MAPPINGS AND MESHES II: CONNECTIONS BETWEEN CONTINUOUS AND DISCRETE GEOMETRY

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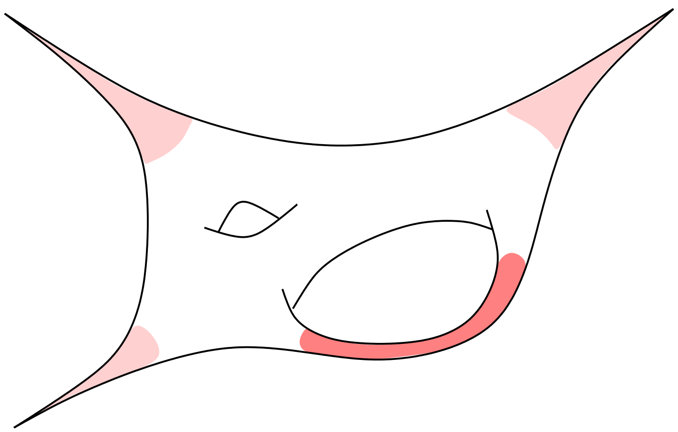
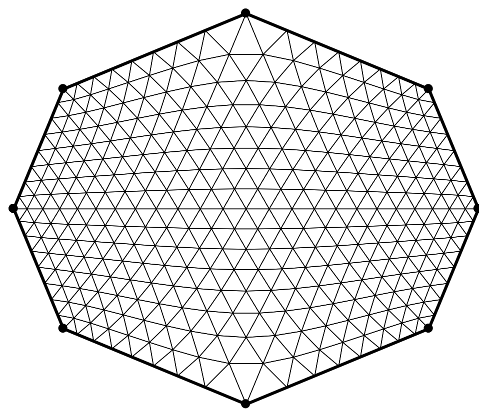
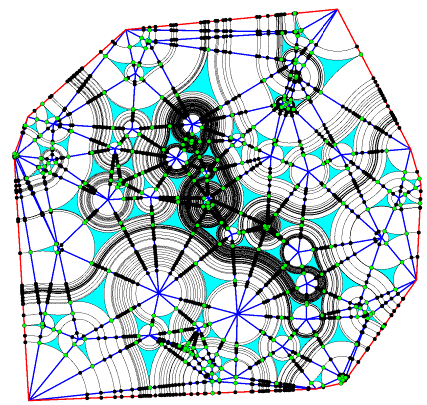
Geometric Methods for Analyzing Discrete Shapes
Center of Mathematical Sciences and Applications
Harvard University, May 7-9, 2021







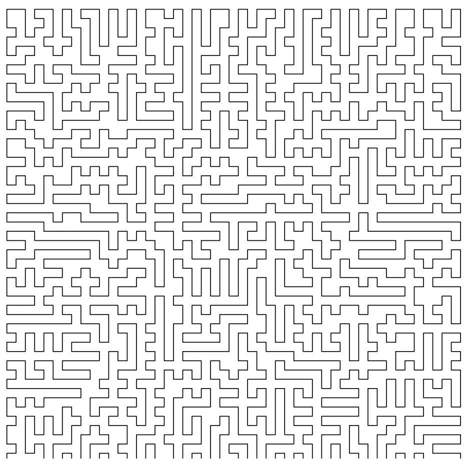
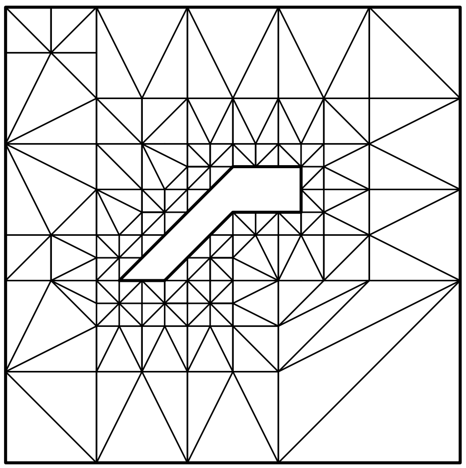
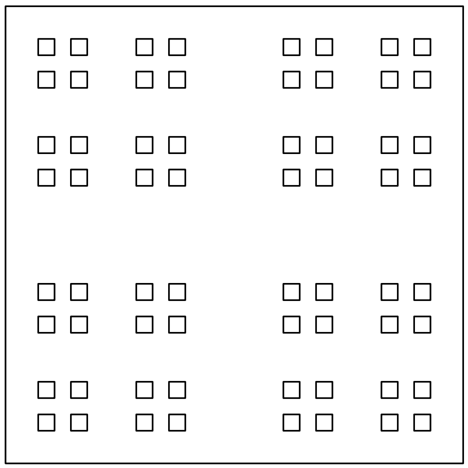
THE PLAN

Lecture 1: Medial axis approximates conformal maps
- Harmonic measure and conformal maps
- The Schwarz-Christoffel formula
- The medial axis
- Convex hulls in hyperbolic space

Lecture 2: Conformal maps give good meshes
- Optimal quad-meshes
- Optimal triangulations of polygons
- Non-obtuse triangulations of PSLGs

Some definitions:

A **planar straight line graph** Γ (or **PSLG**) is finite union of points V and a collection of disjoint edges E with endpoints among these points.

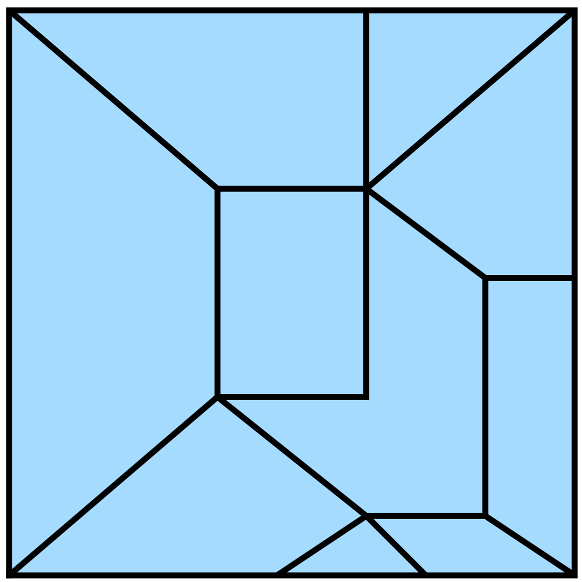
  

Generally let $n = |V|$ be the number of vertices.

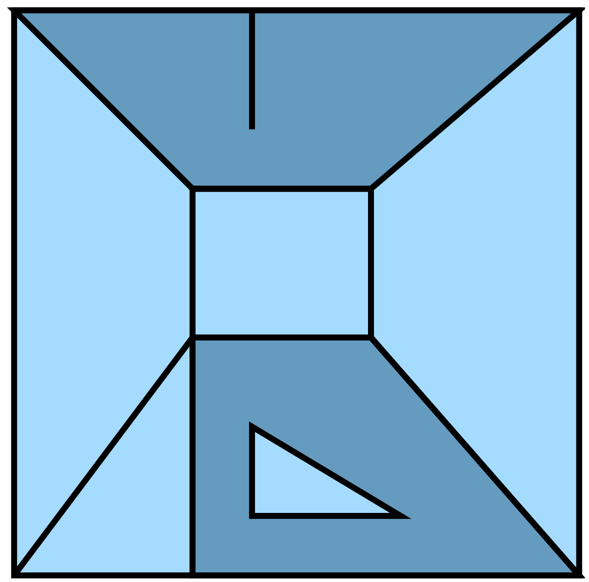
A simple polygon is a PSLG where edges form a closed cycle.

A **face** of a PSLG is a connected component of the complement.

A PSLG is a **mesh** if every face is a Jordan domain.



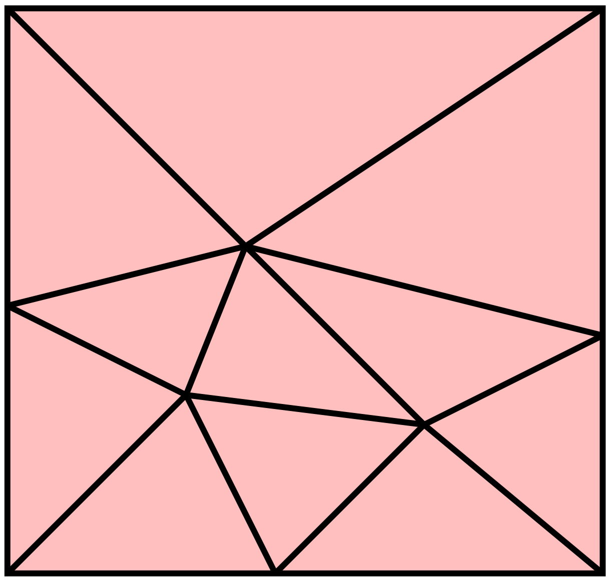
Mesh



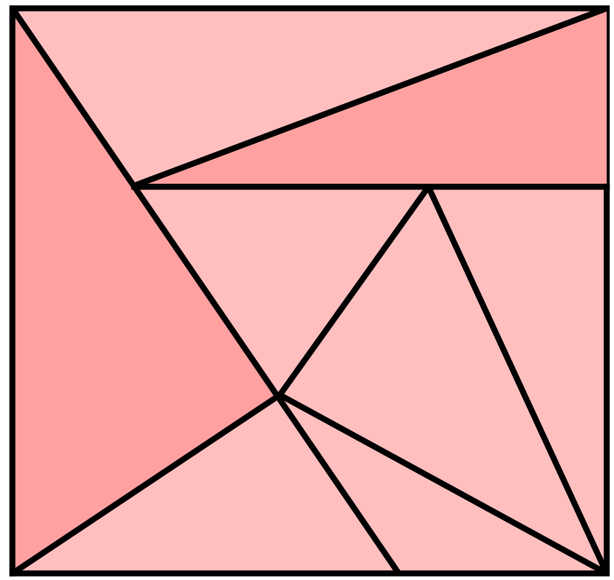
Not a mesh

A PSLG is a **triangulation** if every bounded face is a triangle, i.e., every face is bounded by three edges of the PSLG (= simplicial complex).

A PSLG is a **triangulation dissection** if every bounded face has a triangular shape, but may bound more than 3 edges of the PSLG.

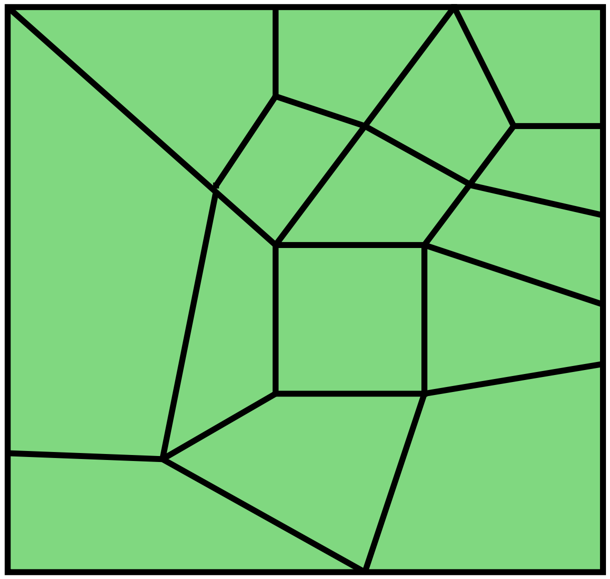


Triangulation

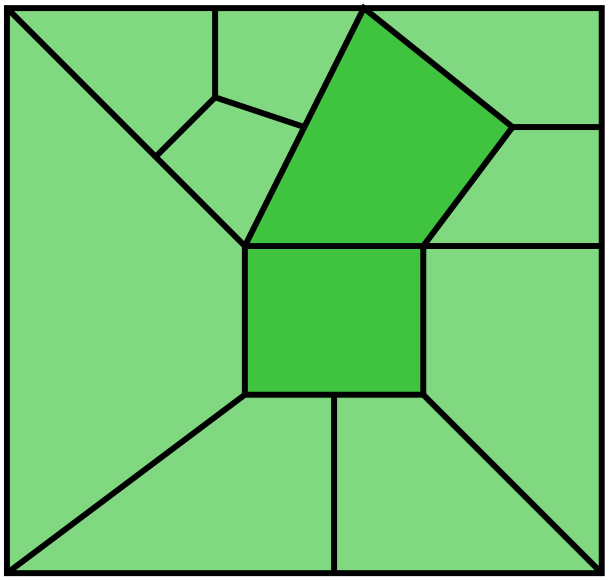


Triangular dissection

Similar definitions for a quadrilateral mesh and qudrilateral dissecttion.



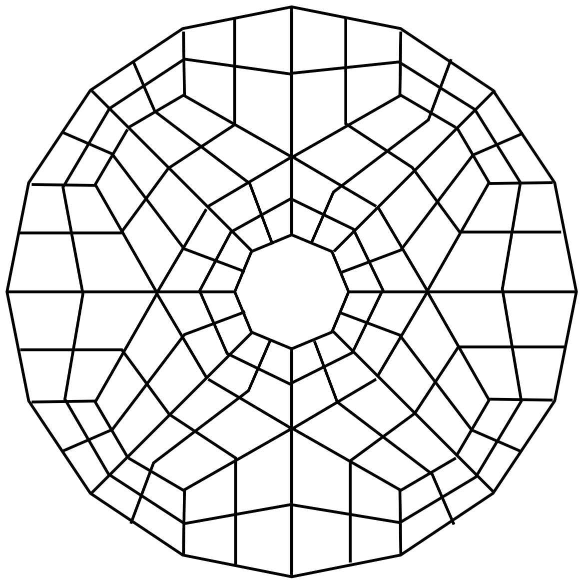
Quad-Mesh

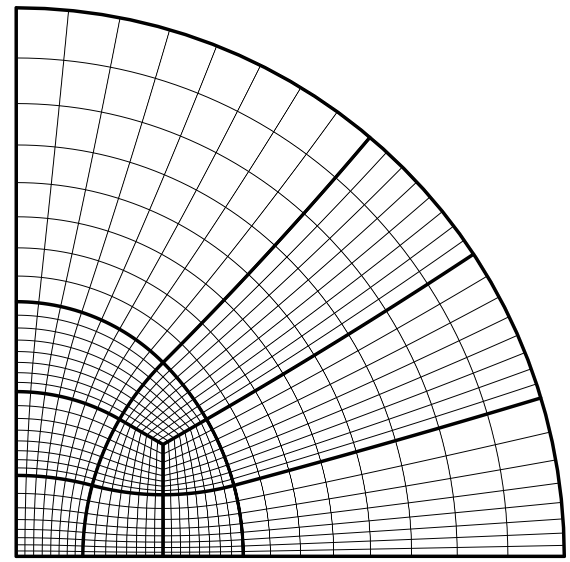


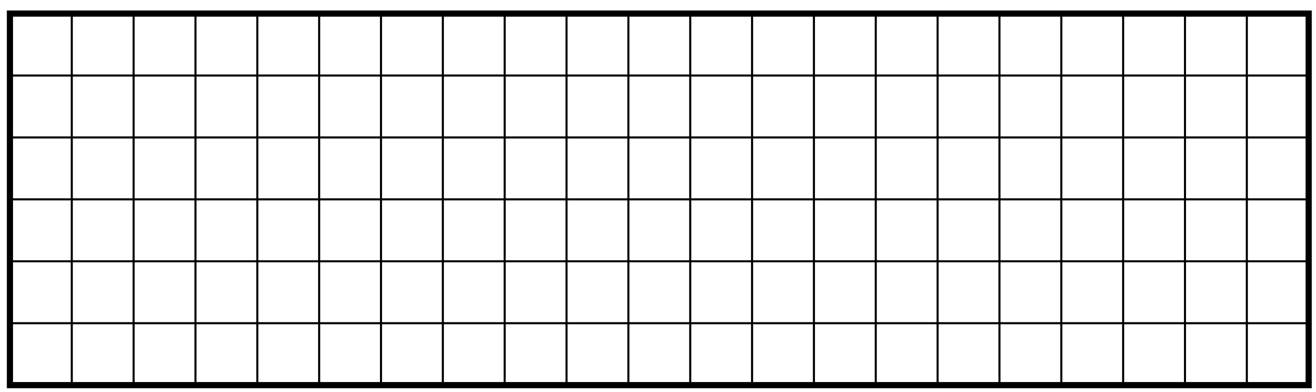
Quad-Dissection

Quad-meshing is "easier" than triangulation.

