# HARMONIC MEASURE, TRUE TREES AND QUASICONFORMAL FOLDING

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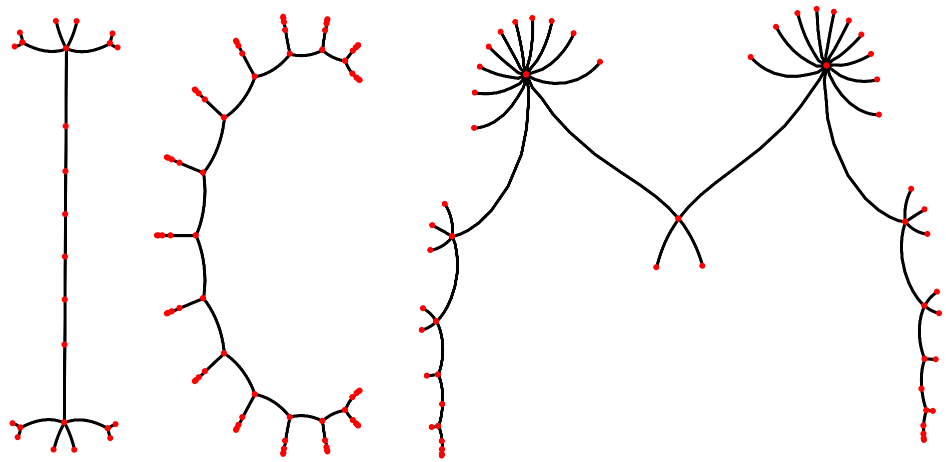
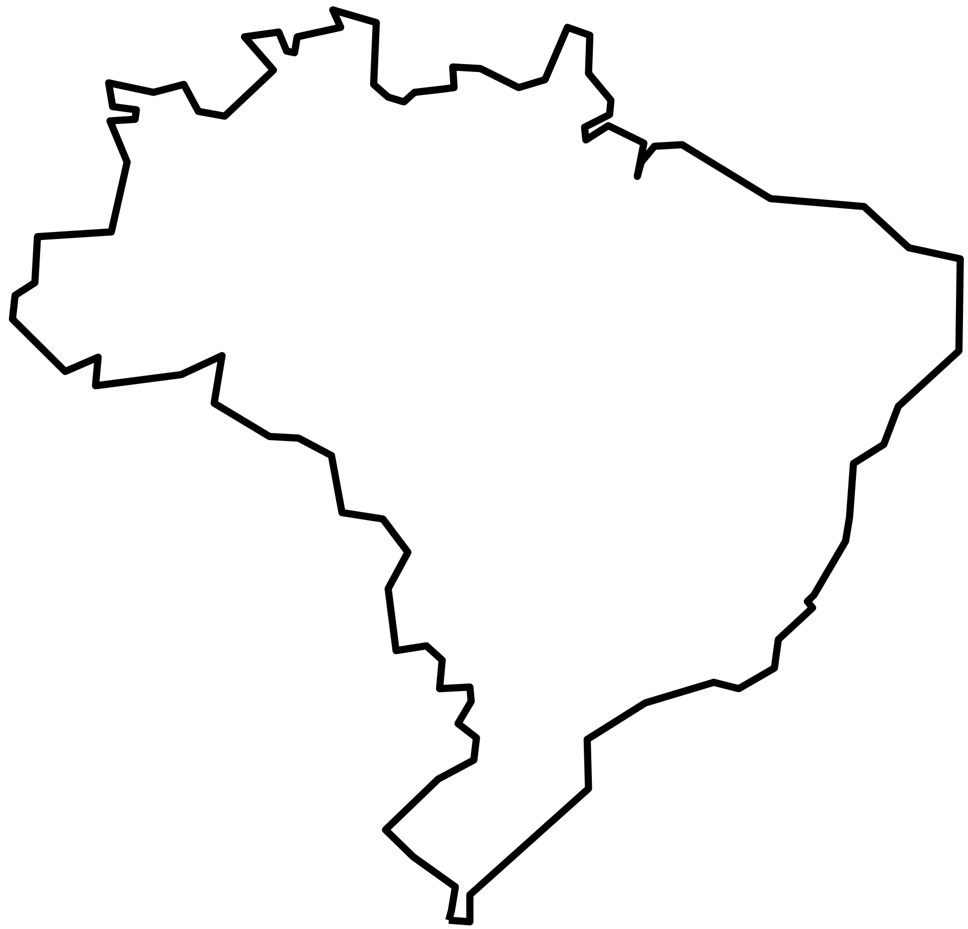
`www.math.sunysb.edu/~bishop/lectures`



**Harmonic measure** = hitting distribution of Brownian motion



Suppose $\Omega$ is a planar Jordan domain.

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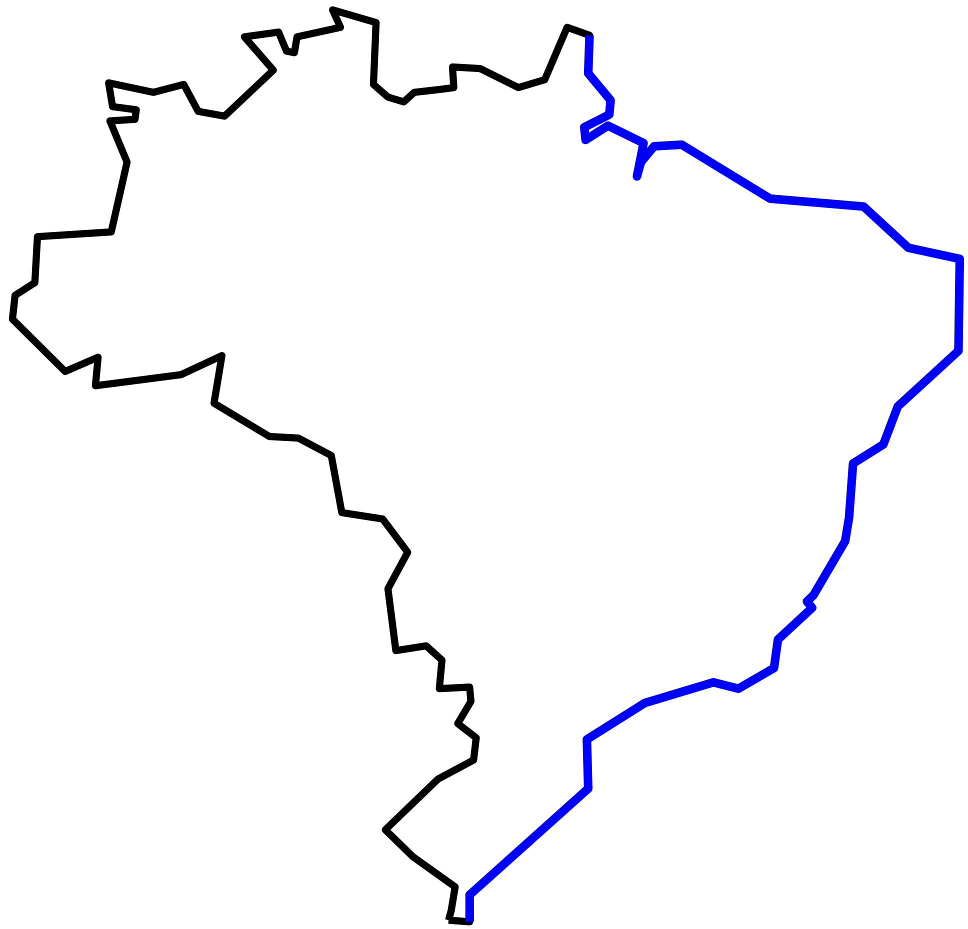

Let $E$ be a subset of the boundary, $\partial\Omega$.

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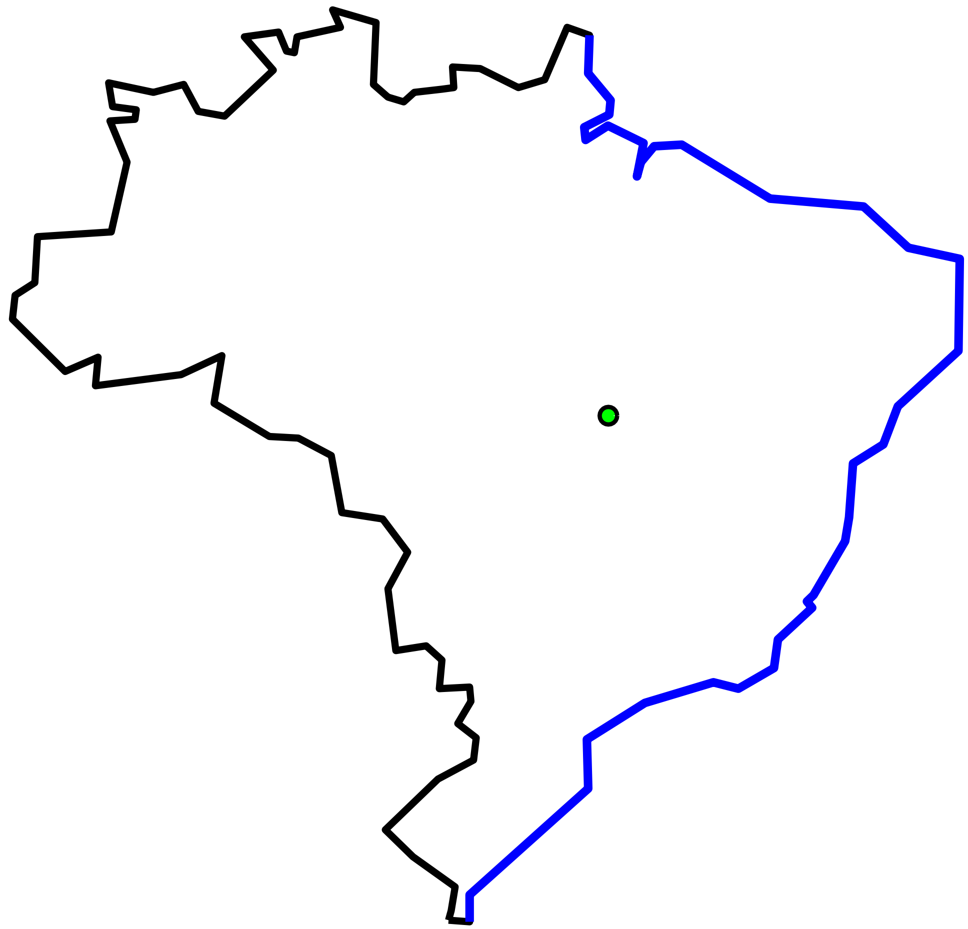

Choose an interior point $z \in \Omega$.

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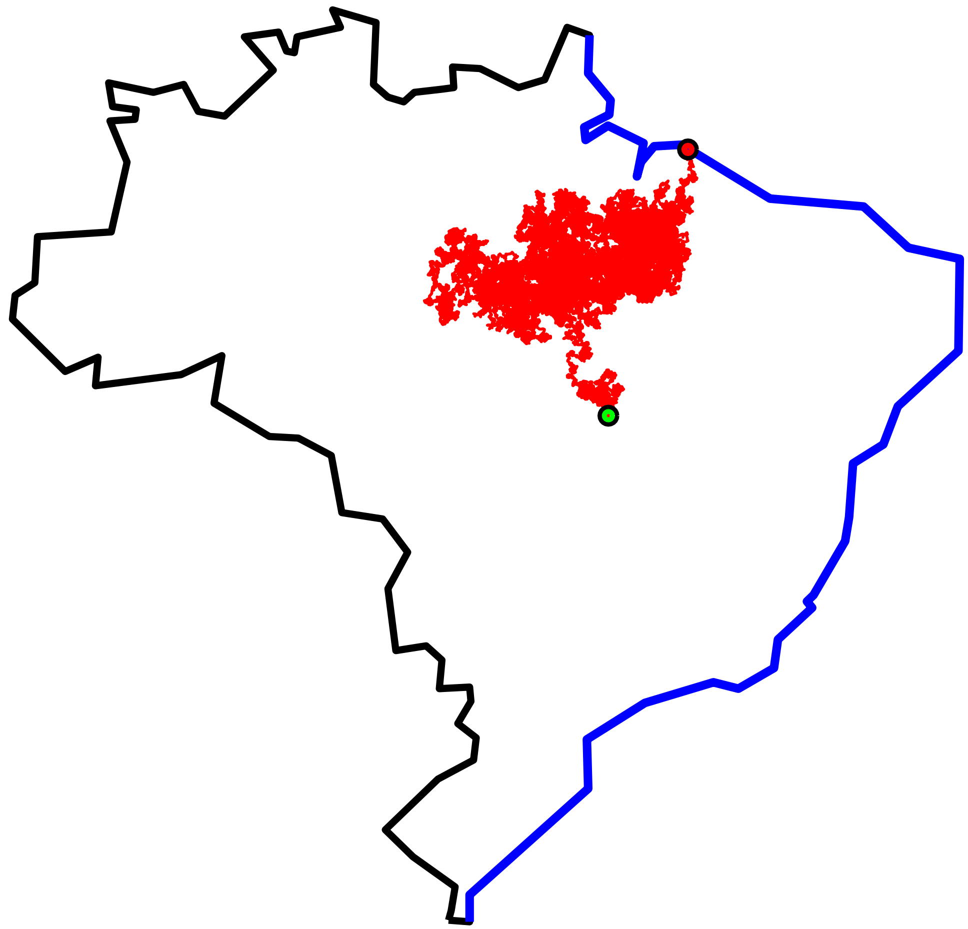

$\omega(z, E, \Omega)$ = probability a particle started at $z$ first hits $\partial\Omega$ in $E$.

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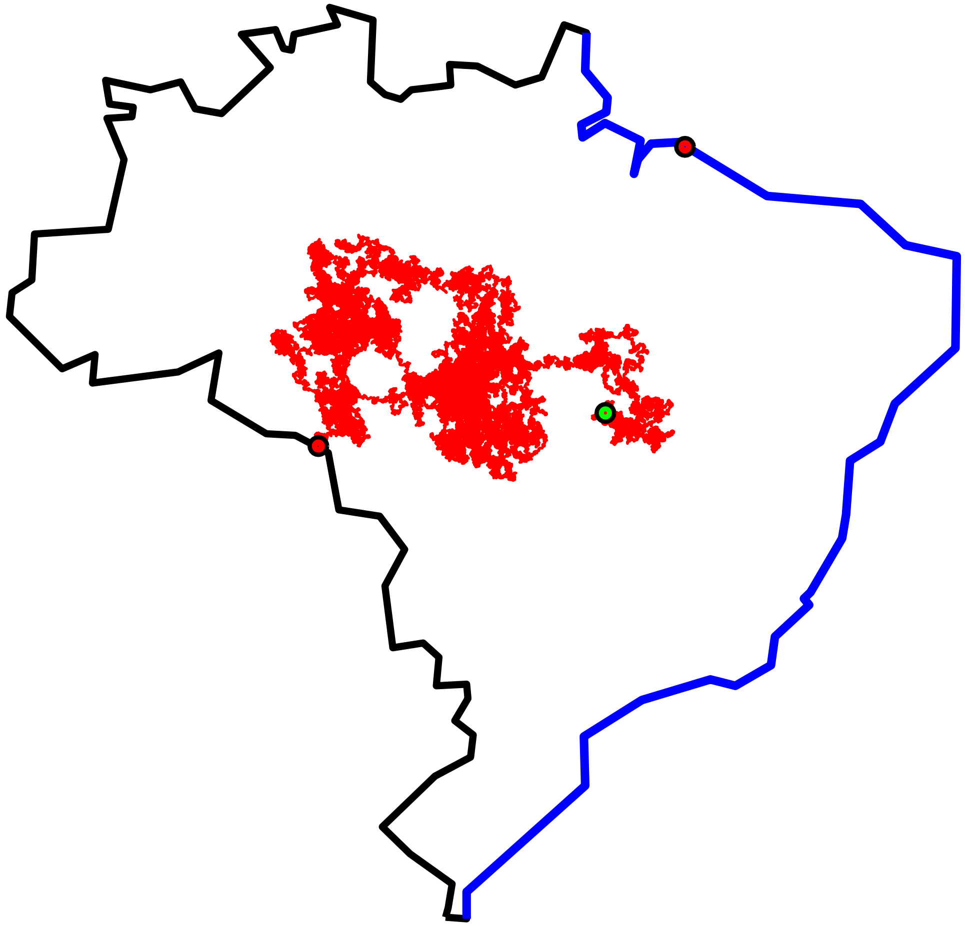

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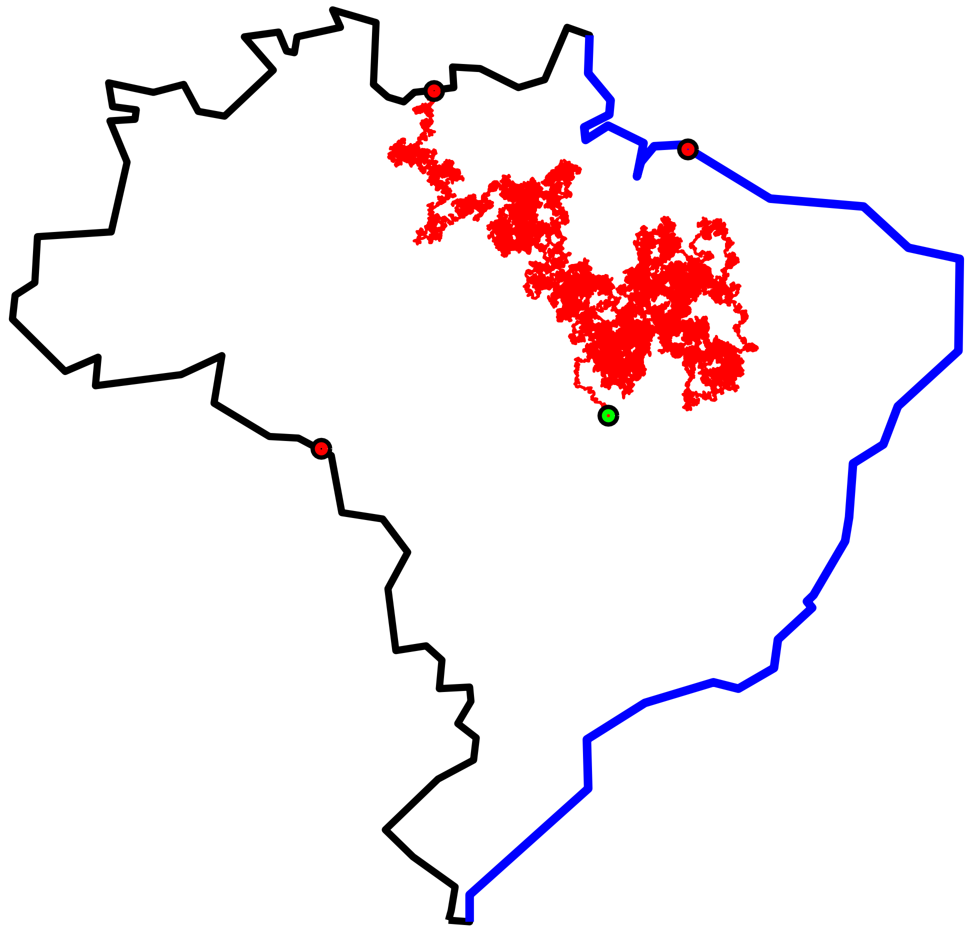

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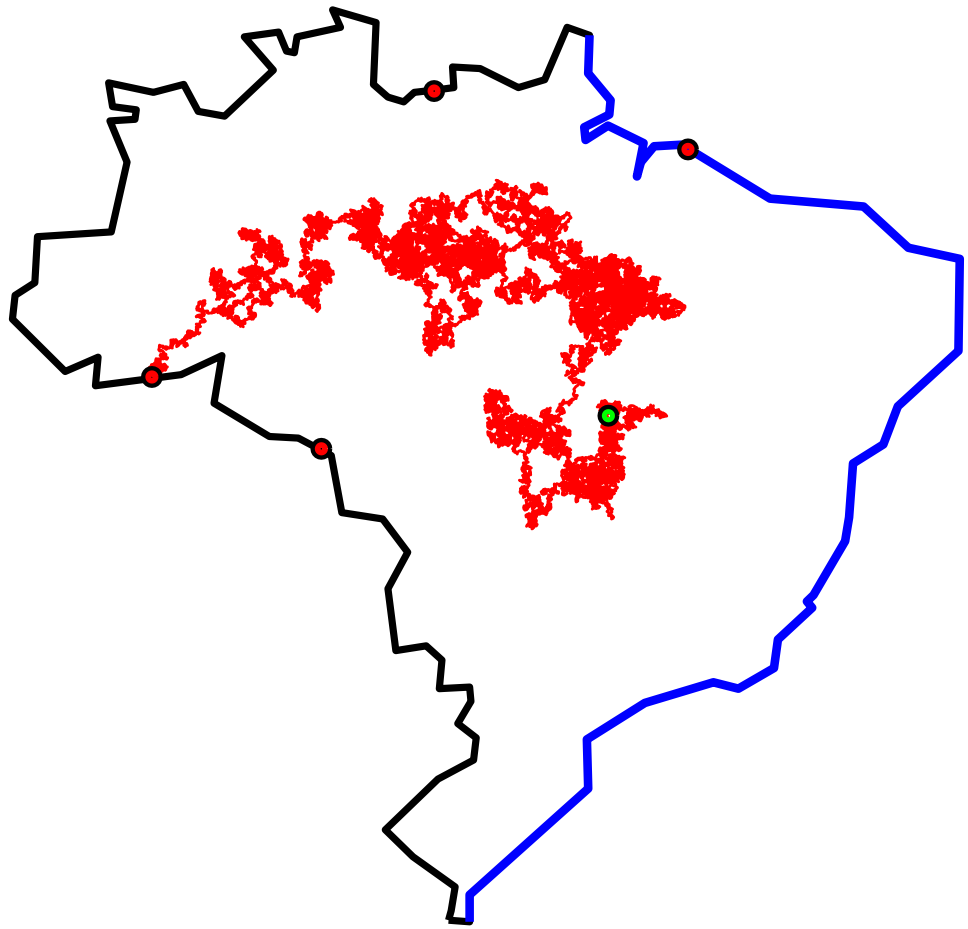

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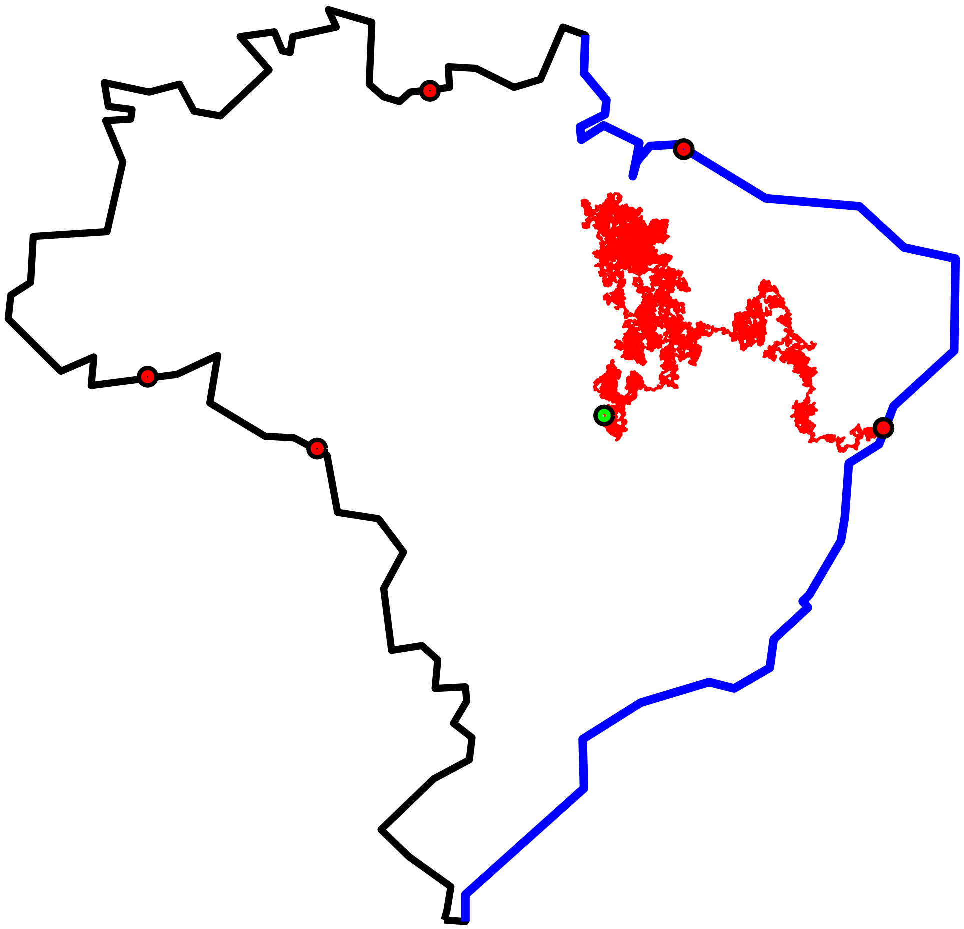

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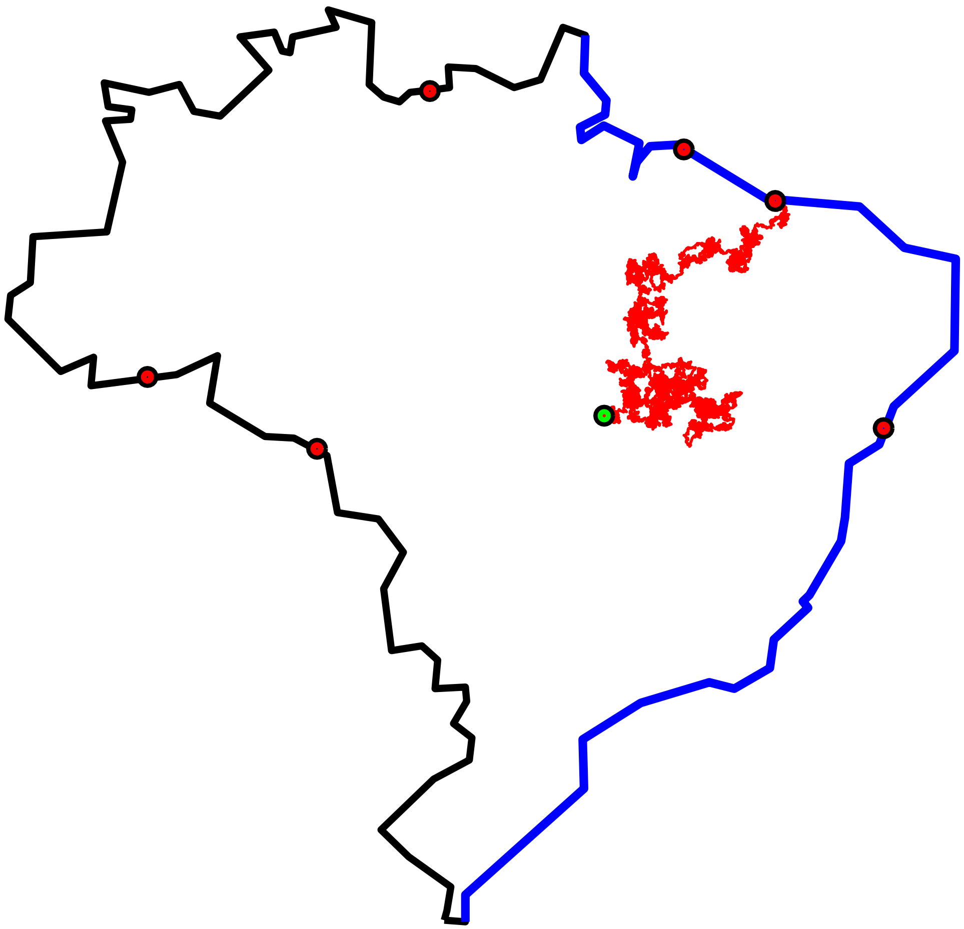

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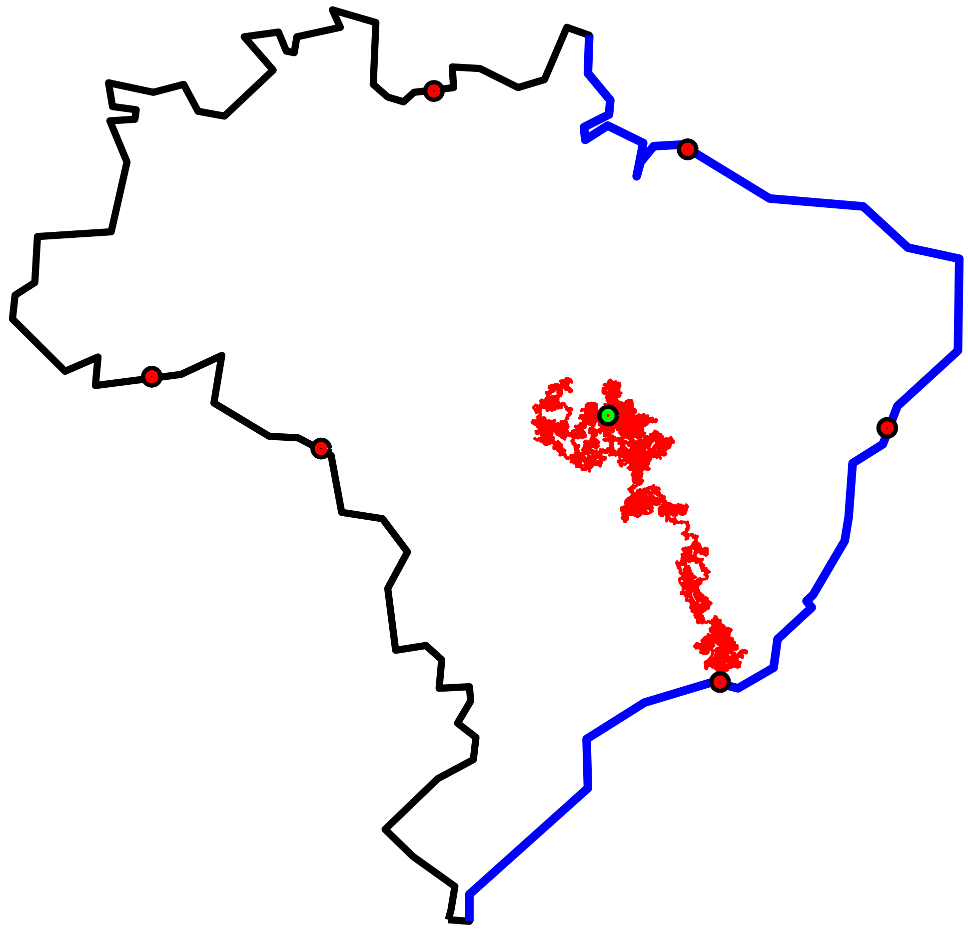

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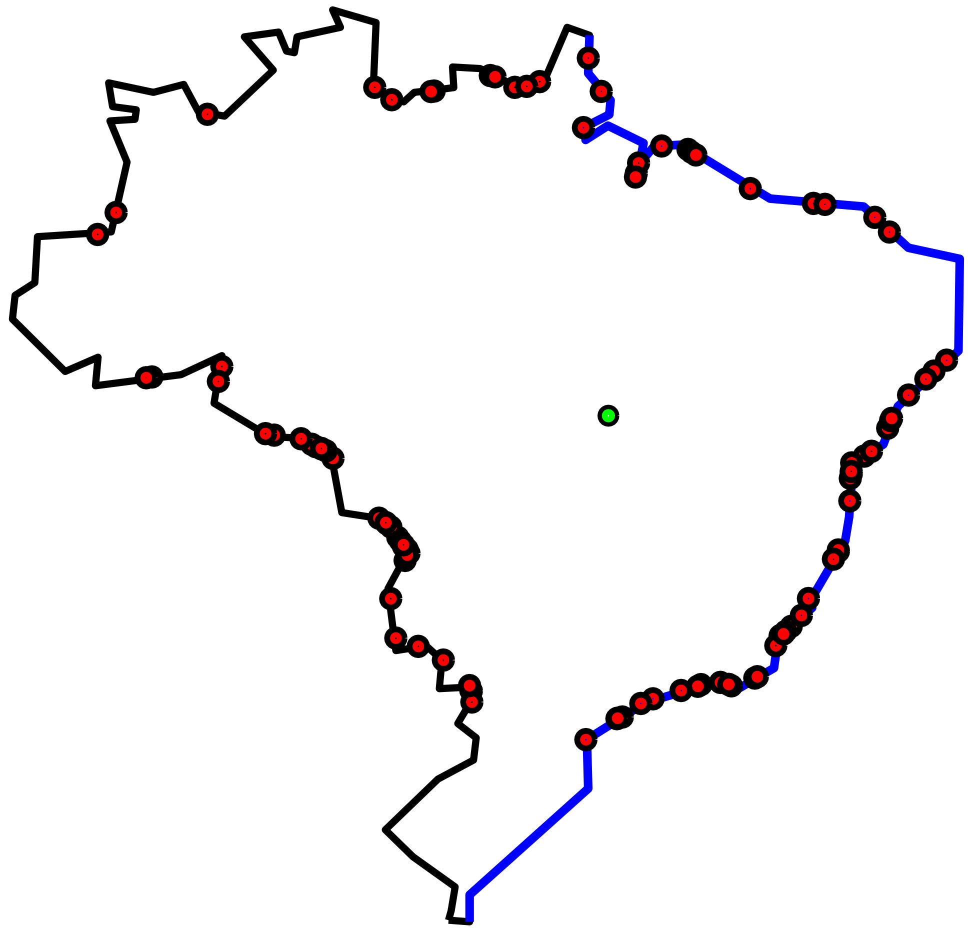

$\omega(z, E, \Omega) \approx 64/100$. What if we move starting point $z$?

