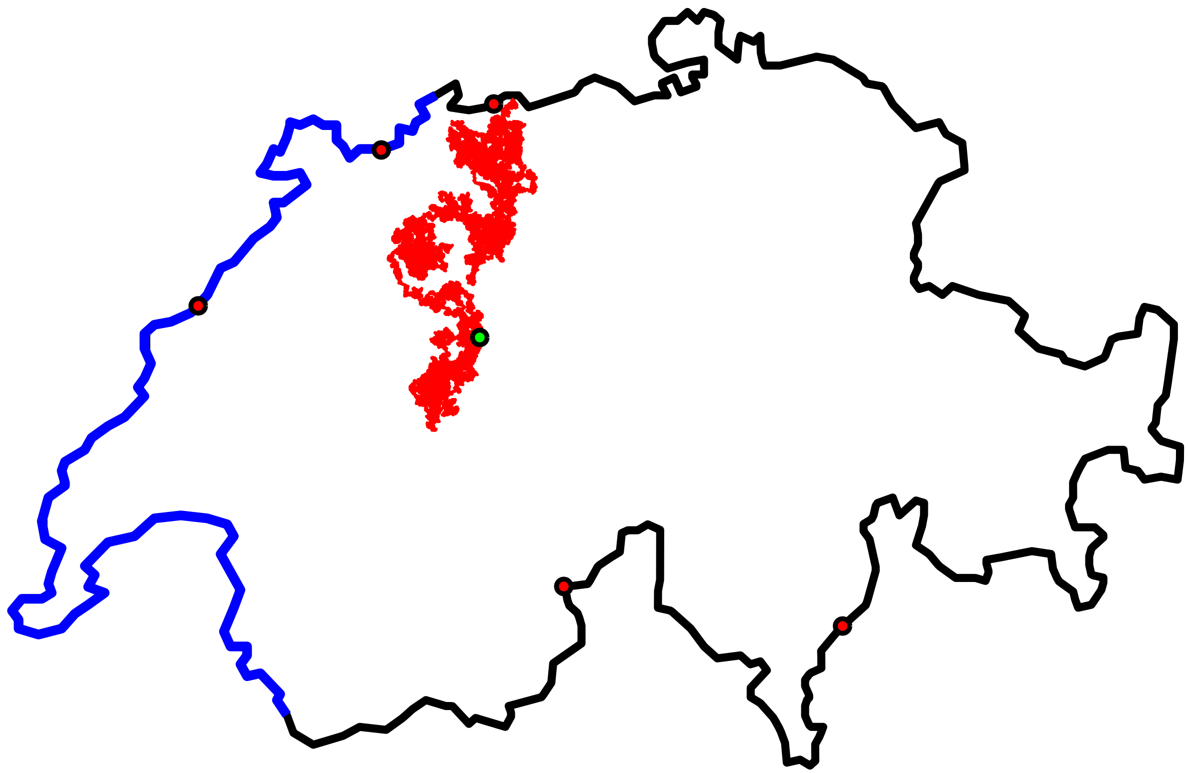

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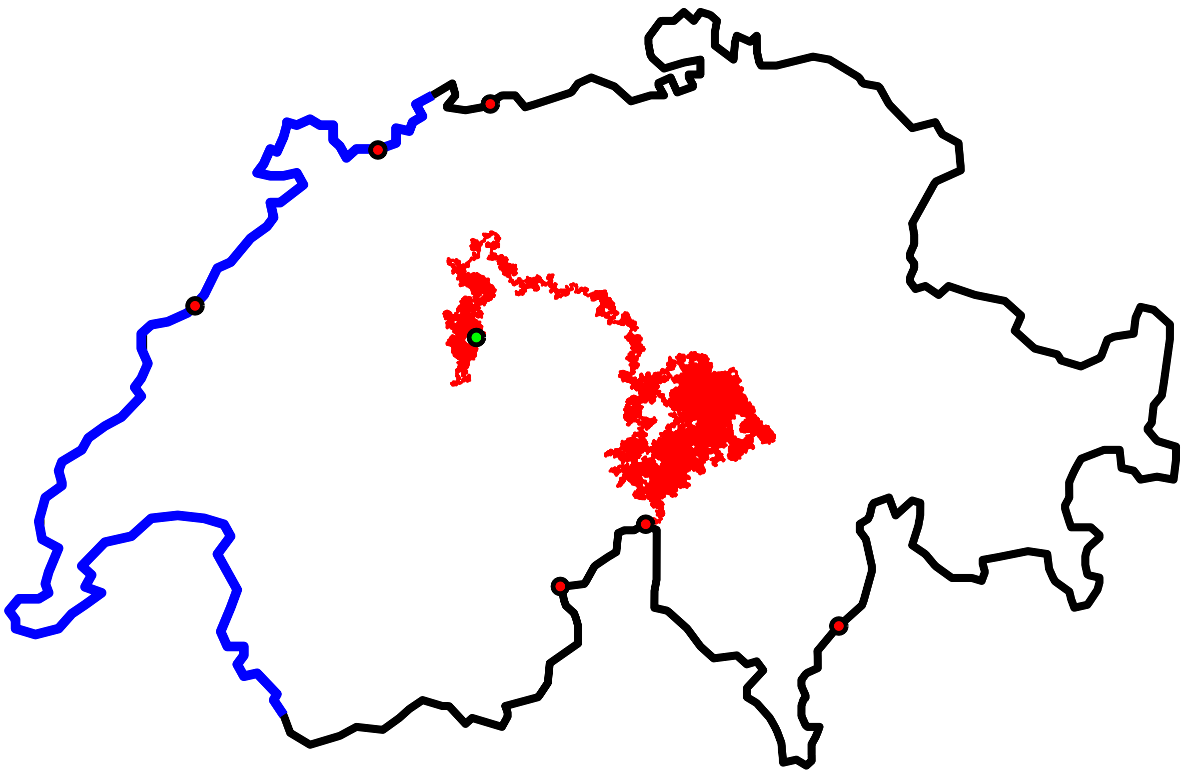

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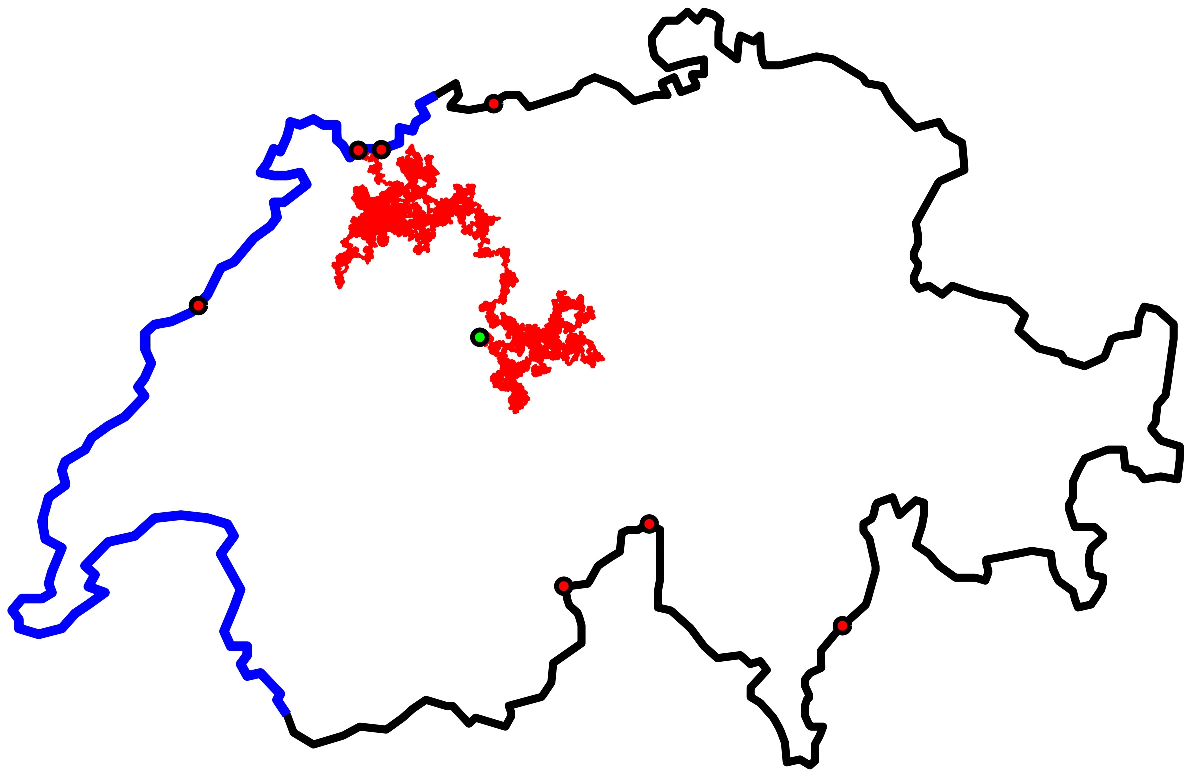

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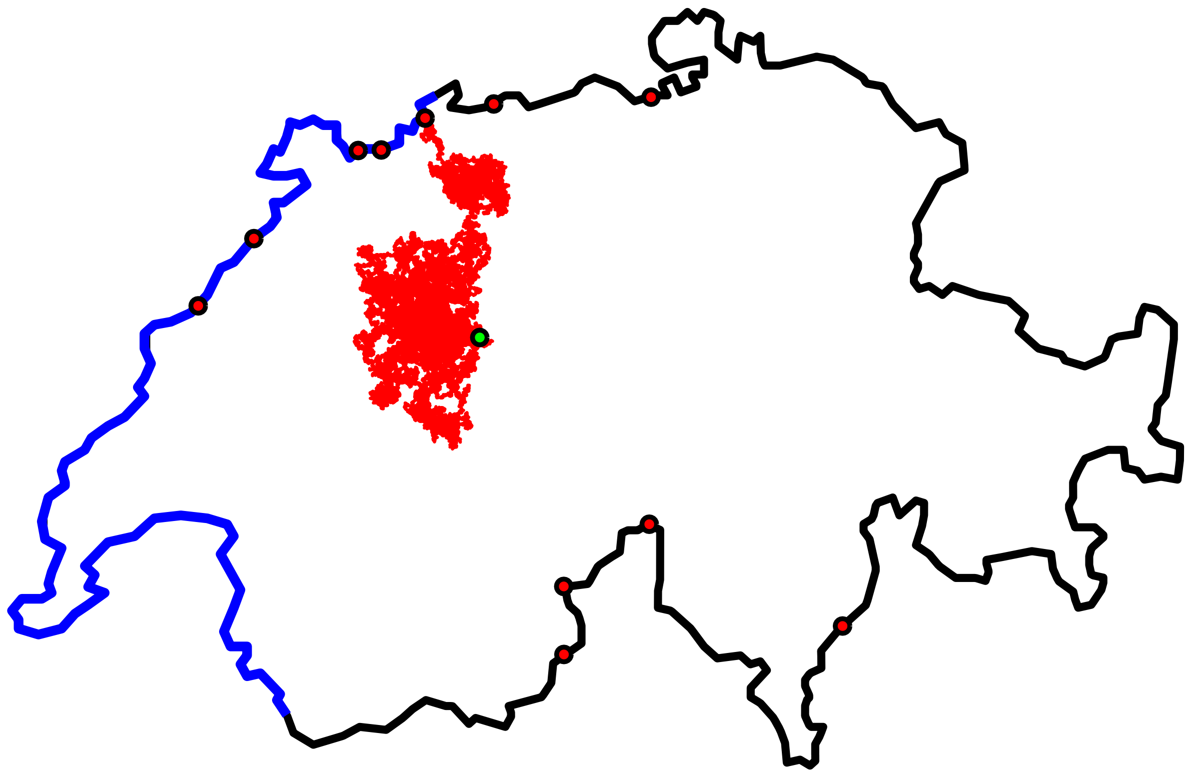

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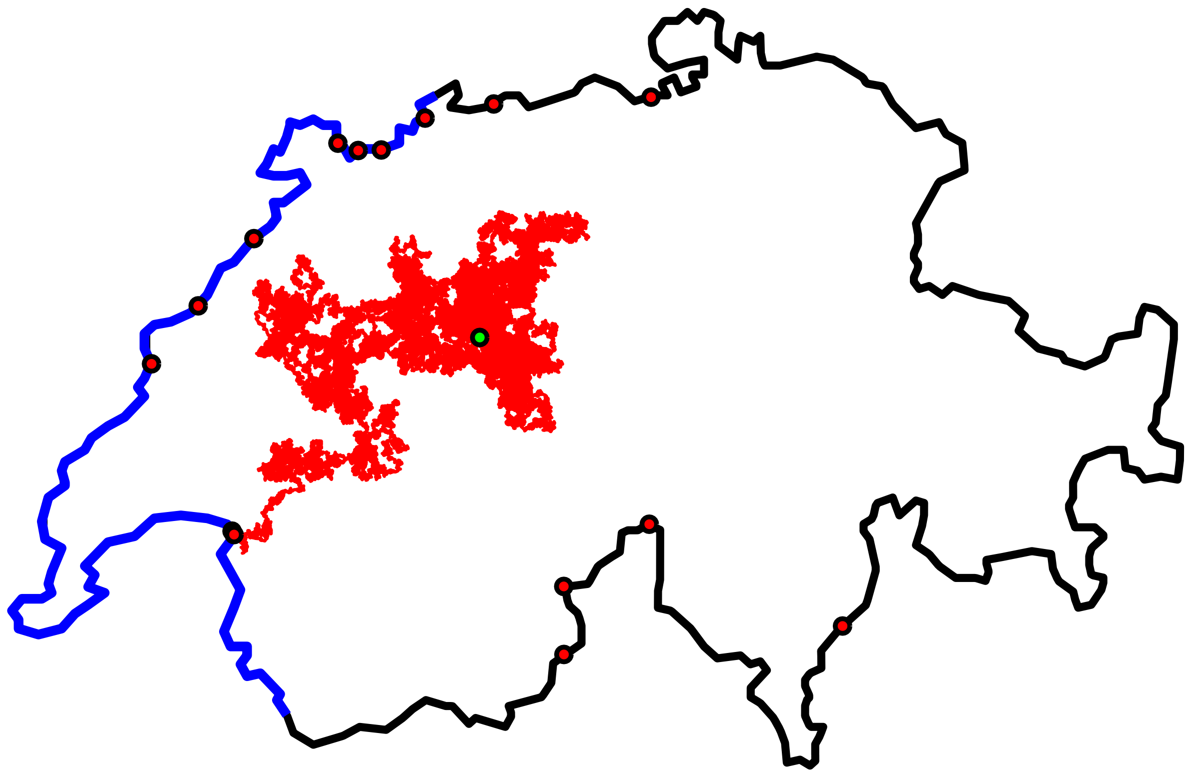

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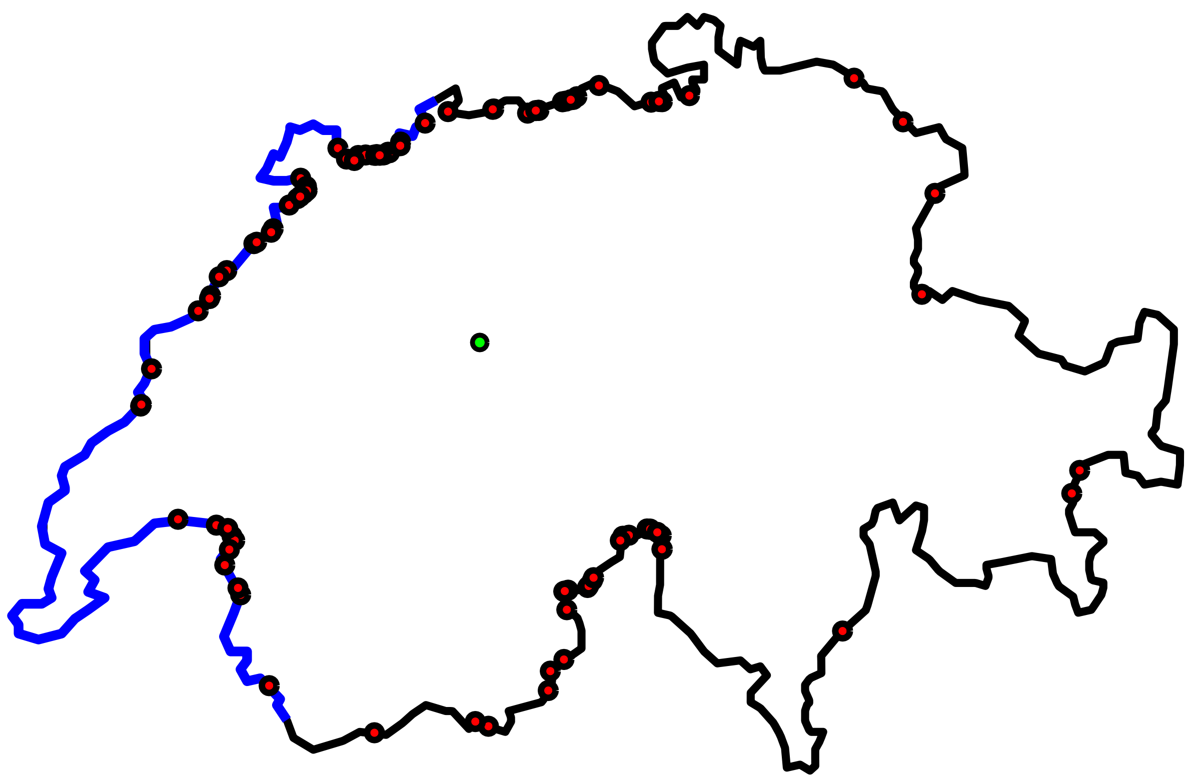
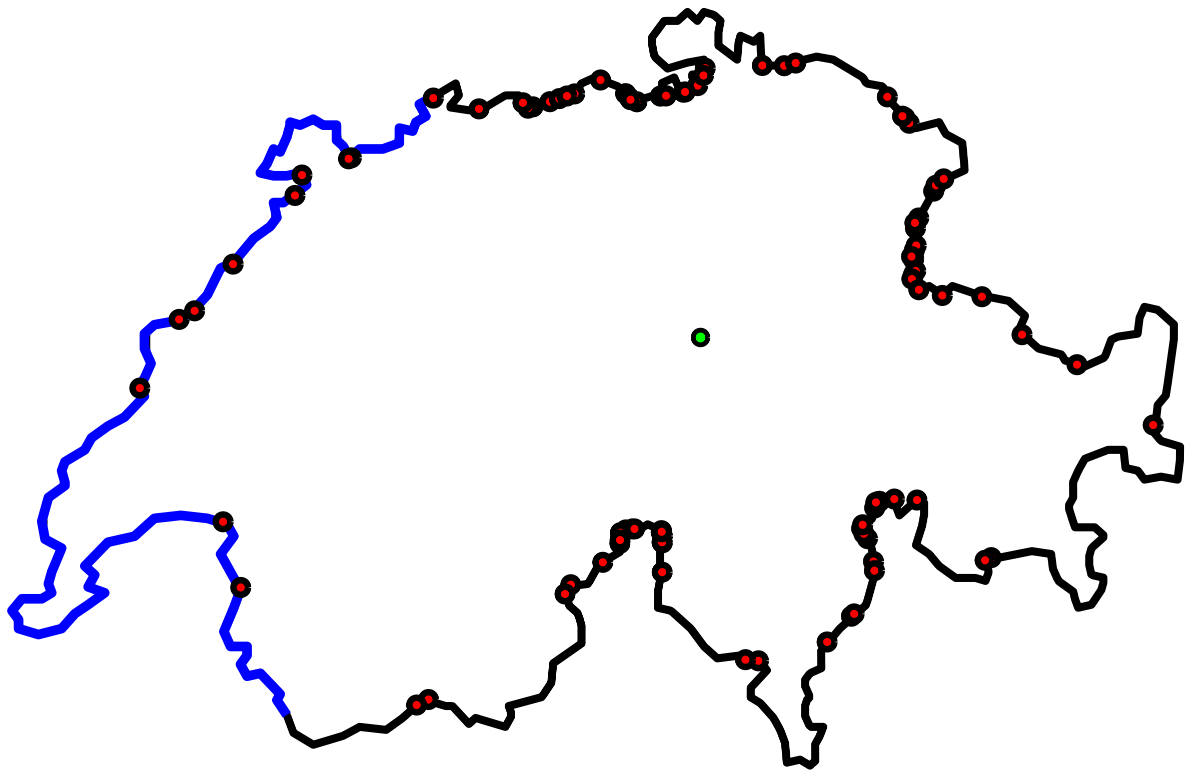


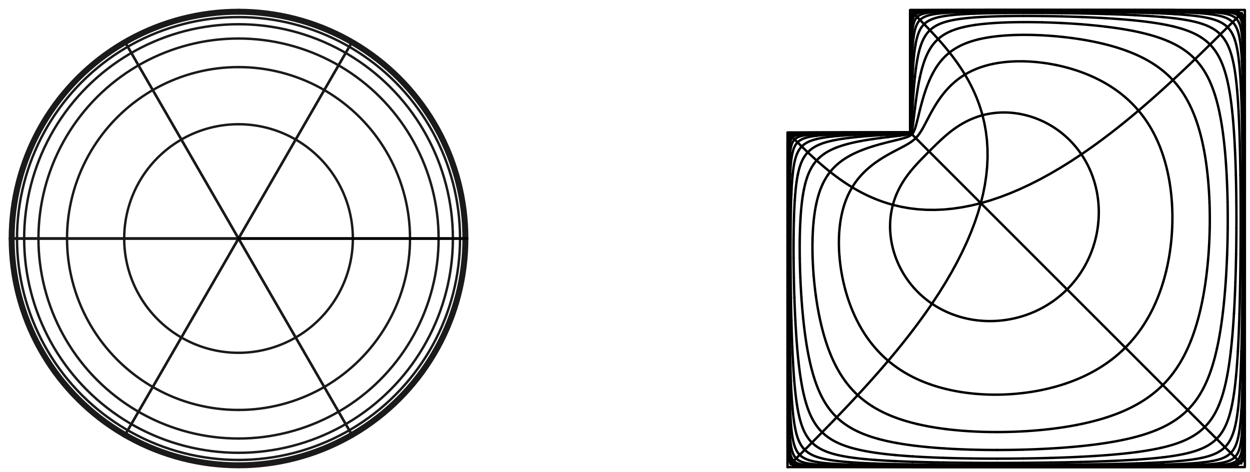
$\omega(z, E, \Omega) \approx 35/100$. What if we move starting point z?

Harmonic measure = hitting distribution of Brownian motion

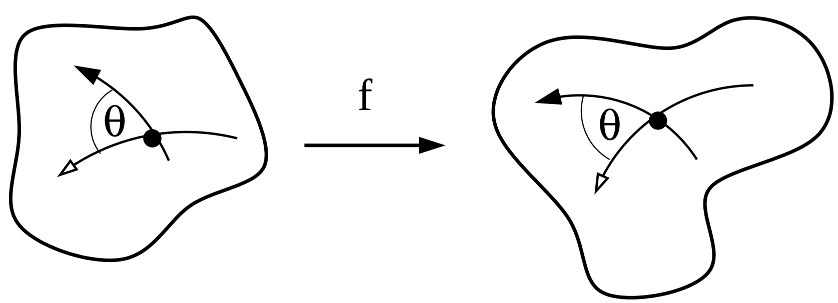


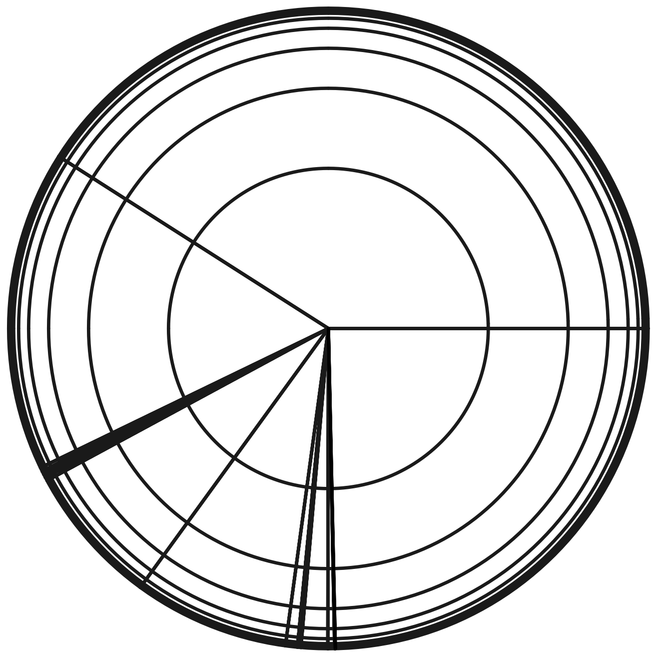
Different z gives $\omega(z, E, \Omega) \approx 10/100$.
ω is harmonic in z with boundary values $\omega = 1$ on E, $\omega = 0$ off E.

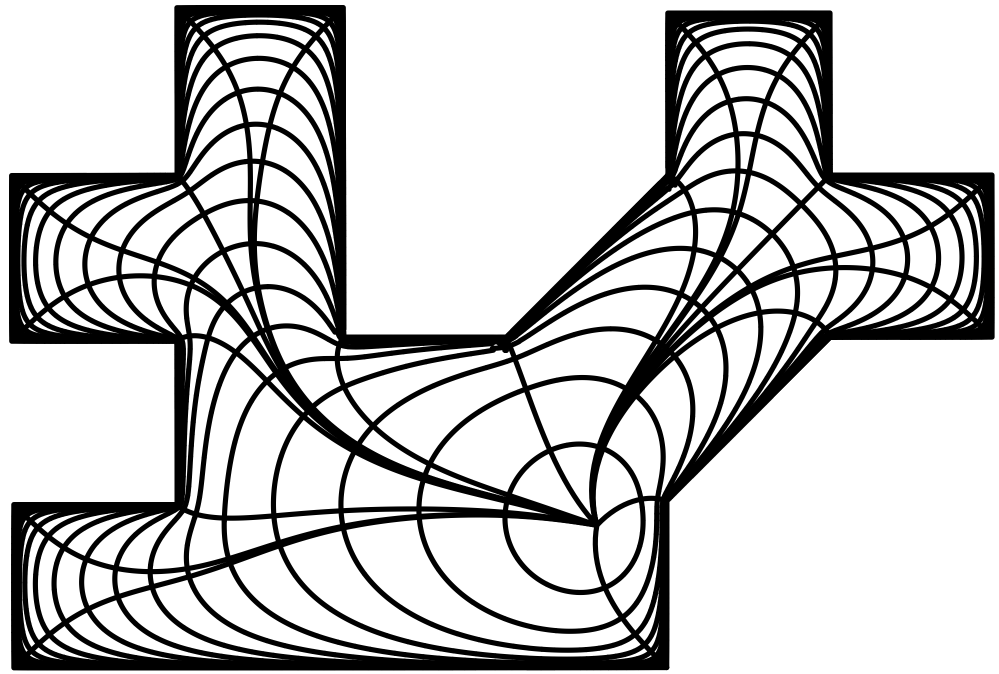
Riemann Mapping Theorem: If $\Omega \subsetneq \mathbb{R}^2$ is simply connected, then there is a conformal map $f : \mathbb{D} \to \Omega$.

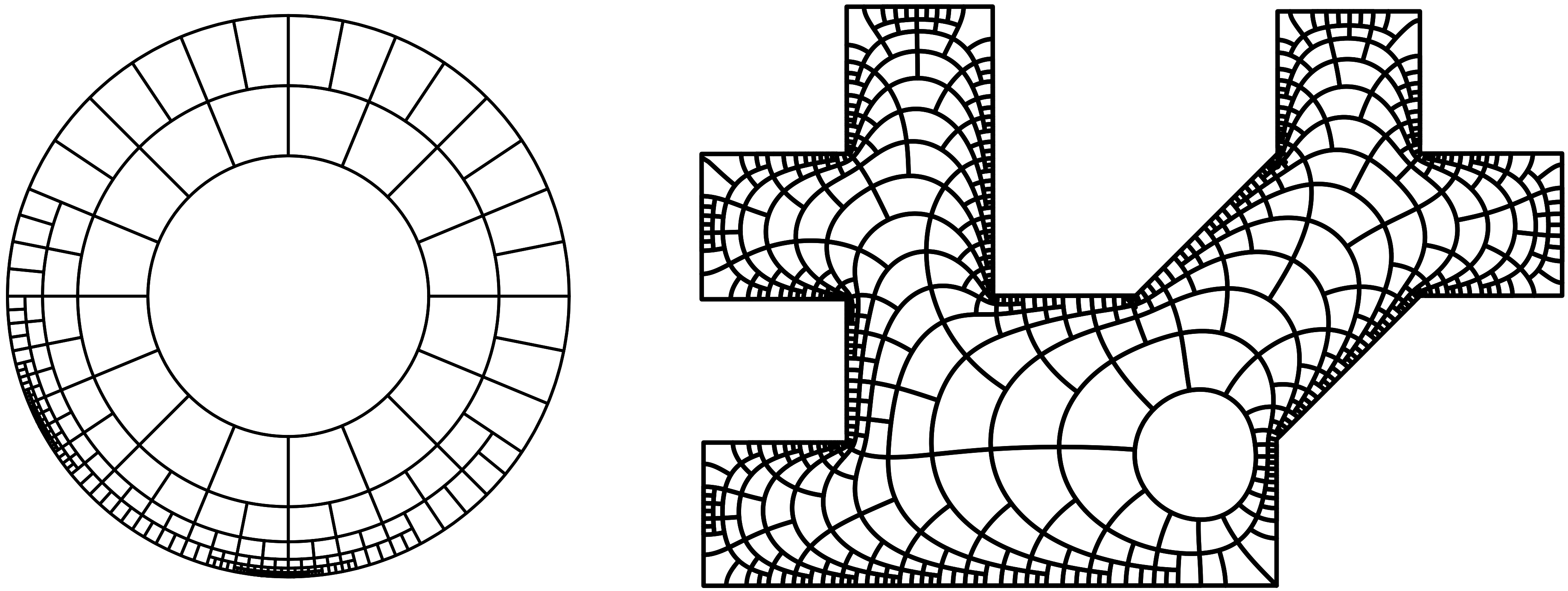


Conformal = angle preserving



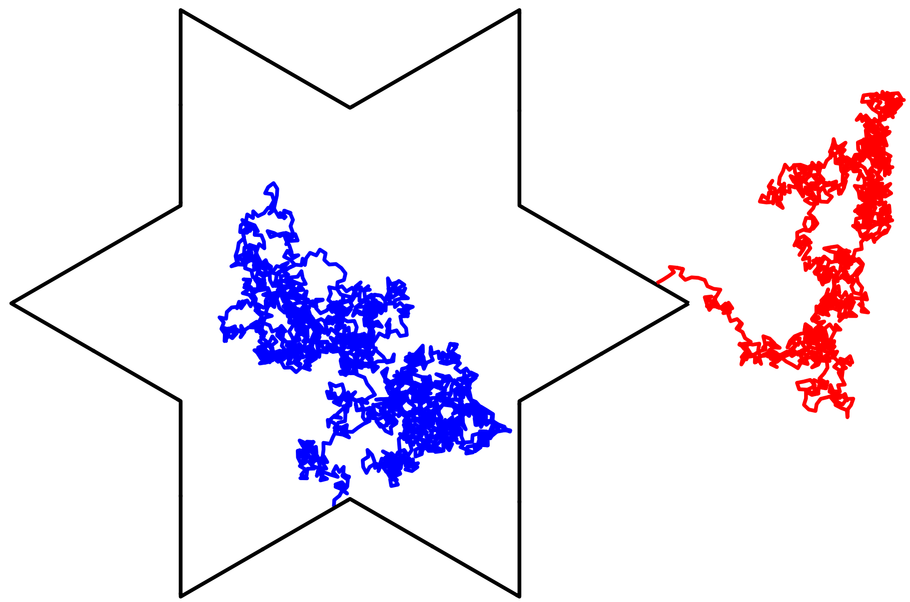






Brownian motion is conformally invariant, so normalized length measure maps to harmonic measure. Fastest way to compute harmonic measure.

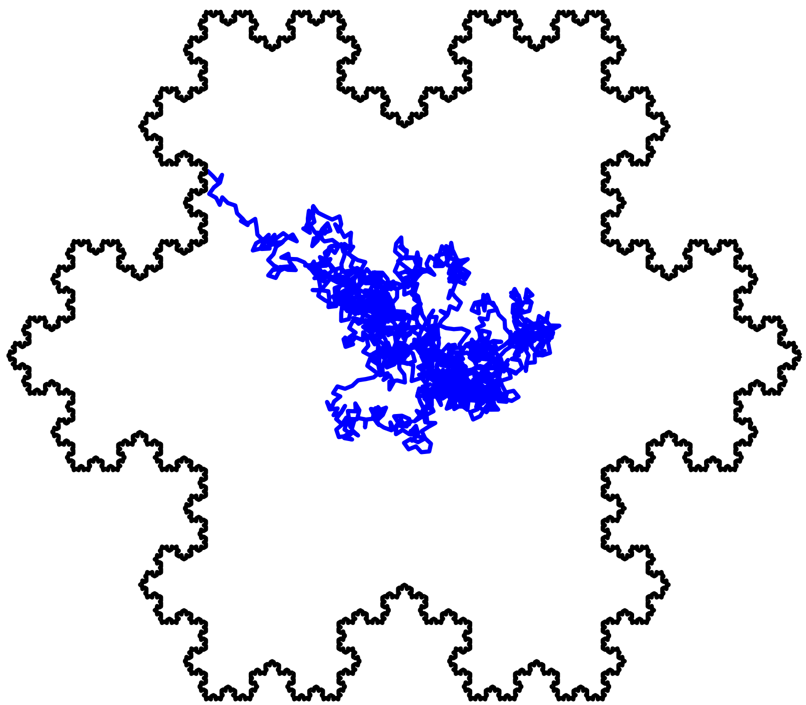
Thm (F & M Riesz 1916):
For rectifiable boundaries, $\omega(E) = 0$ iff E has zero length.



"Inside" and "outside" harmonic measures have same null sets.

Measures are mutually absolutely continuous. Same measure class.

Thm (Makarov 1985):
For fractal domains, ω gives full measure to a set of zero length.



First such examples due to Lavrentiev (1936).

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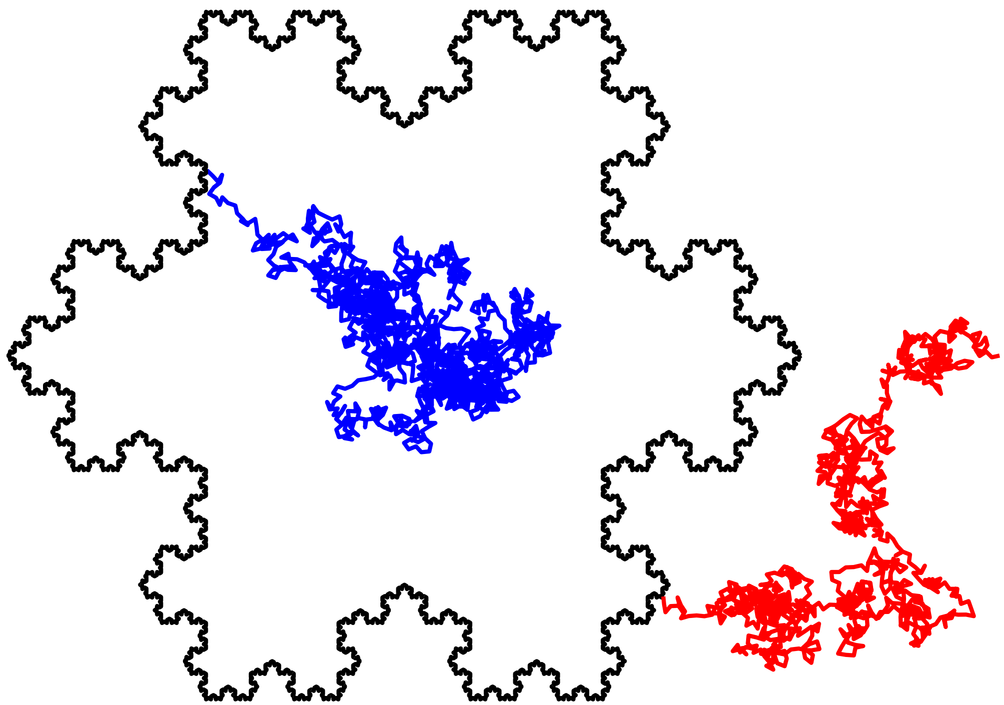
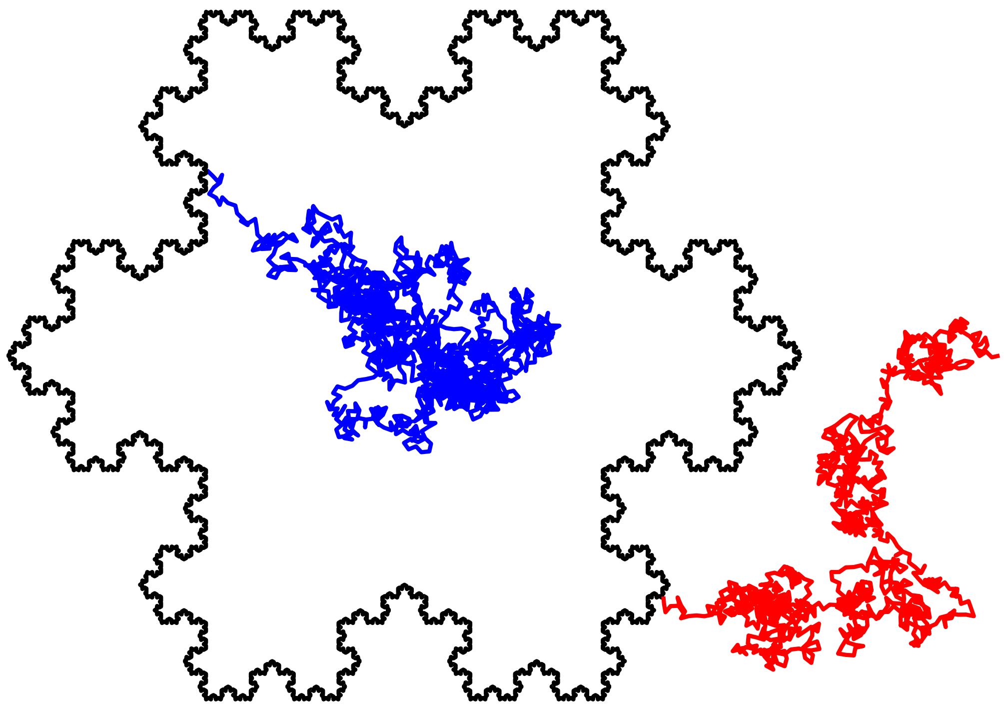


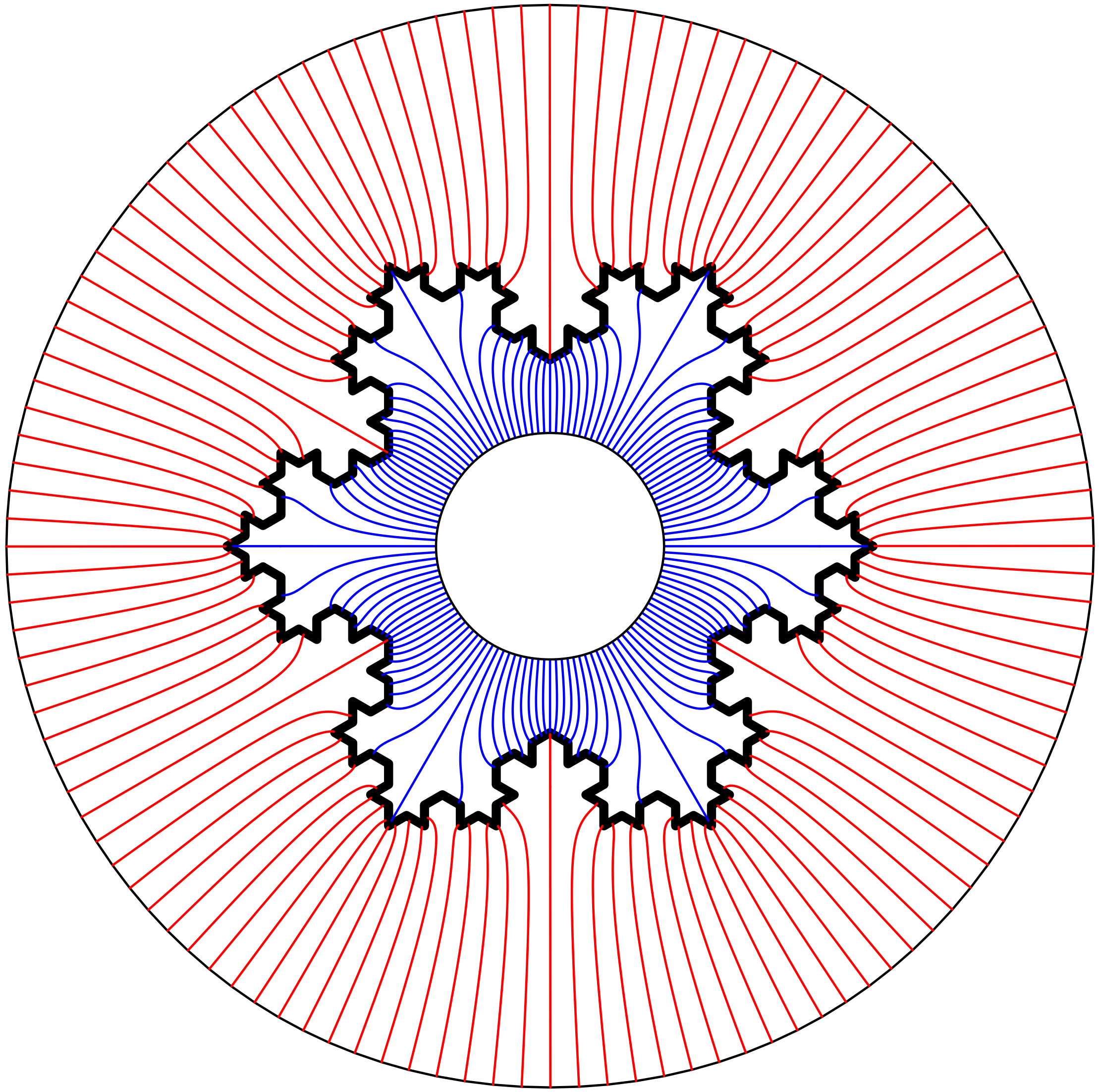
Outside is also fractal. Same set of length zero?

Theorem (B. 1987):

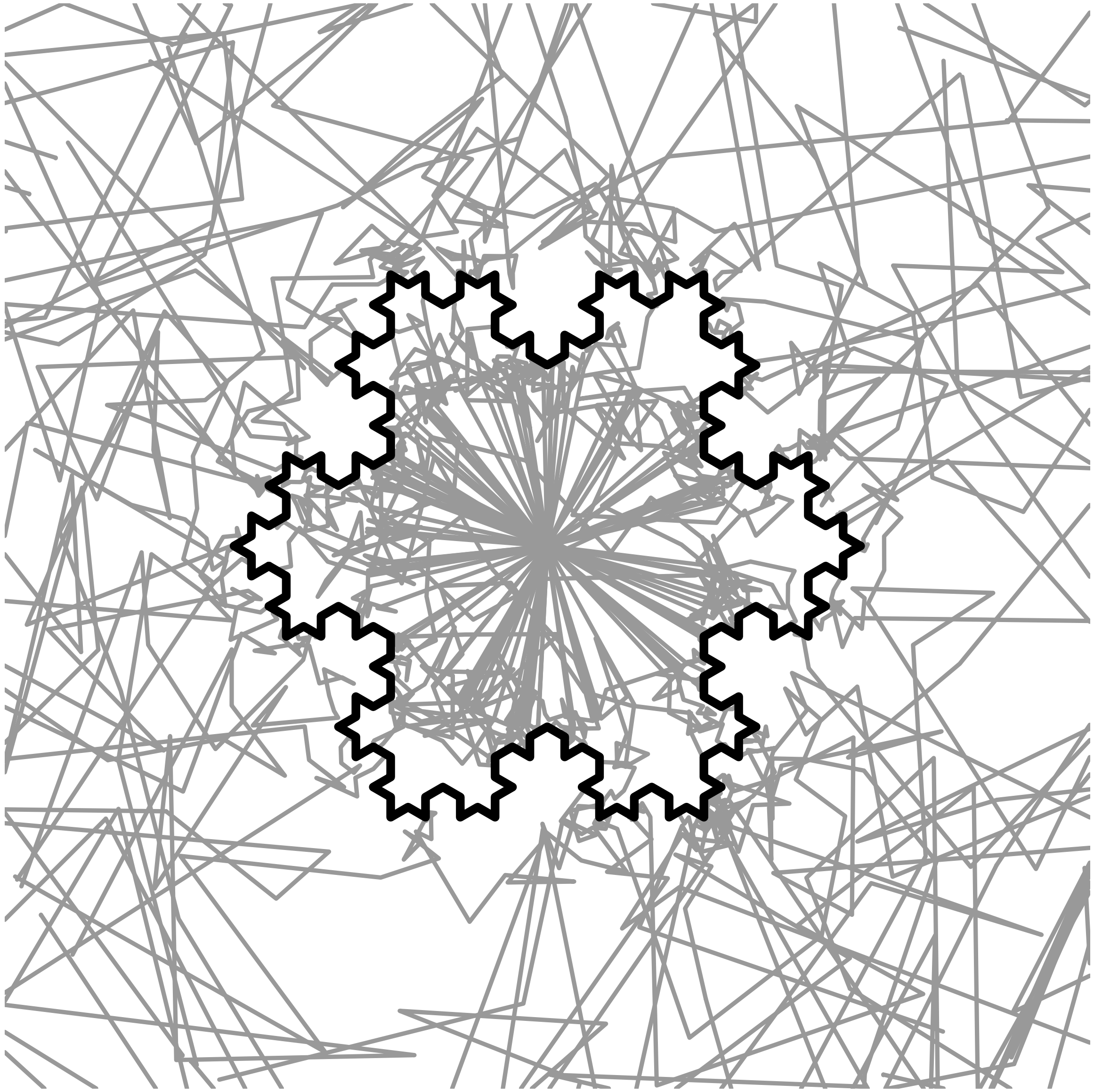
$\omega_1 \perp \omega_2$ iff tangents points have zero length.

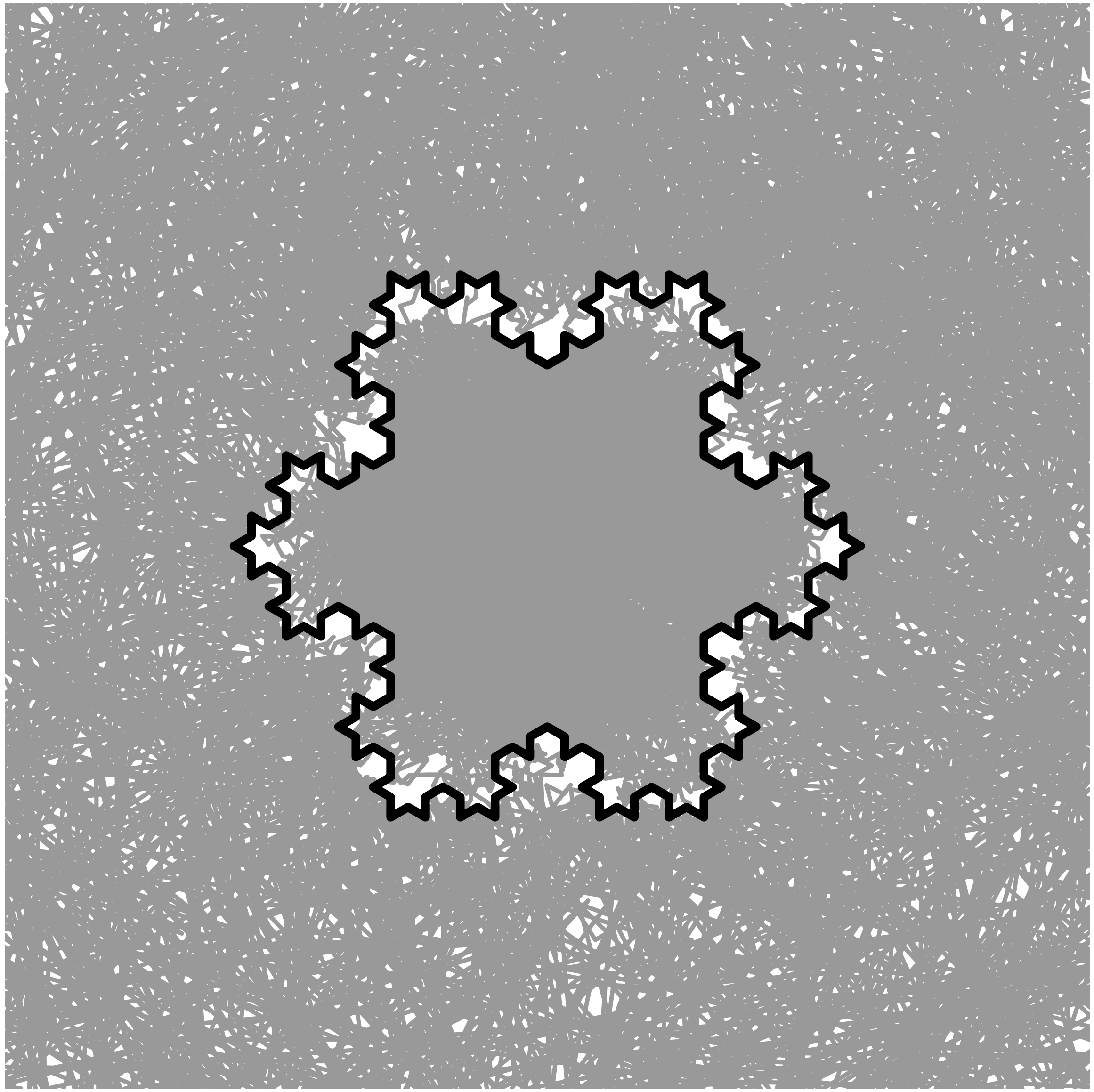


Inside and outside harmonic measures are singular



Images of radial lines for conformal maps to inside and outside.





For which curves is $\omega_1 = \omega_2$?

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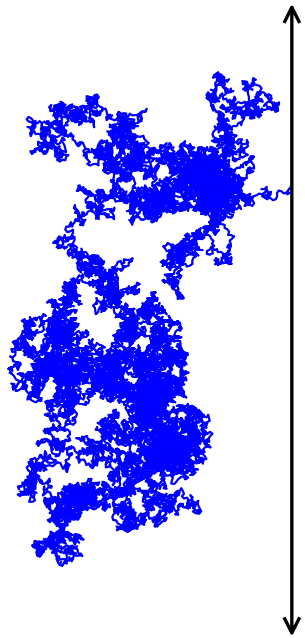

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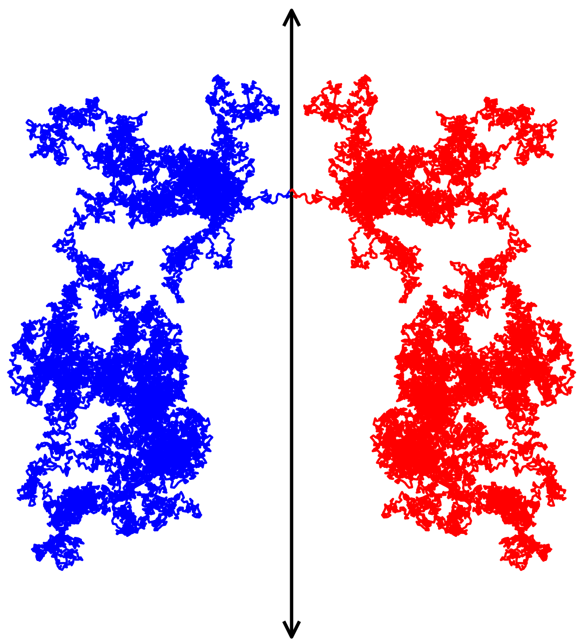

Also for circles (= lines conformally).

Converse? If $\omega_1 = \omega_2$ must Γ be a circle/line?