PART I: OPTIMAL QUAD-MESHES

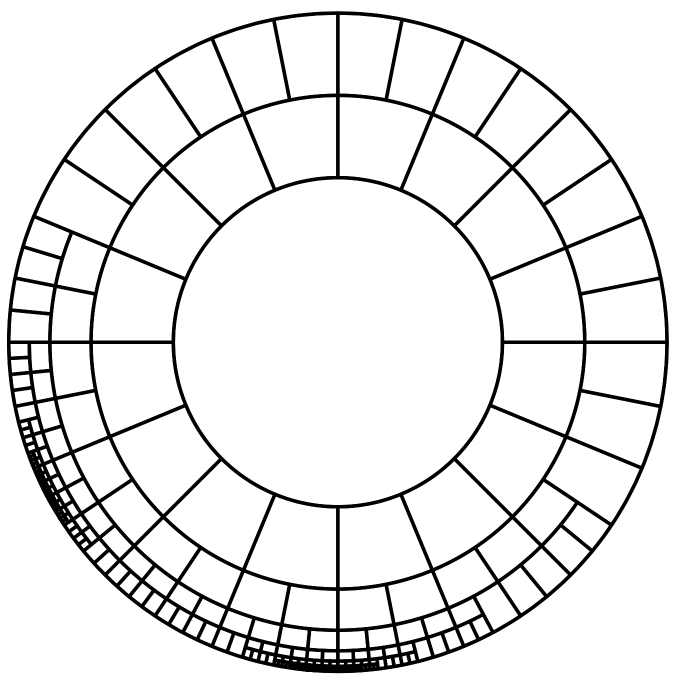


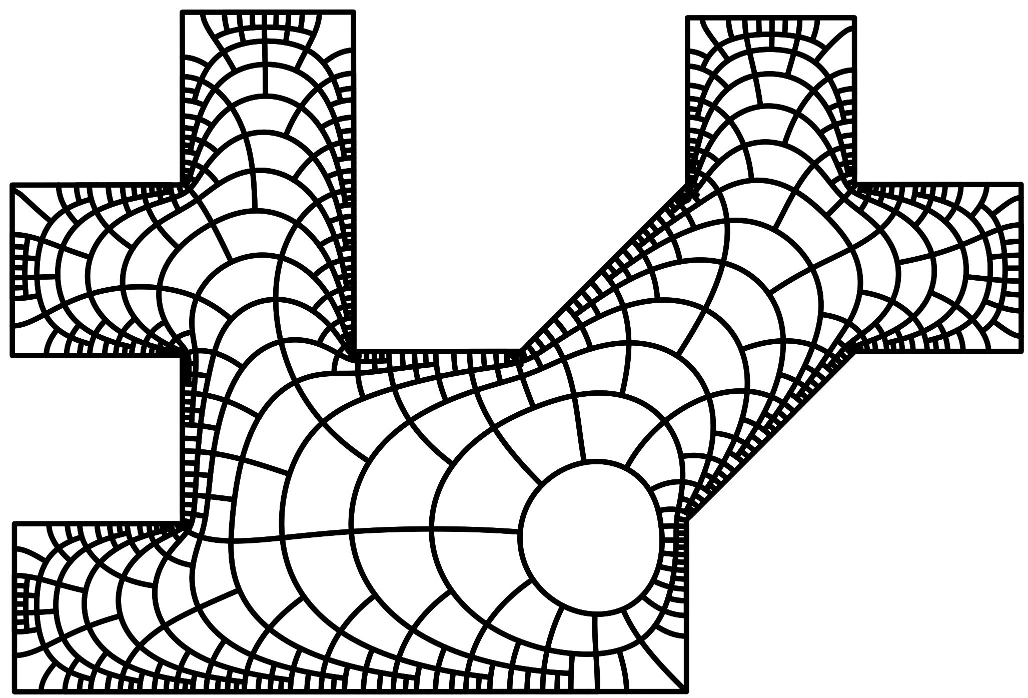




Theorem: Every n-gon has $O(n)$ quad-mesh with all angles $\leq 120°$ and new angles $\geq 60°$. $O(n)$ work.

Original angles $< 60°$ remain unchanged. $60°$ is sharp.





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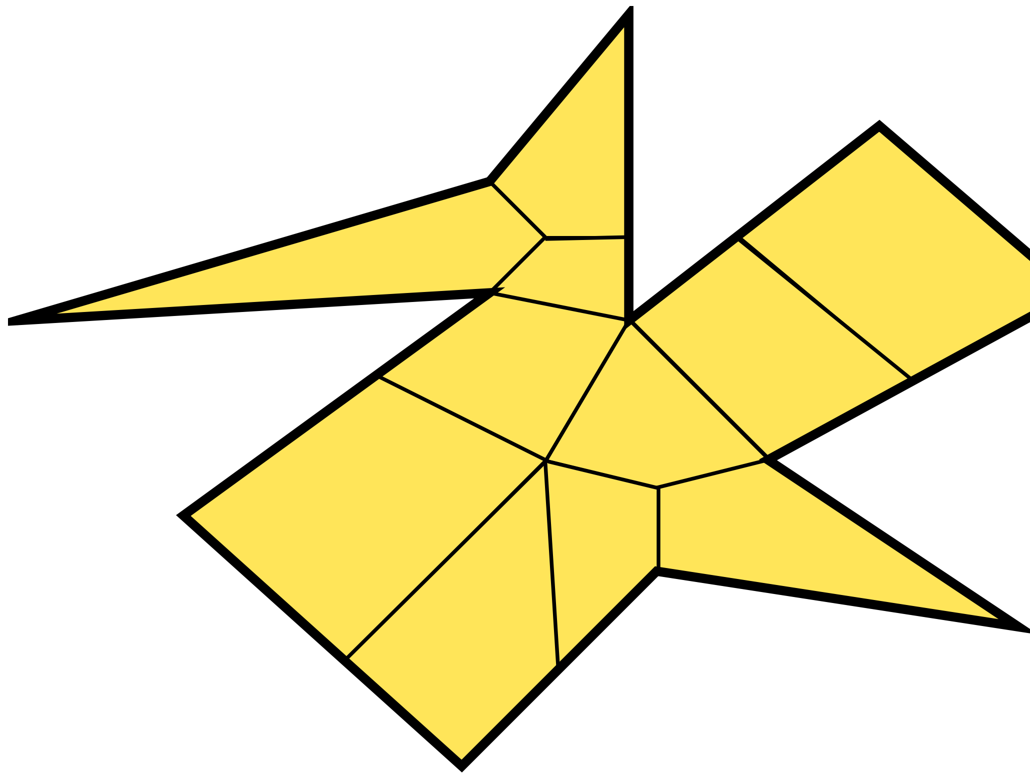

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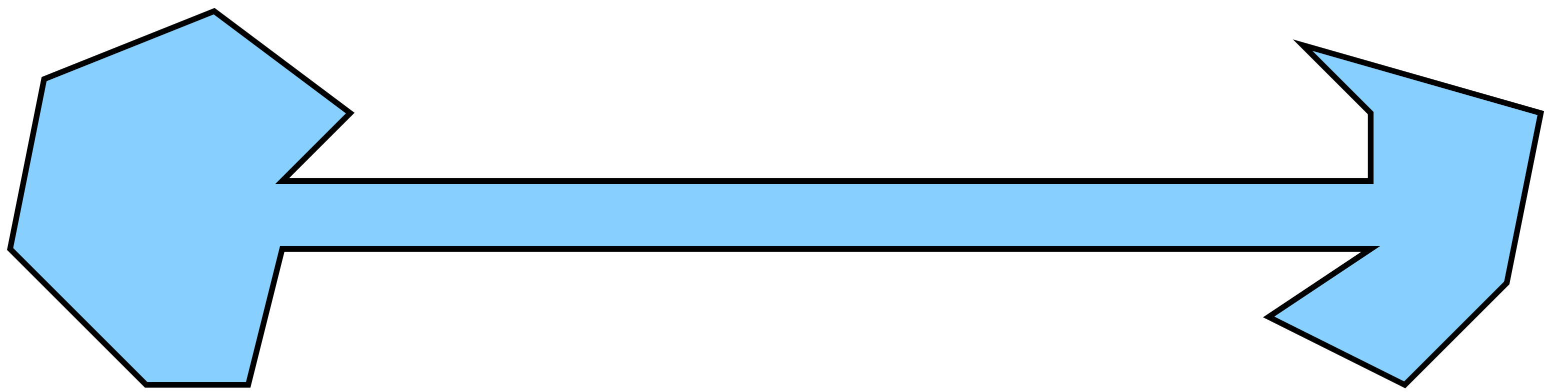
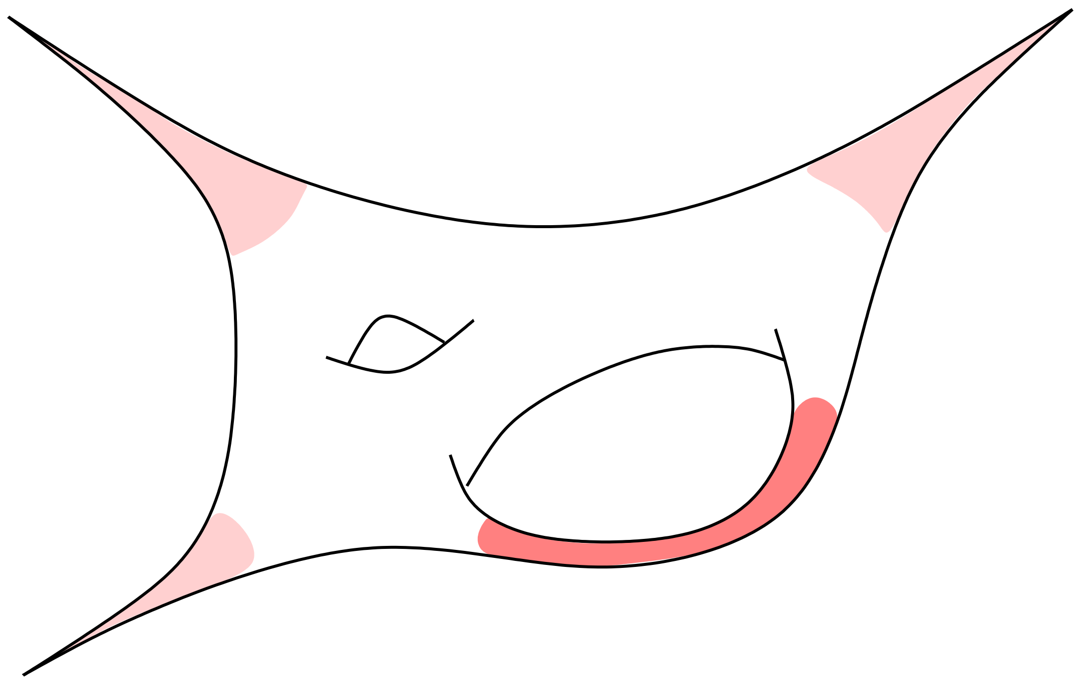


Long, narrow channels require long, narrow quadrilaterals.

Must find all such channels in $O(n)$ time.

Use idea from hyperbolic manifolds: **thick/thin decompositions**.

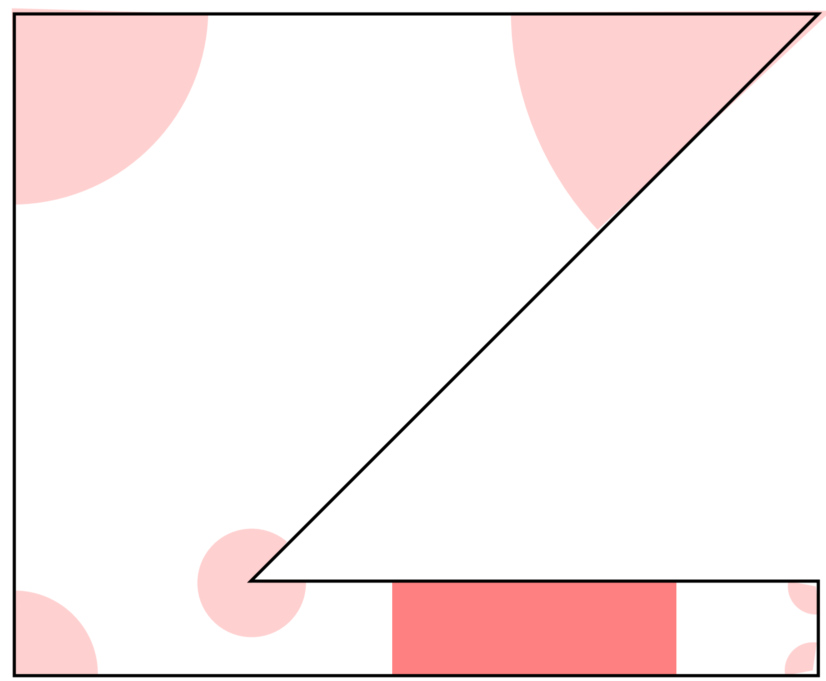
Surface **thin part** is union of short non-trivial loops.



parabolic = puncture, hyperbolic = handle

Thick and Thin parts of a polygon

Thin parts: associated to certain pairs of edges.

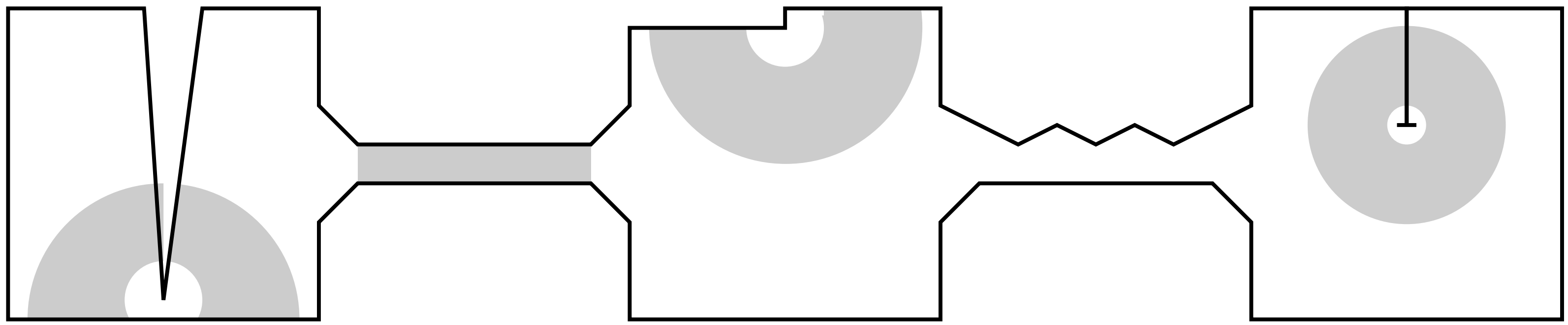


Parabolic = adjacent edges, Hyperbolic = non-adjacent edges

Rough idea: sides I, J so $\mathrm{dist}(I, J) \ll \min(|I|, |J|)$.

Thick parts = remaining components (white)

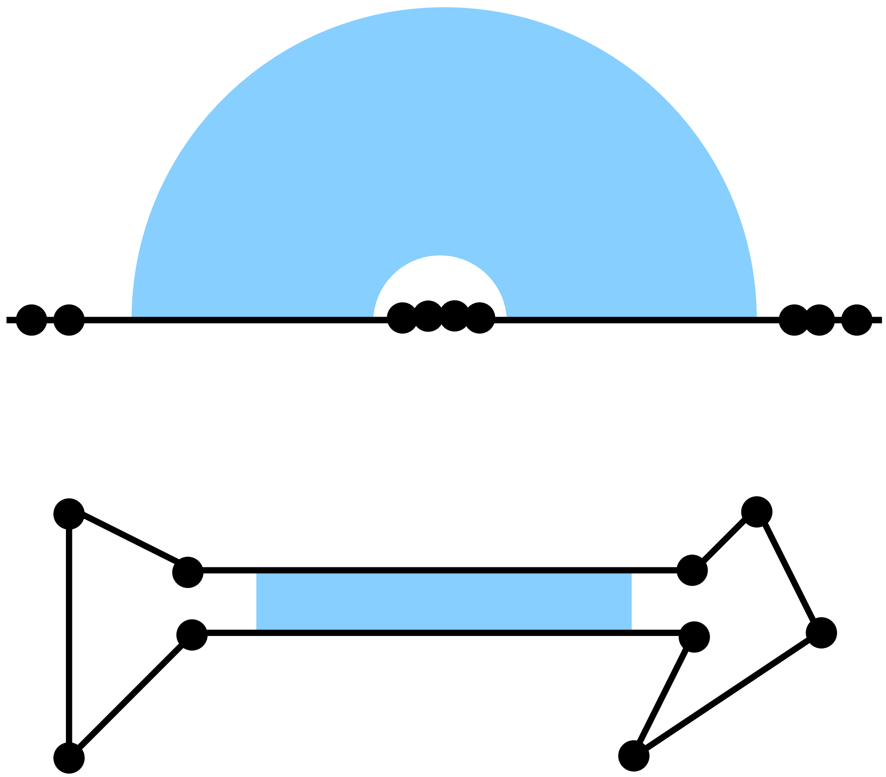
More examples of hyperbolic thin parts.



Inside thick regions (white) conformal pre-vertices are well separated on circle (no clusters).

Thick regions have good estimates for conformal map.

Thick/thin parts can be computed in linear time by computing conformal preimages (= Schwarz-Christoffel parameters).

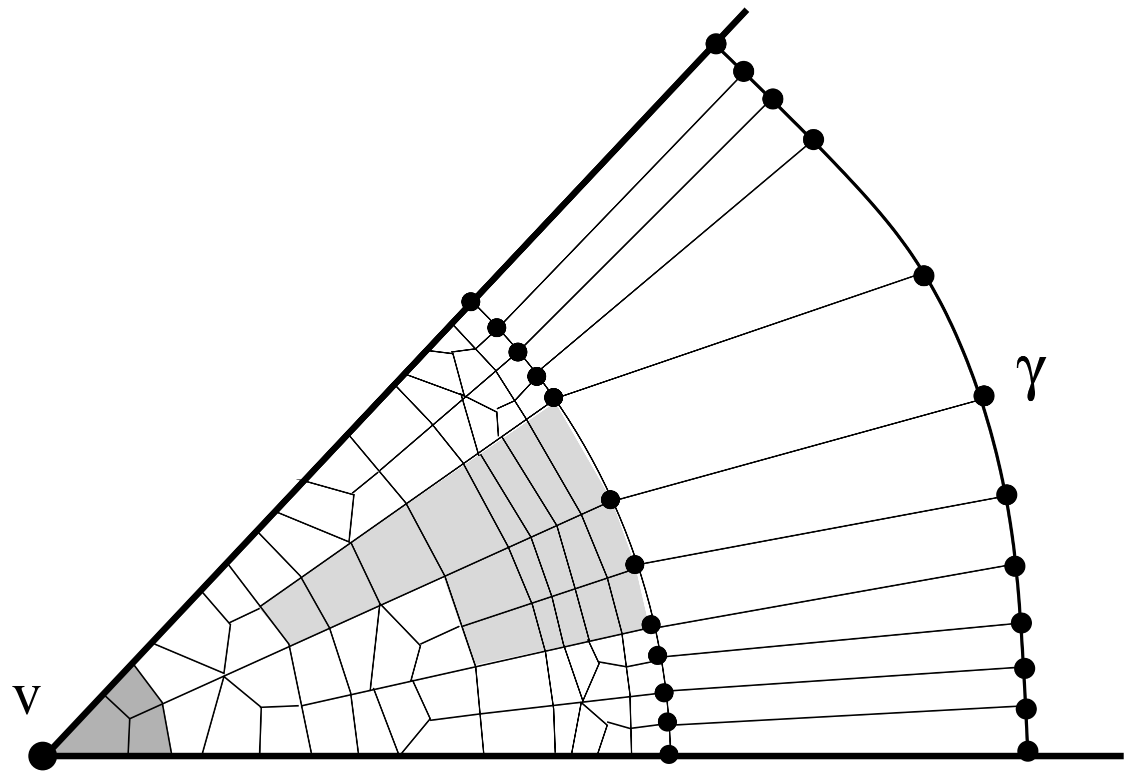


Each thin part creates two widely-separated clusters of preimages on circle. Can found quickly using medial axis.