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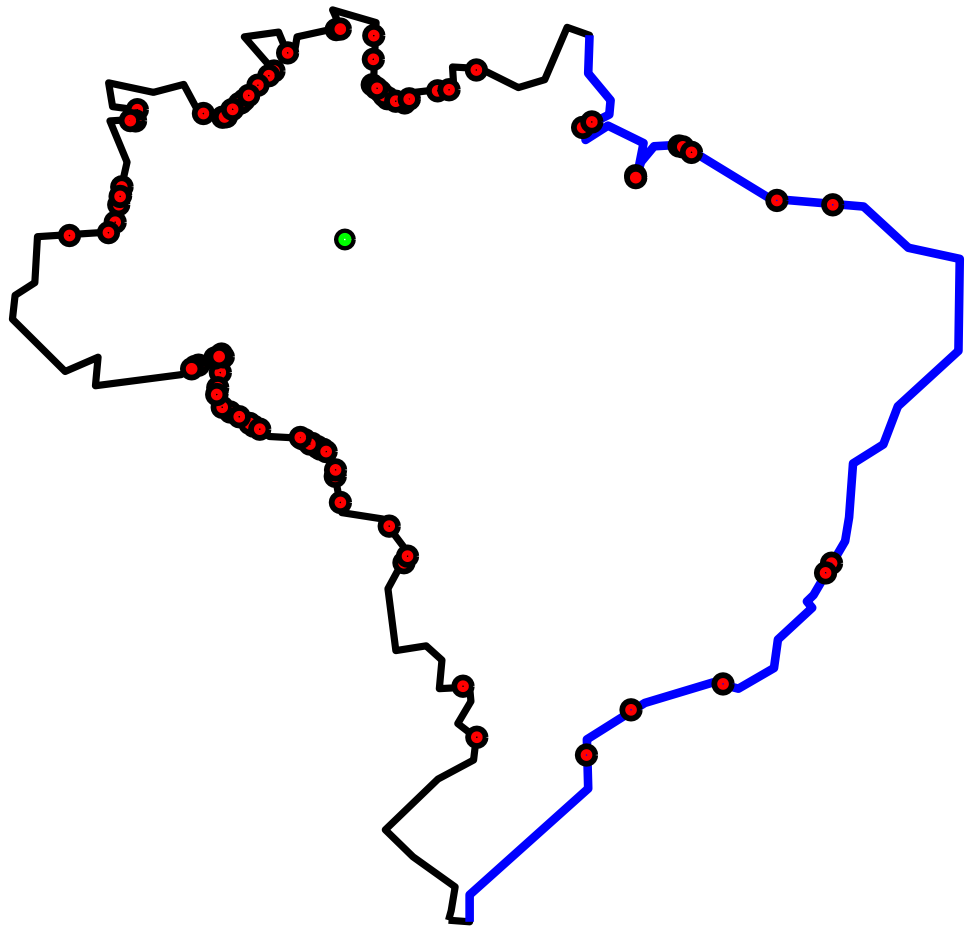
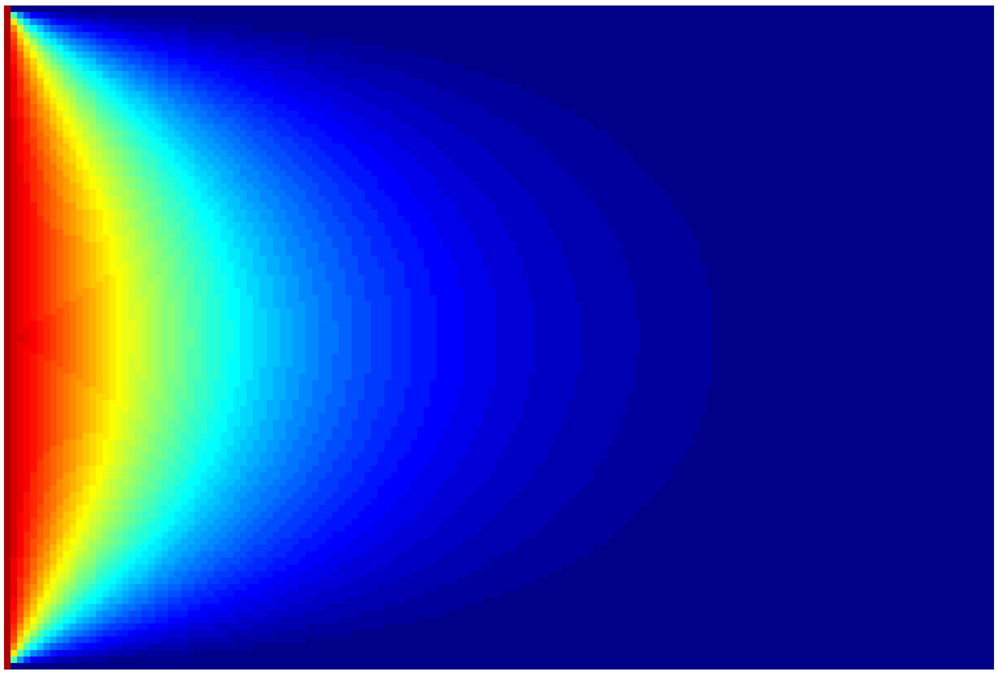


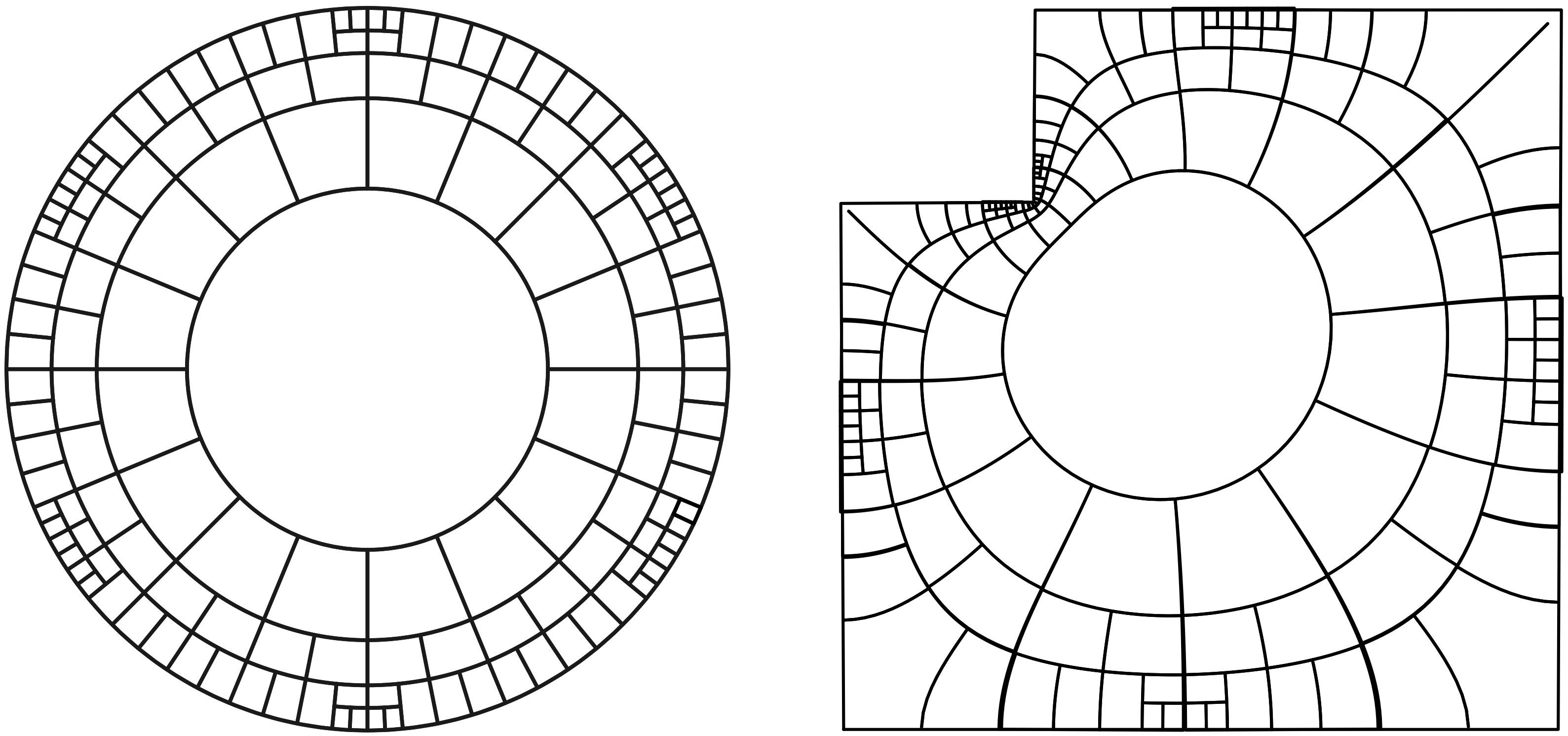
Different $z$ gives $\omega(z, E, \Omega) \approx 12/100$.

$\omega$ is harmonic in $z$ with boundary values $\omega = 1$ on $E$, $\omega = 0$ off $E$.

**Harmonic measure** $\leftrightarrow$ solution of Dirichlet problem:

$$\Delta u = 0, \quad u|_{\partial\Omega} = f, \qquad u(z) = \int_{\partial\Omega} f(x) d\omega_z(x).$$

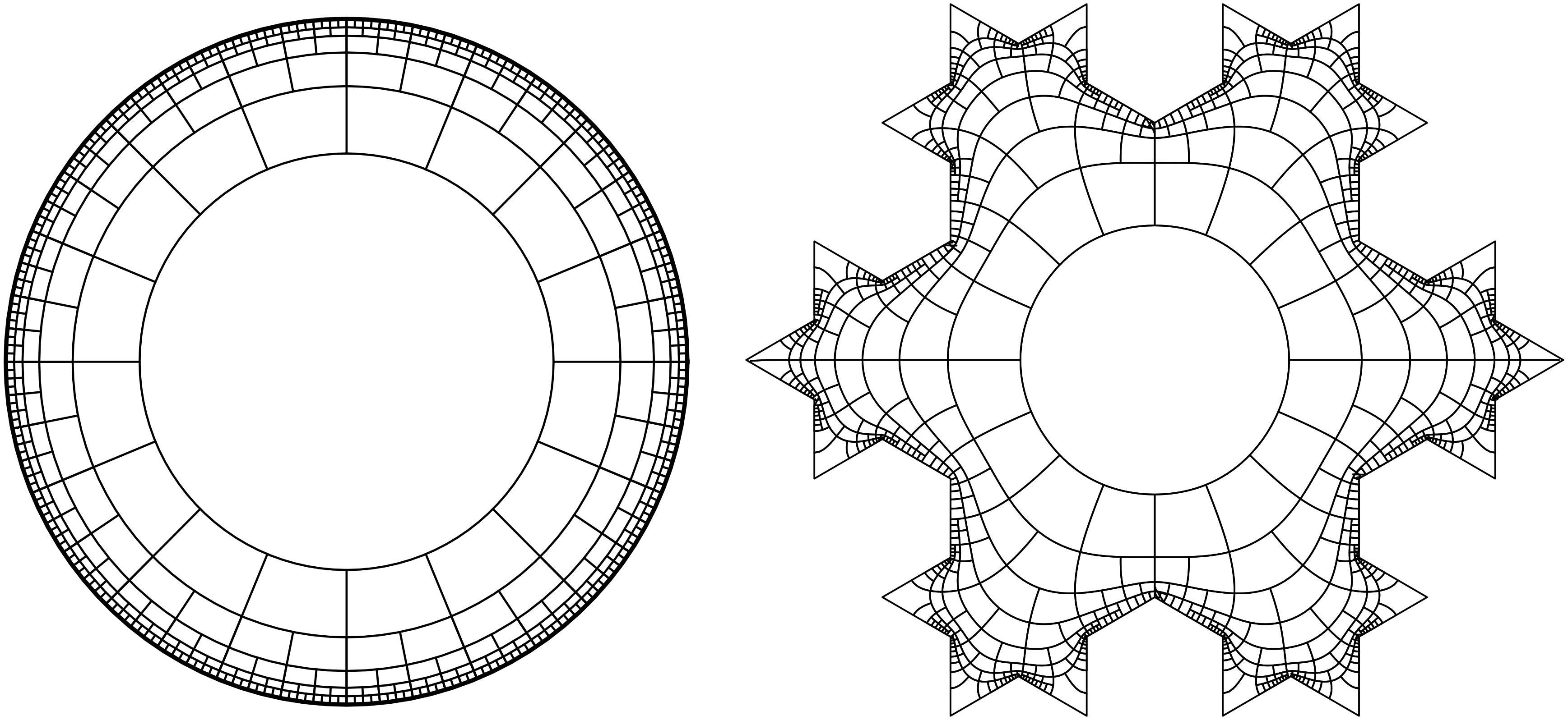




**Conformal map** = holomorphic and 1-1,

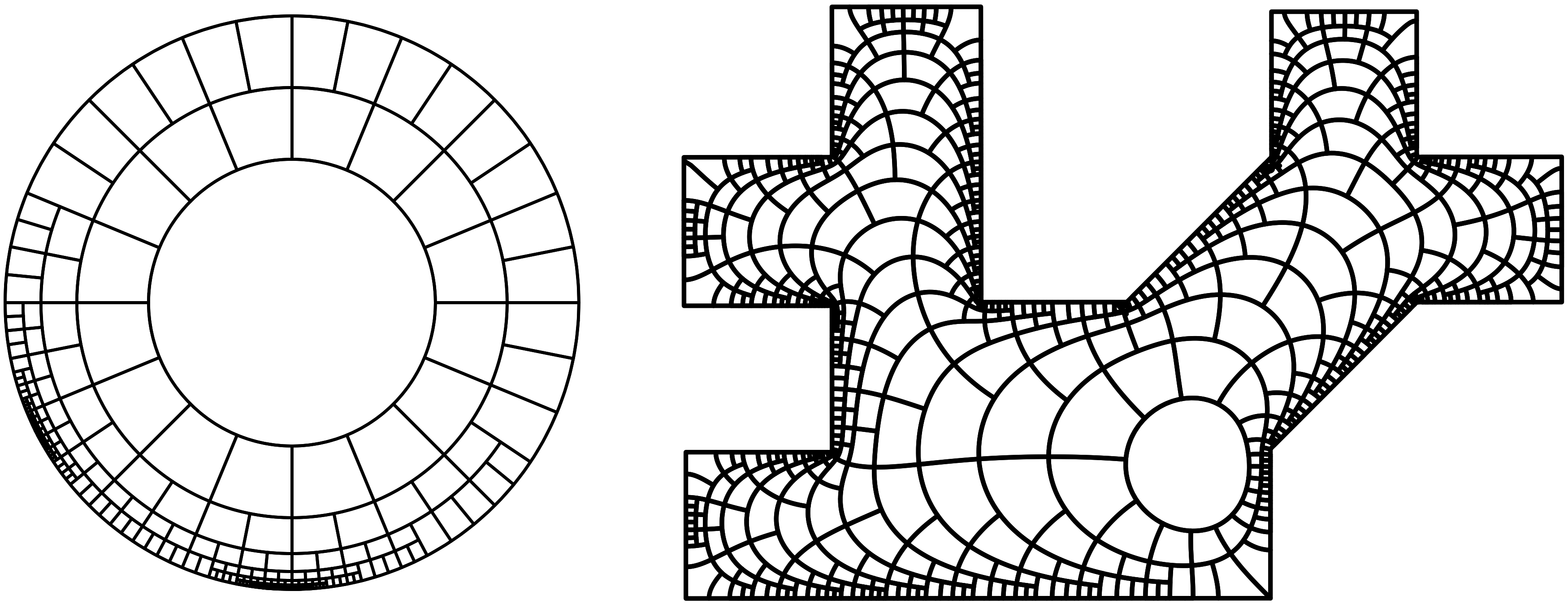
= angle preserving and orientation preserving

= sends infinitesimal circles to circles



**Riemann mapping theorem:**
Any Jordan domain is a conformal image of the unit disk.

**Harmonic measure** = image of normalized length measure.



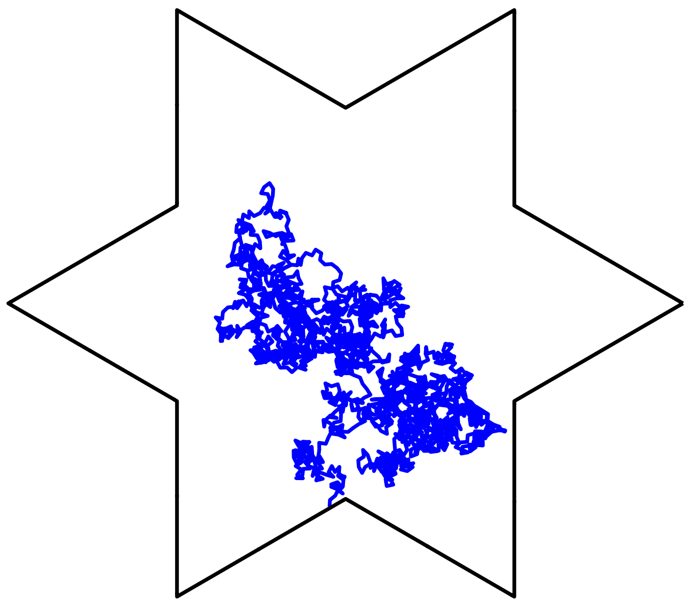
Conformal mapping often fastest way to compute harmonic measure.

Many practical methods (Zipper, CirclePack, SC-Toolbox,...)

Theoretical linear time computation for polygons (B 2010).

Best geometric understanding of harmonic measure in $d = 2$.

**Thm (F & M Riesz 1916):**
For rectifiable boundaries, $\omega(E) = 0$ iff $E$ has zero length.



rectifiable = finite length

length $= \ell(E) =$ Hausdorff 1-measure $= \inf\{\sum r_j : E \subset \cup D(x_j, r_j)\}$

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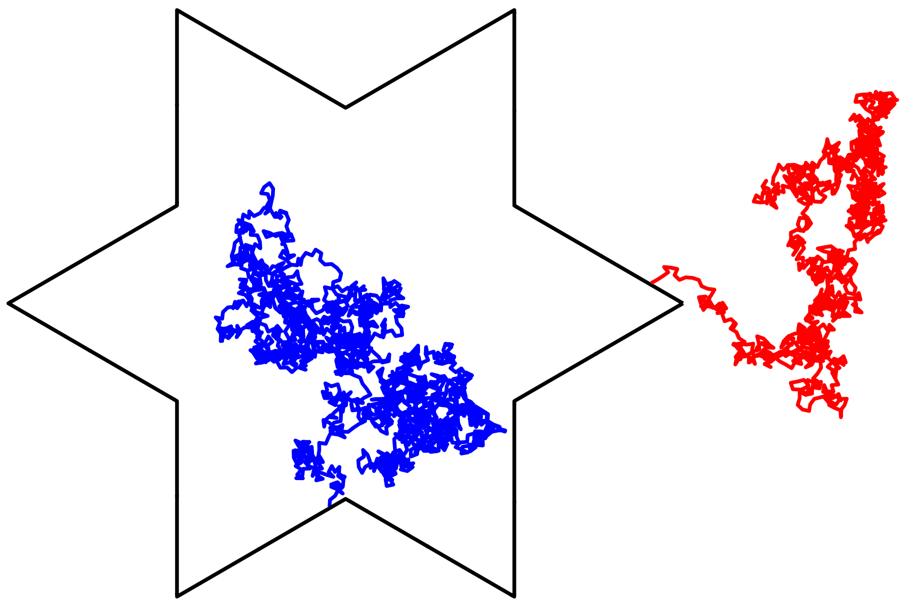
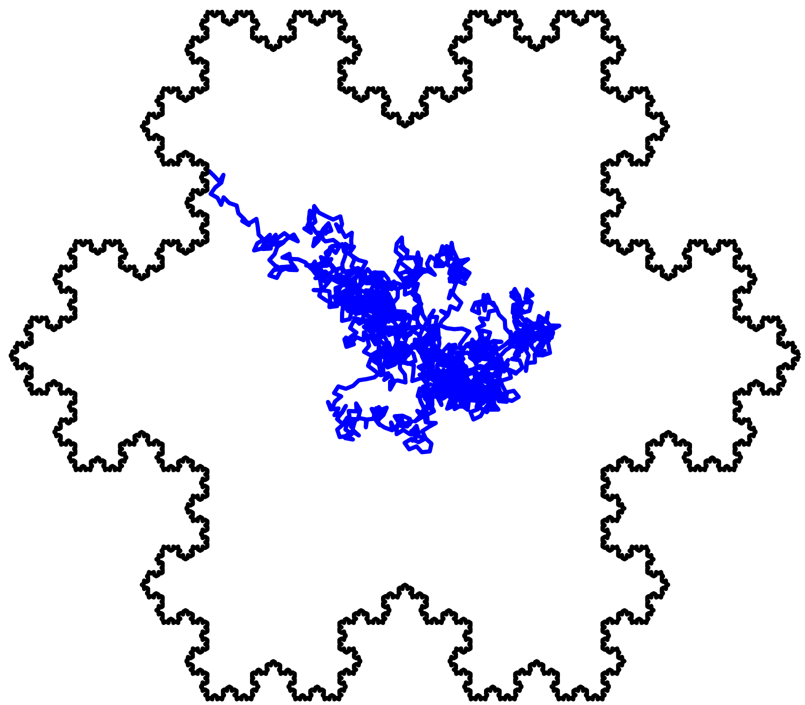


"Inside" and "outside" harmonic measures have same null sets.

Measures are mutually absolutely continuous. Same measure class.

**Thm (Makarov 1985):**
For fractal domains, $\omega$ gives full measure to a set of zero length.



First such examples due to Lavrentiev (1936).

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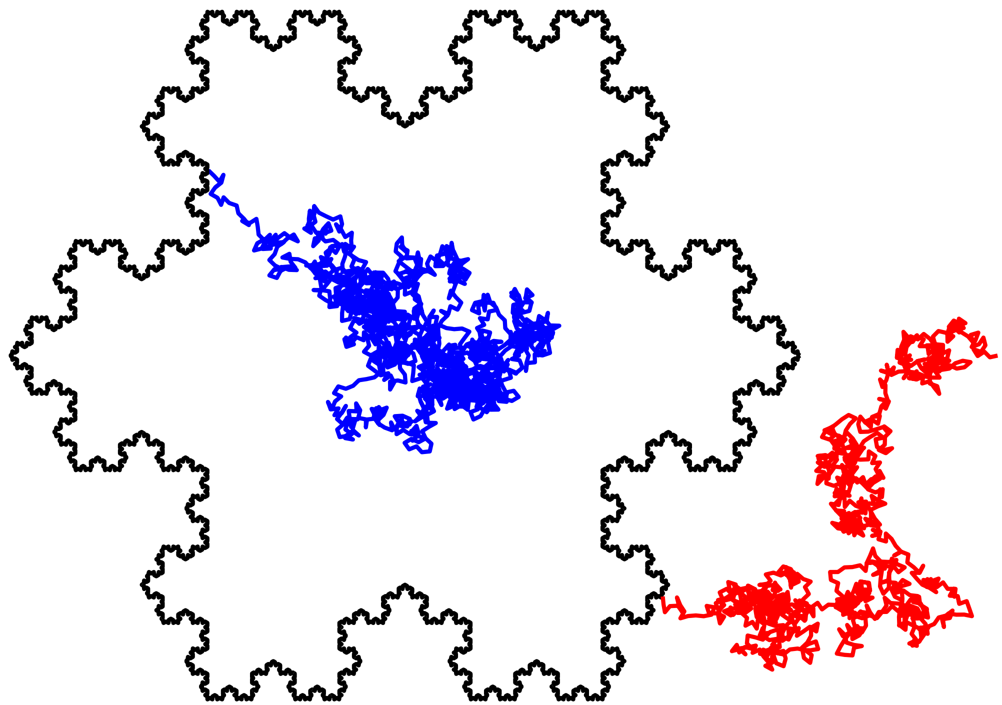
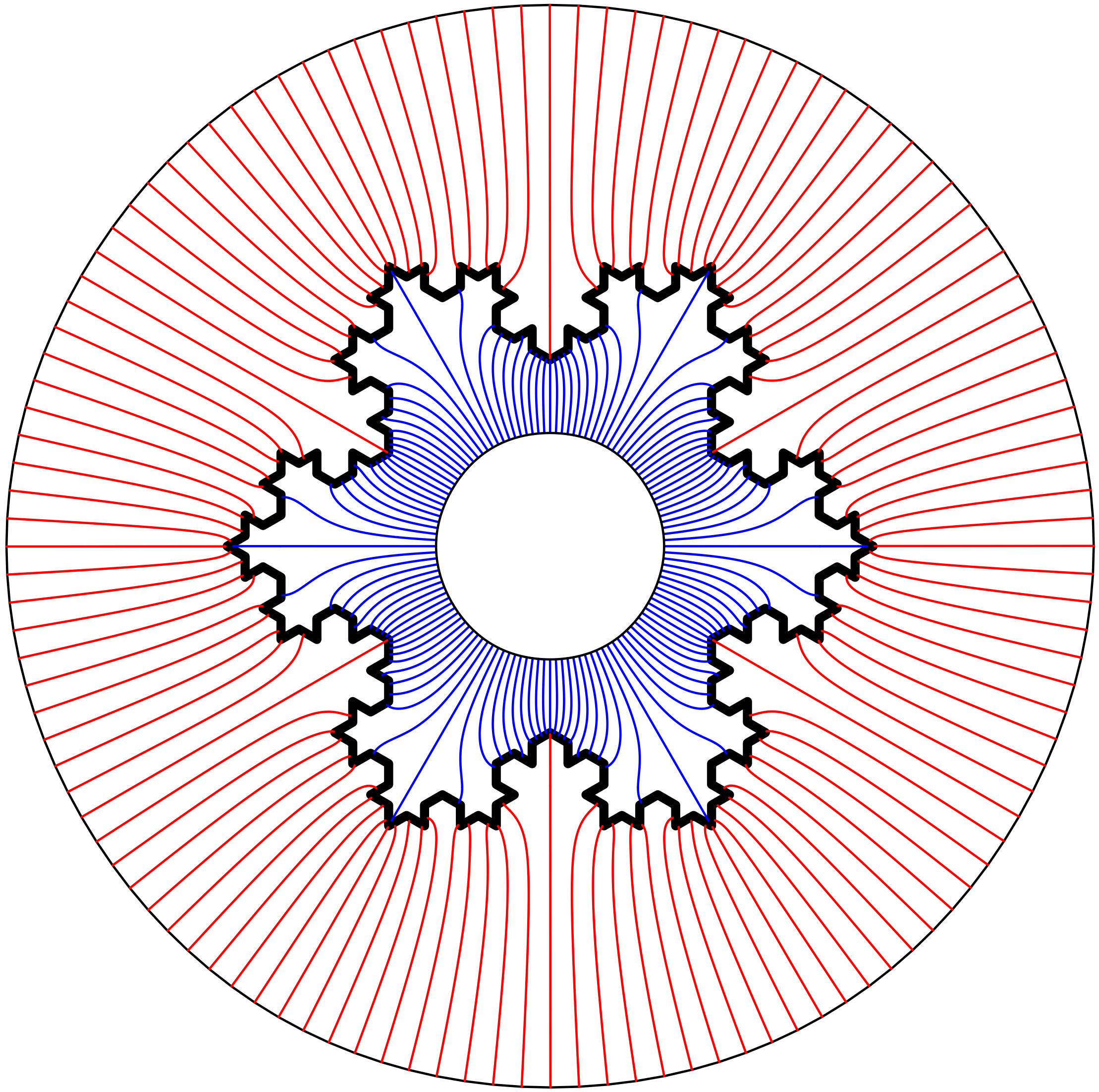
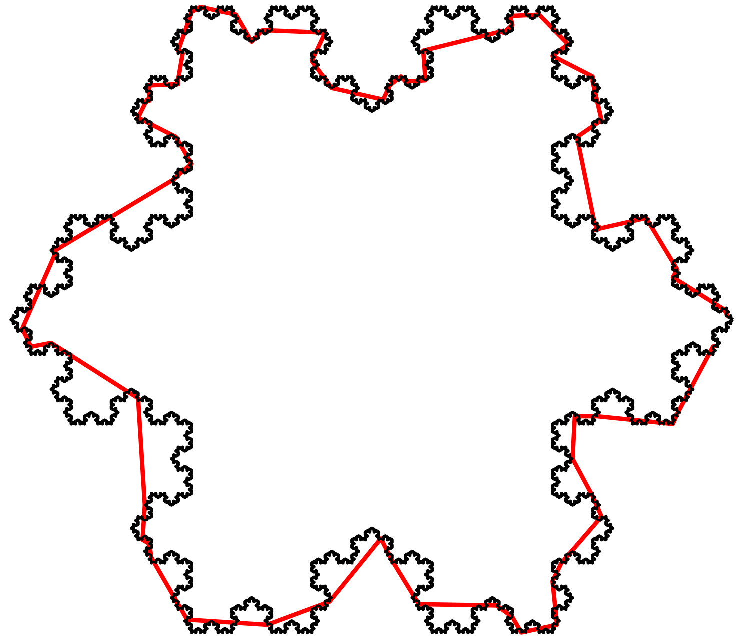


$\omega_1 \ll \omega_2 \ll \omega_1$ on tangent points.

$\omega_1 \perp \omega_2$ iff tangents points have zero length (B 1987).



Images of radial lines for conformal maps to inside and outside.



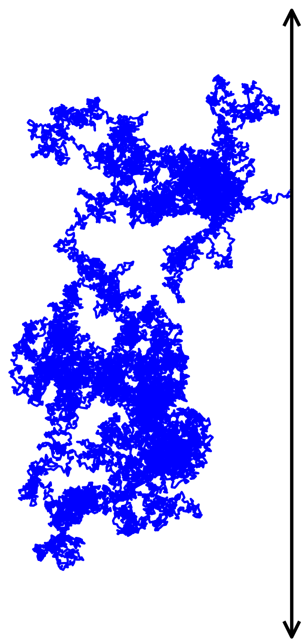
**Thm (B.-Jones 1990):**
Off tangent points, no finite length curve hits positive harmonic measure.

On fractals, $\omega$ has zero length, but is unrectifiable.
Uses "Analyst's traveling salesman theorem".

Generalized to $\mathbb{R}^n$ by Azzam, Hofmann, Martell, Mayboroda, Mourgoglou, Tolsa, Volberg (2016). Uses singular integrals, geometric measure theory.

For which curves is $\omega_1 = \omega_2$?

For which curves is $\omega_1 = \omega_2$? True for lines:



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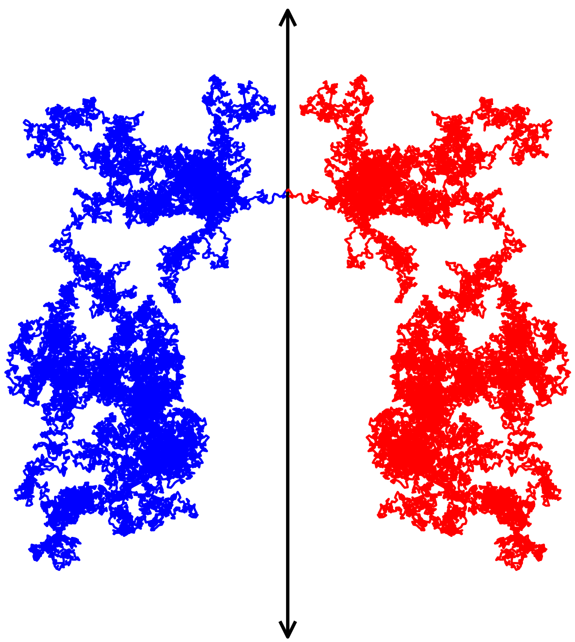
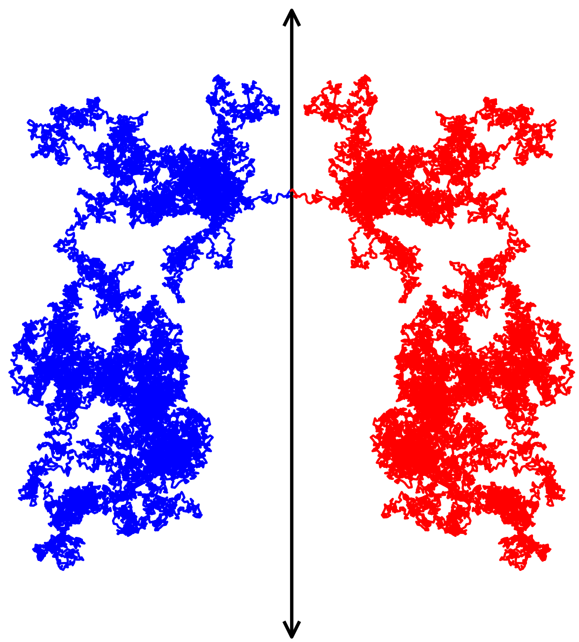


Also for circles ( $=$ lines conformally).

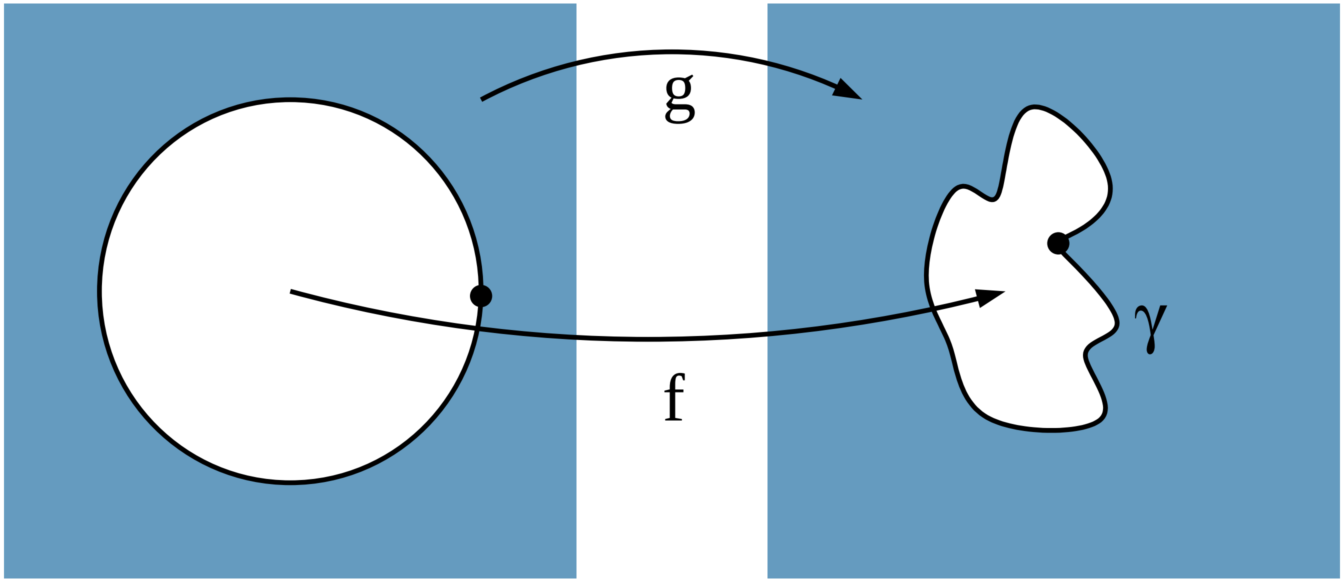
Converse? If $\omega_1 = \omega_2$ must $\Gamma$ be a circle/line?

For which curves is $\omega_1 = \omega_2$? True for lines:



Also for circles ( = lines conformally).

Converse? If $\omega_1 = \omega_2$ must $\Gamma$ be a circle/line? **Yes**




Suppose $\omega_1 = \omega_2$ for a curve $\gamma$.

Conformally map two sides of circle to two sides of $\gamma$ so $f(1) = g(1)$.

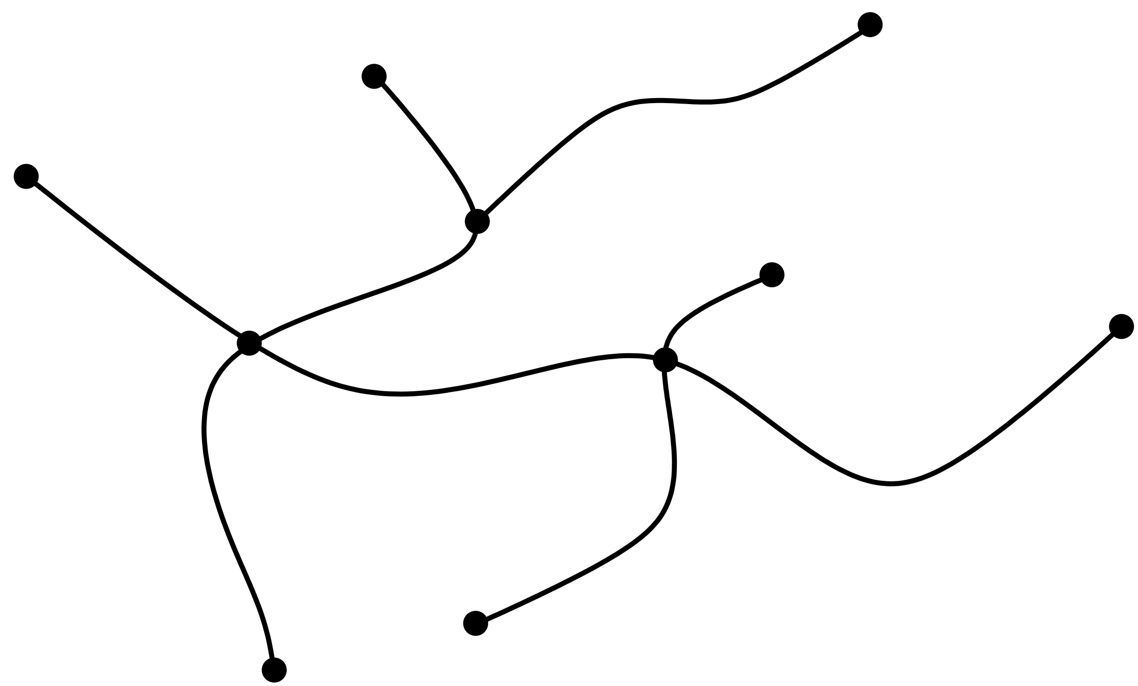
$\omega_1 = \omega_2$ implies maps agree on whole boundary.