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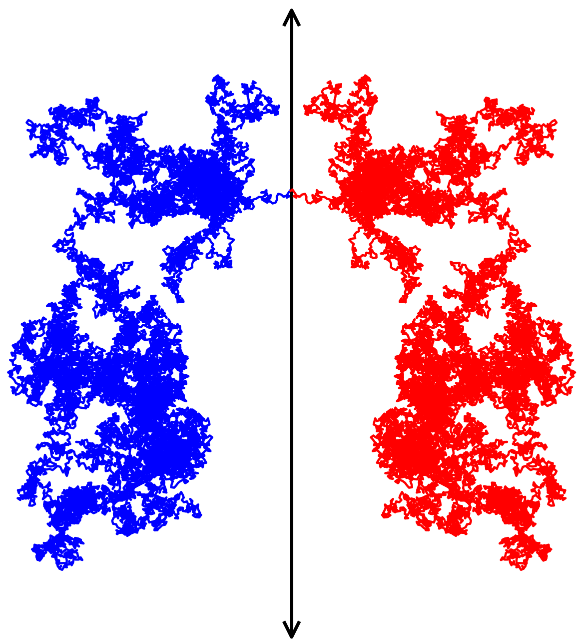
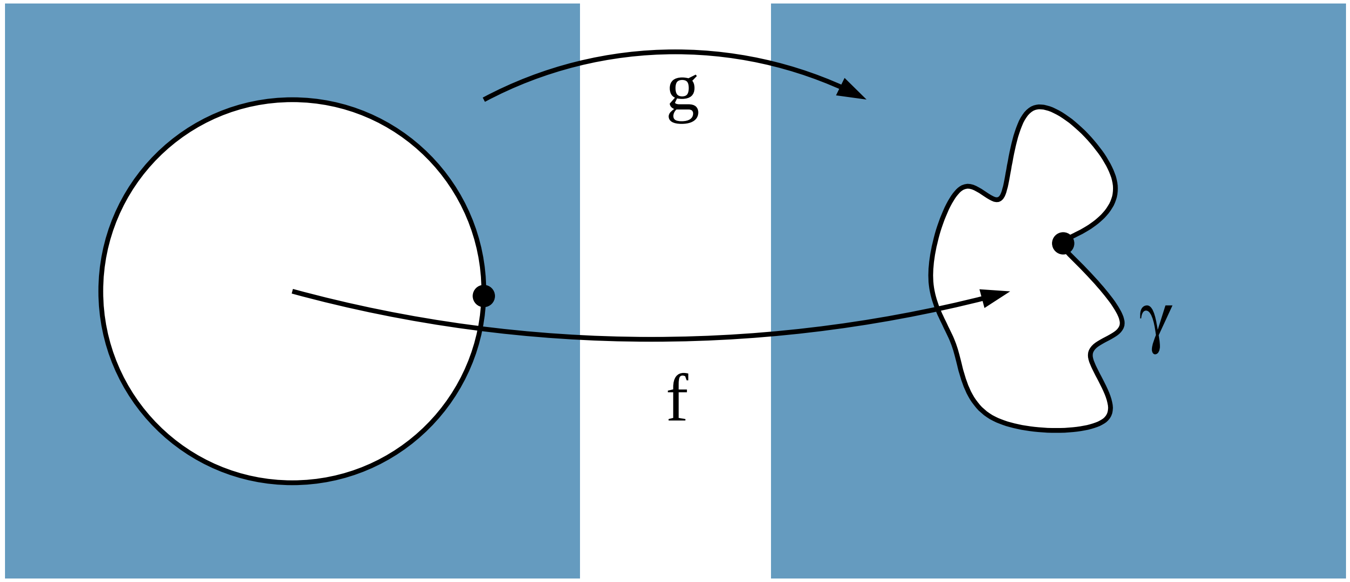


Also for circles ($=$ lines conformally).

Converse? If $\omega_1 = \omega_2$ must Γ be a circle/line? **Yes**



Suppose $\omega_1 = \omega_2$ for a curve γ.

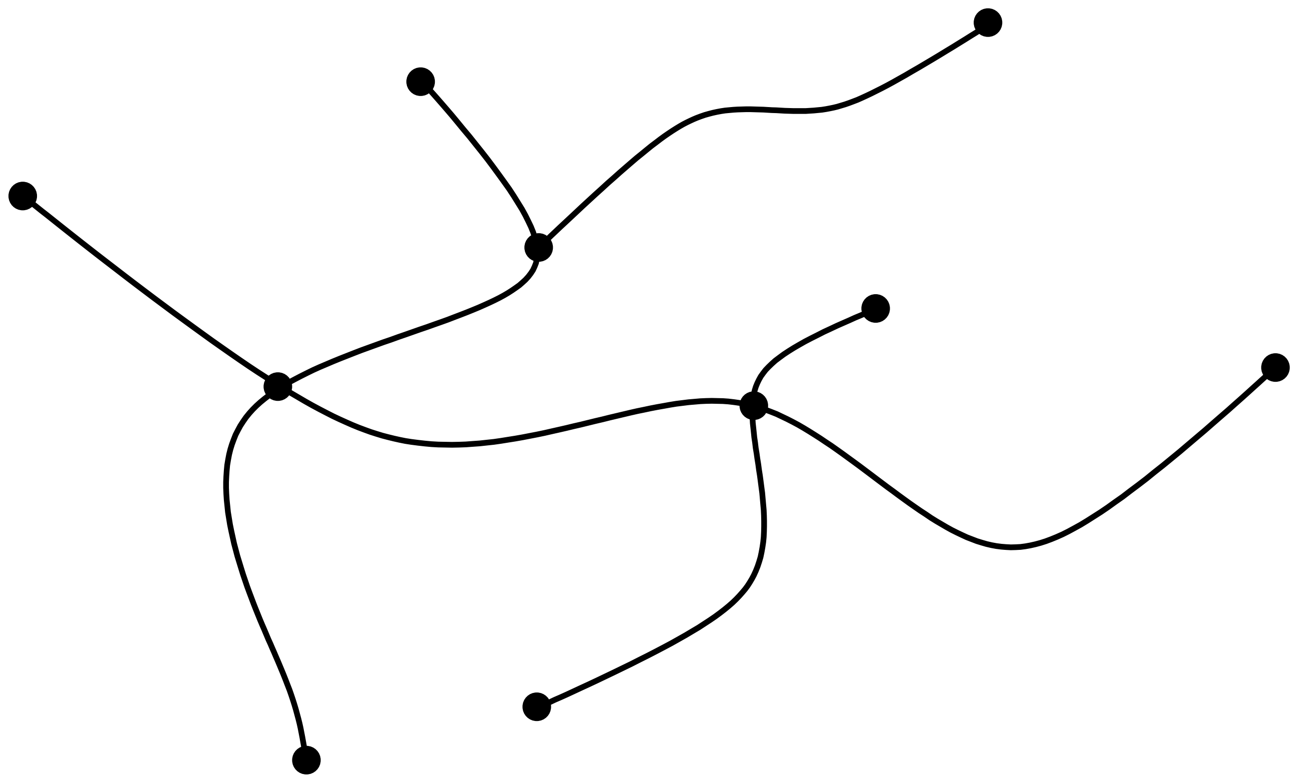
Conformally map two sides of circle to two sides of γ so $f(1) = g(1)$.

$\omega_1 = \omega_2$ implies maps agree on whole boundary.

So f, g define homeomorphism h of plane holomorphic off circle.

Then h is entire by Morera's theorem.

Entire and 1-1 implies h is linear (Liouville's thm), so γ is a circle.



A planar graph is a finite set of points connected by non-crossing edges.

It is a tree if there are no closed loops.

A planar tree is **conformally balanced** if

- every edge has equal harmonic measure from ∞
- edge subsets have same measure from both sides

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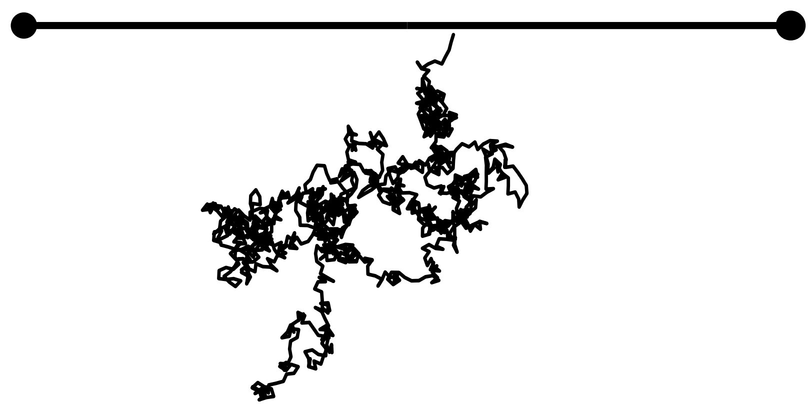
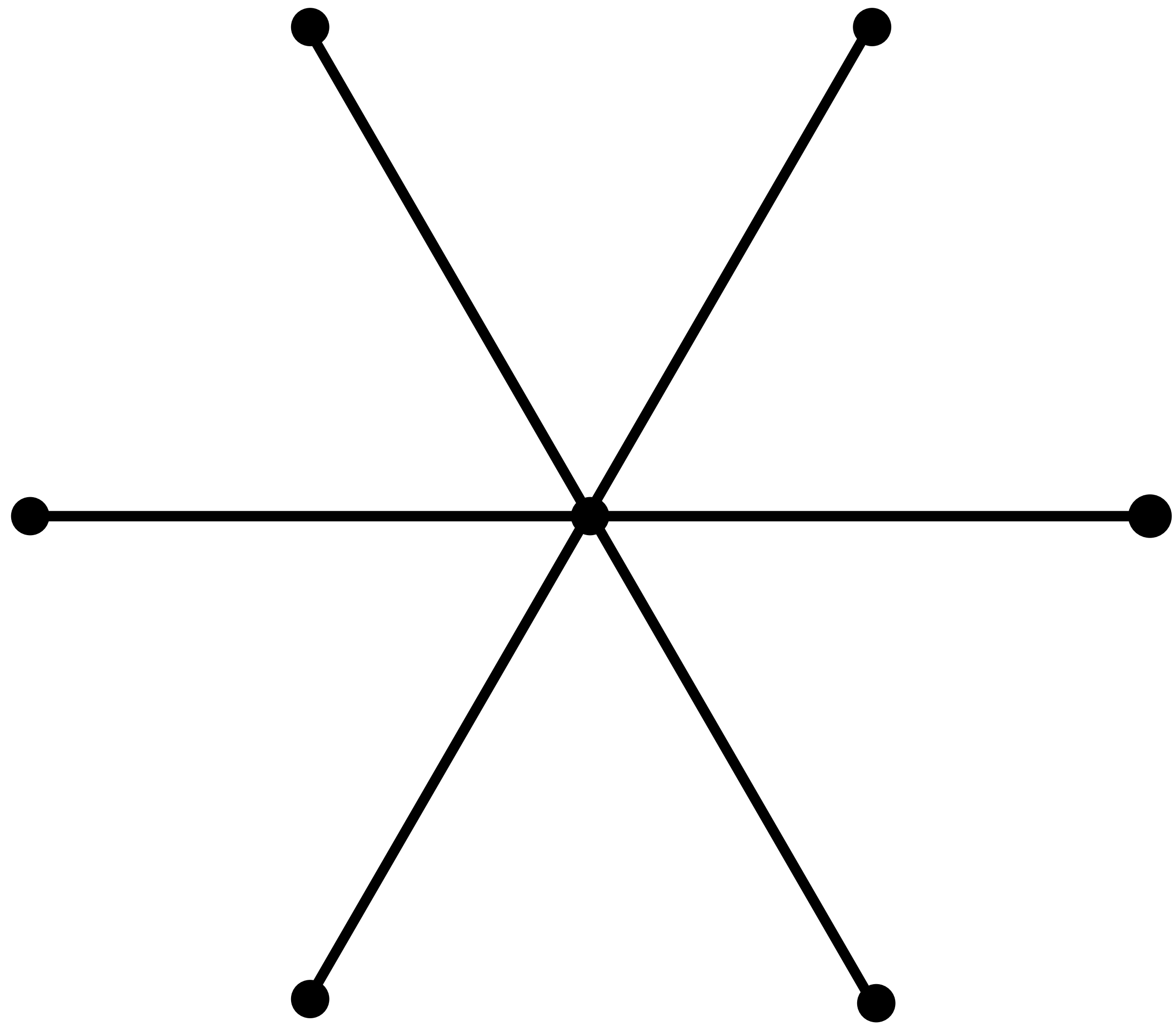
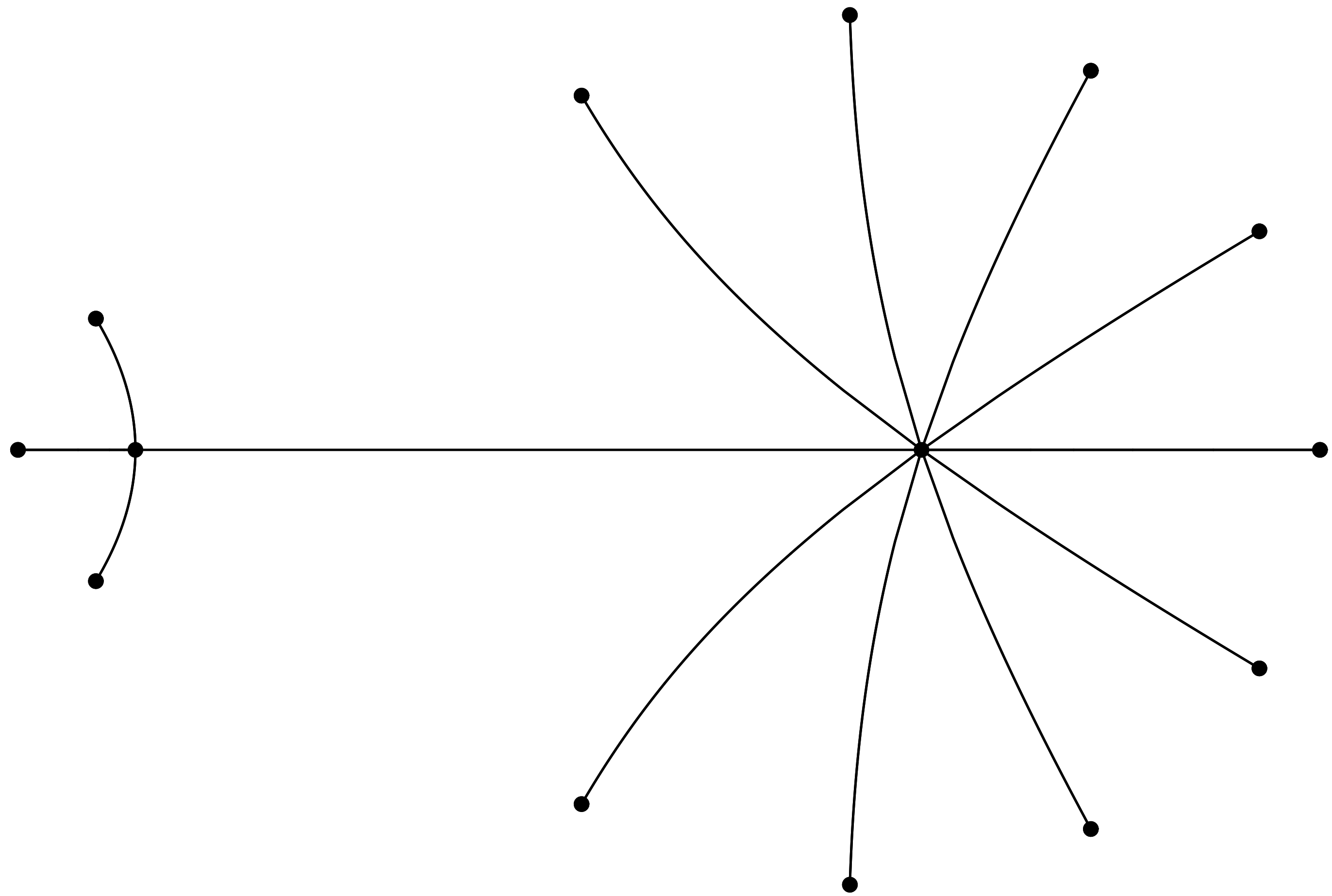




Trivially true by symmetry

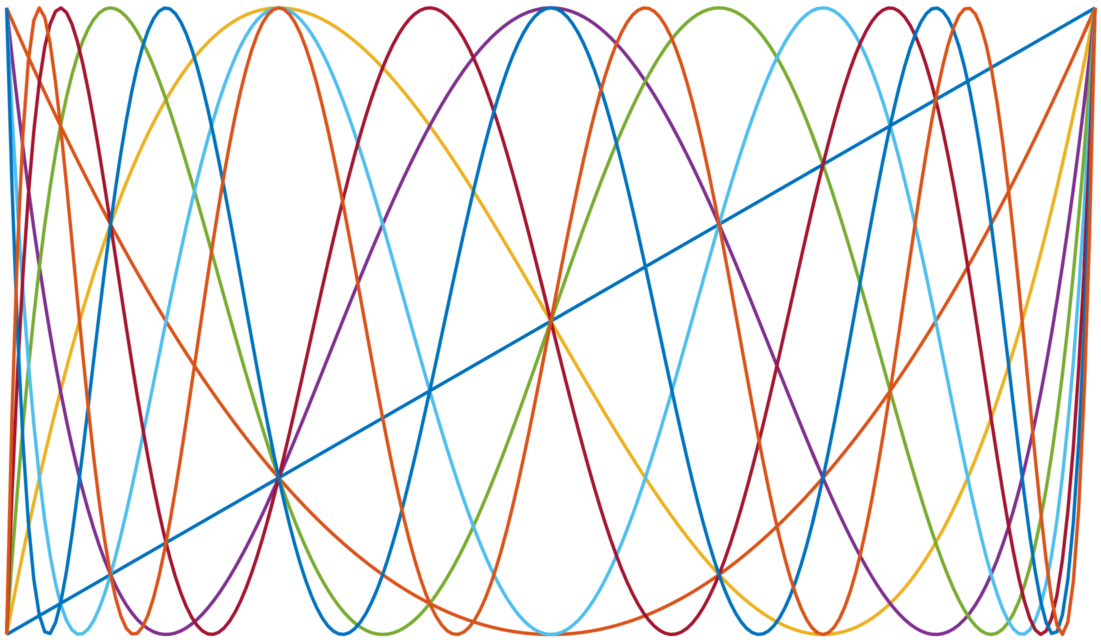


Non-obvious true tree

Definition of critical value: if $p =$ polynomial, then

$$\mathrm{CV}(p) = \{p(z) : p'(z) = 0\} = \text{critical values}$$

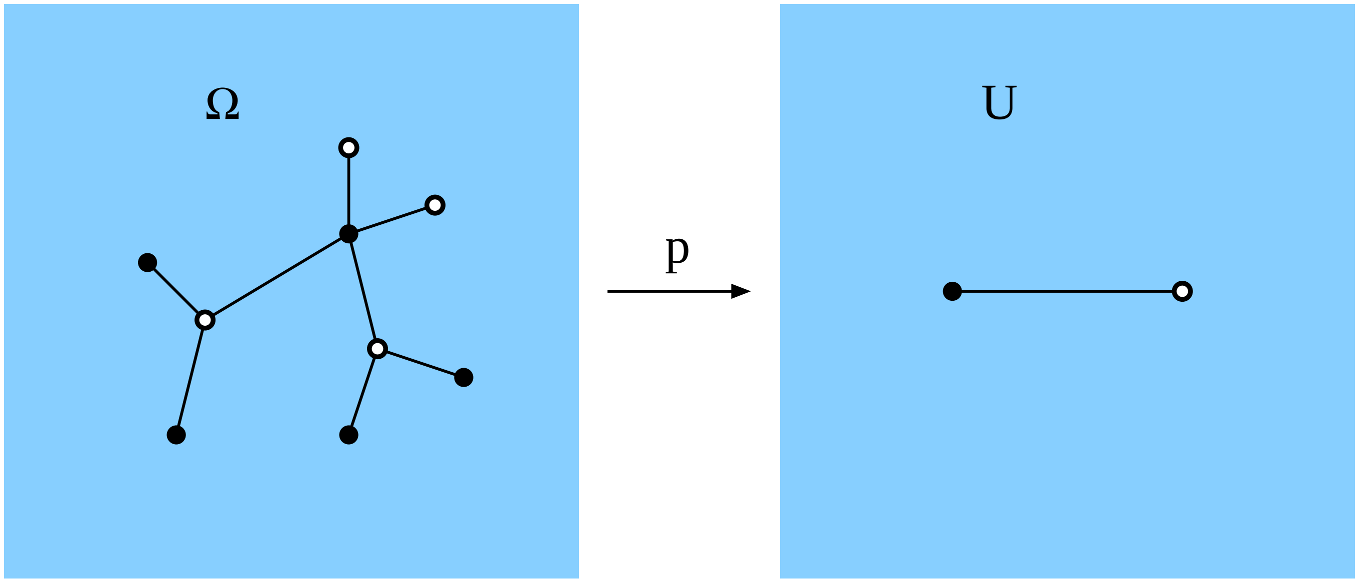
If $\mathrm{CV}(p) = \pm 1$, p is called **generalized Chebyshev** or **Shabat**.



10 classical Chebyshev polynomials

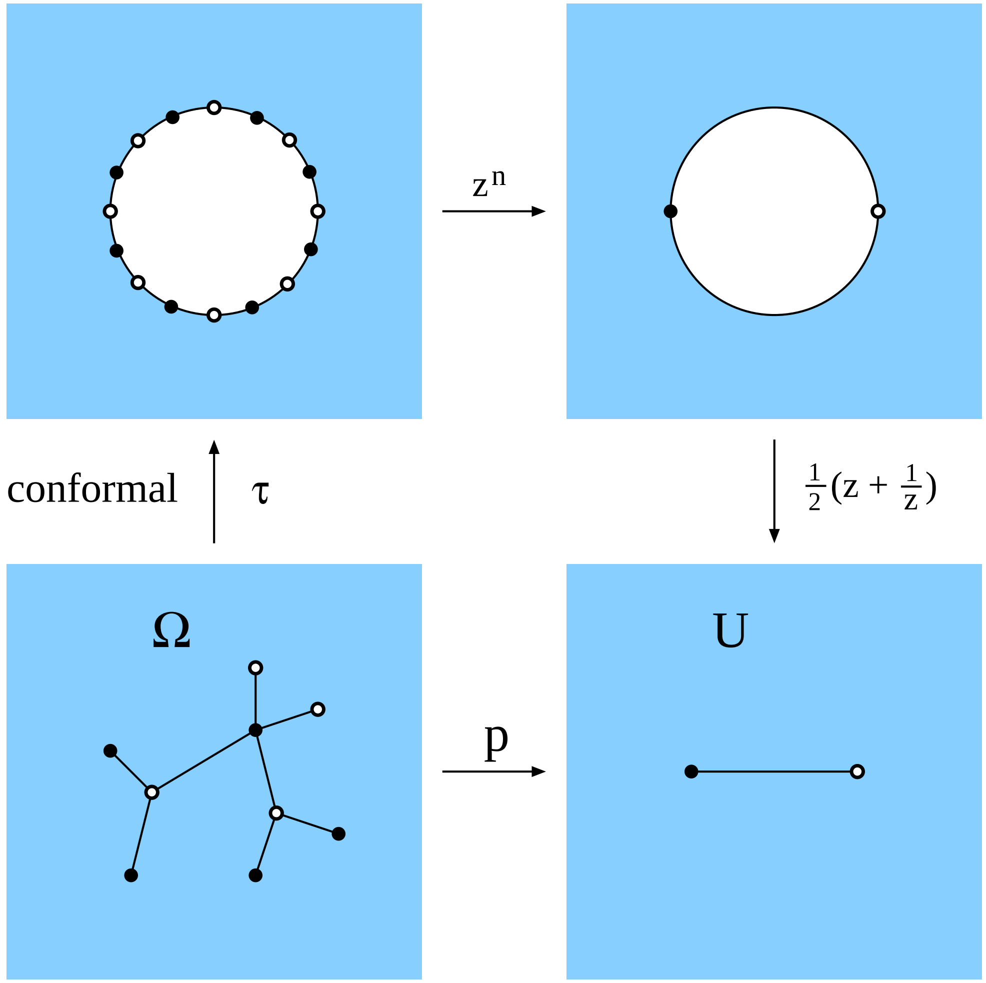
Balanced trees $\leftrightarrow$ Shabat polynomials

Fact: T is balanced iff $T = p^{-1}([-1,1])$, $p =$ Shabat.



$\Omega = \mathbb{C} \setminus T$ $\qquad$ $U = \mathbb{C} \setminus [-1,1]$

T balanced $\Leftrightarrow$ p Shabat.



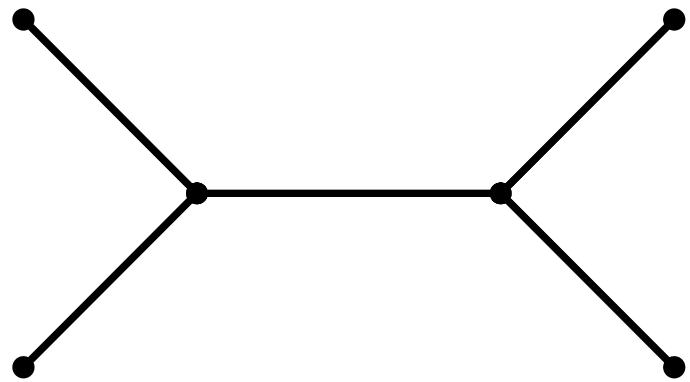
p is entire and n-to-1 $\Leftrightarrow$ p = polynomial.

$\mathrm{CV}(p) \notin U \leftrightarrow p : \Omega \to U$ is covering map.

Theorem: Every finite tree has a true form.

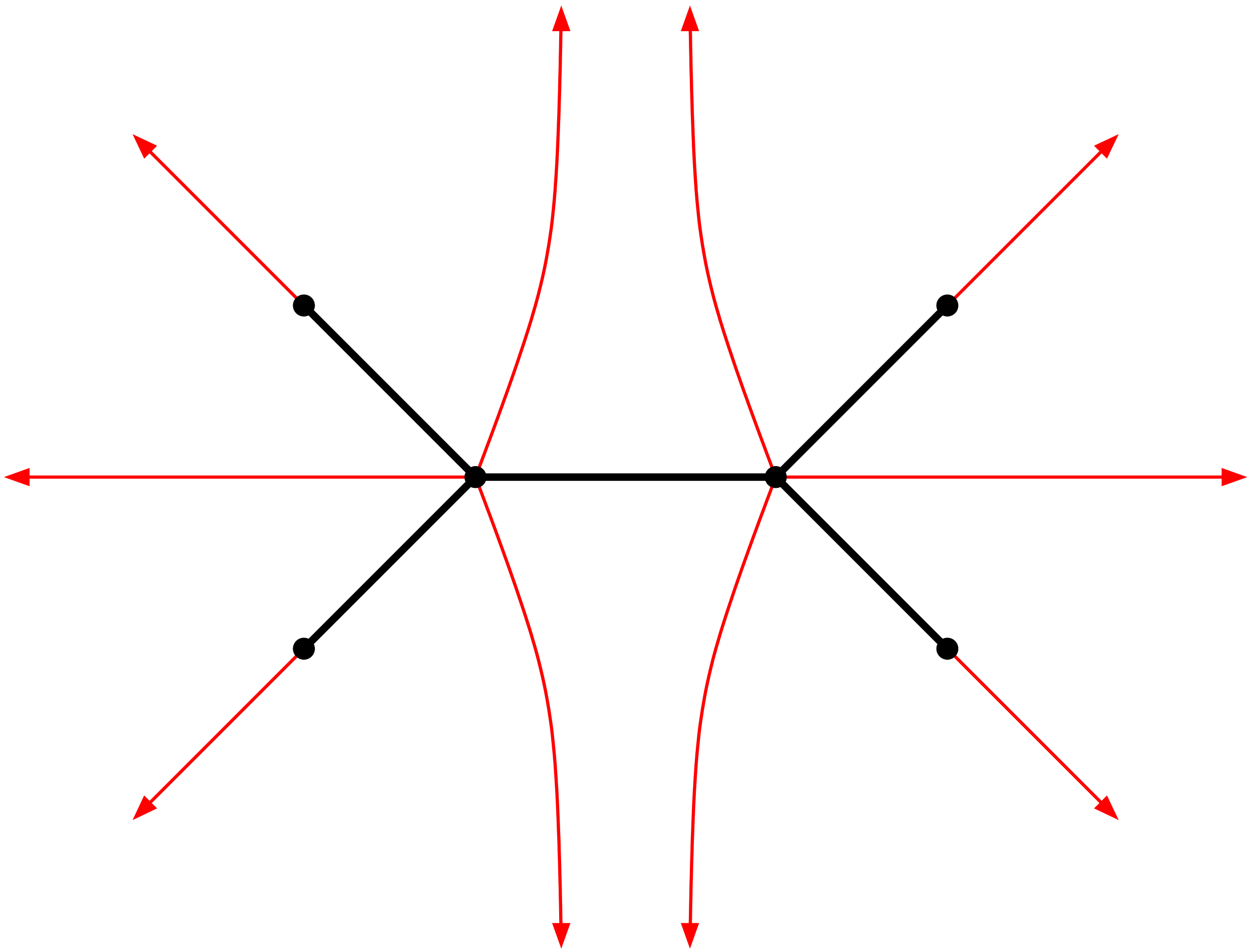
Standard proof uses the uniformization theorem.

Standard proof:



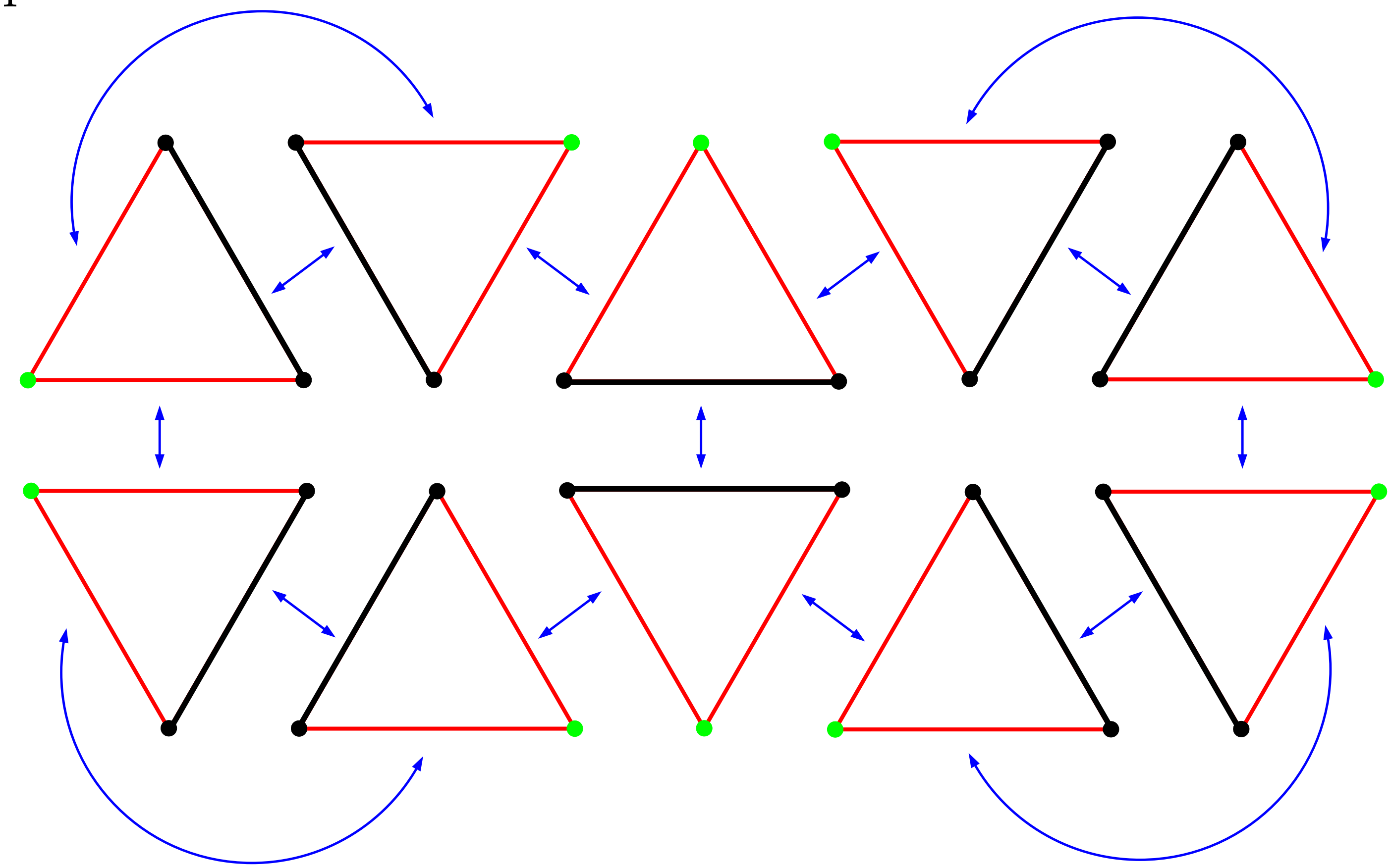
- start with a finite tree.

Standard proof:



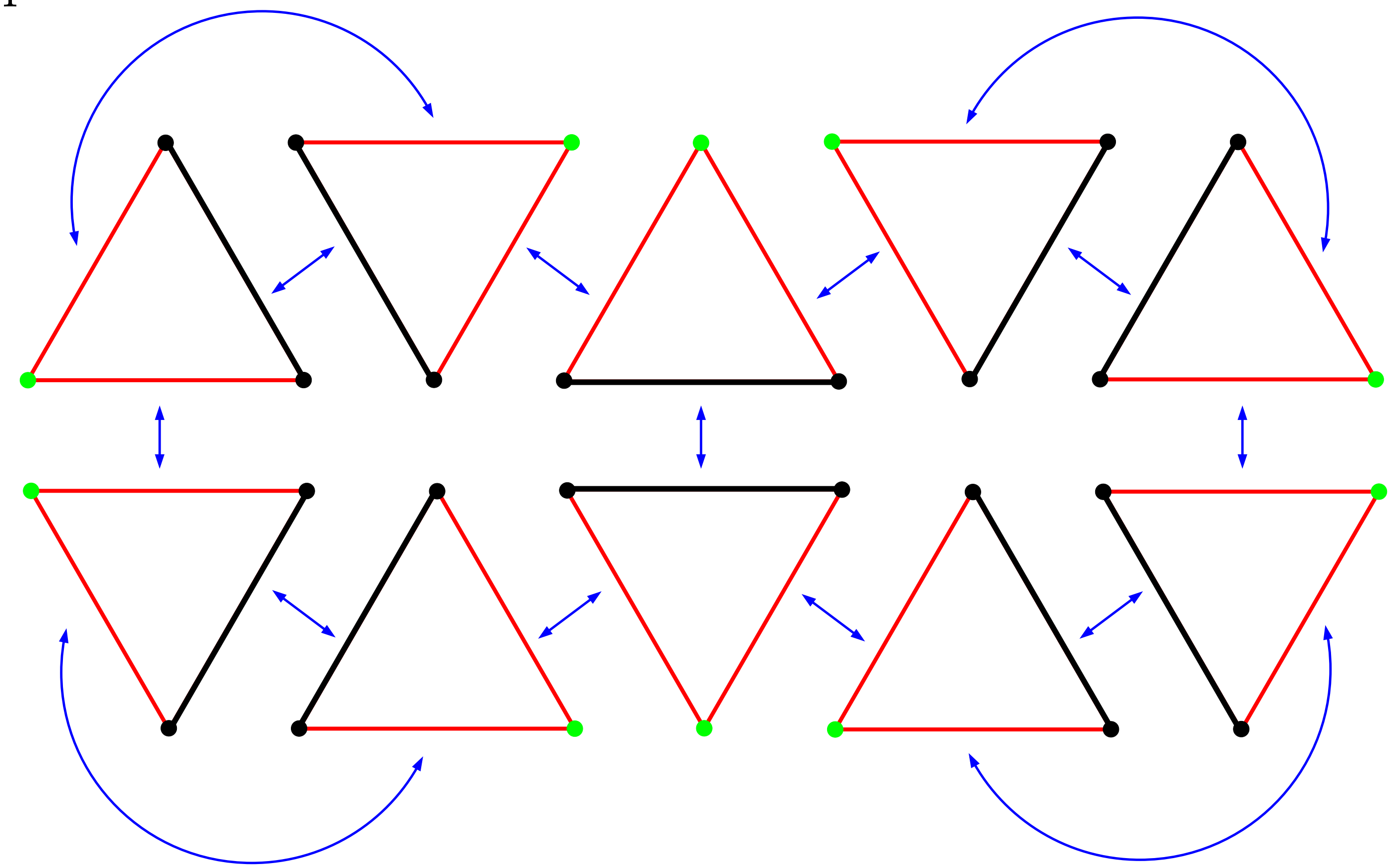
- connect vertices of T to infinity; gives finite triangulation of sphere.
- Defines adjacencies between triangles.

Standard proof:



- Glue equilateral triangles using adjacencies: get a conformal 2-sphere.
- By uniformization theorem, conformal maps to Riemann sphere.
- Can check that tree maps to balanced tree.

Standard proof:

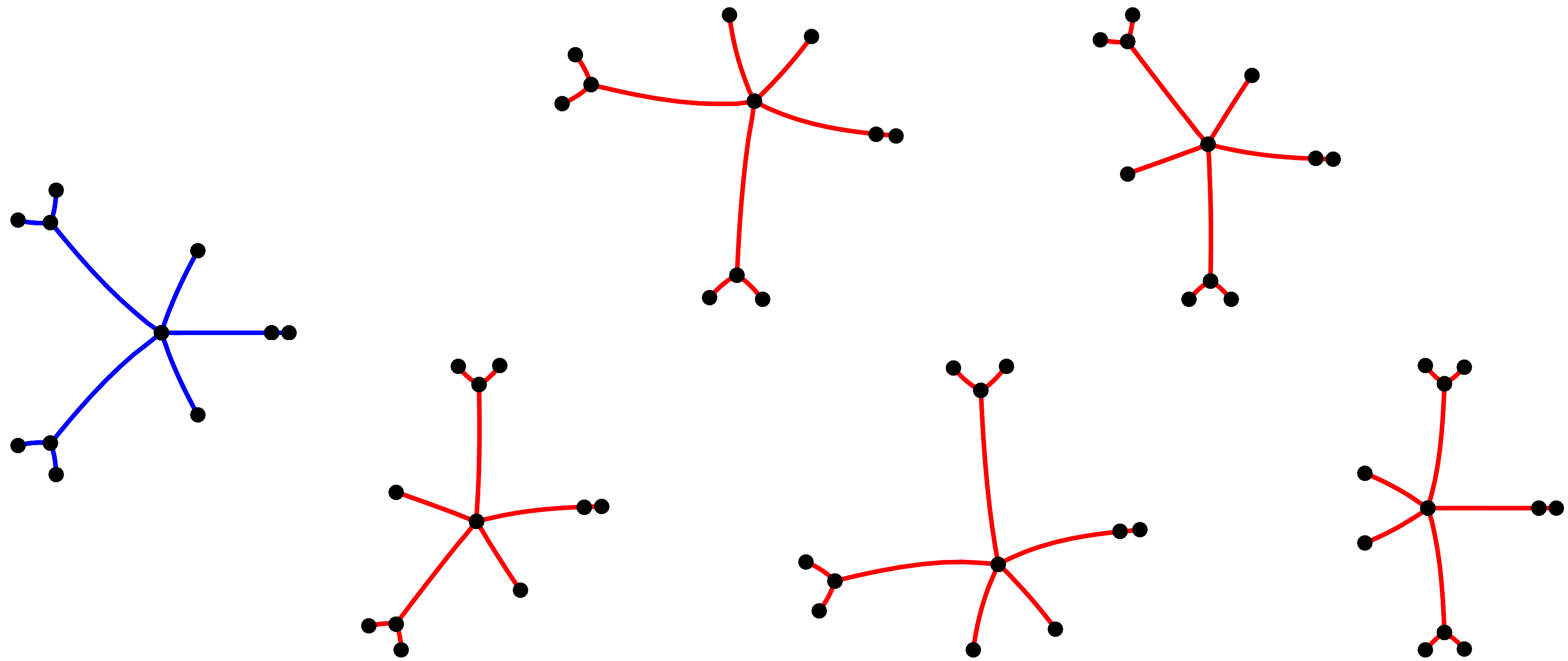


- Note: conformal 2-sphere has many "equilateral triangulations".
- Which other Riemann surfaces can be obtained in this way?
- This is subject of Lasse Rempe-Gillen's talk later.

Algebraic aside:

True trees are examples of Grothendieck's *dessins d'enfants* on sphere.

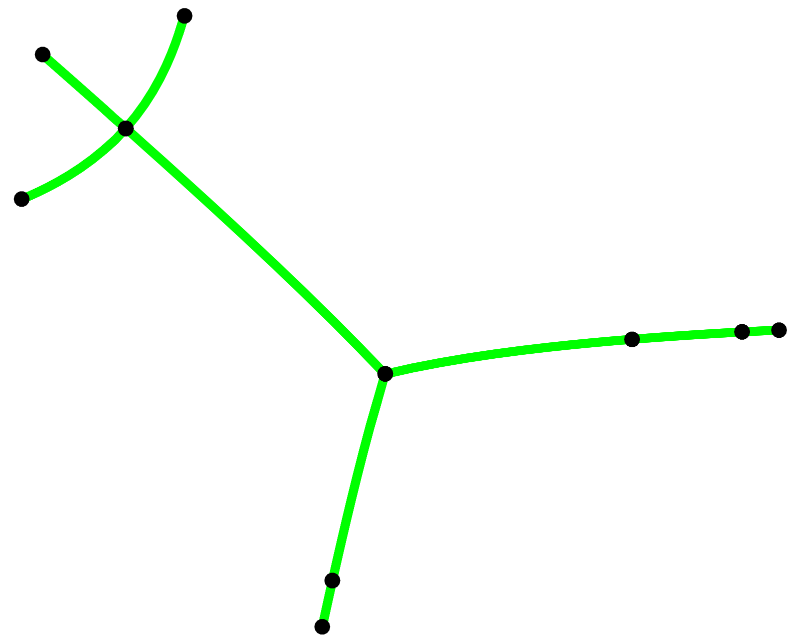
Normalized polynomials are algebraic, so planar trees correspond to number fields. Absolute Galois group acts on trees, but orbits unknown.



Six graphs of type 5 1 1 1 1 1 - 3 3 2 1 1, two orbits.

Even computing number field from tree is difficult.

Kochetkov (2009, 2014): did all trees with 9 and 10 edges.



For example, the polynomial for this 9-edge tree is

$$p(z) = z^4(z^2 + az + b)^2(z - 1),$$

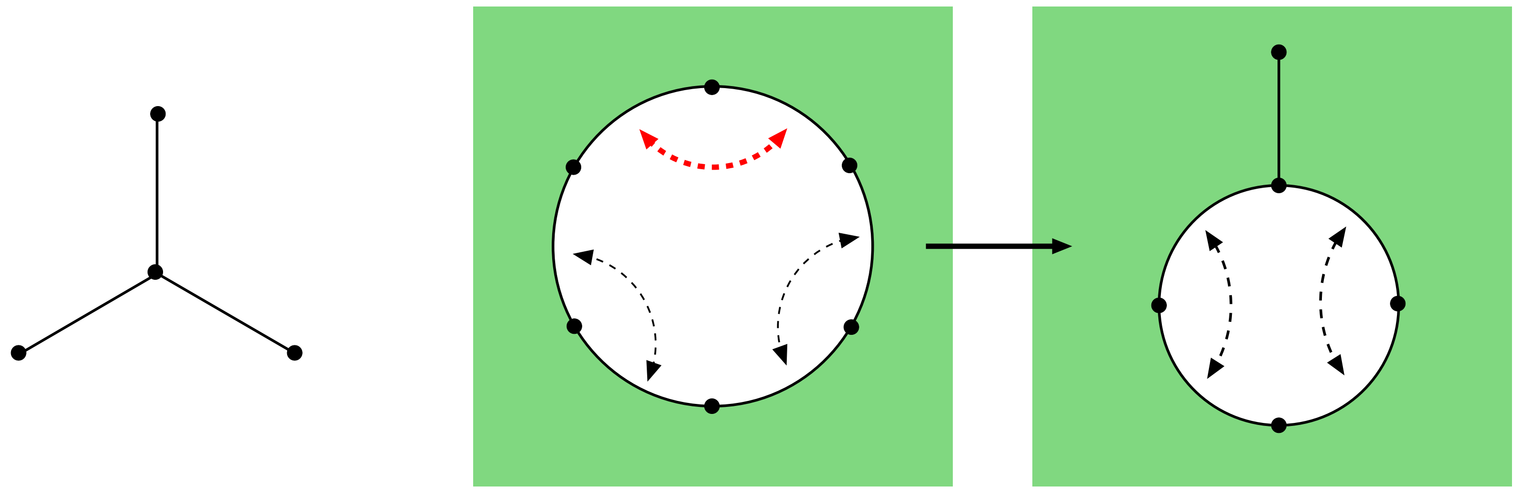
where a is a root of ...

$$
\begin{aligned}
0 = \quad & 126105021875\, a^{15} + 873367351500\, a^{14} \\
& +2340460381665\, a^{13} + 2877817869766\, a^{12} \\
& +3181427453757\, a^{11} - 68622755391456\, a^{10} \\
& -680918281137097\, a^{9} - 2851406436711330\, a^{8} \\
& -7139130404618520\, a^{7} - 12051656256571792\, a^{6} \\
& -14350515598839120\, a^{5} - 12058311779508768\, a^{4} \\
& -6916678783373312\, a^{3} - 2556853615656960\, a^{2} \\
& -561846360735744\, a - 65703906377728
\end{aligned}
$$

This is **not** the most complicated formula in Kochetkov's paper.

However, true form can be drawn without knowing the polynomial.

Don Marshall's `ZIPPER` uses conformal mapping to draw true trees.



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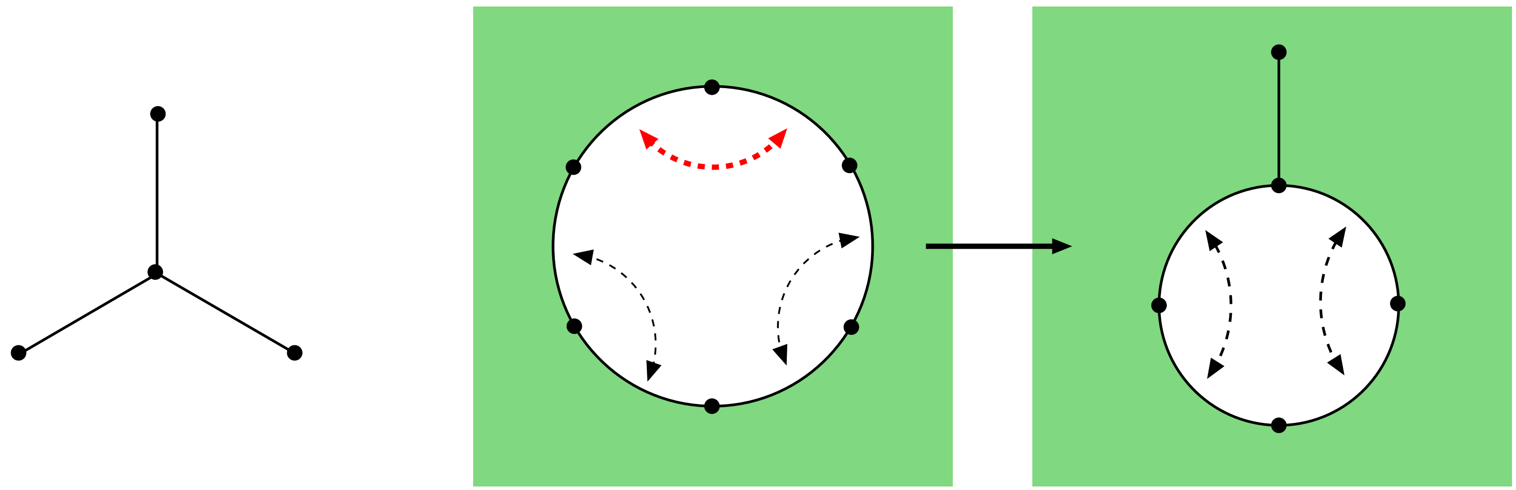
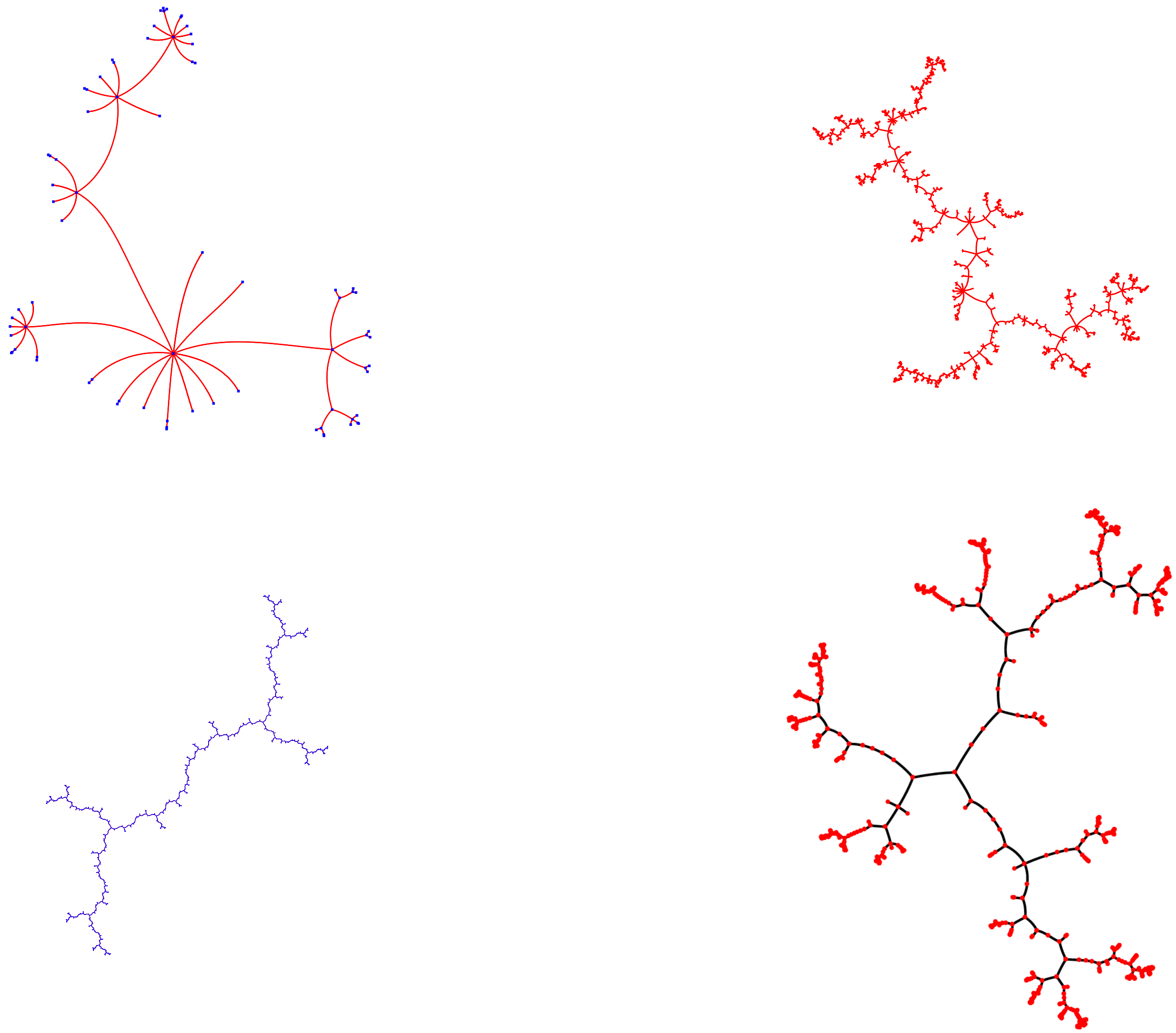


Marshall and Rohde approximated all true trees with ≤ 14 edges.