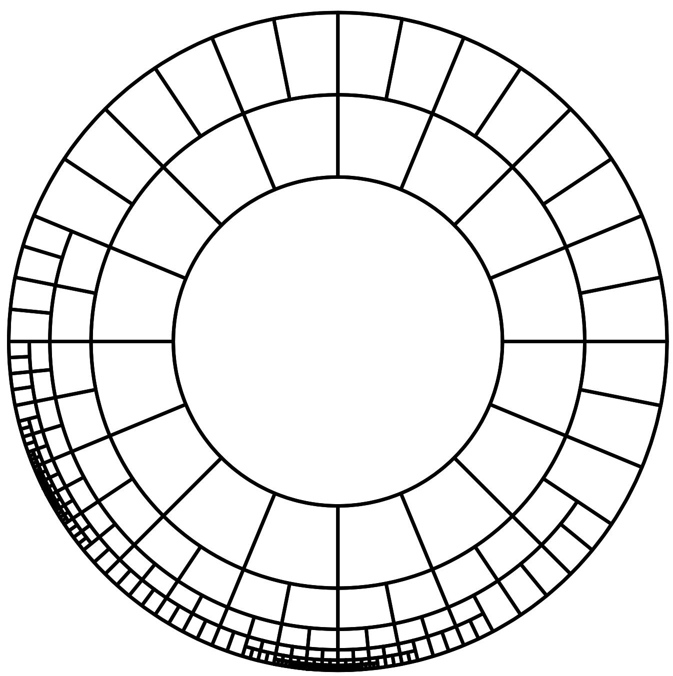
Idea for quad-mesh theorem:

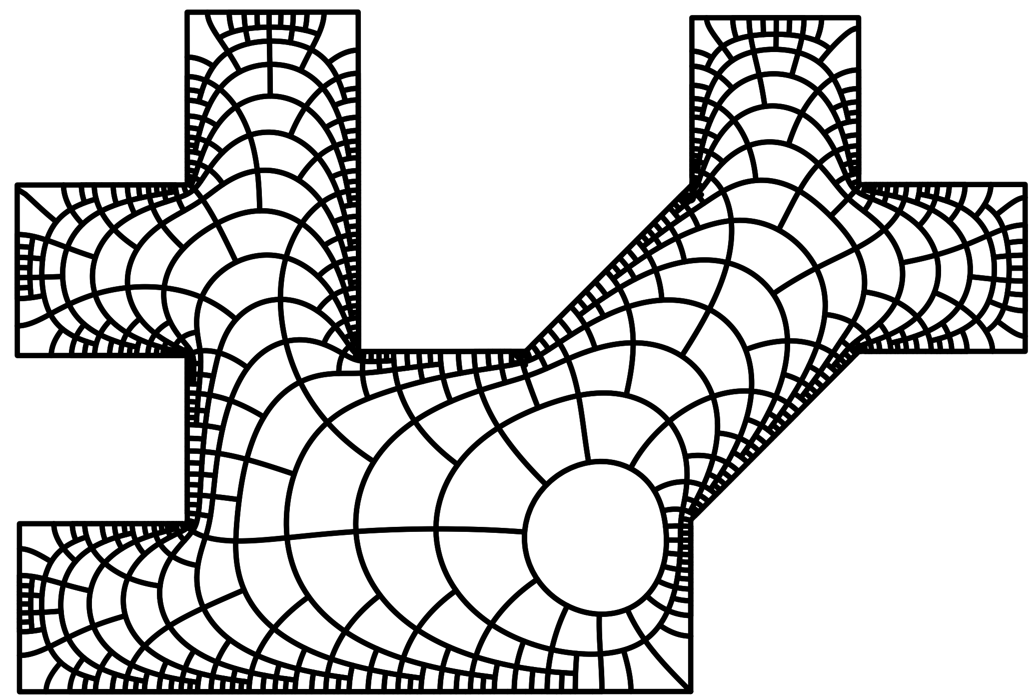
- Decompose polygon into $O(n)$ thick and thin parts.
- Mesh thin parts "by hand".
- Conformally map mesh on disk to thick parts.

Thin parts are meshed by explicit construction (easy).

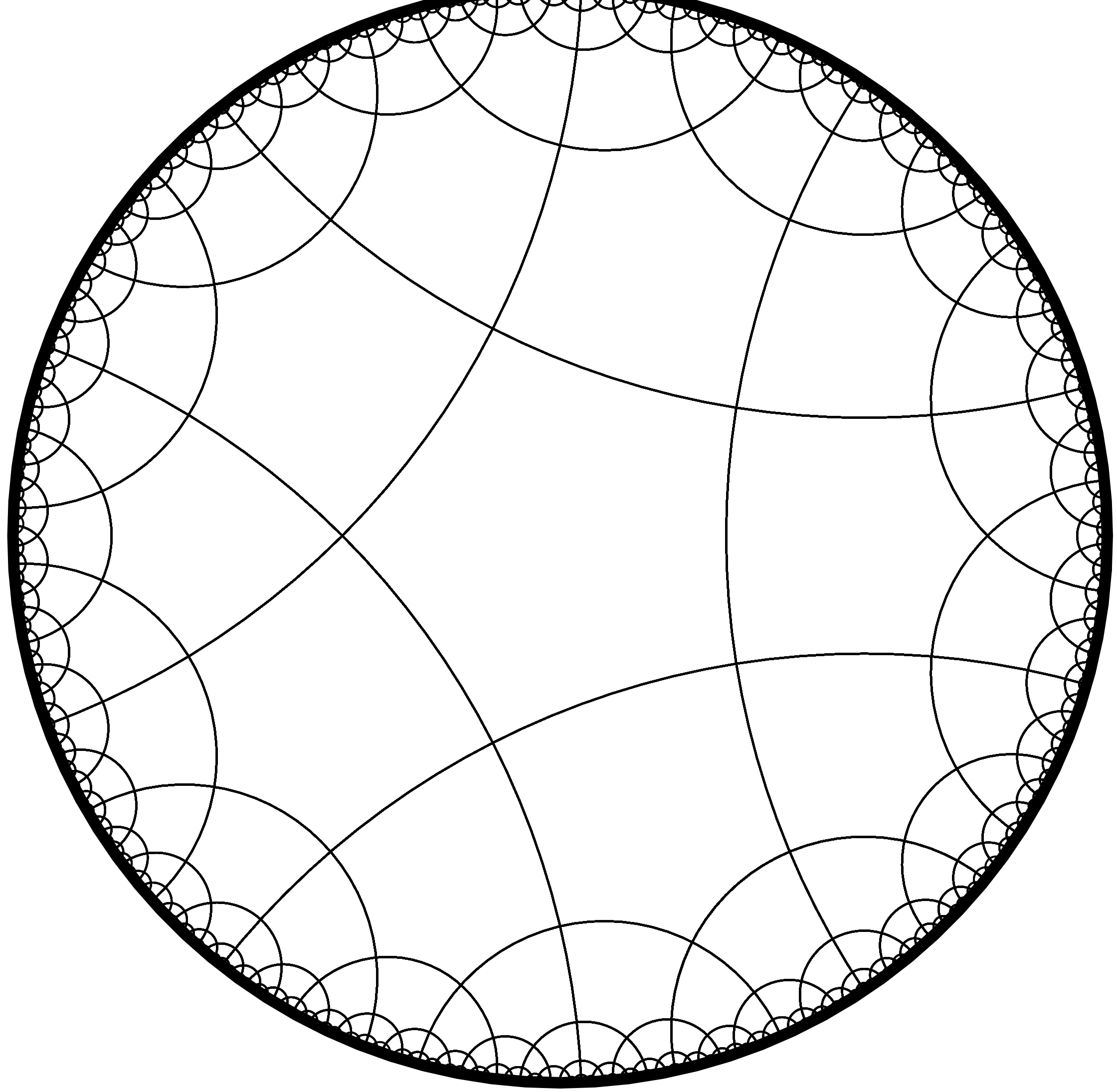


Thick parts: transfer mesh from disk

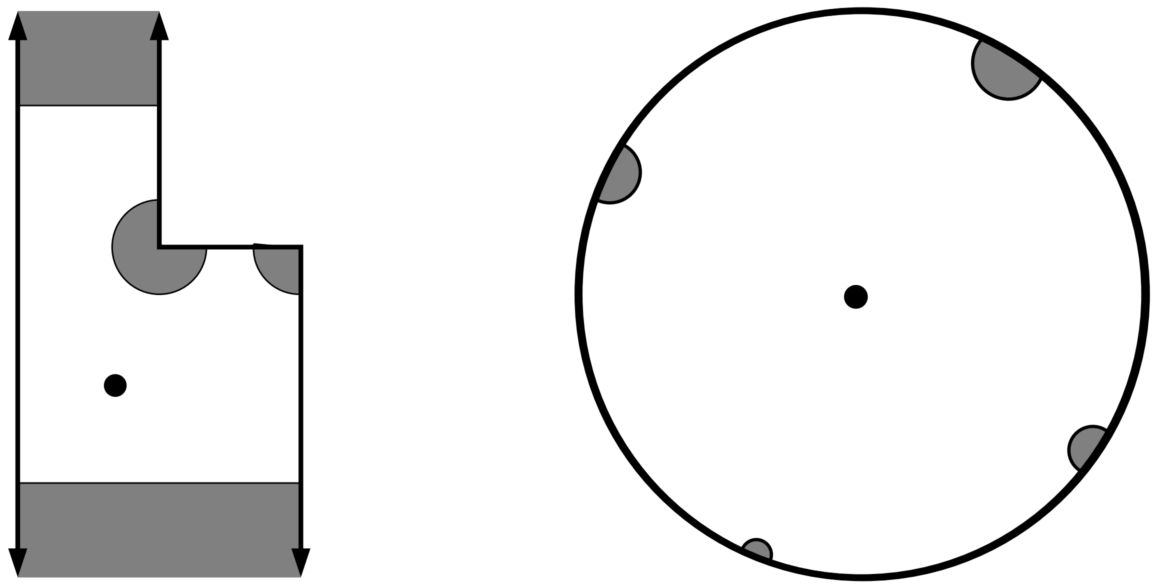




This is not quite the natural geometry.



Conformal map from polygon to disk takes thick and thin parts to disk.



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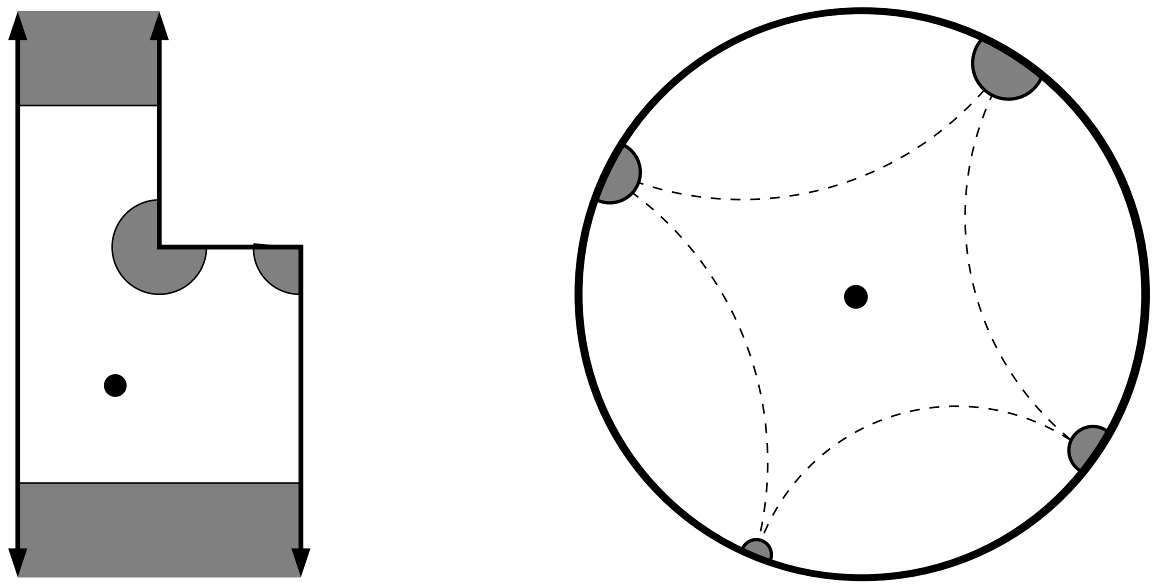
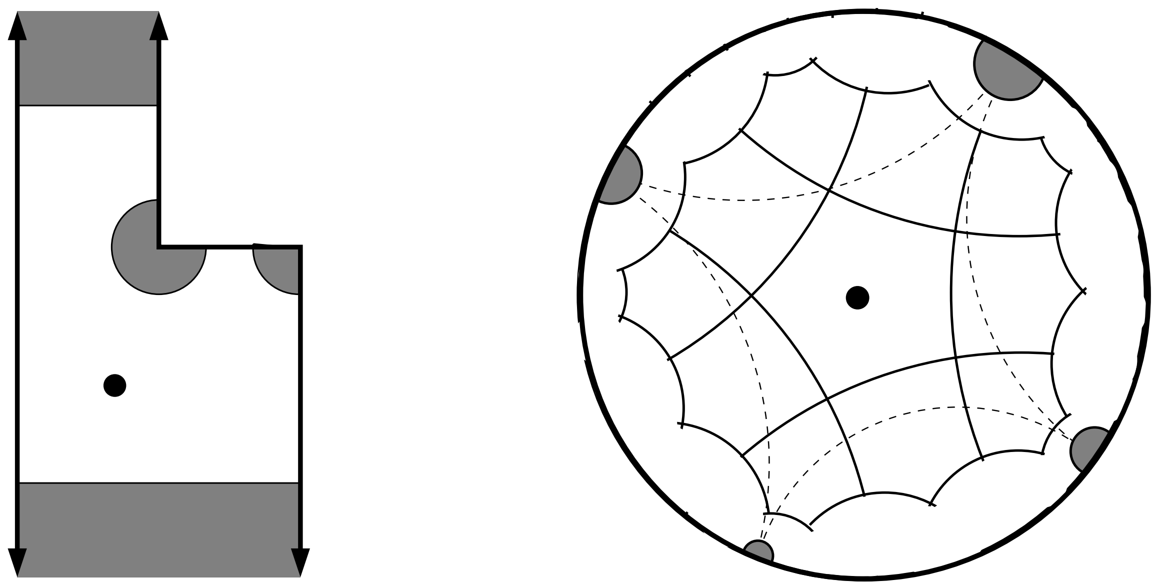


Draw (hyperbolic) convex hull of thin regions.

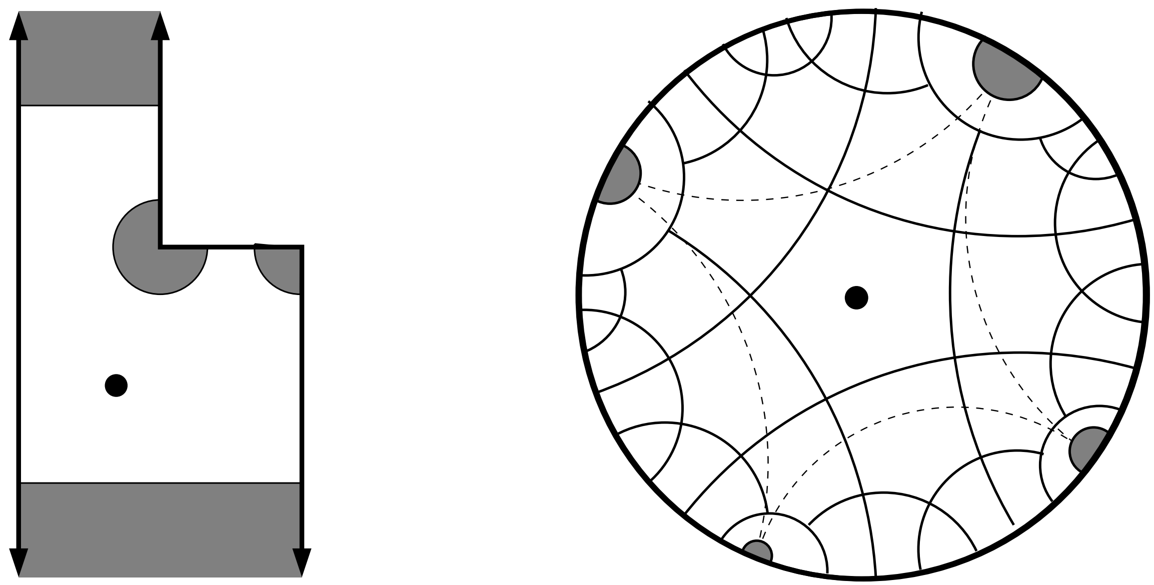
Conformal map from polygon to disk takes thick and thin parts to disk.