So $f, g$ define homeomorphism $h$ of plane holomorphic off circle.

Then $h$ is entire by Morera's theorem.

Entire and 1-1 implies $h$ is linear (Liouville's thm), so $\gamma$ is a circle.

What happens if we replace the closed curve by a tree?



Can we make harmonic measure the same on “both sides” of every edge?

A planar tree is **conformally balanced** if

- every edge has equal harmonic measure from $\infty$
- edge subsets have same measure from both sides

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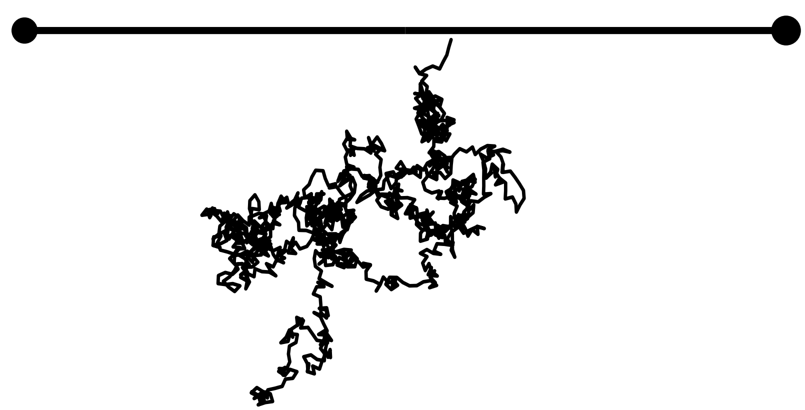
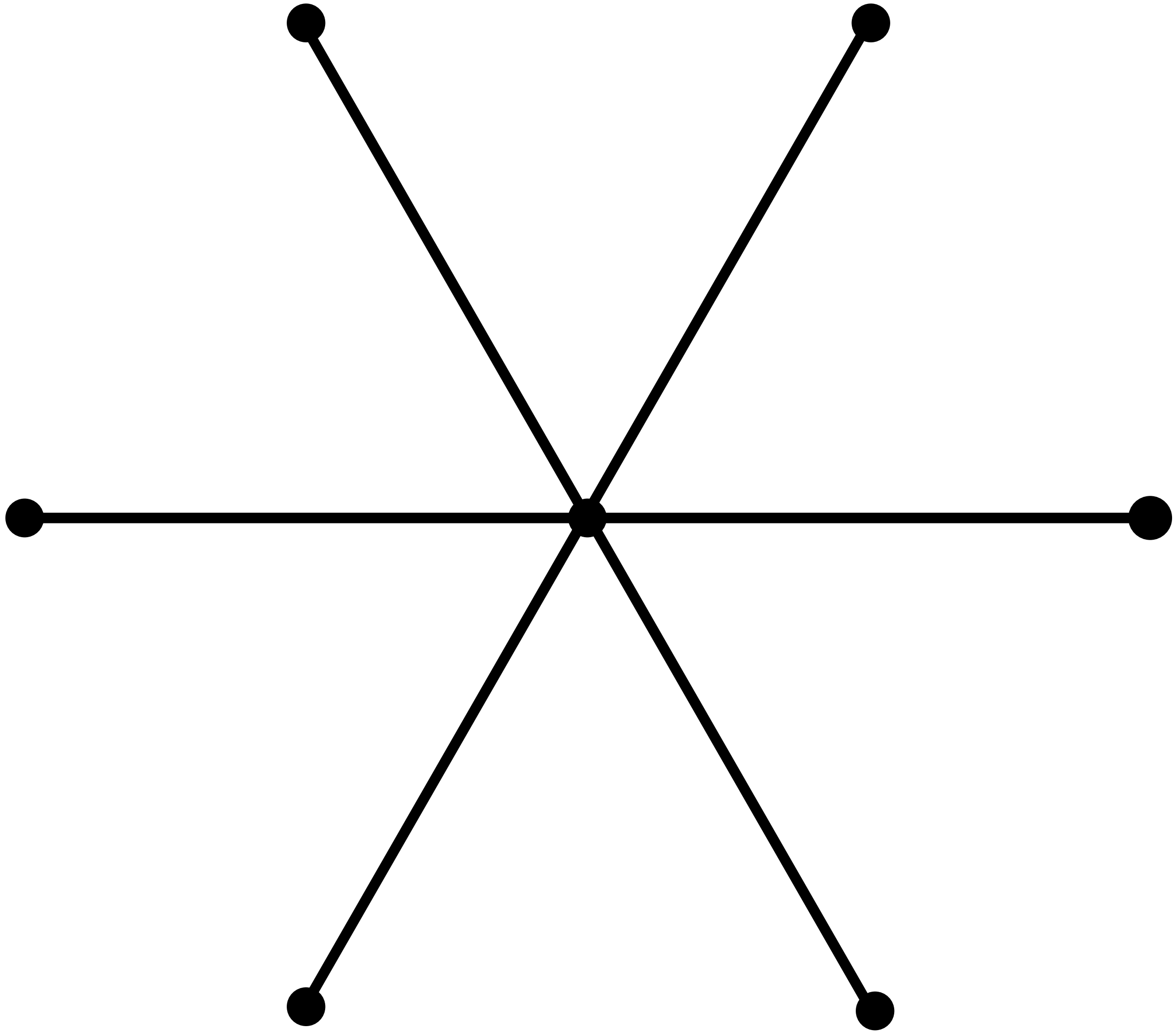
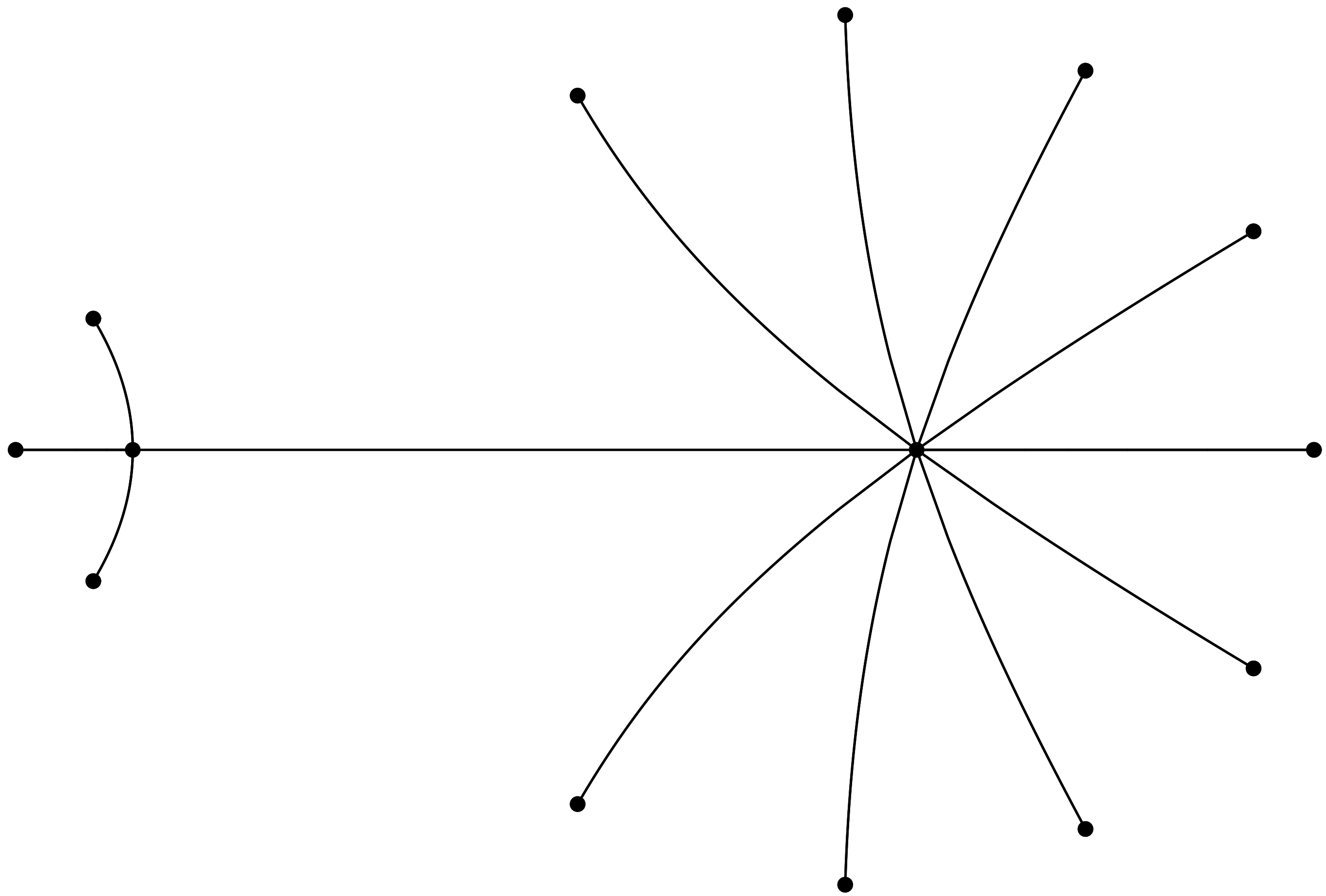




Trivially true by symmetry

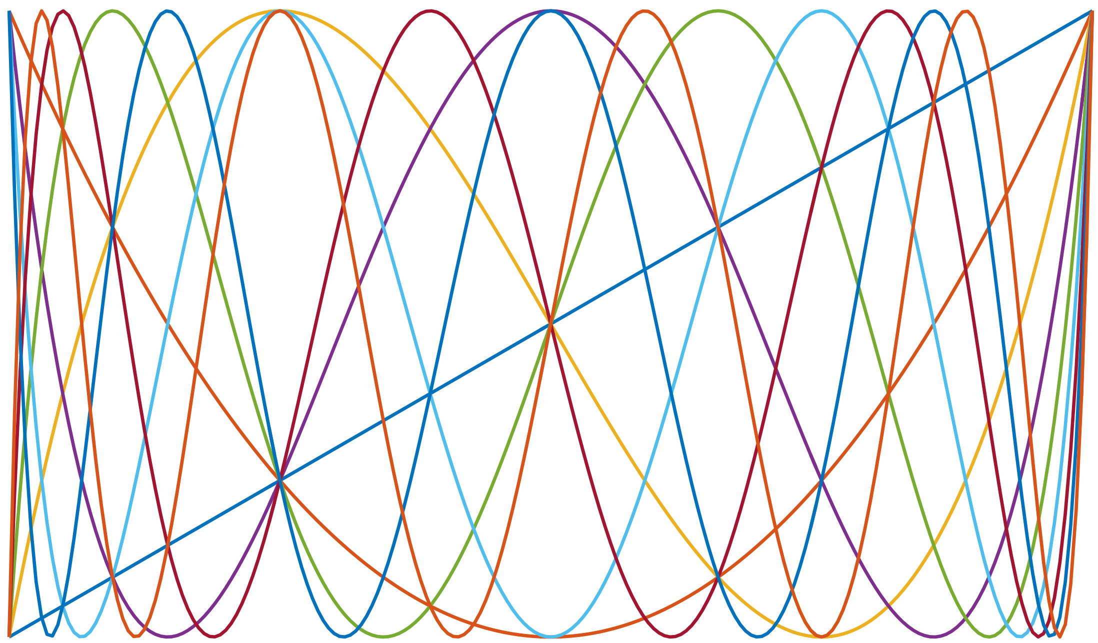


Non-obvious true tree

**Definition of critical value:** if $p$ = polynomial, then

$$\mathrm{CV}(p) = \{p(z) : p'(z) = 0\} = \text{critical values}$$

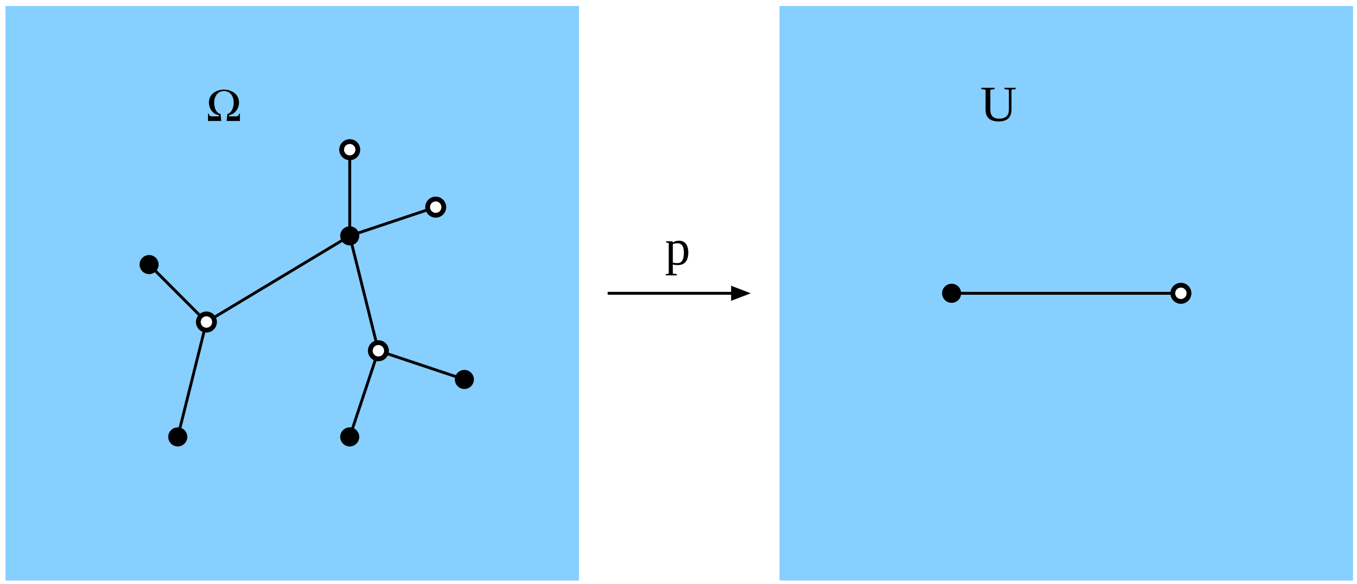
If $\mathrm{CV}(p) = \pm 1$, $p$ is called **generalized Chebyshev** or **Shabat**.



10 classical Chebyshev polynomials

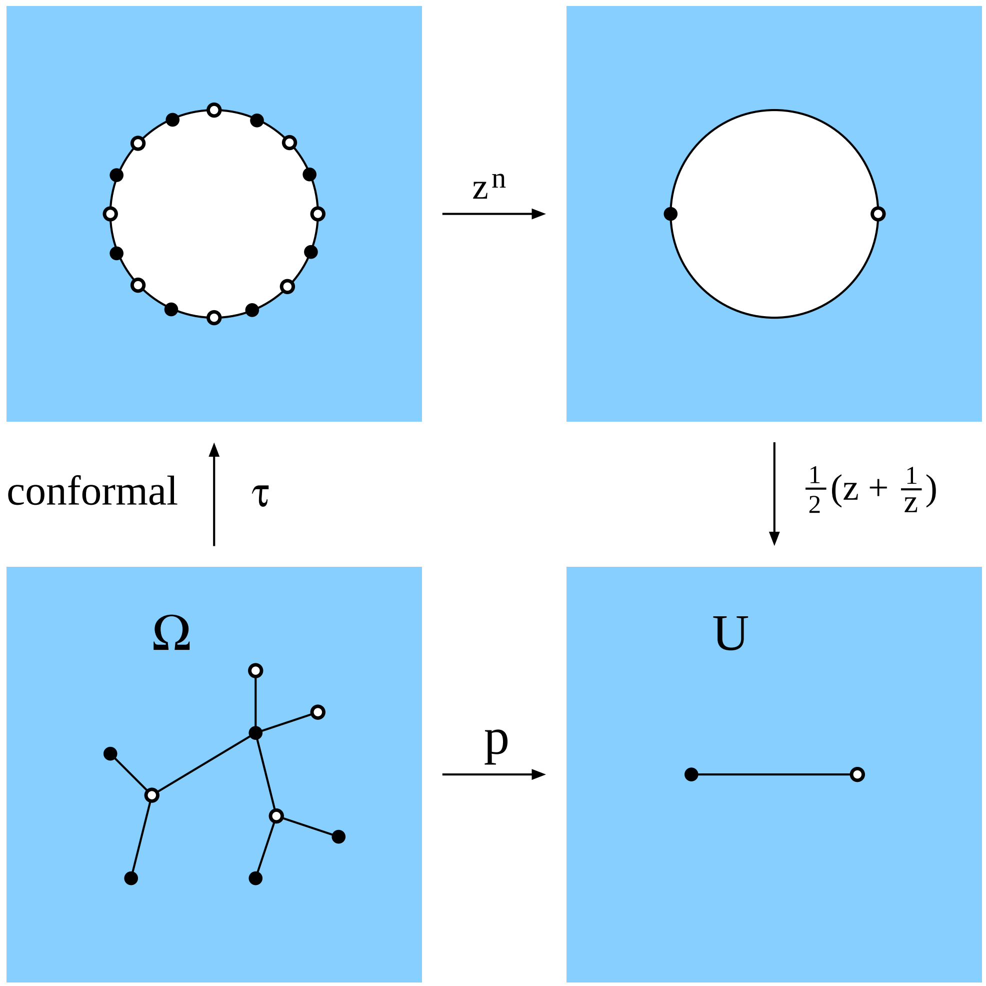
## Balanced trees $\leftrightarrow$ Shabat polynomials

**Fact:** $T$ is balanced iff $T = p^{-1}([-1,1])$, $p =$ Shabat.




$\Omega = \mathbb{C} \setminus T$ $\qquad$ $U = \mathbb{C} \setminus [-1,1]$

$T$ balanced $\Leftrightarrow$ $p$ Shabat.




$p$ is entire and $n$-to-1 $\Leftrightarrow$ $p$ = polynomial.

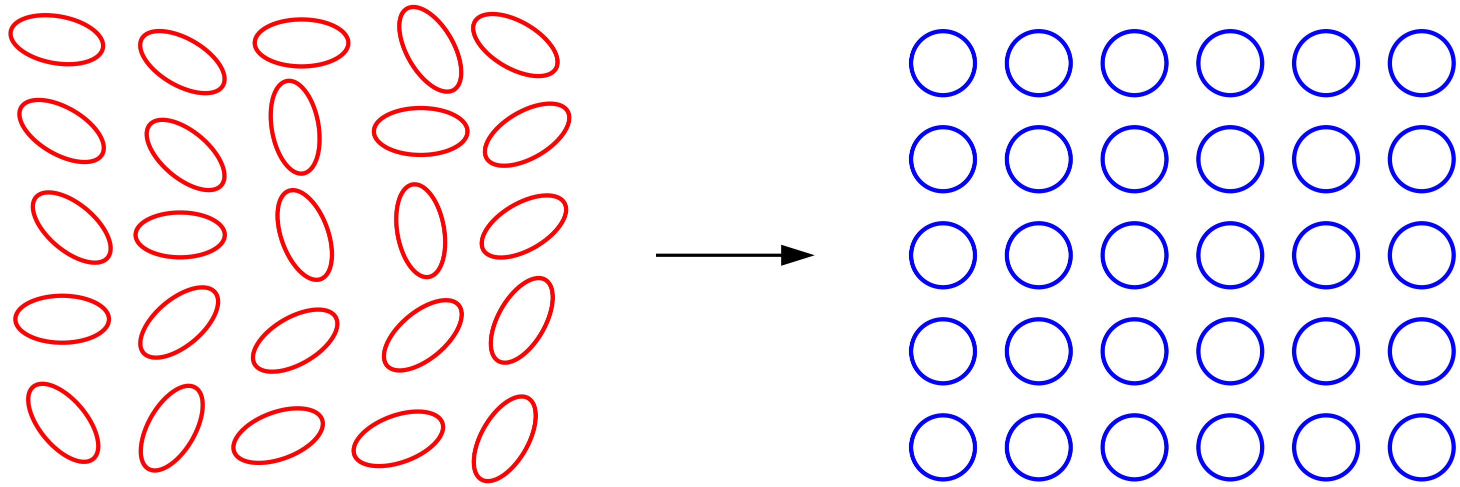
$\mathrm{CV}(p) \not\in U \leftrightarrow p : \Omega \to U$ is covering map.

**Theorem:** Every finite tree has a true form.

Standard proof uses the uniformization theorem.

I will sketch a proof using quasiconformal homeomorphisms.

Quasiconformal (QC) maps send infinitesimal ellipses to circles.



Eccentricity = ratio of major to minor axis of ellipse.

For $K$-QC maps, ellipses have eccentricity $\leq K$

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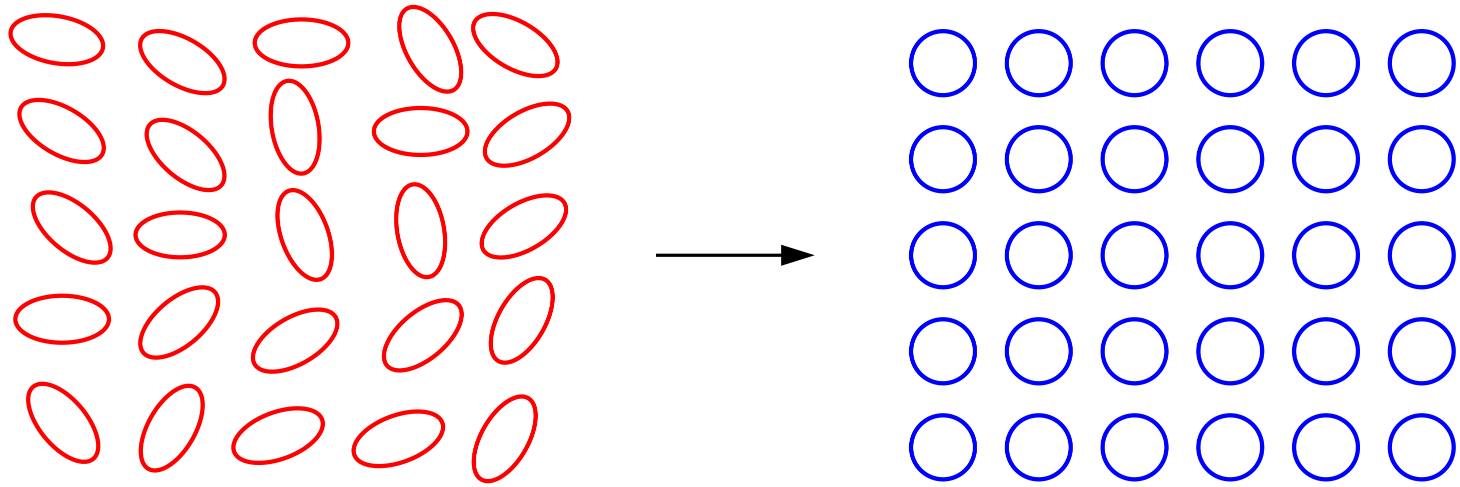

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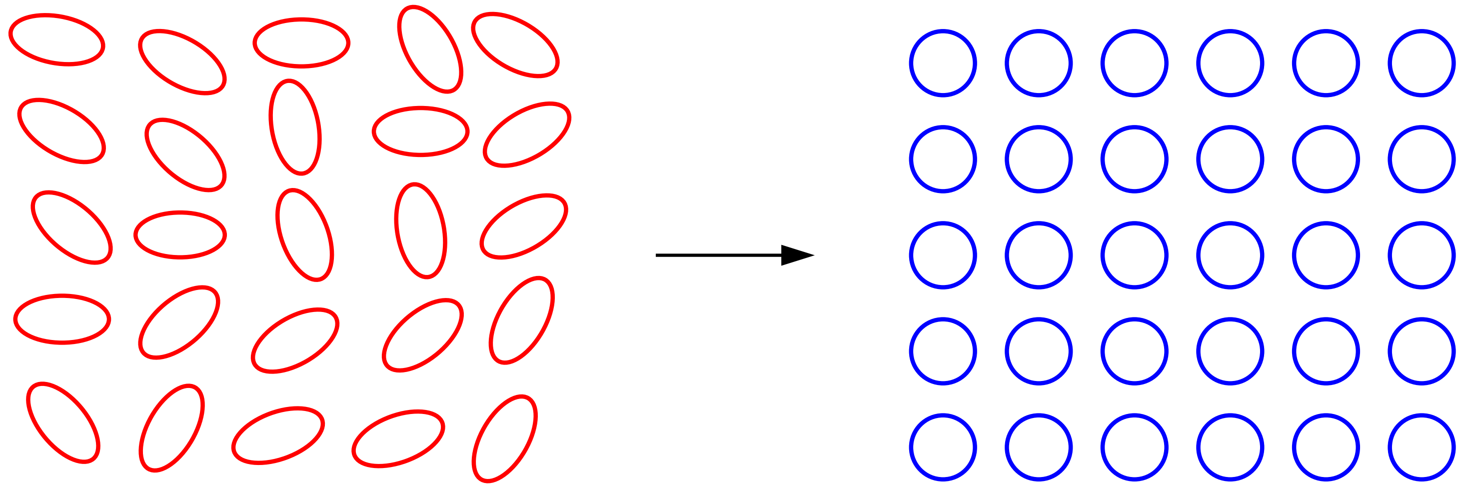
For $K$-QC maps, ellipses have eccentricity $\leq K$

Ellipses determined a.e. by measurable dilatation $\mu = f_{\overline{z}}/f_z$ with

$$|\mu| \leq \frac{K-1}{K+1} < 1.$$

Conversely, . . .

Quasiconformal (QC) maps send infinitesimal ellipses to circles.



**Mapping theorem:** any such $\mu$ comes from some QC map $f$.

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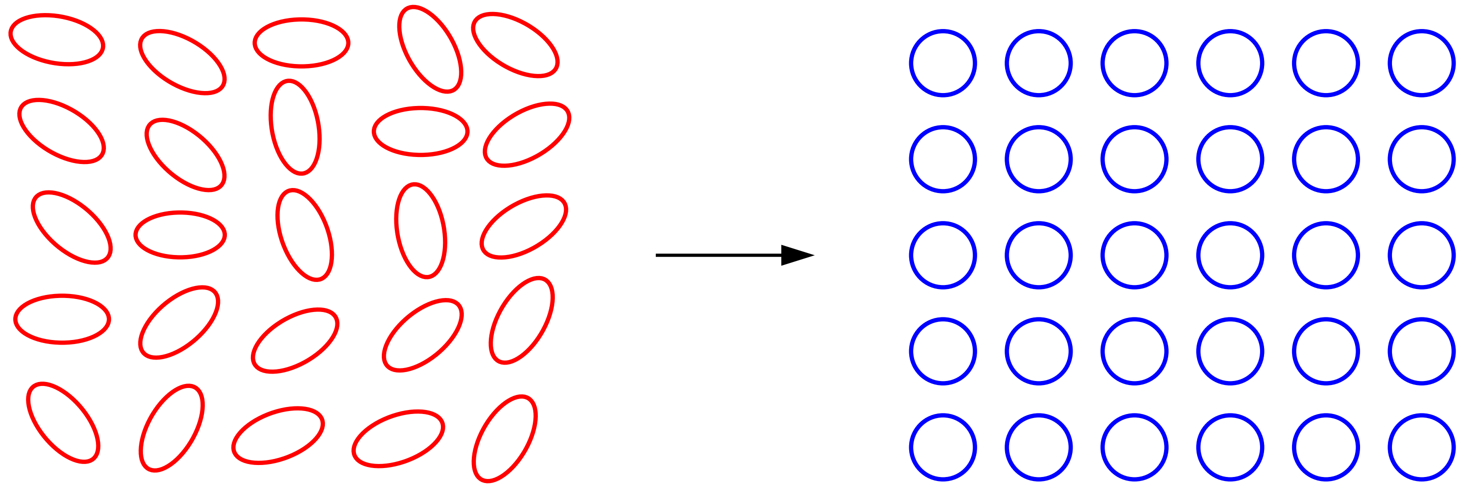

**Mapping theorem:** any such $\mu$ comes from some QC map $f$.

**Cor:** If $f$ is holomorphic and $\psi$ is QC, then there is a QC map $\varphi$ so that $g = \psi \circ f \circ \varphi$ is also holomorphic.

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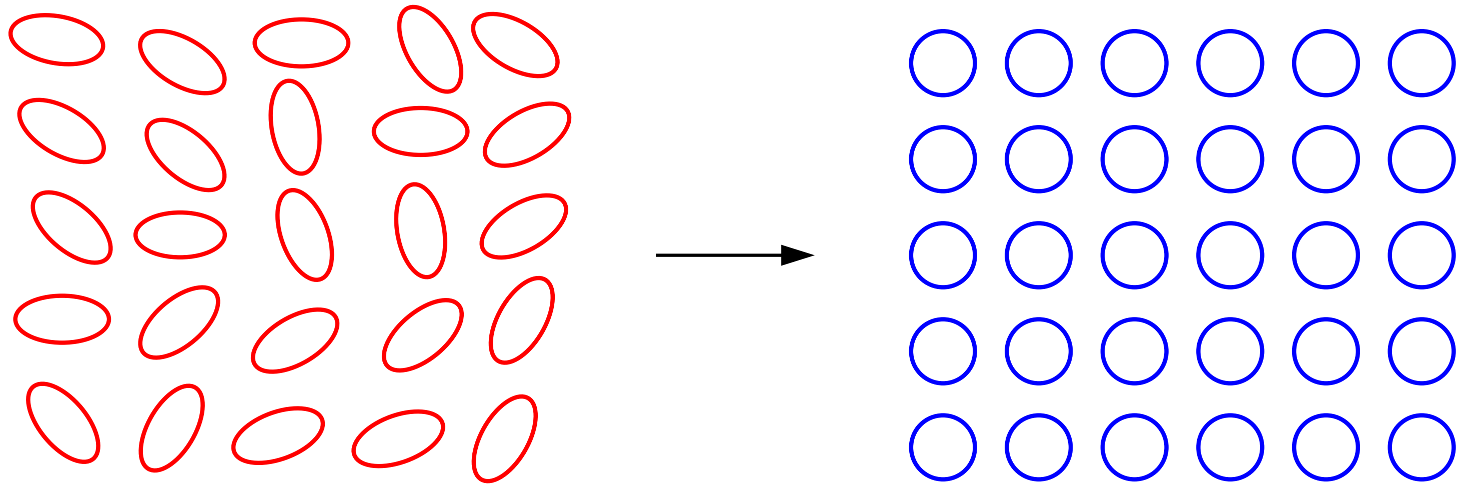

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**Cor:** If $f$ is holomorphic and $\psi$ is QC, then there is a QC map $\varphi$ so that $g = \psi \circ f \circ \varphi$ is also holomorphic.

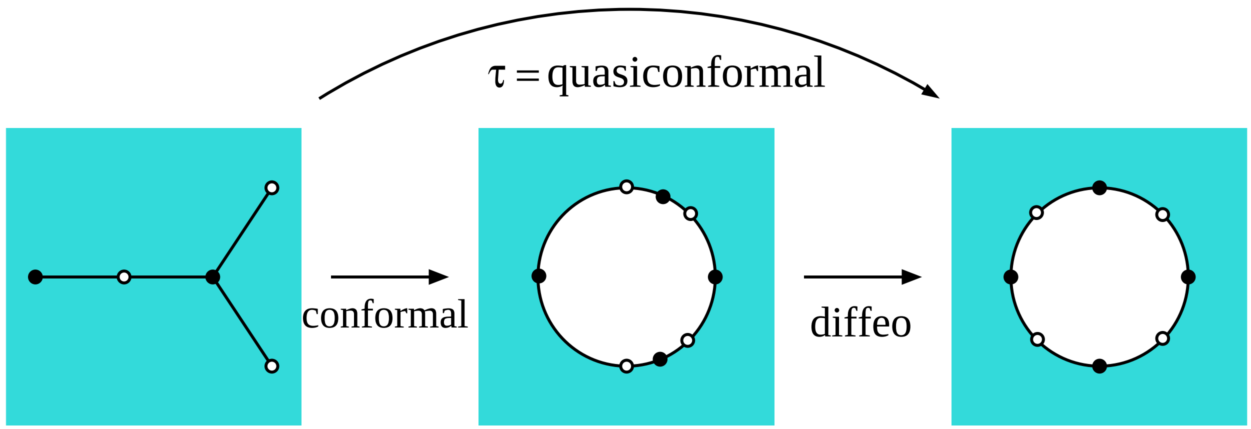
Two such holomorphic functions, $f, g$ are called **QC-equivalent**.

In general, $\psi \circ f \circ \varphi$ is not holomorphic, but is **quasiregular (QR)**.

## Proof that every finite tree has a true form:

Map $\Omega = \mathbb{C} \setminus T$ to $\{|z| > 1\}$ conformally.

"Equalize intervals" by diffeomorphism. Composition is quasiconformal.