They can compute vertices to 1000's of digits of accuracy.

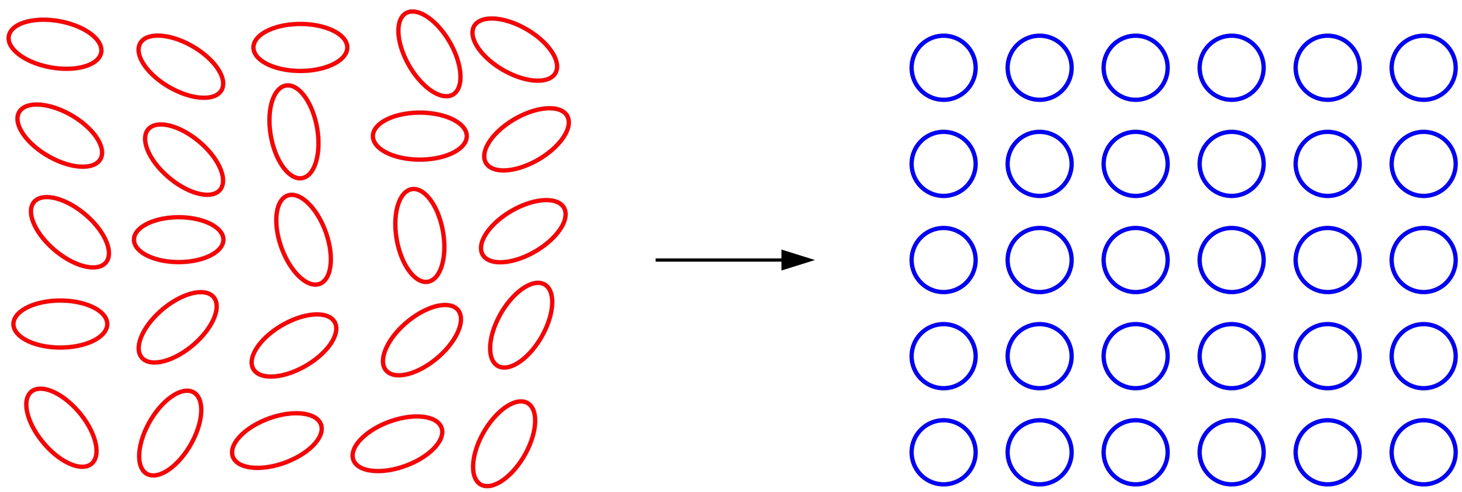
Test if $\alpha \in \mathbb{C}$ is algebraic by seeking integer relationships between $1, \alpha, \alpha^2, \ldots$ using lattice reduction or PSLQ algorithm.



Some true trees, courtesy of Marshall and Rohde

Alternate proof that all trees have true forms using QC mappings.

Quasiconformal (QC) maps send infinitesimal ellipses to circles.



Eccentricity = ratio of major to minor axis of ellipse.

For K-QC maps, ellipses have eccentricity $\leq K$

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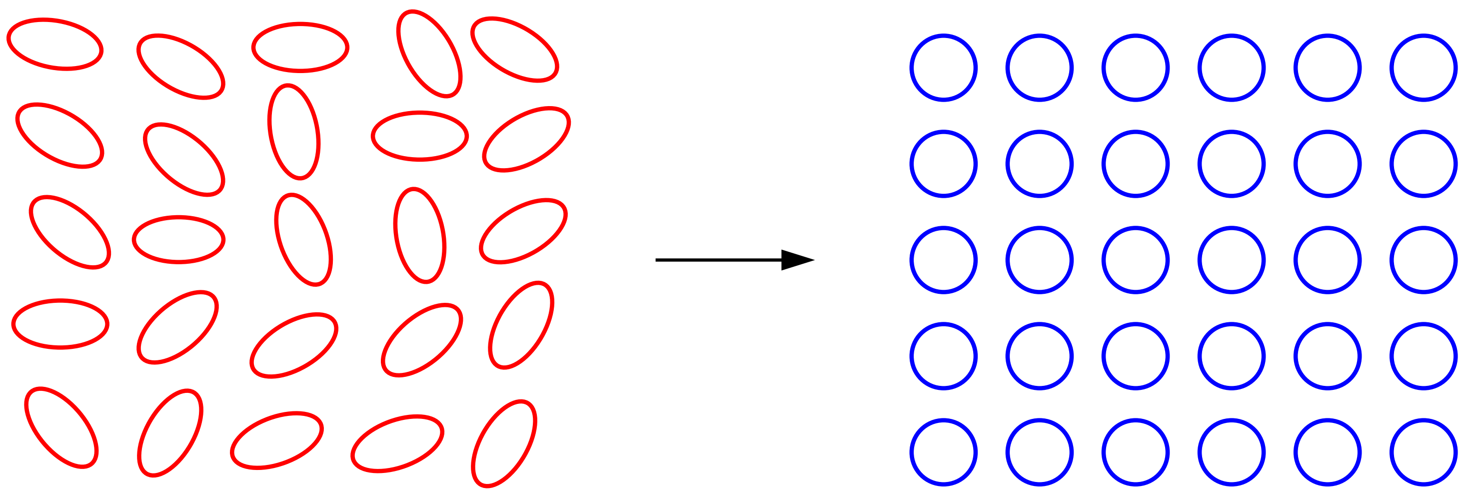


Eccentricity = ratio of major to minor axis of ellipse.

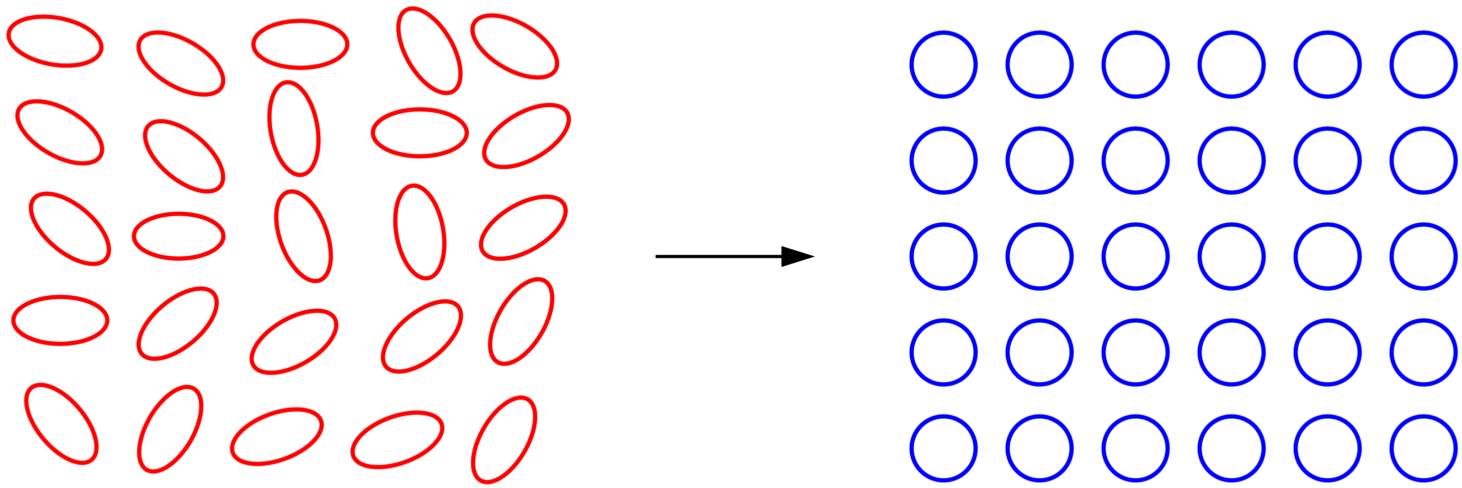
For K-QC maps, ellipses have eccentricity $\leq K$

Ellipses determined a.e. by measurable dilatation $\mu = f_{\overline{z}}/f_z$ with

$$|\mu| \leq \frac{K-1}{K+1} < 1.$$

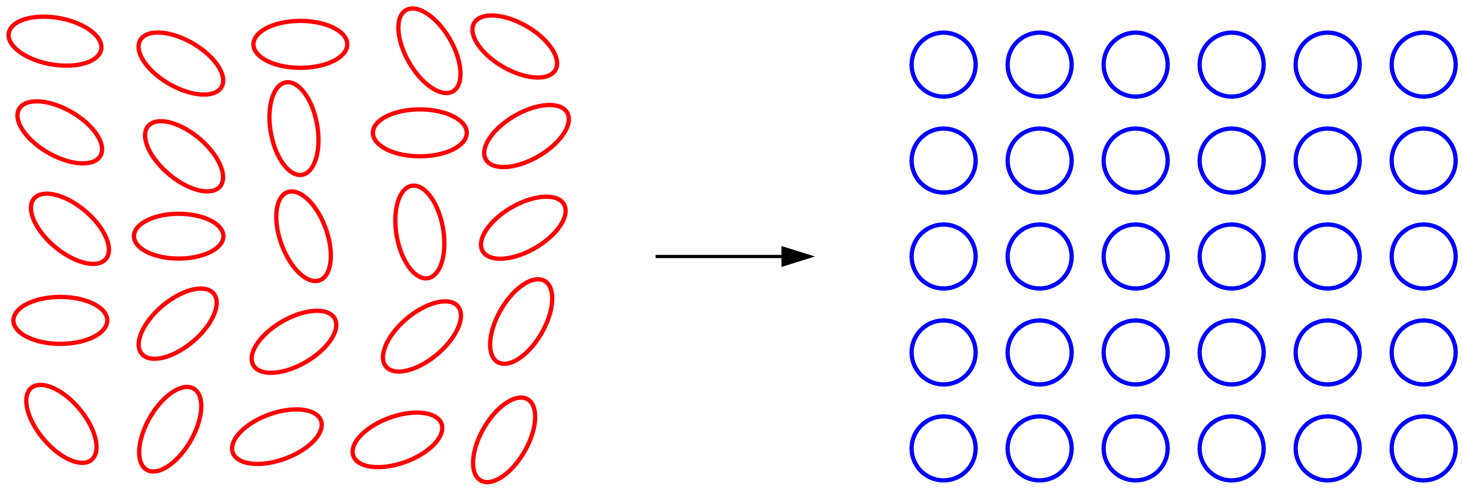
Conversely, ...

Quasiconformal (QC) maps send infinitesimal ellipses to circles.



Mapping theorem: any such μ comes from some QC map f.

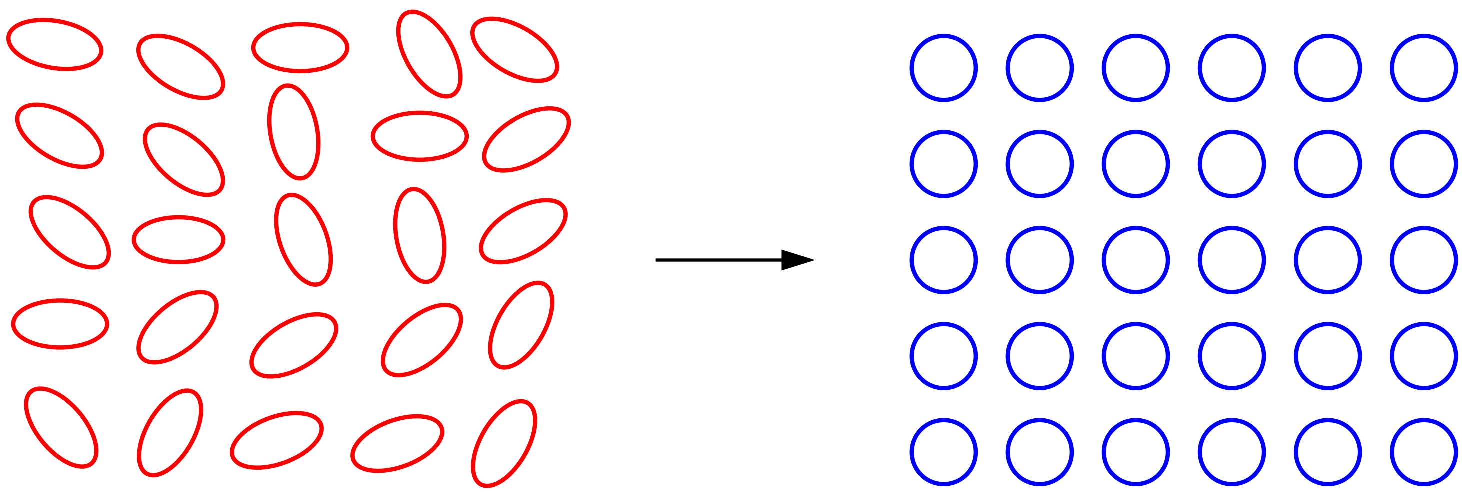
Quasiconformal (QC) maps send infinitesimal ellipses to circles.



Mapping theorem: any such μ comes from some QC map f.

Cor: If f is holomorphic and ψ is QC, then there is a QC map φ so that $g = \psi \circ f \circ \varphi$ is also holomorphic.

Quasiconformal (QC) maps send infinitesimal ellipses to circles.



Mapping theorem: any such μ comes from some QC map f.

Cor: If f is holomorphic and ψ is QC, then there is a QC map φ so that $g = \psi \circ f \circ \varphi$ is also holomorphic.

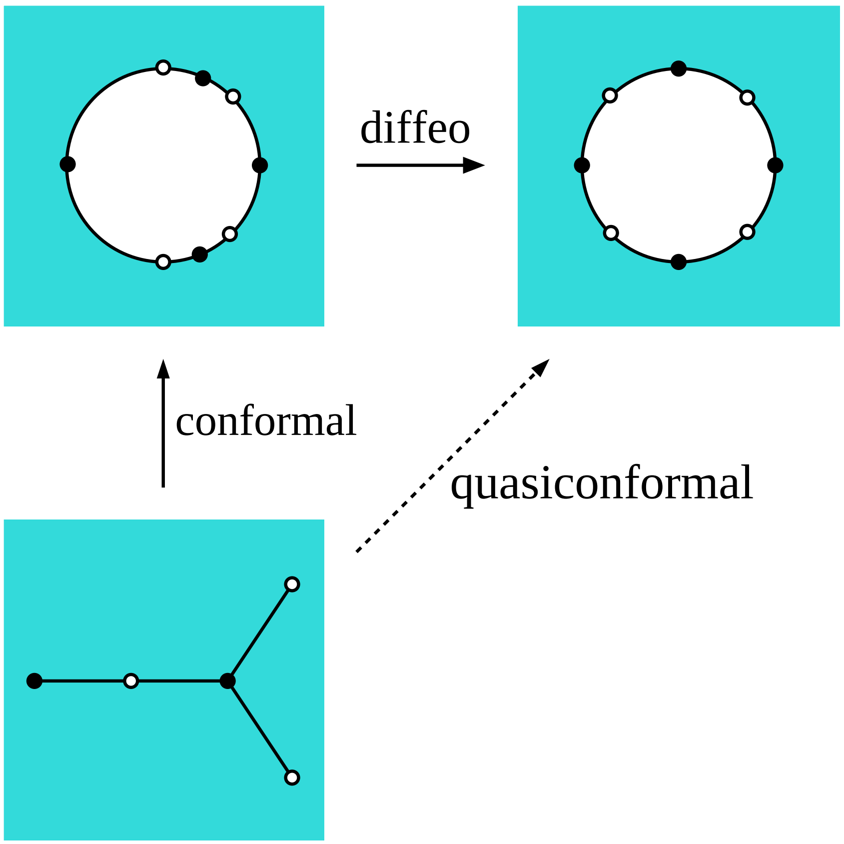
Such f, g are called **QC-equivalent** (weaker than being conjugate).

In general, $\psi \circ f \circ \varphi$ is not holomorphic, but is **quasiregular (QR)**.

QC proof that every finite tree has a true form:

Map $\Omega = \mathbb{C} \setminus T$ to $\{|z| > 1\}$ conformally.

"Equalize intervals" by diffeomorphism. Composition is quasiconformal.



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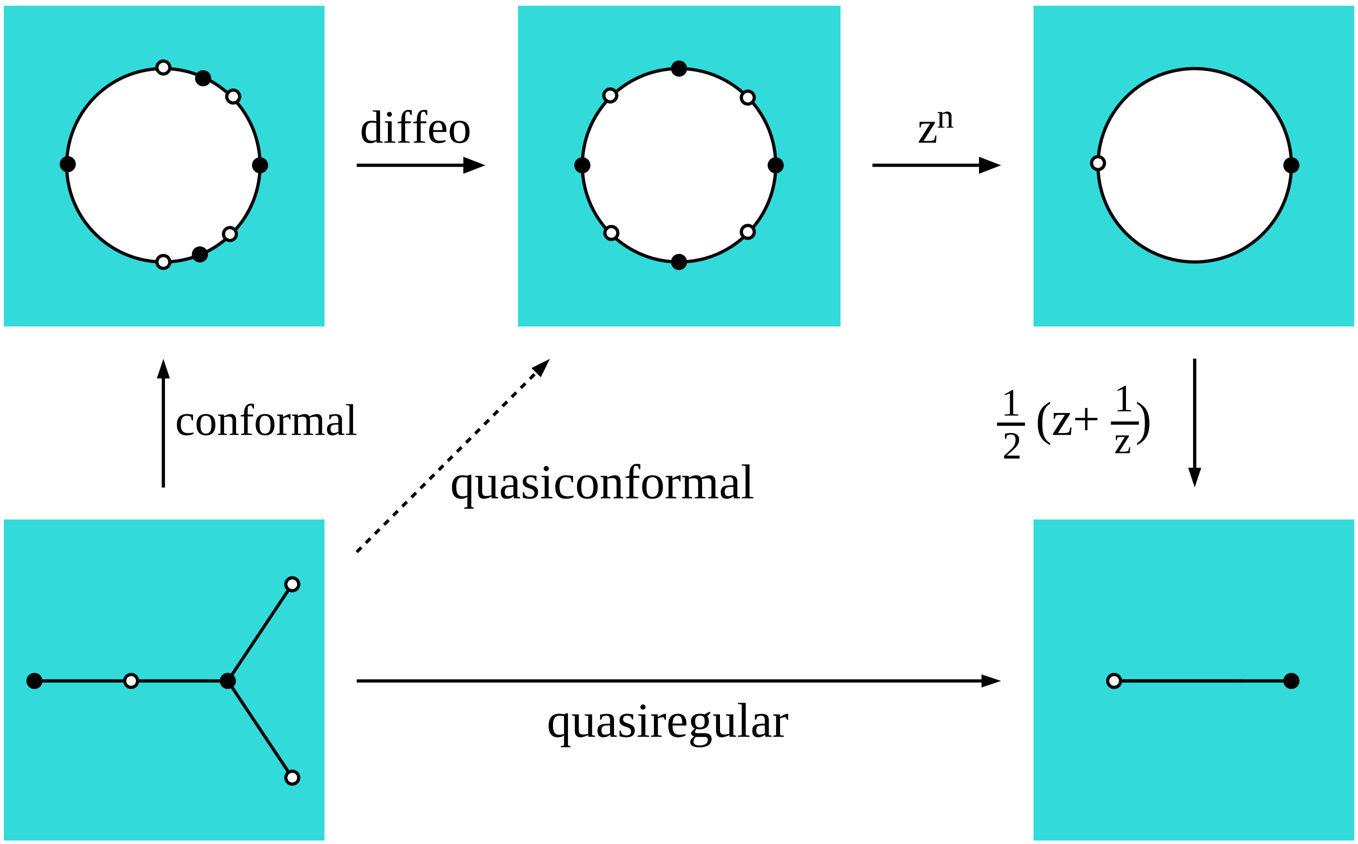
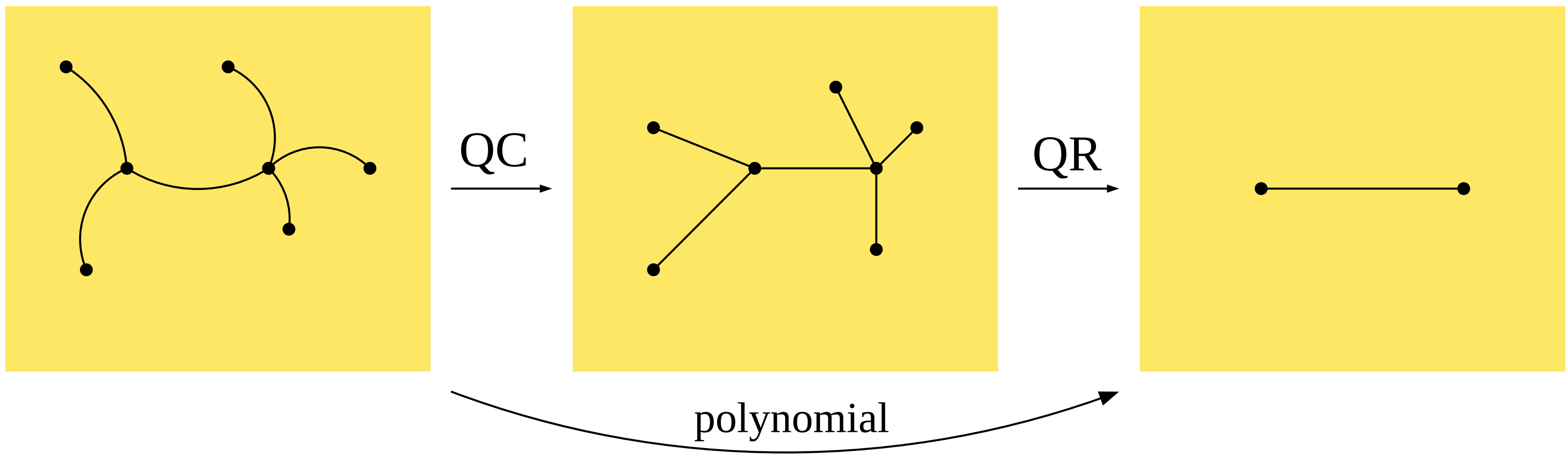


Mapping theorem implies there is a QC φ so $p = q \circ \varphi$ is a polynomial.

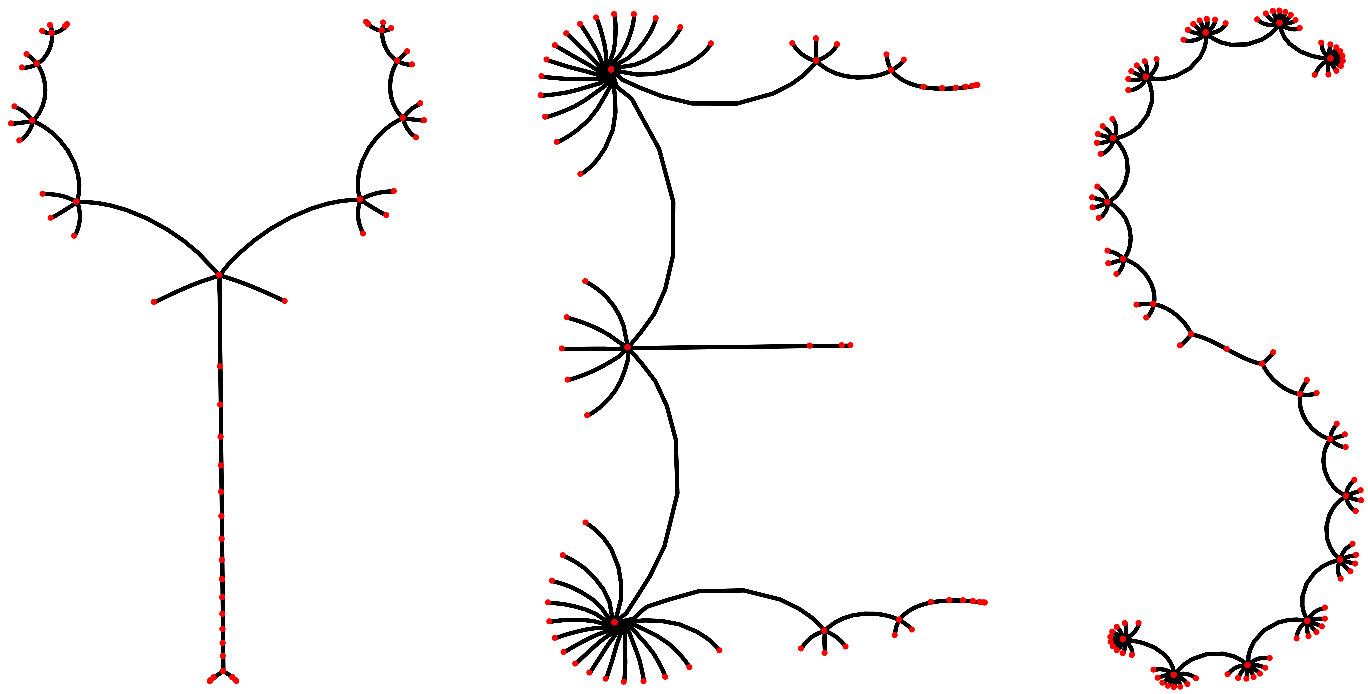


Only possible critical points are vertices of tree; these map to ± 1.

Thus every planar tree has a true form.

Do true trees approximate all possible shapes?

Do true trees approximate all possible shapes?



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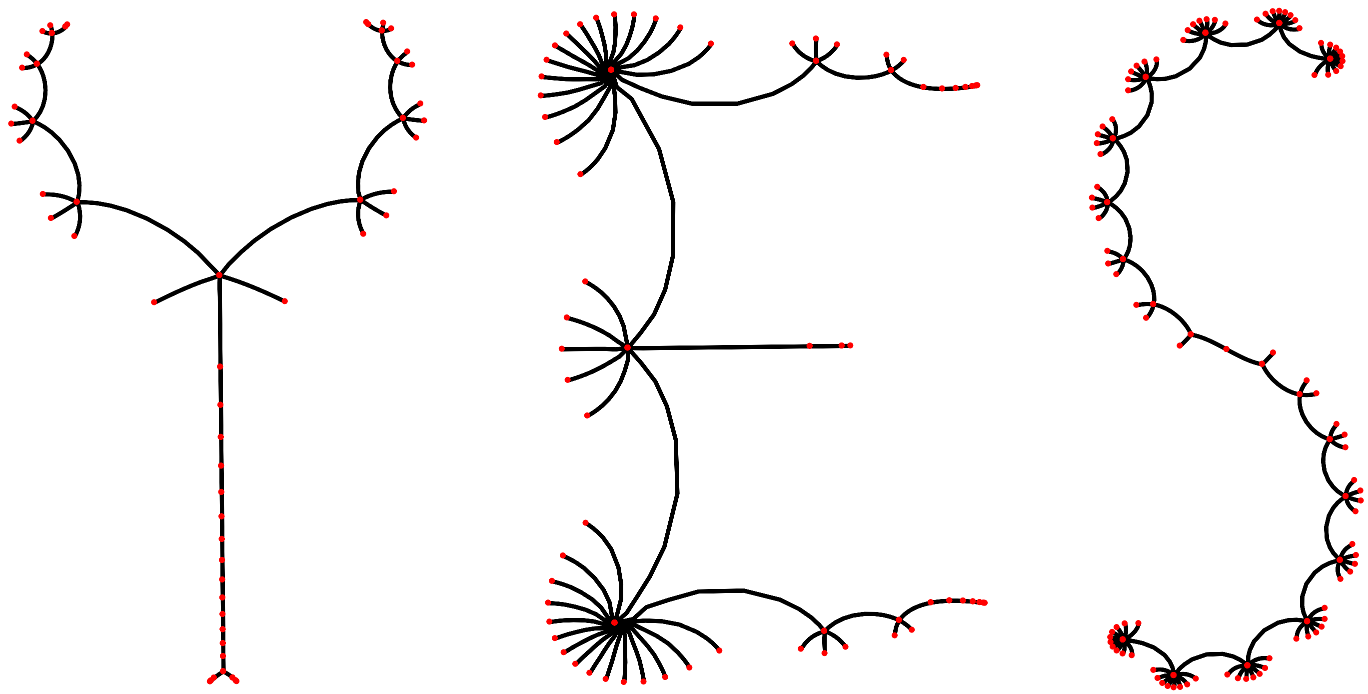
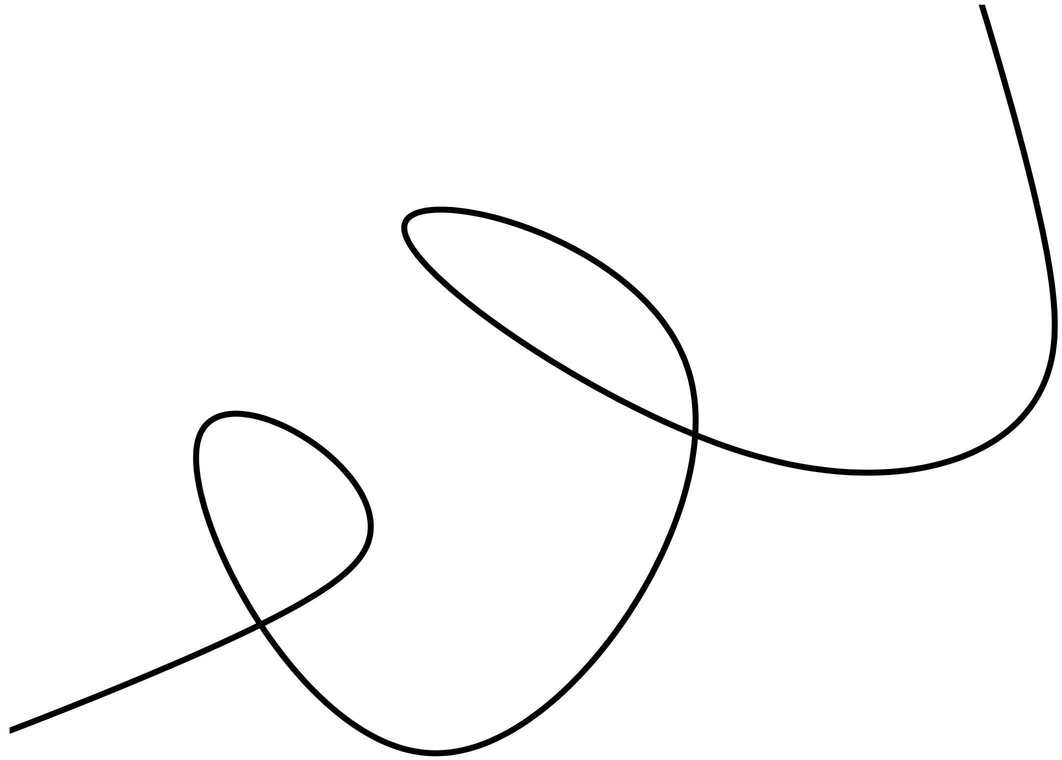
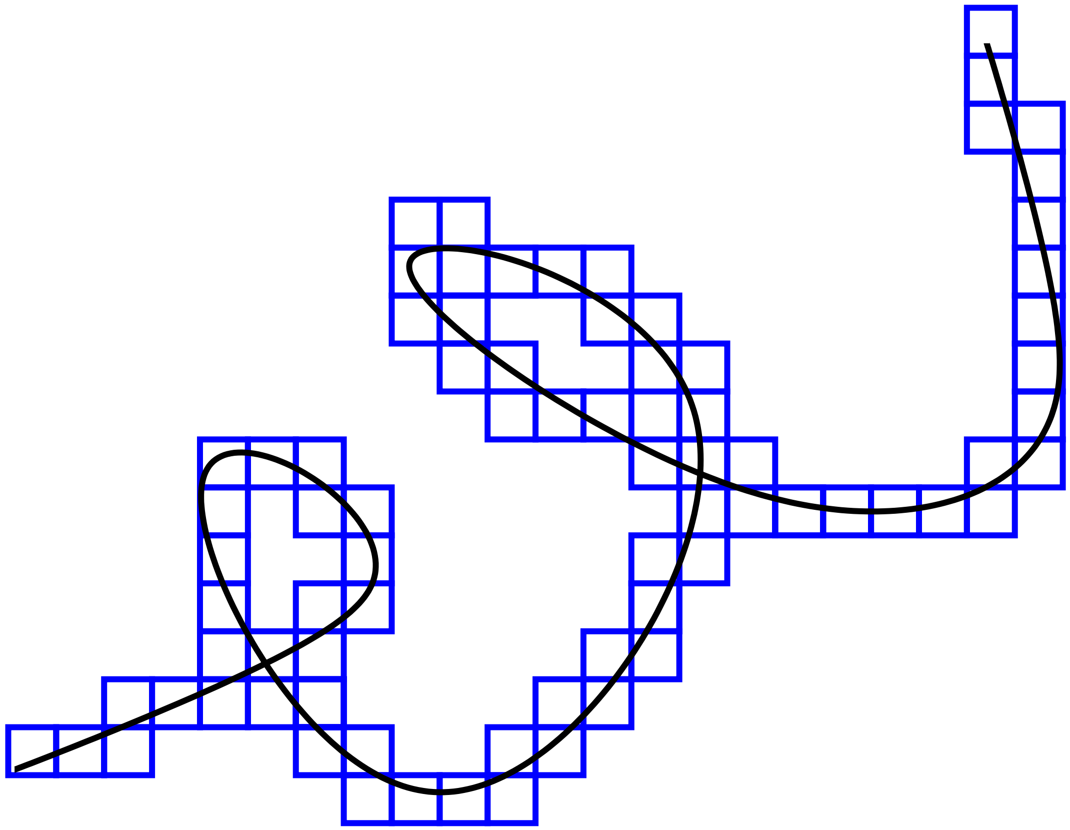


Thm: (B. 2013) Every planar continuum is Hausdorff limit of true trees.



Suffices to approximate subtrees of a grid.



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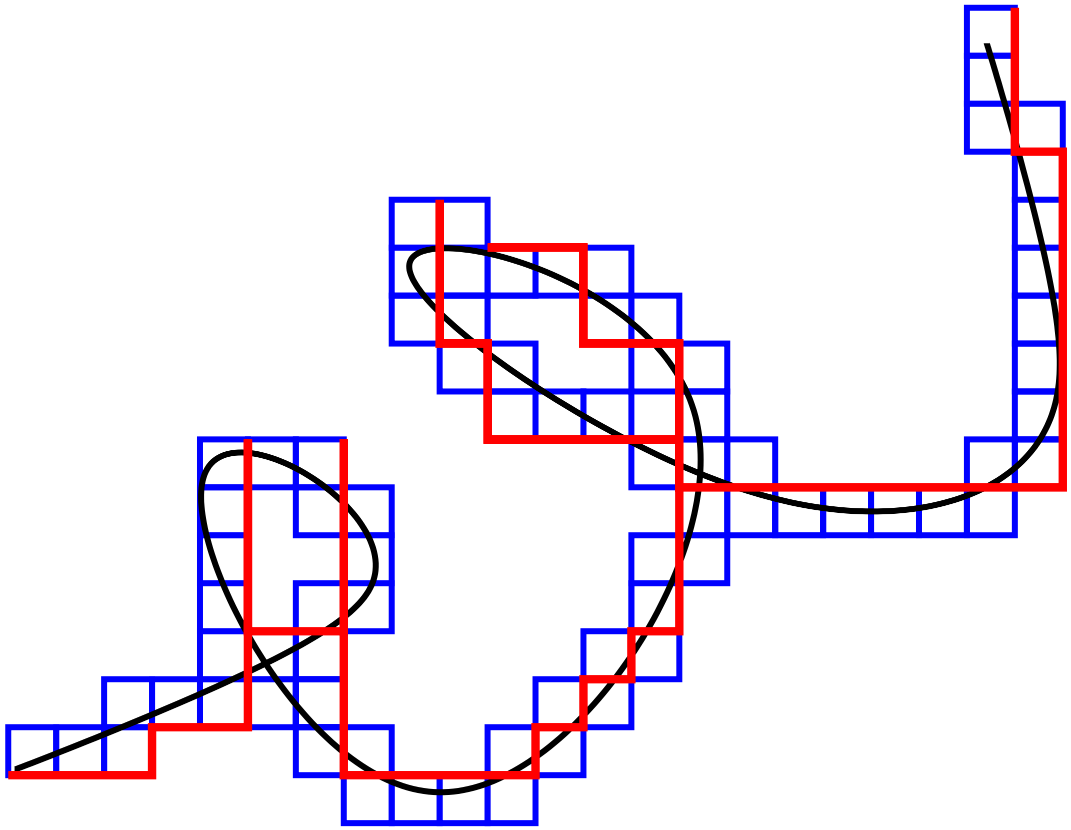

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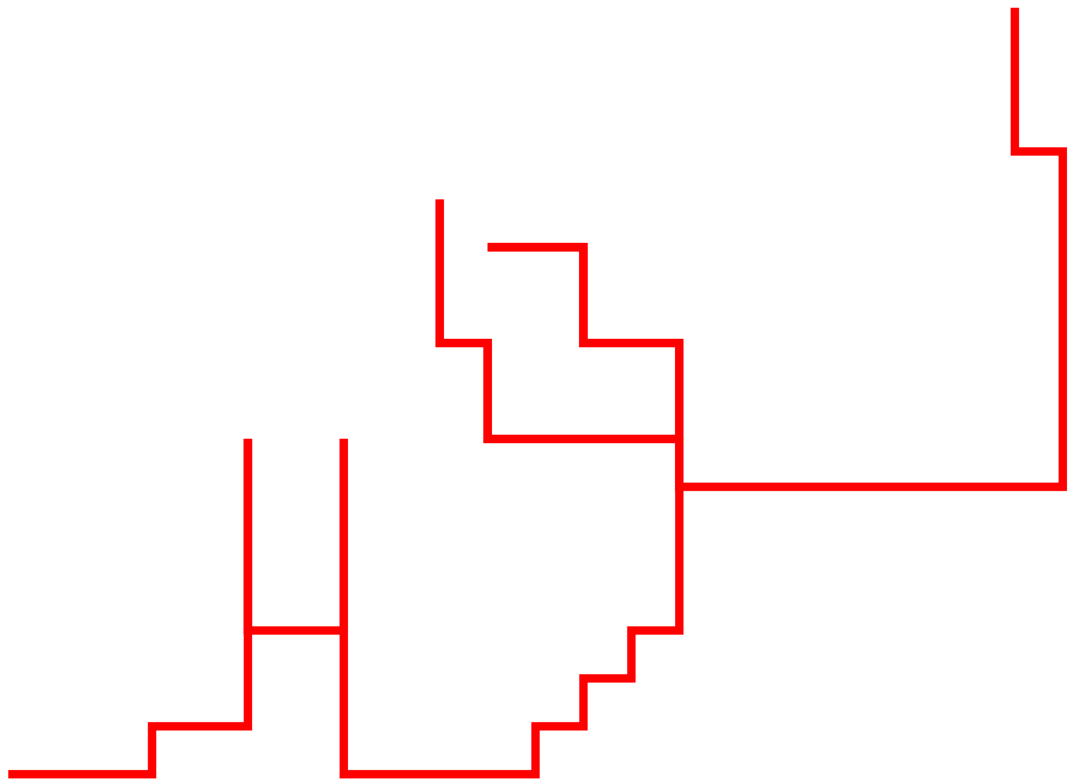
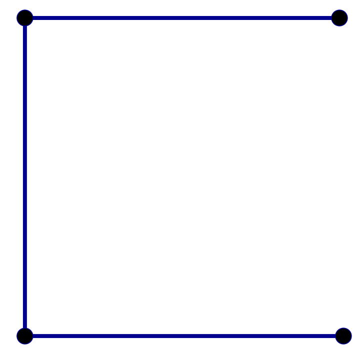


Suffices to approximate subtrees of a grid.

Theorem: Every planar continuum is a limit of true trees.

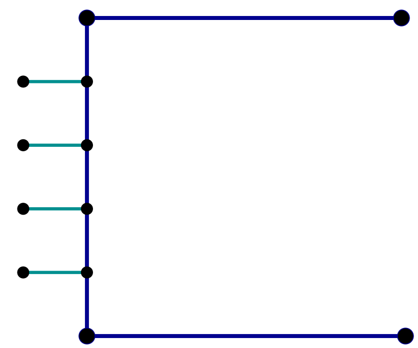
Idea of Proof: reduce harmonic measure ratio by adding edges.



Vertical side has much larger harmonic measure from left.

Theorem: Every planar continuum is a limit of true trees.

Idea of Proof: reduce harmonic measure ratio by adding edges.



"Left" harmonic measure is reduced (roughly 3-to-1).

New edges are approximately balanced (universal constant).

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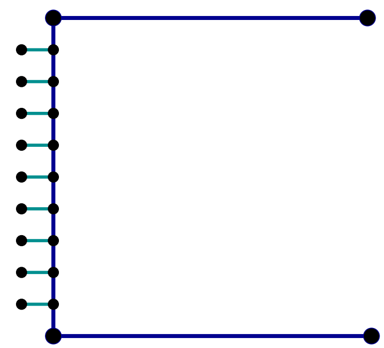


"Left" harmonic measure is reduced (roughly 3-to-1).

New edges are approximately balanced (universal constant).

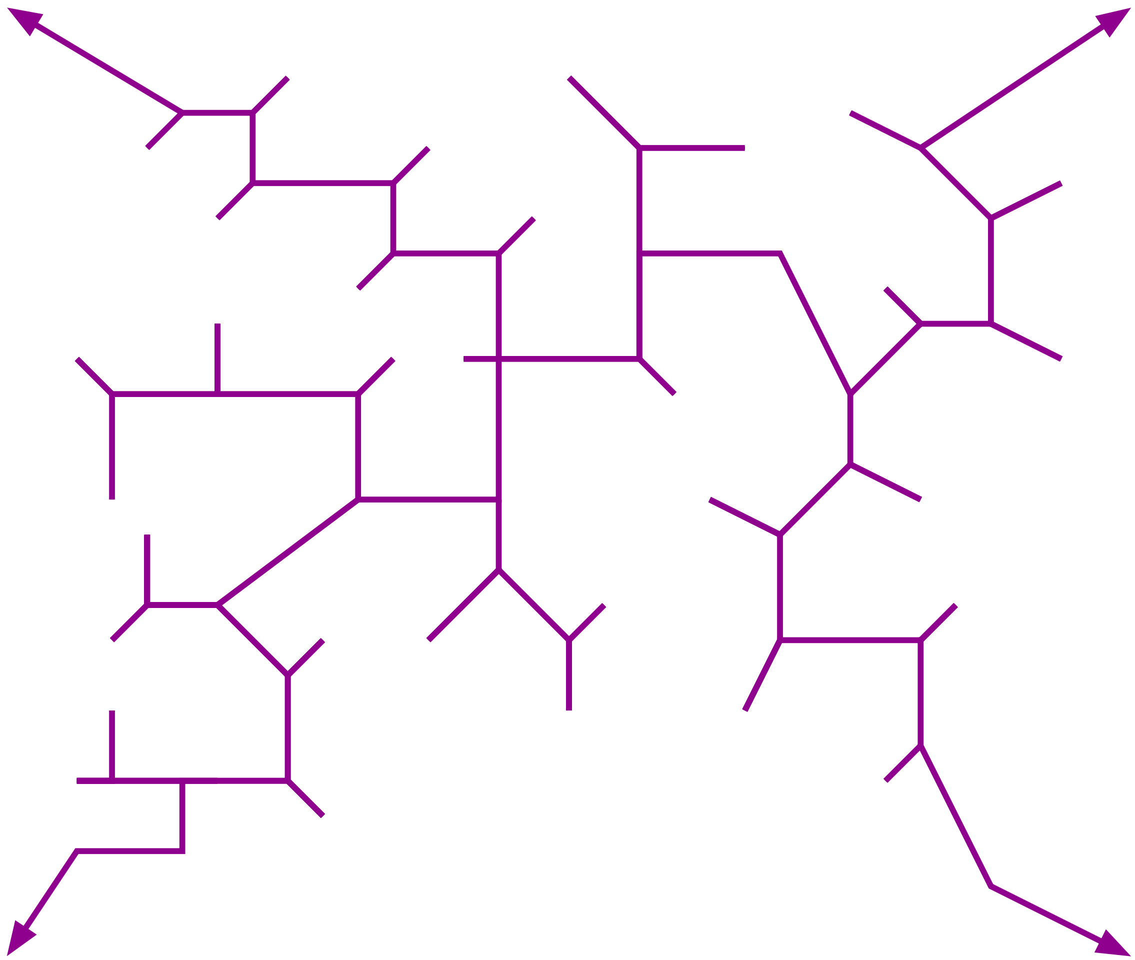
Mapping theorem gives exactly balanced.

QC correction map is near identity if "spikes" are short.

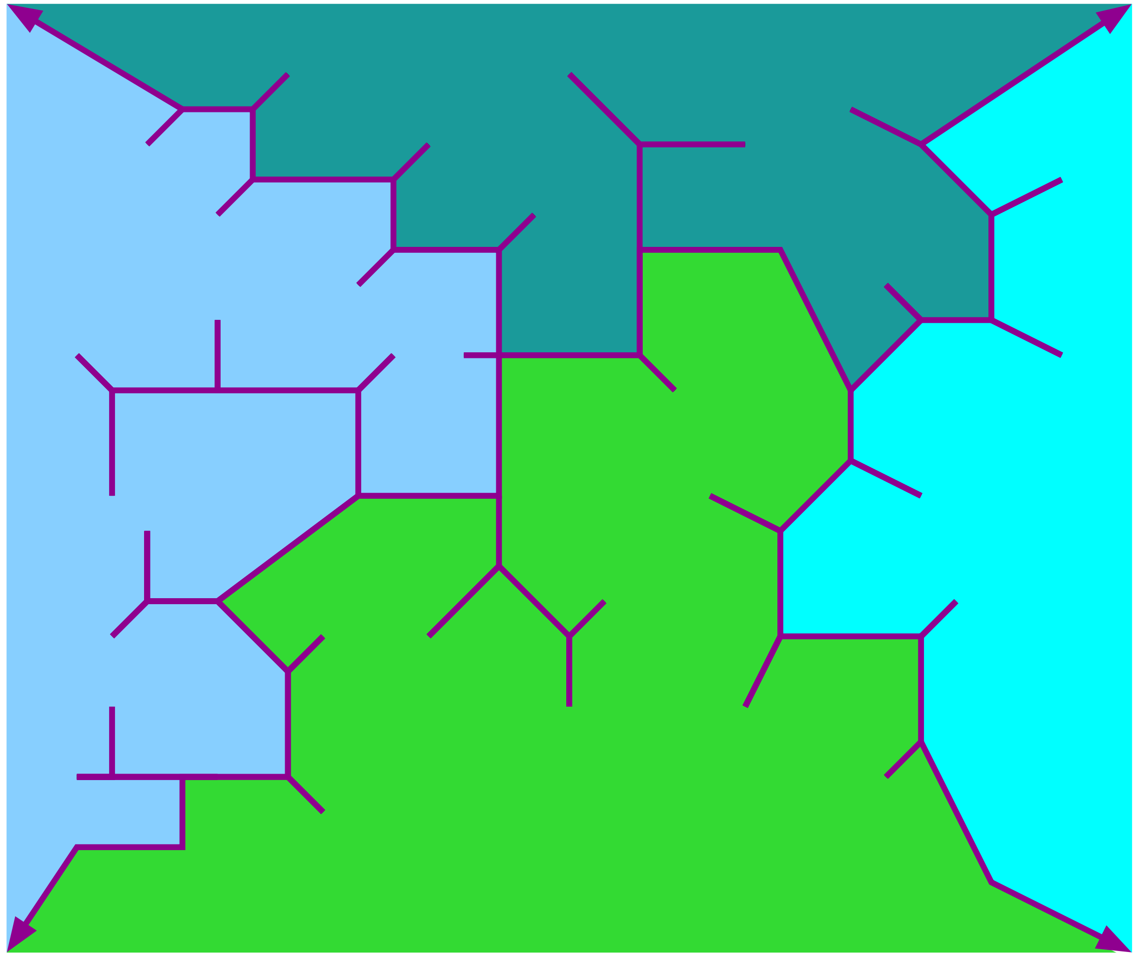
New tree approximates shape of old tree; different combinatorics.

Corollary (B., Pilgrim): Every planar continuum is a limit of Julia sets of polynomials with two post-critical points.

Also see work of Ivrii, Lindsey, Younsi,...



Do infinite trees correspond to entire functions with 2 critical values?