Draw (hyperbolic) convex hull of thin regions.

Take pentagons from tesselation hitting convex hull but missing thin parts.

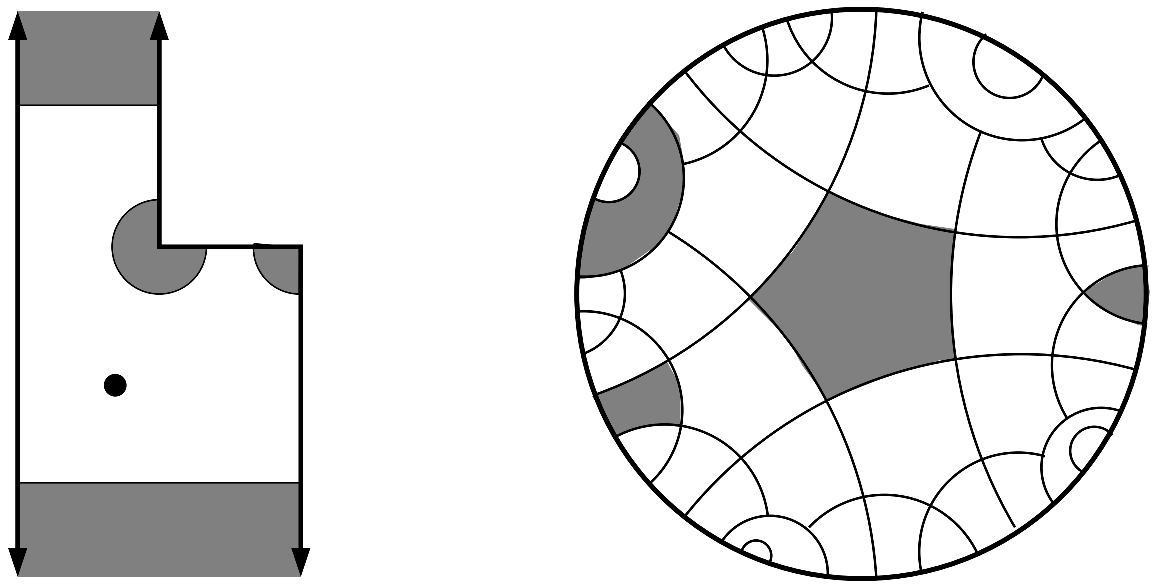
Conformal map from polygon to disk takes thick and thin parts to disk.



Draw (hyperbolic) convex hull of thin regions.

Take pentagons from tesselation hitting convex hull but missing thin parts.
Extend pentagon edges to boundary.

Conformal map from polygon to disk takes thick and thin parts to disk.



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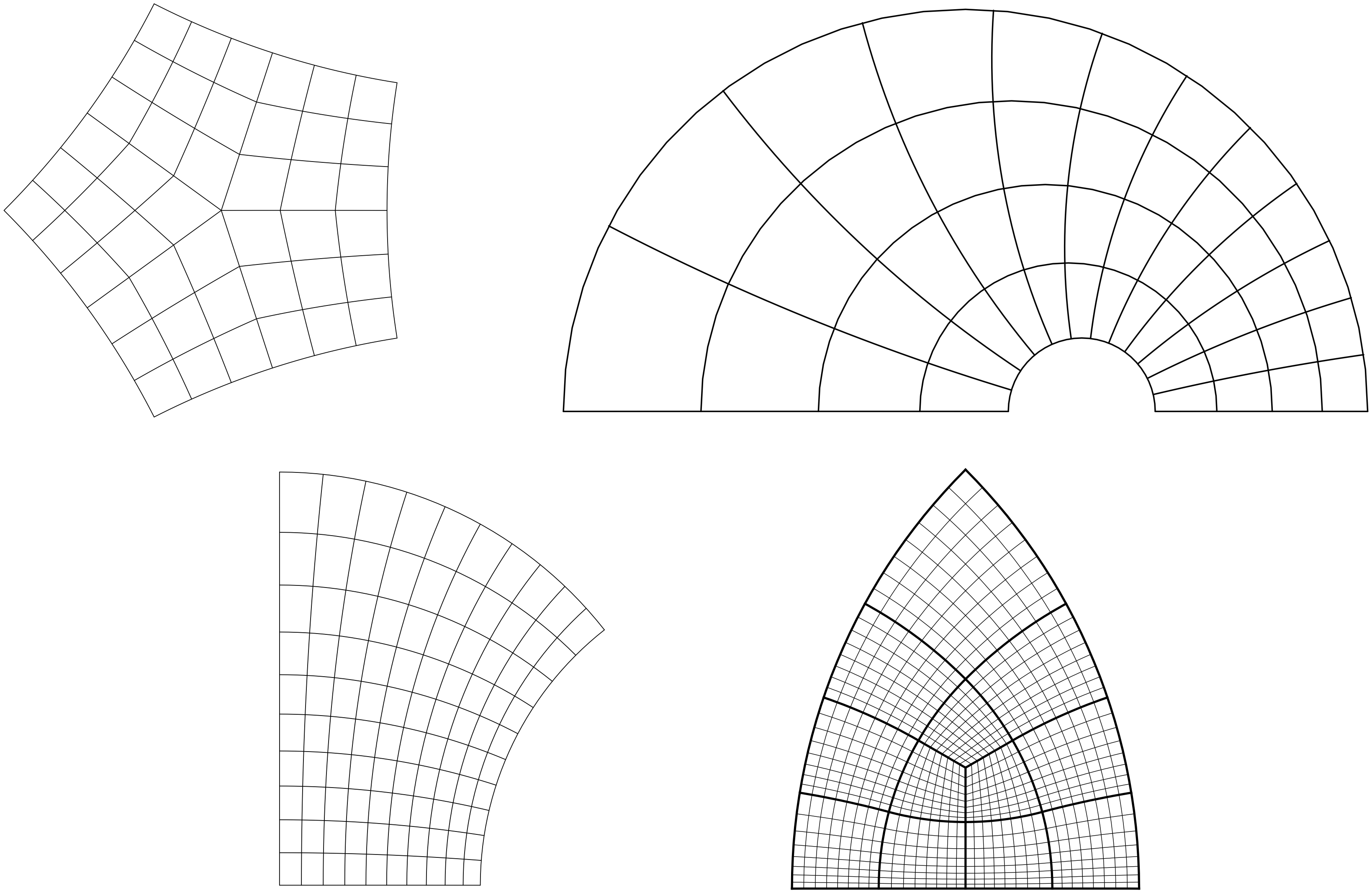
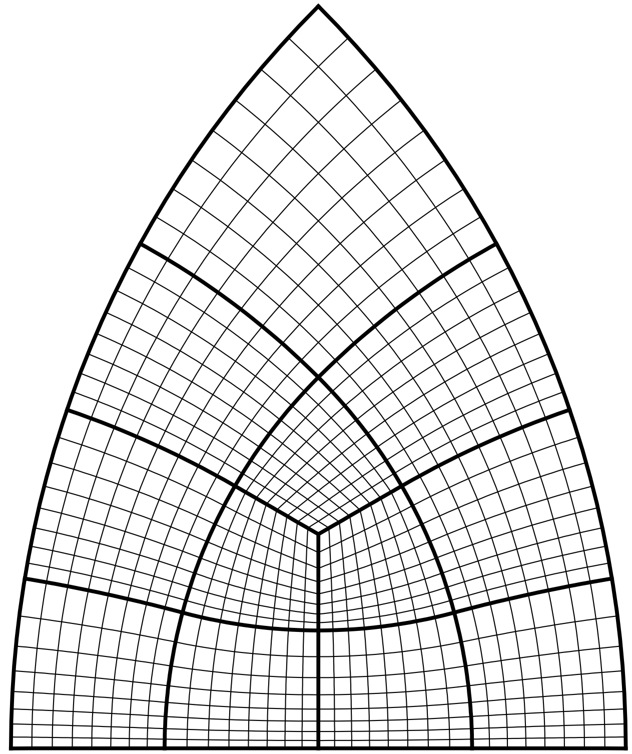
Extend pentagon edges to boundary.

Pentagons, quadrilaterals, triangles and half-annuli.



Meshes designed to match along common edges.
Conformal map only changes angles slightly.

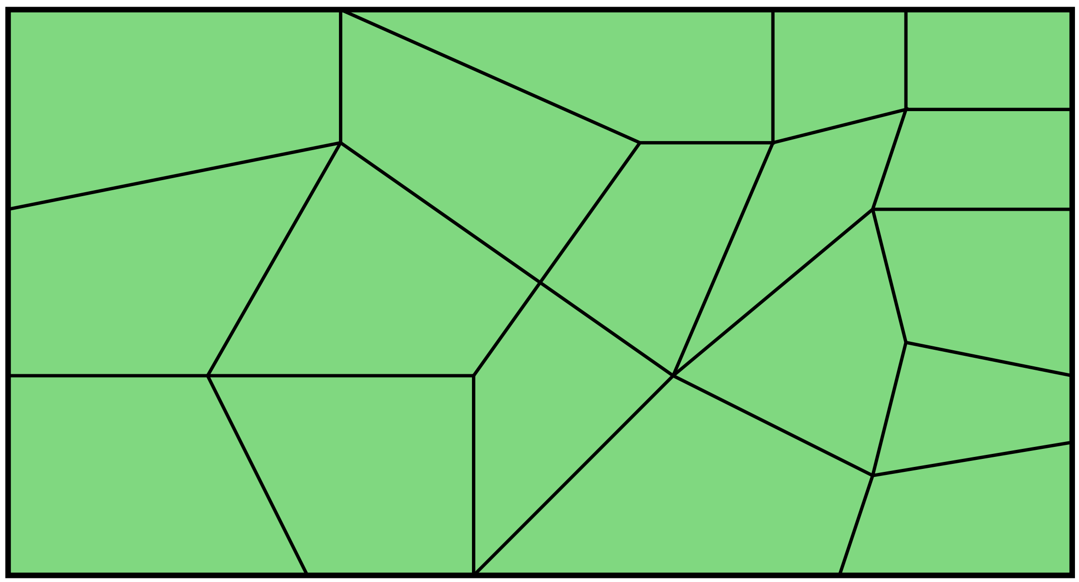


All angles strictly between 60° and 120°, except at center of triangles. Replace conformal map by linear to get the exact angle bounds.

PART II: OPTIMAL TRIANGULATION OF POLYGONS

Cor of Part I: Every polygon has a $O(n)$ 120°-triangulation.

Proof: add diagonals to quad-mesh.



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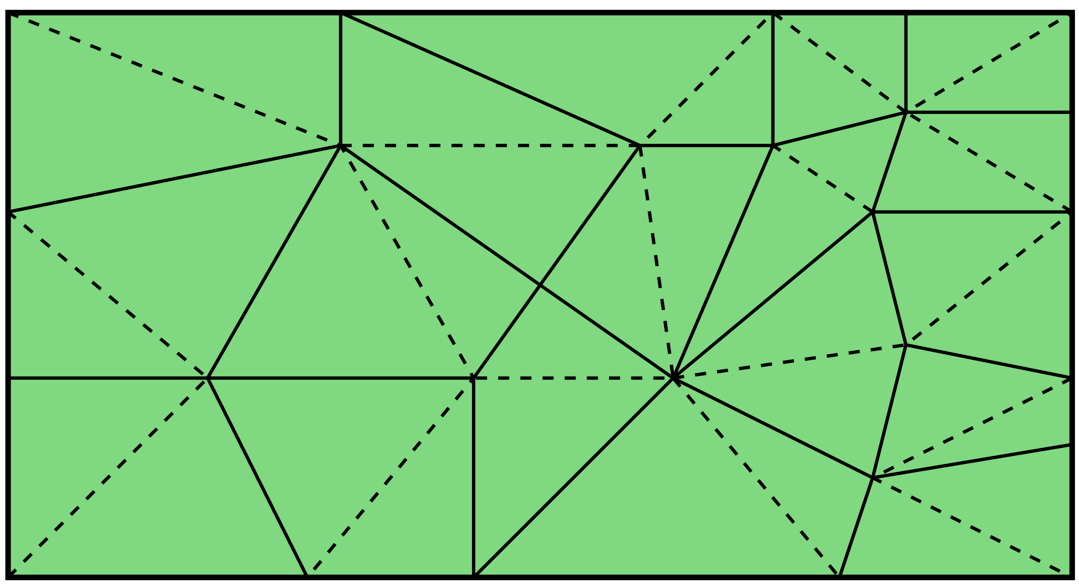


Better results known: every polygon has a 90°-triangulation.

= NOT = **N**on**O**btuse **T**riangulation

- Acute triangulation always possible (no bound): Burago, Zalgaller 1960.
- Rediscovered: Baker, Grosse, Rafferty, 1988.
- $O(n)$ for points sets: Bern, Eppstein, Gilbert 1990
- $O(n^2)$ for polygons: Bern, Eppstein, 1991
- $O(n)$ for polygons: Bern, S. Mitchell, Ruppert, 1994
- nonobtuse $\Rightarrow$ acute refinement, comparable complexity Maehara 2002.

See also Yuan 2005, Saraf 2009.

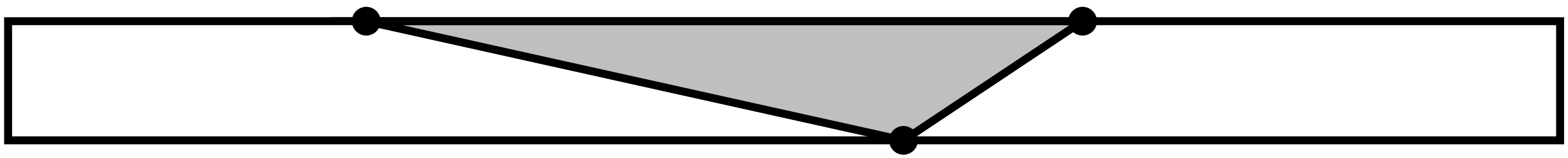
Can we do better? Lower angle bound? Improve 90° upper bound?

Answer depends on complexity.

Complexity bound $\Rightarrow$ no lower angle bound.

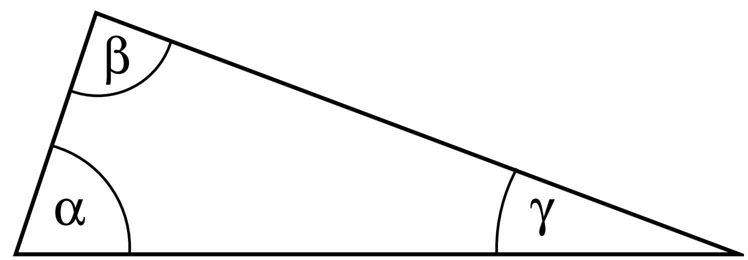
For $1 \times R$ rectangle.

$$\text{number of triangles} \gtrsim R \times (\text{smallest angle})$$



If we bound number of triangles in terms of number of vertices, no uniform lower angle bound is possible.

Complexity bound ⇒ no lower bound ⇒ 90° is optimal.



$$\alpha, \beta < (90^\circ - \epsilon) \Rightarrow \gamma = 180^\circ - \alpha - \beta \geq 2\epsilon.$$

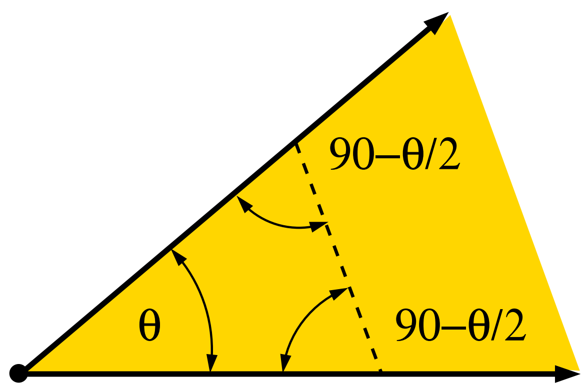
If we bound number of triangles in terms of number of vertices, no upper bound better than 90° is possible.

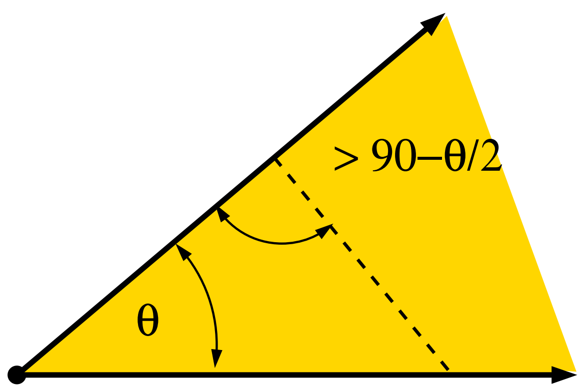
But what if we don't care about complexity bound?

Optimal angle bound depends on P:

If P has angle θ at v, any triangle $v \in T \subset P$ has angle $\leq \theta$.

Opposite angles sum to $\geq 180° - \theta$, so one is $\geq 90° - \theta/2$.





Remarkably, for small θ this is the **only restriction**.

Theorem: Suppose the minimal interior angle of P is θ. Then P has a ϕ-triangulation with $\phi = 90^\circ - \min(\theta, 36^\circ)$.

In other words:
(1) if $\theta \leq 36^\circ$ then P has a ϕ-triangulation with $\phi = 90^\circ - \theta/2$.
(2) if $\theta \geq 36^\circ$, then P has a 72°-triangulation.