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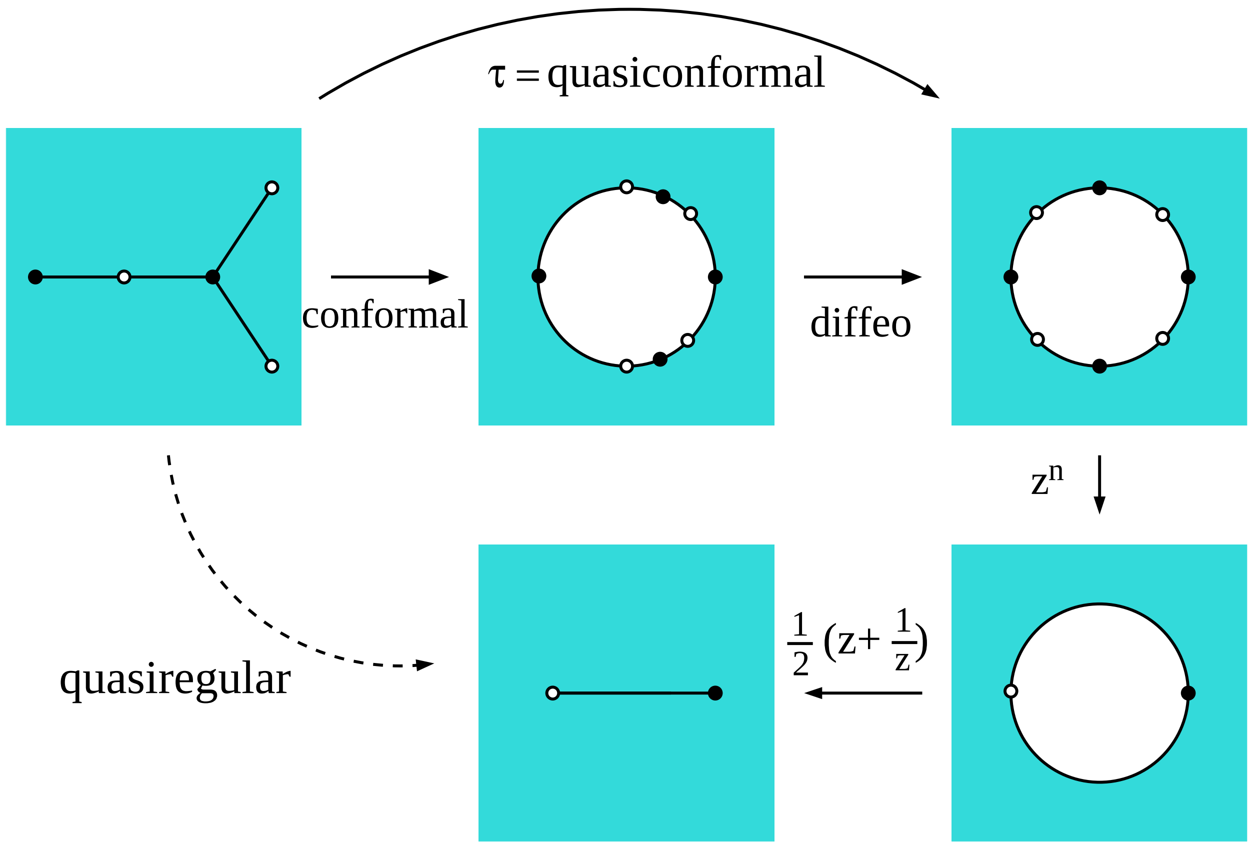
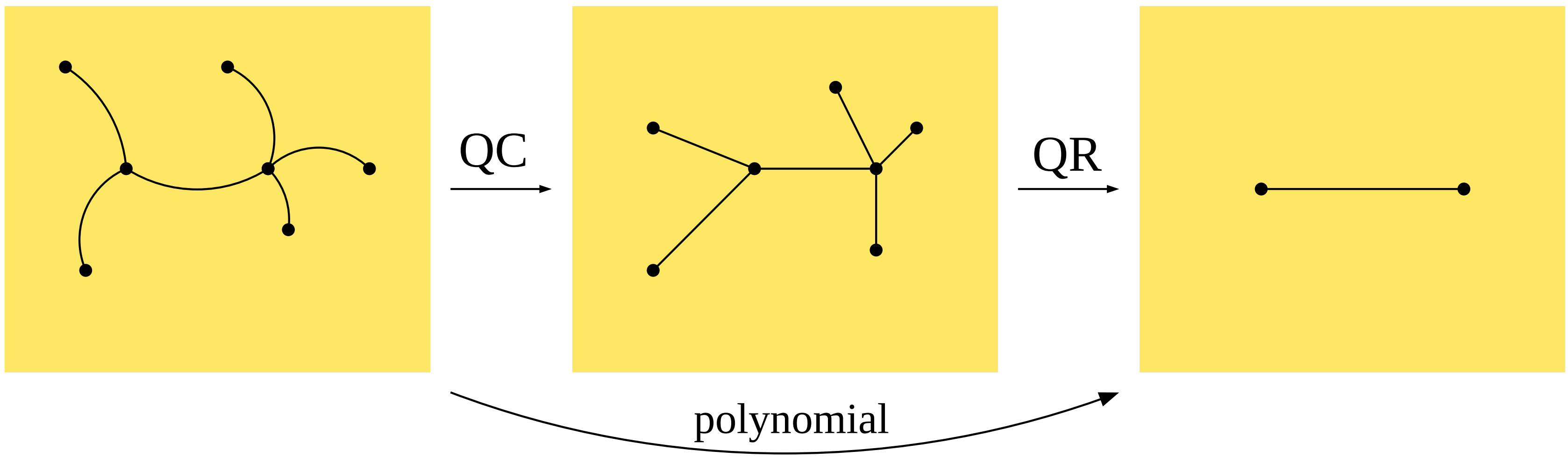


Mapping theorem implies there is a QC $\varphi$ so $p = q \circ \varphi$ is a polynomial.



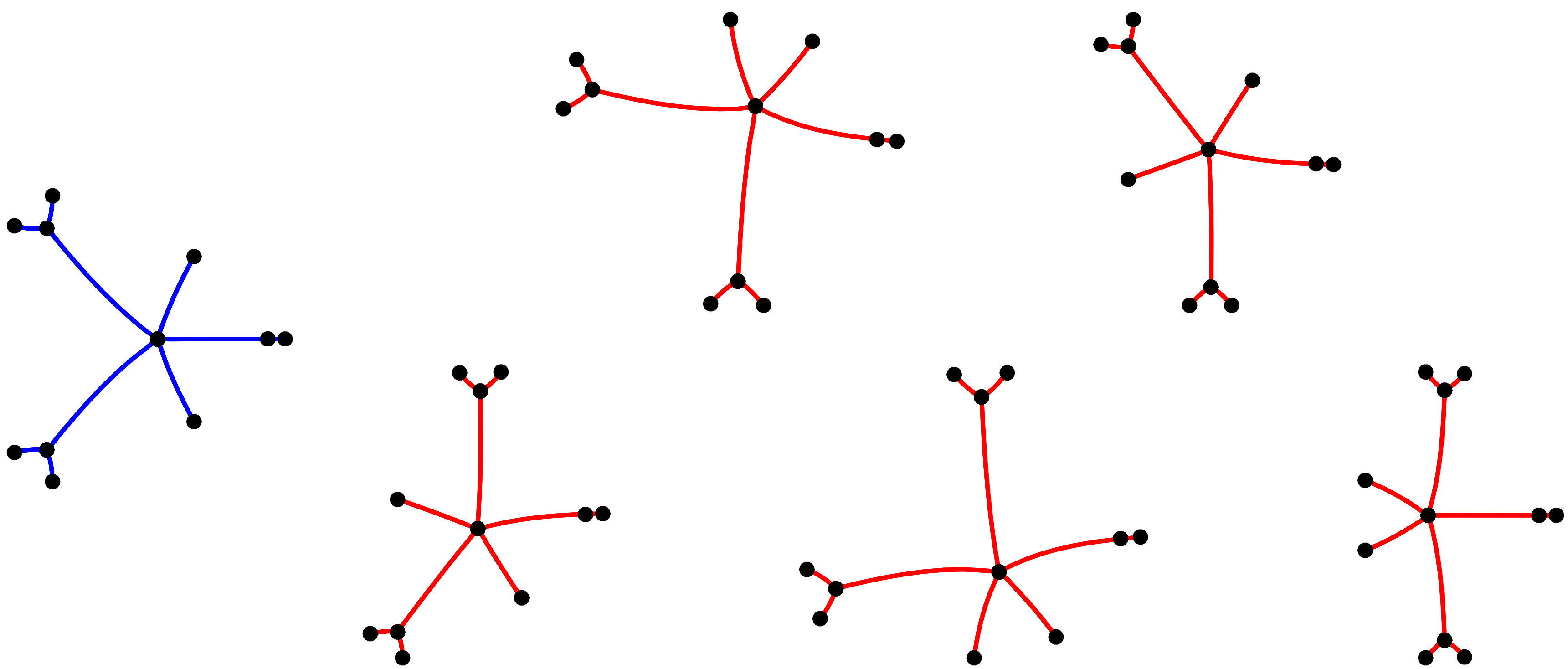

Only possible critical points are vertices of tree; these map to $\pm 1$.

Thus every planar tree has a true form.

## Algebraic aside:

True trees are examples of Grothendieck's *dessins d'enfants* on sphere.

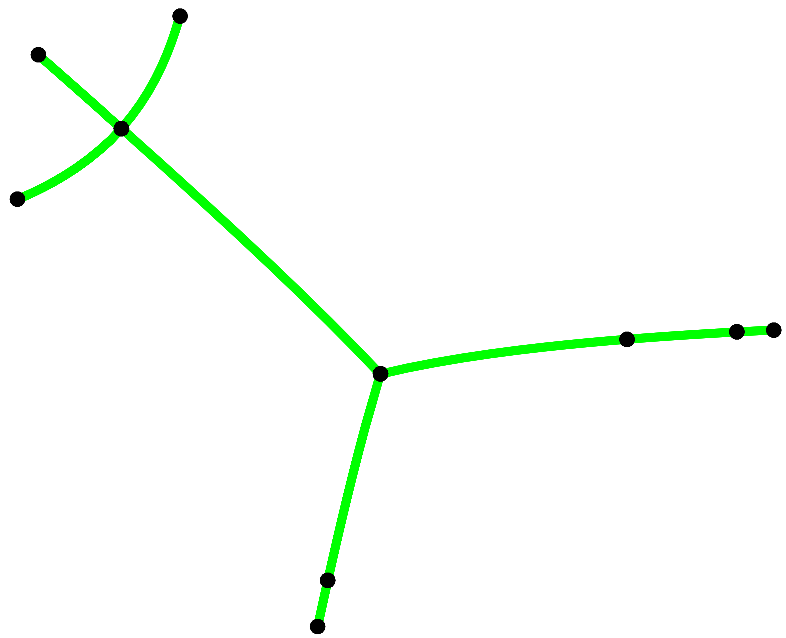
Normalized polynomials are algebraic, so planar trees correspond to number fields. Absolute Galois group acts on trees, but orbits unknown.



Six graphs of type 5 1 1 1 1 1 - 3 3 2 1 1, two orbits.

Even computing number field from tree is difficult.

Kochetkov (2009, 2014): did all trees with 9 and 10 edges.



For example, the polynomial for this 9-edge tree is

$$p(z) = z^4(z^2 + az + b)^2(z - 1),$$

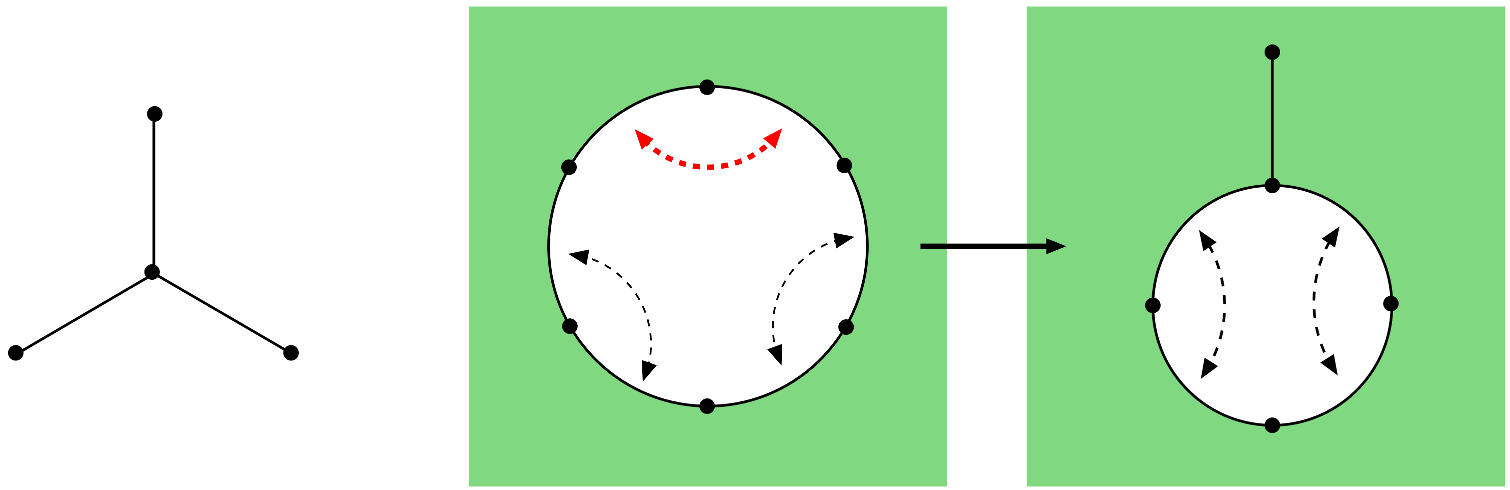
where $a$ is a root of ...

$$
\begin{aligned}
0 = \quad & 126105021875\, a^{15} + 873367351500\, a^{14} \\
& +2340460381665\, a^{13} + 2877817869766\, a^{12} \\
& +3181427453757\, a^{11} - 68622755391456\, a^{10} \\
& -680918281137097\, a^{9} - 2851406436711330\, a^{8} \\
& -7139130404618520\, a^{7} - 12051656256571792\, a^{6} \\
& -14350515598839120\, a^{5} - 12058311779508768\, a^{4} \\
& -6916678783373312\, a^{3} - 2556853615656960\, a^{2} \\
& -561846360735744\, a - 65703906377728
\end{aligned}
$$

This is **not** the most complicated formula in Kochetkov's paper.

However, true form can be drawn without knowing the polynomial.

Don Marshall's `ZIPPER` uses conformal mapping to draw true trees.



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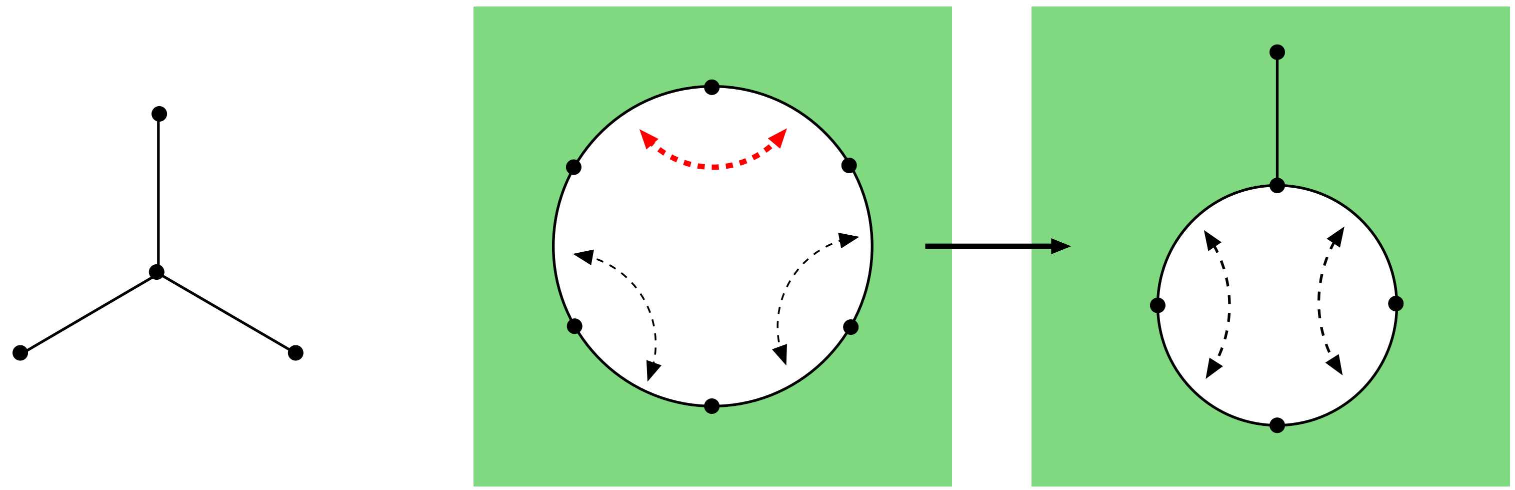
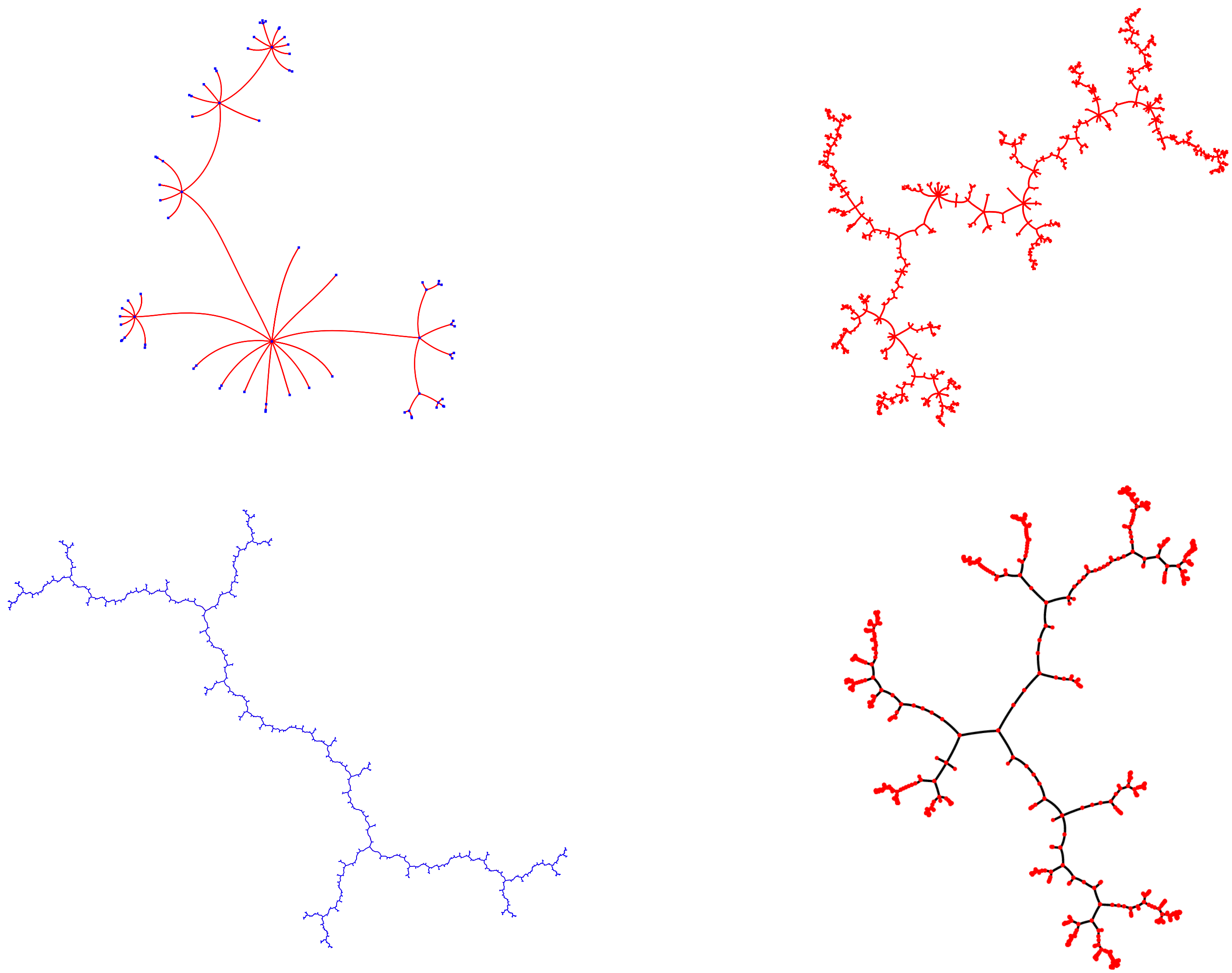


Marshall and Rohde approximated all true trees with $\leq 14$ edges.

They can compute vertices to 1000's of digits of accuracy.

One can test if $\alpha \in \mathbb{C}$ is algebraic by looking for integer relationships between $1, \alpha, \alpha^2, \ldots$ using lattice reduction or Ferguson's PSLQ.



Some true trees, courtesy of Marshall and Rohde

Every planar tree has a true form.

In other words, all possible **combinatorics** occur (countably many).

What about all possible **shapes**?

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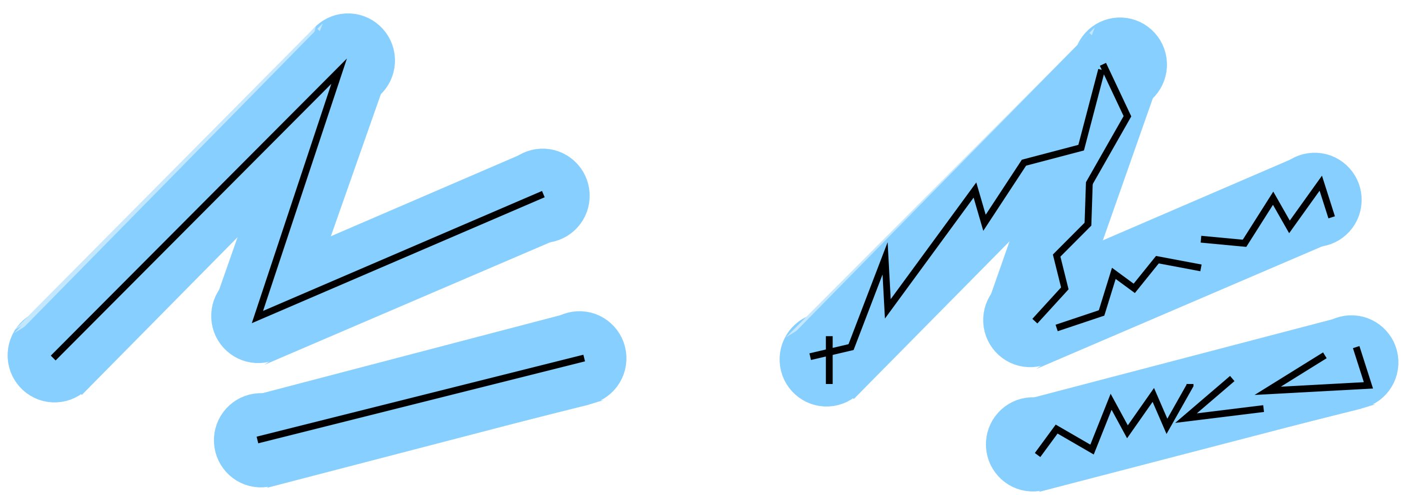
In other words, all possible **combinatorics** occur (countably many).

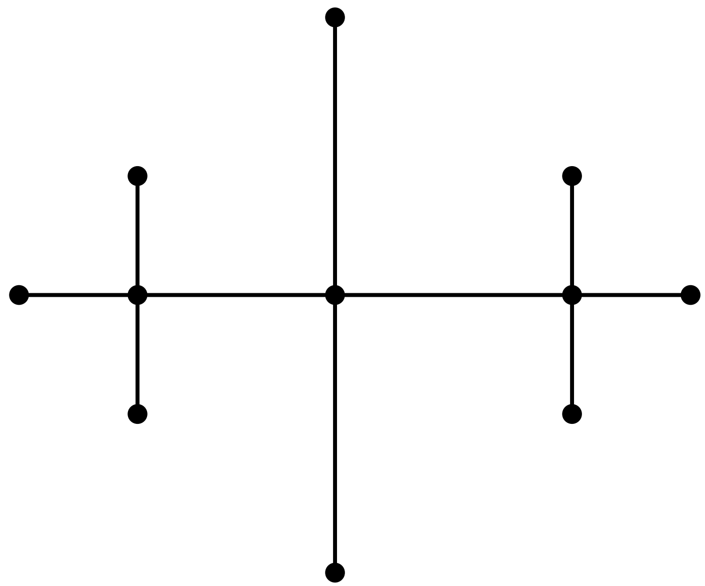
What about all possible **shapes**?

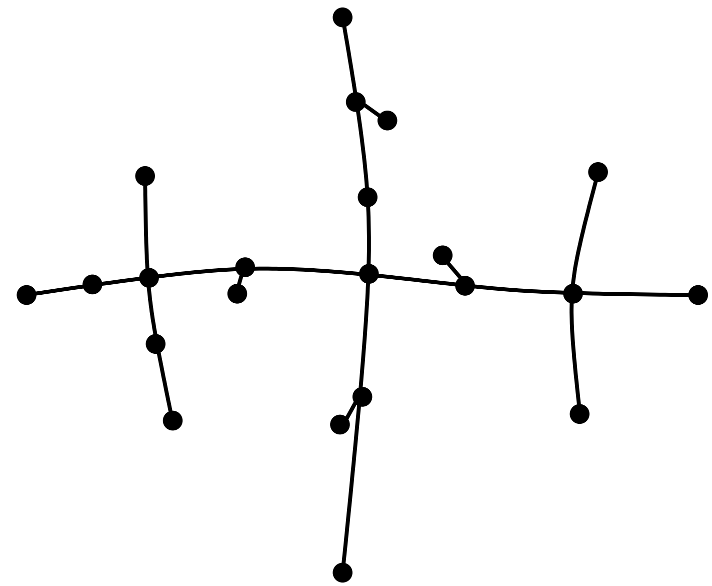
**Hausdorff metric**: if $E$ is compact,

$$E_\epsilon = \{z : \mathrm{dist}(z, E) < \epsilon\} = \epsilon\text{-neighborhood of } E$$

$$\mathrm{dist}(E, F) = \inf\{\epsilon : E \subset F_\epsilon, F \subset E_\epsilon\}.$$





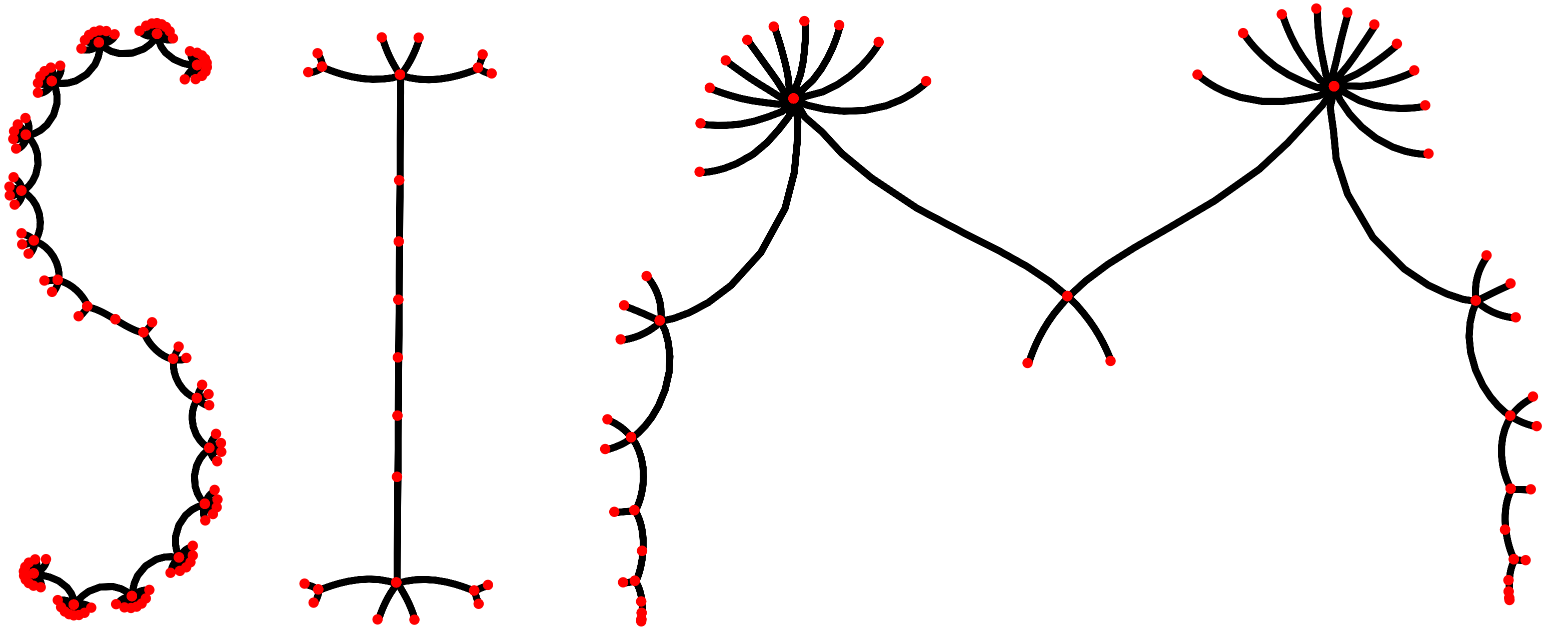


Different combinatorics, similar shapes

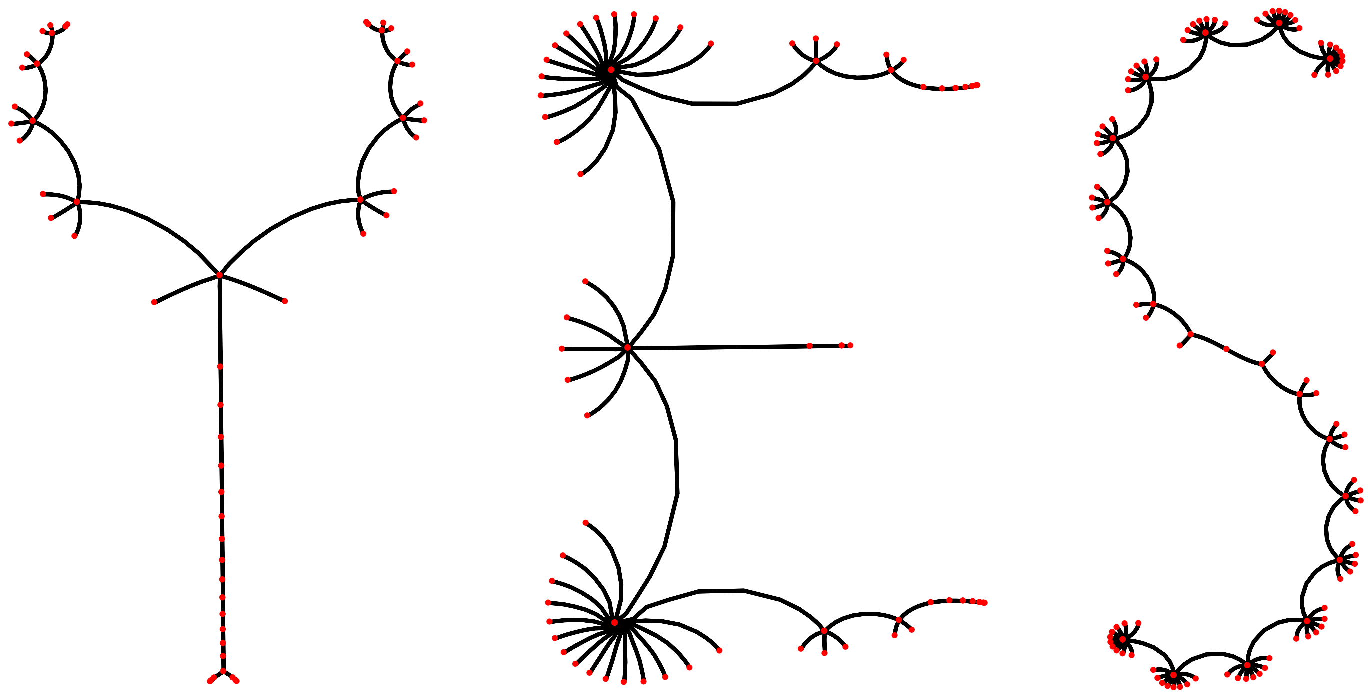
Close in Hausdorff metric

Do true trees approximate all possible shapes?

Do true trees approximate all possible shapes?

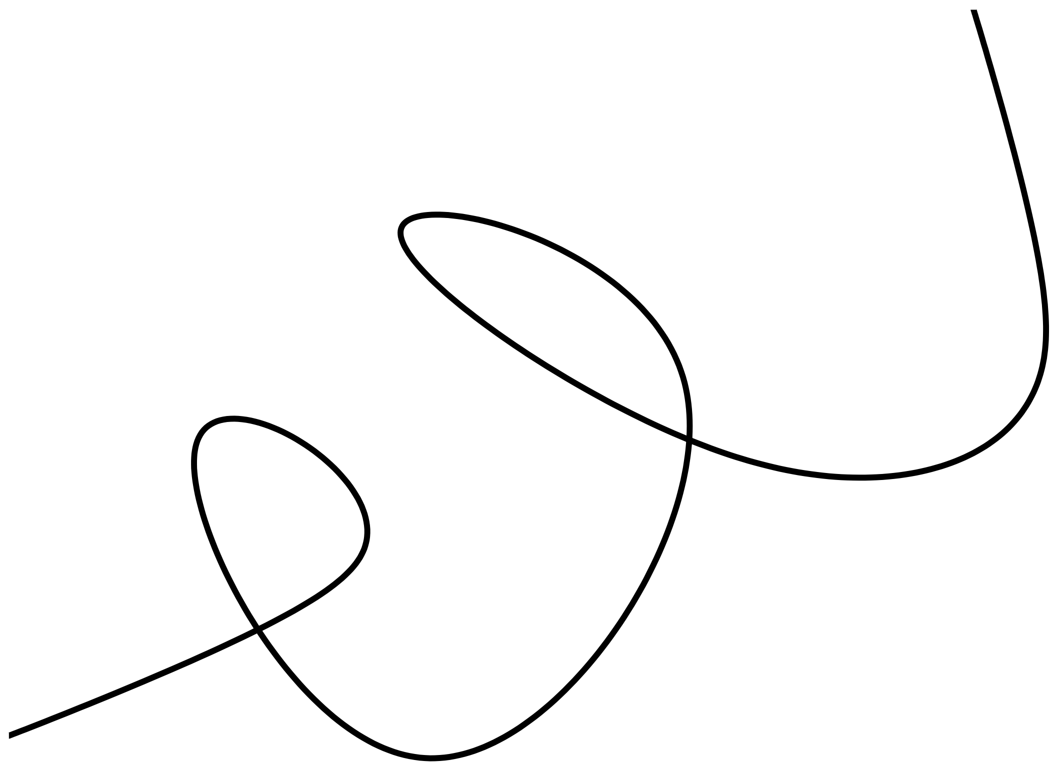


Do true trees approximate all possible shapes?