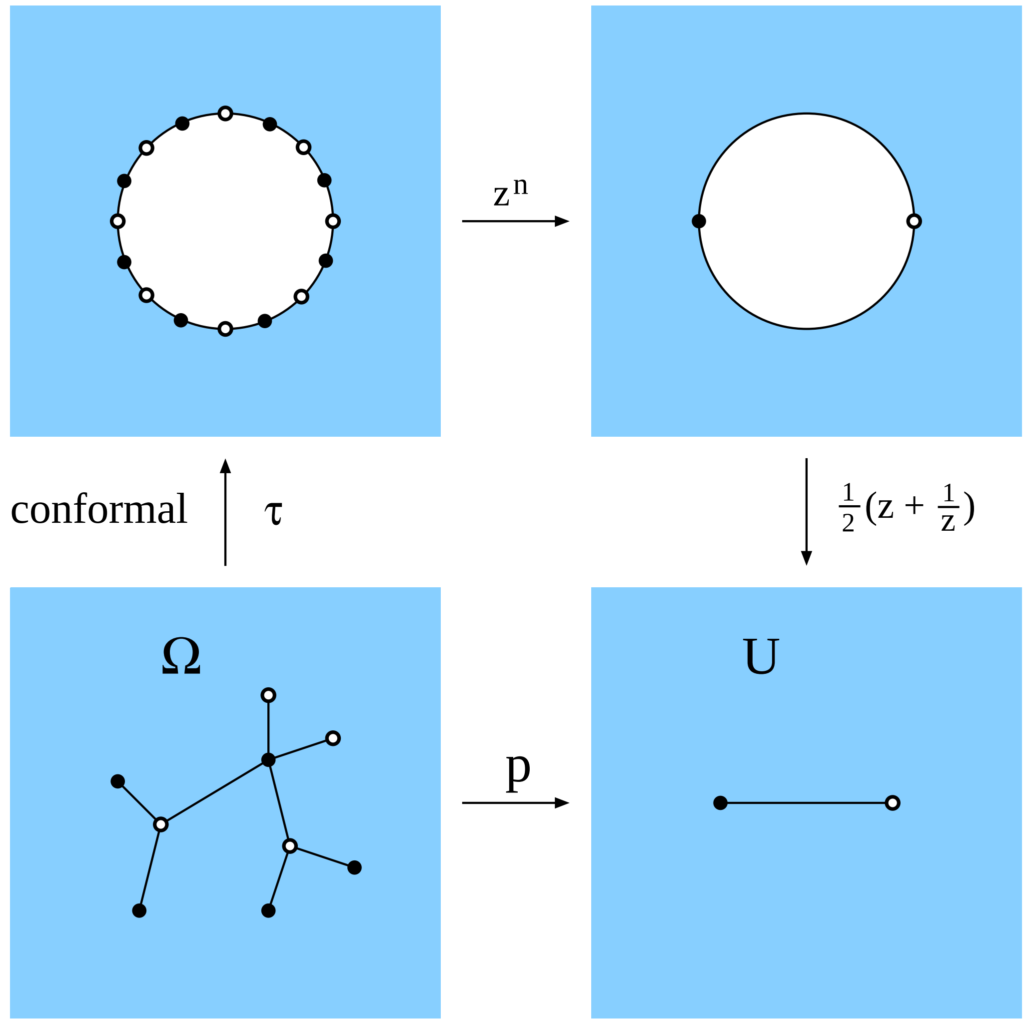


Main difference:

$\mathbb{C}\setminus$ finite tree = one topological annulus

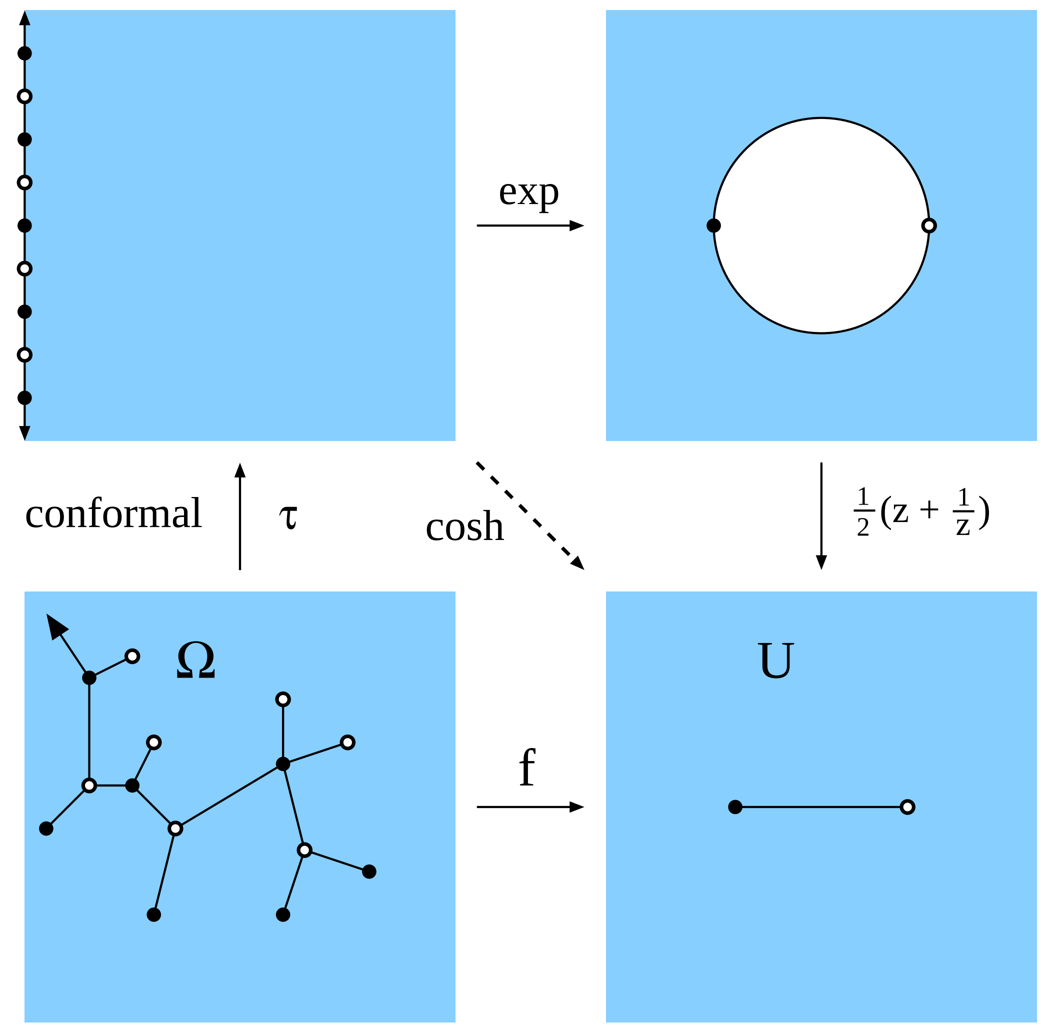
$\mathbb{C}\setminus$ infinite tree = many simply connected components

Recall finite case

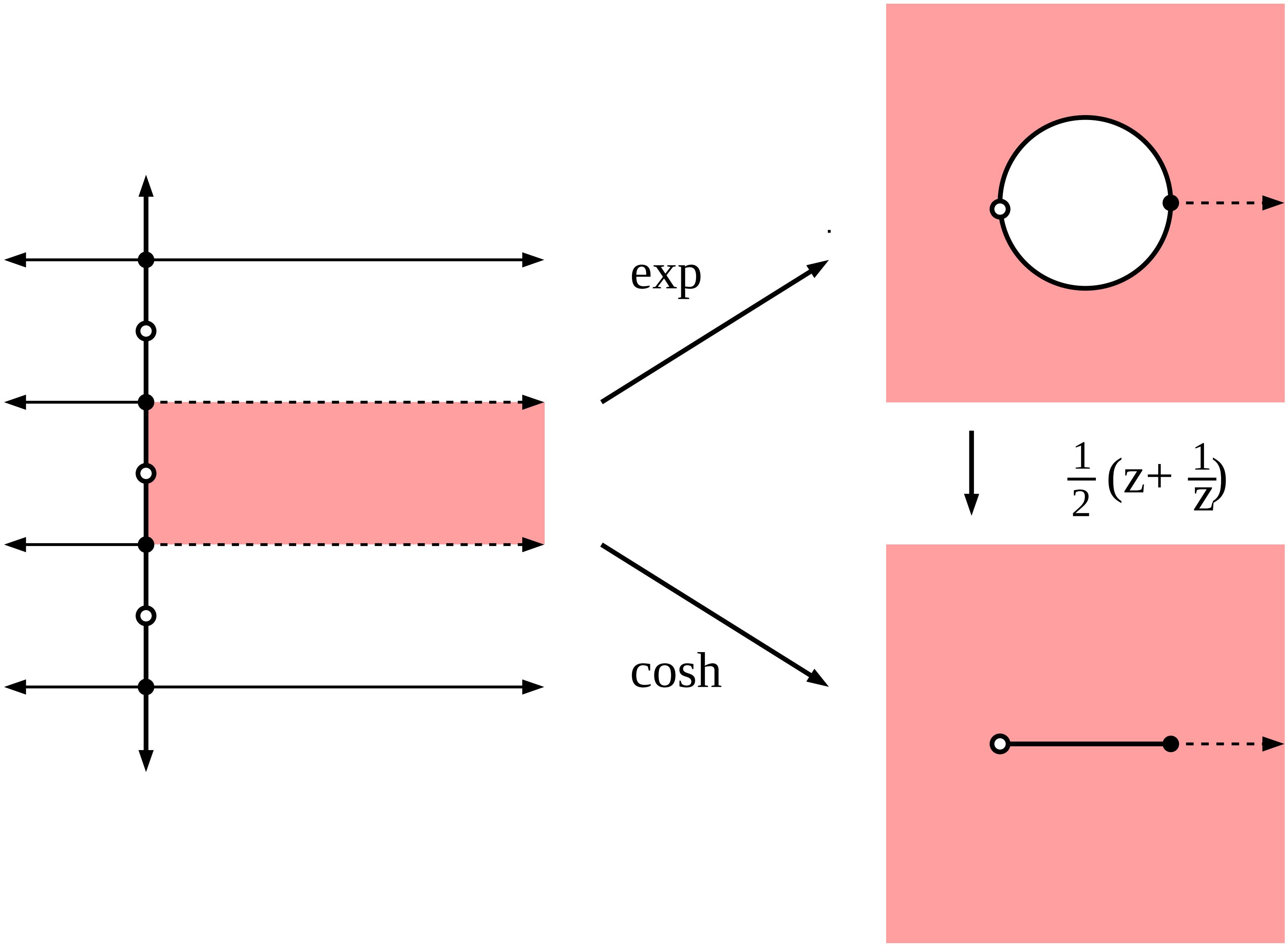


T is true tree $\Leftrightarrow p = \frac{1}{2}(\tau^n + 1/\tau^n)$ is continuous across T.

Infinite case



Infinite balanced tree $\Leftrightarrow$ $f = \cosh \circ \tau$ is continuous across T.



Definitions of exp and cosh.

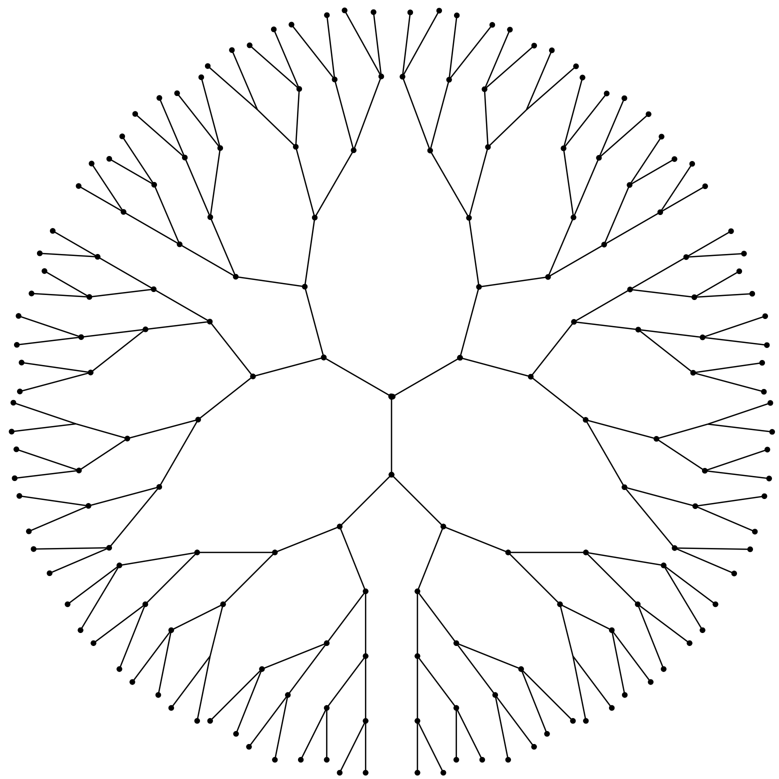
Do all infinite planar trees have true forms?

Do infinite true trees approximate any shape?

Do all infinite planar trees have true forms? **No.**

Do infinite true trees approximate any shape? **Yes (sort of).**

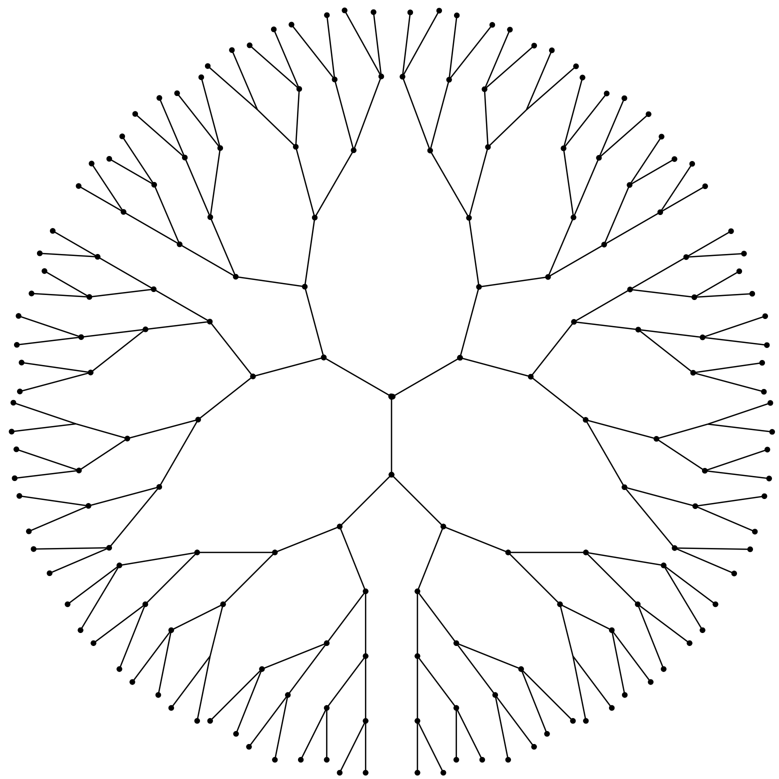
The infinite 3-regular tree has no true form in plane.



Reason 1 (W. Cui, 2018): Corresponding f would contradict a theorem of Nevanlinna: $\sum_{i=1}^{q}(1-\frac{1}{m_i}) \leq 2$, $q=3, m_1=m_2=3, m_3=\infty$.

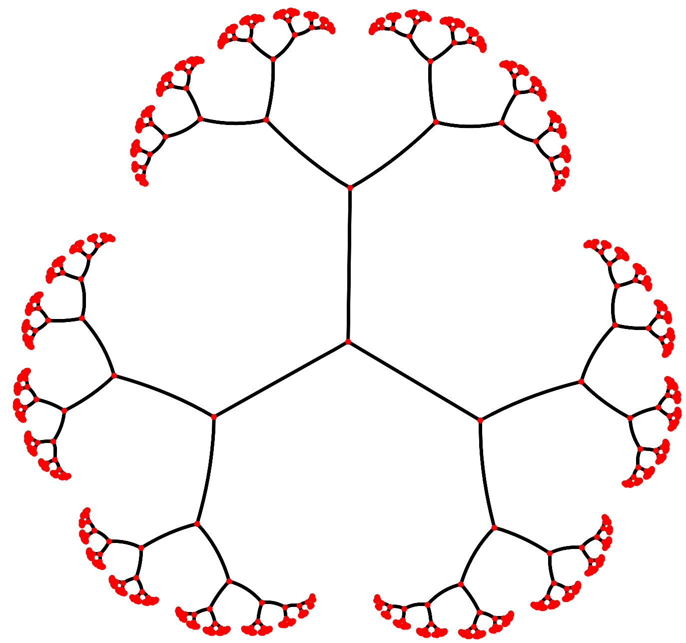
Cui also gives sufficient conditions for an infinite tree to have a true form.

The infinite 3-regular tree has no true form in plane.

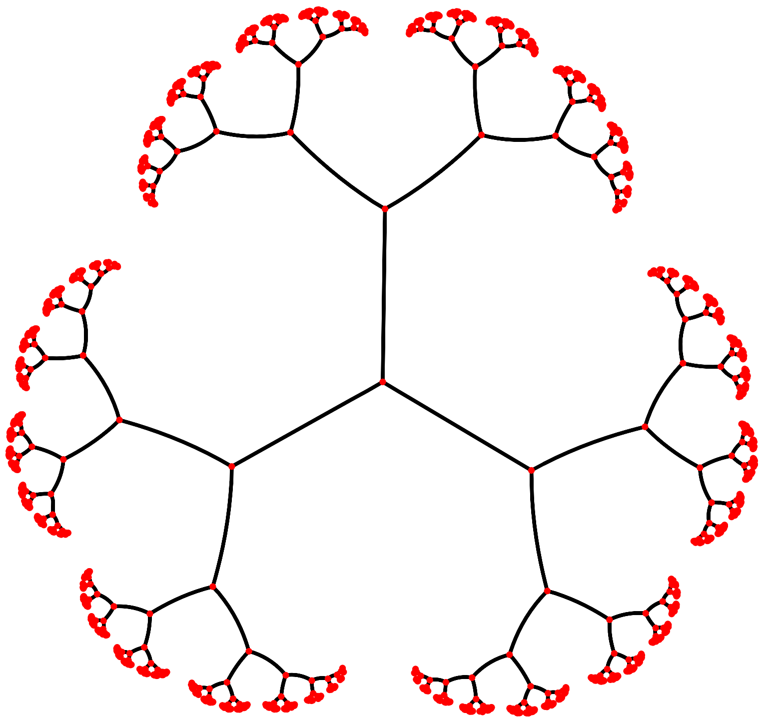
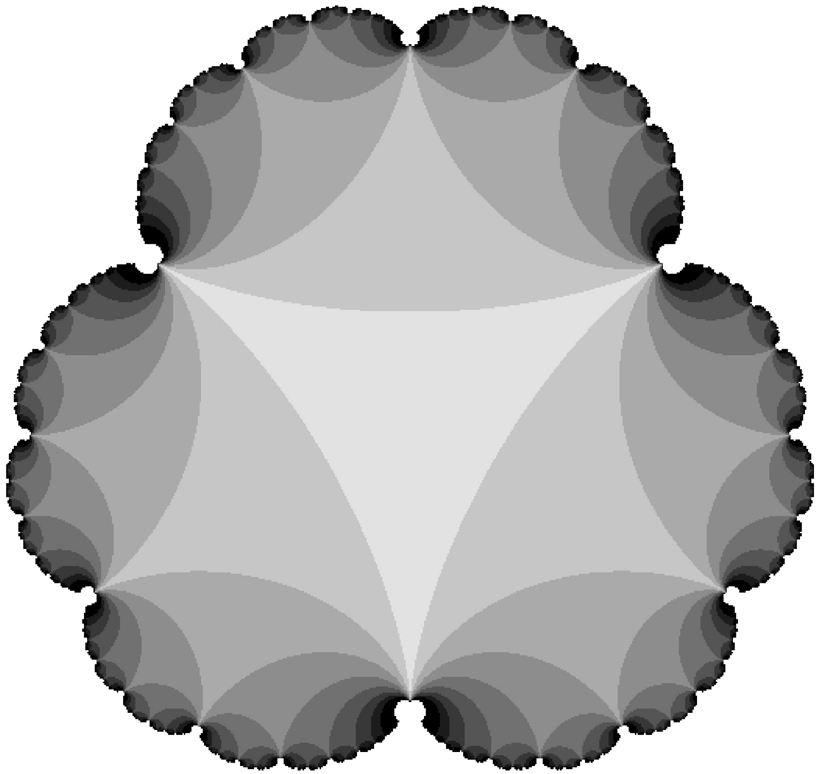


Reason 2: Planar homeomorphisms mapping tree to itself permute the complementary components. Corresponding conformal maps are continuous across tree, hence linear, hence isometries. But this group has exponential growth: “too large” to fit inside isometries of plane.

The infinite 3-regular tree has no true form in plane.



"Reason" 3: Computation of true form of truncated regular tree indicate convergence to bounded set. (Image due to Marshall and Rohde.)

Coincidence of shapes noted by Rohde and Werness.
Deltoid fractal studied by Lee, Lyubich, Makarov, Mukherjee.
Arises in anti-holomorphic dynamics.

Now, the second question: which shapes can occur?

We will introduce two assumptions that substitute for finiteness, but first we define a certain neighborhood $T(r)$ of an infinite tree.

(Replaces Hausdorff ϵ-neighborhood in finite case.)

If e is an edge of T and $r > 0$ let

$$e(r) = \{z : \mathrm{dist}(z, e) \le r \cdot \mathrm{diam}(e)\}$$

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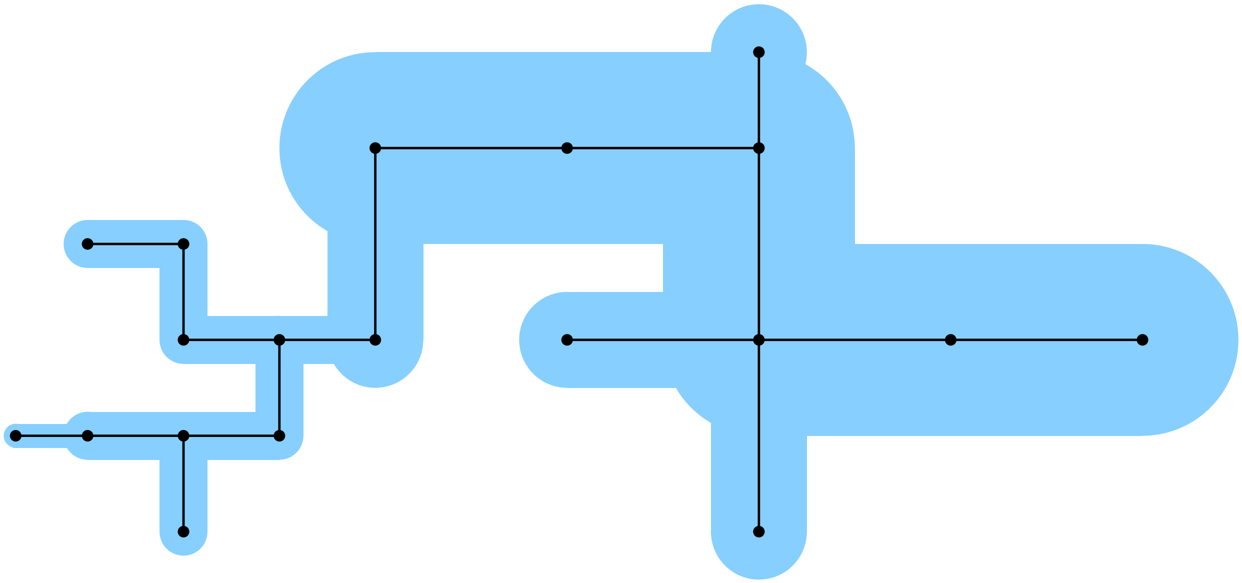
Define neighborhood of T: $T(r) = \cup\{e(r) : e \in T\}$.



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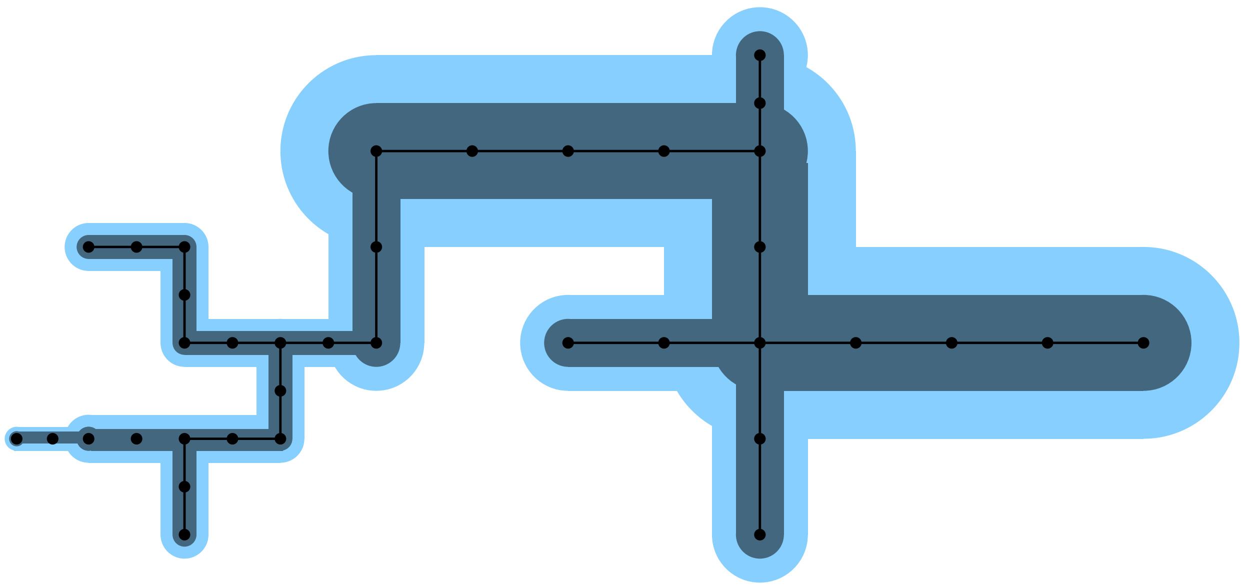
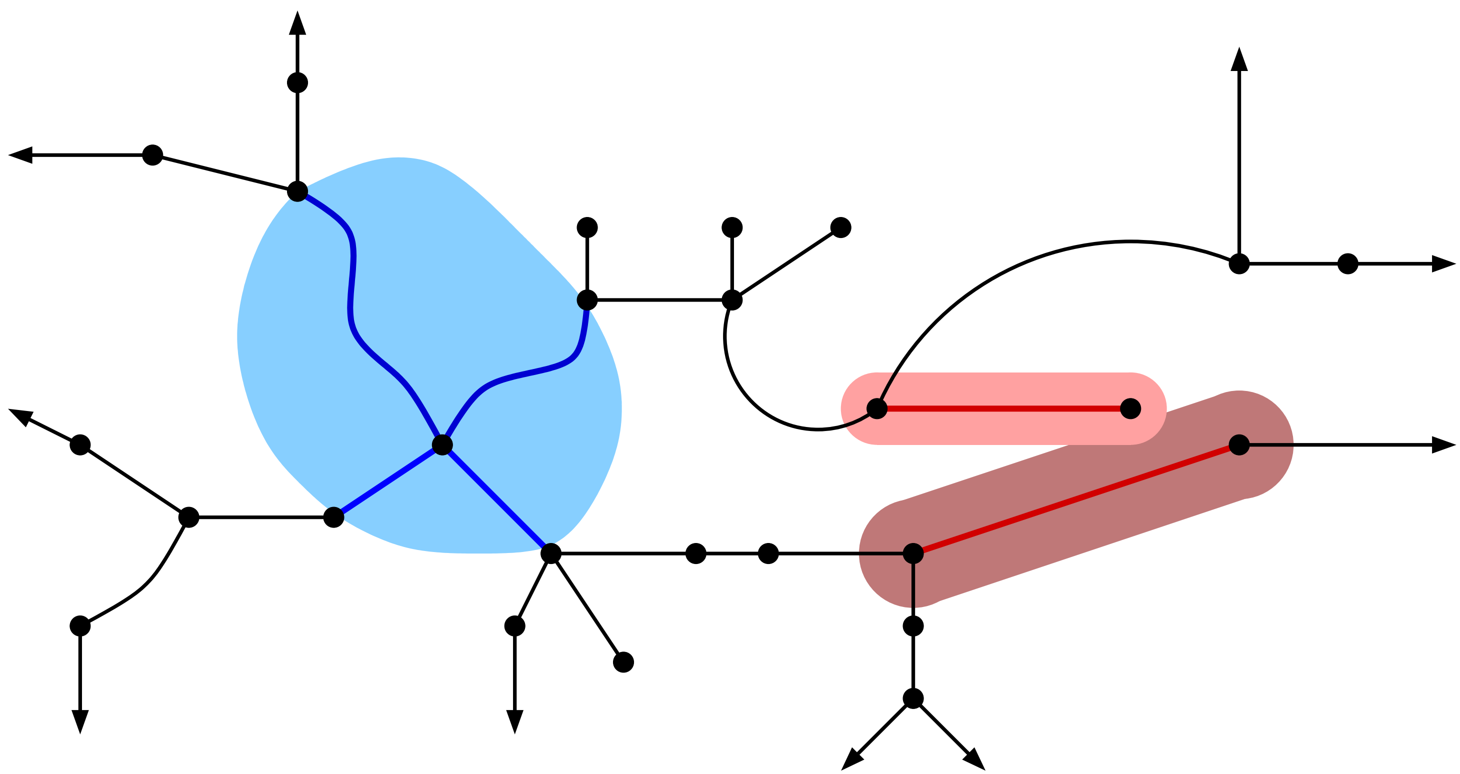


Adding vertices reduces $T(r)$. Useful scaling property.

(1) Bounded Geometry (local condition; easy to verify):

- edges are uniformly smooth.
- adjacent edges form bi-Lipschitz image of a star $= \{z^n \in [0, r]\}$
- non-adjacent edges are well separated,

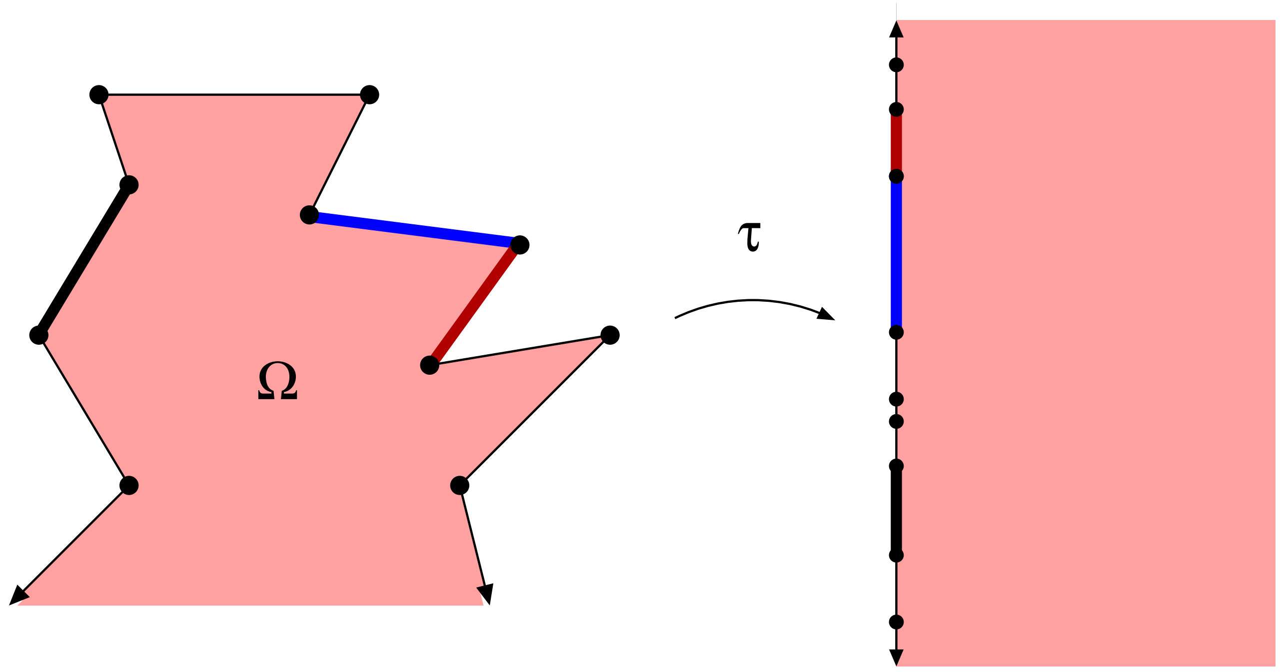
$$\mathrm{dist}(e, f) \geq \epsilon \cdot \min(\mathrm{diam}(e), \mathrm{diam}(f)).$$



(2) τ-Lower Bound (global condition; harder to check):

Complementary components of tree are simply connected.

Each can be conformally mapped to right half-plane. Call map τ.

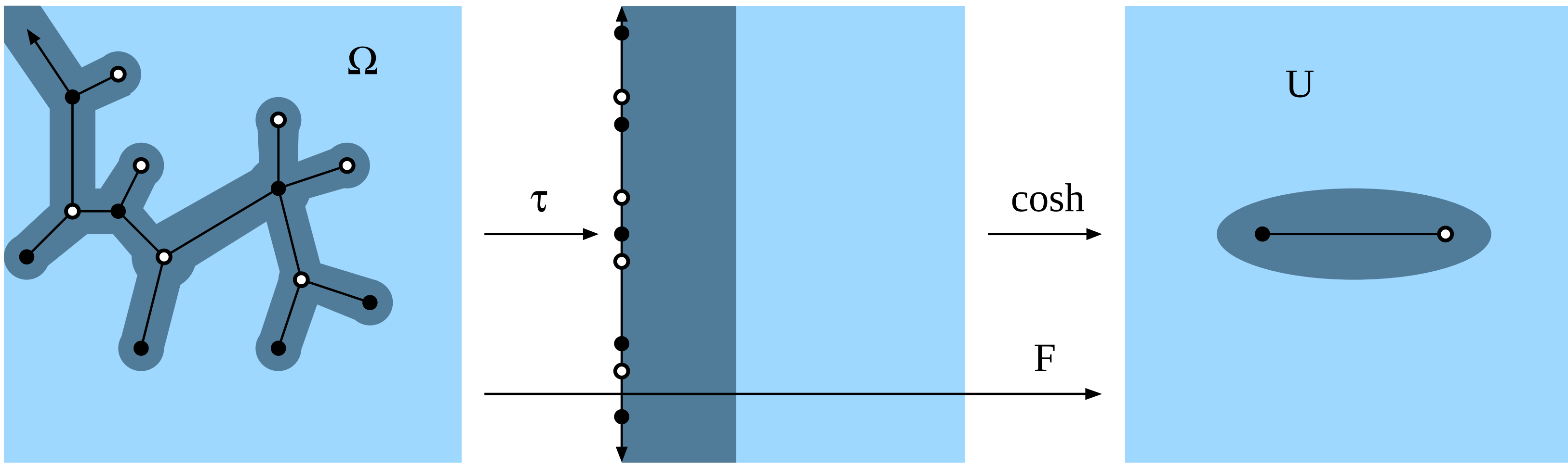


We assume all images have length $\geq \pi$.

Need positive lower bound; actual value usually not important.

Components are "thinner" than half-plane near ∞.

QC-Folding Theorem (B 2015): If T has bounded geometry and the τ-lower bound, then T can be approximated by a true tree in the following sense. Let $F = \cosh \circ \tau$. Then there is a K-quasiregular g and $r > 0$ such that $g = F$ off $T(r)$ (shaded) and $\mathrm{CV}(g) = \pm 1$.

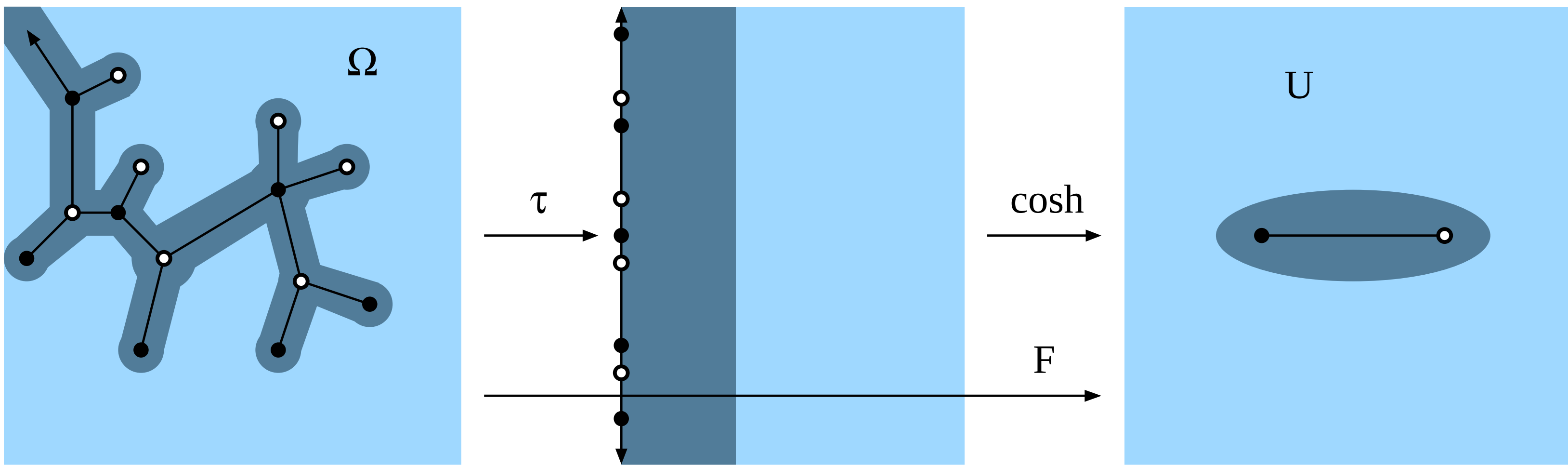


K and r only depend on the bounded geometry constants.

$g = F$ on light blue.

F may be discontinuous across T. g is continuous everywhere.

QC-Folding Theorem (B 2015): If T has bounded geometry and the τ-lower bound, then T can be approximated by a true tree in the following sense. Let $F = \cosh \circ \tau$. Then there is a K-quasiregular g and $r > 0$ such that $g = F$ off $T(r)$ (shaded) and $\mathrm{CV}(g) = \pm 1$.

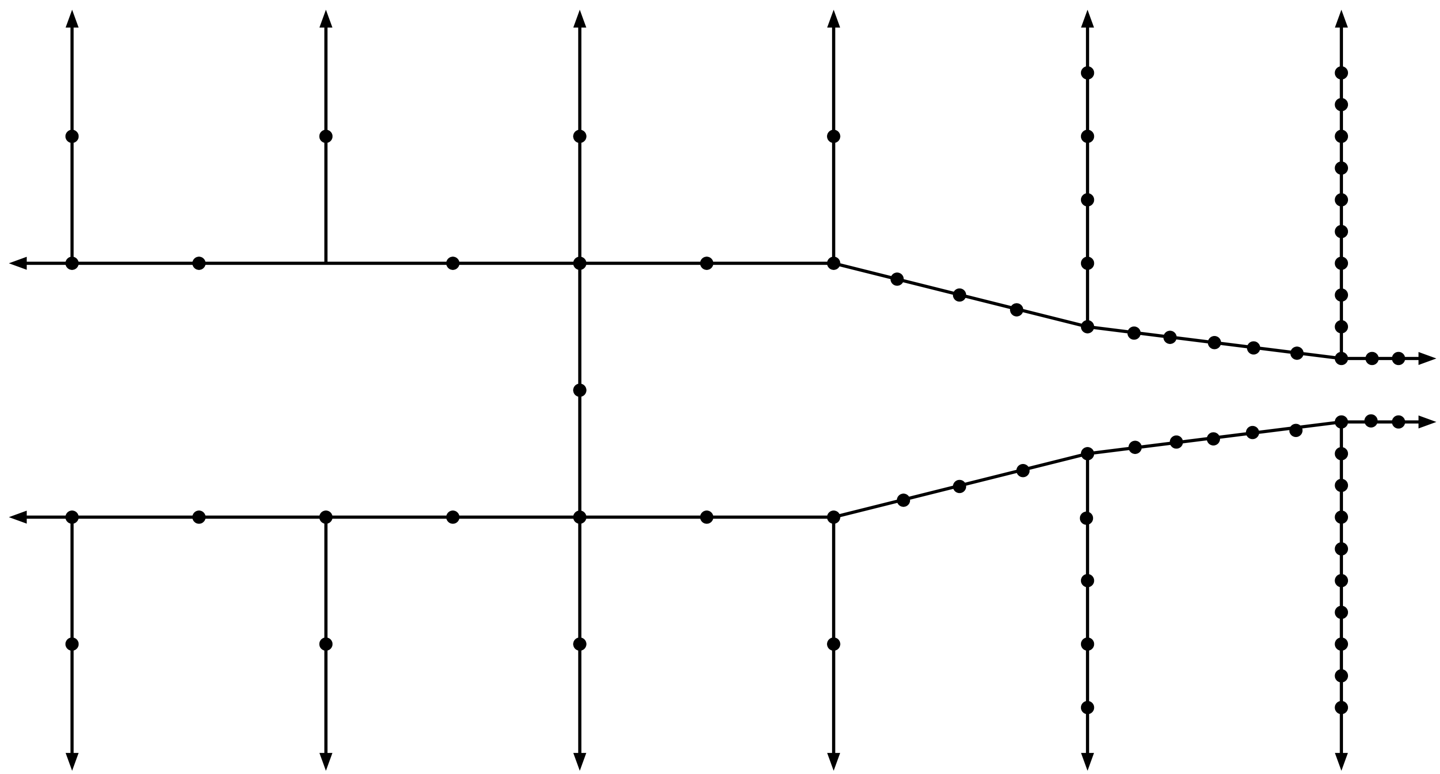


K and r only depend on the bounded geometry constants.

Cor: Any T as above is approximated by $\varphi(T')$ where $T' = f^{-1}([-1, 1])$, f is entire with $\mathrm{CV}(f) = \pm 1$, φ is QC and conformal off $T(r)$.
(In many applications, φ is close to identity, so $T \approx T'$.)

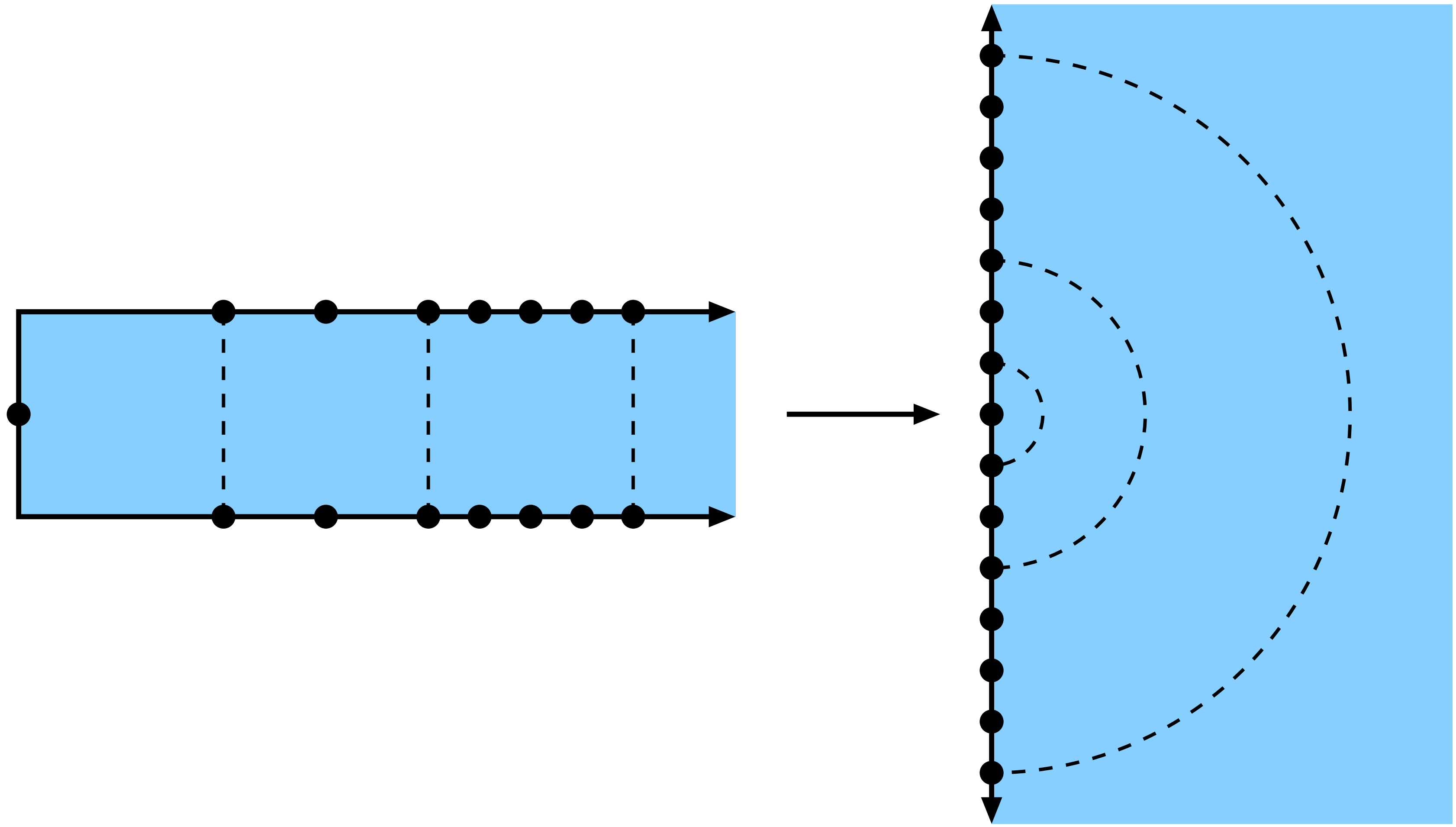
First, a sample application.

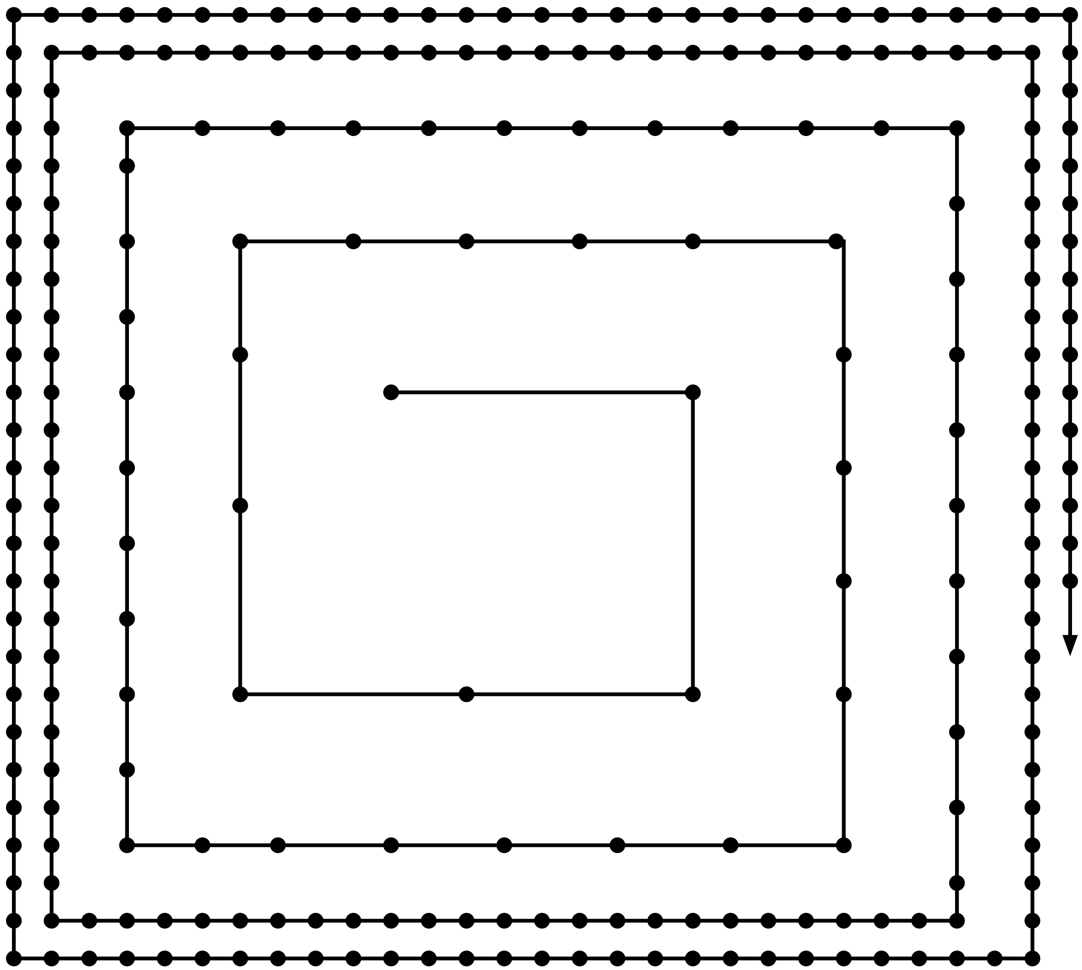
Rapid increase in Speiser class



f has 2 singular values, $f(x) \nearrow \infty$ as fast as we wish.

Similar examples due to Sergei Merenkov 2008 (3 singular values).



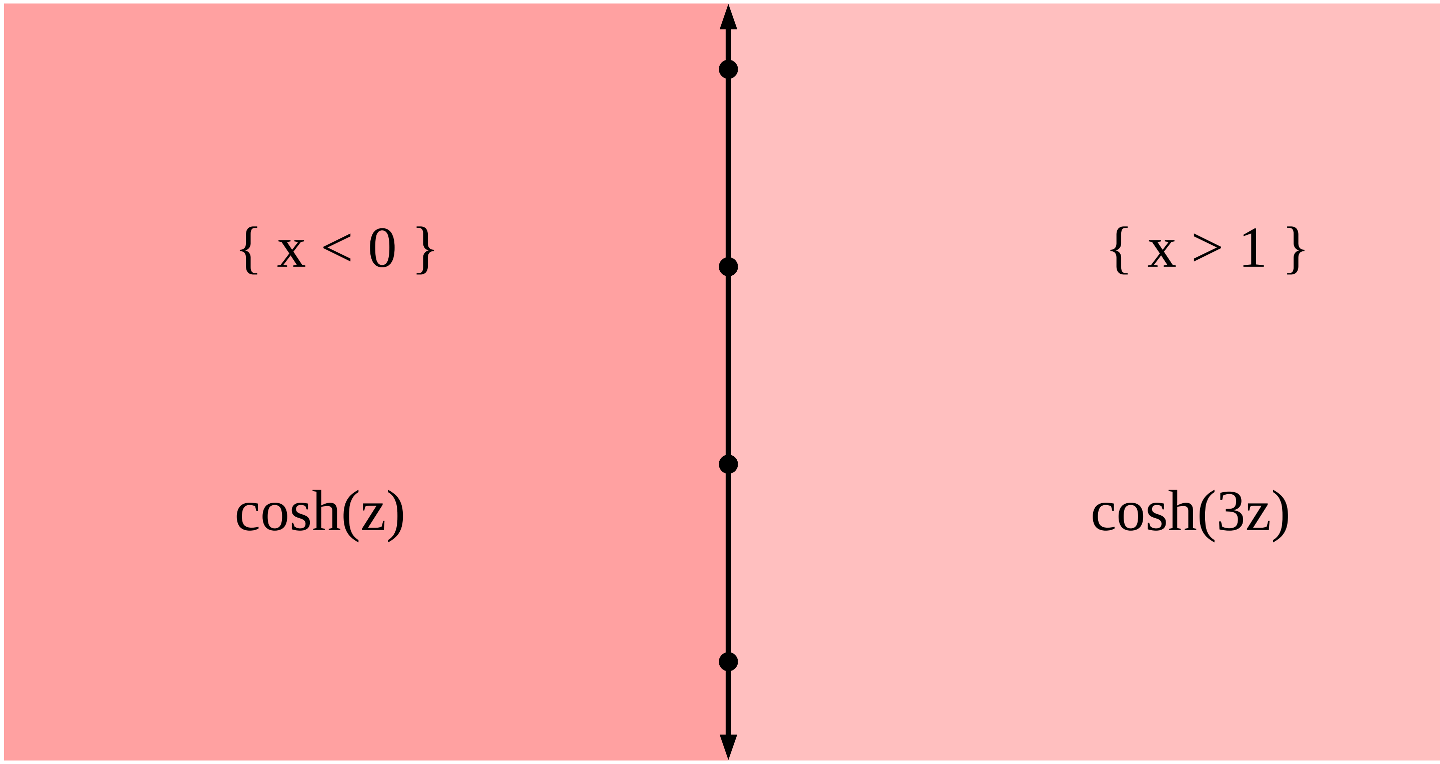


Speiser example with single tract; rapid growth and spiraling.
Previous examples had logarithmic spirals.

Next, the idea of the proof of folding in a simple case.

The tree is a vertical line with even spaced vertices.

The τ map are different linear maps on each side.

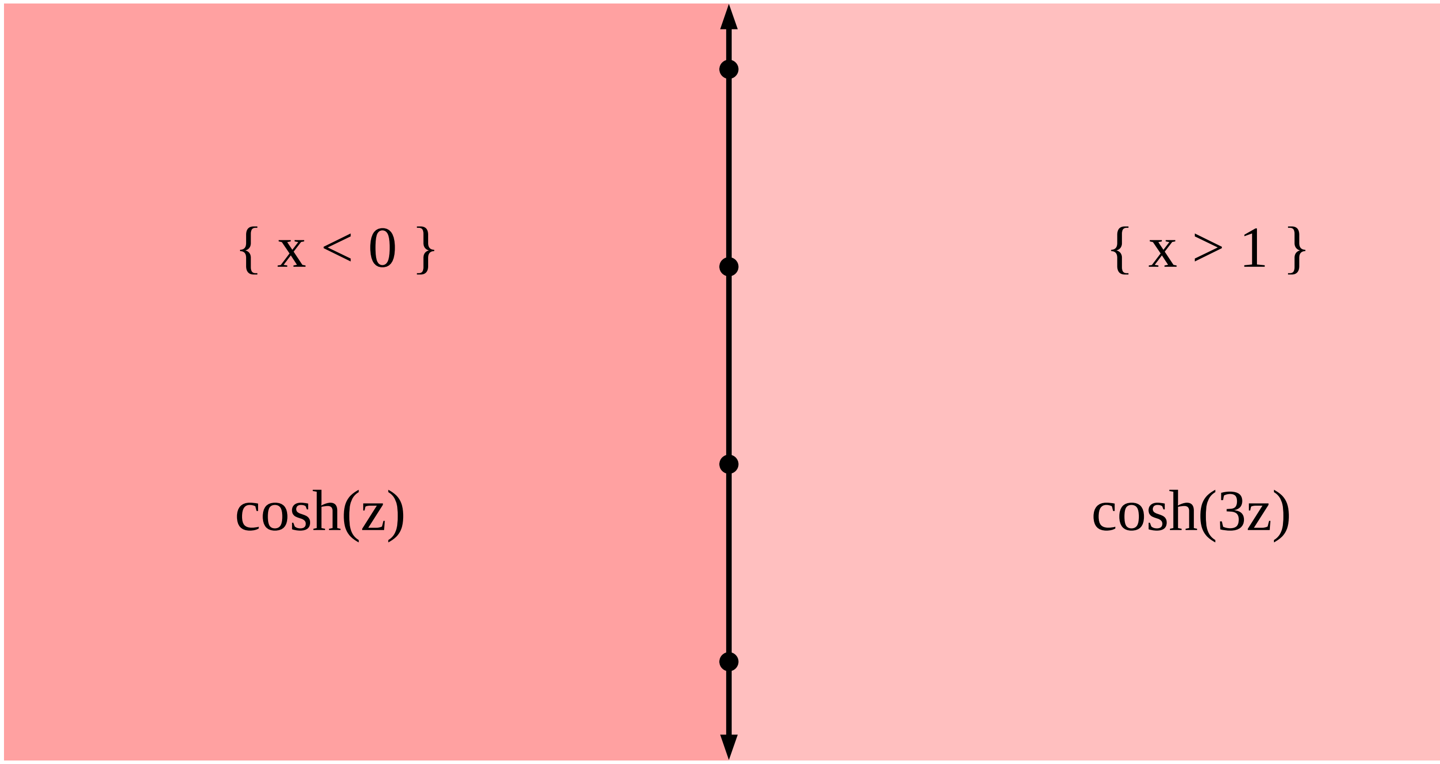


Construct an entire function F so that

$$F(z) = \cosh(z) \text{ on left half-plane } = \mathbb{H}_l = \{z = x + iy : x < 0\}$$

$$F(z) = \cosh(3z) \text{ on right half-plane } = \mathbb{H}_r = \{z = x + iy : x > 0\}$$

$\tau(z) = z$ on left, $\tau(z) = 3z$ on right.