We already saw (1) is sharp.

In (2), 72° not always sharp; optimal angle is computable (more later).

Idea of proof:

- Conformally map P to some P' that has a equilateral triangulation.
- Transfer mesh from P' back to P.

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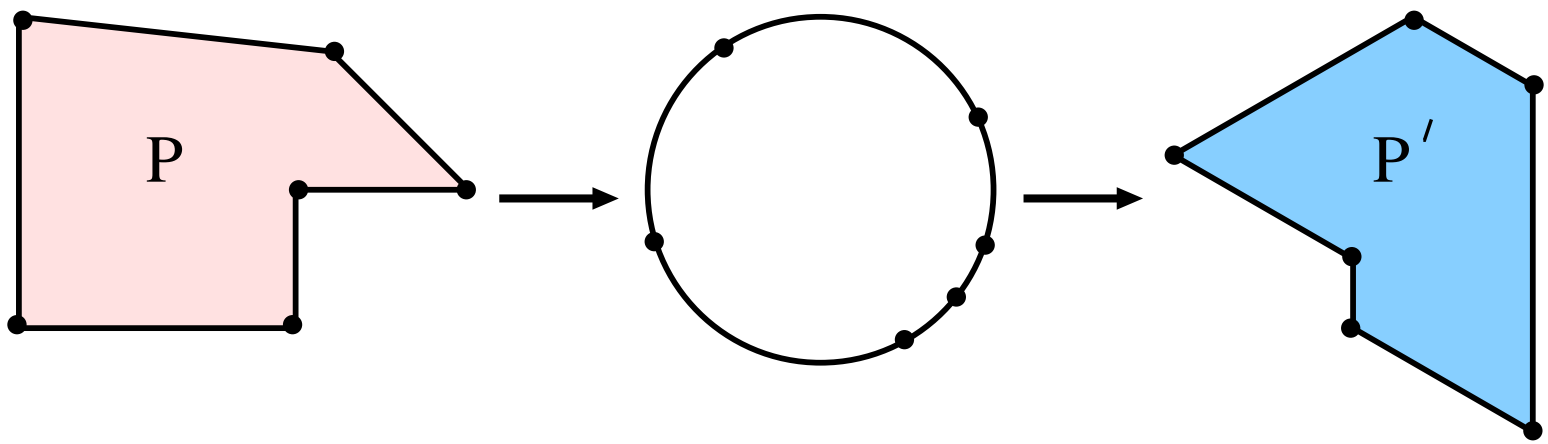
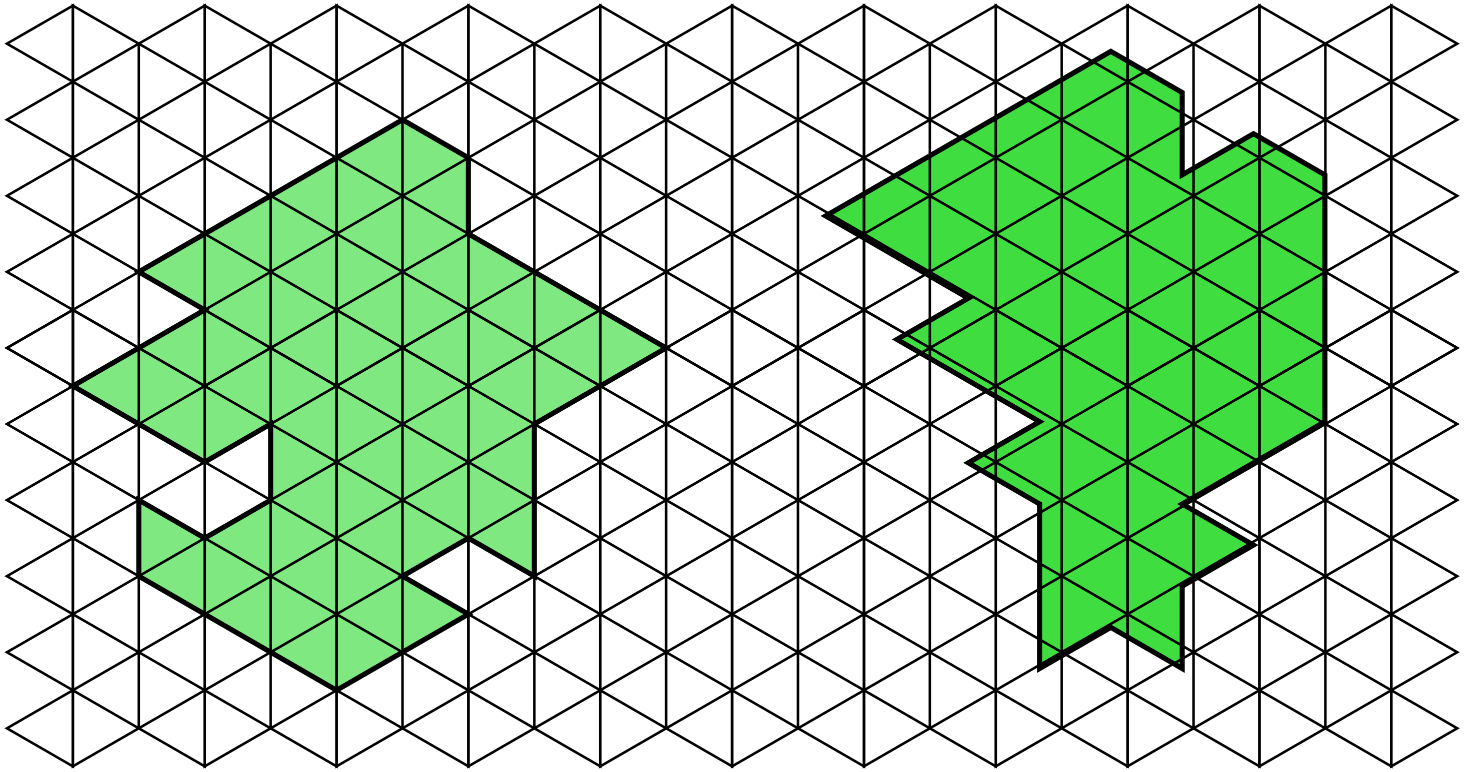


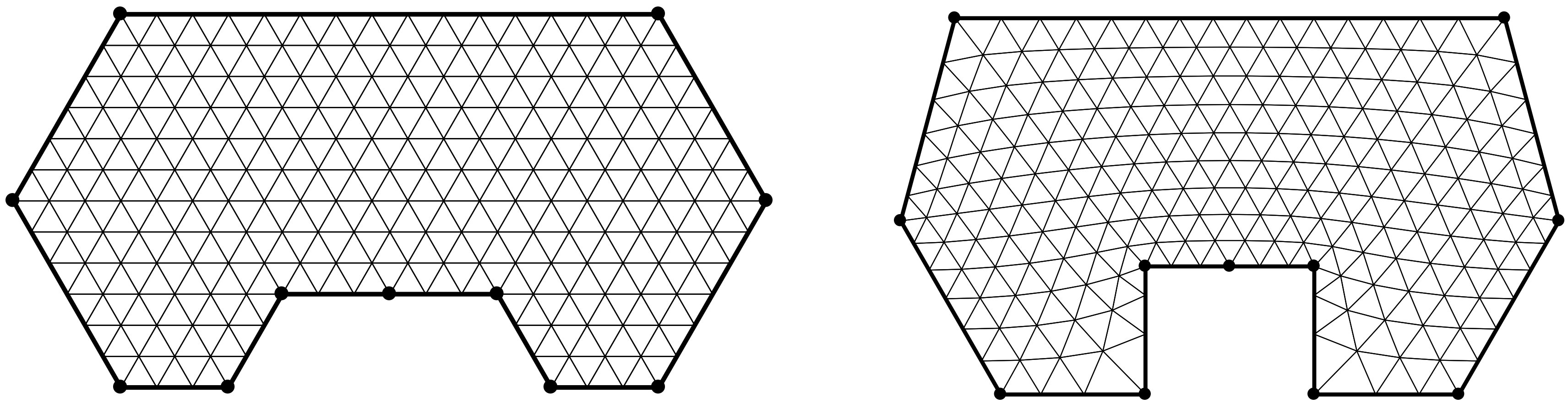
We can map P to disk conformally, then use Schwarz-Christoffel.
Can choose any angles that sum to $(n-2) \cdot 180°$

Choose $P' \approx P$ to have (nearly) equilateral triangulation.



Lemma: Equilateral triangulation $\Rightarrow$ angles of P are multiples of $60°$.

Partial converse: Any $60°$-polygon (angles in $60° \cdot \mathbb{N}$) has a $(60° + \epsilon)$-triangulation for all $\epsilon > 0$.

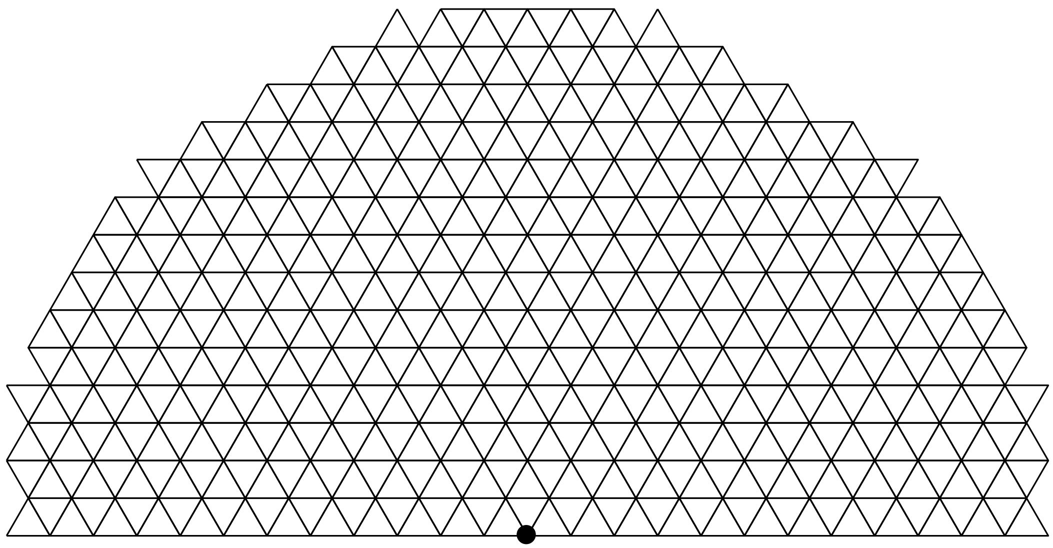
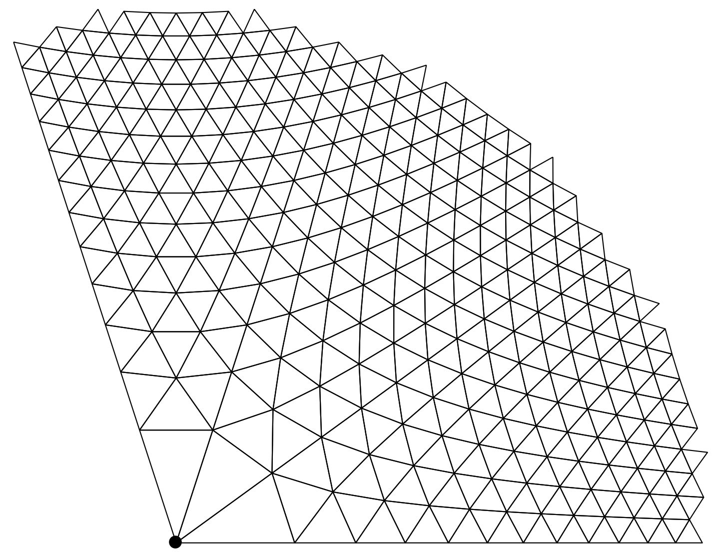


Under a conformal map angles are preserved infinitesimally.

For a triangle T, its angles change by $O\,(\mathrm{diam}(T)/\mathrm{dist}(T,V))$

Take triangles much smaller than edges of P.

Image triangulation nearly equilateral except near vertices.

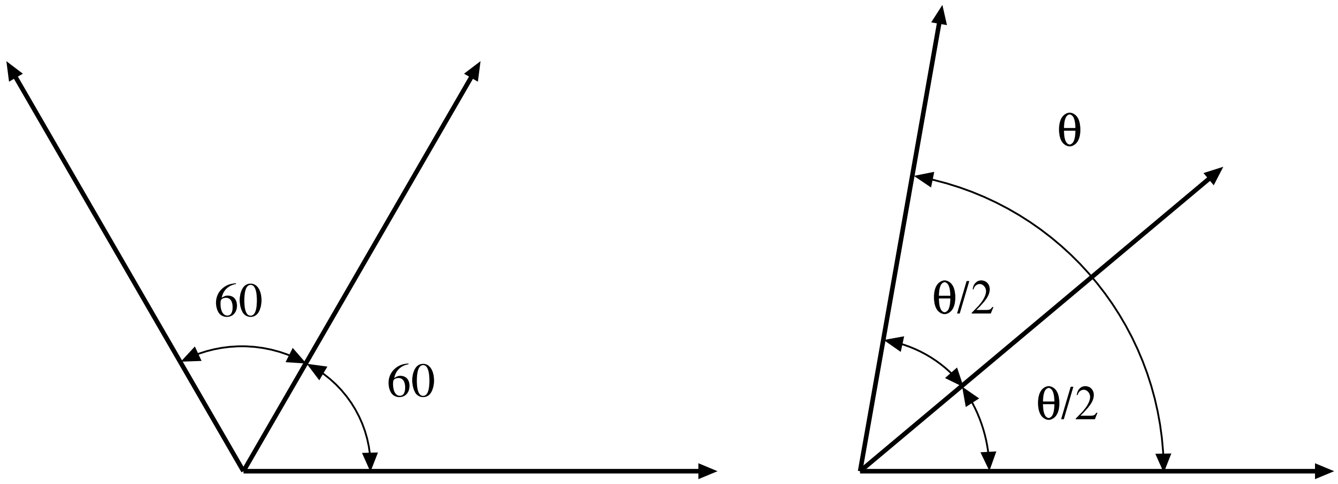
 

Near vertices, conformal map looks like power map.

Here 60° angles at vertex are mapped to $\frac{3}{5} \cdot 60^\circ = 36^\circ$.

The worst angle distortion occurs at the vertices.

Suppose P has angle θ corresponding to $\psi = k \cdot 60^\circ$ in P'.



Thus we want

$$36^\circ \leq \frac{\theta}{k} \leq 72^\circ$$

or equivalently

$$\frac{\theta}{72^\circ} \leq k \leq \frac{\theta}{36^\circ}.$$

θ range in P	allowable ψ in P'
0–72	60
72–108	120
108–144	120, 180
144–180	180, 240
180–216	180, 240, 300
216–288	240, 300, 360
288–360	300, 360

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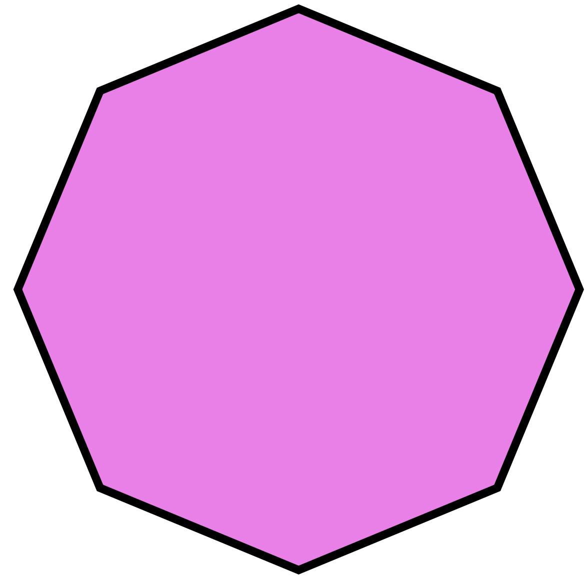
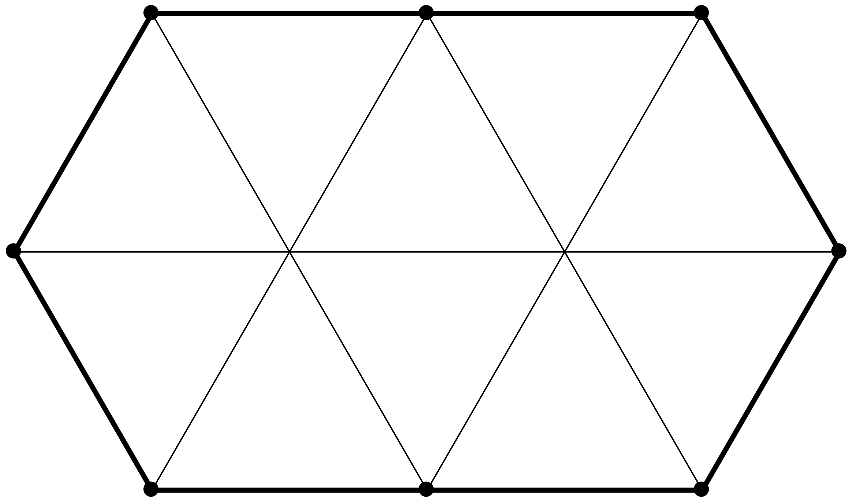


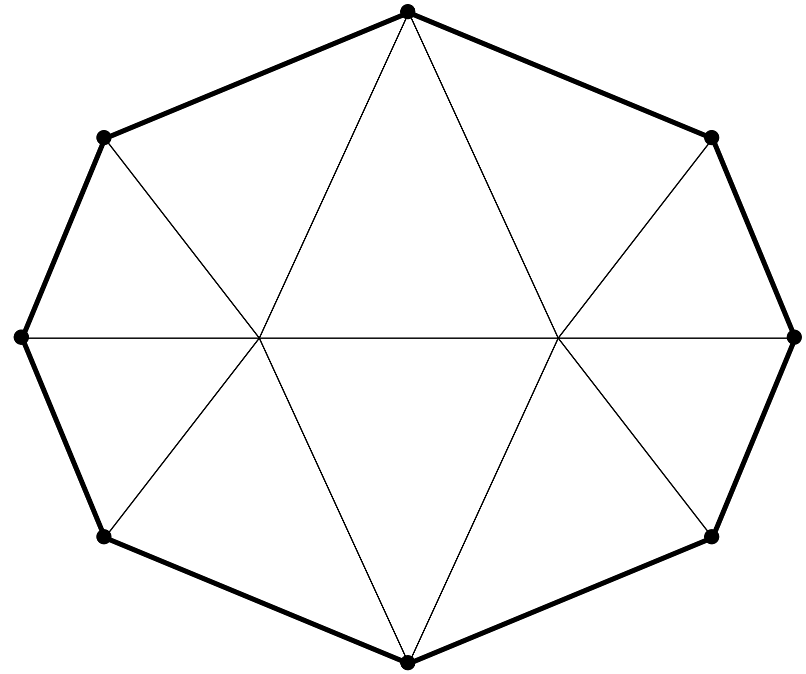
Easy case, a regular octagon: 8 angles of 135°.