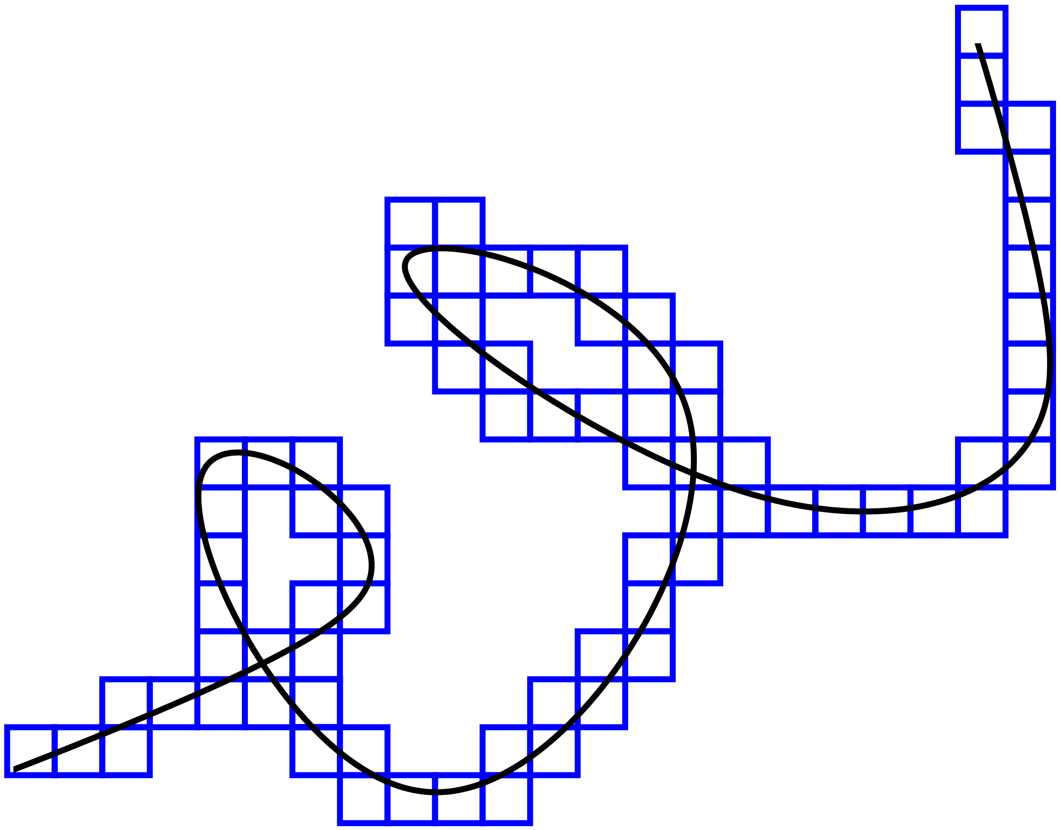


**Thm:** (B 2013) Every planar continuum is Hausdorff limit of true trees.

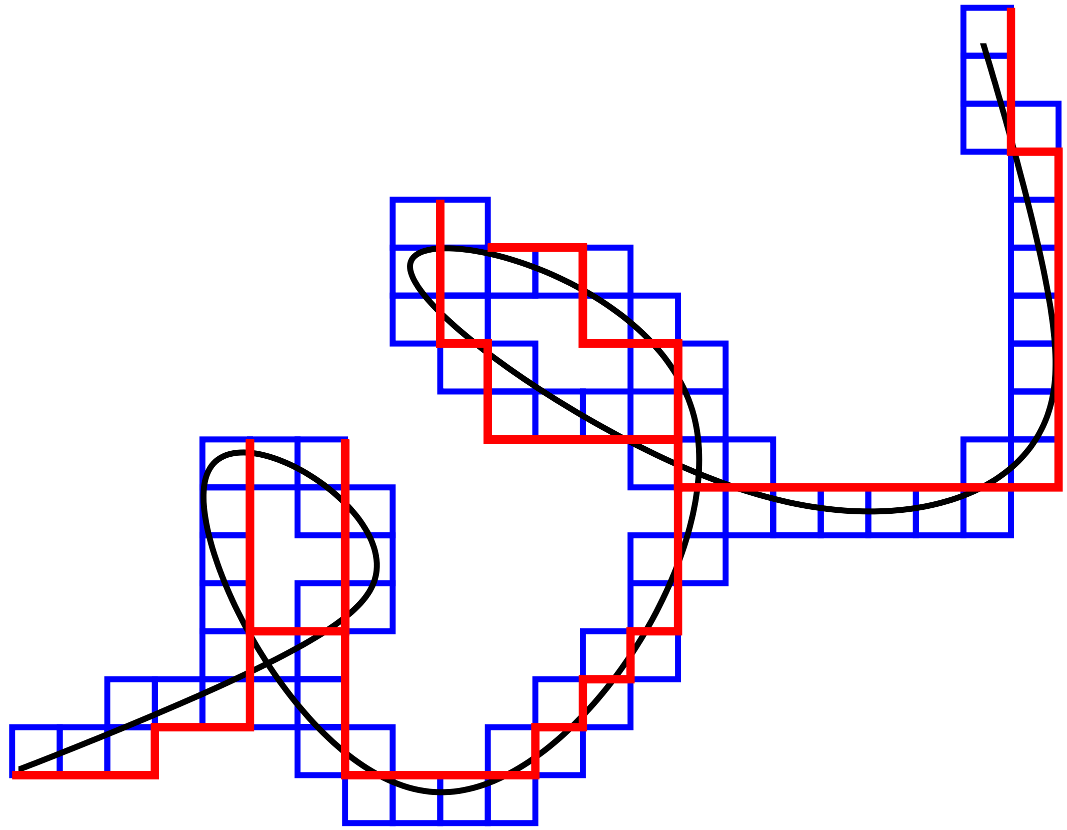
Answers question of Alex Eremenko. "True trees are dense".



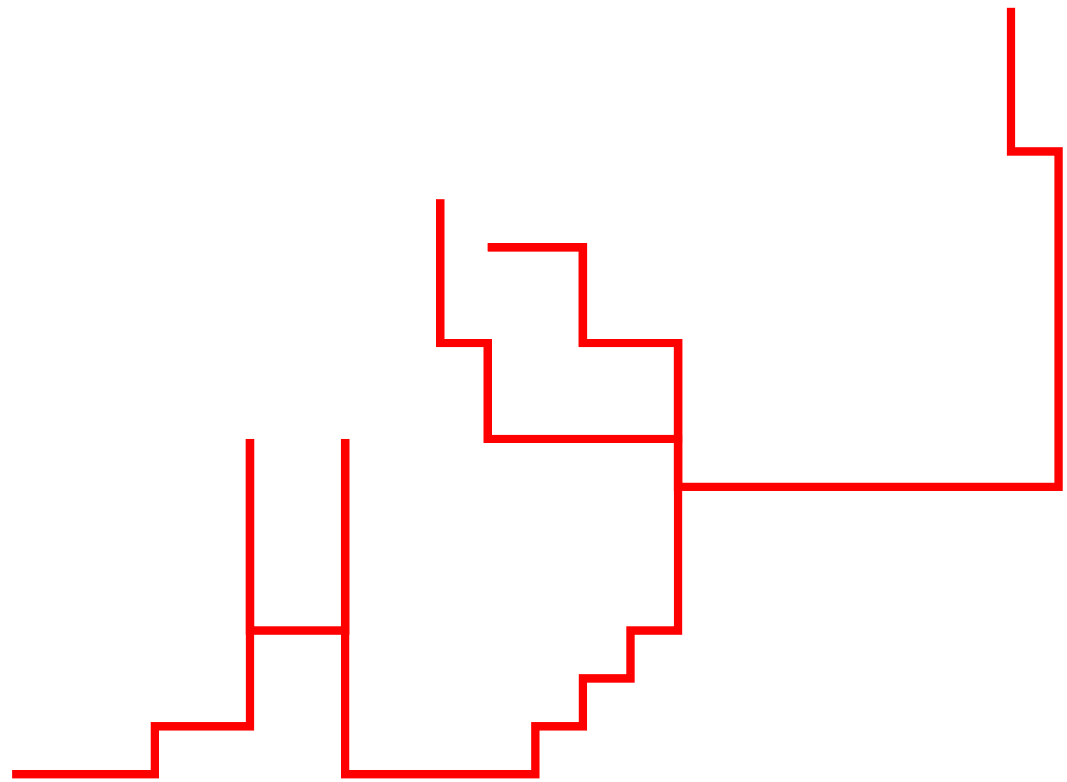
Suffices to approximate subtrees of a grid.



Suffices to approximate subtrees of a grid.



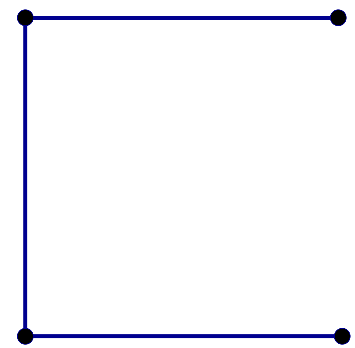
Suffices to approximate subtrees of a grid.



Suffices to approximate subtrees of a grid.

**Theorem:** Every planar continuum is a limit of true trees.

**Idea of Proof:** reduce harmonic measure ratio by adding edges.



Vertical side has much larger harmonic measure from left.

**Theorem:** Every planar continuum is a limit of true trees.

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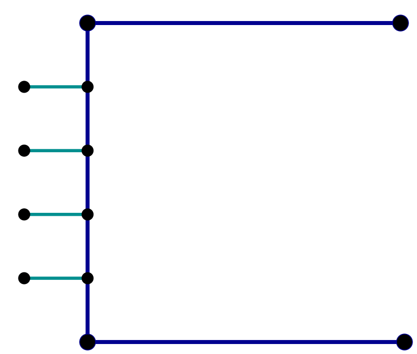

"Left" harmonic measure is reduced (roughly 3-to-1).

New edges are approximately balanced (universal constant).

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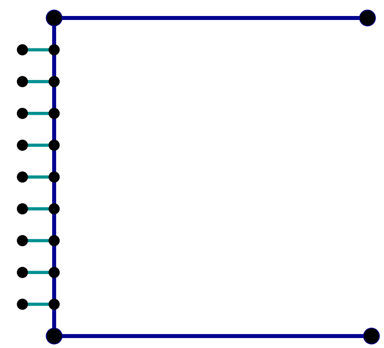


"Left" harmonic measure is reduced (roughly 3-to-1).

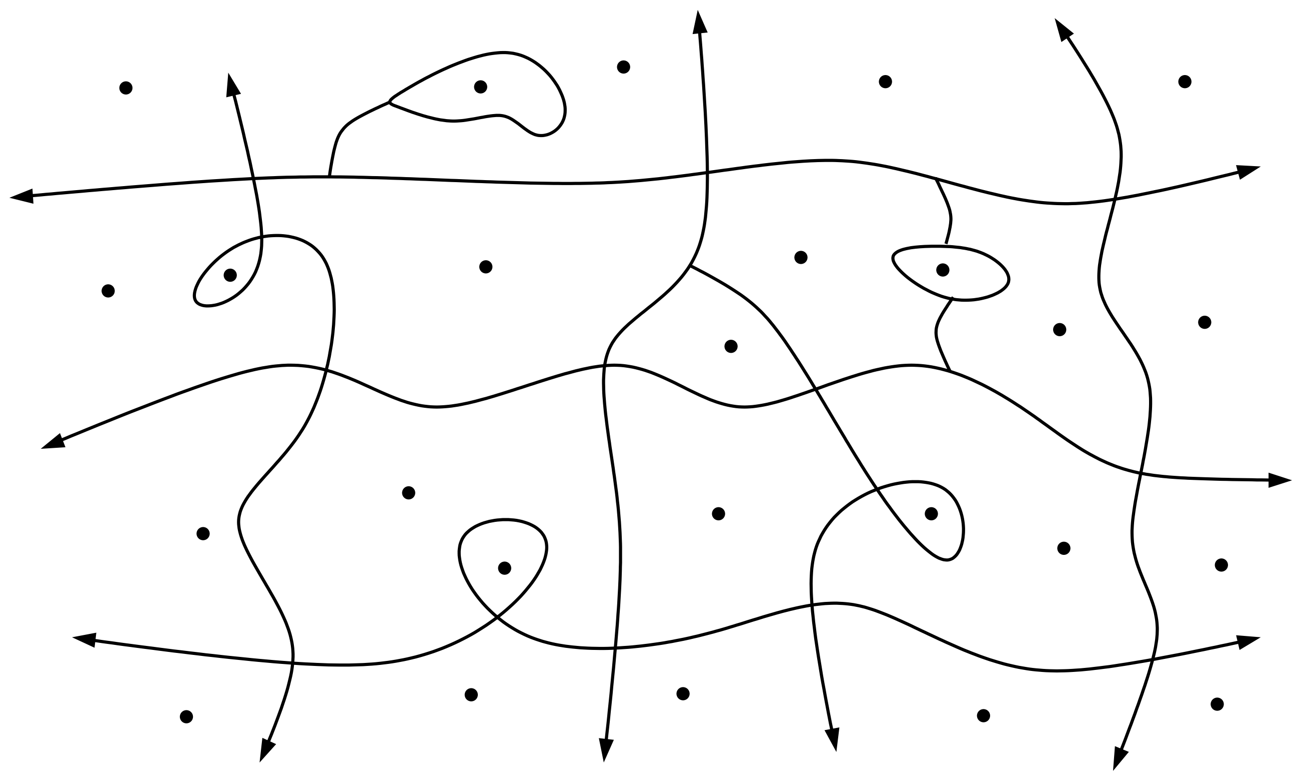
New edges are approximately balanced (universal constant).

Mapping theorem gives exactly balanced.

QC correction map is near identity if "spikes" are short.

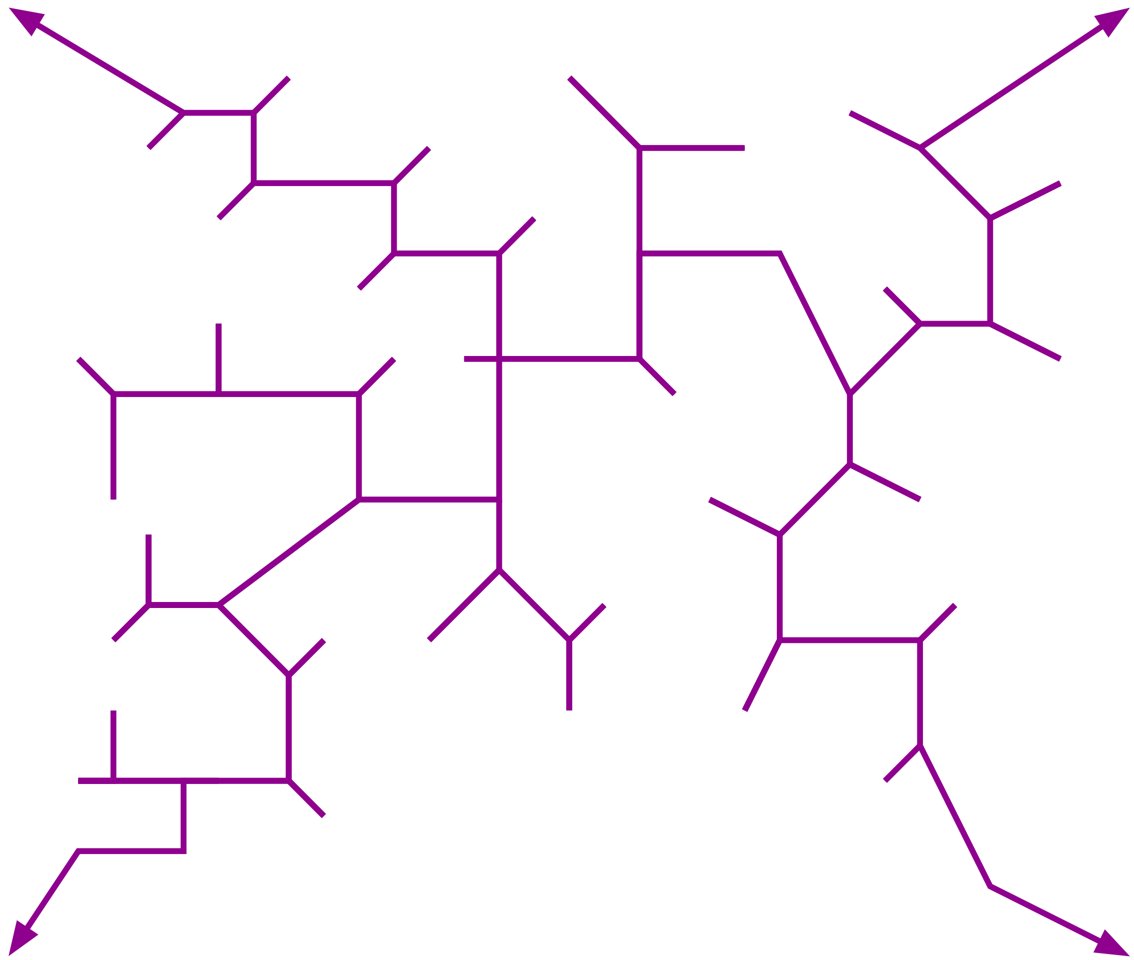
New tree approximates shape of old tree; different combinatorics.

Can replace trees by meshes. Get rational or meromorphic functions.

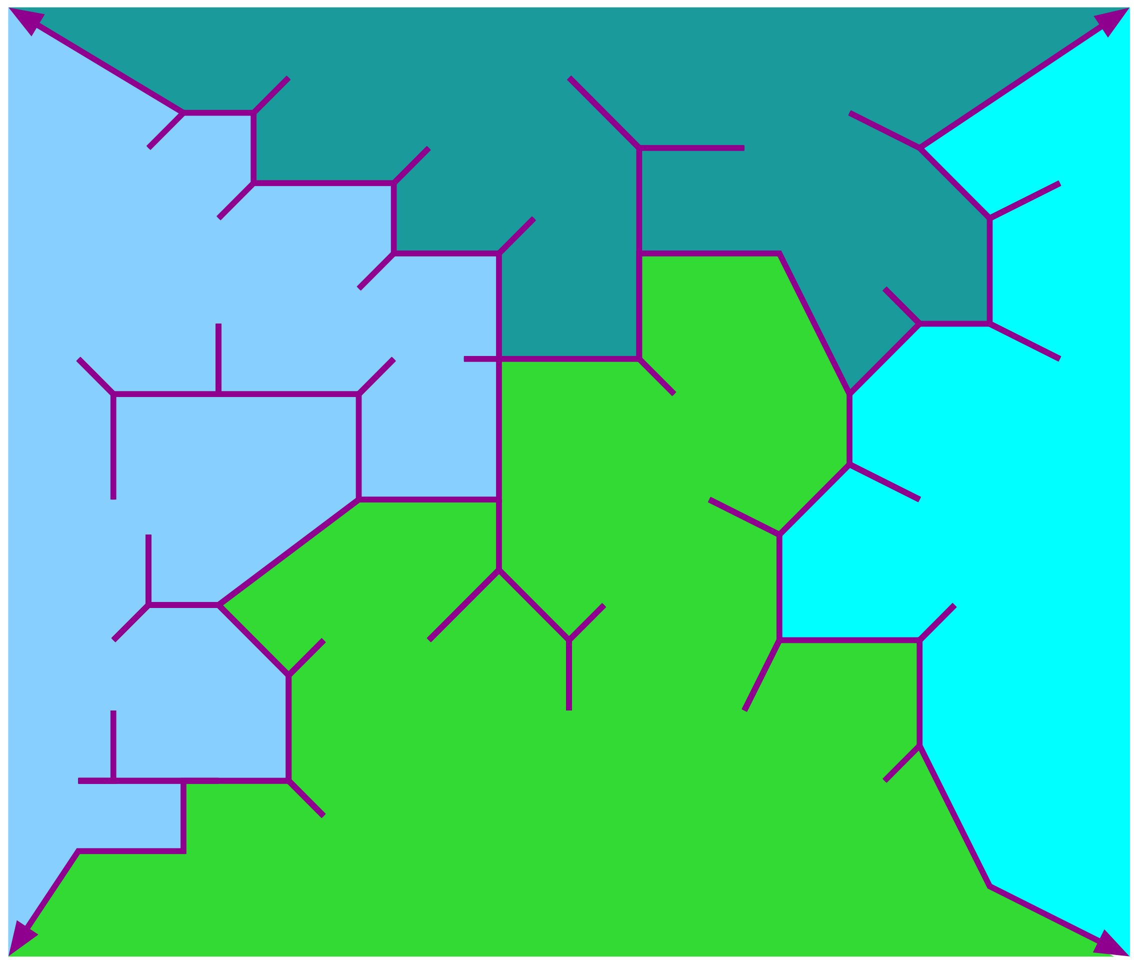


Random “true triangulations” related to Liouville quantum gravity.

We will go in a different direction: infinite trees.



Do infinite trees correspond to entire functions with 2 critical values?

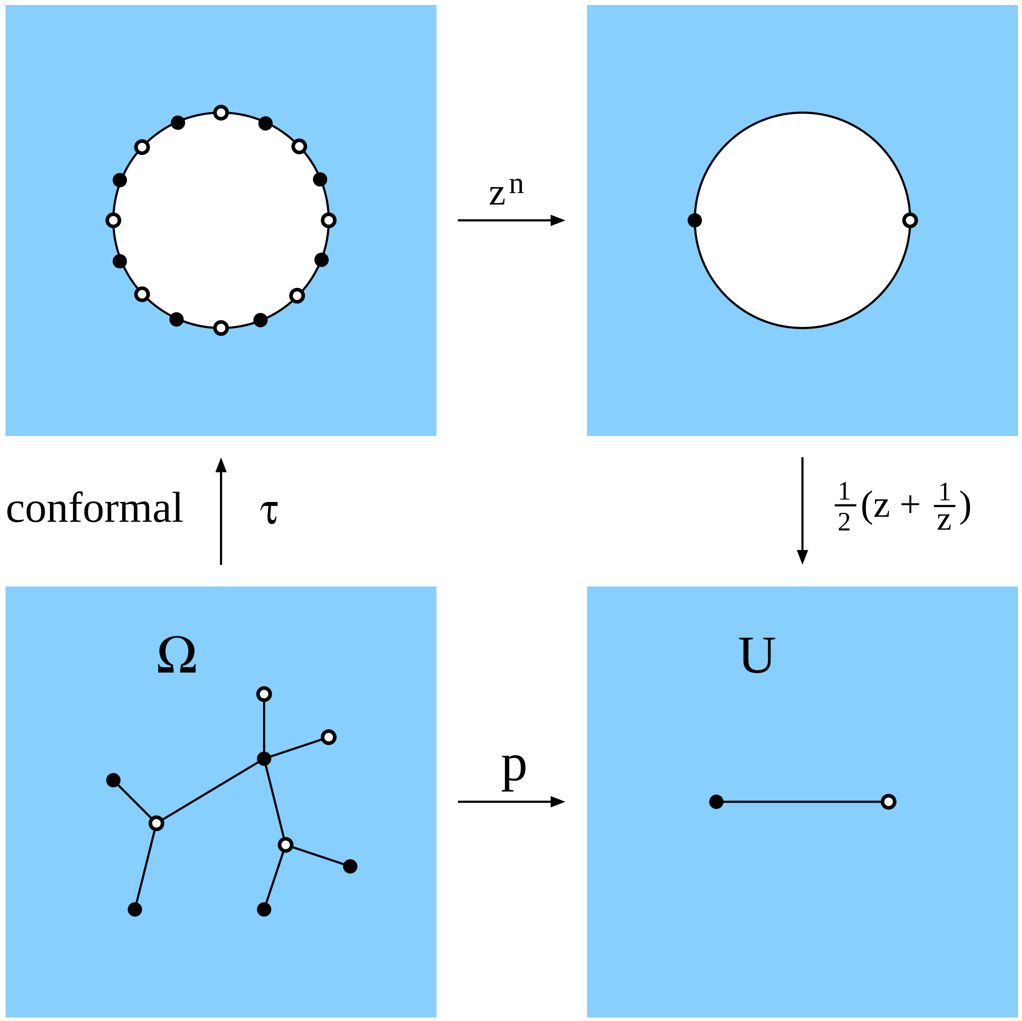


Main difference:

$\mathbb{C} \setminus$ finite tree = one topological annulus

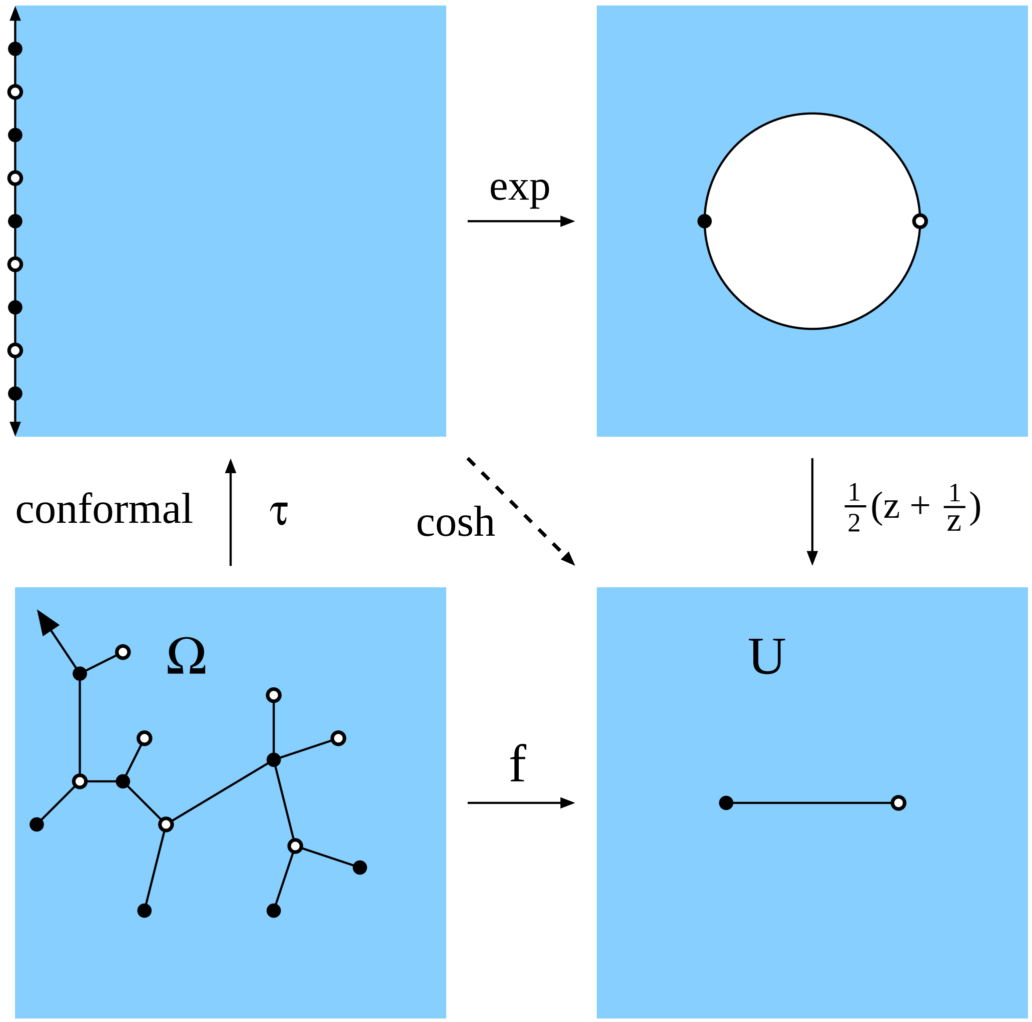
$\mathbb{C} \setminus$ infinite tree = many simply connected components

# Finite case




$T$ is true tree $\Leftrightarrow p = \frac{1}{2}(\tau^n + 1/\tau^n)$ is continuous across $T$.

# Infinite case




Infinite balanced tree $\Leftrightarrow$ $f = \cosh \circ \tau$ is continuous across $T$.

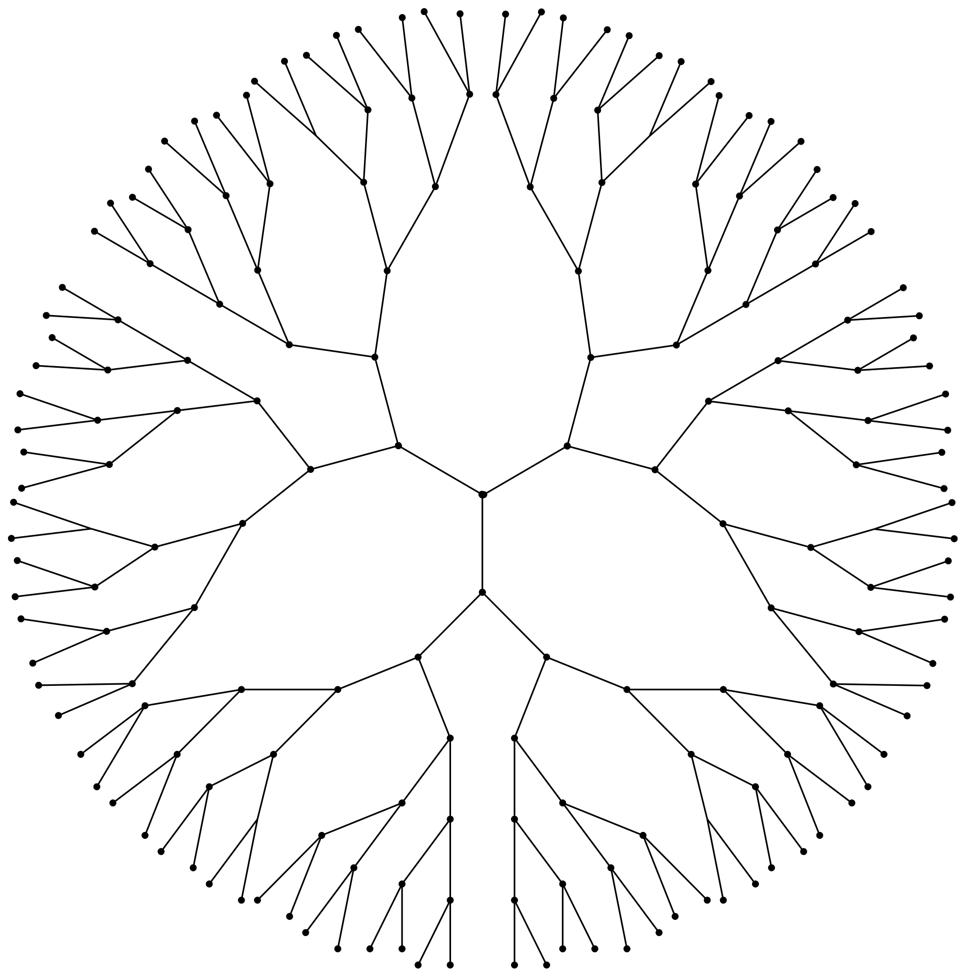
Do all infinite planar trees have true forms?

Do infinite true trees approximate any shape?

Do all infinite planar trees have true forms? **No.**

Do infinite true trees approximate any shape? **Yes (sort of).**

The infinite 3-regular tree has no true form in plane.



Automorphisms are induced by linear maps. Group is "too large" to fit inside isometries of plane. There is a true form in the hyperbolic disk.

Which infinite planar trees can be approximated by true trees?

We need two assumptions that substitute for finiteness.

First we define a neighborhood $T(r)$ of an infinite tree.

(Replaces Hausdorff $\epsilon$-neighborhood in finite case.)

If $e$ is an edge of $T$ and $r > 0$ let

$$e(r) = \{z : \mathrm{dist}(z, e) \leq r \cdot \mathrm{diam}(e)\}$$

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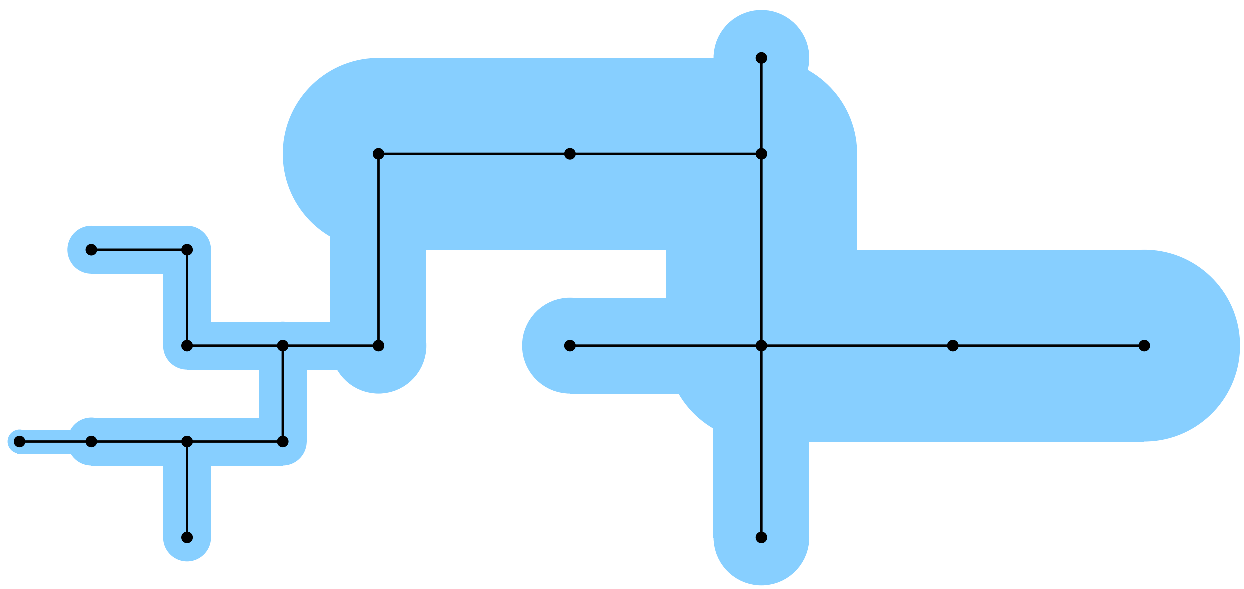
Define neighborhood of $T$: $T(r) = \cup\{e(r) : e \in T\}$.



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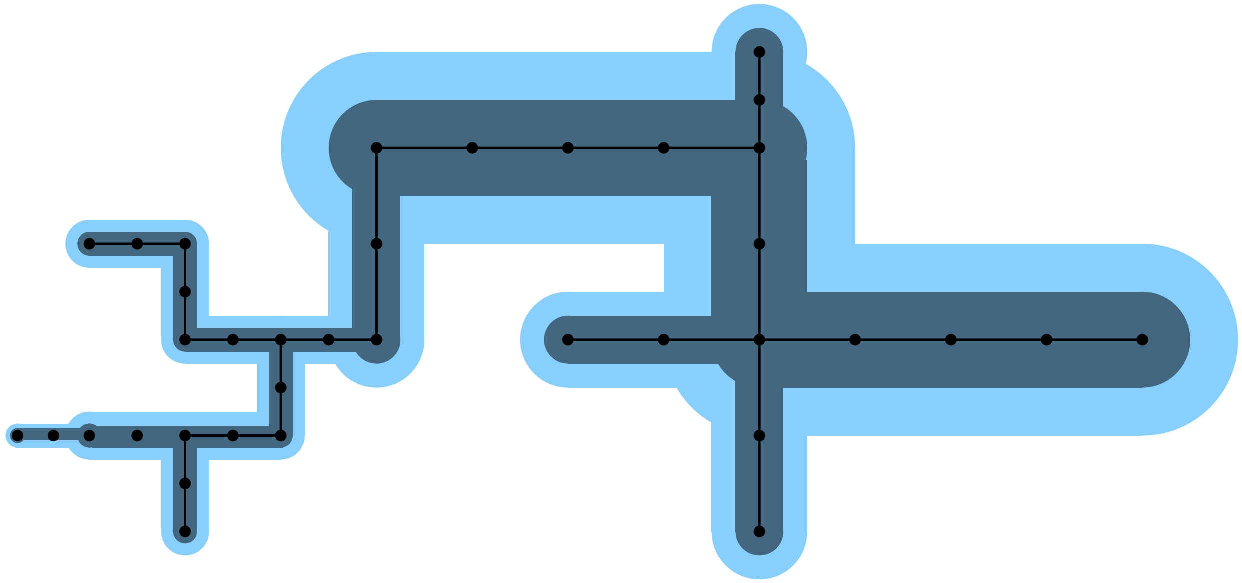
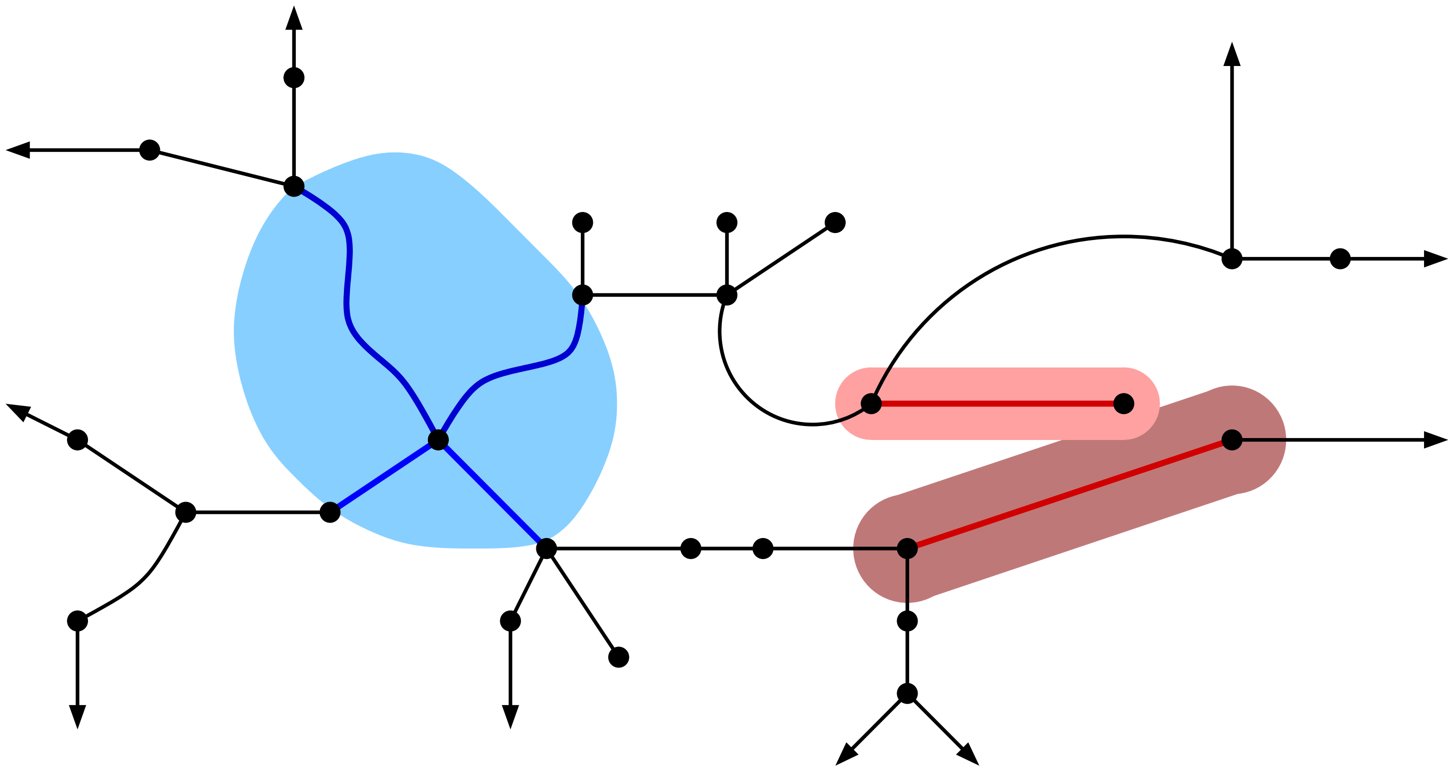


Adding vertices reduces $T(r)$. Useful scaling property.