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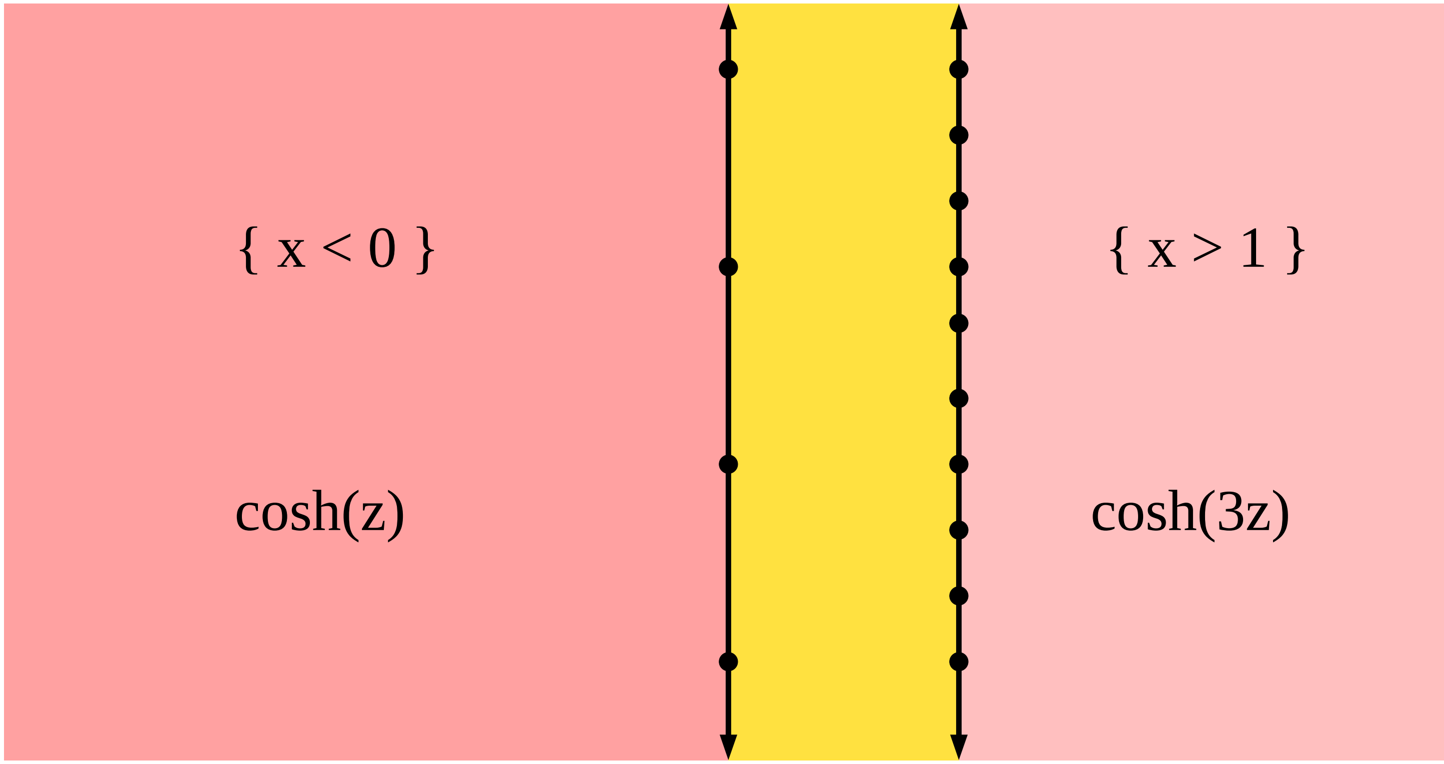
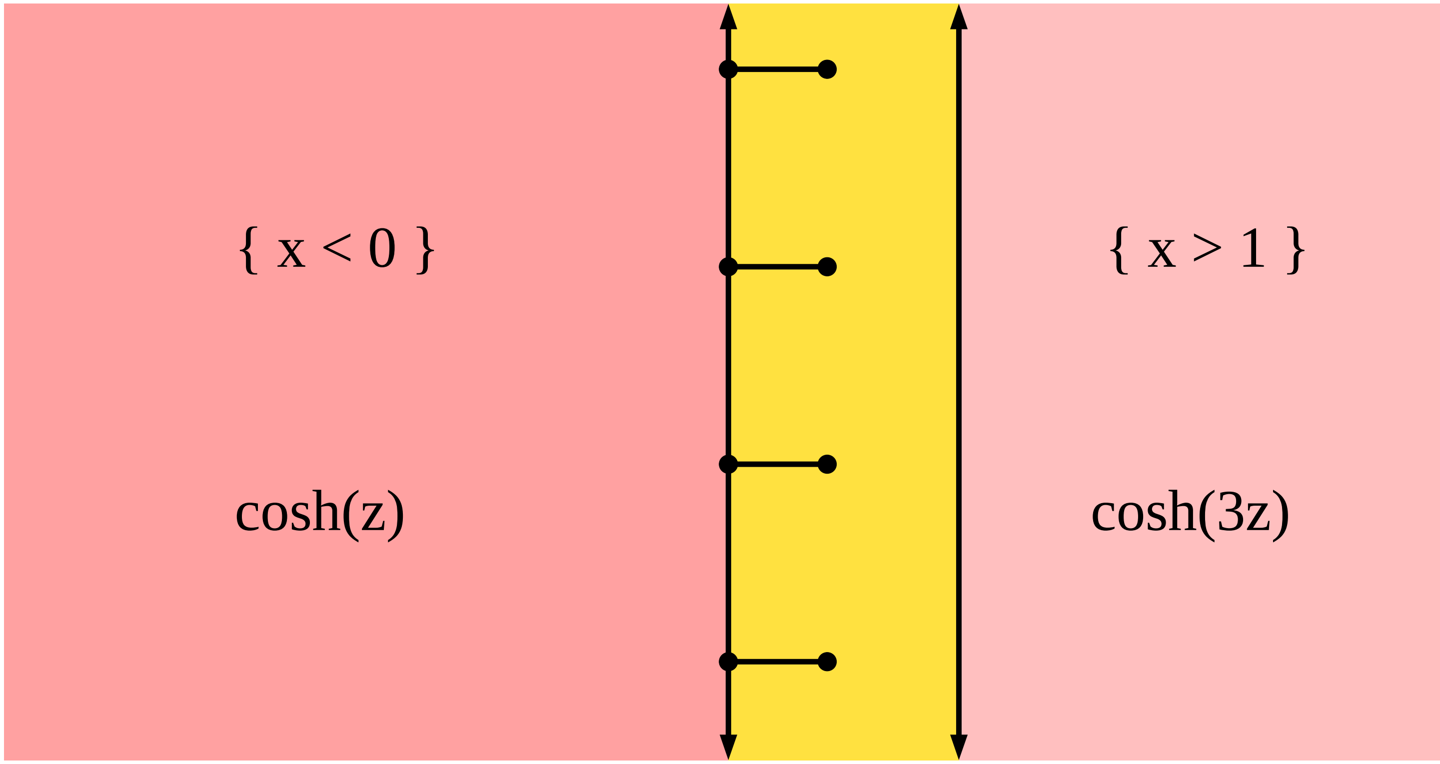
Impossible: violates unique continuation



Construct an **quasiregular** function F so that

$$F(z) = \cosh(z) \text{ on } \{x < 0\}$$

$$F(z) = \cosh(3z) \text{ on } \{x > 1\}$$



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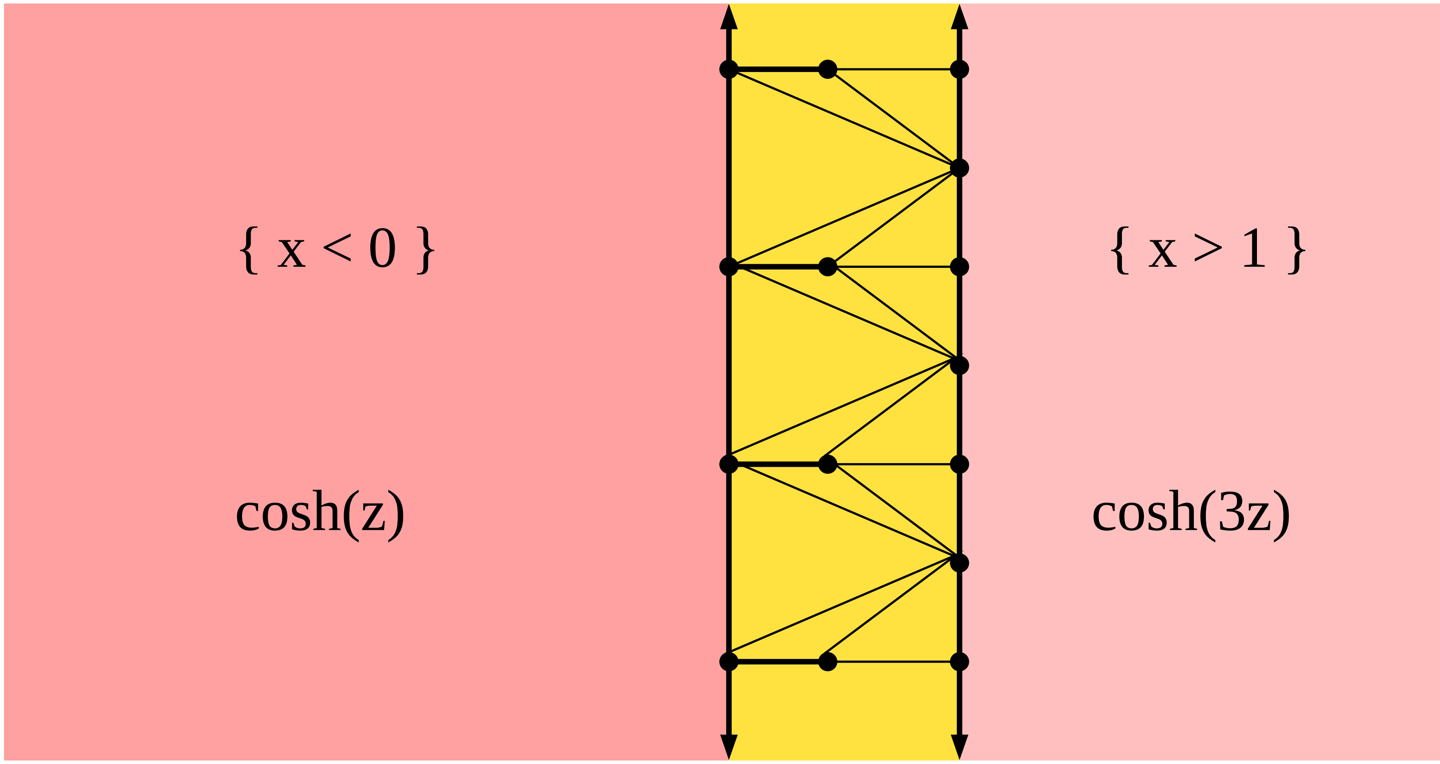
This is easy. First add some spikes in the strip.



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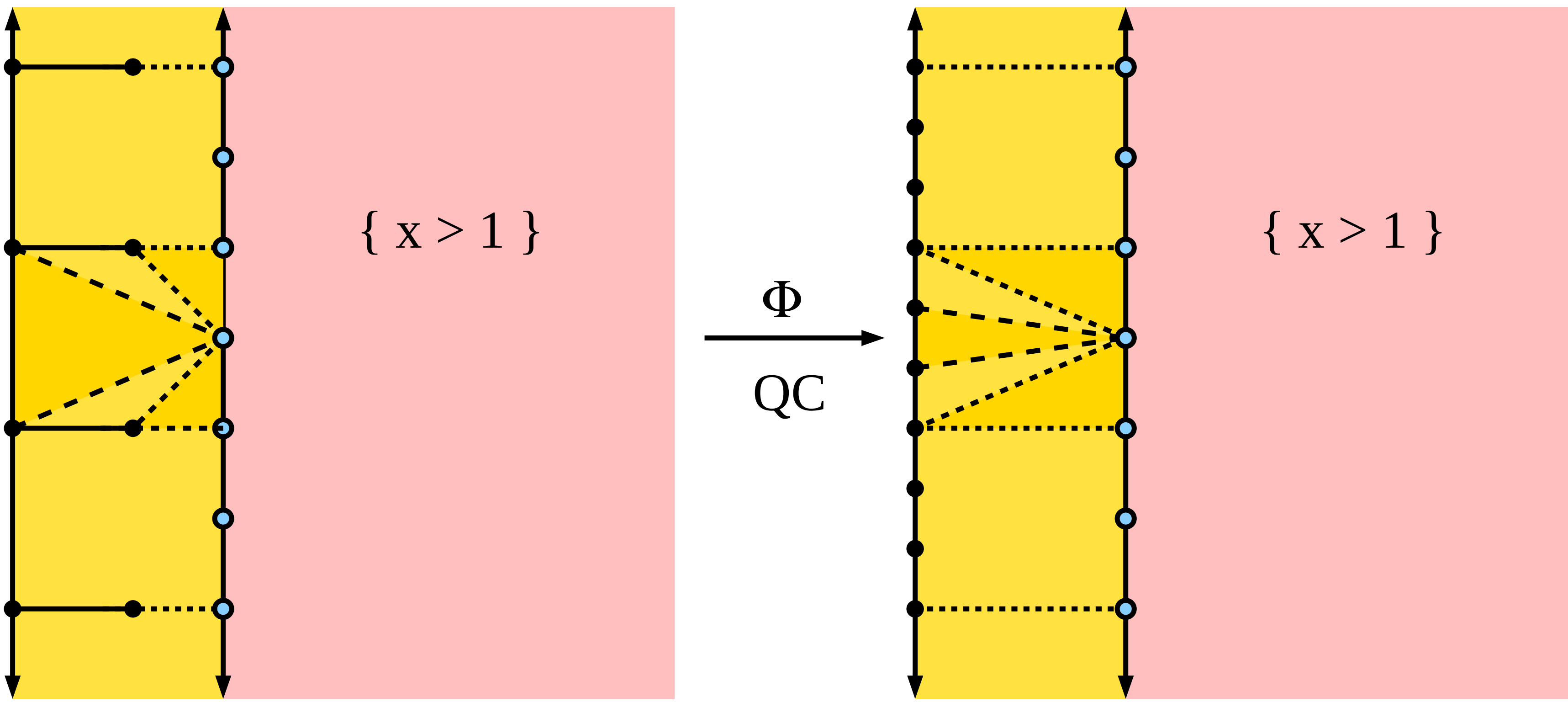
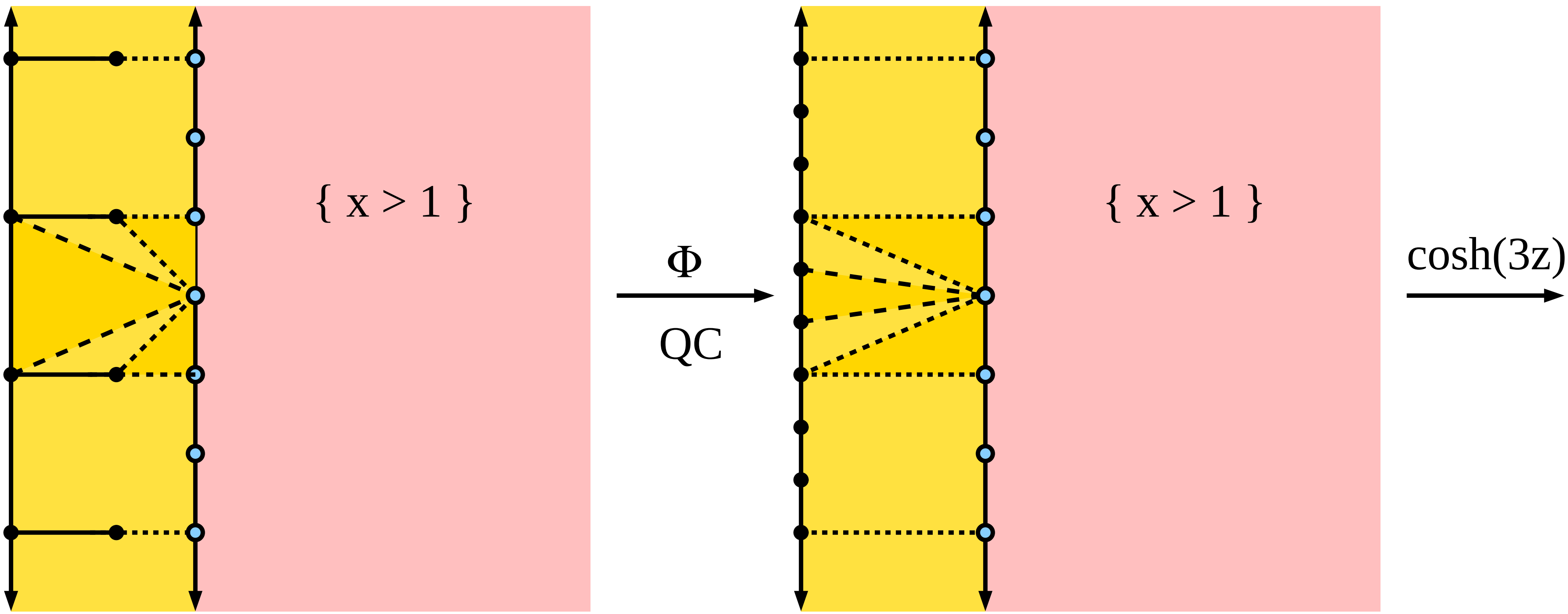
$$F(z) = \cosh(3z) \text{ on } \{x > 1\}$$

Triangulate the folded strip.



Let Φ be piecewise affine map: folded strip $\rightarrow$ un-folded strip.

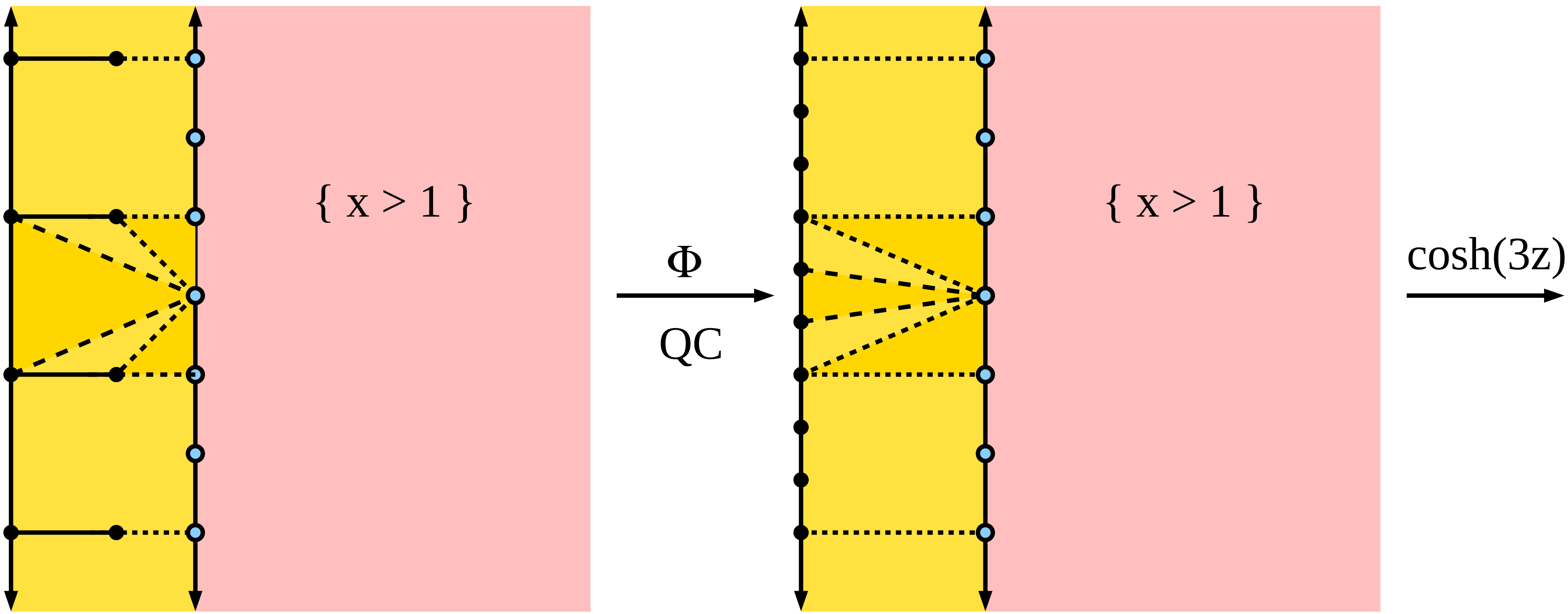
Take $\Phi =$ identity map off the strip.



$G(z) = \cosh(3 \cdot \Phi(z))$ satisfies:

- equals $\cosh 3z$ on $\{x \geq 1\}$
- equals $\cosh z$ on $\{x = 0\}$
- Is quasiregular (composition of QC and holomorphic)

Measurable Riemann mapping theorem gives entire $F = G \circ \varphi$.



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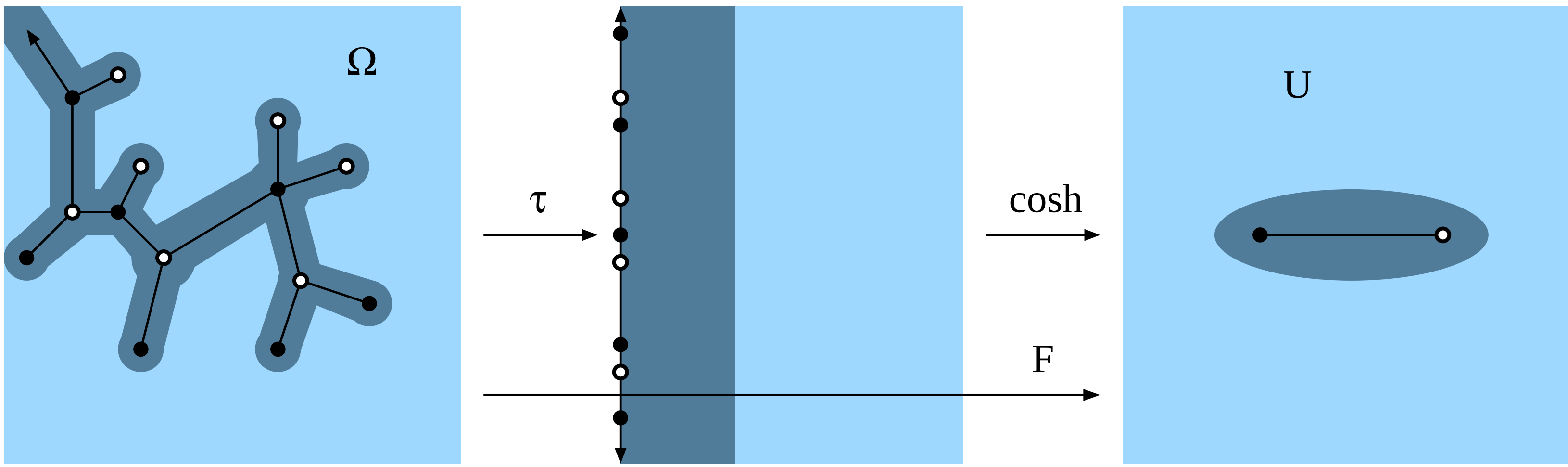
- equals $\cosh 3z$ on $\{x \geq 1\}$
- equals $\cosh z$ on $\{x = 0\}$
- Is quasiregular (composition of QC and holomorphic)

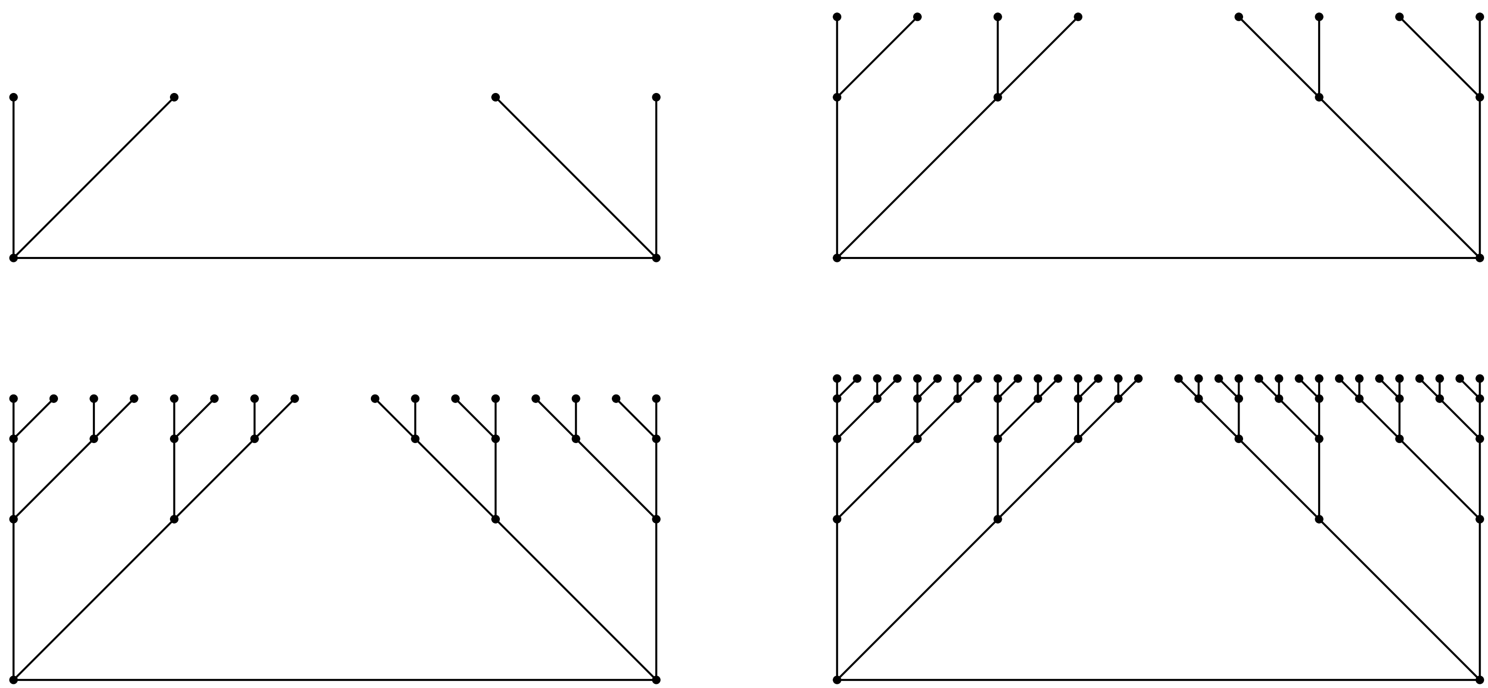
Can replace 3, by any n and keep QC constant bounded.

QC-Folding Theorem: If T has bounded geometry and the τ-lower bound, then there is a K-quasiregular g and $r > 0$ such that $g = \cosh \circ \tau$ off $T(r)$ (shaded) and $\mathrm{CV}(g) = \pm 1$.

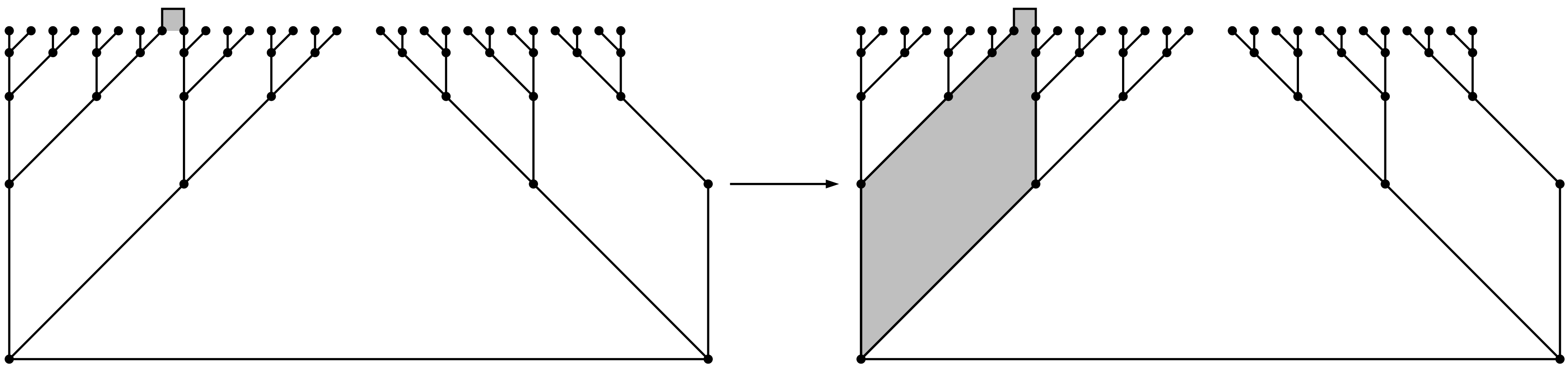


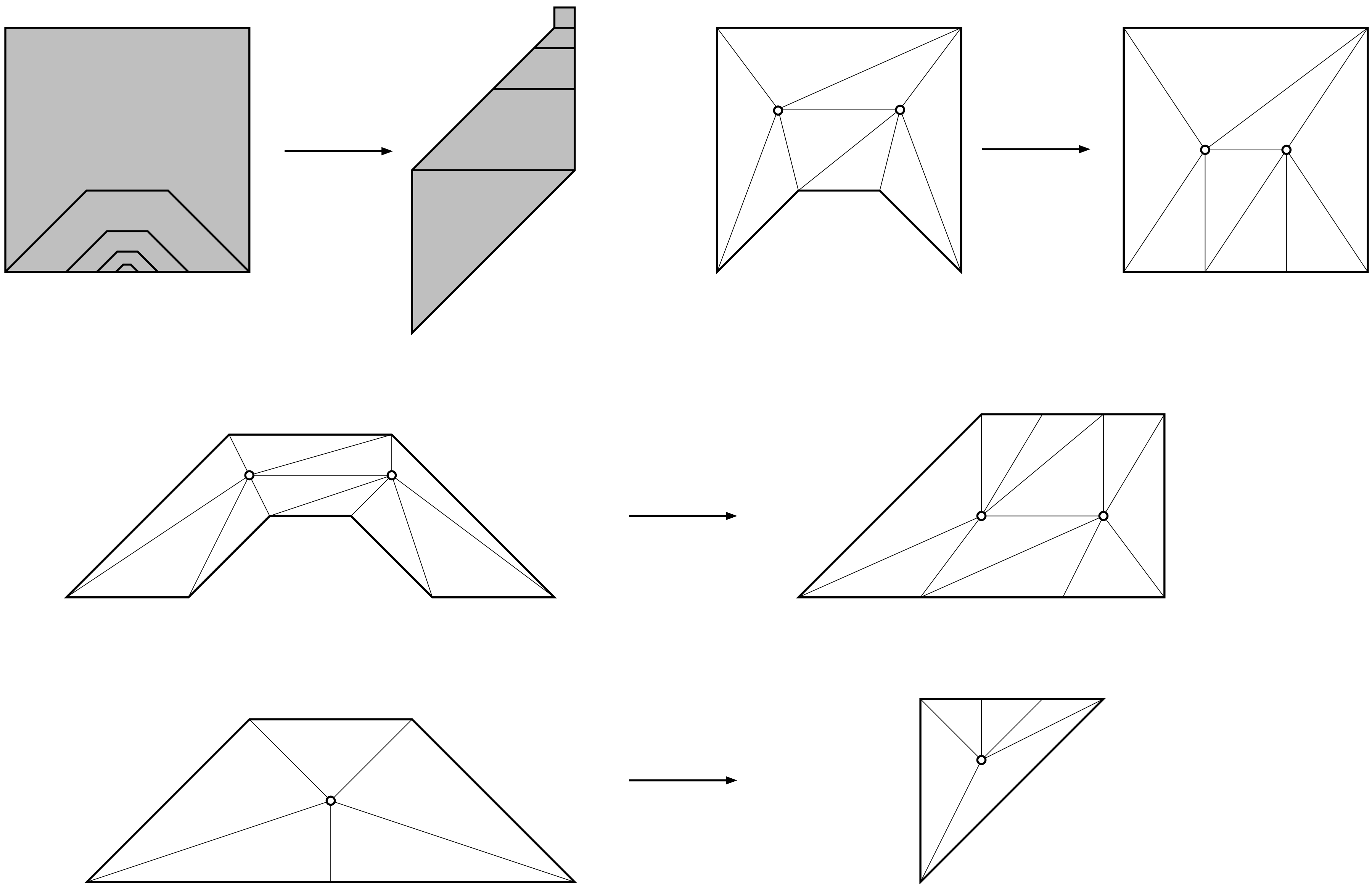
K and r only depend on the bounded geometry constants.

Proof of Folding Theorem: add finite trees (instead of spikes) to sides of T, so one edge is glued to tree edge; other are folded onto each other.

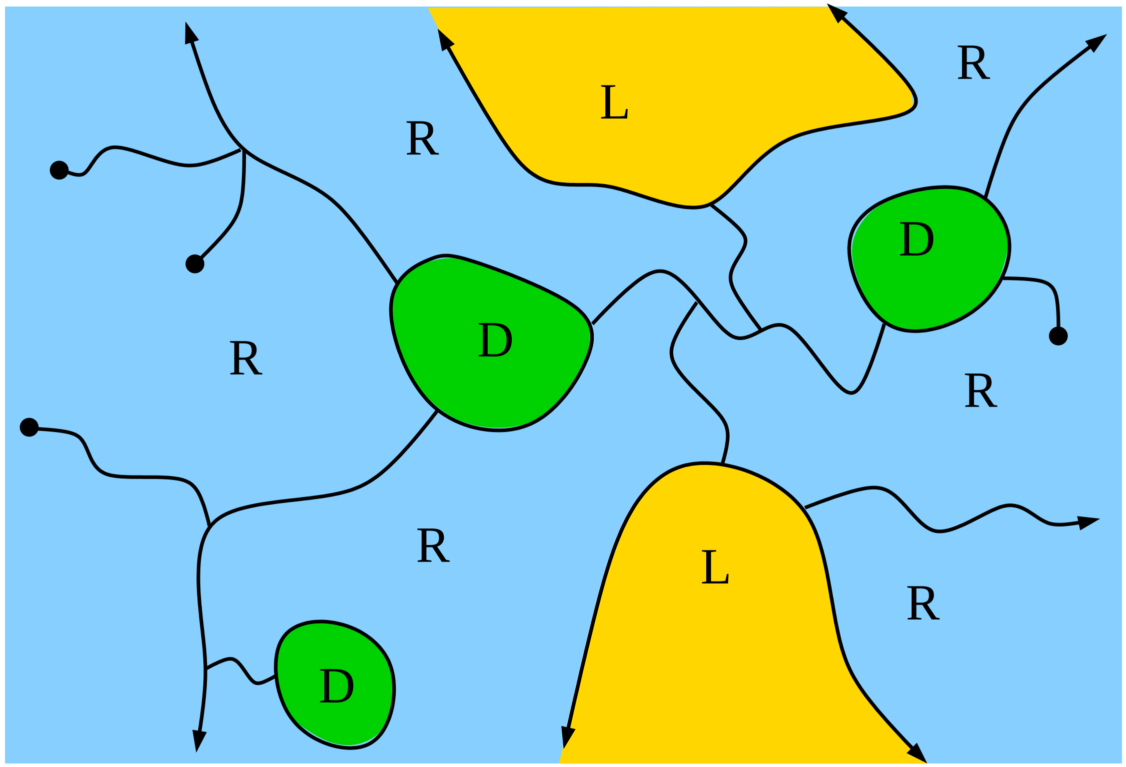


This allows arbitrarily large foldings with uniform QC bounds.





More general folding theorem: replace tree by graph, faces labeled D,L,R.



D = bounded Jordan domains (z^n, high degree critical points)

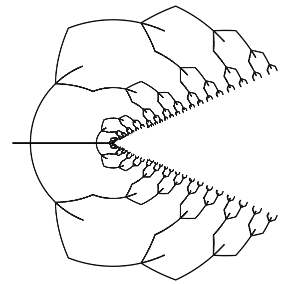
L = unbounded Jordan domains ($e^{-\tau(z)}$, finite asymptotic values)

RR-edges map to $[-1,1]$, other edges map to $\mathbb{T}$.

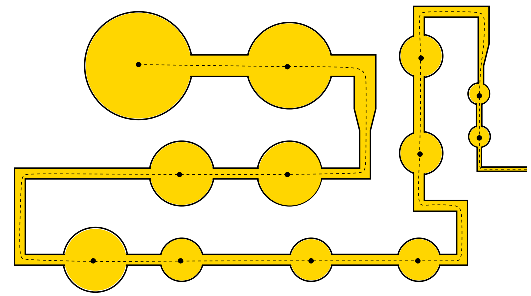
Another variation inverts the D-components, introduce poles.

APPLICATIONS OF QC-FOLDING

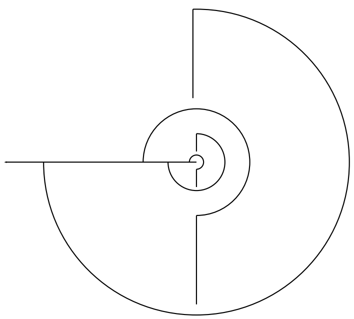
EL Models



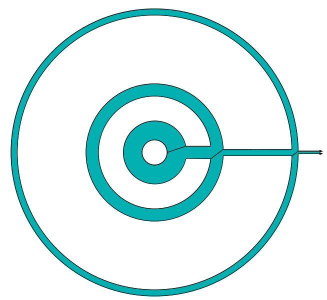
Order Conjecture



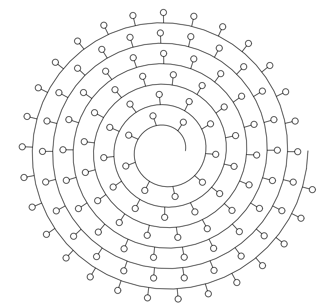
Post-singular dynamics



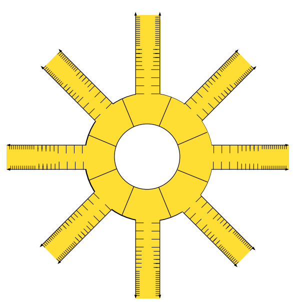
Eremenko Conjecture



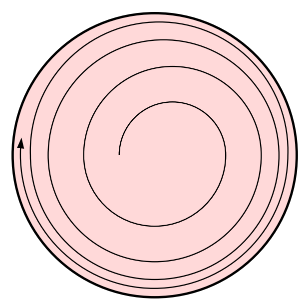
Near zero



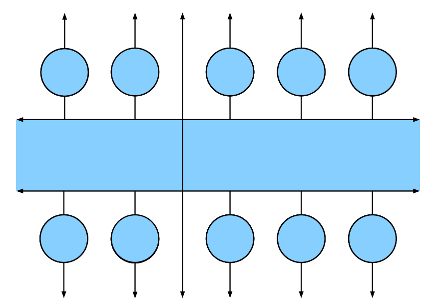
Wiman's Conjecture



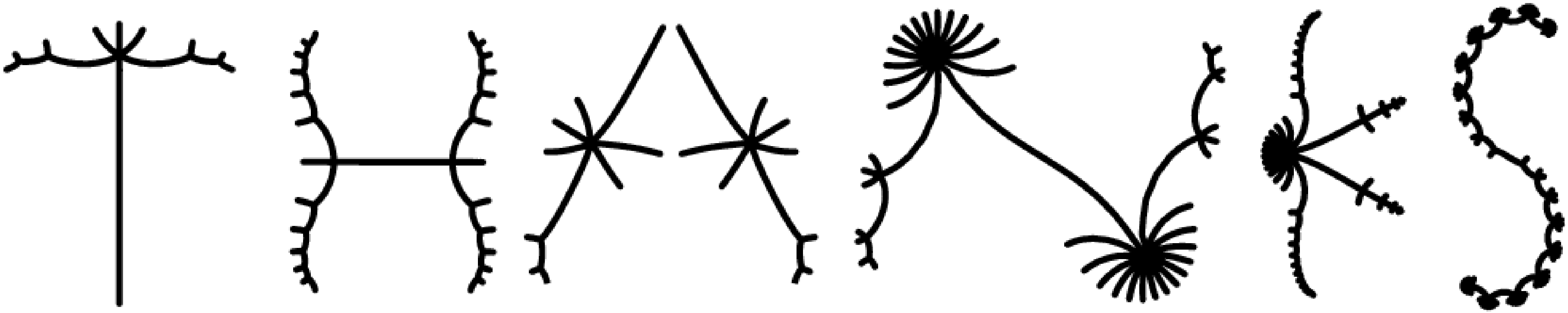
Dimension near 1



Folding in disk



Wandering domain



“ In the old days, when people invented a new function they had something useful in mind.”

–Henri Poincaré

$$
\begin{aligned}
\textbf{Singular set} &= \text{closure of critical values and finite asymptotic values} \\
&= \text{smallest set so that } f \text{ is a covering map onto } \mathbb{C} \setminus S
\end{aligned}
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Eremenko-Lyubich class = bounded singular set $= \mathcal{B}$

Speiser class = finite singular set $= \mathcal{S} \subset \mathcal{B}$

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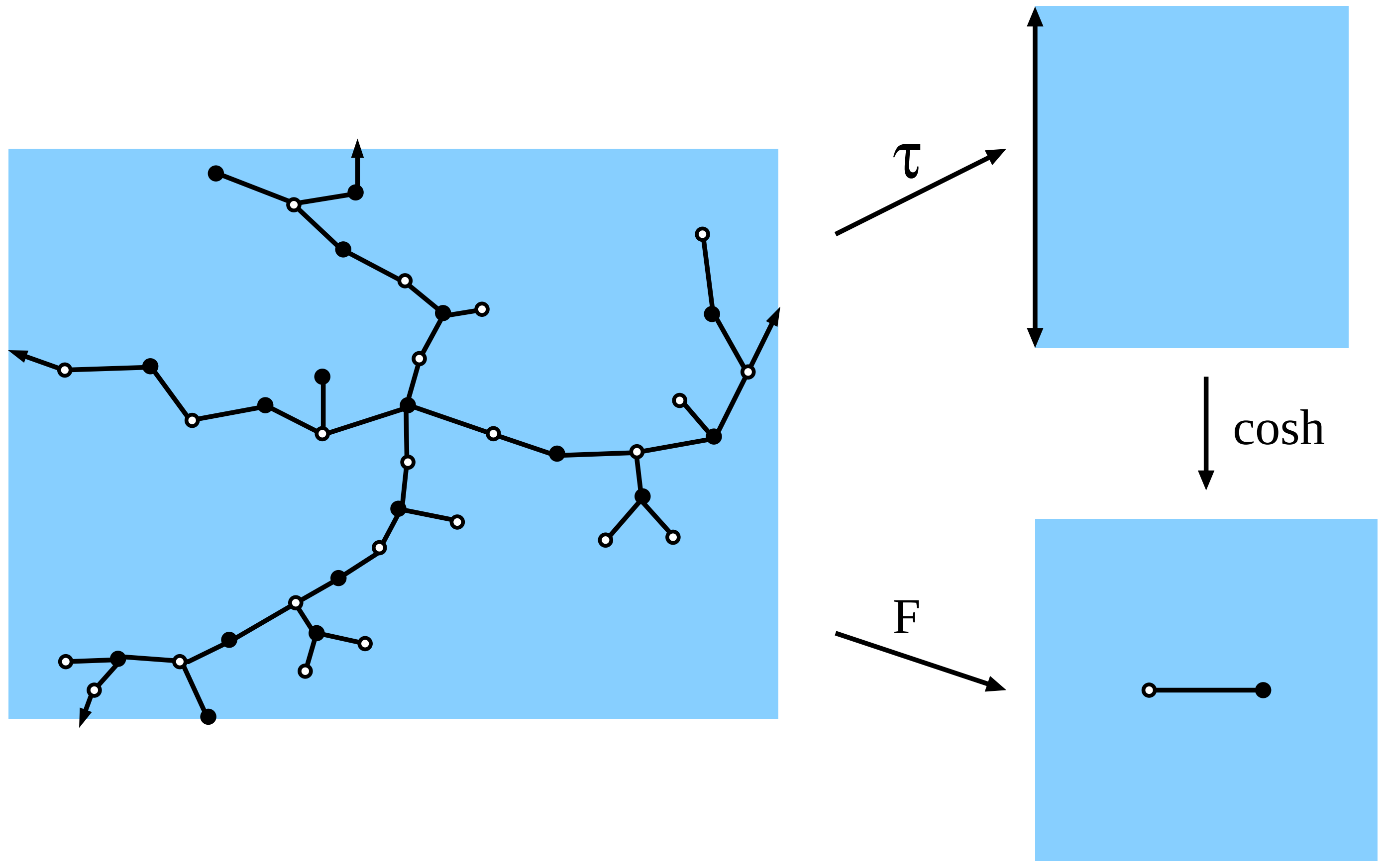
Eremenko-Lyubich class = bounded singular set = $\mathcal{B}$

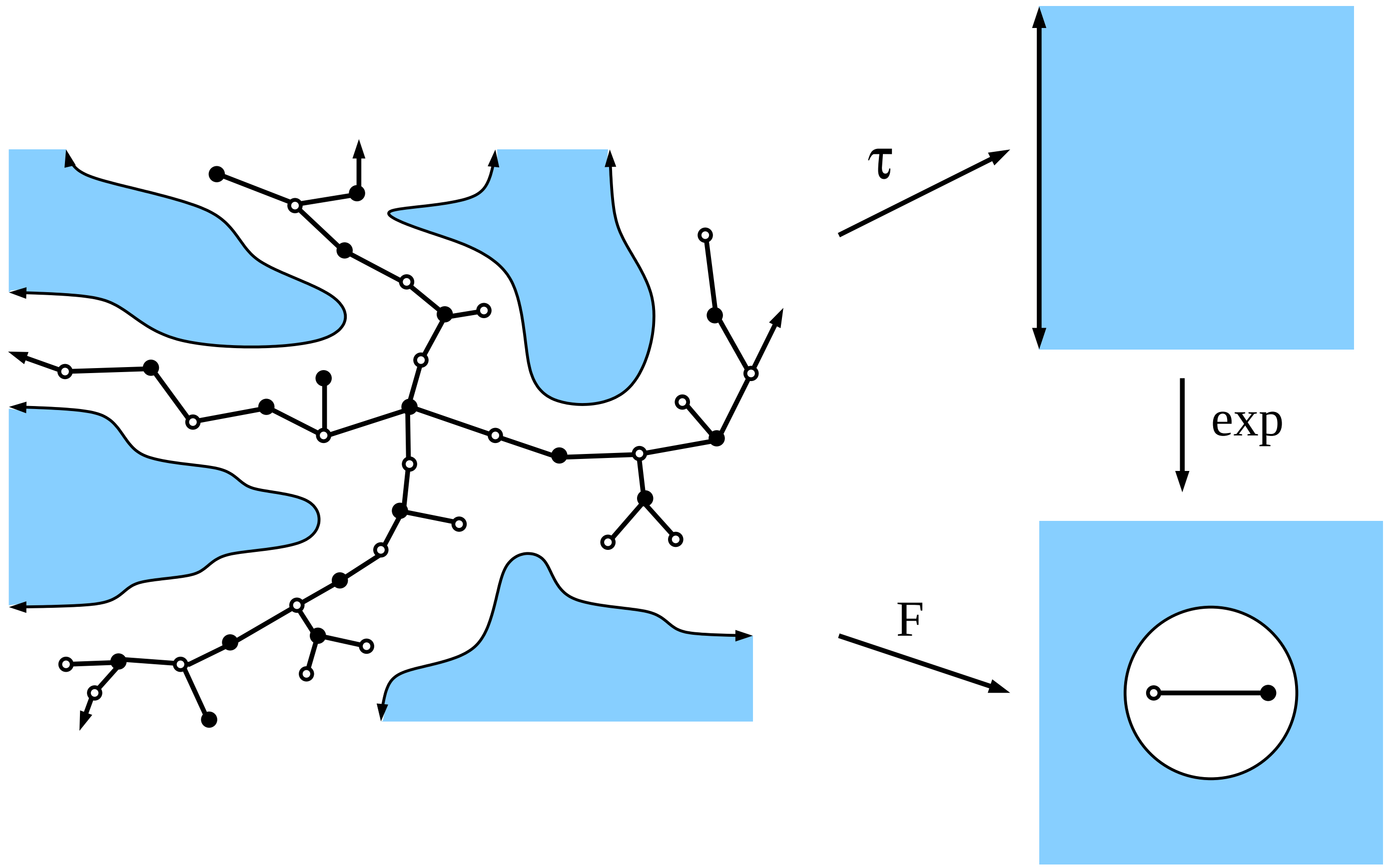
Speiser class = finite singular set = $\mathcal{S} \subset \mathcal{B}$

Folding produces functions in the smaller Speiser class.

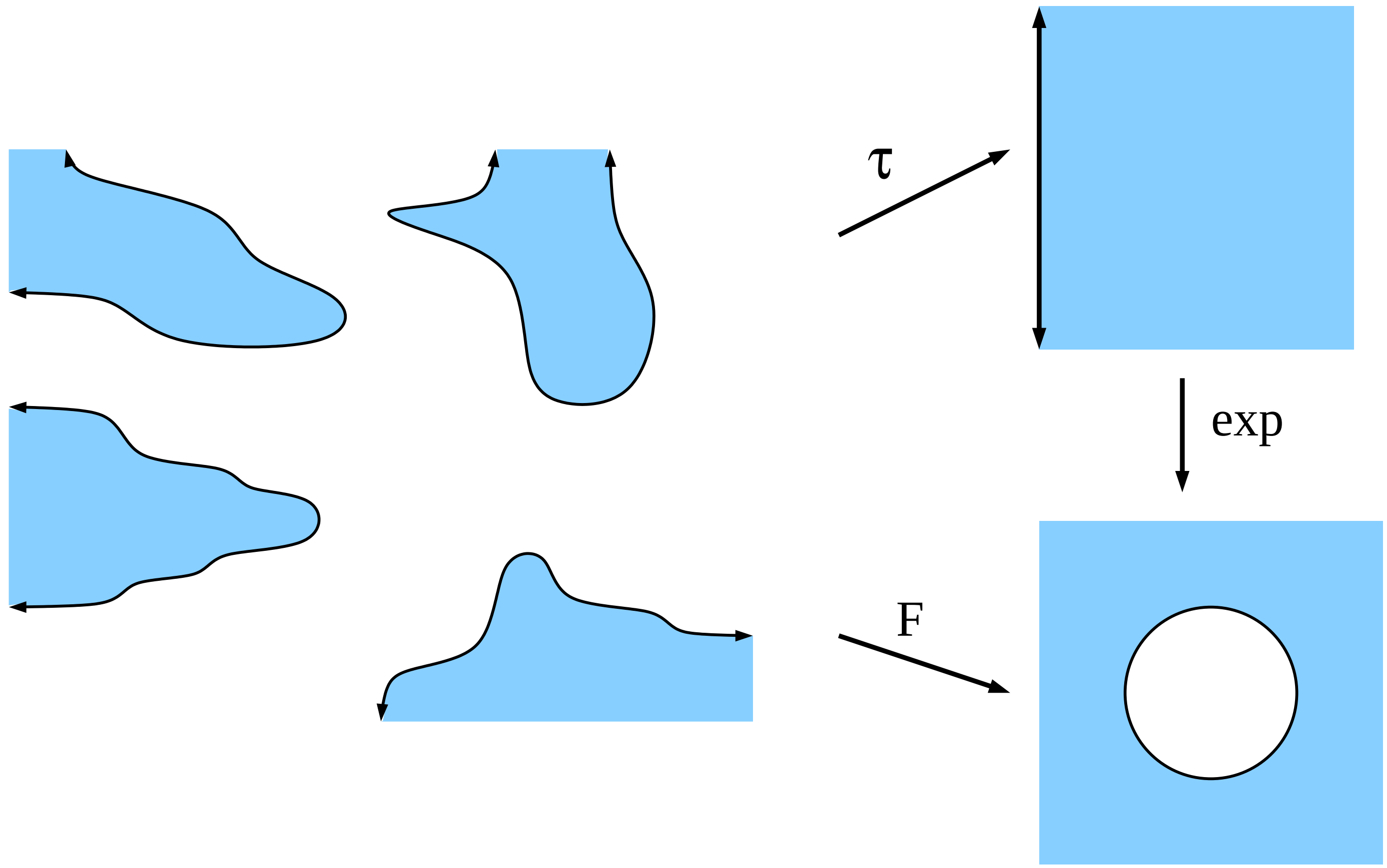
There is an "easier" version for the Eremenko-Lyubich class. It uses fewer assumptions and has a simpler proof.