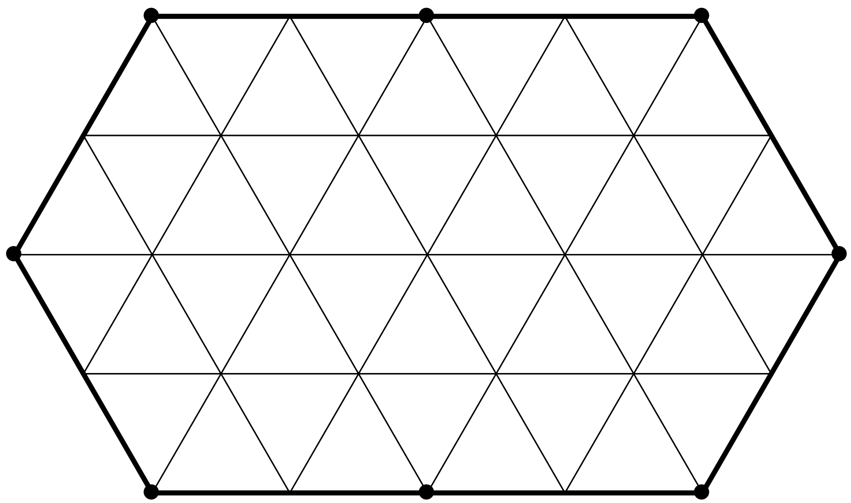
These should map to 120° or 180°.

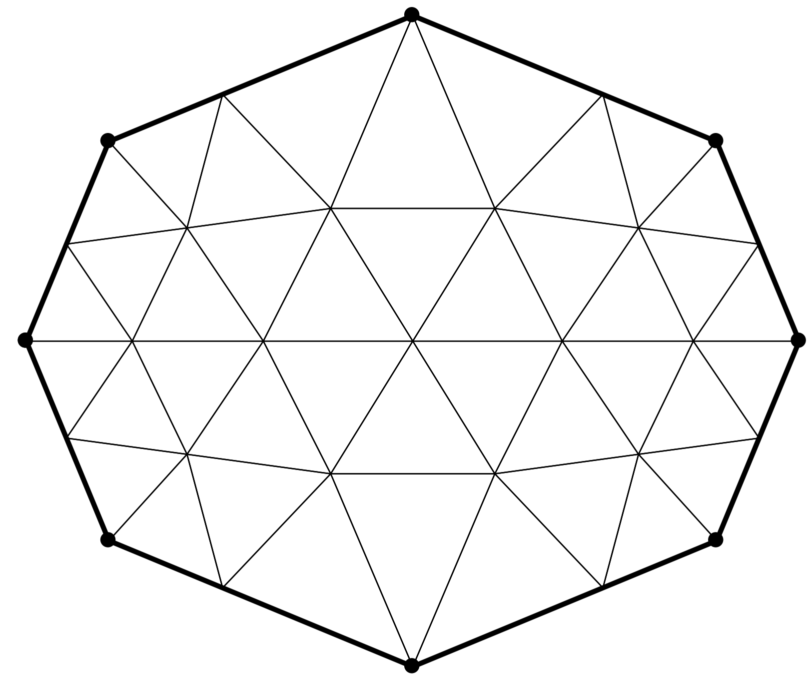
Sum should be $(8-2)\cdot 180^\circ = 1080^\circ$.

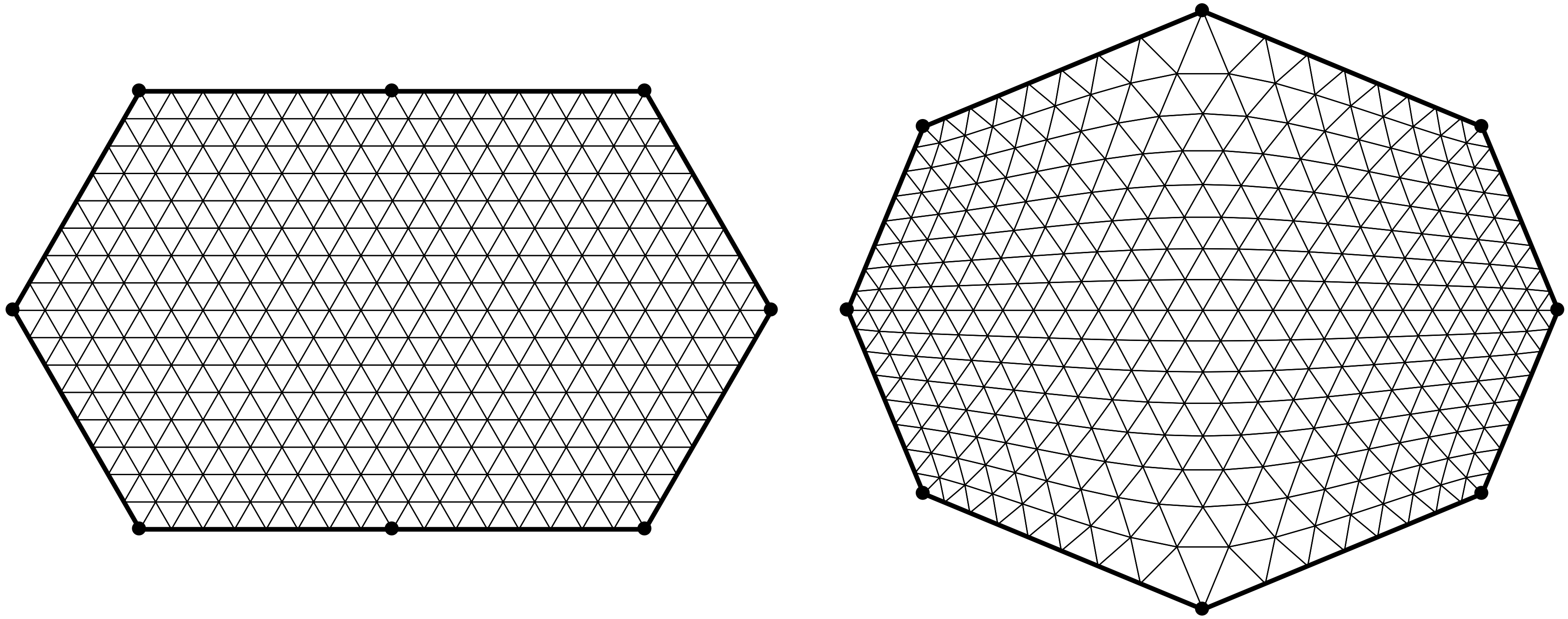
Taking 6 angles of 120° and 2 of 180° gives correct sum.









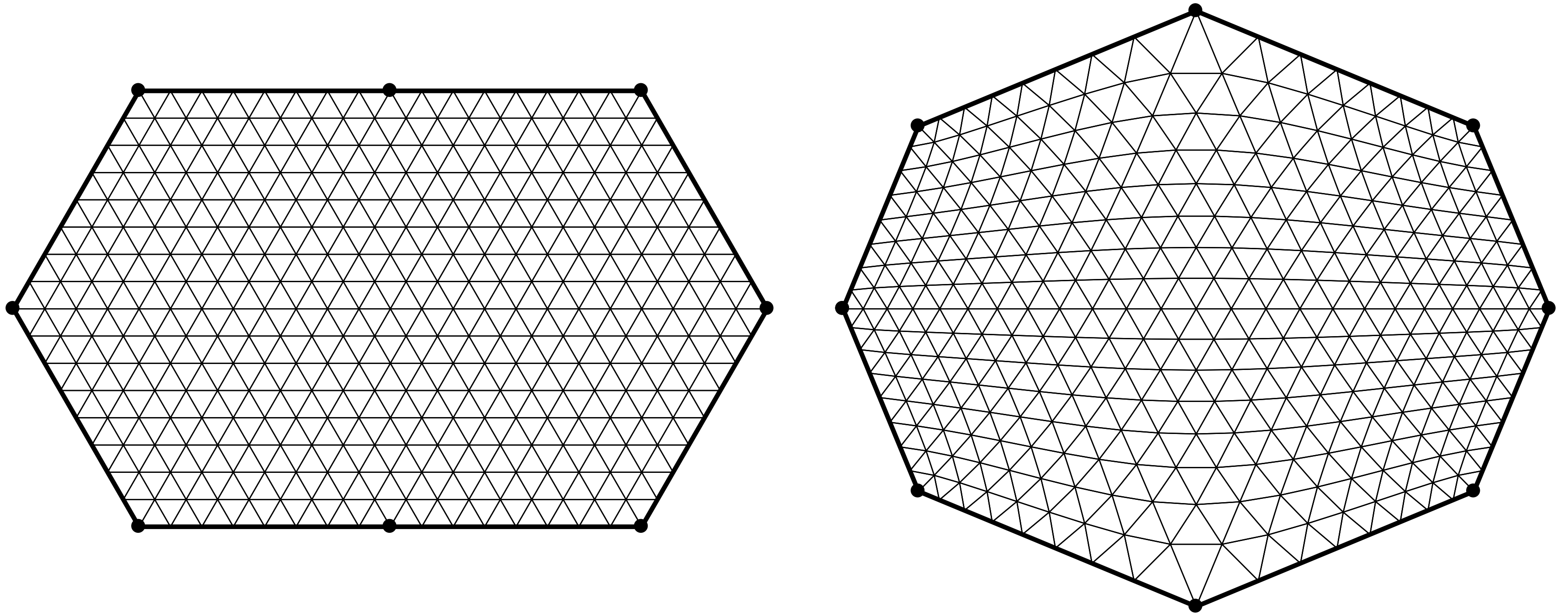


Worst distortion is near corners; angles $45^\circ = 135^\circ/3$ and $67.5^\circ = 135^\circ/2$.

As grid gets finer, the max angle tends to $67.5^\circ = 135^\circ/2$.

Small modification and 67.5° can be attained.

Is 67.5° optimal for octagon?



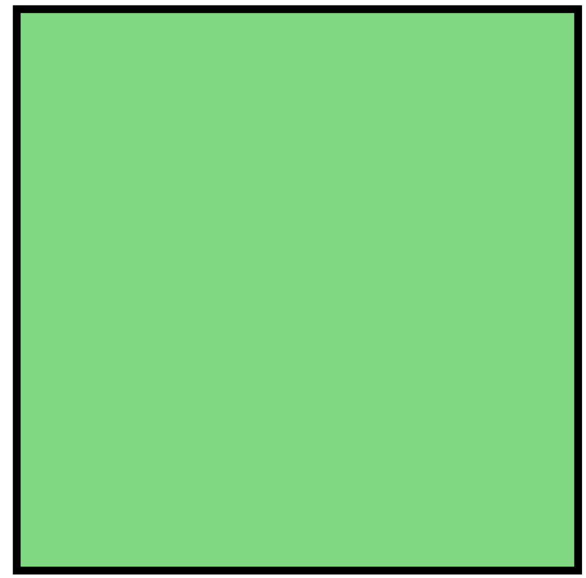
Worst distortion is near corners; angles $45^\circ = 135^\circ/3$ and $67.5^\circ = 135^\circ/2$.

As grid gets finer, the max angle tends to $67.5^\circ = 135^\circ/2$.

Small modification and 67.5° can be attained.

Is 67.5° optimal for octagon? **Yes.**

θ range in P	allowable ψ in P'
0–72	60
72–108	120
108–144	120, 180
144–180	180, 240
180–216	180, 240, 300
216–288	240, 300, 360
288–360	300, 360



Harder case, a square: table says choose P' with all angles $= 120°$.

Angle sum would be $4 \times 120° = 480° \neq 360°$.

No way to assign angles so P' exists. What to do?

How to compute optimal angle bound for a polygon P.

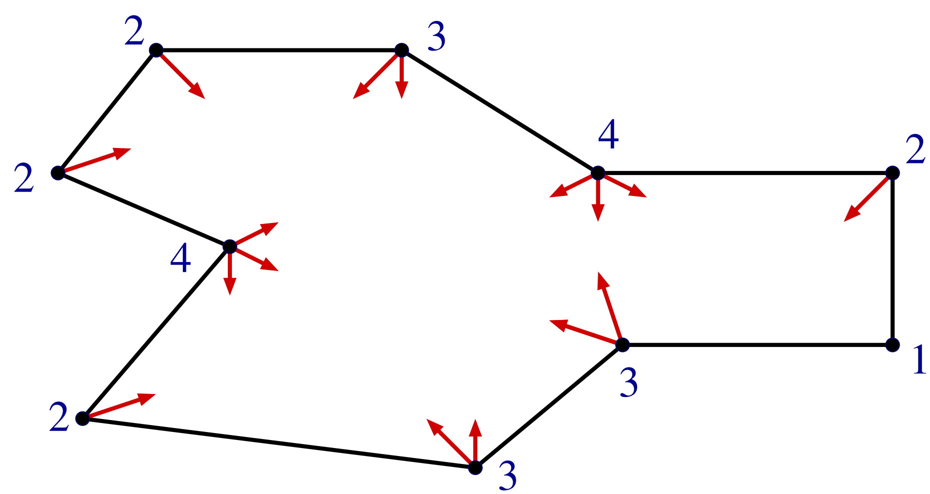
P = polygon, V = vertices, θ_v = interior angle at $v \in V$.

For $\phi \in [60°, 90°]$ define the interval $I(\phi) = [180 - 2\phi, \phi]$.

Any ϕ-triangulation must have all of its angles in $I(\phi)$.

If $L(v)$ is number of triangles containing v, then

$$\theta_v \in L(v) \cdot I(\phi) \quad \text{for all } v \in V. \qquad (*)$$



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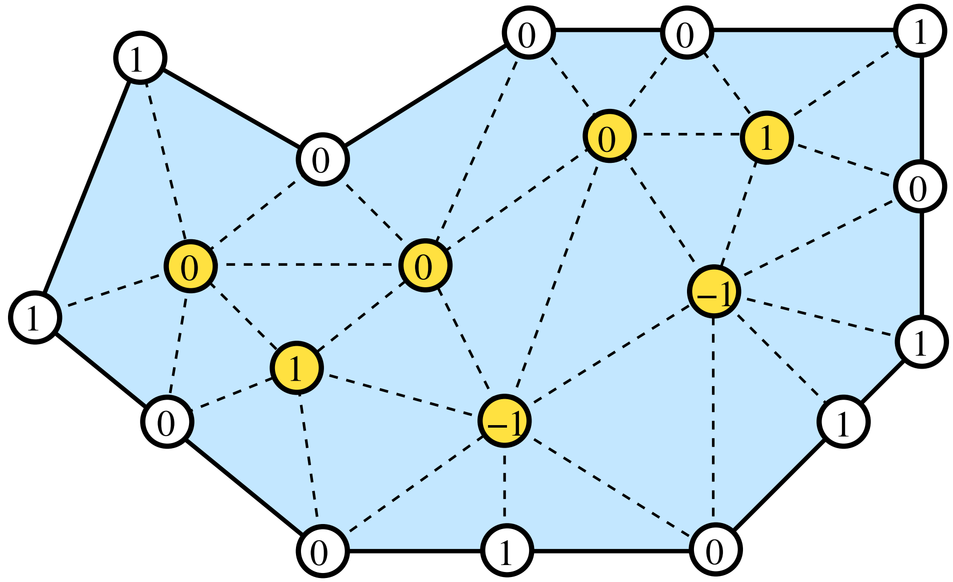
If $L(v)$ is number of triangles containing v, then

$$\theta_v \in L(v) \cdot I(\phi) \quad \text{for all } v \in V. \tag{$*$}$$

We say a labeling $L : V \to \mathbb{N} = \{1, 2, \ldots\}$ is ϕ-admissible if $(*)$ holds.

$\exists$ ϕ-triangulation $\Rightarrow$ $\exists$ admissible labeling. If $\phi \geq 72^\circ$, converse holds.

Otherwise need stronger conditions involving discrete curvature.