**(1) Bounded Geometry (local condition; easy to verify):**

- edges are uniformly smooth.
- adjacent edges form bi-Lipschitz image of a star $= \{z^n \in [0, r]\}$
- non-adjacent edges are well separated,

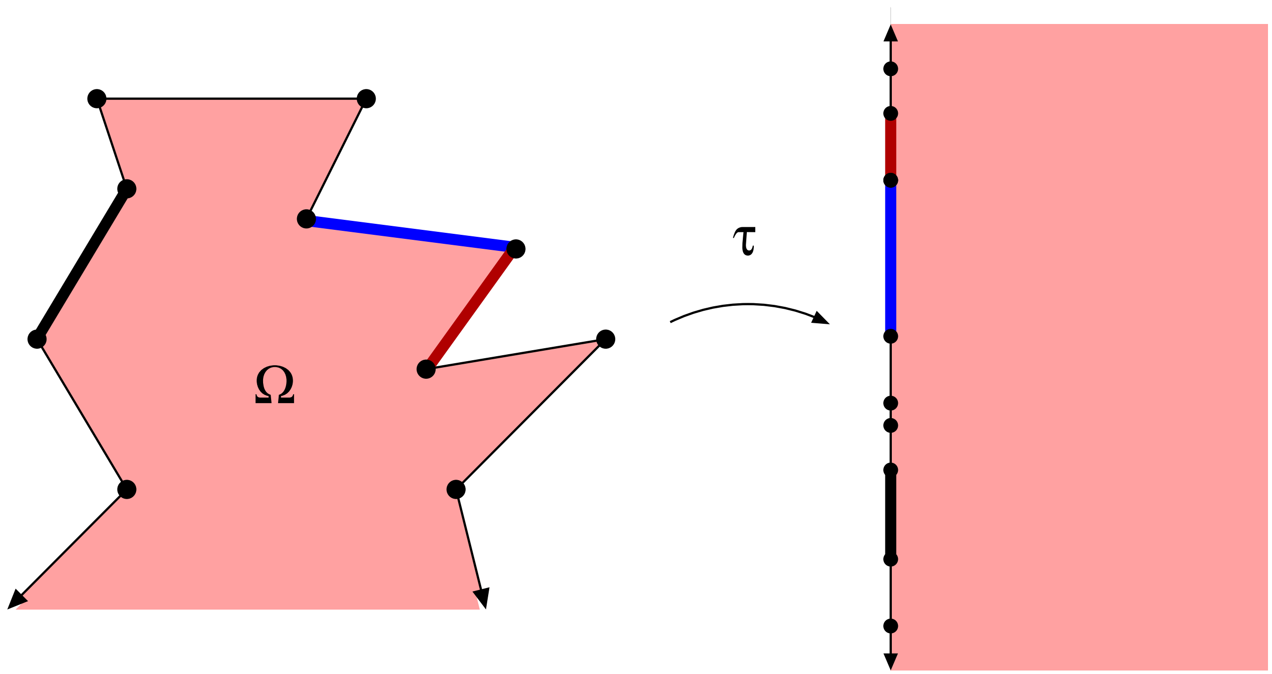
$$\mathrm{dist}(e, f) \geq \epsilon \cdot \min(\mathrm{diam}(e), \mathrm{diam}(f)).$$



**(2) $\tau$-Lower Bound (global condition; harder to check):**

Complementary components of tree are simply connected.

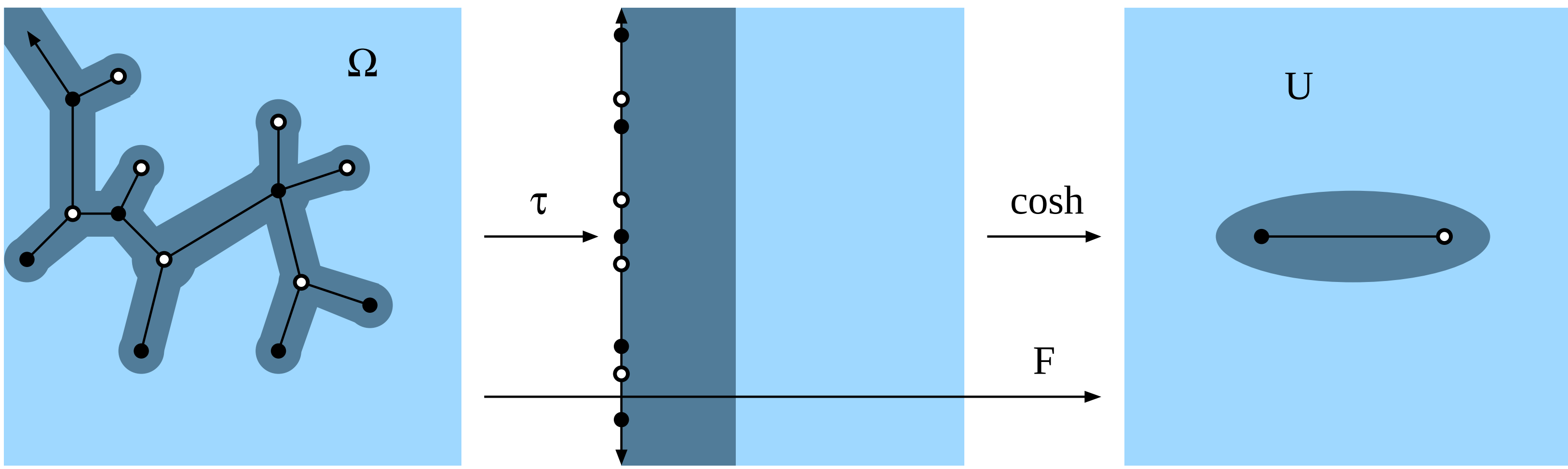
Each can be conformally mapped to right half-plane. Call map $\tau$.




We assume all images have length $\geq \pi$.

Need positive lower bound; actual value usually not important.

**QC-Folding Theorem (B 2015):** If $T$ has bounded geometry and the $\tau$-lower bound, then $T$ can be approximated by a true tree in the following sense. Let $F = \cosh \circ \tau$. Then there is a $K$-quasiregular $g$ and $r > 0$ such that $g = F$ off $T(r)$ (shaded) and $\mathrm{CV}(g) = \pm 1$.

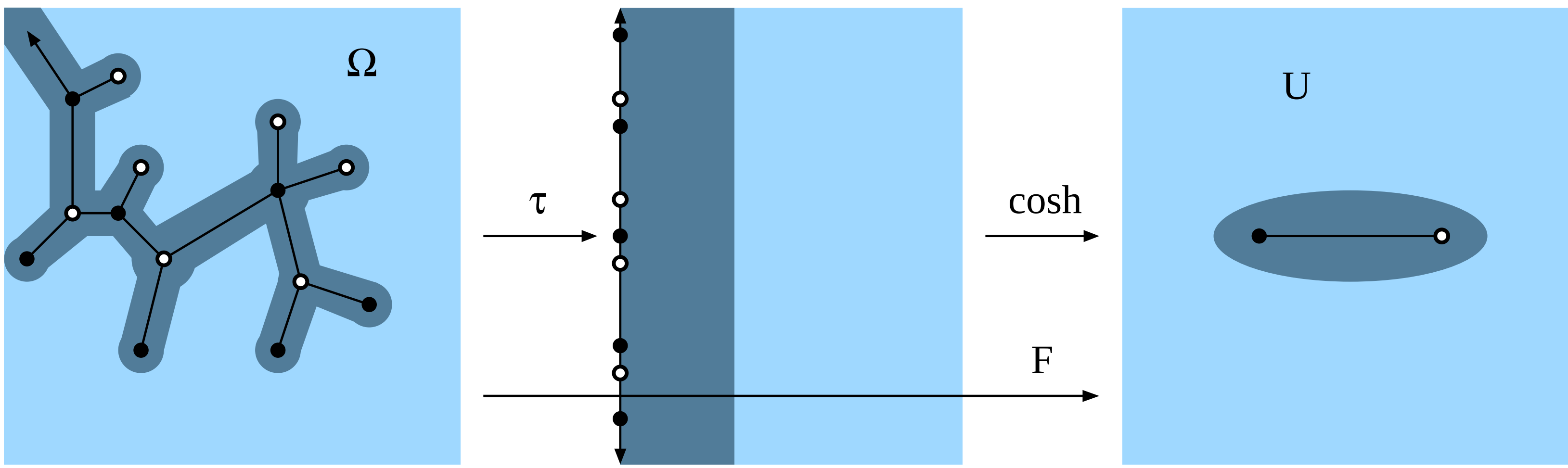



$K$ and $r$ only depend on the bounded geometry constants.

$g = F$ on light blue.

$F$ may be discontinuous across $T$. $g$ is continuous everywhere.

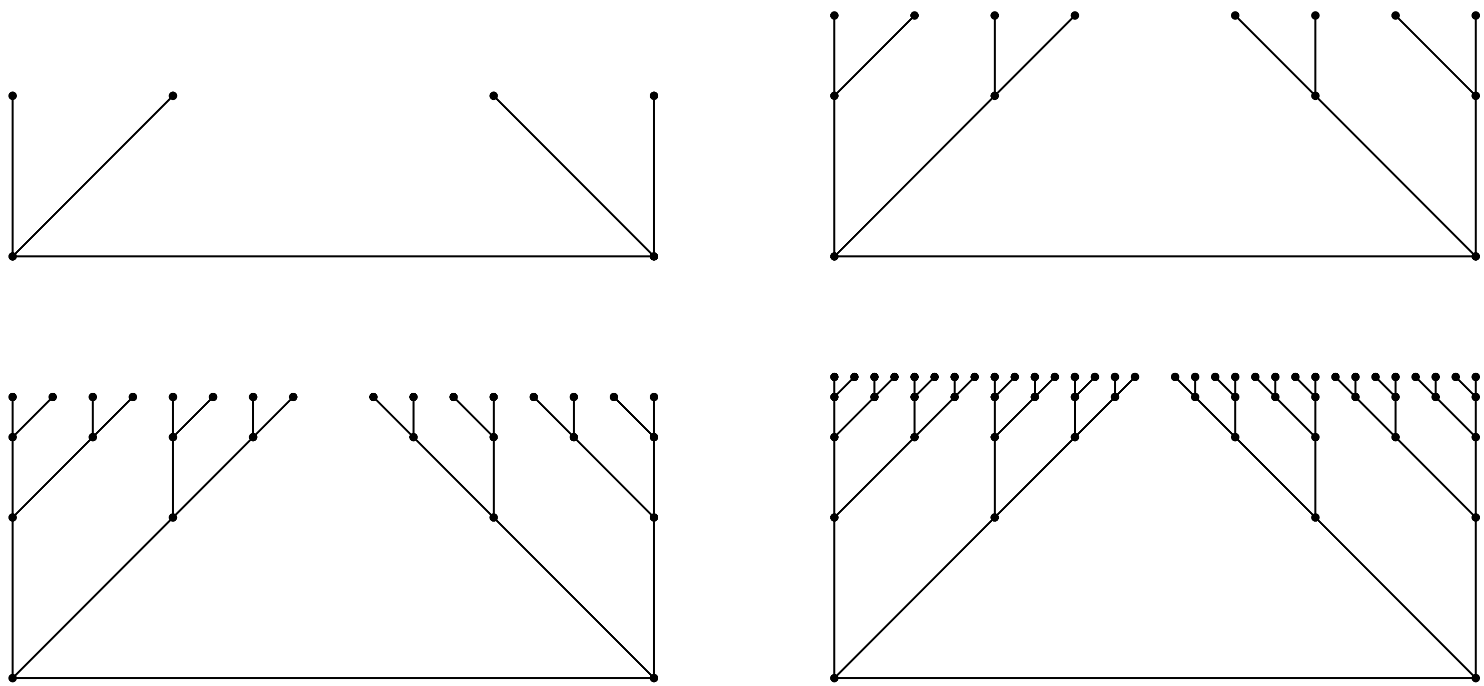
**QC-Folding Theorem (B 2015):** If $T$ has bounded geometry and the $\tau$-lower bound, then $T$ can be approximated by a true tree in the following sense. Let $F = \cosh \circ \tau$. Then there is a $K$-quasiregular $g$ and $r > 0$ such that $g = F$ off $T(r)$ (shaded) and $\mathrm{CV}(g) = \pm 1$.



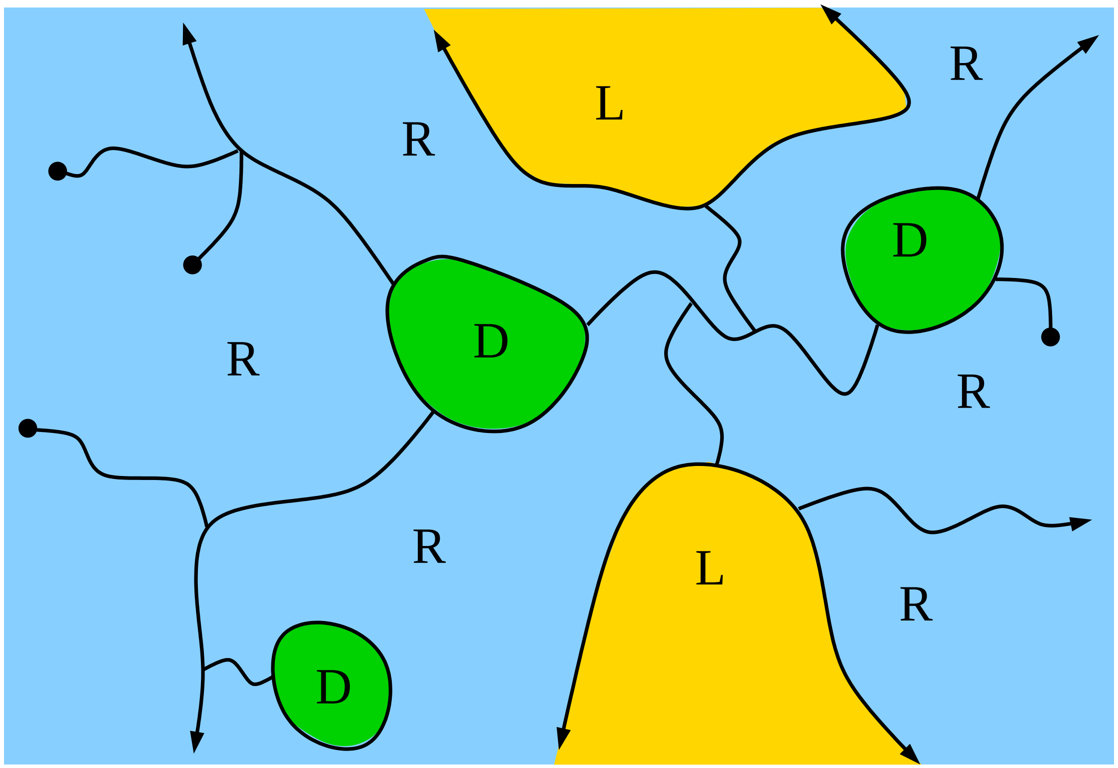

$K$ and $r$ only depend on the bounded geometry constants.

**Cor:** Any $T$ as above is approximated by $\varphi(T')$ where $T' = f^{-1}([-1,1])$, $f$ is entire with $\mathrm{CV}(f) = \pm 1$, $\varphi$ is QC and conformal off $T(r)$. (In many applications, $\varphi$ is close to identity, so $T \approx T'$.)

**Proof of Folding Theorem:** similar to finite case: add finite trees (instead of spikes) to sides of $T$, in order make two sides almost balanced.



More bells and whistles: Replace tree by graph, faces labeled D,L,R.




D = bounded Jordan domains (high degree critical points)

L = unbounded Jordan domains (finite asymptotic values)

Original version uses only R-components.

**Transcendental** entire functions = non-polynomials.

**Singular set** = closure of critical values and asymptotic values
= points where not all branches of $f^{-1}$ are defined.

**Speiser class** = $\mathcal{S}$ = finite singular set

**Eremenko-Lyubich class** = $\mathcal{B}$ = compact singular set

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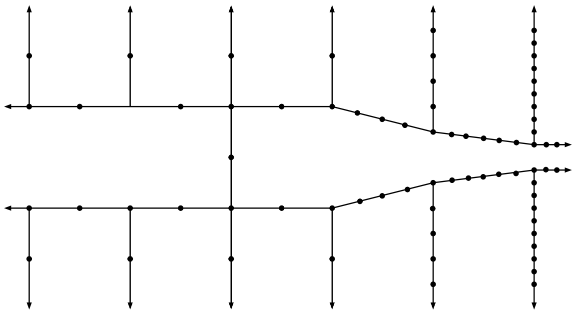
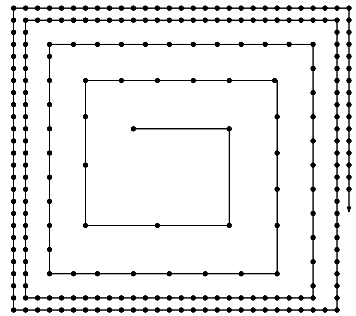
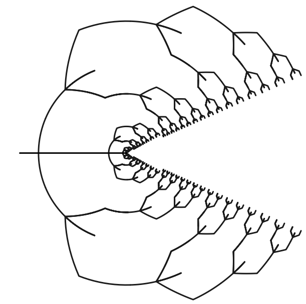
- Many previous examples based on formulas, e.g., $\sin(z)$, $e^z$, ...
- QC-Folding is a geometric alternative that is simple and flexible.
- Gives precise control of singular values; good for dynamics.
- Many applications, such as ...



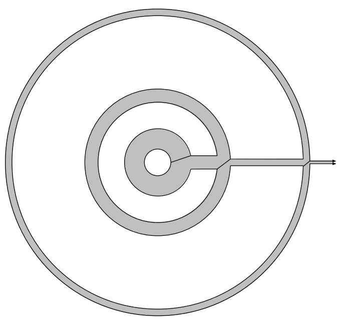
Rapid increase



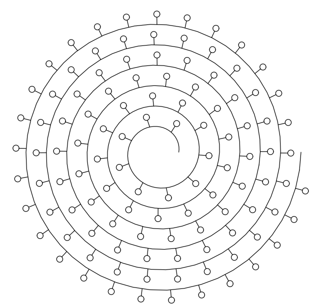
Spirals



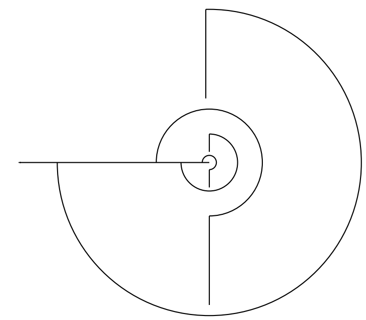
Order Conjecture



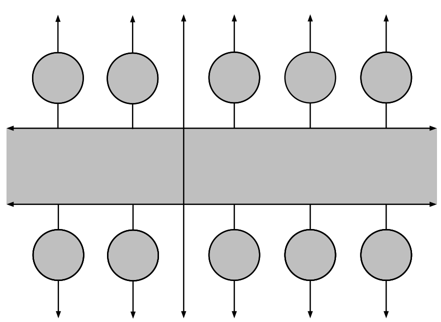
Area Conjecture



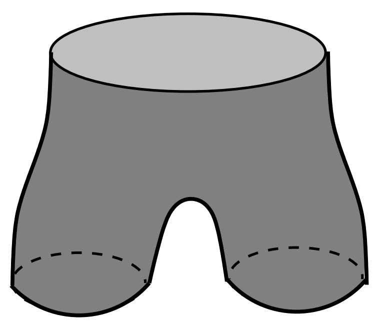
Wiman's Conjecture



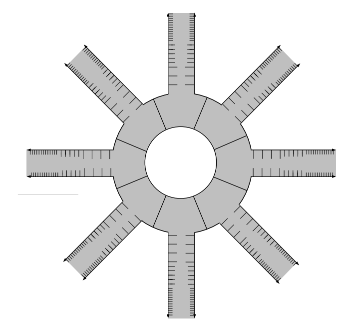
Eremenko Conjecture



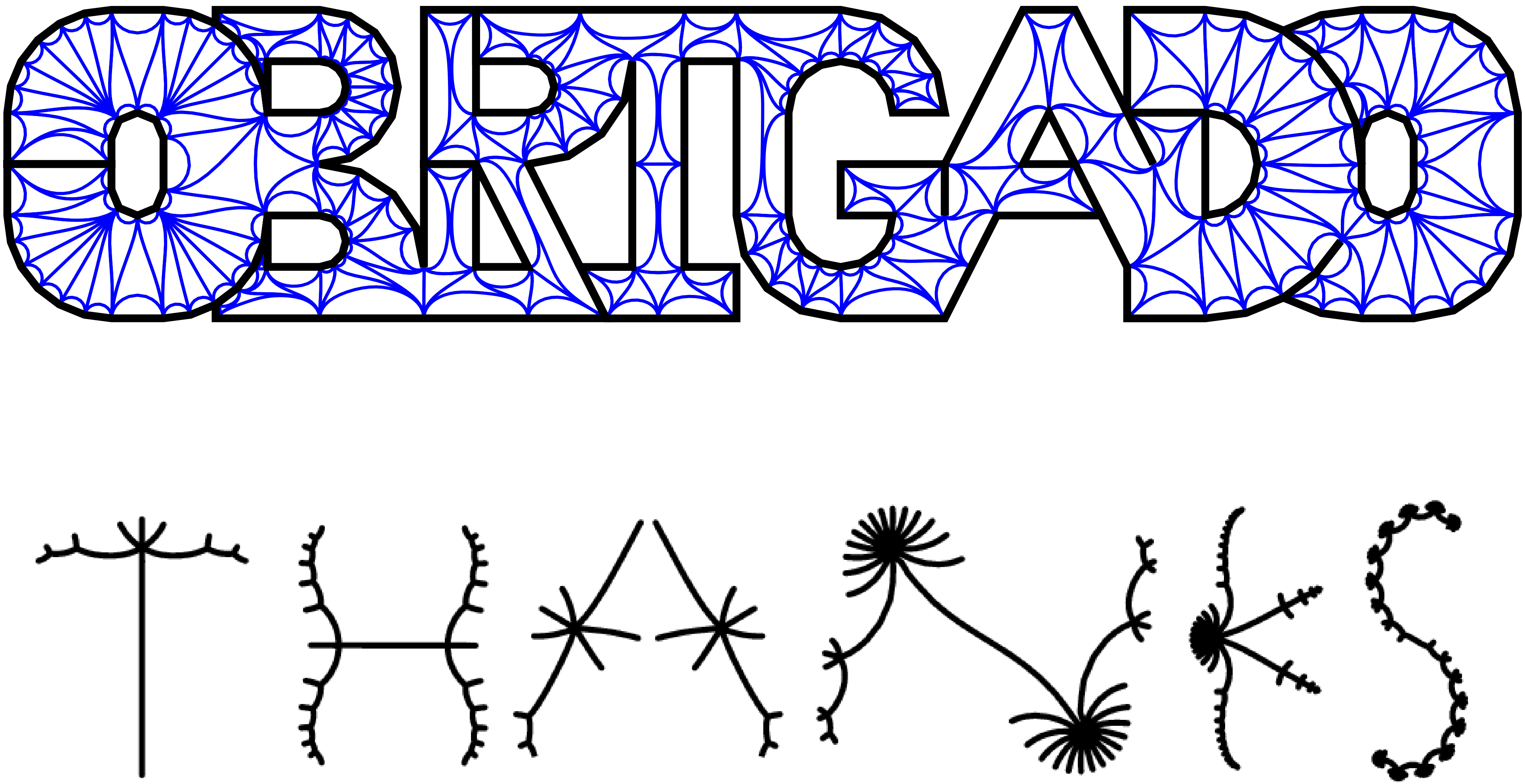
Wandering domain



Folding a pair-of-pants



Dimension near 1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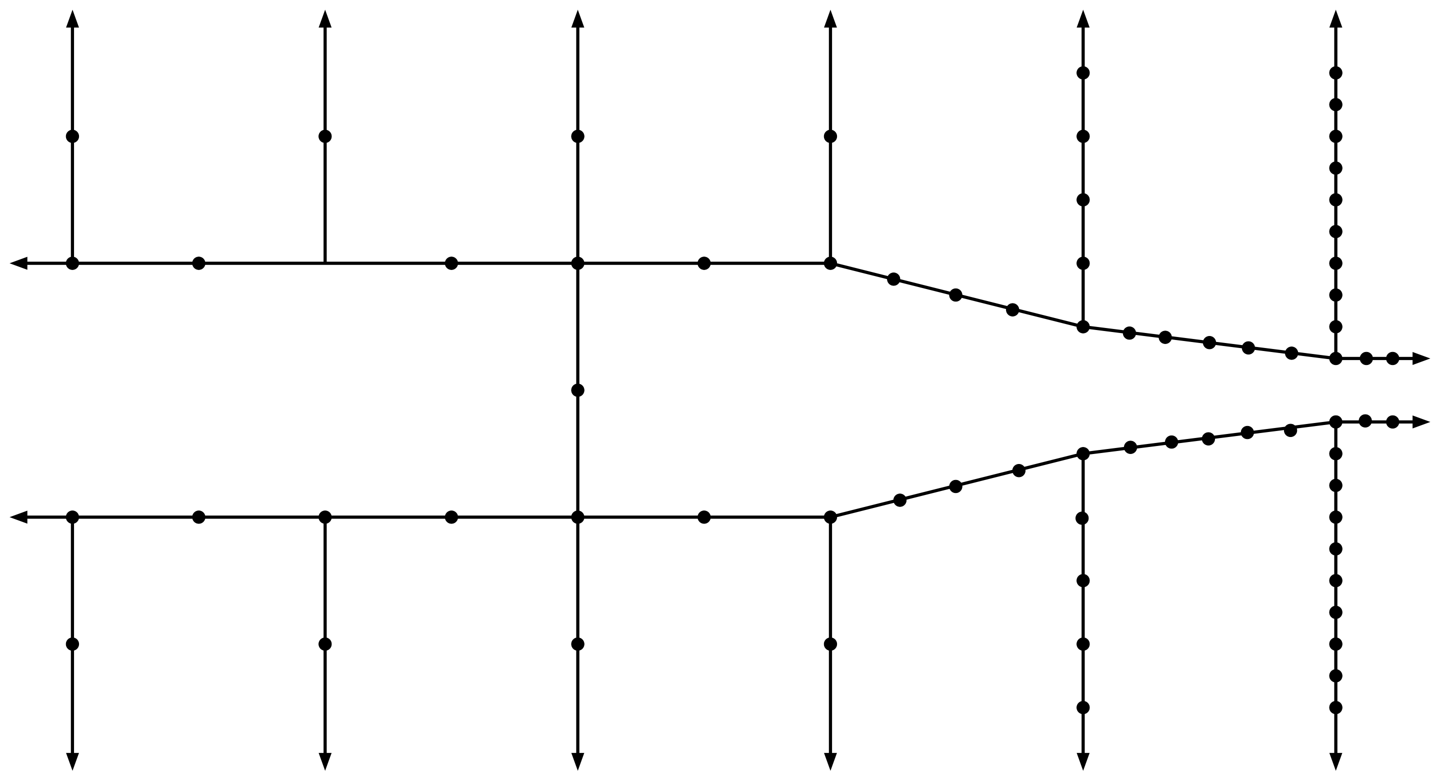
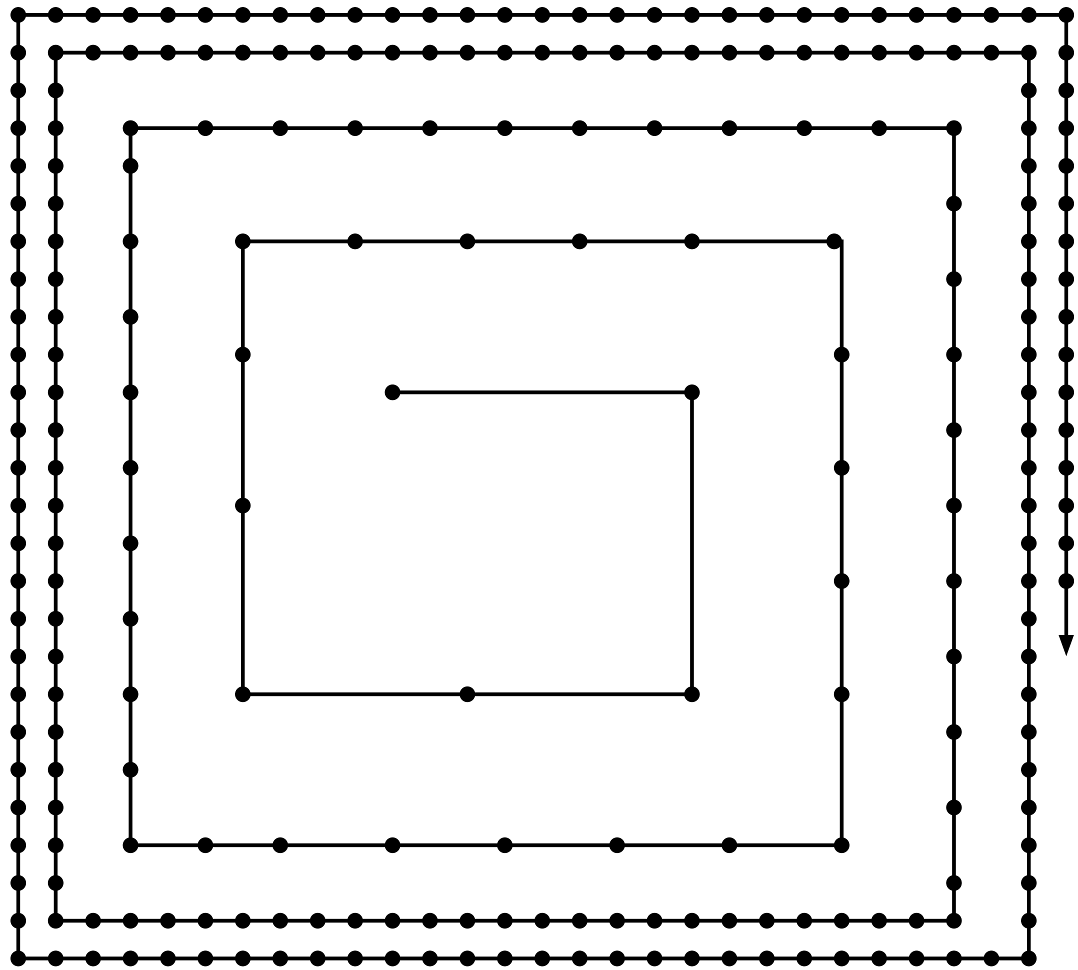
THANKS

## Rapid increase in Speiser class



$f$ has 2 singular values, $f(x) \nearrow \infty$ as fast as we wish.

Similar examples due to Sergei Merenkov 2008 (3 singular values).



Speiser example with single tract; rapid growth and spiraling.