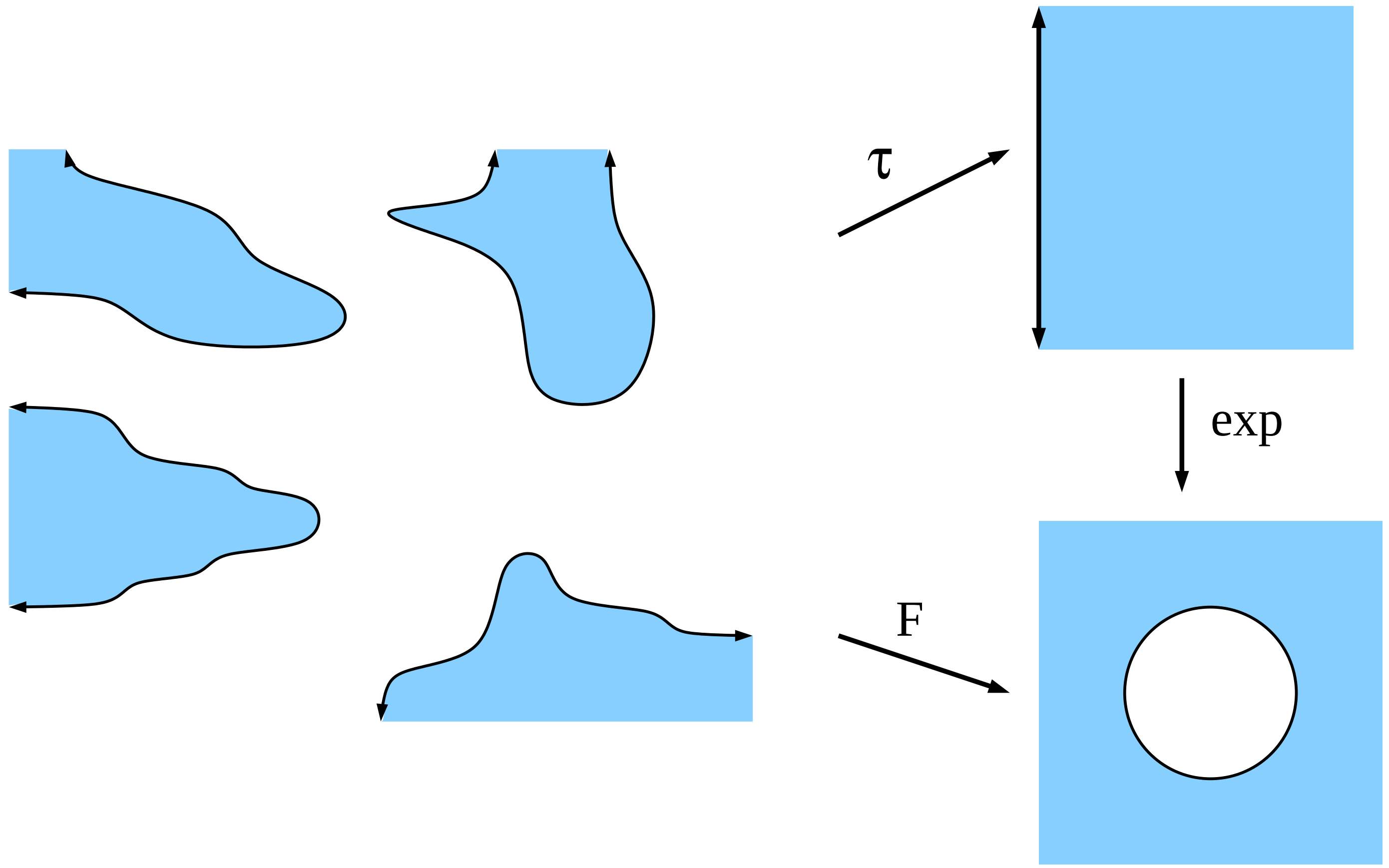
For Speiser class, the tree is preimage of a segment.



The preimage of a surrounding circle is a union of infinite Jordan domains.

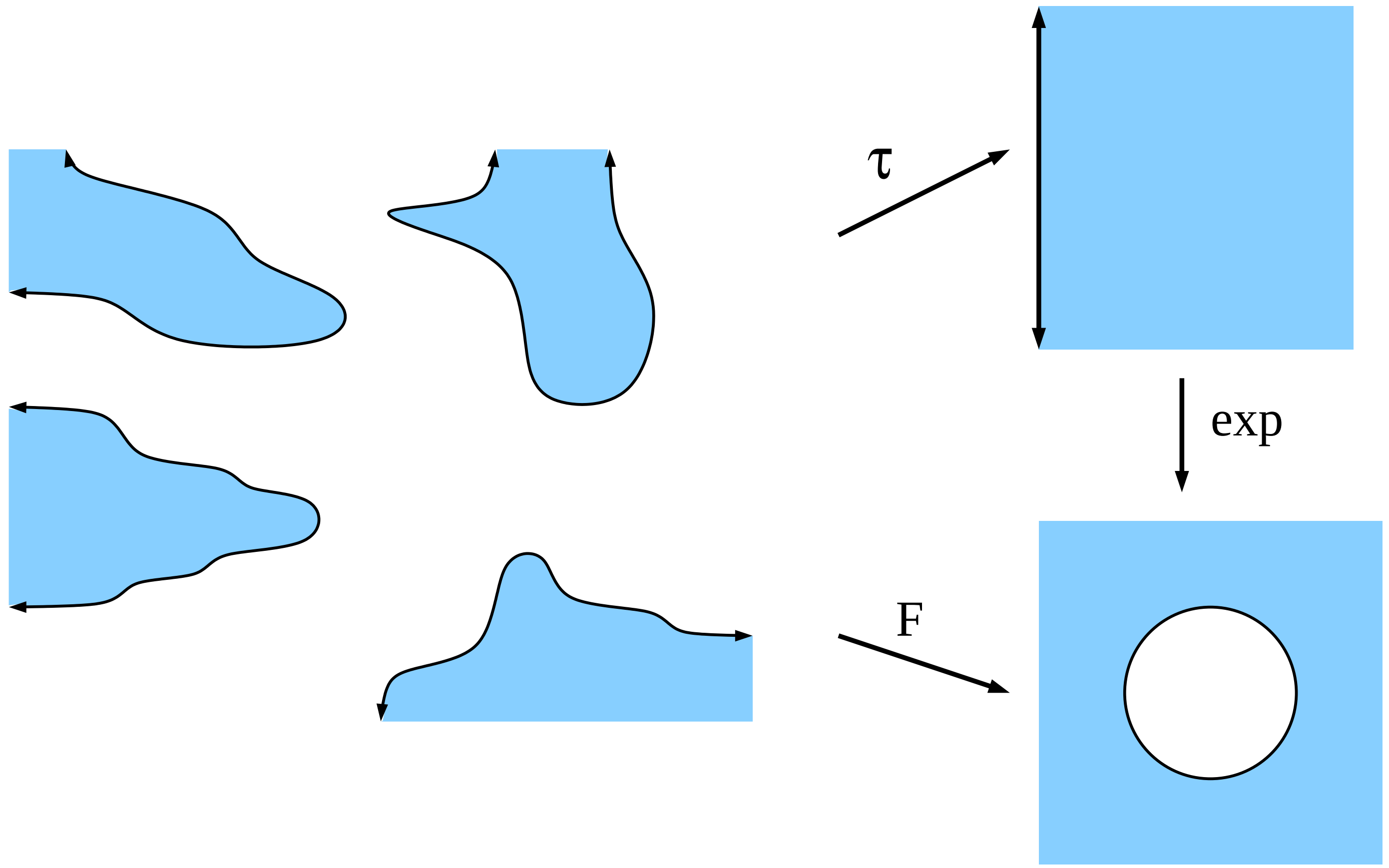


This preimage makes sense as long as all singular values are strictly inside the circle, i.e., for Eremenko-Lyubich functions.



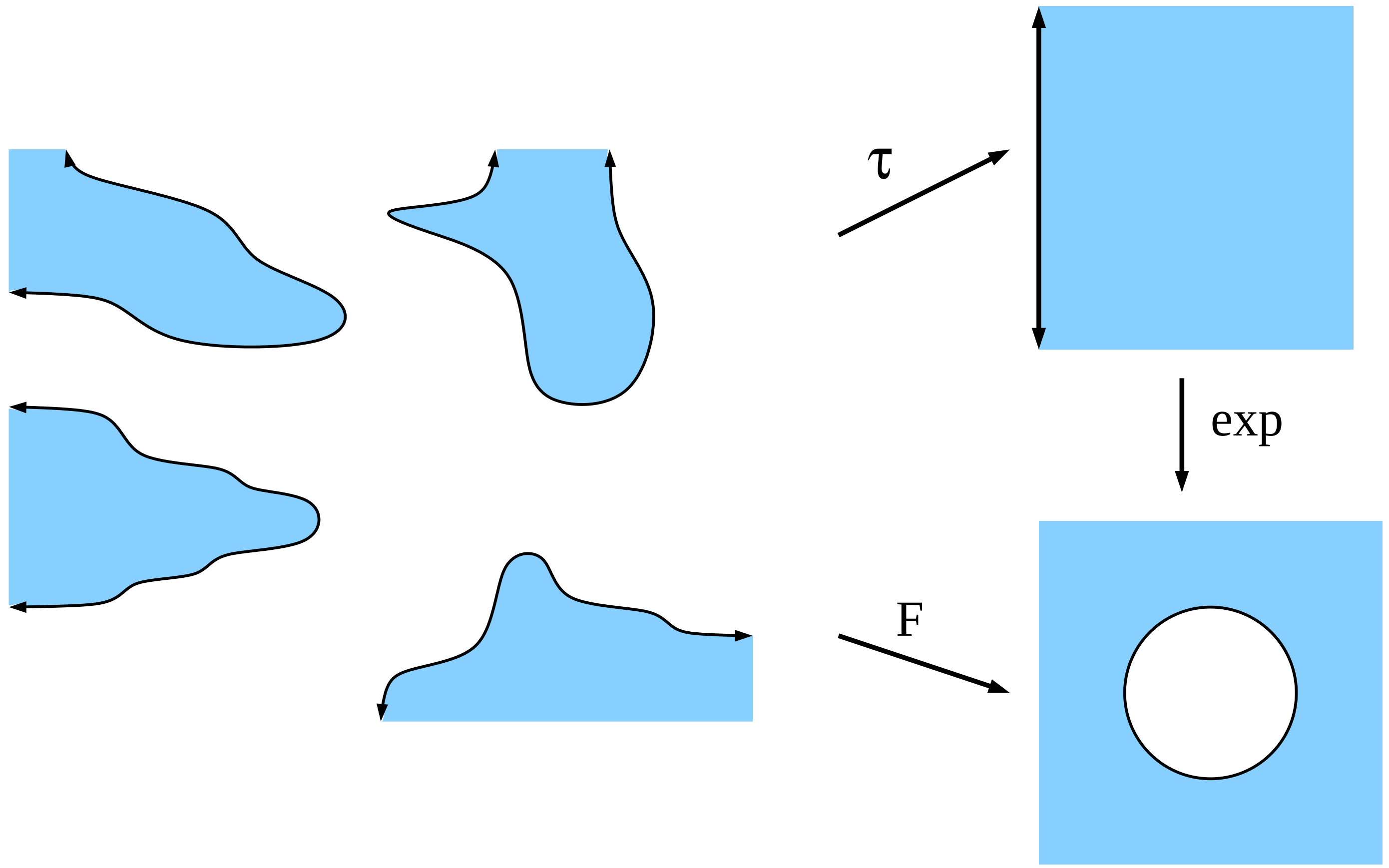
A **model** is a pair (Ω, τ) where

- $\Omega = \cup\, \Omega_j$ is a disjoint union of tracts (unbounded Jordan domains).
- τ is conformal from each Ω_j to $\mathbb{H}_r$ $(\infty \to \infty)$.

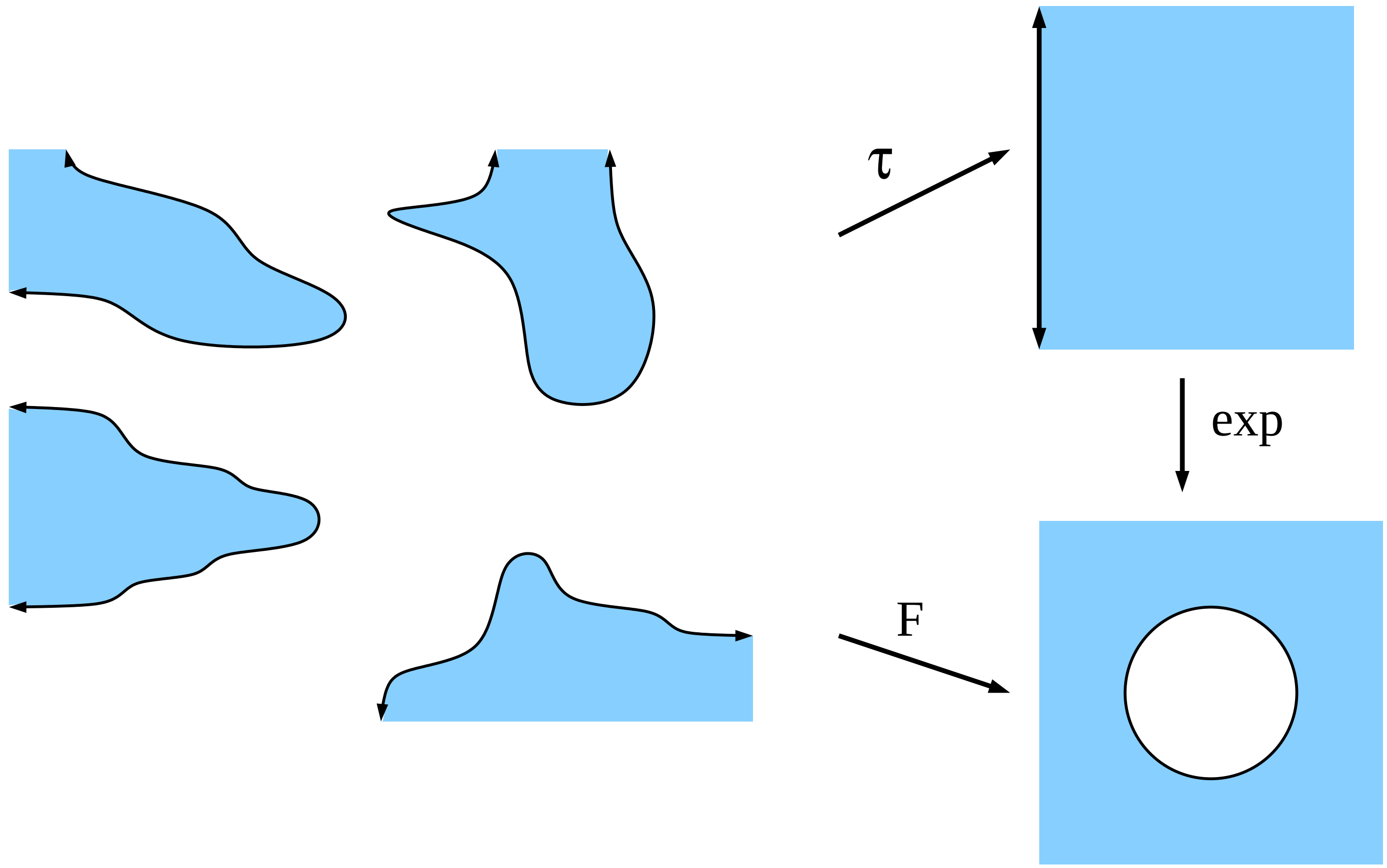


Every Eremenko-Lyubich function F gives a model with $\Omega = \{|F| > R\}$.

Does every model give an Eremenko-Lyubich function?



Given **any** model (Ω, τ) can we extend $F = e^{\tau}$ holomorphically to $\mathbb{C}$?

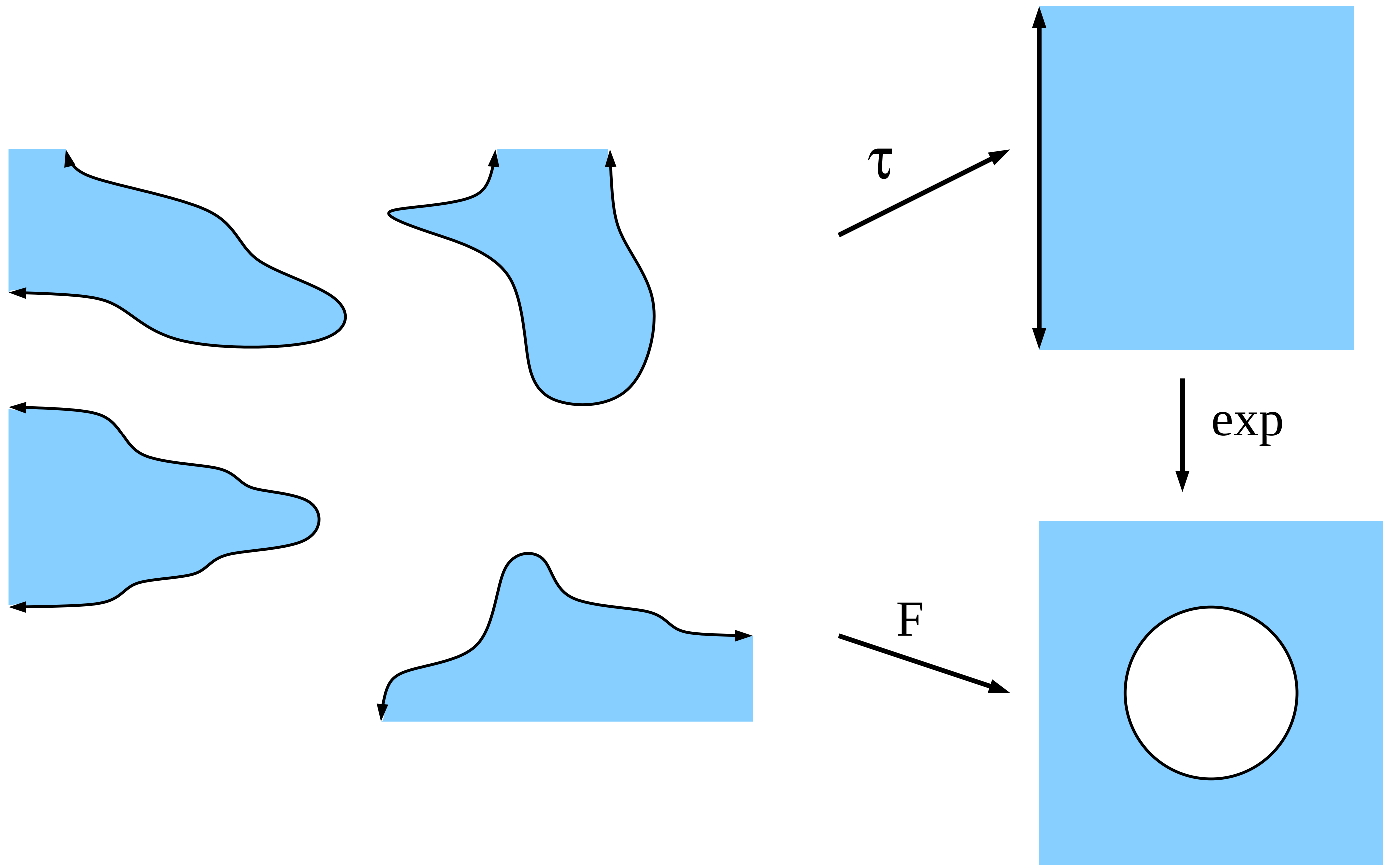


Given **any** model (Ω, τ) can we extend $F = e^{\tau}$ holomorphically to $\mathbb{C}$?

Obviously impossible:

(1) violates unique continuation.

(2) $\partial\Omega$ must be analytic if F is.



Given **any** model (Ω, τ) can we extend $F = e^{\tau}$ holomorphically to $\mathbb{C}$?

Obviously impossible:

(1) violates unique continuation. **Use QR function.**

(2) $\partial\Omega$ must be analytic if F is. **Use conformal approximation.**

Theorem: Suppose (Ω, τ) is a model and $\rho > 0$. Define

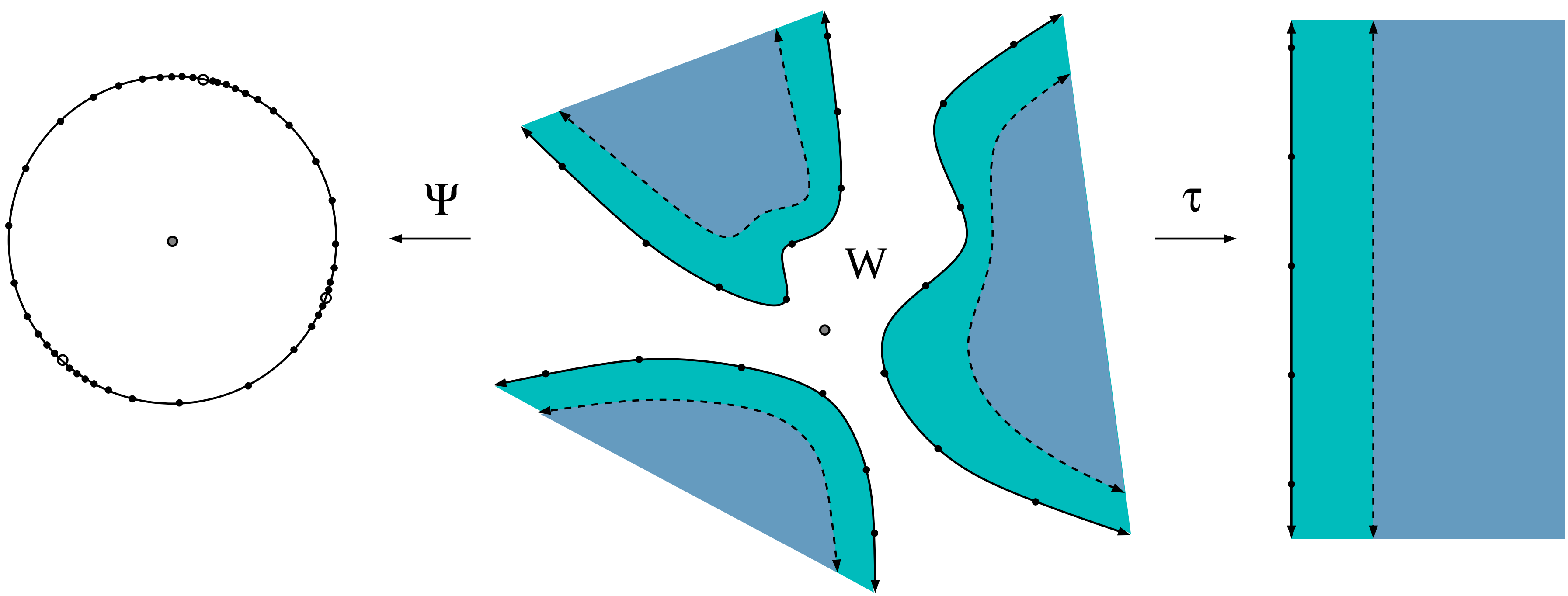
$$\Omega(\rho) = \tau^{-1}(\{x + iy : x > \rho\}) \subset \Omega.$$

Then there is a quasiregular g so that

(1) $g = e^{\tau}$ on $\Omega(2\rho)$,

(2) $|g| \le e^{2\rho}$ off $\Omega(2\rho)$.

(3) g is holomorphic off $\Omega(\rho)$.



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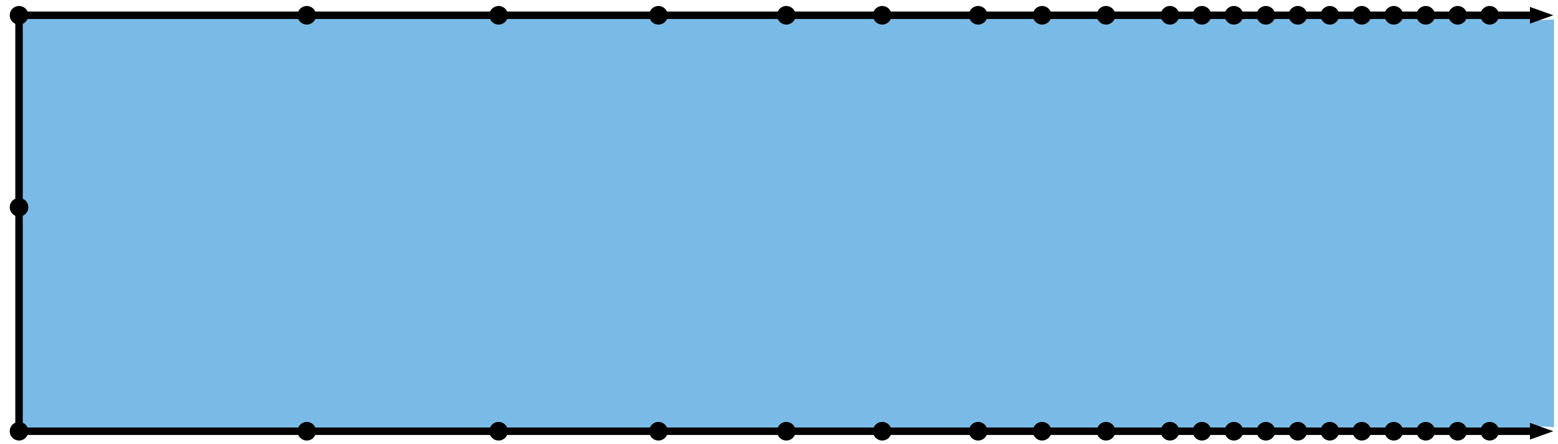
(3) g is holomorphic off $\Omega(\rho)$.

The QR constant depends on ρ, but not on Ω.

There is a quasiconformal φ so that $f = g \circ \varphi \in \mathcal{B}$.

g is holomorphic except on $\Omega(\rho) \setminus \Omega(\rho/2)$ (often has finite area).

Formulation inspired by Lasse Rempe-Gillen.

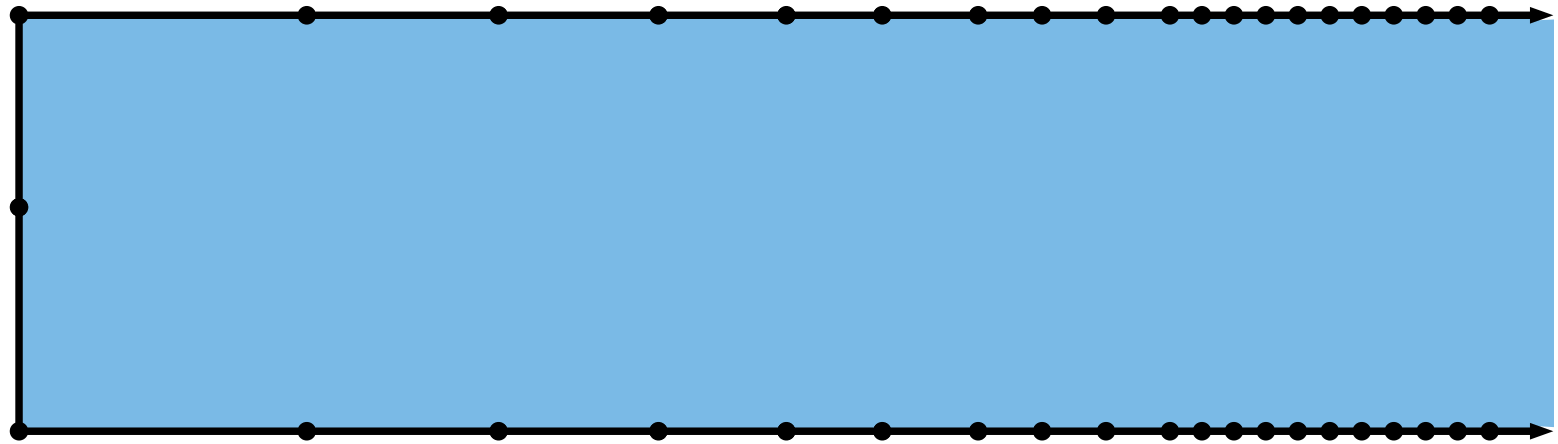


If Ω = half-strip, the τ-partition for inside decays exponentially.

But τ-partition for outside grows like $\sqrt{n}$.

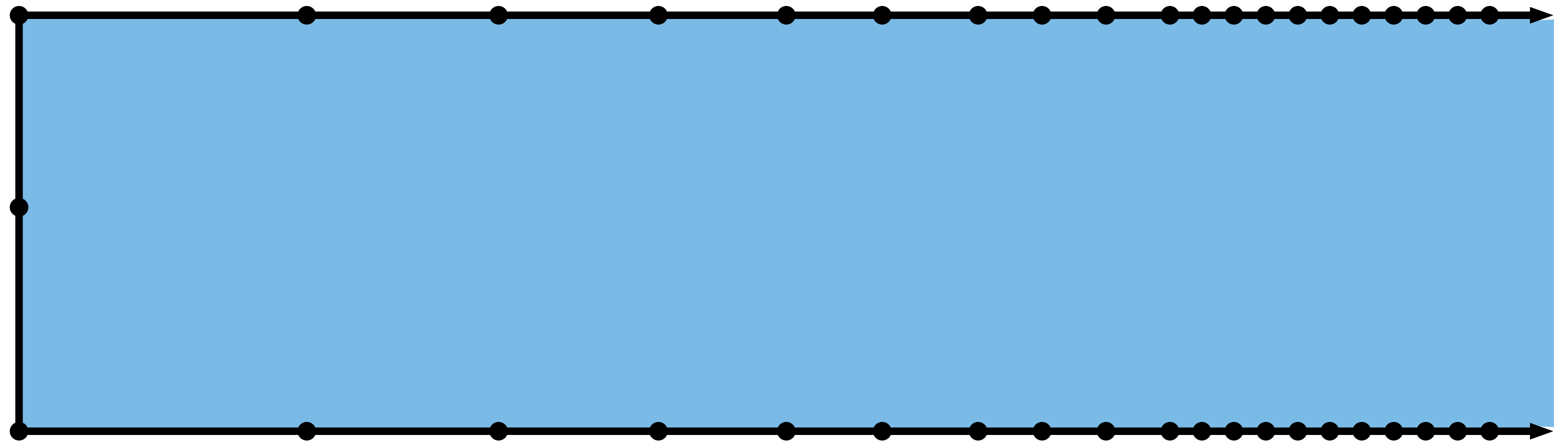
Can't choose vertices with both bounded geometry and τ lower bound.

So QC-folding theorem does not apply here.



Indeed, no Speiser class function with a single tract can have that tract approximate a half-strip.

However, the models theorem does apply: there are Eremenko-Lyubich functions with tracts approximating the half-strip.

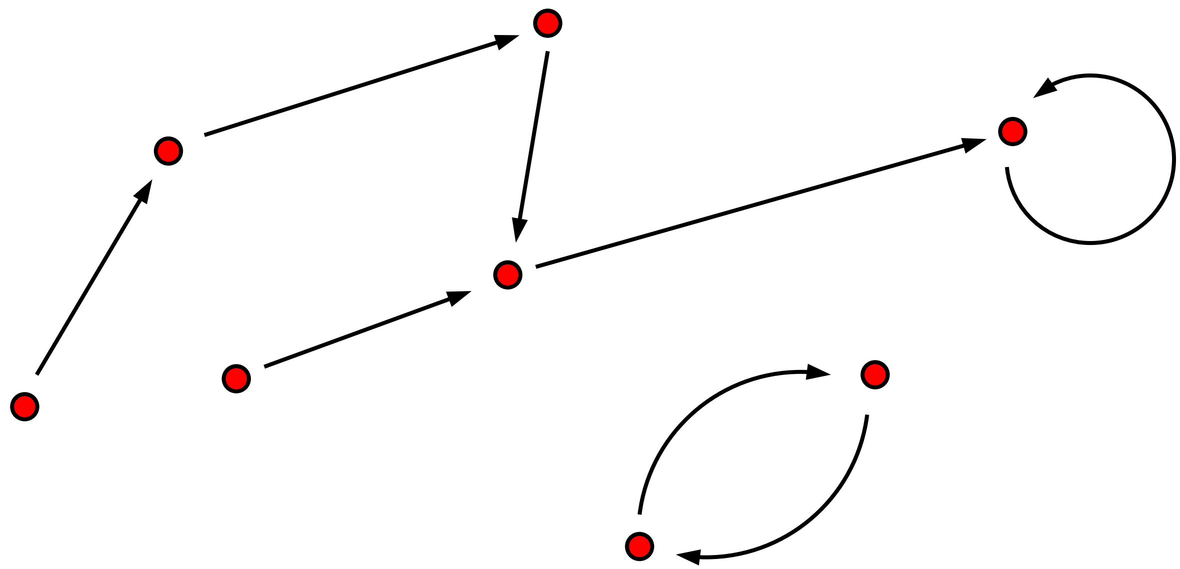


In fact, such examples give a theorem of Stallard:

Theorem: There are Eremenko-Lyubich functions whose Julia sets have Hausdorff dimension close to 1.

DeMarco, Koch and McMullen proved that a post-critically finite rational map can have **any** given dynamics.

More precisely, given any $\epsilon > 0$, any finite set X and any map $h : X \to X$ there is a rational r whose post-singular set $P(r)$ is the same size as X, $P(r)$ ϵ-approximates X in Hausdorff metric, and $|r - h| < \epsilon$ on P.



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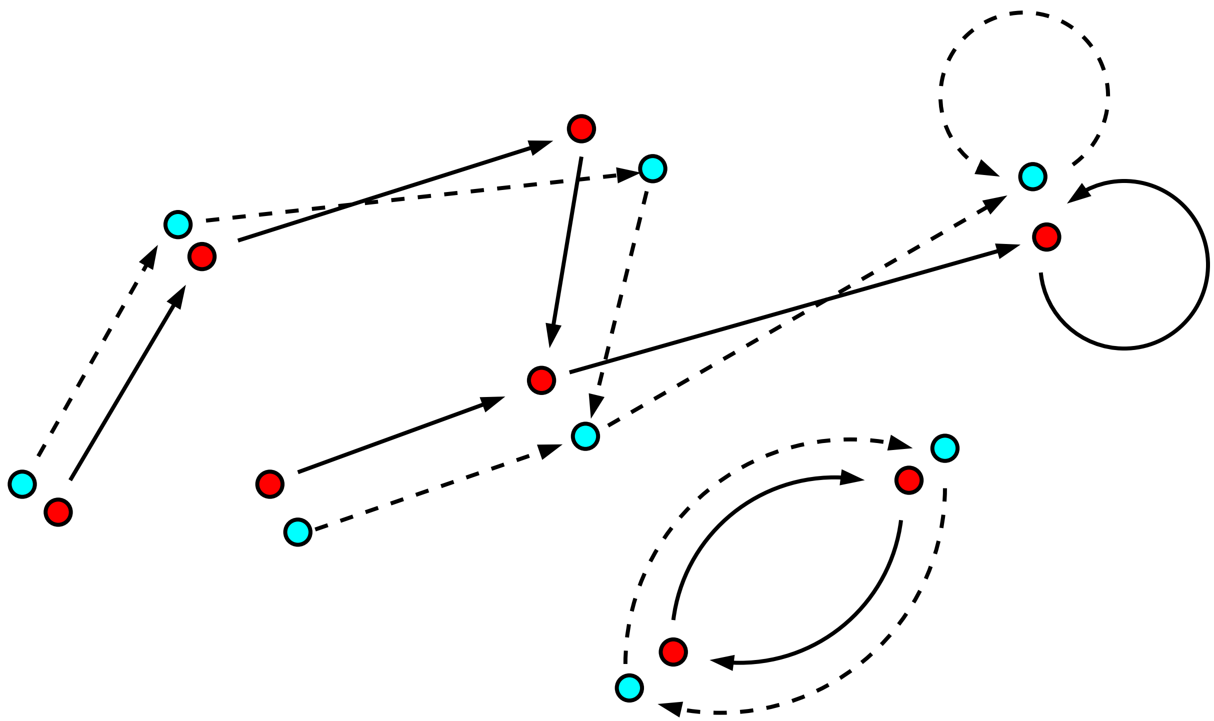
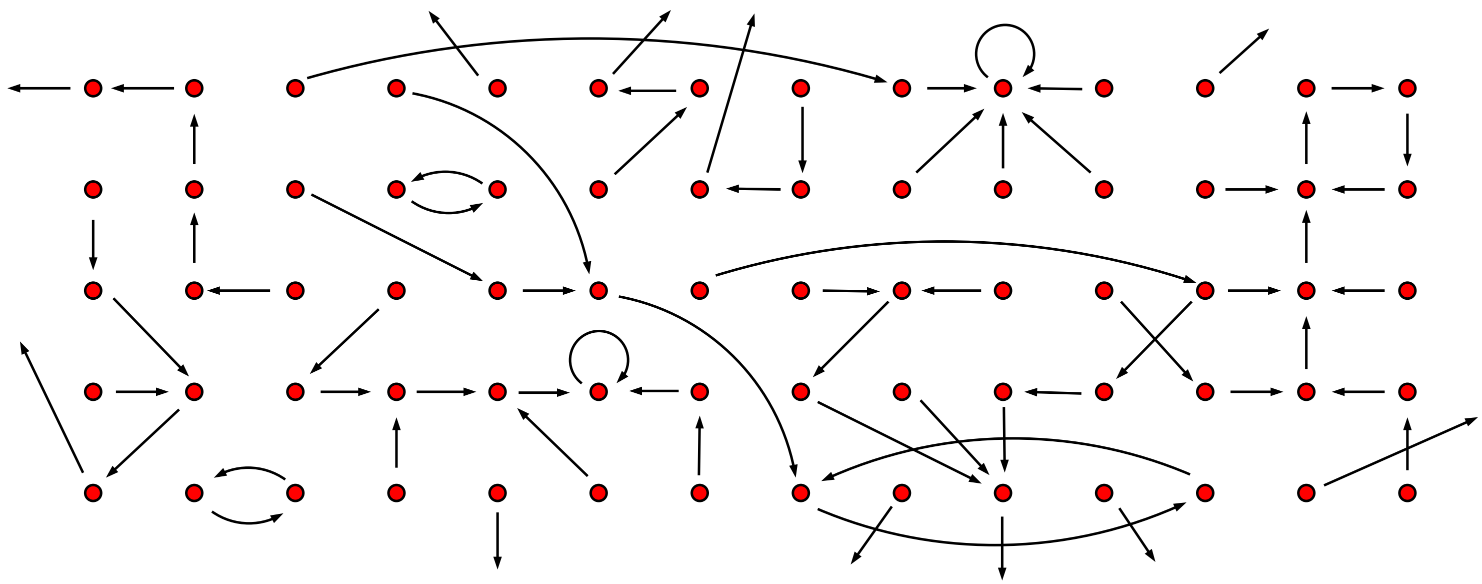
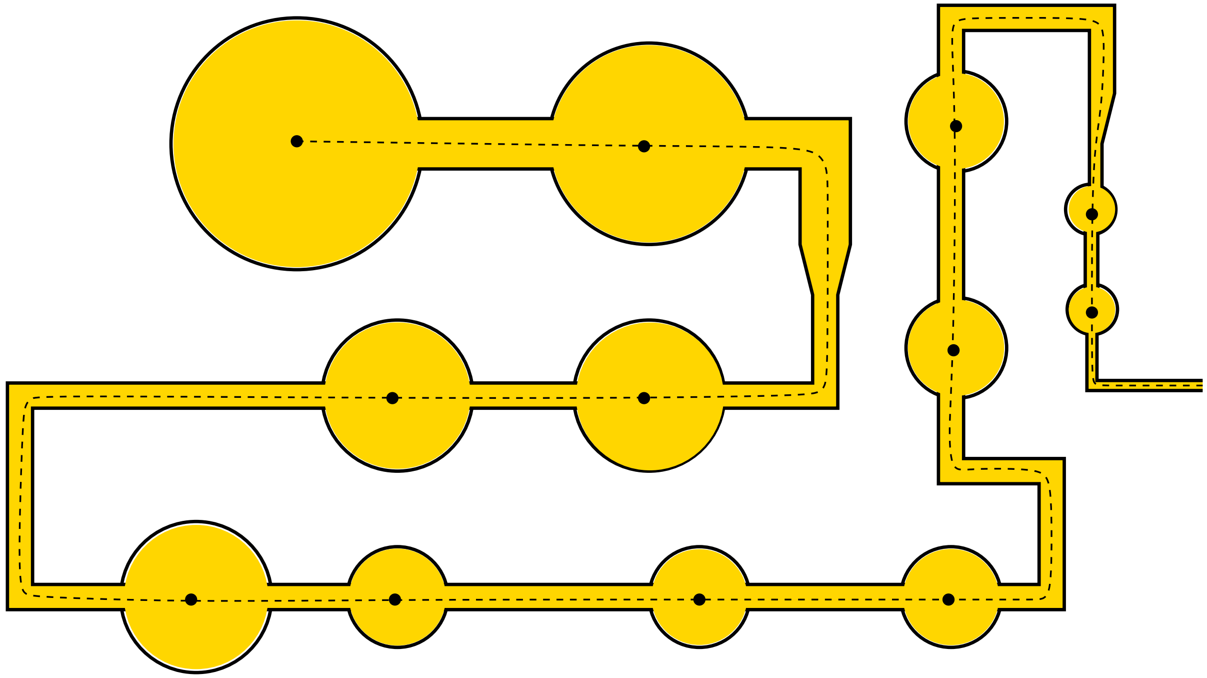




Theorem (Lazebnik, B.): Let X be discrete (≥ 4 points), let $h : X \to X$ be any map, and let $\epsilon > 0$. Then there is a transcendental meromorphic function f and a bijection $\psi : X \to P(f)$ so that

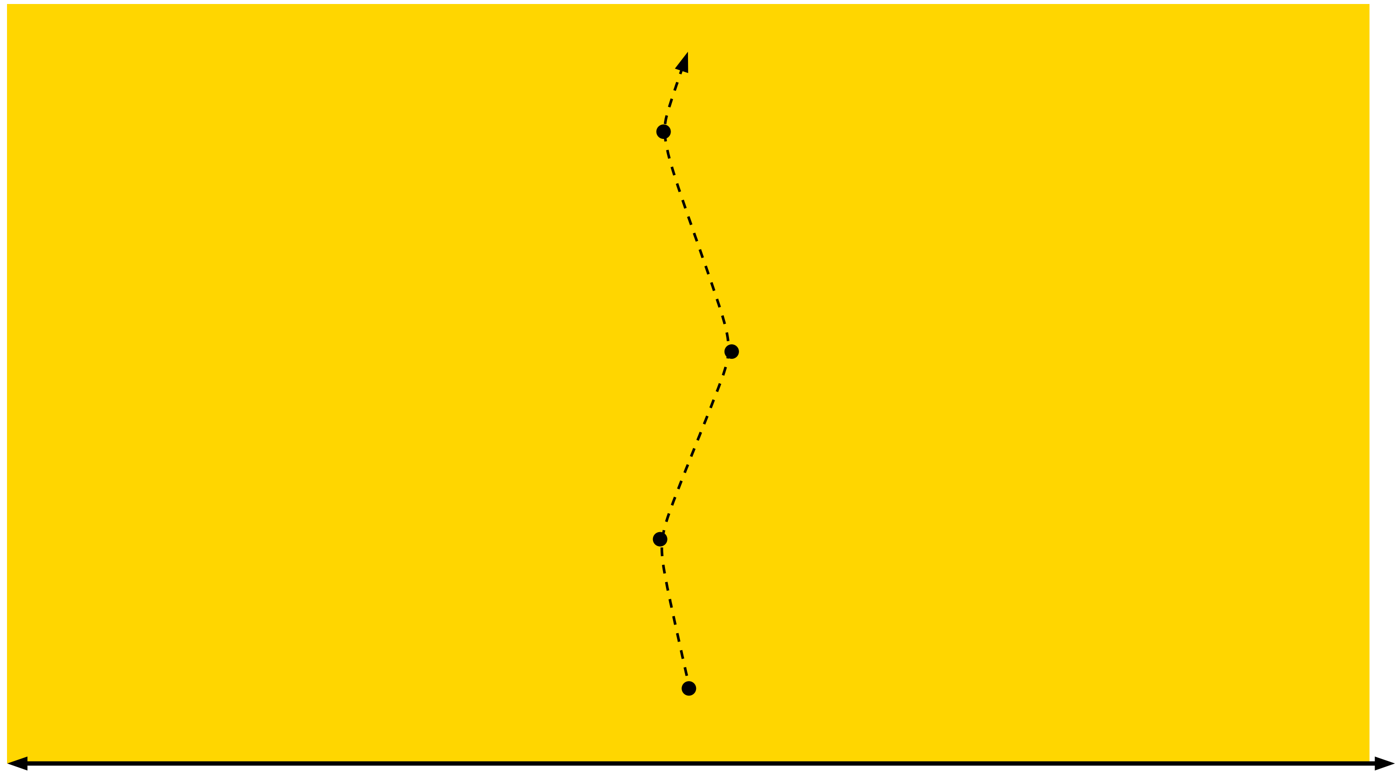
$$|\psi(z) - z| \leq \max(\epsilon, o(1)) \quad \text{and} \quad f = \psi \circ h \circ \psi^{-1} \text{ on } P(f).$$

Main argument (folding + fixed point thm) due to Kirill.

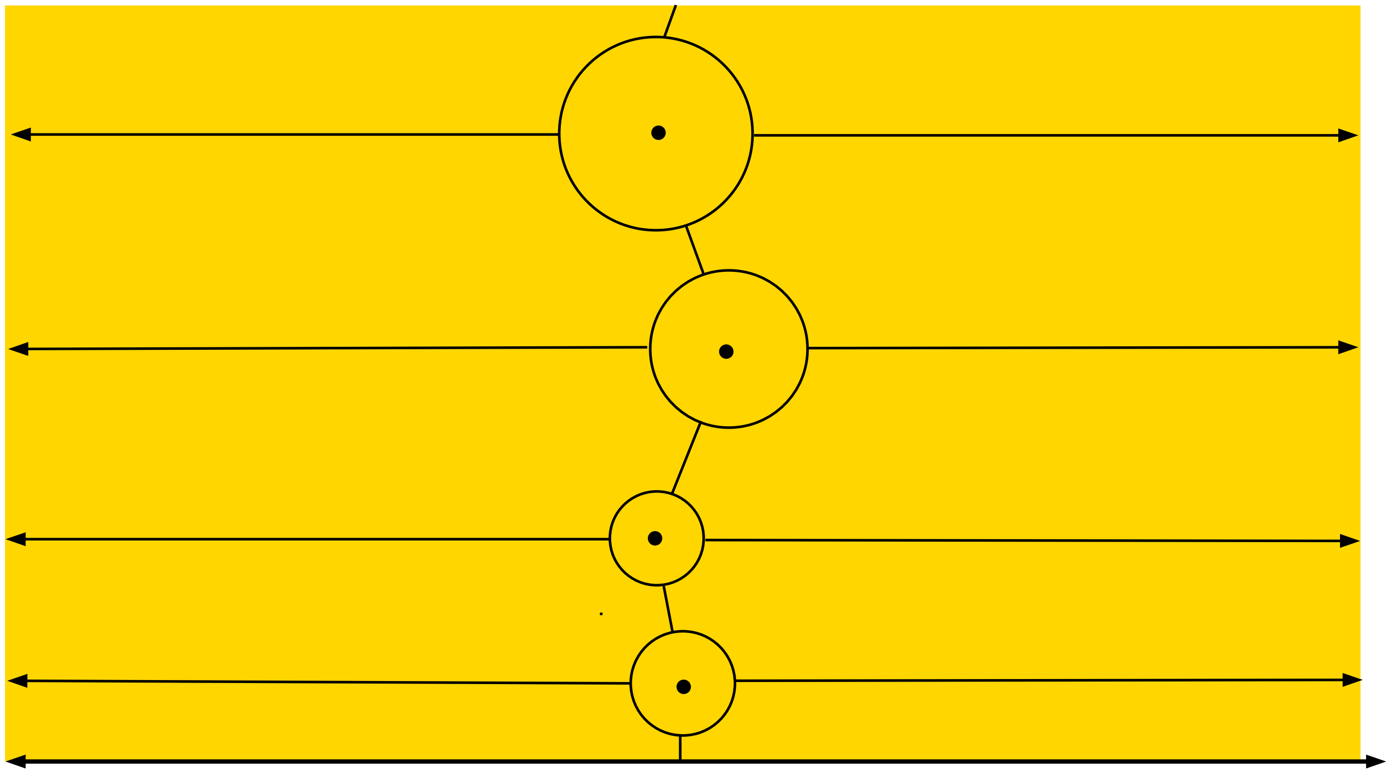


Reduce to case where ± 1 are in post-singular set and all other post-singular points are off the unit circle.

Construct a domain W that contains the given set, and so that these points lie very close to hyperbolic geodesic to ∞.



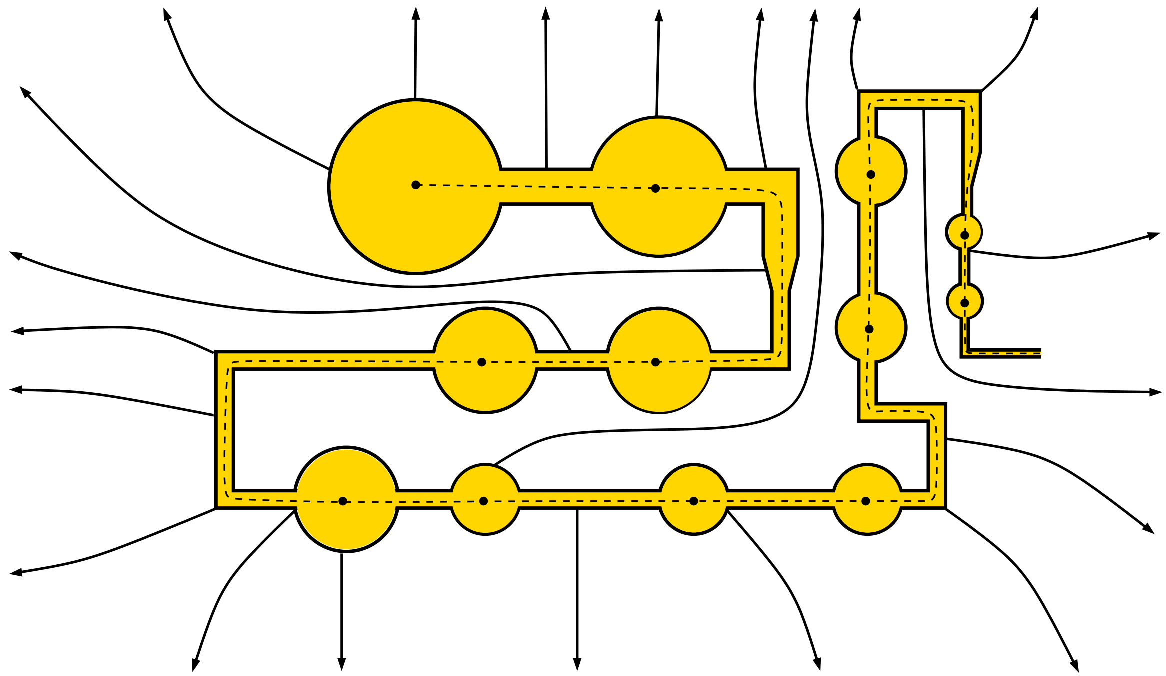
Conformally map W to upper half-plane; points almost on vertical ray.