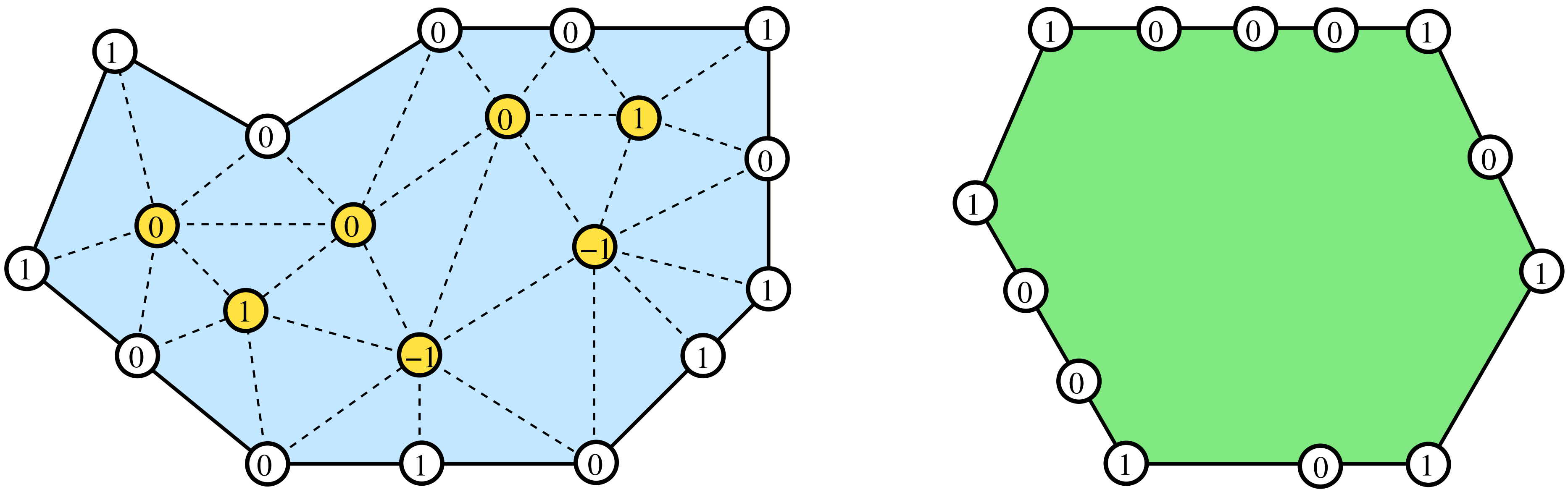
The discrete curvature a vertex of a triangulation is

$$\kappa(v) = 3 - L(v), \qquad \kappa(v) = 6 - L(v).$$

for boundary/interior vertices.

Rewrite Euler's formula to look like Gauss-Bonnet:

$$\sum_{v \in \text{interior}} \kappa(v) = 6 - \sum_{v \in \text{boundary}} \kappa(v) \equiv \kappa(L)$$



Given labeling for P we want a 60°-polygon P' with same labeling.

P' has angle $L(v) \cdot 60^\circ$ at v, so angle sum for P' is

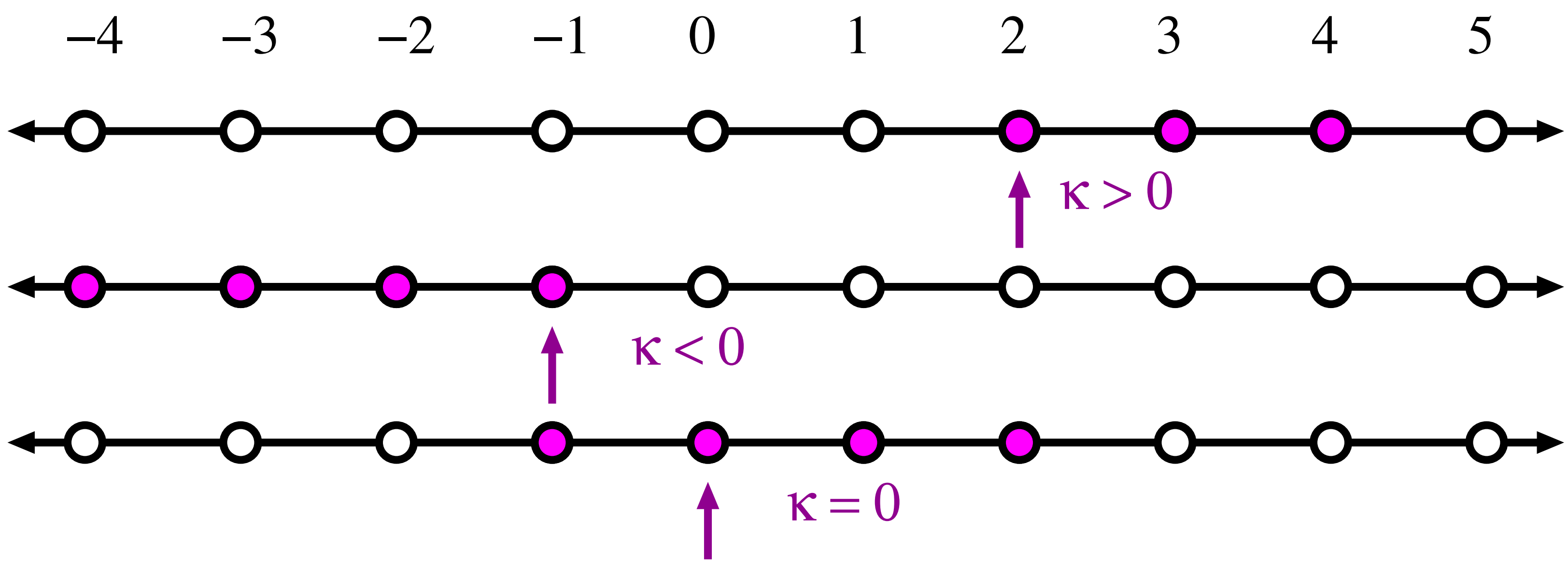
$$60^\circ \cdot \sum_{v \in V} L(v) = 180^\circ \cdot (|V| - 2) + 60^\circ \cdot \kappa(L).$$

If $\kappa(L) = 0$, then P' exists. Plan works. Otherwise minimize $|\kappa(L)|$

Let $\mathcal{K}(\phi)$ be set of $\kappa(L)$ over admissible labelings ($= \infty$ if none).

$\mathcal{K}(\phi)$ is either ∞ or a non-empty interval of integers.

Let $\kappa(\phi) \in \mathcal{K}(\phi)$ be closest element to 0 (easy to compute).



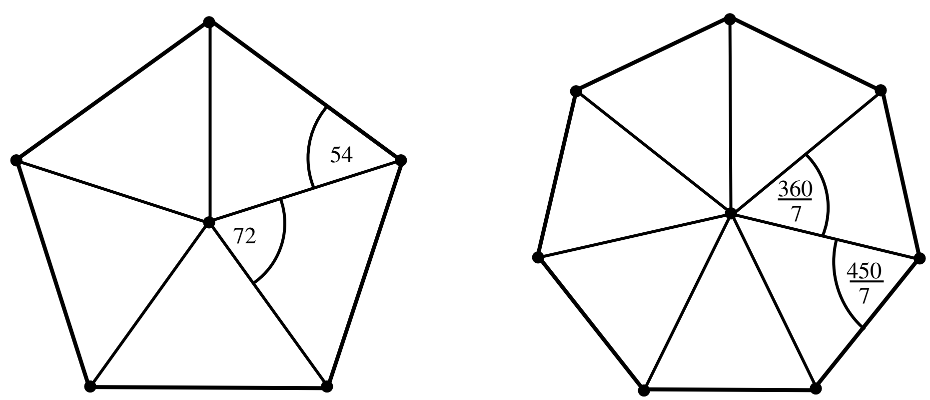
Theorem: For $60^\circ < \phi < 90^\circ$, a polygon P has a ϕ-triangulation iff

1. $72^\circ \leq \phi < 90^\circ$ and $\kappa(\phi) < \infty$
2. $\frac{5}{7} \cdot 90^\circ \leq \phi < 72^\circ$, and $\kappa(\phi) \leq 0$
3. $60^\circ < \phi < \frac{5}{7} \cdot 90^\circ$, and $\kappa(\phi) = 0$

All interior vertices are degree 6, except:

when $\kappa(\phi) > 0$, $\kappa(\phi)$ vertices have degree 5,

when $\kappa(\phi) < 0$, $|\kappa(\phi)|$ vertices have degree 7.



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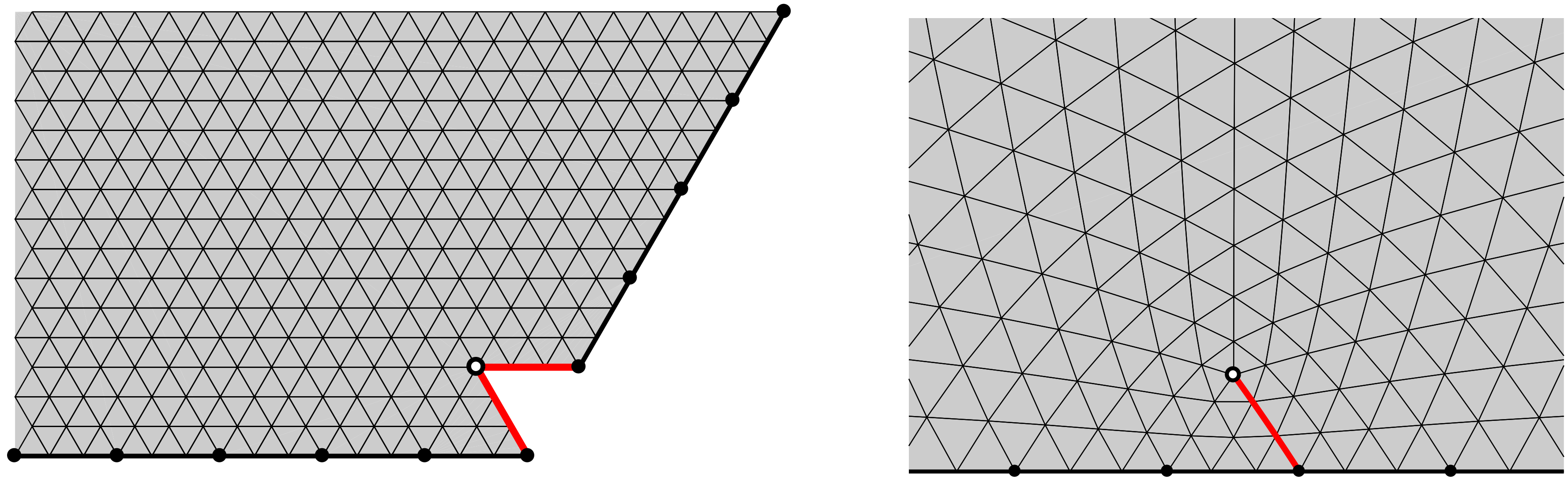
2. $\frac{5}{7} \cdot 90^\circ \leq \phi < 72^\circ$, and $\kappa(\phi) \leq 0$
3. $60^\circ < \phi < \frac{5}{7} \cdot 90^\circ$, and $\kappa(\phi) = 0$

Cor 1: Optimal upper bound only depends on (unordered) angles of P.

Cor 2: Optimal bound is attained if $\phi > 60^\circ$.

Cor 3: For $\phi > 60^\circ$, P has a ϕ-triangulation iff it has a ϕ-dissection.

Gerver (1984) proved ϕ-dissection implies (1)-(3) using Euler's formula.

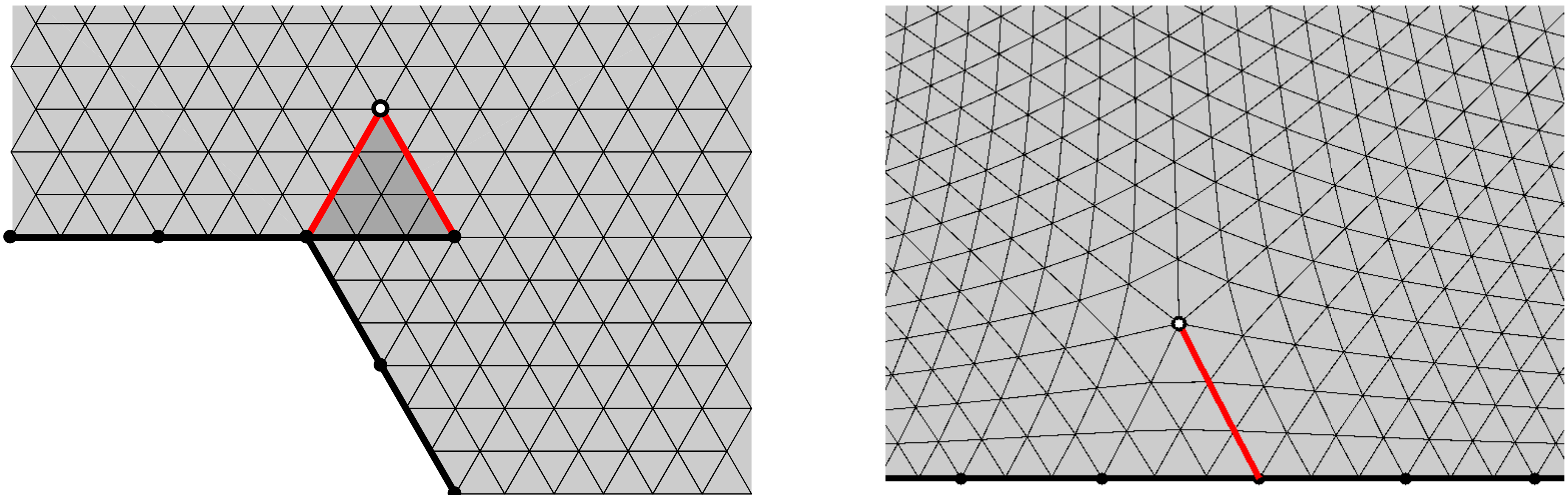


When $\kappa(\phi) > 0$, cut slits in P, open them to get 60°-polygon P'.

Choose conformal map so triangles in P' "match up" along paired edges.

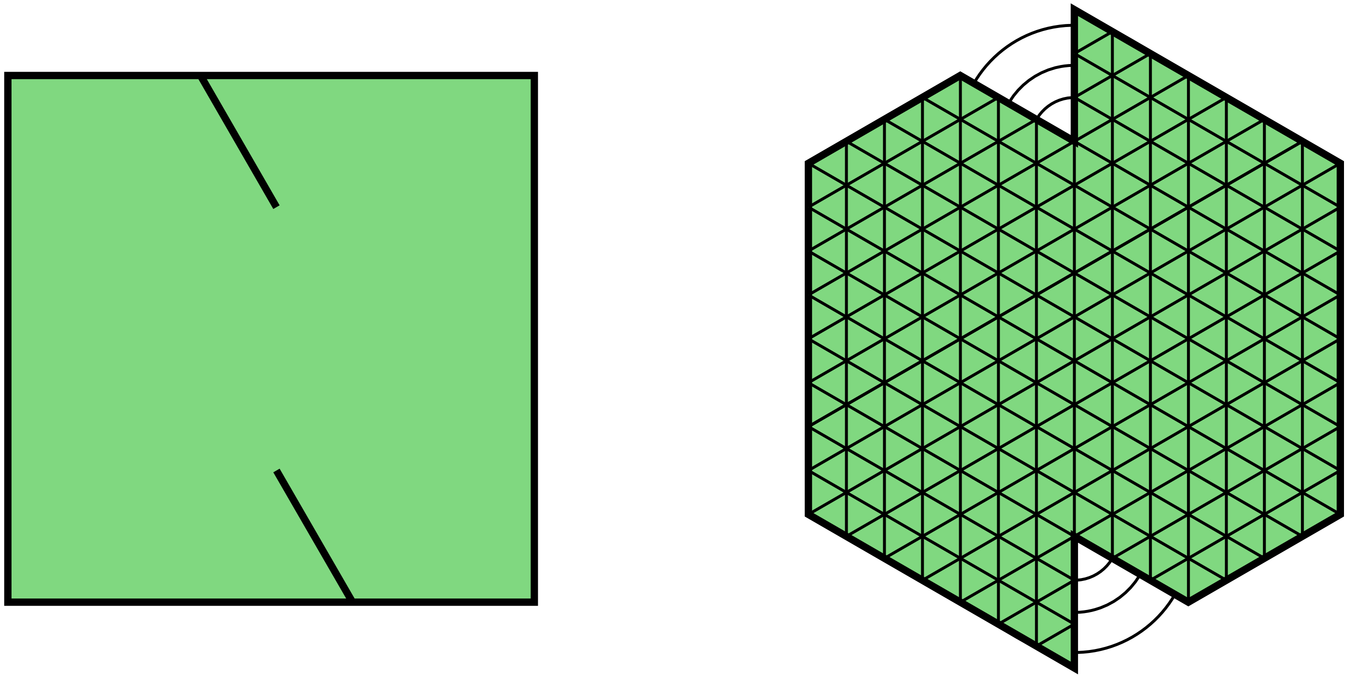
Creates an interior vertex of degree 5 in P, hence angle $\geq 72^\circ$.

Transfers curvature between interior and boundary. Decreases $\kappa(\phi)$.



When $\kappa(\phi) < 0$, add slit to increase it, but P' is a Riemann surface.