**Order of growth of entire function:**

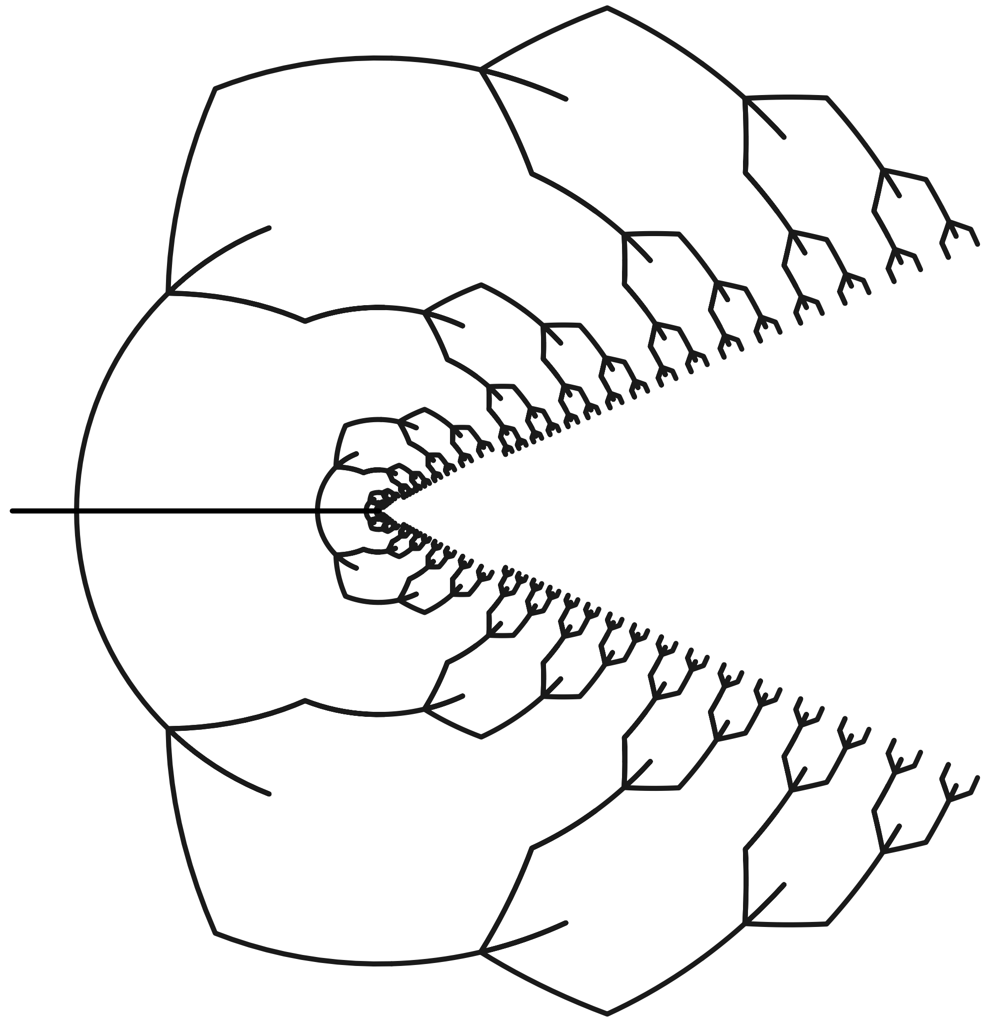
$$\rho(f) = \limsup_{|z|\to\infty} \frac{\log\log|f(z)|}{\log|z|}, \qquad \rho(e^{z^d}) = d.$$

**QC-equivalent:** $f \sim g$ if $g = \psi \circ f \circ \phi$ for quasiconformal $\psi, \phi$.

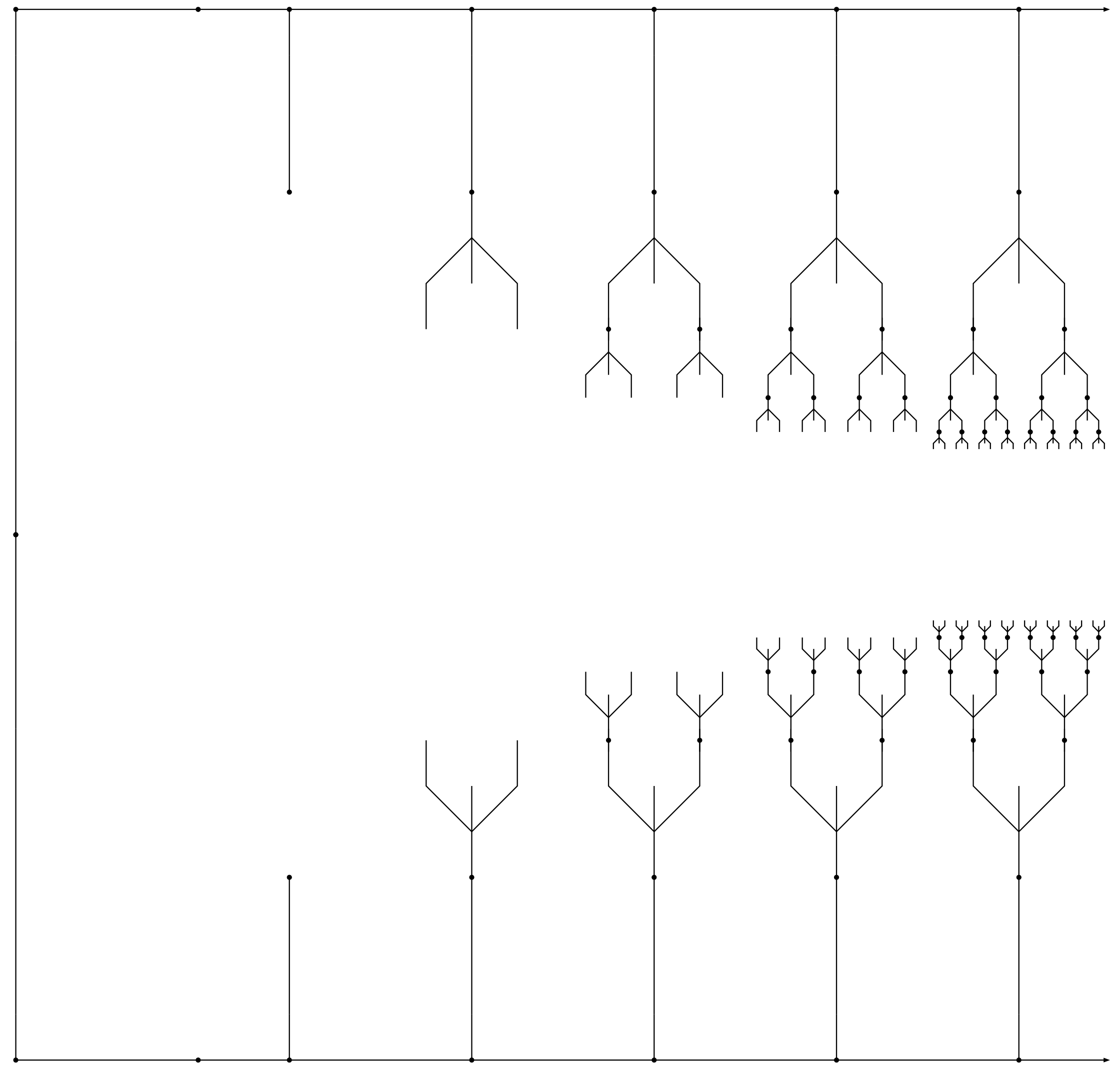
Speiser class QC-equivalence classes $M_f$ are finite dimensional.

**Order conjecture:** Is $\rho$ constant on $M_f$?

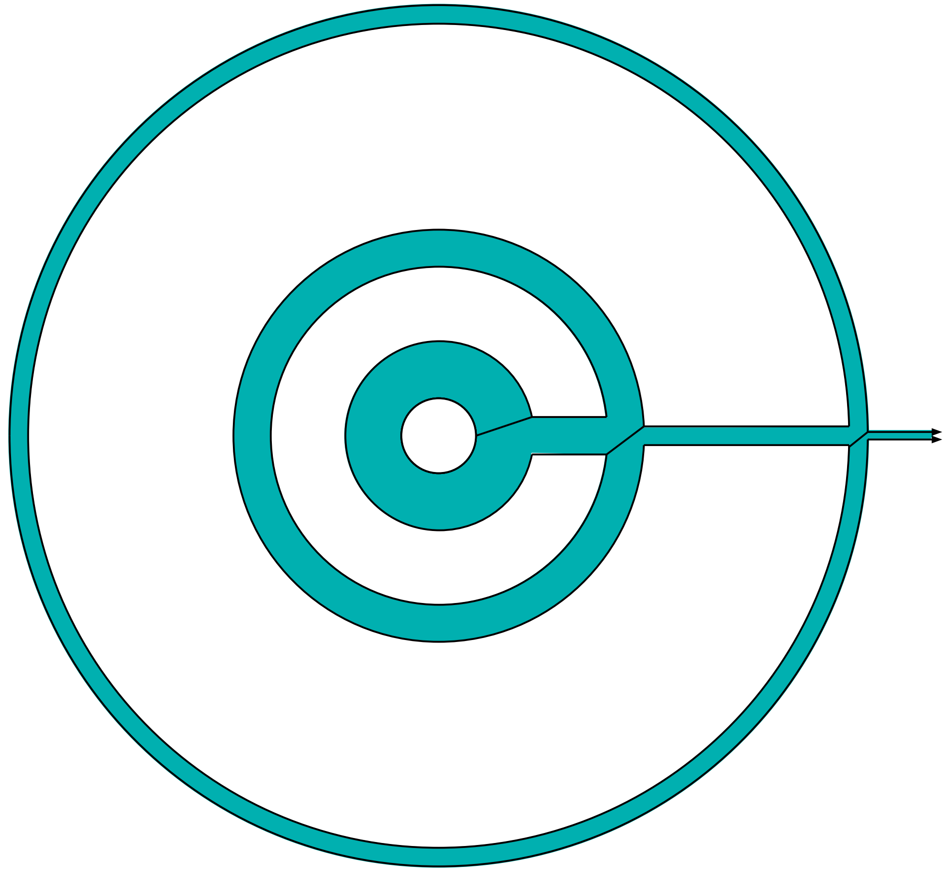
Adam Epstein observed this holds in many cases where order is determined by "combinatorial data". He and Rempe-Gillen gave EL counterexample.



Speiser class counterexample with 3 singular values.



Same tree in logarithmic coordinates.



Speiser function that is "statistically zero": for all $\epsilon > 0$,

$$\text{area}(\{|f| > \epsilon\}) < \infty.$$

Counterexample to Eremenko-Lyubich area conjecture.

**Wiman's conjecture:**

Define $\quad m(r) = \min_{|z|=r} |f(z)|, \qquad M(r) = \max_{|z|=r} |f(z)|.$

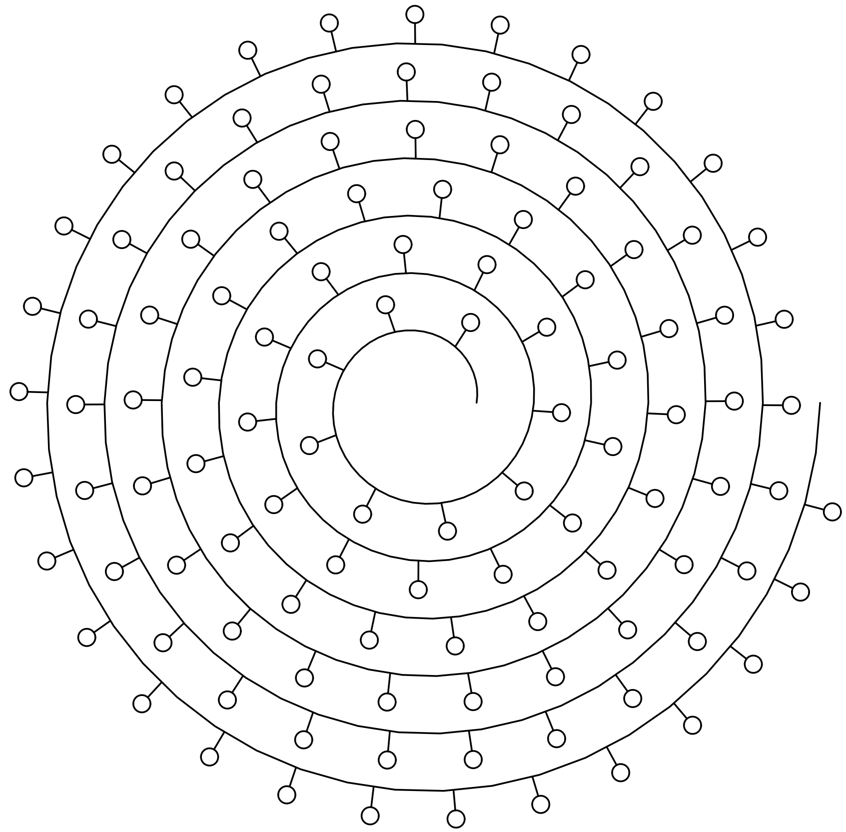
Wiman (1916) conjectured that for all entire functions

$$\limsup_{r \nearrow \infty} \frac{\log m(r,f)}{\log M(r,f)} \geq -1.$$

Sharp for $f(z) = \exp(z)$.

True in special cases, e.g., $|f(r)| = m(r)$ (Beurling, 1949).

False in general (Hayman, 1952).



Speiser class counterexample to Wiman's conjecture

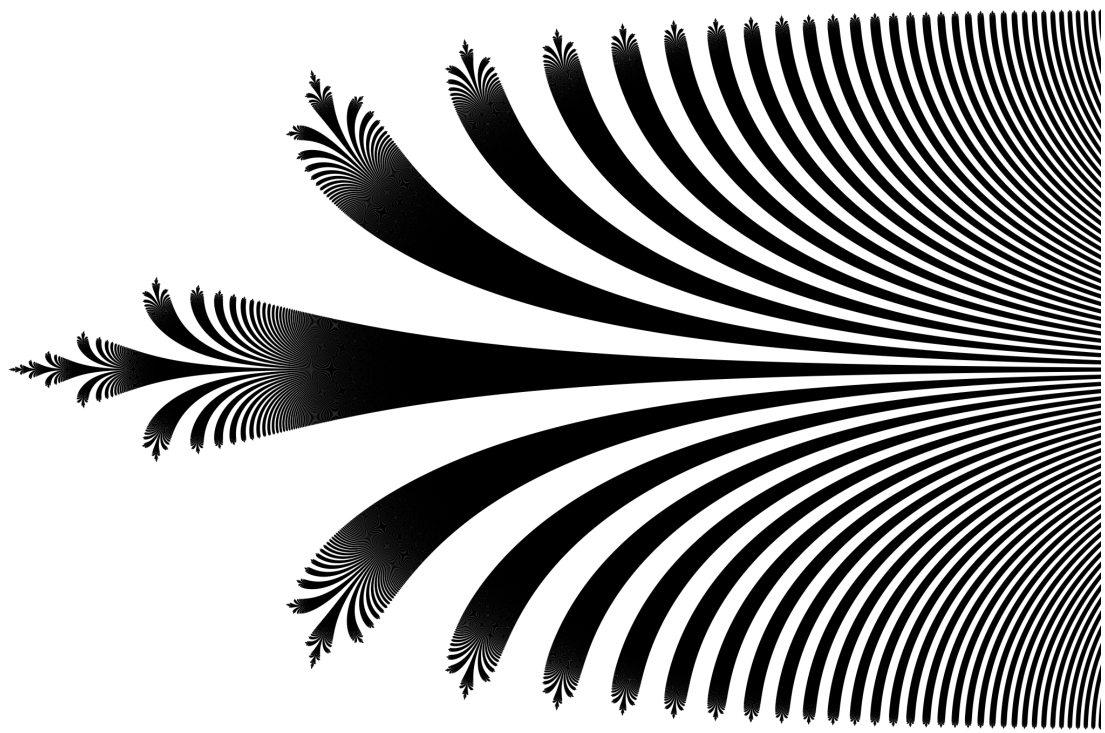
$$\frac{\log m(r,f)}{\log M(r,f)} \leq -C \cdot \log\log\log M(r,f)$$

Given an entire function $f$,

**Fatou set** $= \mathcal{F}(f) =$ open set where iterates are normal family.

**Julia set** $= \mathcal{J}(f) =$ complement of Fatou set.

Julia set is usually fractal. What is its (Hausdorff) dimension?



$\mathcal{J}((e^z - 1)/2)$, courtesy of Arnaud Chéritat

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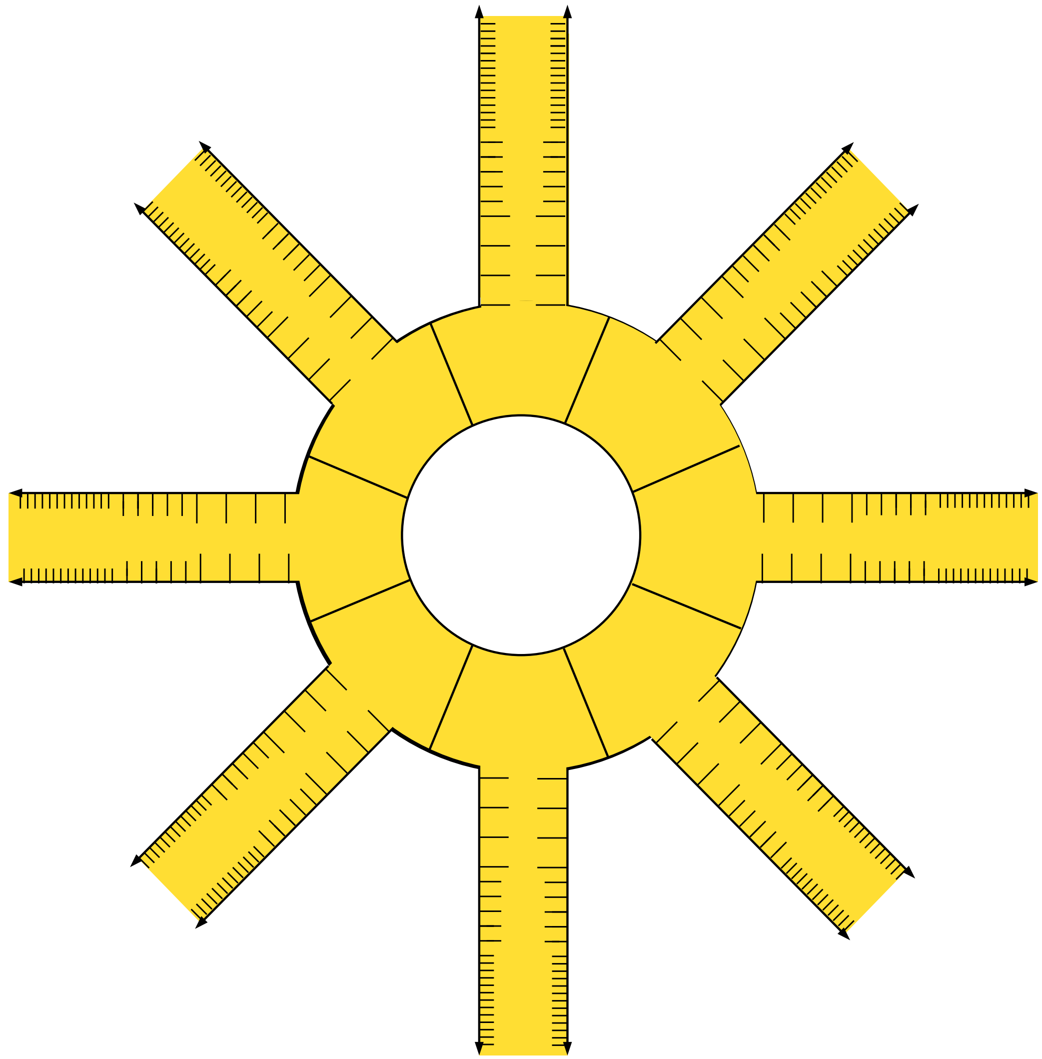
- Baker (1975): $f$ transcendental $\Rightarrow \mathcal{J}$ contains a continuum $\Rightarrow$ dim $\geq 1$.

- Misiurewicz (1981), McMullen (1987) dim =2 occurs (is common)

- Stallard (1997, 2000): $\{\dim(\mathcal{J}(f)) : f \in \mathcal{B}\} = (1, 2]$.

- B (2018) dim $= 1$ can occur (example outside $\mathcal{B}$).

Is dimension$< 2$ possible in Speiser class?



$$\inf\{\dim(\mathcal{J}(f)) : f \in \mathcal{S}\} = 1 \qquad \text{(B.-Albrecht, 2018).}$$

Given an entire function $f$,

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$f$ permutes components of its Fatou set.

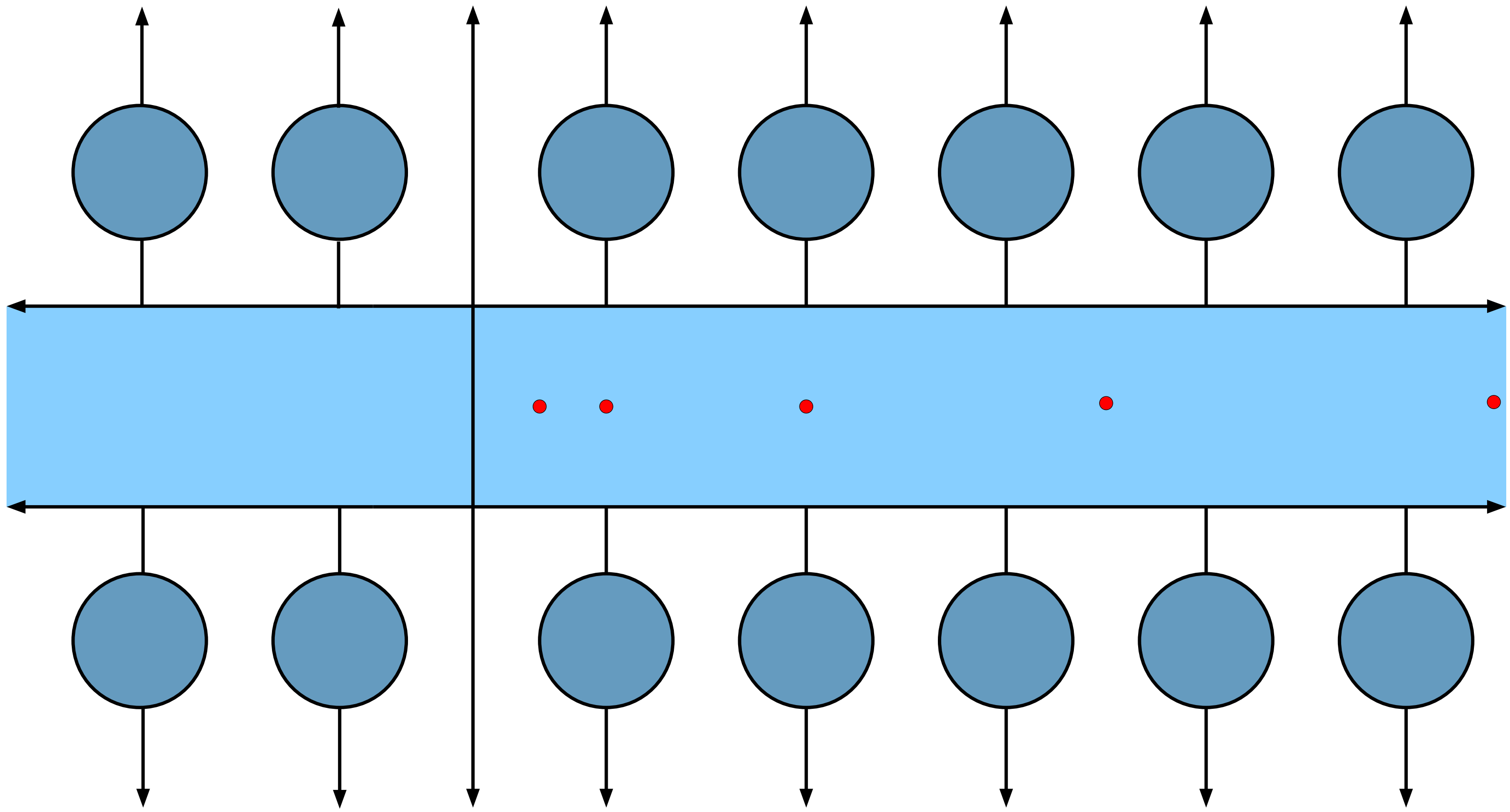
**Wandering domain =** Fatou component with infinite orbit.

- Entire functions can have wandering domains (Baker 1975).

- No wandering domains for rational functions (Sullivan 1985).

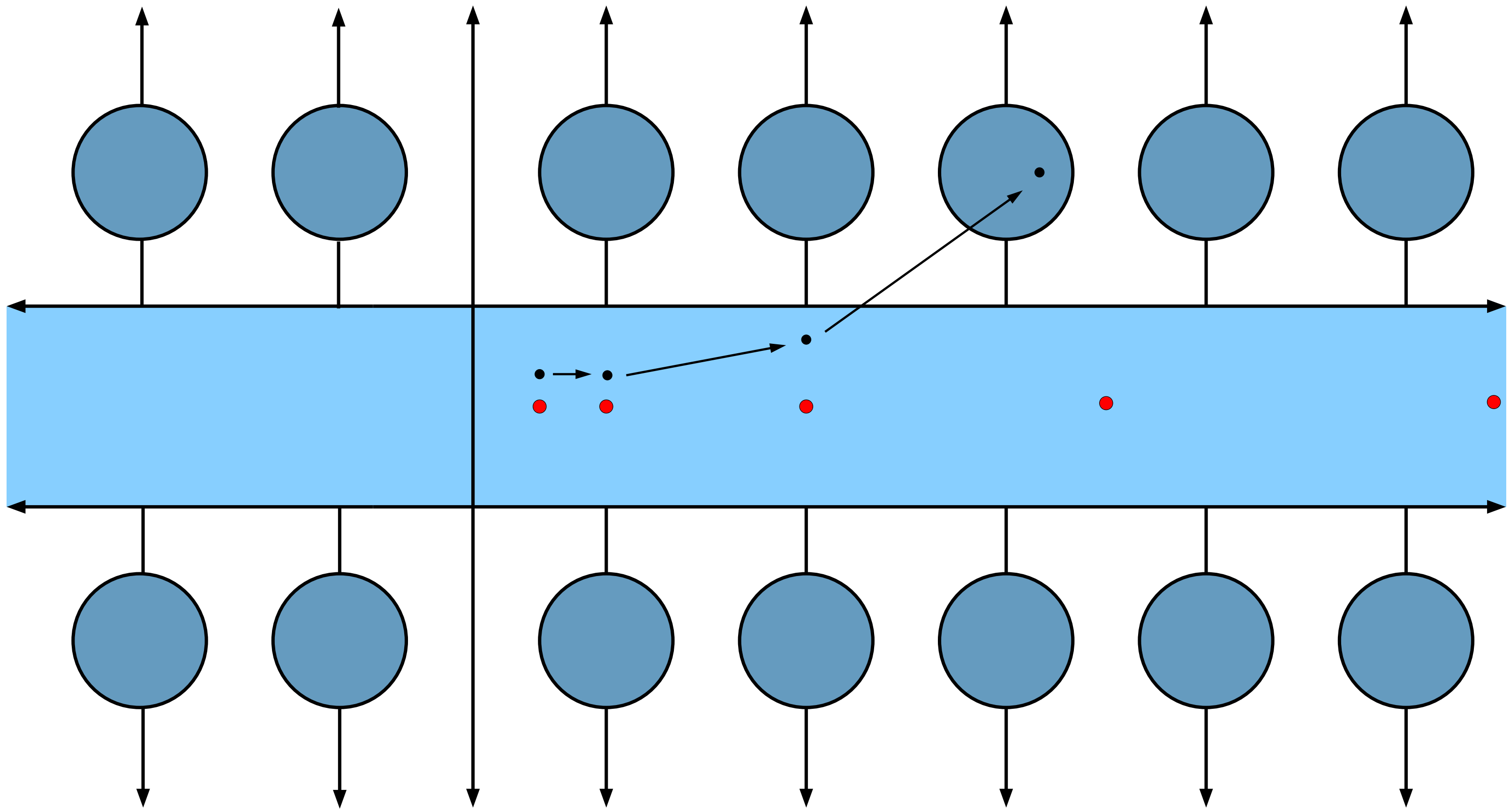
- Also none in Speiser class (Eremenko-Lyubich, Goldberg-Keen).

Are there wandering domains in Eremenko-Lyubich class?



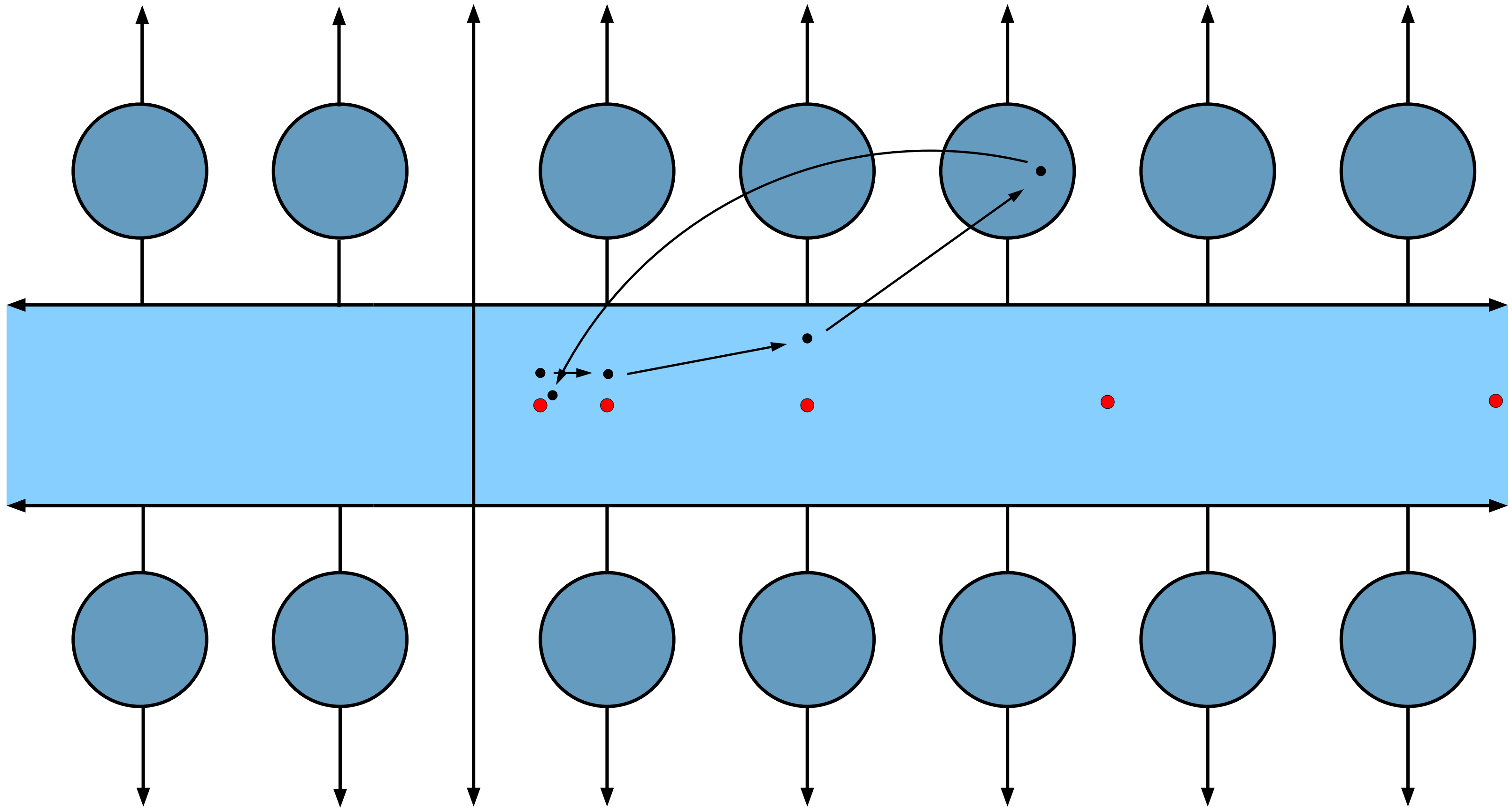
Graph giving wandering domain in Eremenko-Lyubich class.

Original proof corrected by Marti-Pete and Shishikura.



Graph giving wandering domain in Eremenko-Lyubich class.

Variations by Lazebnik, Fagella-Godillon-Jarque, Osborne-Sixsmith.



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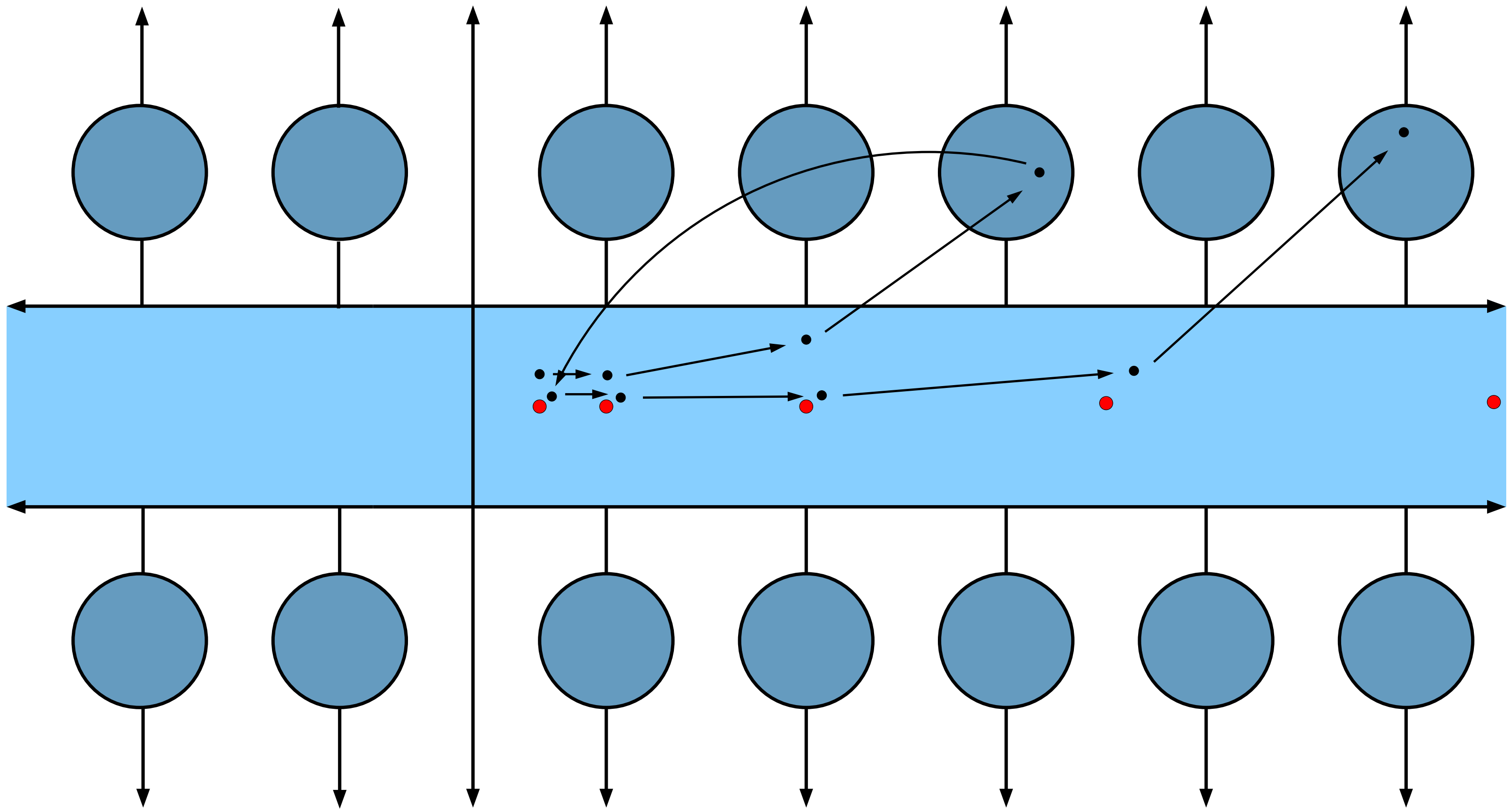


Graph giving wandering domain in Eremenko-Lyubich class.

Variations by Lazebnik, Fagella-Godillon-Jarque, Osborne-Sixsmith.

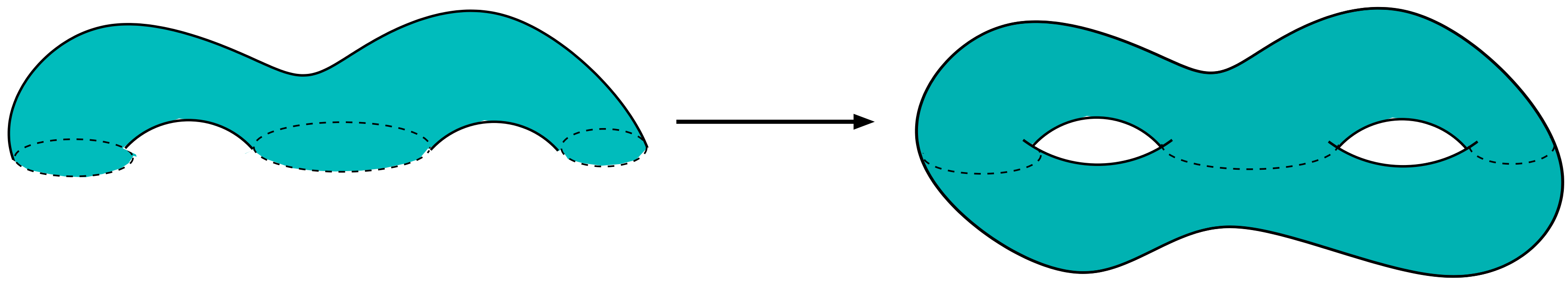
## Folding a pair of pants

For Riemann surfaces, A. Epstein defines a non-constant $f : Y \to X$ to be **finite type** if set of singular values is finite. If $Y \subset X$, $f$ can be iterated until an orbit leaves $Y$.