Divide upper half-plane into R and D components.

Folding map sends centers anywhere we want in $\mathbb{D}$.

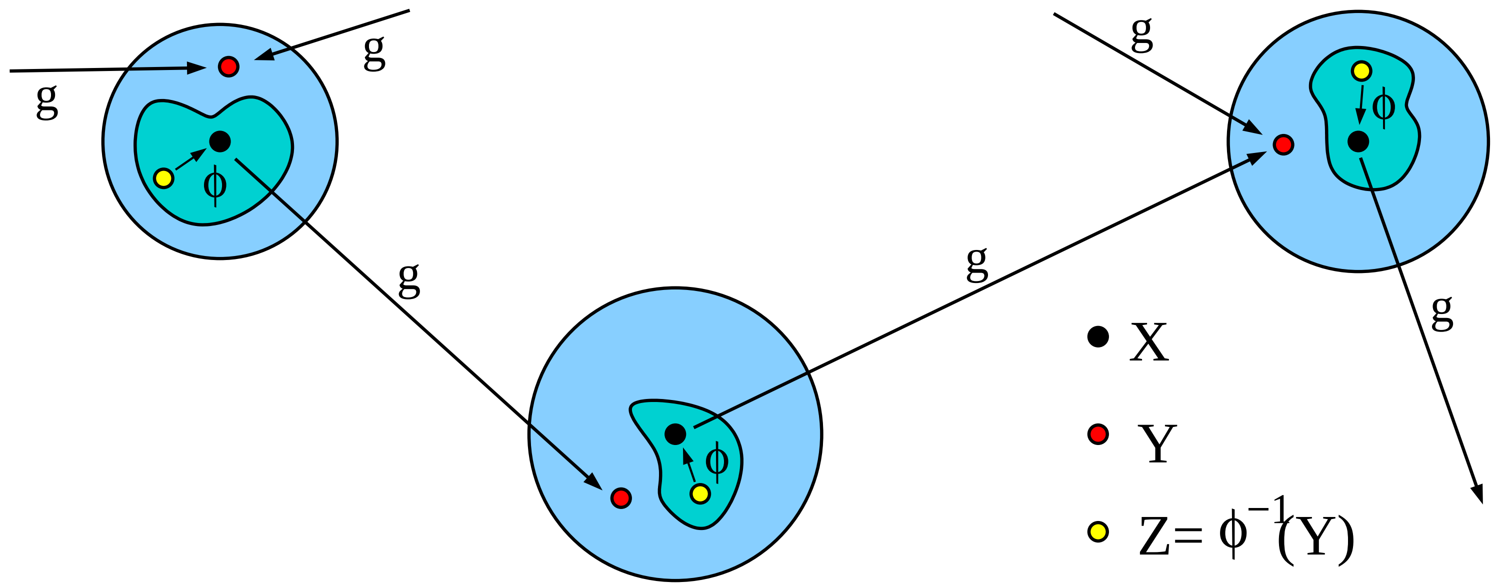
Analogous construction maps centers outside $\mathbb{D}$; introduces poles.



Transfer graph back to W. Decompose complement of W into R-components (not hard) and define map on whole plane by folding.

Gives quasi-meromorphic map g with desired post-singular behavior.

Mapping theorem gives a meromorphic $f = g \circ \phi$. However, because of ϕ, the post-singular set might not be invariant under f. How to fix this?



Replace X (black) by a union of disjoint disks (blue), and let $Y \approx X$ be one point from each disk (red). As above, construct g so $X \to Y$. Get meromorphic $f = g \circ \phi$ that has singular points at $Z = \phi^{-1}(Y)$ (yellow).

Z (yellow) is continuous function of Y (red). Because ϕ is close to the identity, Z remain inside the disks. An infinite-dimensional fixed point theorem gives choice of Y so that $Z = Y$, as desired.

Order of growth of entire function:

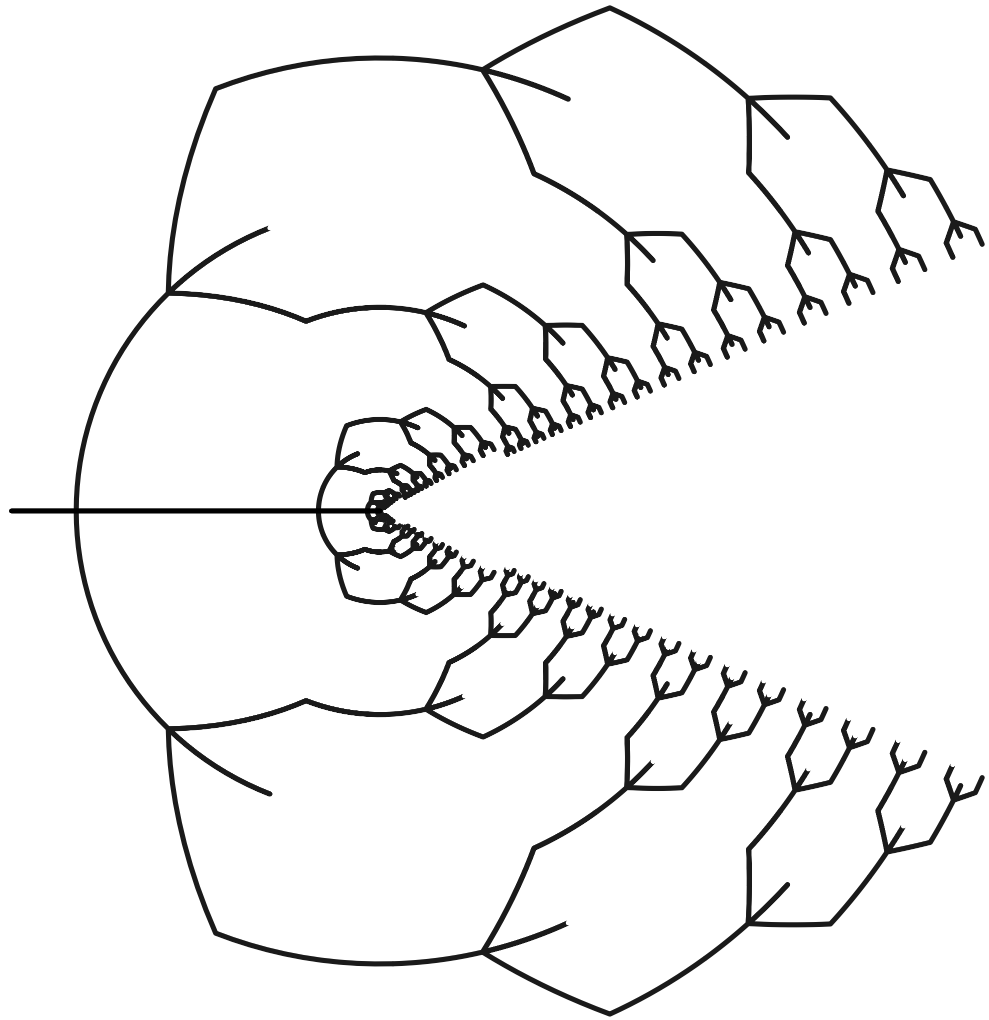
$$\rho(f) = \limsup_{|z|\to\infty} \frac{\log\log|f(z)|}{\log|z|}, \qquad \rho(e^{z^d}) = d.$$

QC-equivalent: $f \sim g$ if $g = \psi \circ f \circ \phi$ for quasiconformal ψ, ϕ.

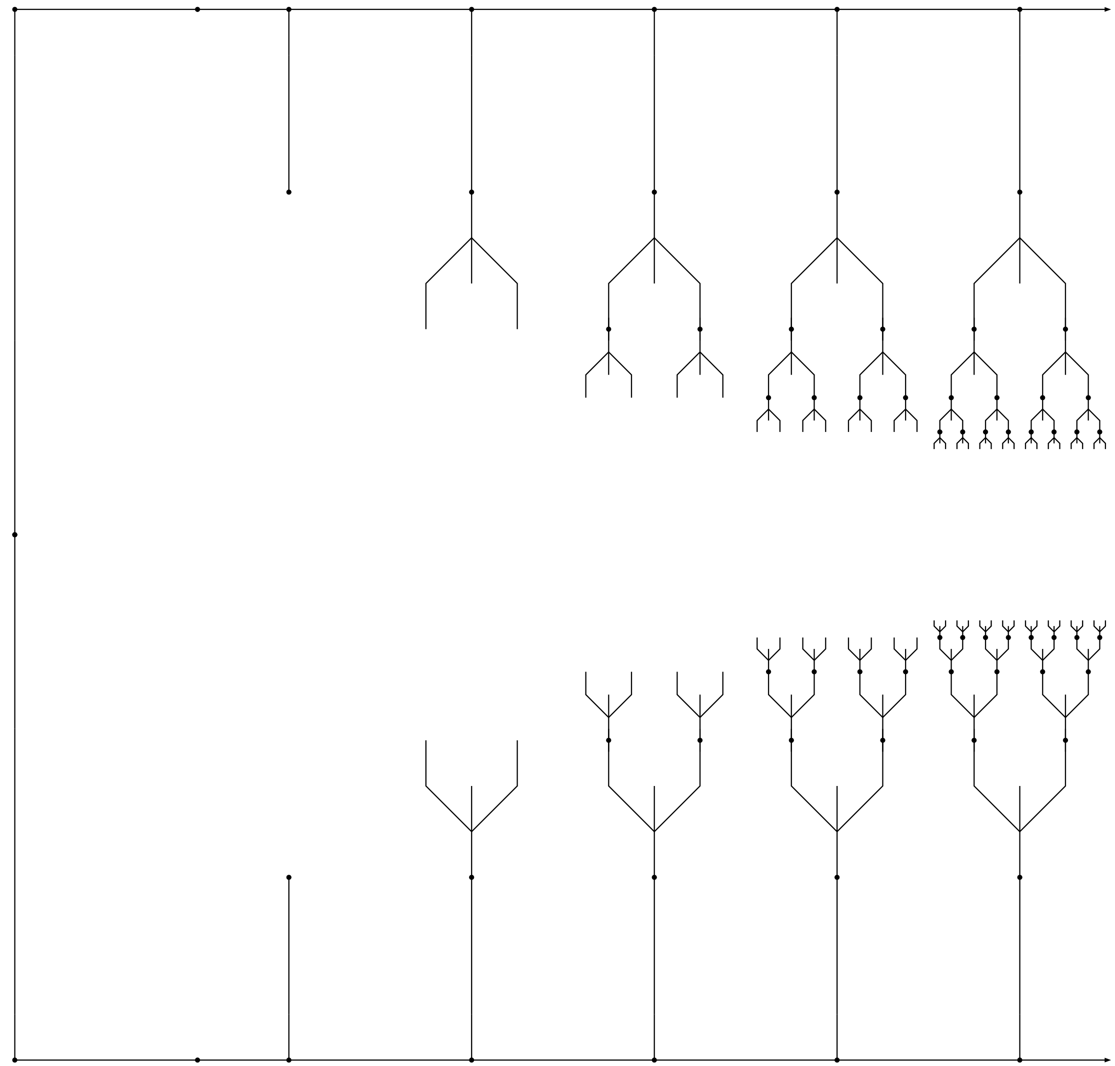
Speiser class QC-equivalence classes M_f are finite dimensional.

Order conjecture: Is ρ constant on M_f?

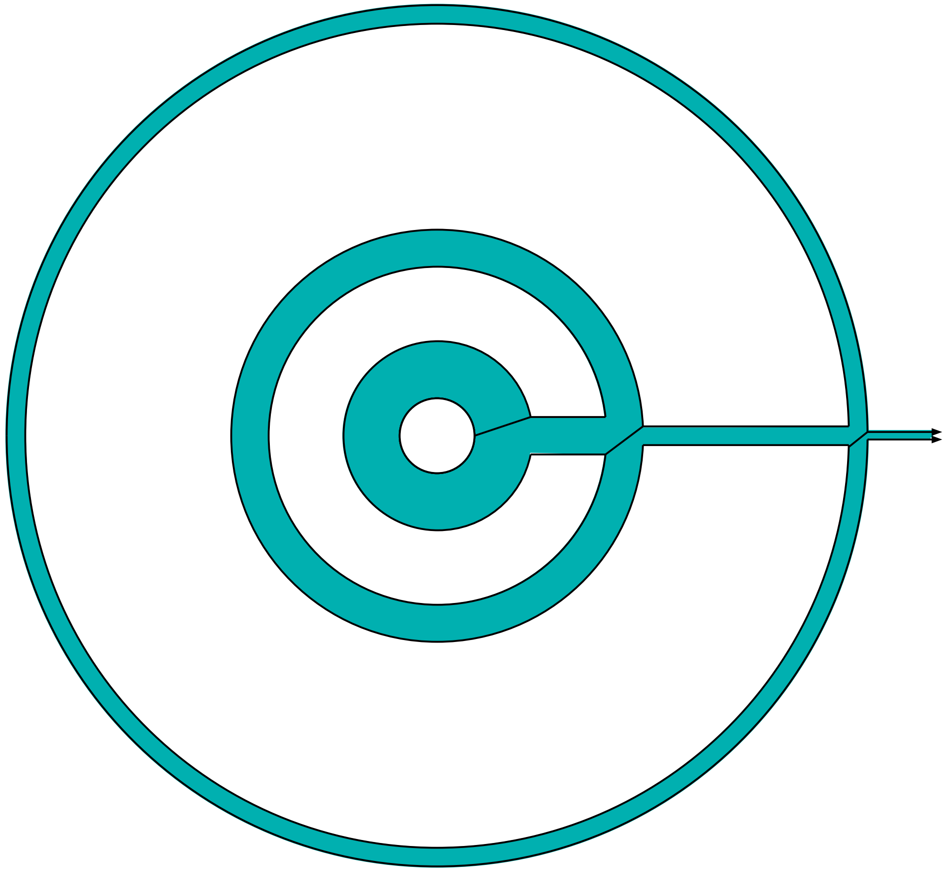
Adam Epstein observed this holds in many cases where order is determined by "combinatorial data". He and Rempe-Gillen gave EL counterexample.



Speiser class counterexample with 3 singular values.



Same tree in logarithmic coordinates.



Speiser function that is "statistically zero": for all $\epsilon > 0$,

$$\text{area}(\{|f| > \epsilon\}) < \infty.$$

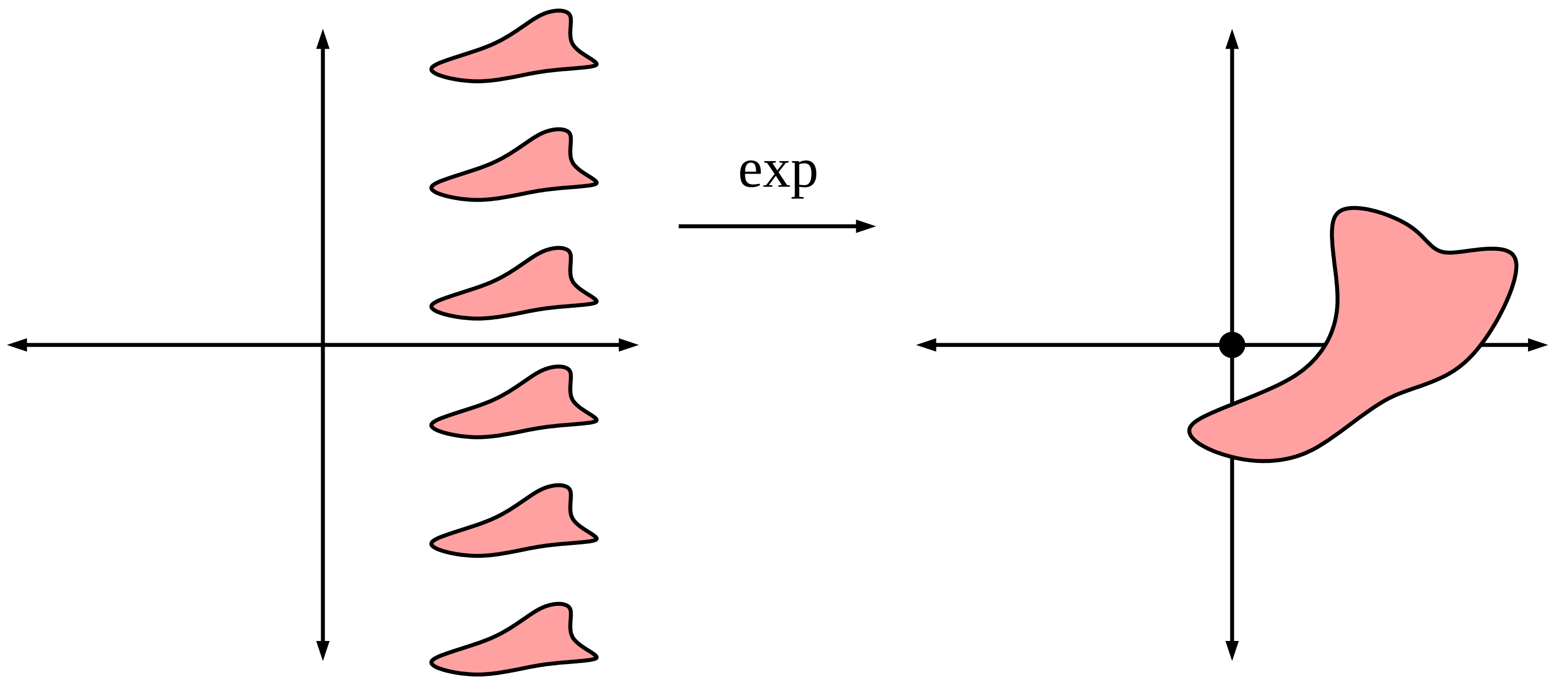
Related to area conjecture of Eremenko and Lyubich.

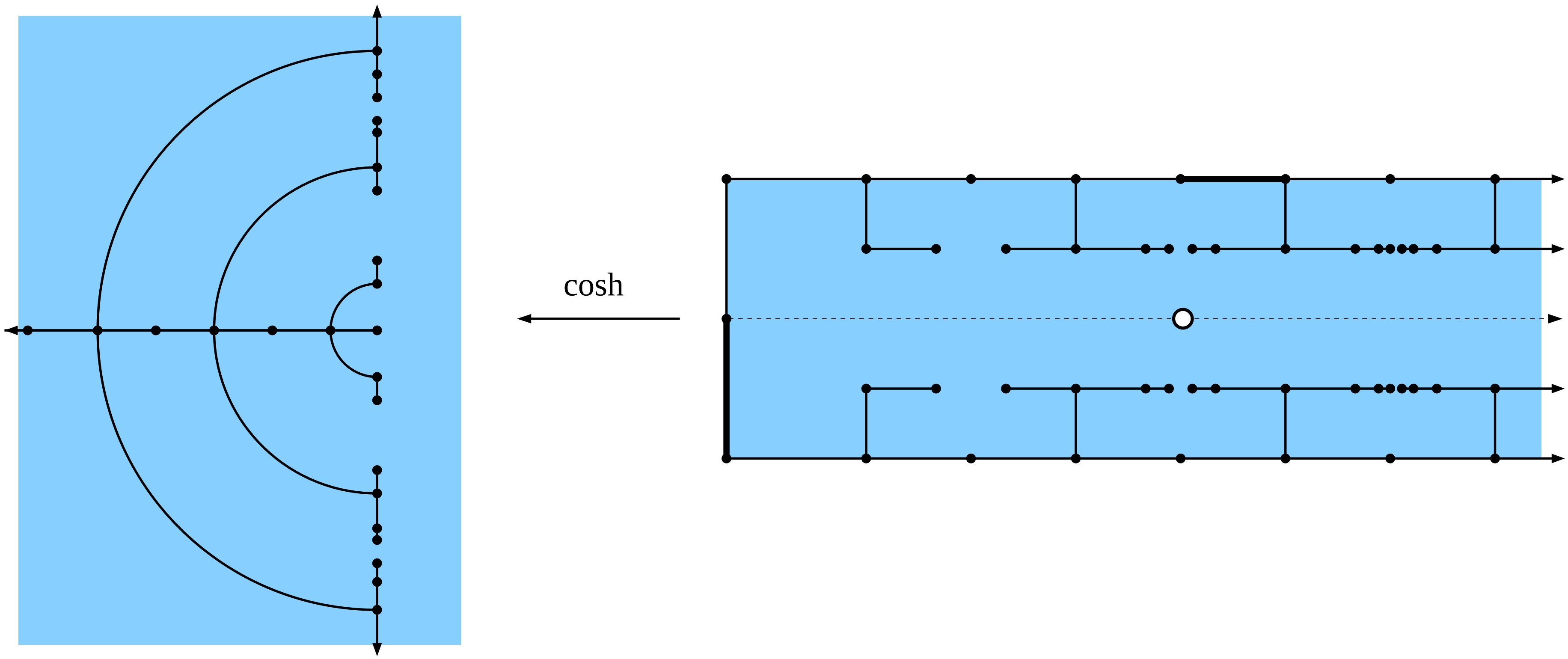
The logarithmic area of a set E is

$$\int_E \frac{dxdy}{x^2+y^2}.$$

The **area conjecture** asks if $f^{-1}(K)$ has finite logarithmic area whenever K is a compact subset of the complement of the singular set of K.

This happens for e^z. Preimages have log-area $\simeq n^{-2}$.





Counterexample to area conjecture

Openings are chosen so top edges have τ length $\simeq 1$

Each "room" contributes fixed area to the preimage of some disk.

Wiman's conjecture:

Define $m(r) = \min_{|z|=r} |f(z)|, \qquad M(r) = \max_{|z|=r} |f(z)|.$

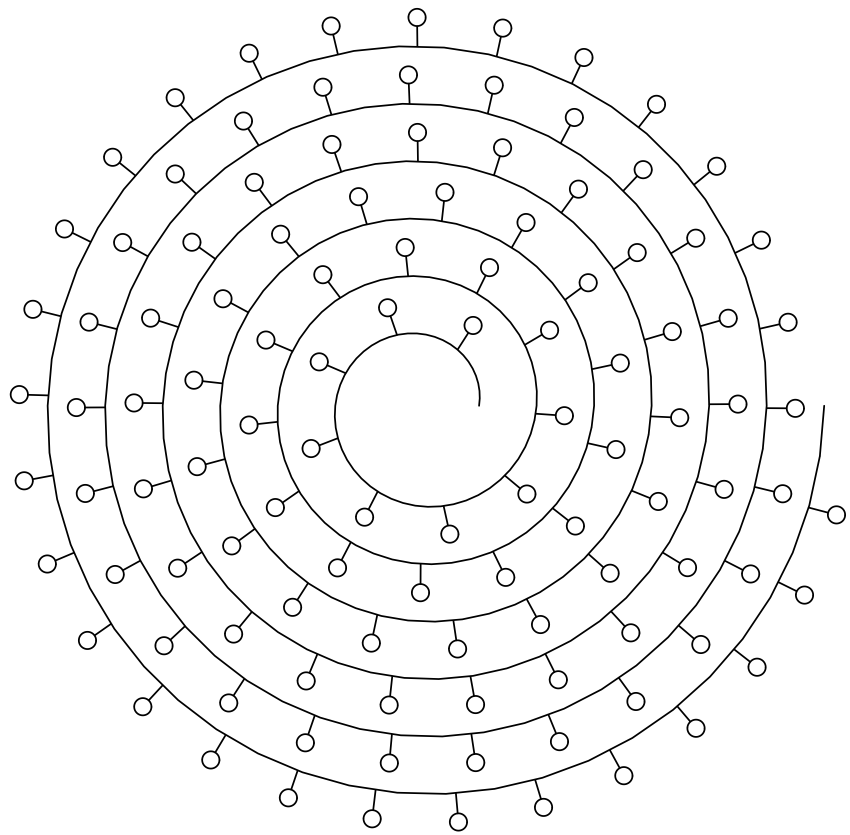
Wiman (1916) conjectured that for all entire functions

$$\limsup_{r \nearrow \infty} \frac{\log m(r,f)}{\log M(r,f)} \geq -1.$$

Sharp for $f(z) = \exp(z)$.

True in special cases, e.g., $|f(r)| = m(r)$ (Beurling, 1949).

False in general (Hayman, 1952).



Speiser class counterexample to Wiman's conjecture

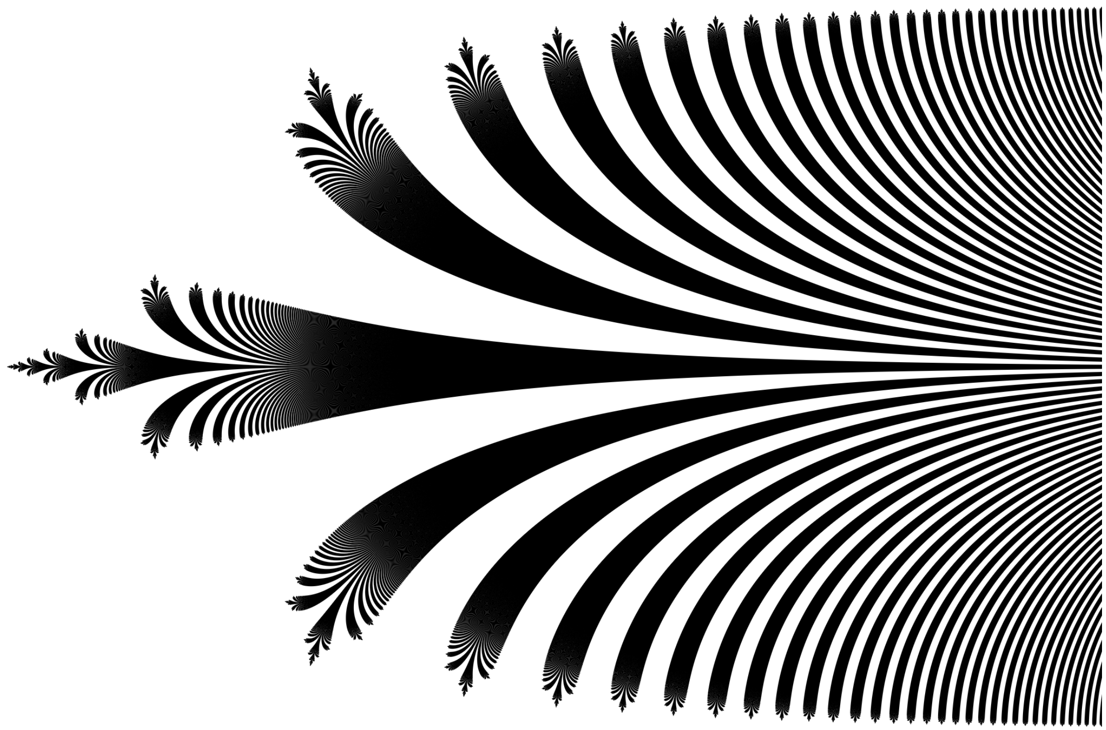
$$\frac{\log m(r,f)}{\log M(r,f)} \leq -C \cdot \log\log\log M(r,f)$$

Given an entire function f,

Fatou set $= \mathcal{F}(f) =$ open set where iterates are normal family.

Julia set $= \mathcal{J}(f) =$ complement of Fatou set.

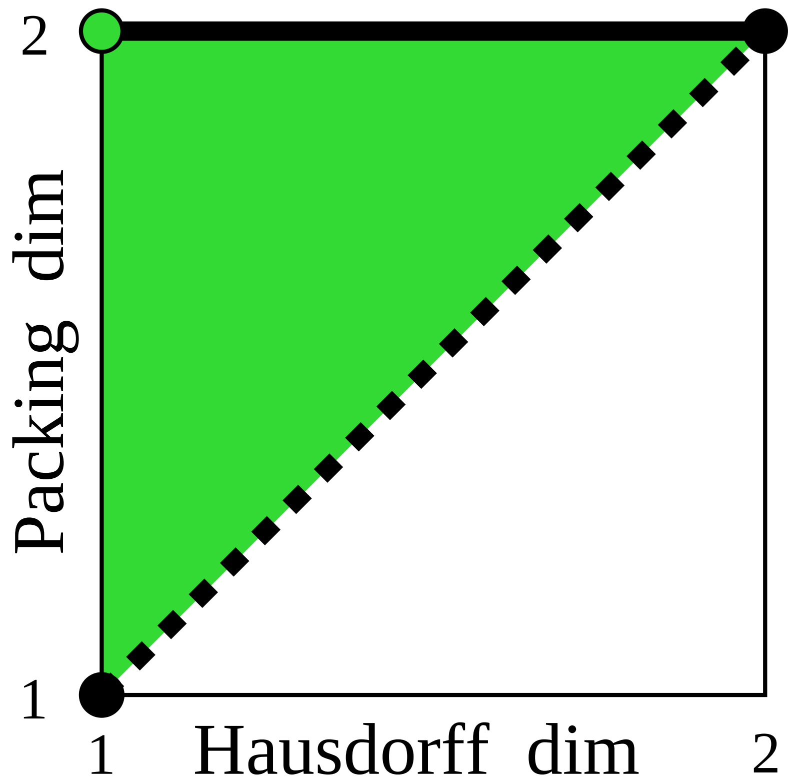
Julia set is usually fractal. What is its (Hausdorff) dimension?



$\mathcal{J}((e^z - 1)/2)$, courtesy of Arnaud Chéritat

A short (and incomplete) history:

- Baker (1975): f transcendental $\Rightarrow \mathcal{J}$ contains a continuum $\Rightarrow$ dim ≥ 1.
- Misiurewicz (1981), McMullen (1987) dim =2 occurs (is common)
- Stallard (1997, 2000): $\{\dim(\mathcal{J}(f)) : f \in \mathcal{B}\} = (1, 2]$.
- Rippon-Stallard (2005) Packing dim = 2 for $f \in \mathcal{B}$.
- Albrecht-B. $1 <$ H-dim < 2 can also occur in Speiser class
- B (2018) H-dim = P-dim = 1 can occur (example outside $\mathcal{B}$).
- Burkart (2019) $1 < \mathrm{P} - \dim < 2$ can occur (outside $\mathcal{B}$).
- Many other results known, e.g., relating growth rate to dimension.

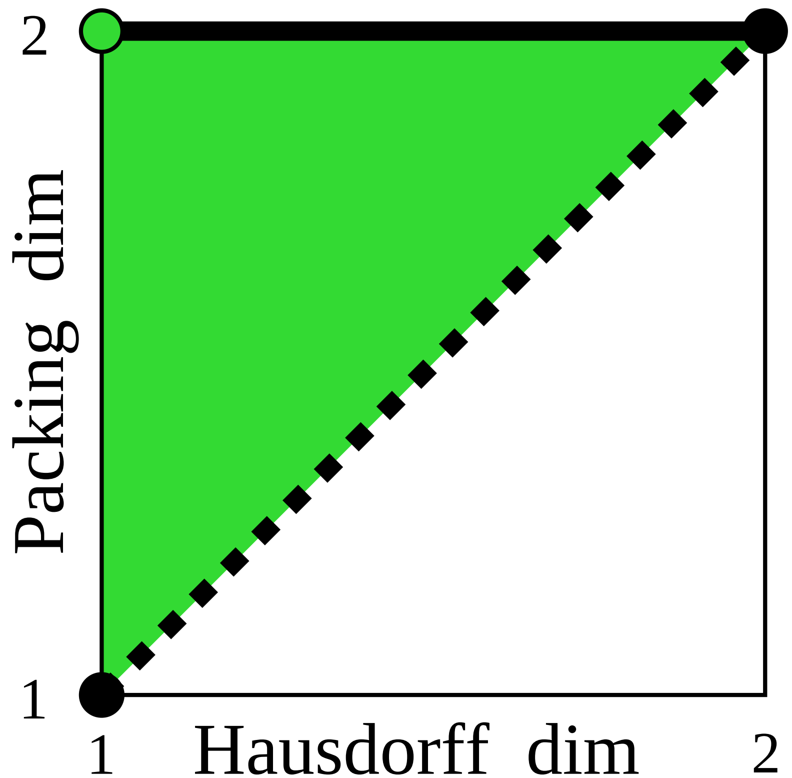


Hausdorff, upper Minkowski and packing dimension are defined as

$$\mathrm{Hdim}(K) = \inf\{s : \inf\{\textstyle\sum_j r_j^s : K \subset \cup_j D(x_j, r_j)\} = 0\},$$

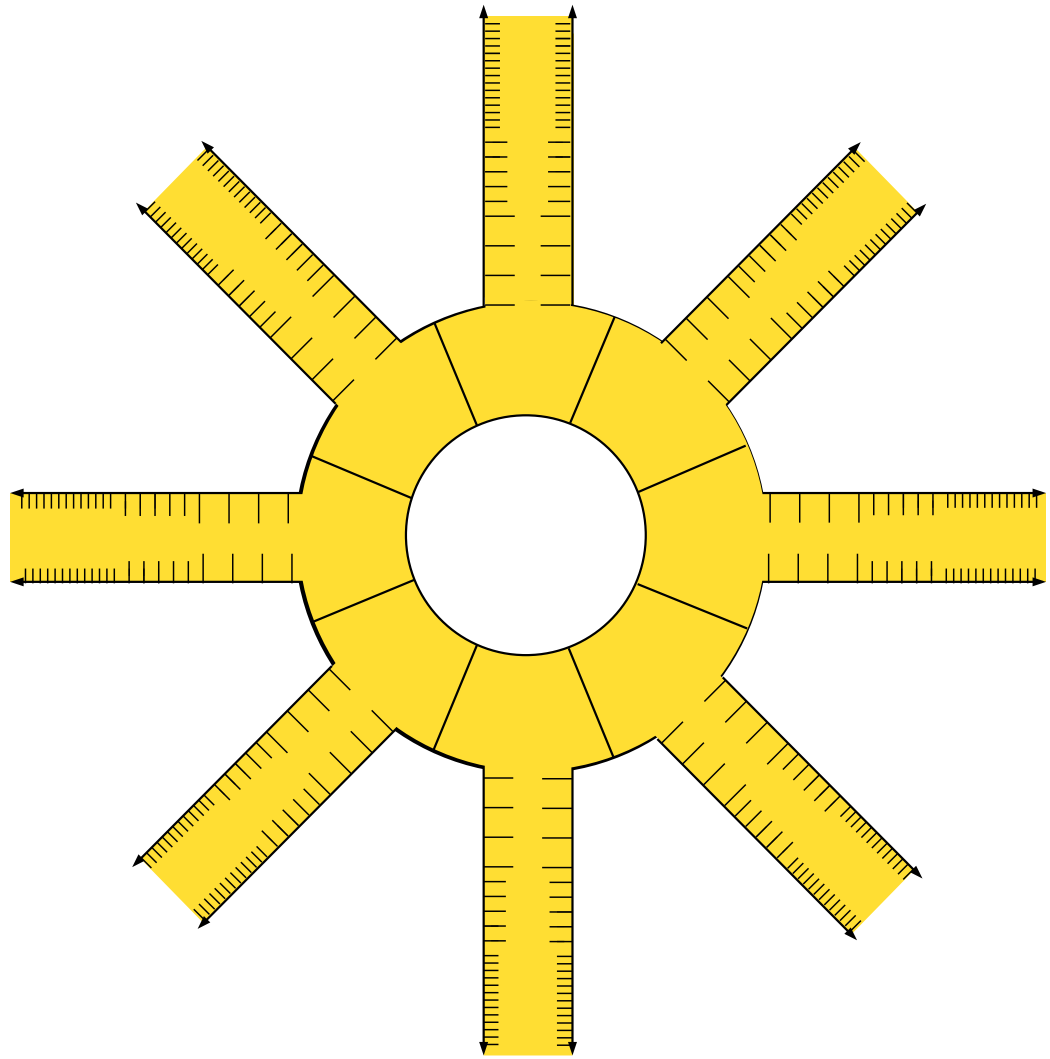
$$\overline{\mathrm{Mdim}}(K) = \inf\{s : \limsup_{r\to 0} \inf_N N r^s = 0 : K \subset \cup_{j=1}^N D(x_j, r)\},$$

$$\mathrm{Pdim}(K) = \inf\{s : K \subset \cup_{j=1}^\infty K_j : \overline{\mathrm{Mdim}}(K_j) \le s \text{ for all } j\},$$



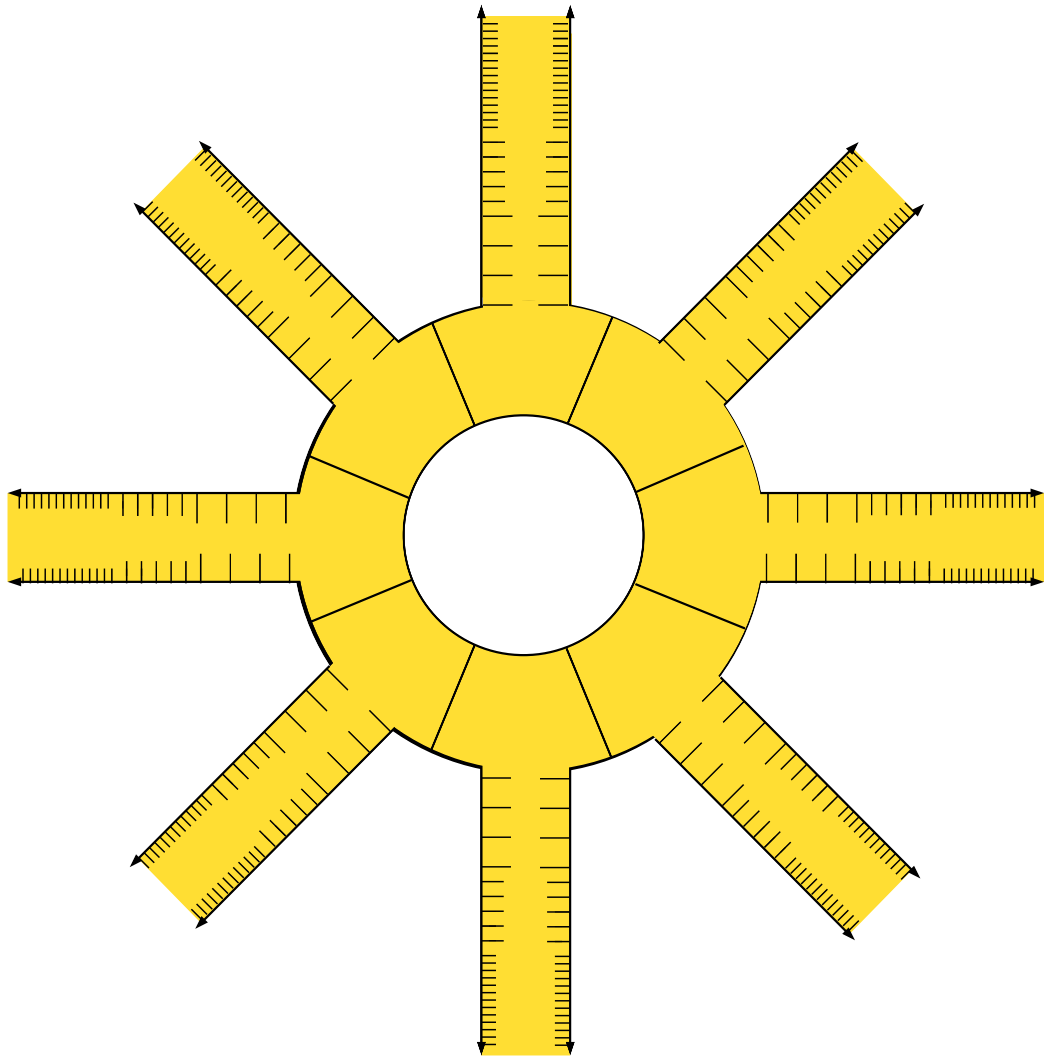
In polynomial dynamics it is difficult to construct examples with large dimension or positive area (Shishikura, Buff, Cheritat).

For entire functions, it is harder to find small Julia sets (dimension 1, finite length, rectifiable).

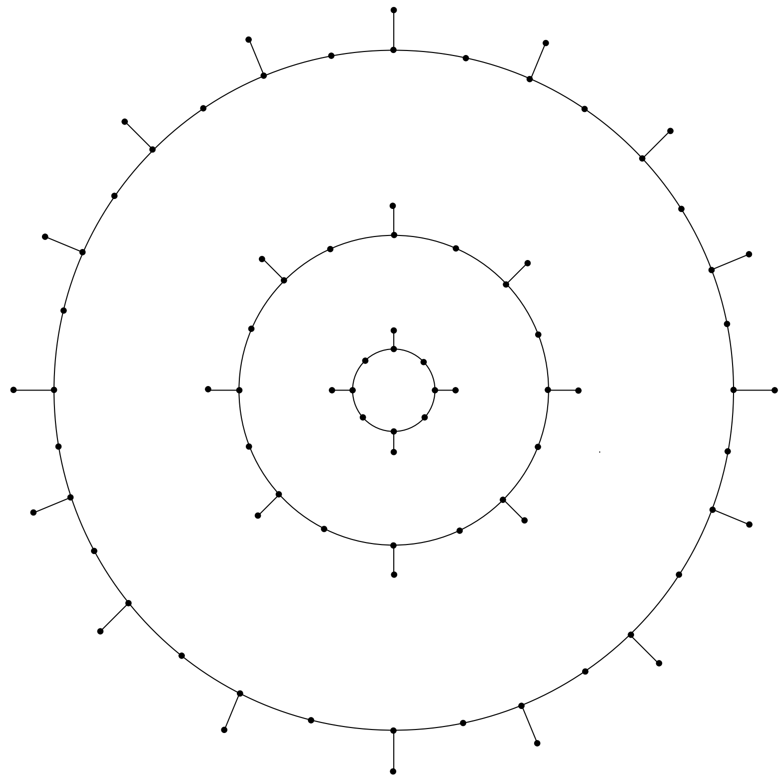


$$\inf\{\dim(\mathcal{J}(f)) : f \in \mathcal{S}\} = 1 \qquad \text{(B.-Albrecht, 2018).}$$

Uses upper and lower τ-bounds to control dimension.



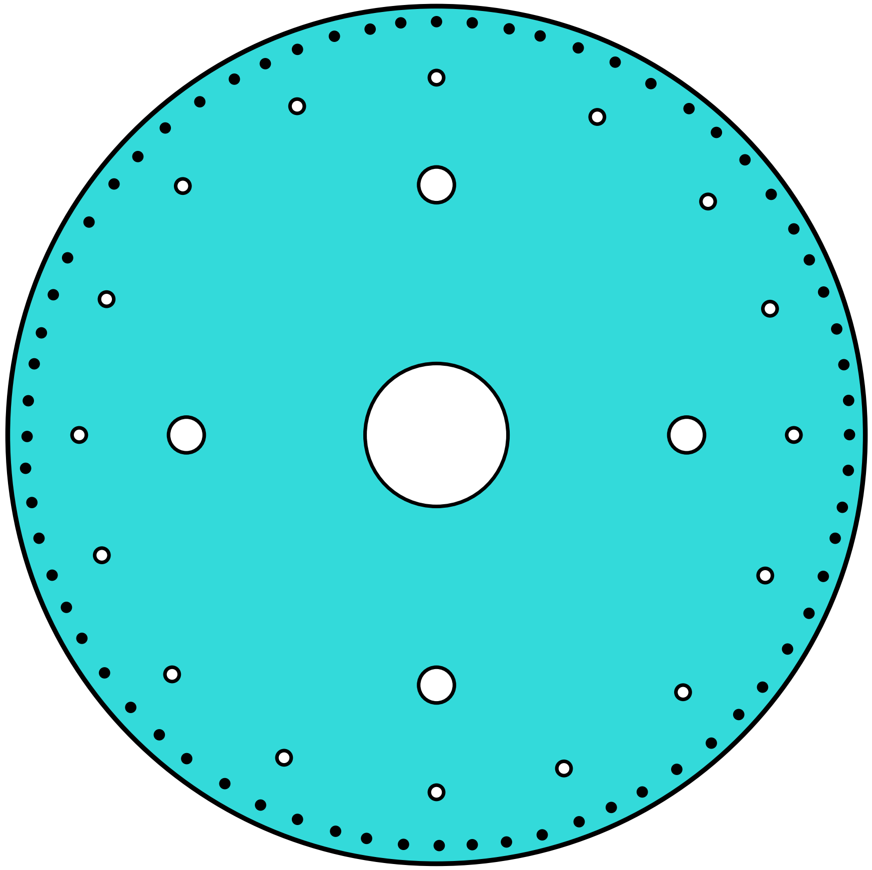
Radial edges have exponential imbalance. “Spikes” are chosen to give bounded imbalance. Must do this explicitly to control number and sizes of preimages of a disk.



Inspiration for dim $= 1$ example.

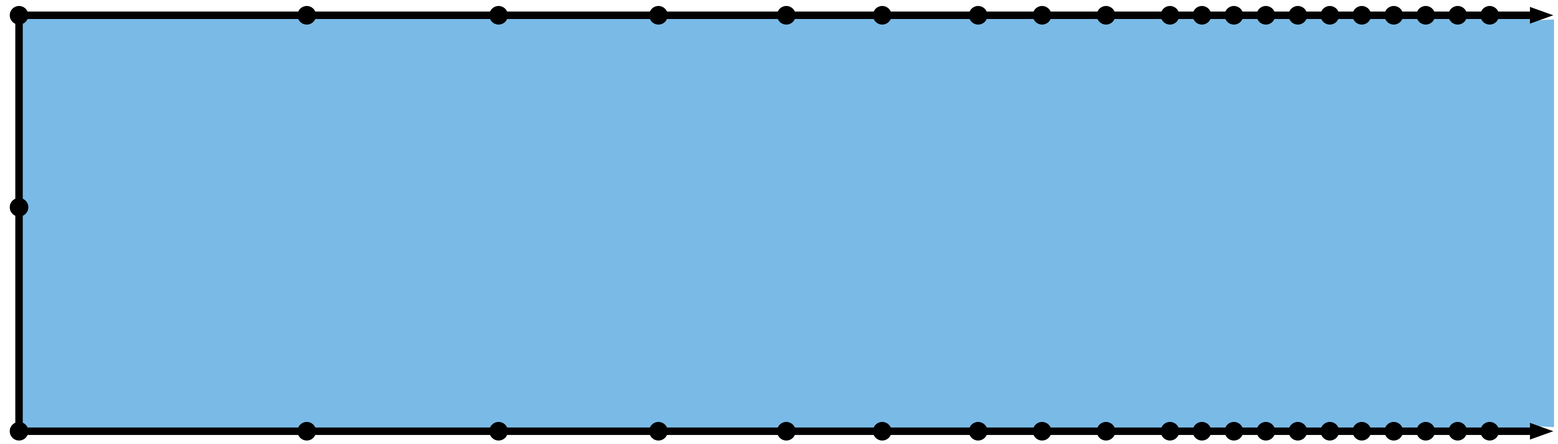
Actual proof uses infinite product with zeros on concentric circles.

$$f(z) = f_0(z) \cdot \prod_k \left(1 - \tfrac{1}{2}\,(z/R_k)^{n_k}\right)$$



Fatou component in "Dim = 1" example: C^1 boundaries
First multiply connected Fatou component where dynamics is understood.

Generalized in thesis of Markus Baumgartner under Bergweiler.
(Many thanks to Markus and Walter for help on my "dim =1" paper.)



Easier to build dim $= 1 + \epsilon$ examples in Eremenko-Lyubich class than in Speiser class. Recall EL-models theorem applies to half-strip.

Theorem (Stallard): There are Eremenko-Lyubich functions whose Julia sets have Hausdorff dimension close to 1.

We will define an entire function f that has an attracting fixed point at 0 and the basin contains a large disk $D(0, R)$. Julia set is outside this disk.

$$\mathcal{J}(f) \subset \bigcap X_n,$$

where

$$X_n = \{z : |f^k(z)| \geq R, k = 1, \dots, n\}.$$

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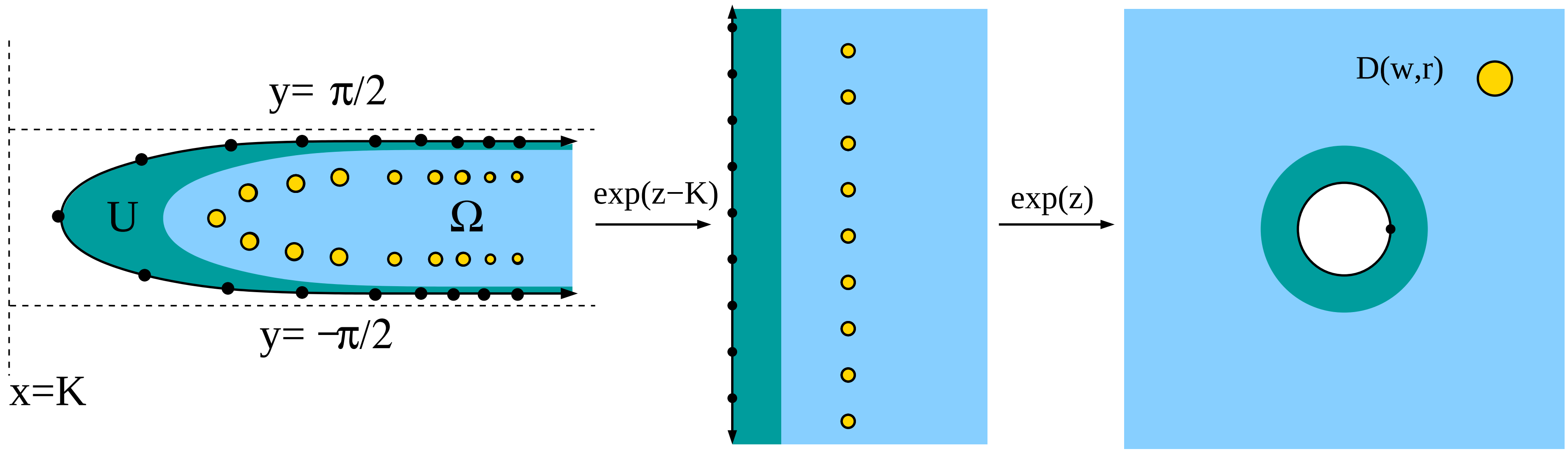
To prove $\dim(\mathcal{J}) \leq 1 + \delta$, it Suffices to cover X_1 by disks $\{D_j\}$ so that

$$\sum_j \operatorname{diam}(D_j)^{1+\delta} < \infty,$$

and prove that if D is a disk that hits $\mathcal{J}$ and is contained in $\{|z| > R\}$, then the preimages satisfy

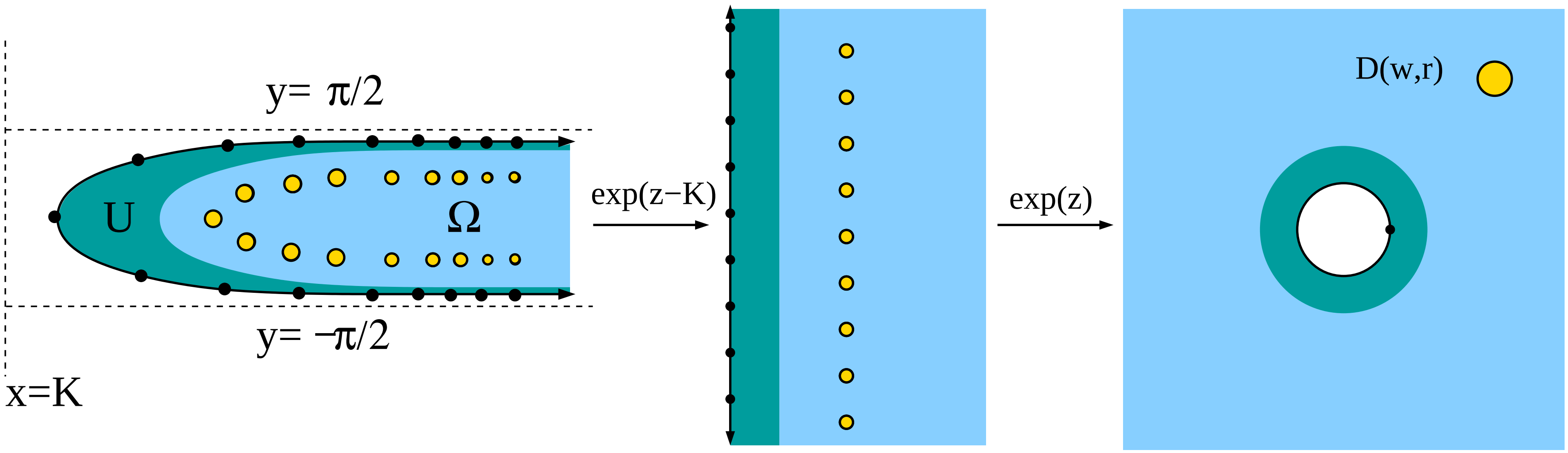
$$\sum_{W_j \in f^{-1}(D)} \operatorname{diam}(W_j)^{1+\delta} \leq \epsilon \cdot \operatorname{diam}(D)^{1+\delta}.$$



The models theorem applies to this tract.