- Rational maps $S^2 \to S^2$,
- Speiser class $\mathbb{C} \to \mathbb{C}$,
- Covering map $\mathbb{D} \to X$.

Are there examples where domain is not simply connected?

Folding creates non-trivial examples, where $Y \subset X$ is a "pair of pants".



Pair of pants = sphere minus three disks.

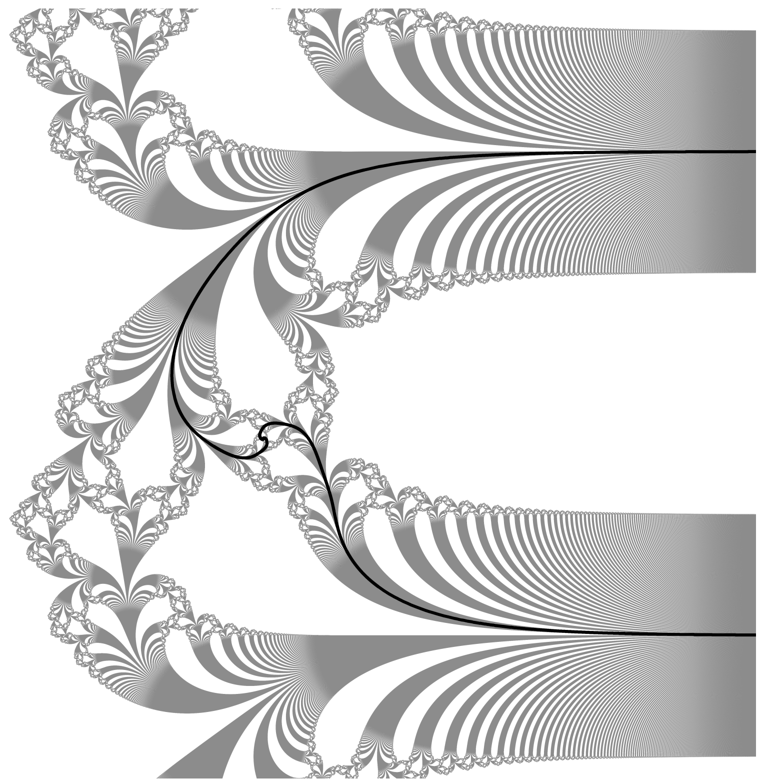
Every surface is union of these.

Does every compact surface have such self-maps? If not, which do?

For $f$ entire, **escaping set** is $I(f) = \{z : f^n(z) \to \infty\}$.

Known that $\mathcal{J}(f) = \partial I(f)$.

Fatou observed $I(f)$ often consists of curves to $\infty$.



Curve in escaping set, courtesy of Lasse Rempe-Gillen

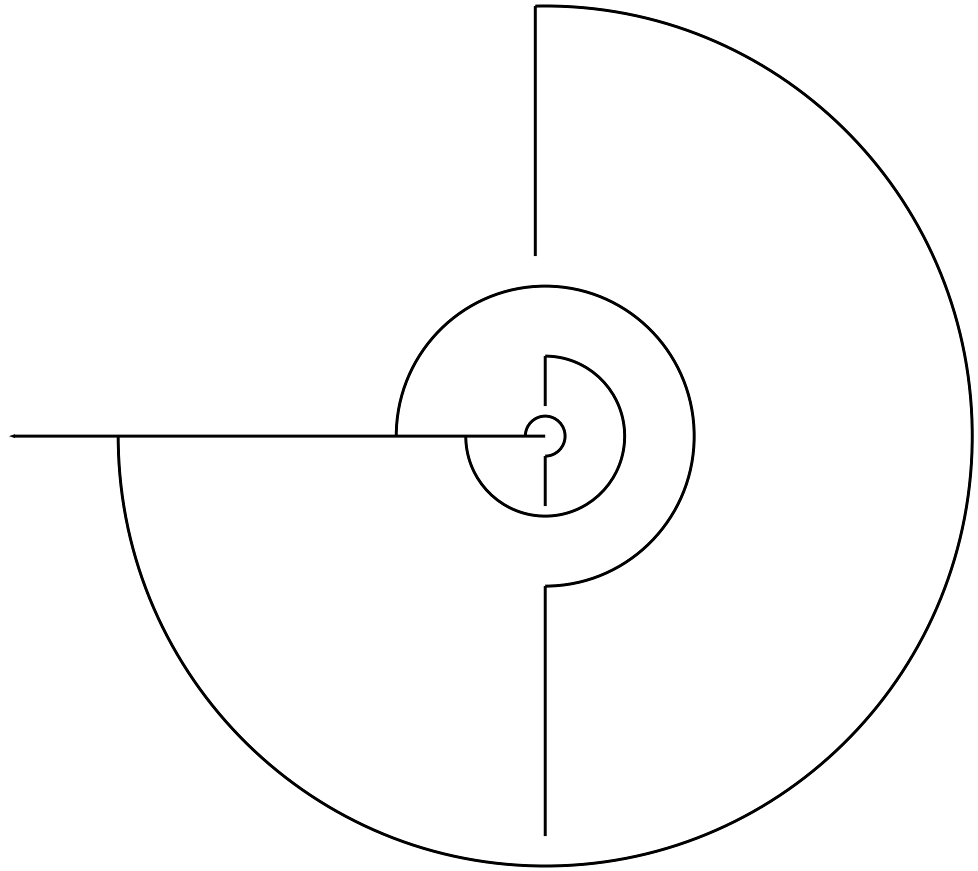
**Eremenko Conj:** components of $I(f)$ are unbounded (still open).

**Strong Eremenko Conj (SEC):** path components are unbounded.

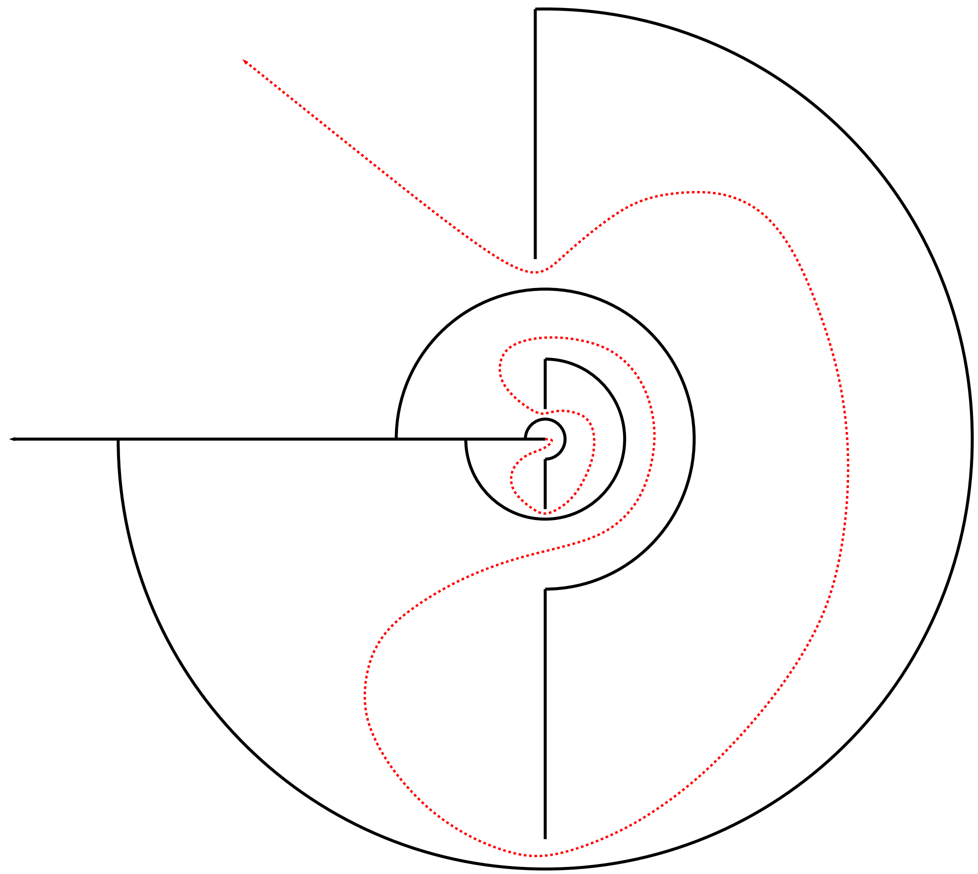
Rottenfusser, Rüchert, Rempe-Gillen and Schleicher (2011) proved:

- SEC true for EL functions with finite order of growth.
- SEC false for some EL functions with infinite order.
- Examples with trivial path components.

QC-Folding gives examples in Speiser class.



Speiser class counterexample to SEC.
Path components of escaping set can be points.
Other exotic examples by Rempe-Gillen.



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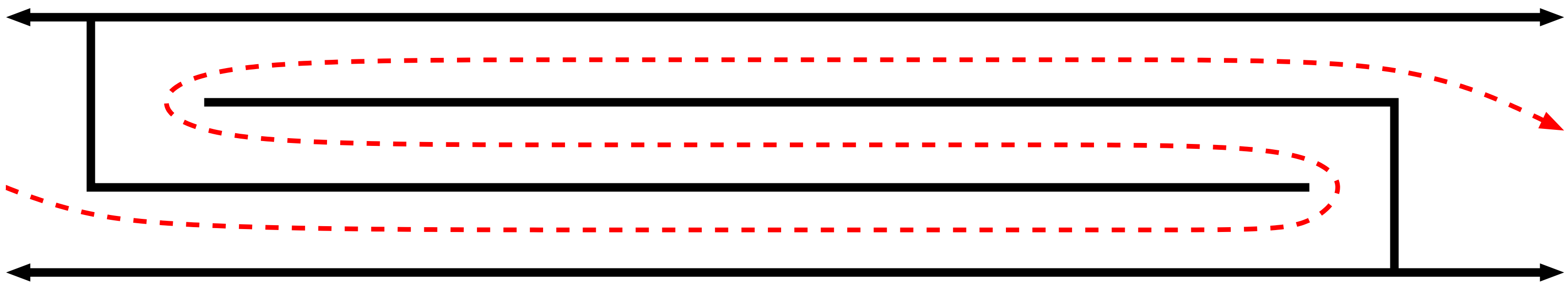
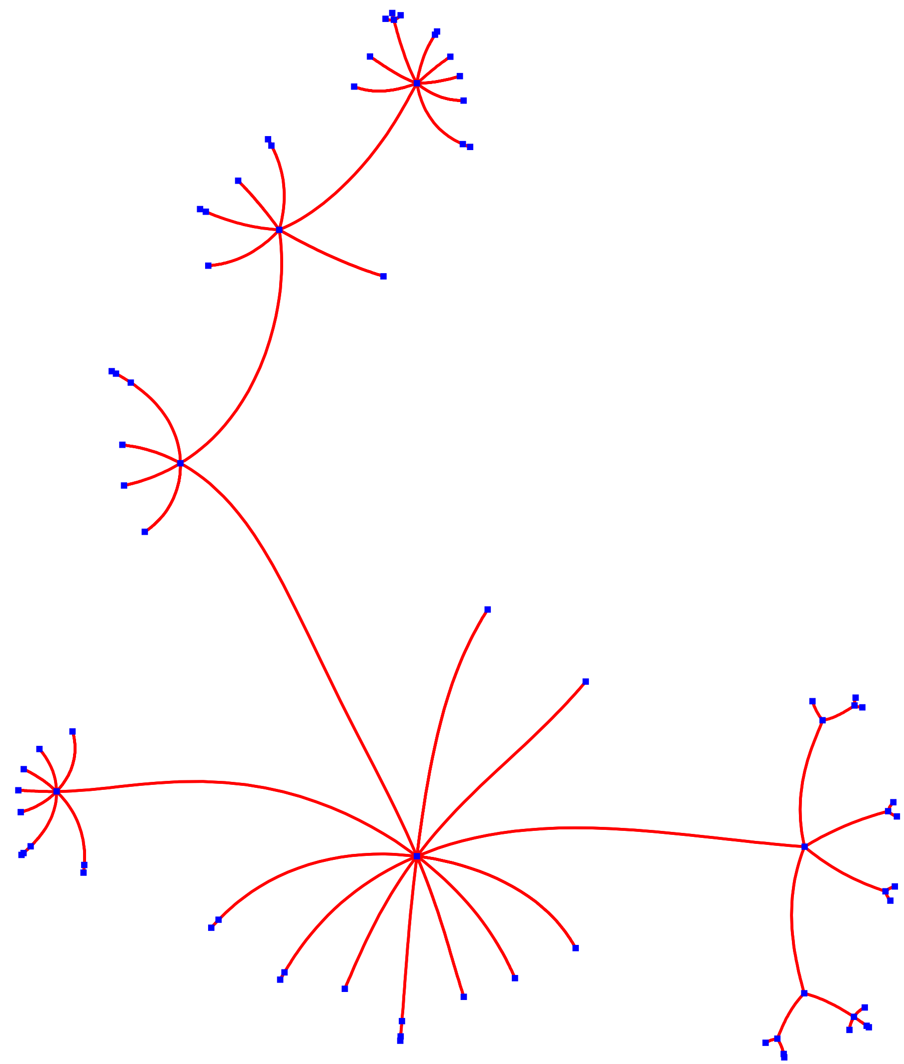
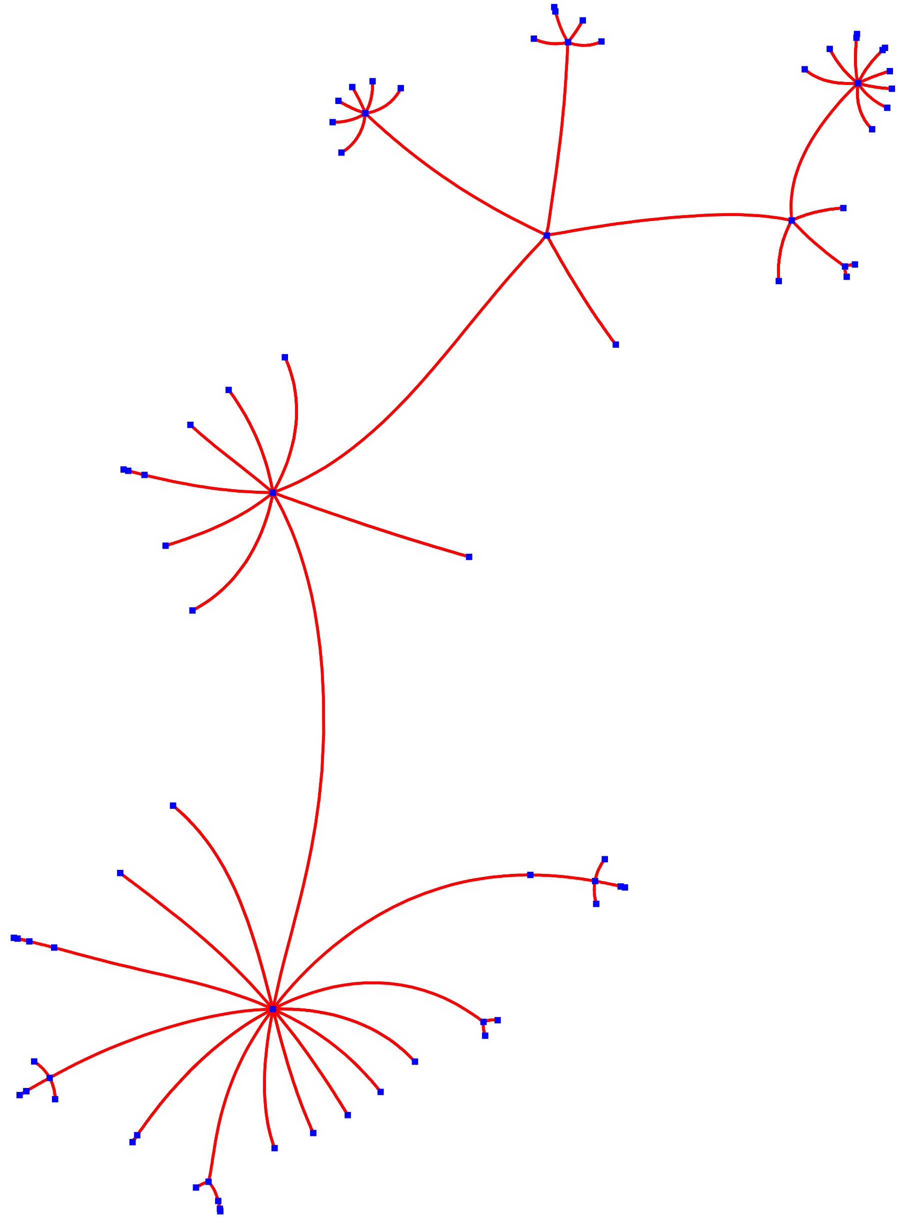
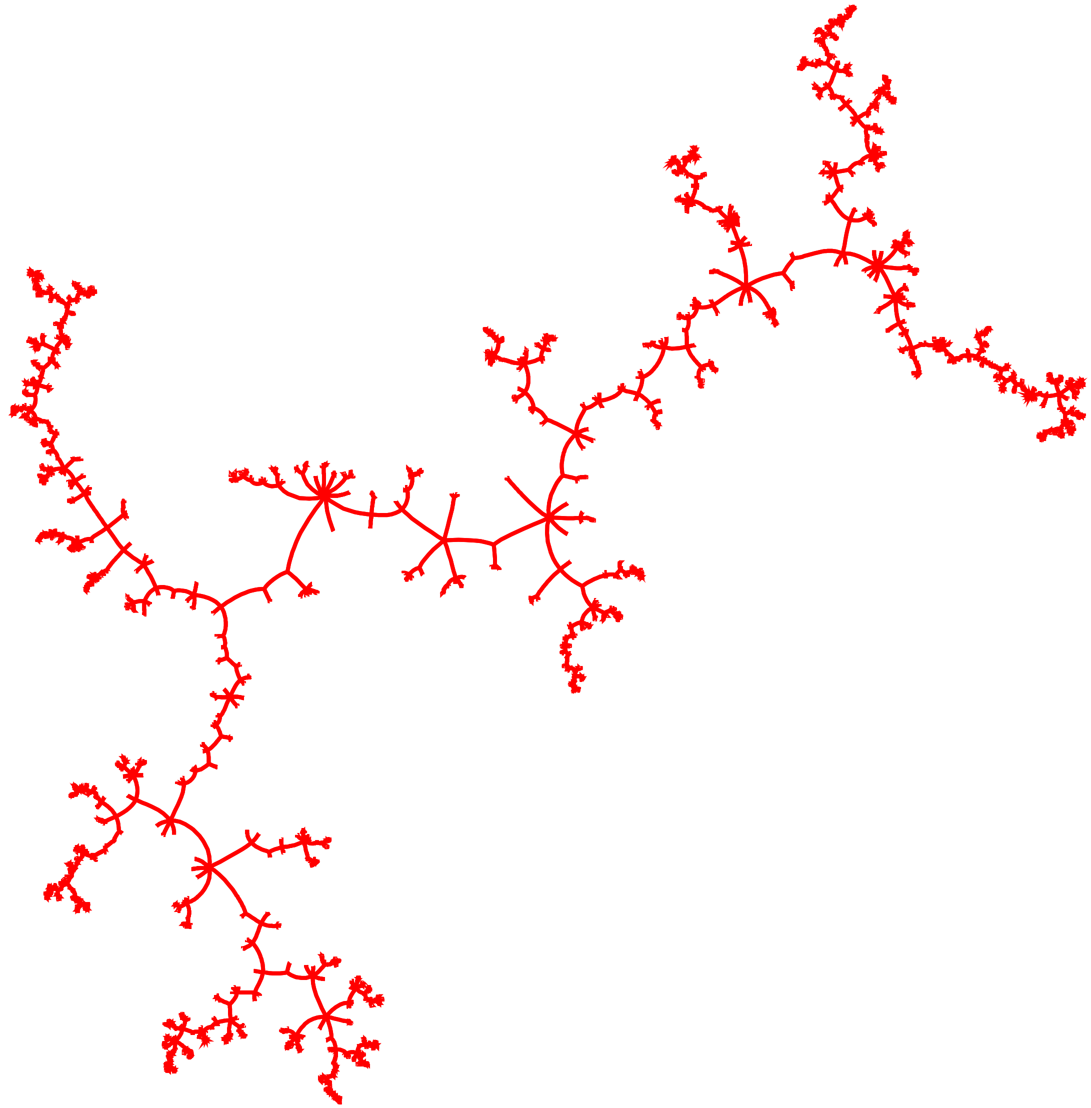
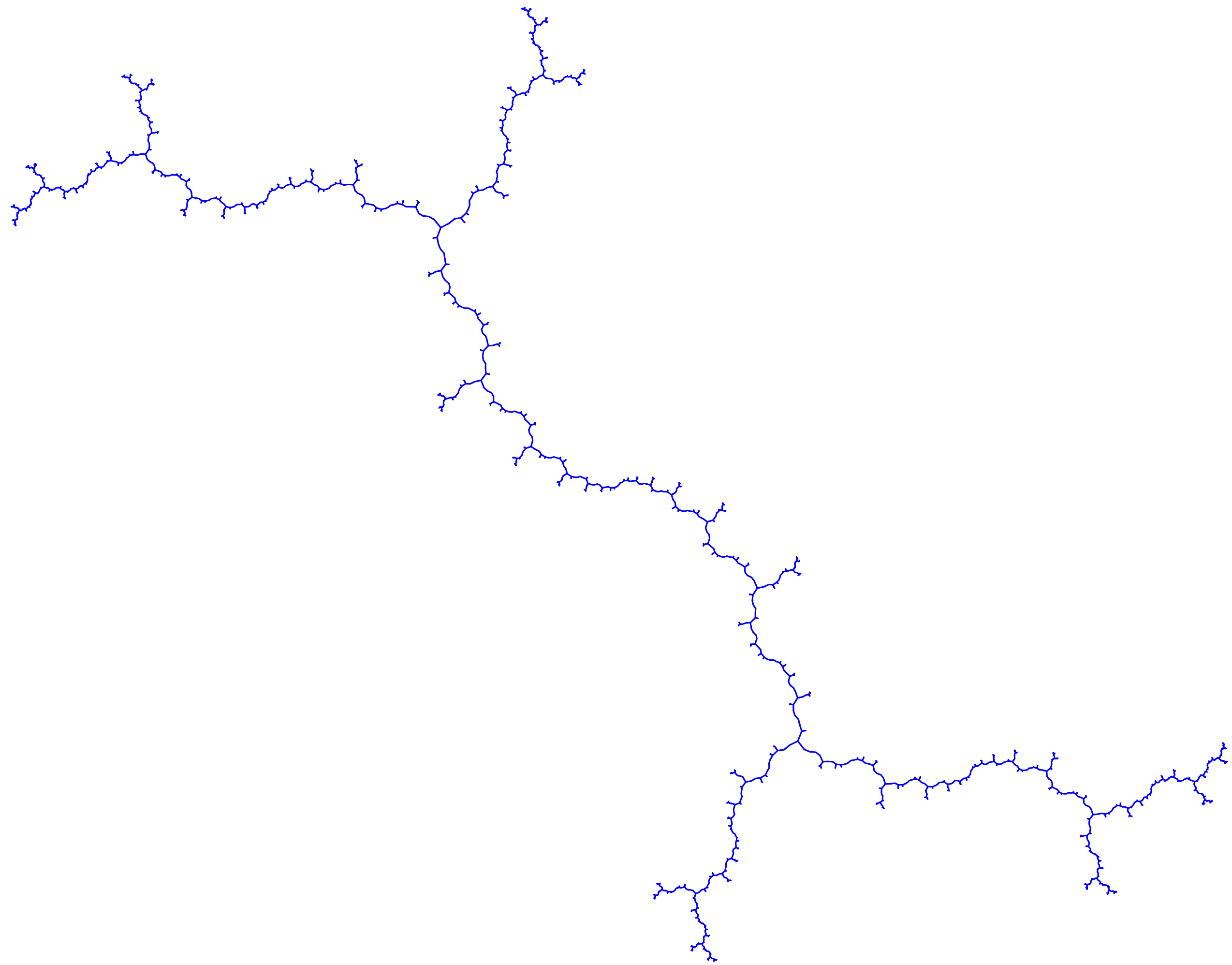


SEC counterexample in logarithmic coordinates

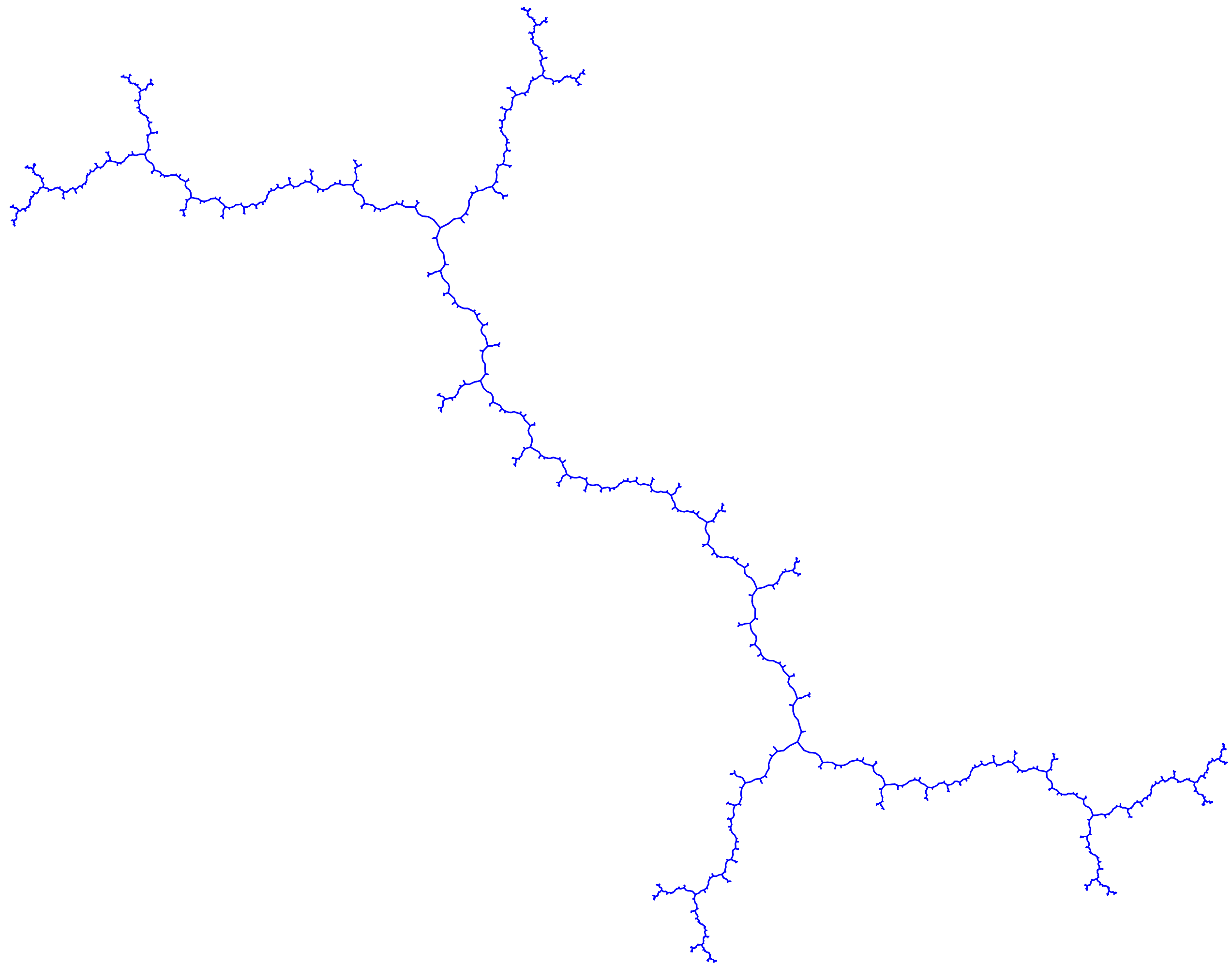




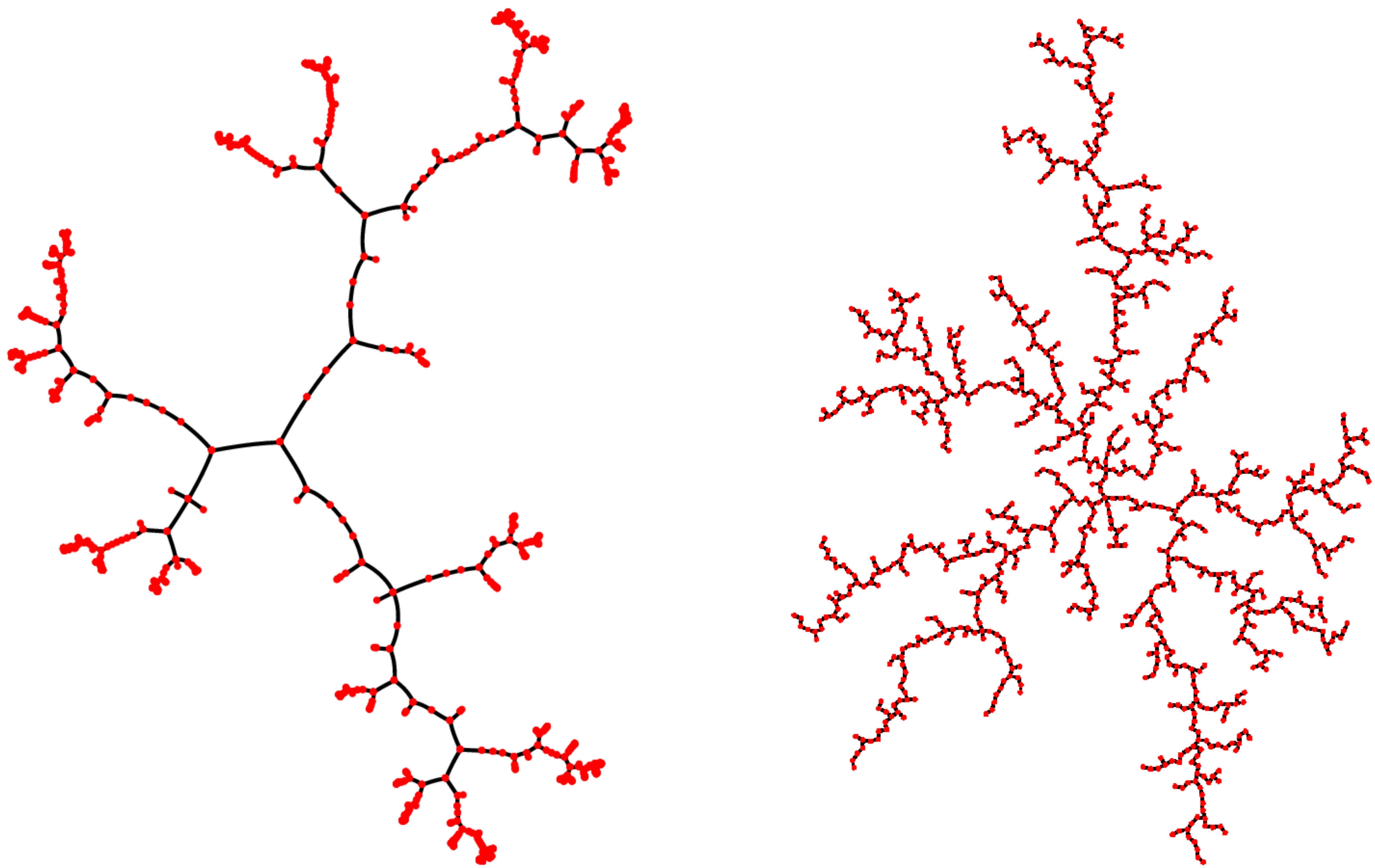




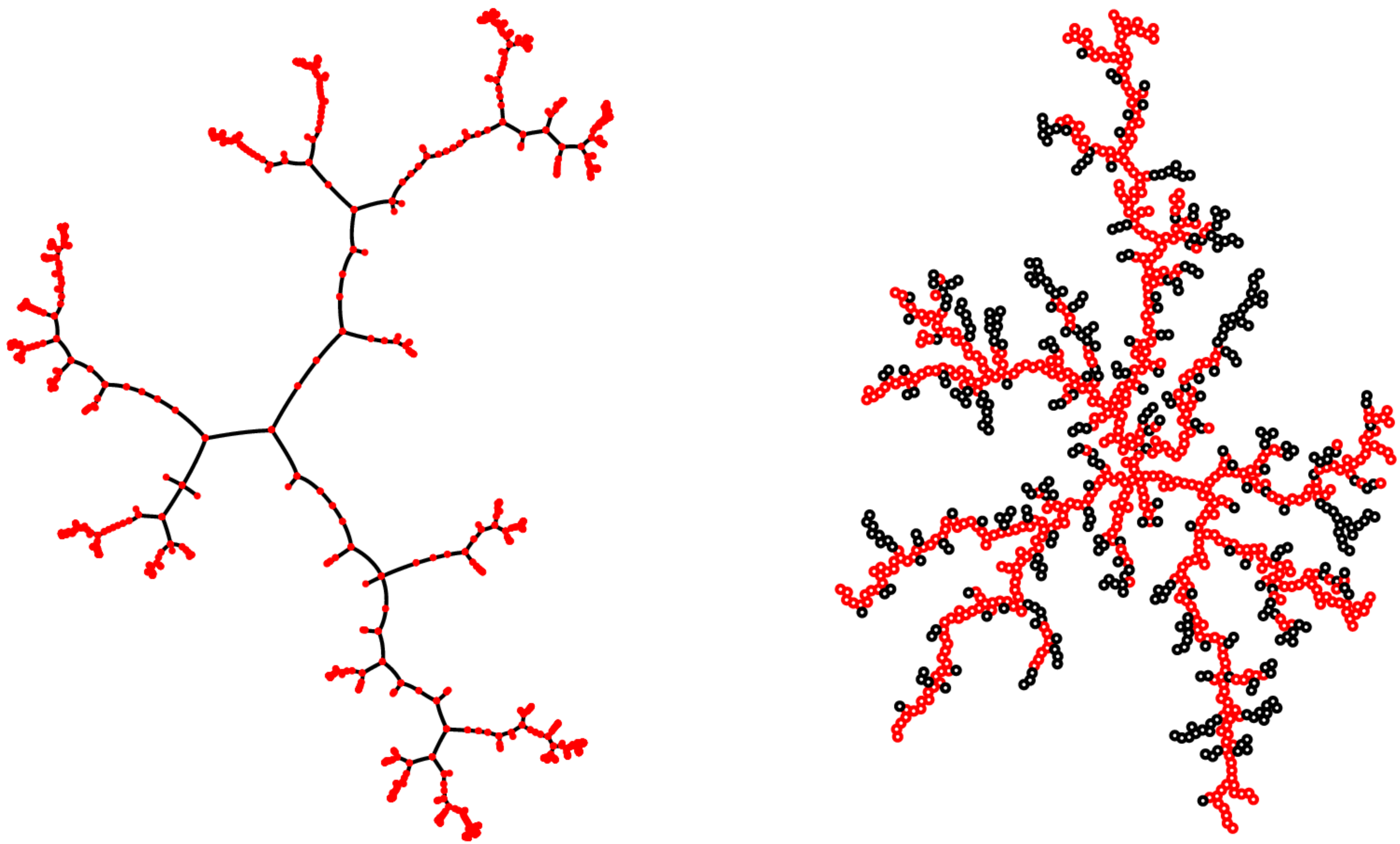
Ceci n'est pas un ensemble de Julia.



True tree based on combinatorics of Julia set of $z^2 + i$.
Example of "rigidity": combinatorics determines geometry.



“True DLA” (Diffusion Limited Aggregation)



“True DLA” (Diffusion Limited Aggregation)