Implies there is a QR extension of $\exp(\exp(z - K))$ from tract to plane, that is bounded off the tract.

Outside of tract maps compactly into itself, hence has bounded iterates, hence is in Fatou set. Thus Julia set is contained in the (blue) tract.

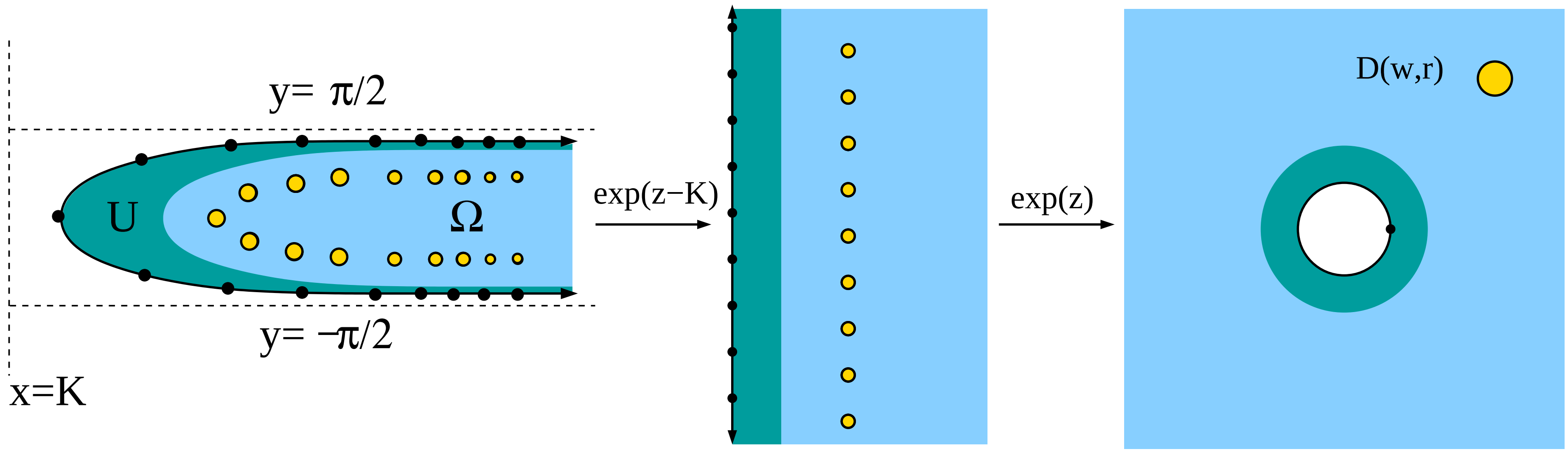


Preimage of gold disk $D = D(w, r)$ defined in two steps:

- stack of regions of diameter $O(r/|w|)$ on line $\{x = \log|w|\}$.
- region at height $2\pi k$ in stack has single preimage U_k of diameter

$$O\left(\frac{r}{|w|(\log|w| + 2\pi|k|)}\right).$$

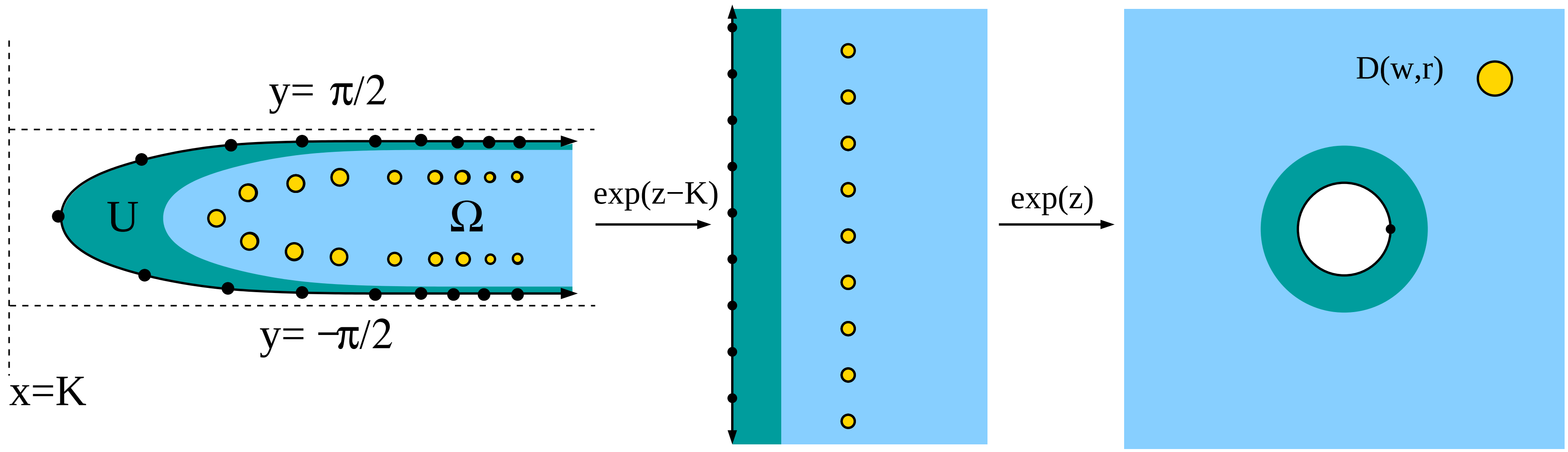
These estimates only use $(\log z)' = 1/z$.



If $\delta > 0$ is fixed and R is large enough, then

$$\sum_k \operatorname{diam}(U_k)^{1+\delta} \lesssim \left(\frac{r}{|w|}\right)^{1+\delta} \sum_k \frac{1}{(\log|w| + 2\pi|k|)^{1+\delta}}$$

$$\lesssim \left|\frac{r}{w}\right|^{1+\delta} \frac{1}{\delta \log^{1+\delta}|w|} \ll \left|\frac{r}{w}\right|^{1+\delta} \ll r^{1+\delta}$$



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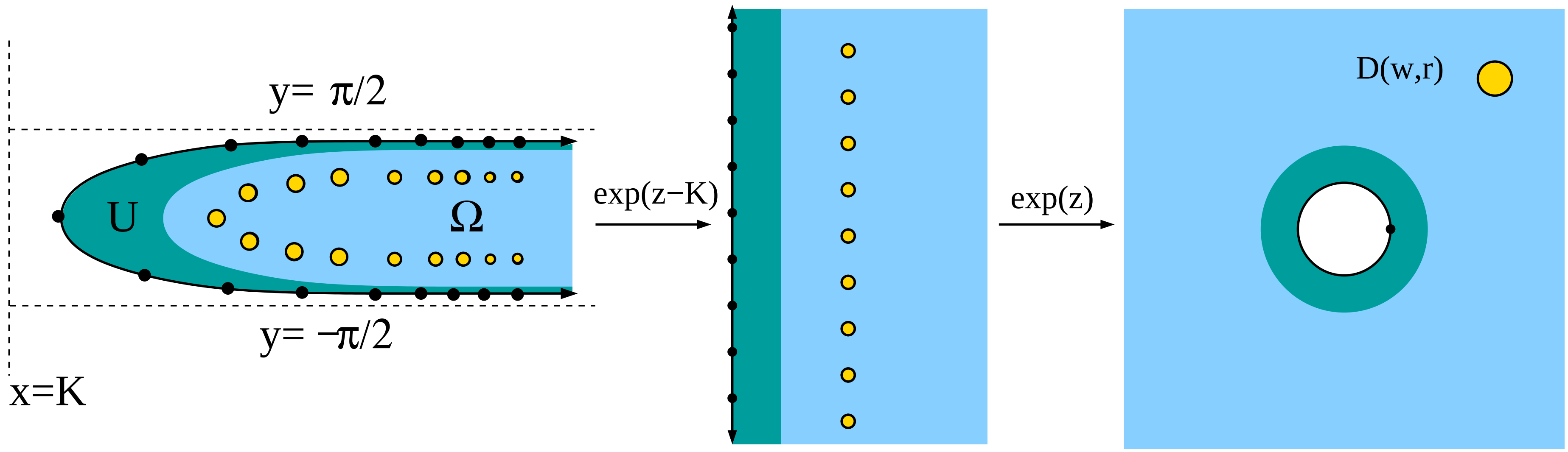
This almost proves $\dim(\mathcal{J}) \leq 1 + \delta$.

Proof above applies to quasiregular g. What about entire $f = g \circ \varphi$?



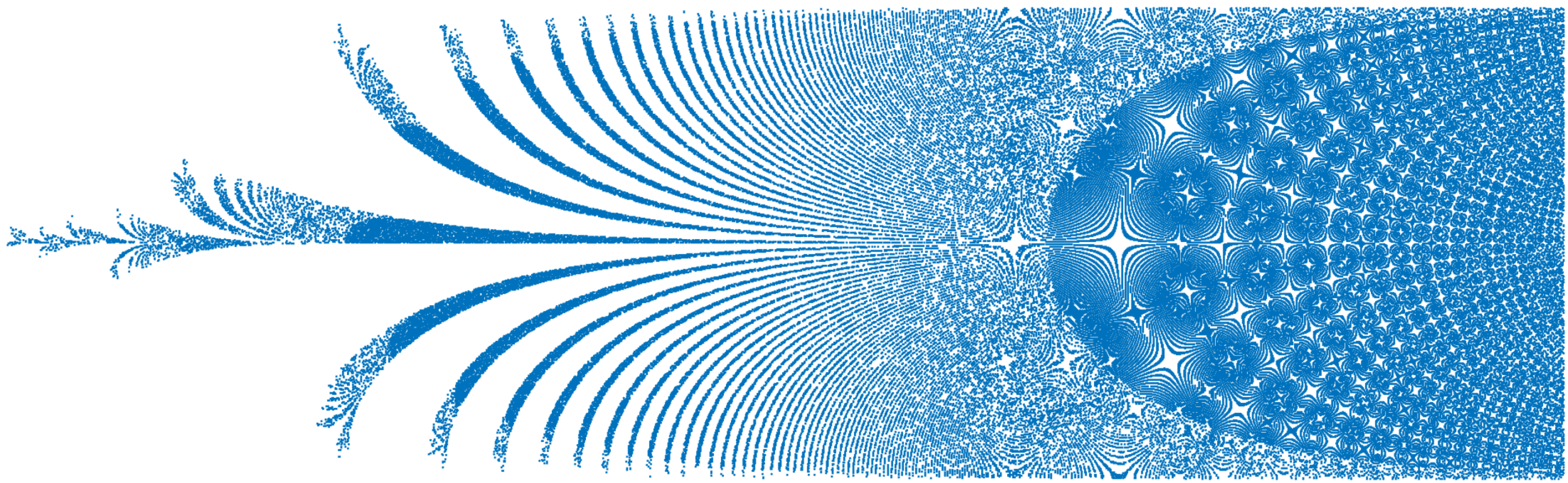
Lemma: Suppose A, B are disjoint, planar sets and

$$\int_A \frac{dxdy}{|z-w|^2} \leq C < \infty,$$

for all $w \in B$. If φ is a K-QC map that is conformal off A, then φ is biLipschitz on B with constant $M(C, K)$.

Follows from work of Bojarski, Lehto, Tecihmüller and Wittich.

In our case, φ is non-conformal on exponentially thin region (cyan) and hence is bi-Lipschitz near the Julia set (gold disks).



Given an entire function f,

Fatou set $= \mathcal{F}(f) =$ open set where iterates are normal family.

Julia set $= \mathcal{J}(f) =$ complement of Fatou set.

f permutes components of its Fatou set.

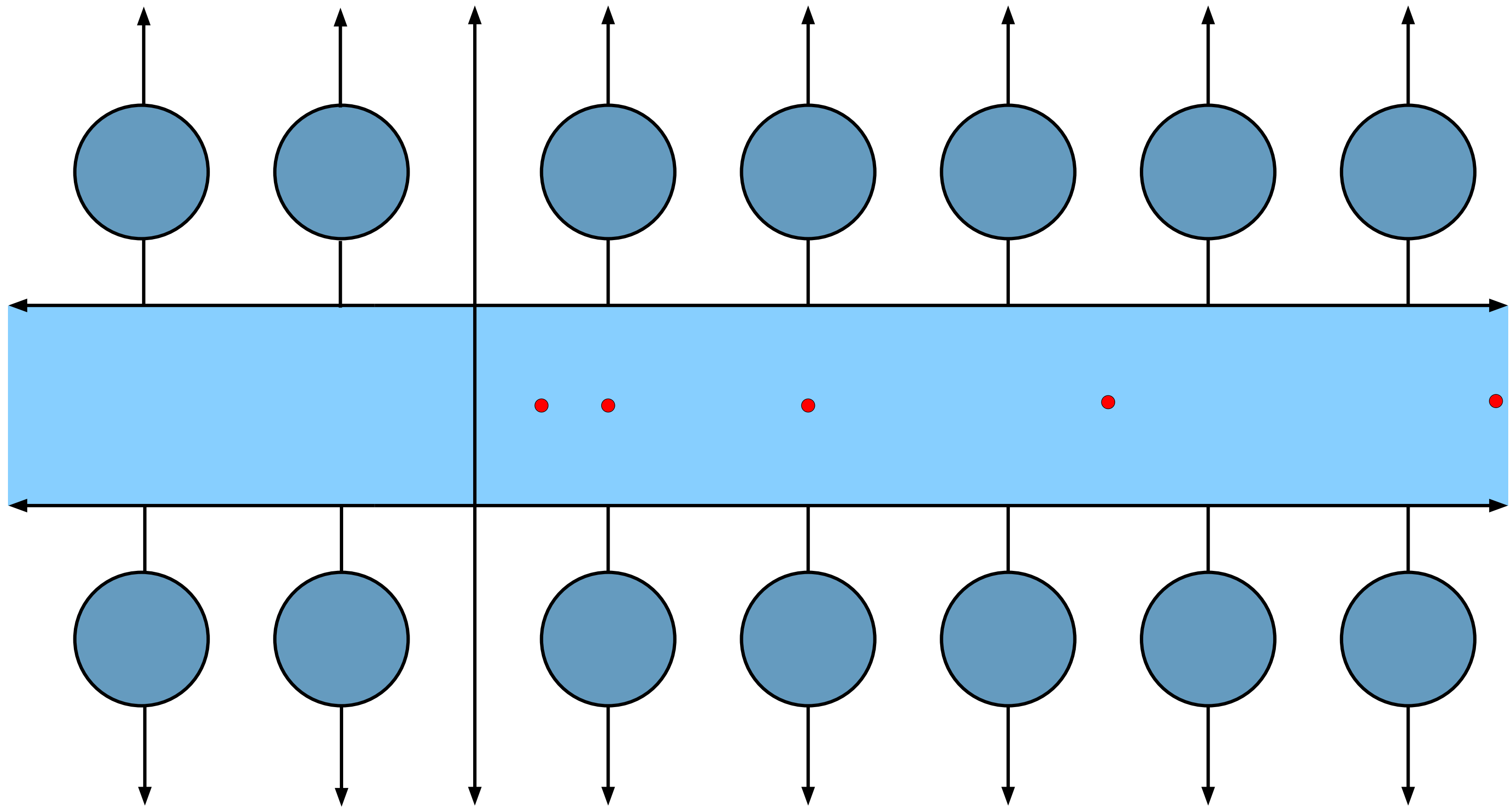
Wandering domain $=$ Fatou component with infinite orbit.

- Entire functions can have wandering domains (Baker 1975).

- No wandering domains for rational functions (Sullivan 1985).

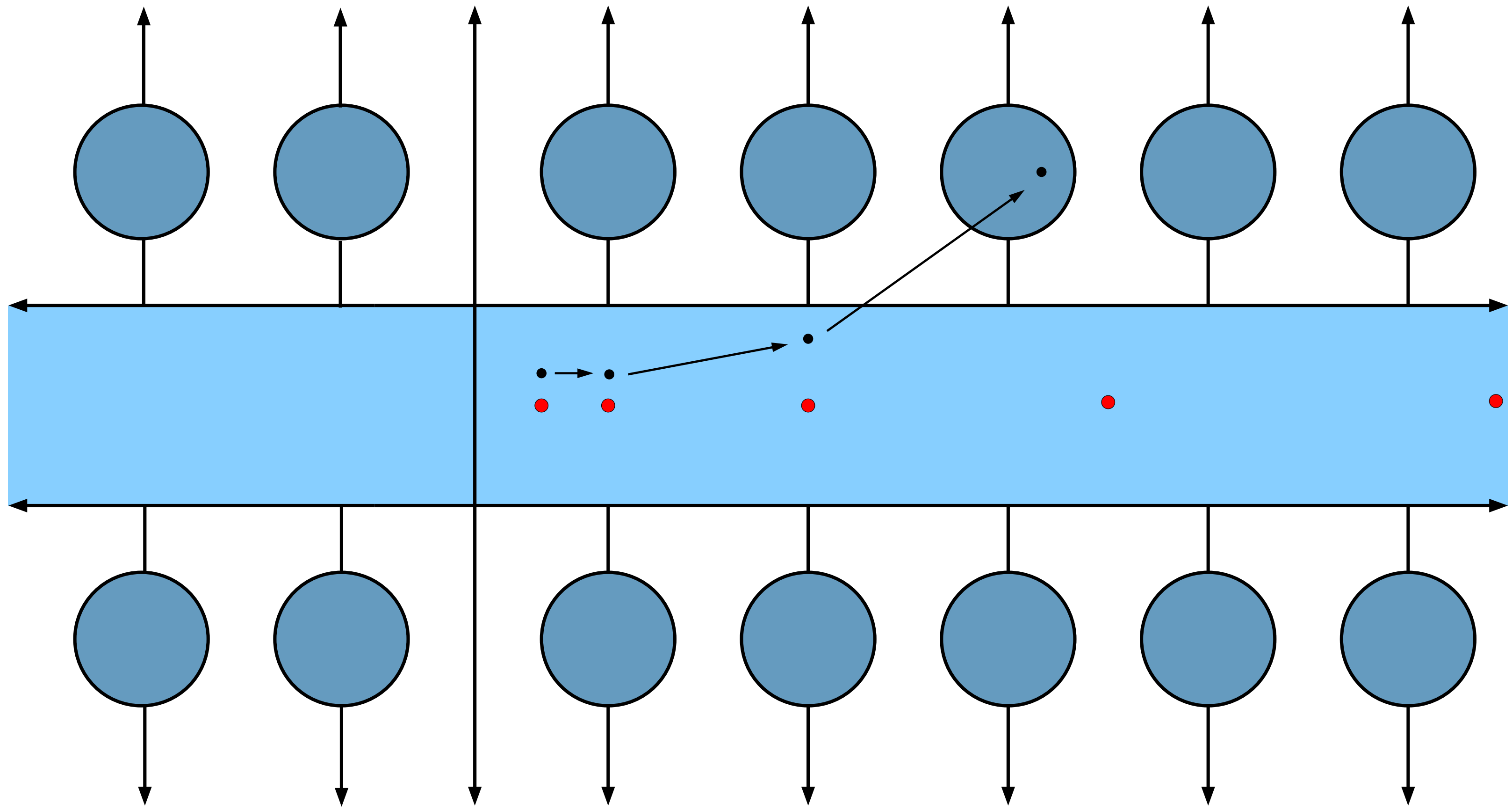
- Also none in Speiser class (Eremenko-Lyubich, Goldberg-Keen).

Are there wandering domains in Eremenko-Lyubich class?



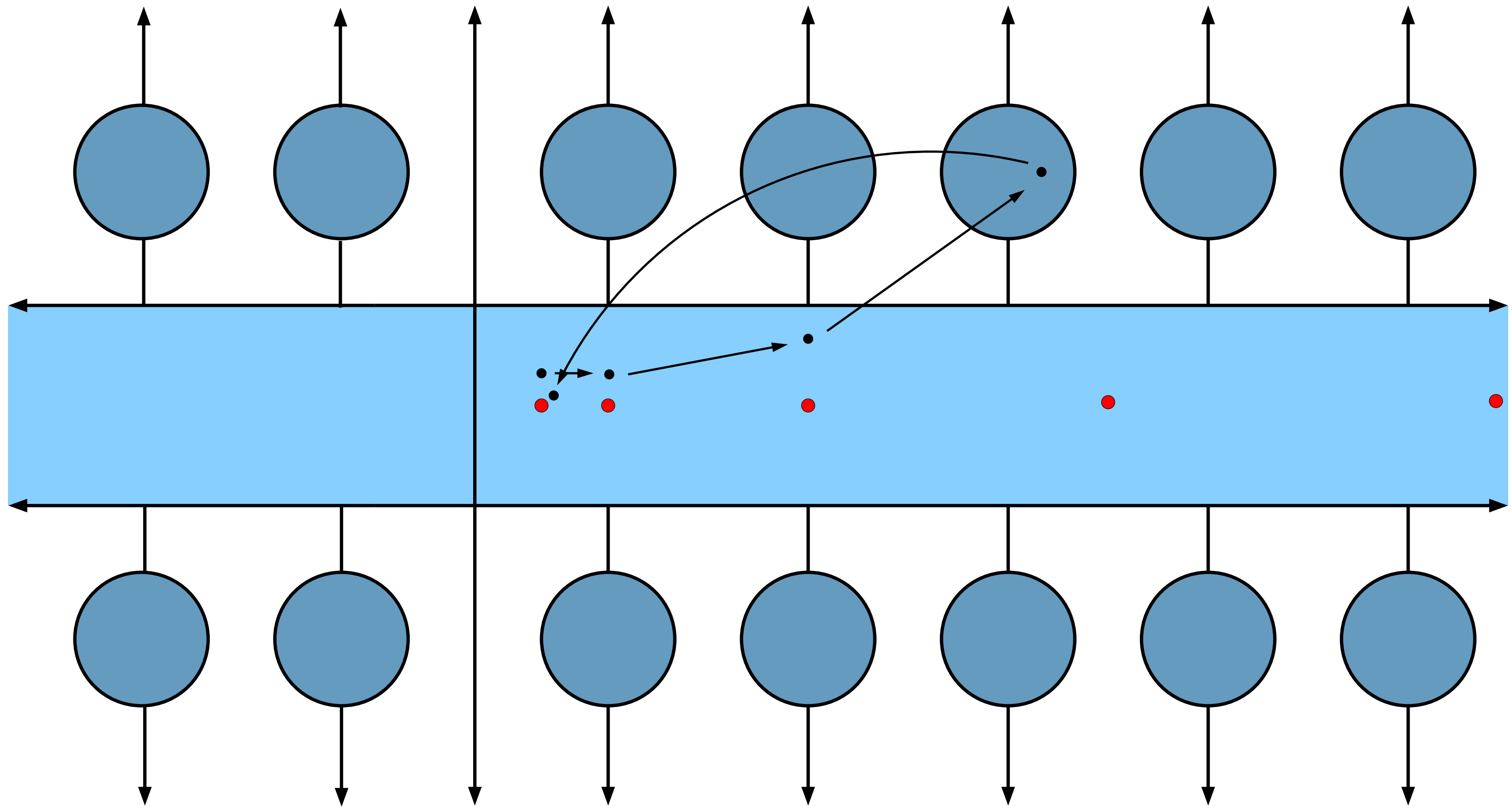
Graph giving wandering domain in Eremenko-Lyubich class.

Original proof corrected by Marti-Pete and Shishikura, who also give alternate construction.



Graph giving wandering domain in Eremenko-Lyubich class.

Variations by Lazebnik, Fagella-Godillon-Jarque, Osborne-Sixsmith.



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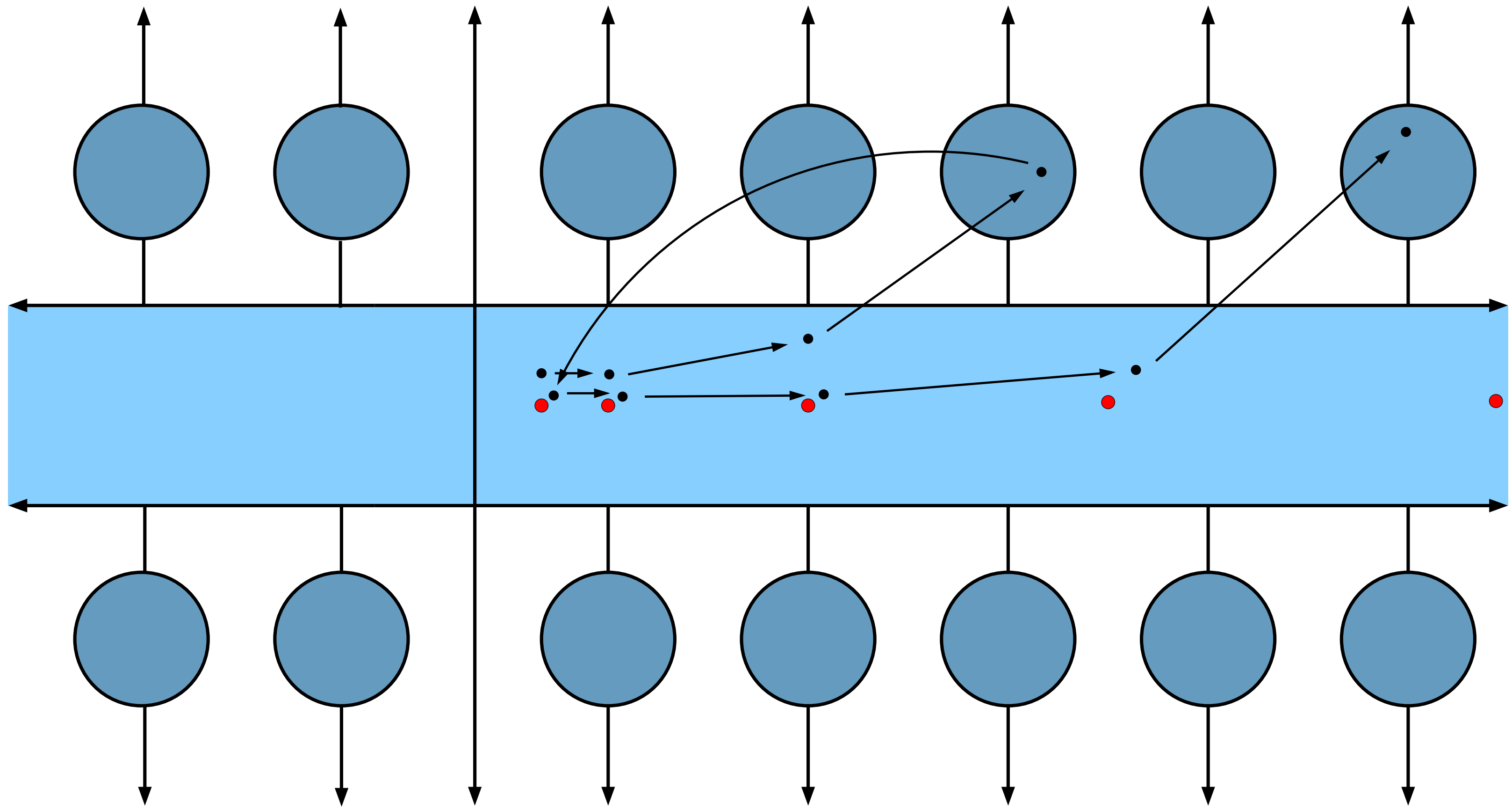

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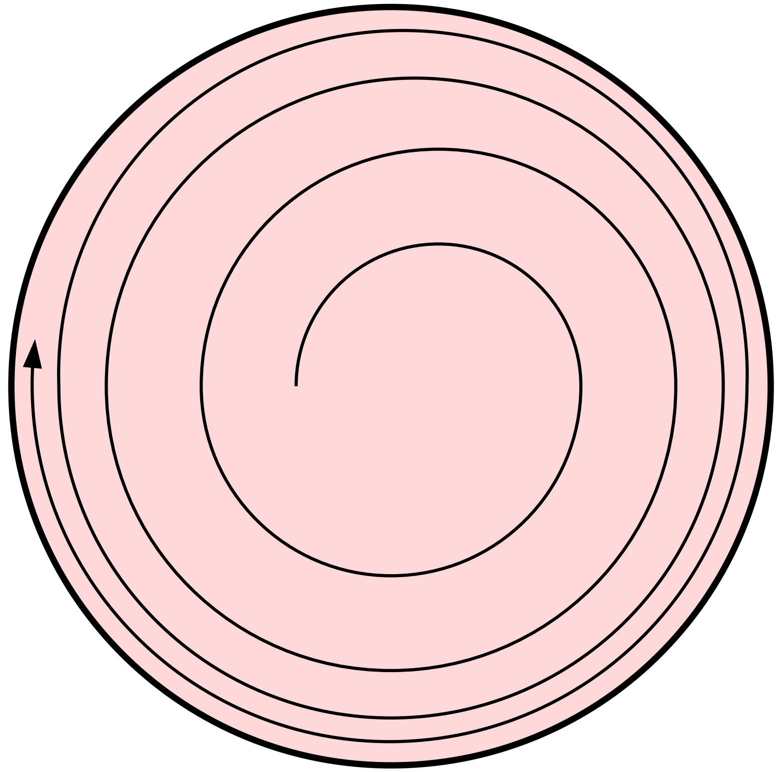
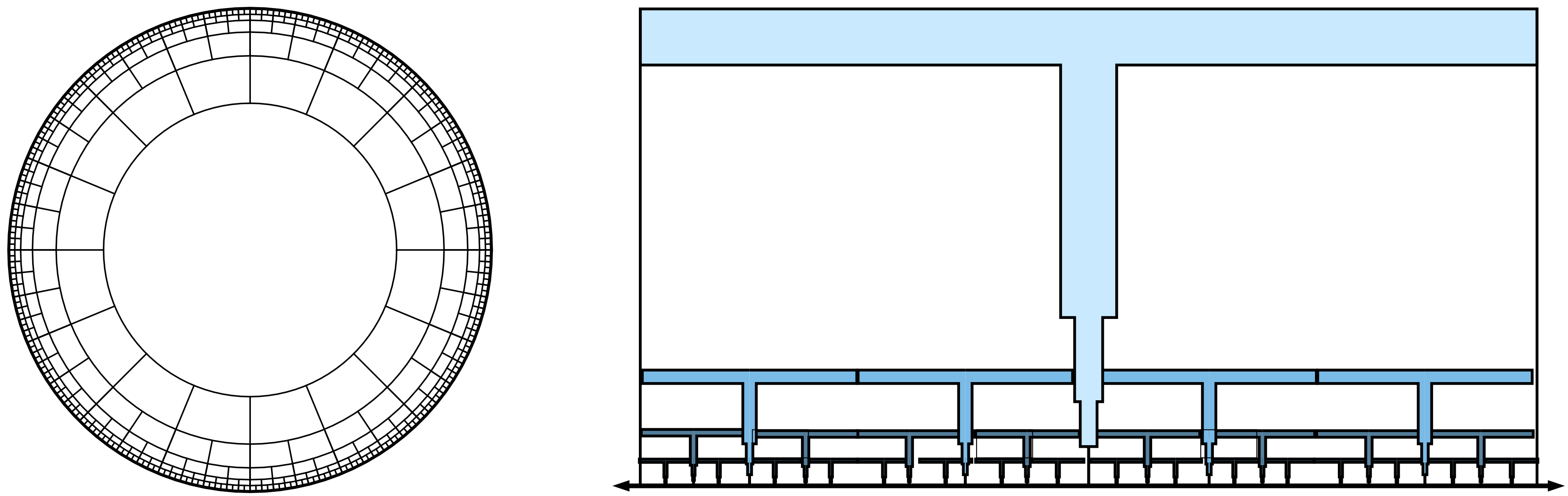


A curve spiraling out the boundary of the disk. We can add vertices to make this bounded geometry and satisfy the τ-lower bound.

The folding construction gives a function on the unit disk with two critical values so that $\{|f| > 2\}$ spirals out to disk (first constructed by Valiron).



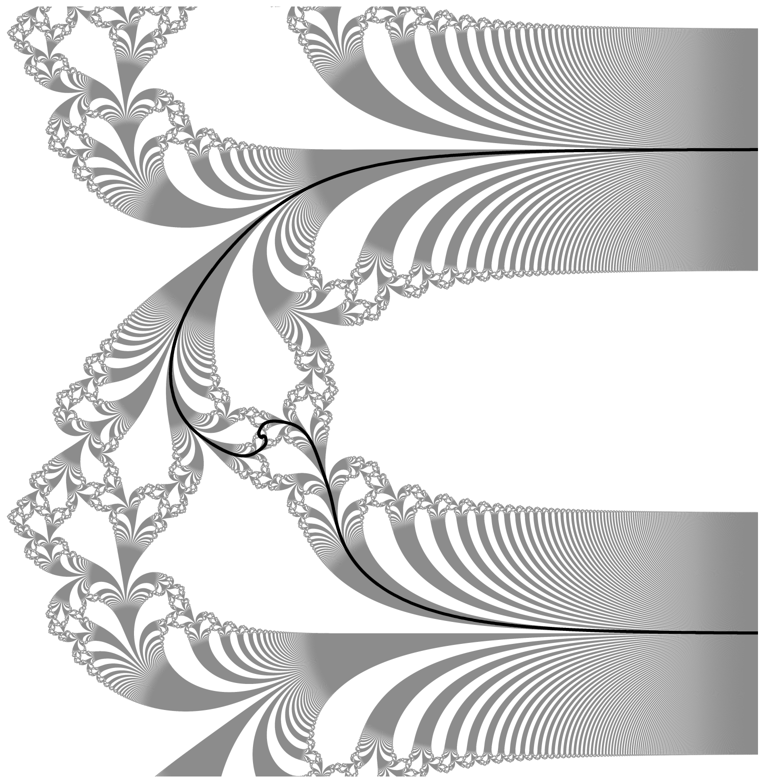
A holomorphic function on the disk with two critical values so that the set $\{z : |f(z)| > \epsilon\}$ has finite hyperbolic area for any $\epsilon > 0$.

Decompose $\mathbb{D}$ into Whitney boxes. Thicken the edges to form disjoint R-components, each tending to boundary. The remaining boxes are D-components, where f is close to zero (except very near the boundary), if we place a high degree critical point in each.

For f entire, **escaping set** is $I(f) = \{z : f^n(z) \to \infty\}$.

Known that $\mathcal{J}(f) = \partial I(f)$.

Fatou observed $I(f)$ often consists of curves to ∞.



Curve in escaping set, courtesy of Lasse Rempe-Gillen

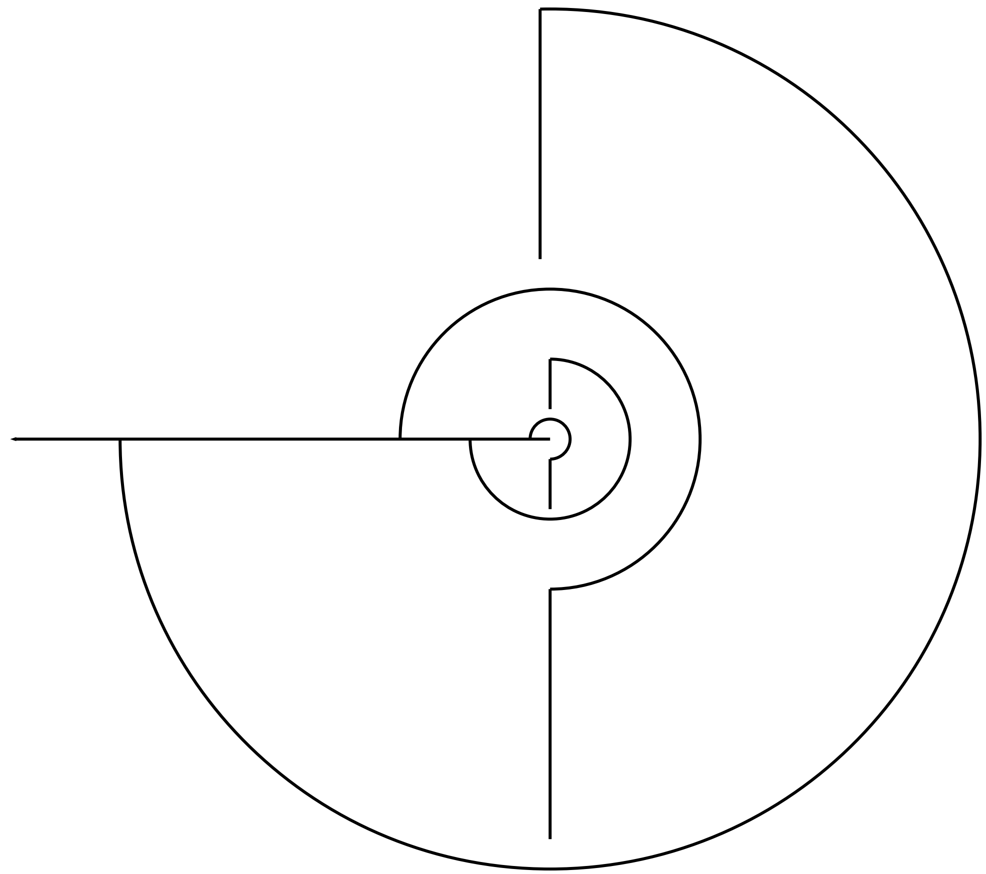
Eremenko Conj: components of $I(f)$ are unbounded (still open).

Strong Eremenko Conj (SEC): path components are unbounded.

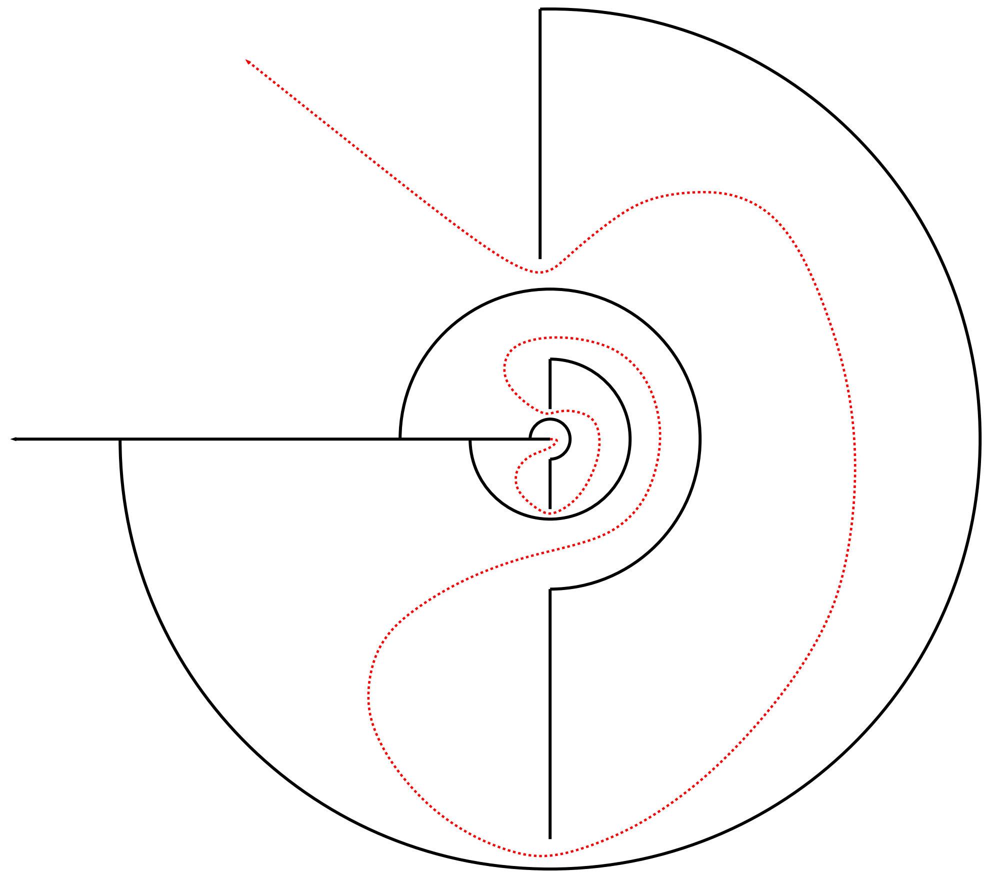
Rottenfusser, Rüchert, Rempe-Gillen and Schleicher (2011) proved:

- SEC true for EL functions with finite order of growth.
- SEC false for some EL functions with infinite order.
- Examples with trivial path components.

QC-Folding gives examples in Speiser class.



Speiser class counterexample to SEC.
Path components of escaping set can be points.
Other exotic examples by Rempe-Gillen.



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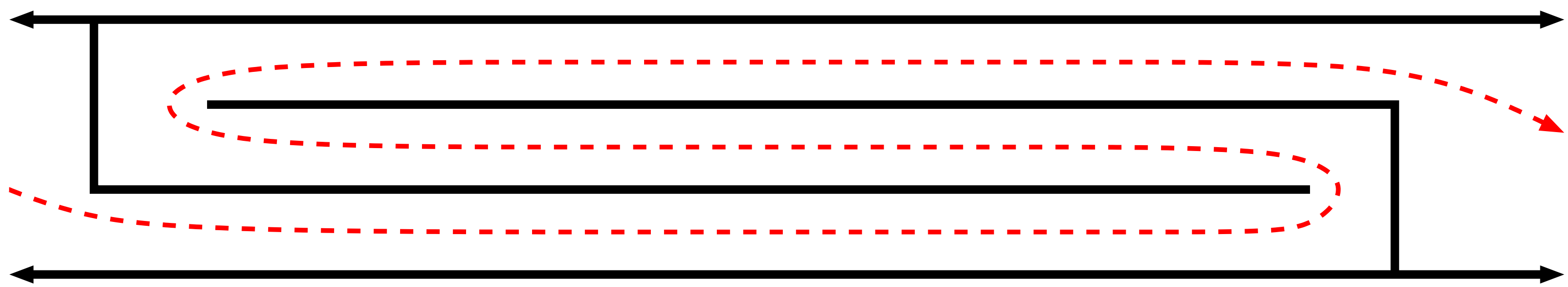
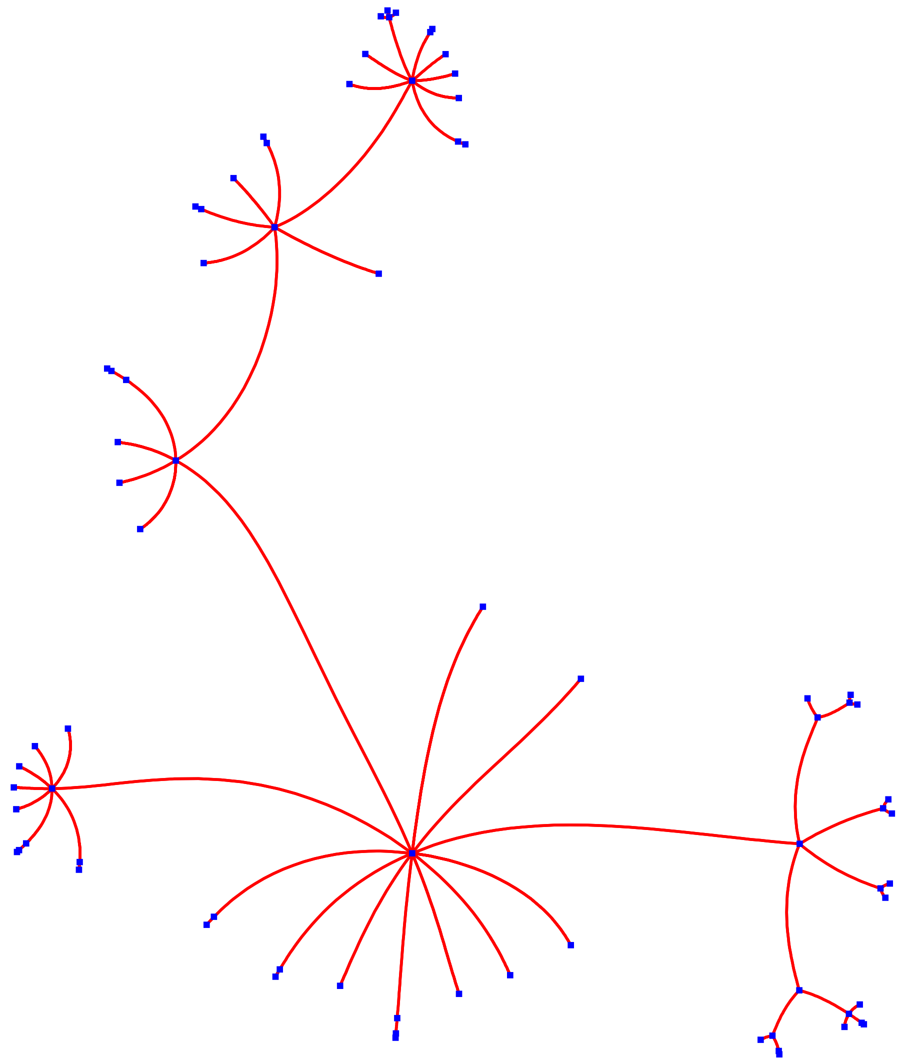
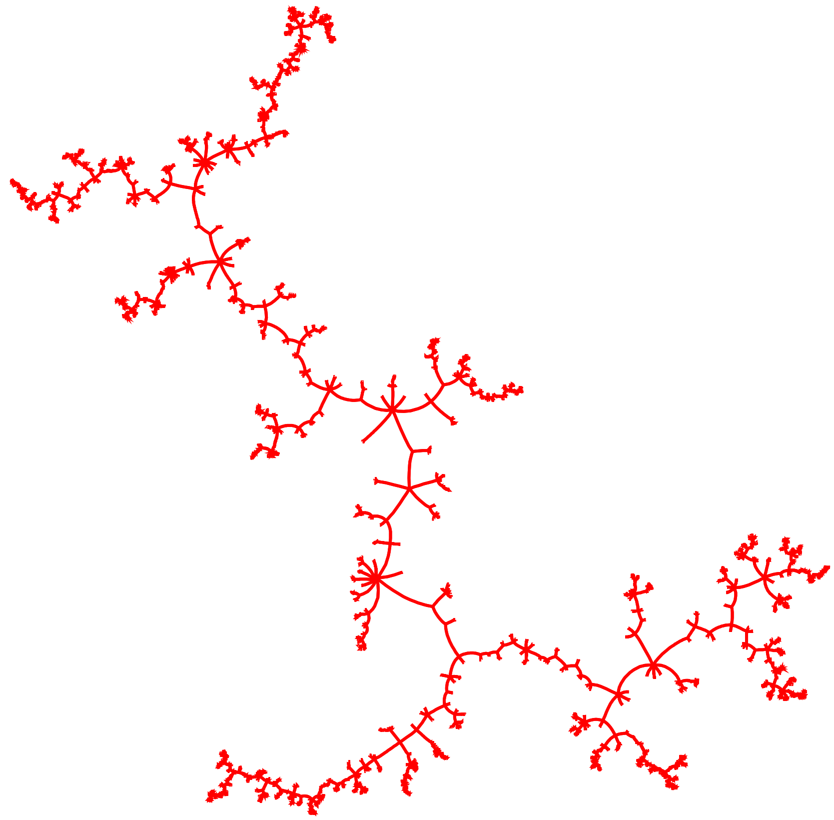
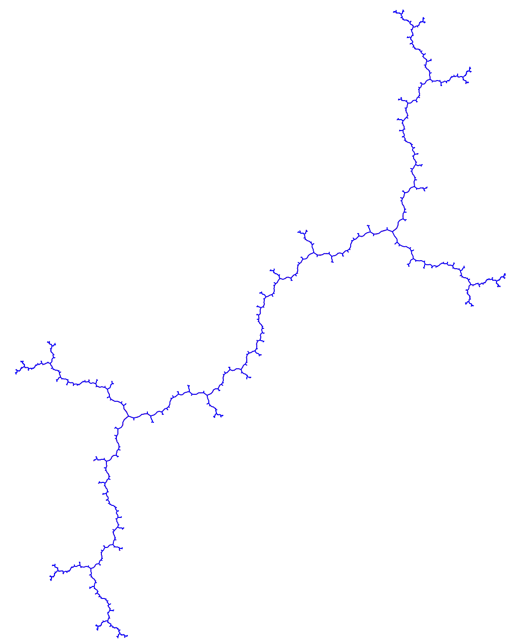


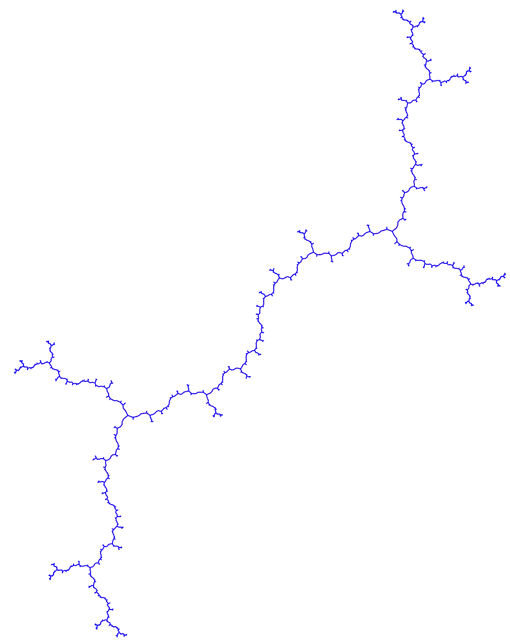
SEC counterexample in logarithmic coordinates



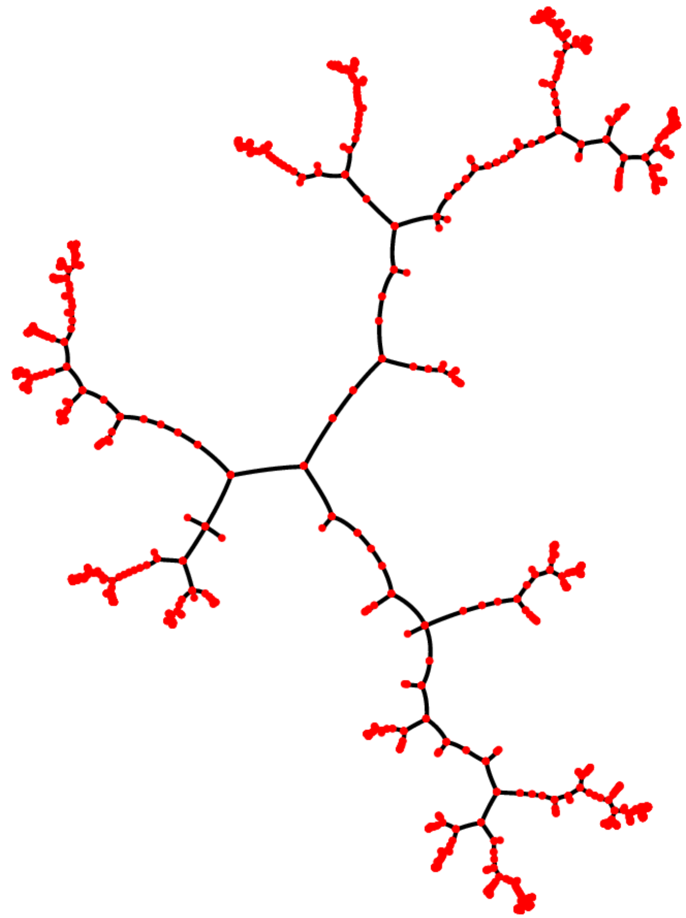




Ceci n'est pas un ensemble de Julia.



True tree based on combinatorics of Julia set of $z^2 + i$.
Example of “rigidity”: combinatorics determines geometry.



True DLA