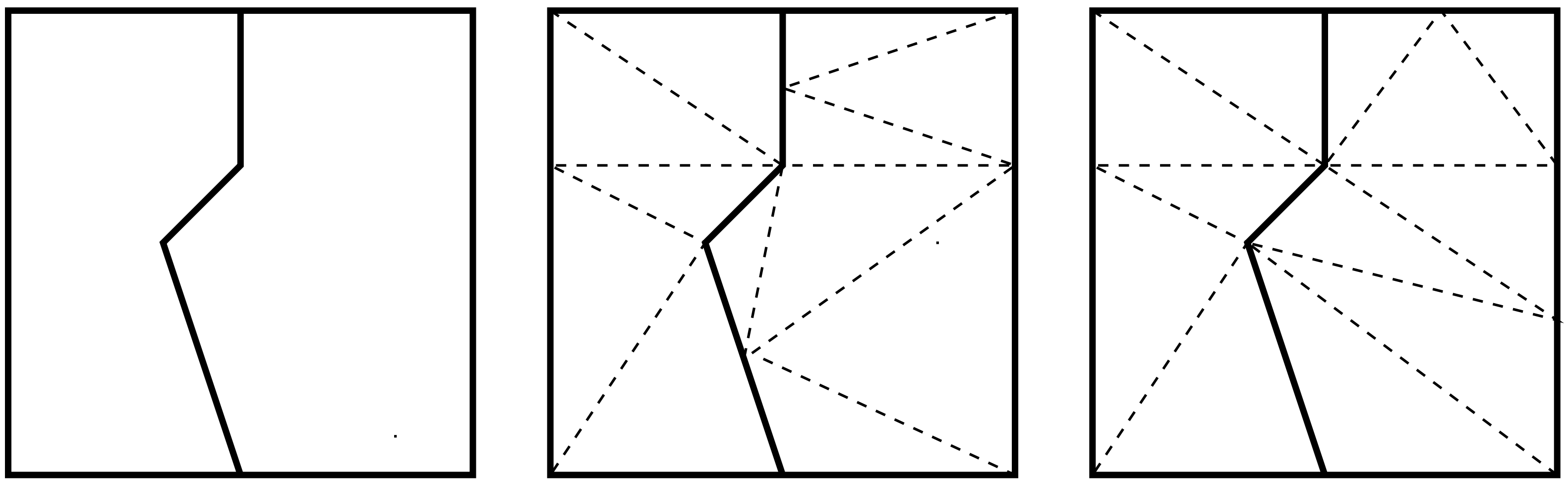
Creates an internal vertex of degree 7, hence an angle $\geq 64.28^\circ = 450^\circ/7$.

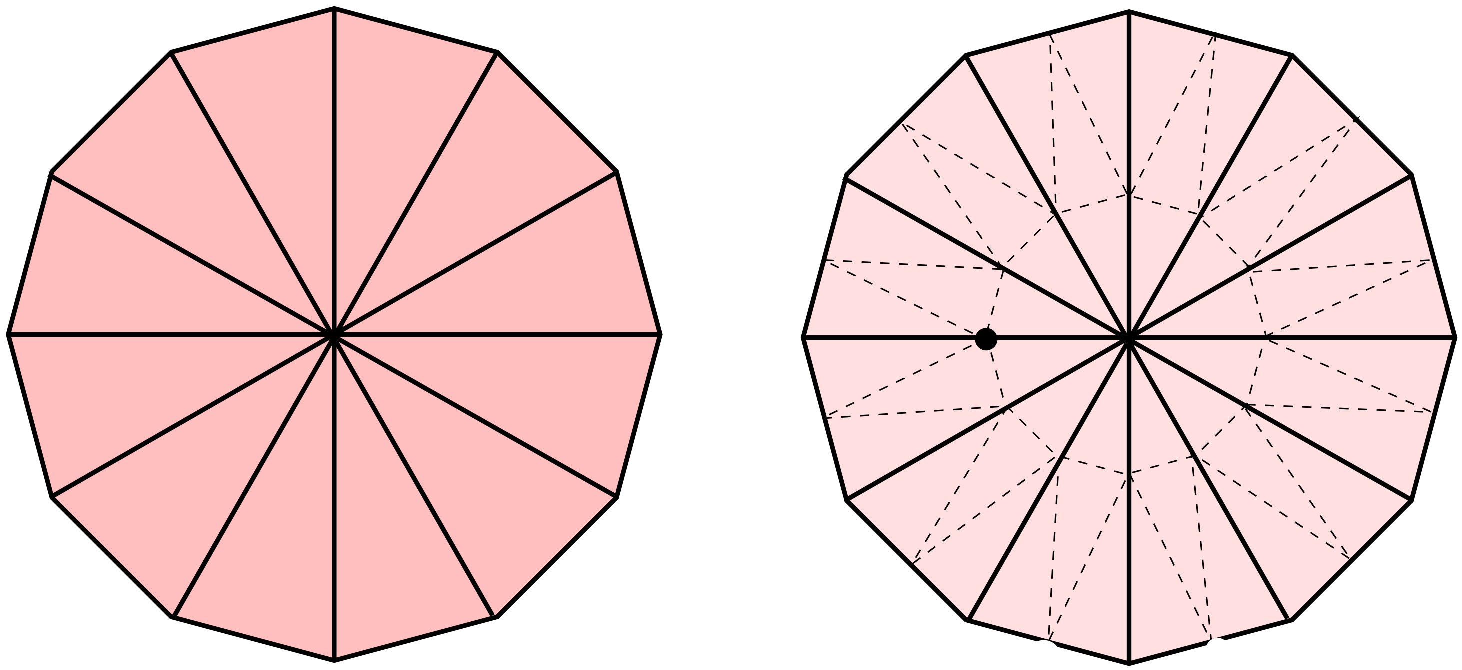


Square has $\kappa(72^\circ) = 2$, requires two slits.

PART III: THE NOT THEOREM

A **conforming triangulation** of a PSLG is a triangulation of each face, consistent across edges of the PSLG.





Adding one point may require many more triangles.
Some PSLGS with n vertices have minimal NOT $\simeq n^2$.
NOT = Non-Obtuse Triangulation = all angles $\leq 90^\circ$.
Any NOT is automatically a conforming Delaunay triangulation.

S. Mitchell, 1993: Every PSLG has a $157.5°$-triangulation, size $O(n^2)$.

Tan, 1996: Every PSLG has a $132°$-triangulation, size $O(n^2)$.

Burago-Zalgaller, 1960: Every PSLG has an NOT (no size bound).

S. Mitchell, 1993: Every PSLG has a 157.5°-triangulation, size $O(n^2)$.

Tan, 1996: Every PSLG has a 132°-triangulation, size $O(n^2)$.

Burago-Zalgaller, 1960: Every PSLG has an NOT (no size bound).

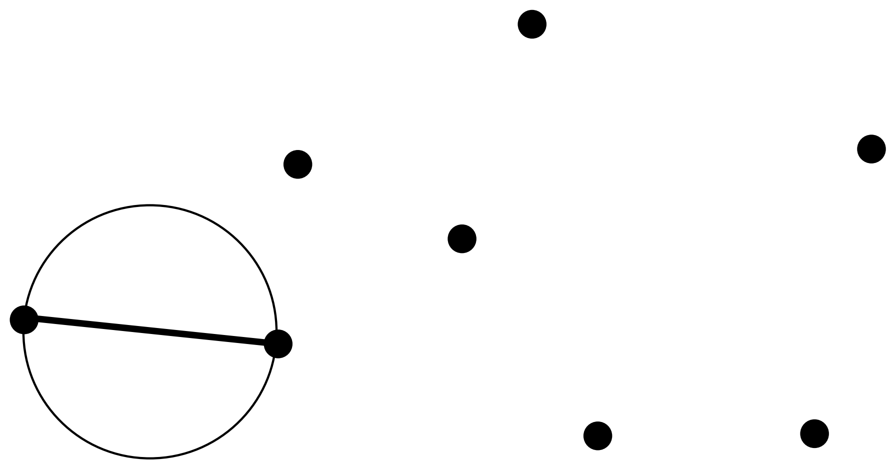
NOT-Theorem: Every PSLG has a NOT with $O(n^{2.5})$ elements.

Gap remains between n^2 example and $O(n^{2.5})$ algorithm.

Improves $O(n^3)$ for Delaunay triangulation by Edelsbrunner, Tan (1993).

Proof uses Gabriel edges of a point set (related to Delaunay edges).

The segment $[v, w]$ is a **Gabriel** edge if it is the **diameter** of a disk containing no other points of V.



Gabriel edge.

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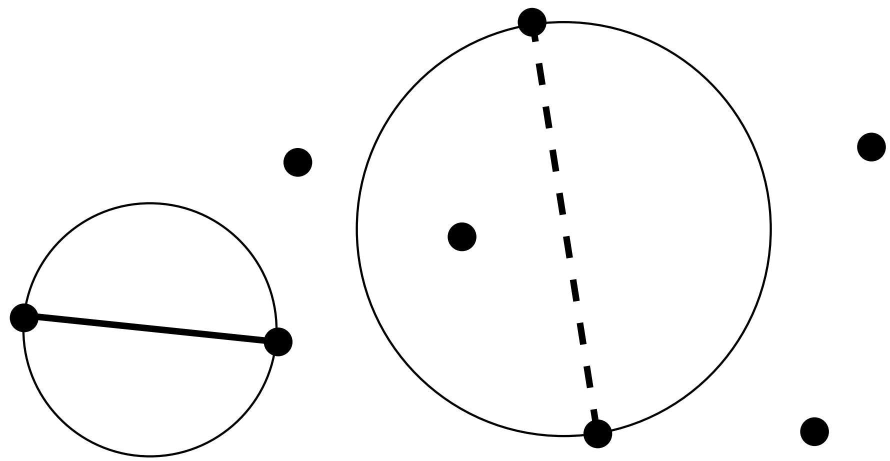
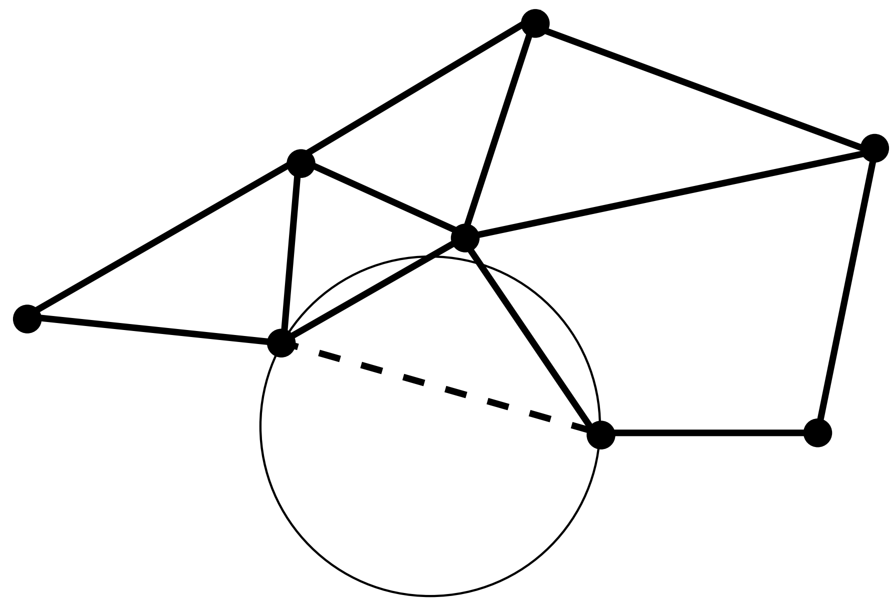


Not a Gabriel edge.

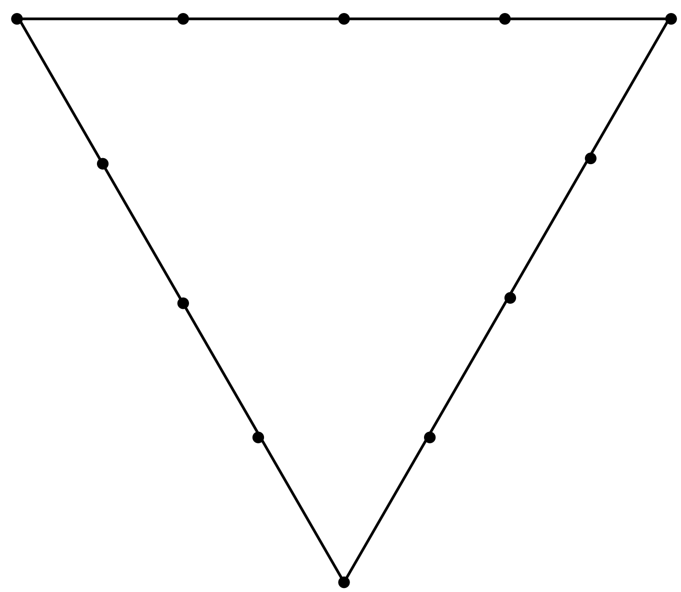
Gabriel edge is a special case of a **Delaunay** edge: $[v, w]$ is a **chord** of an open disk not hitting V.



Gabriel edges $\Rightarrow$ non-obtuse triangulation

Bern-Mitchell-Ruppert (1994)

BMR Lemma: Add k vertices to sides of triangle (at least one per side) so all edges become Gabriel, then add all midpoints. Resulting polygon has a $O(k)$ NOT, with **no additional vertices** on boundary.



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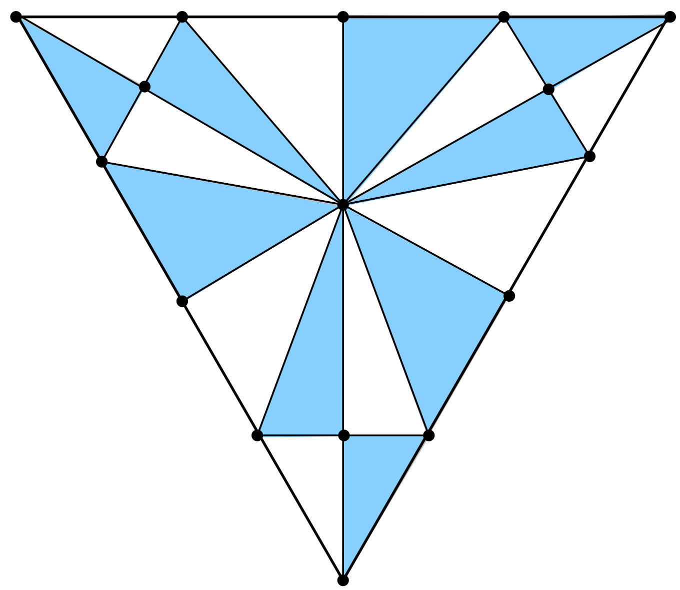

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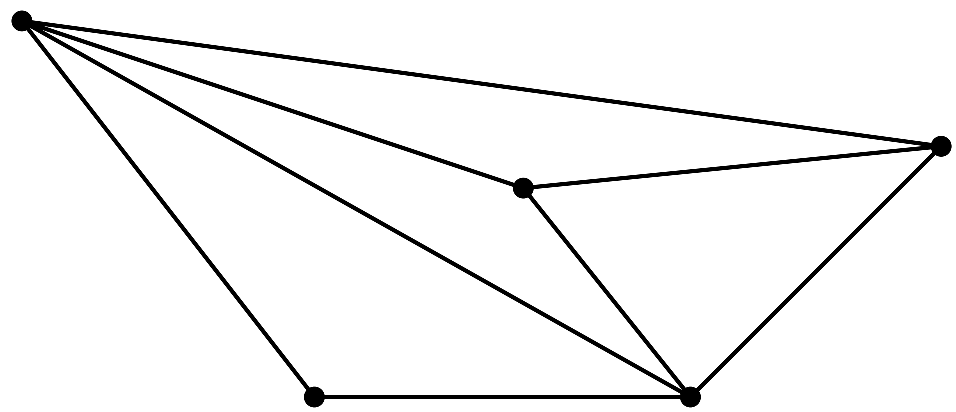
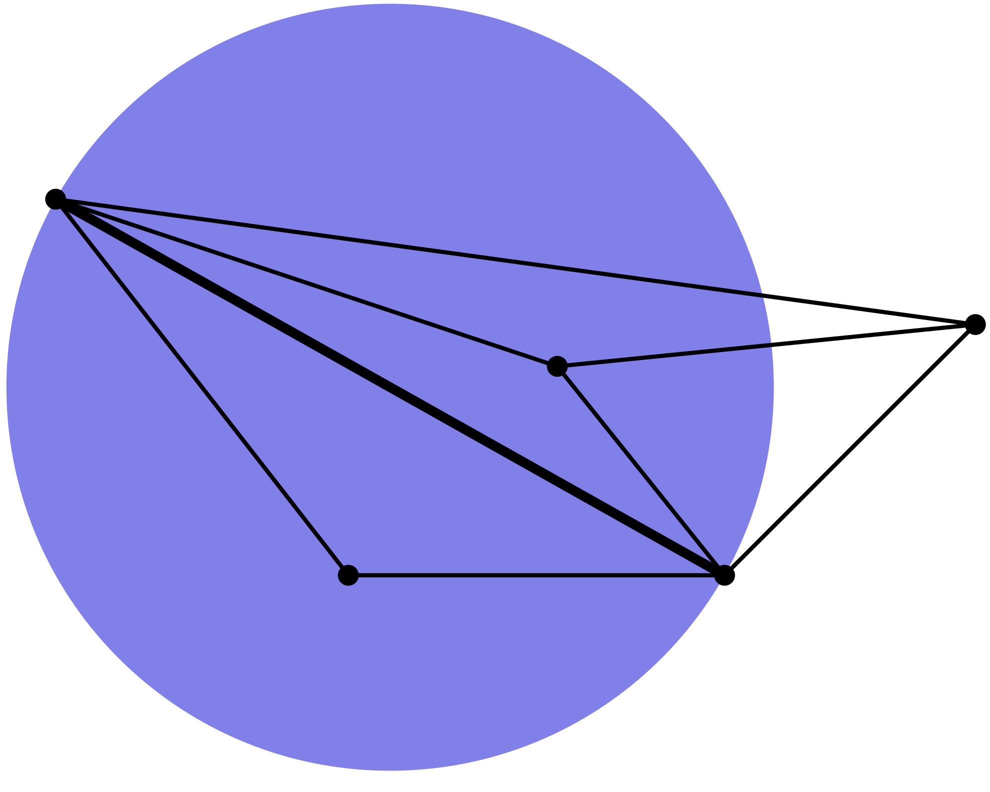
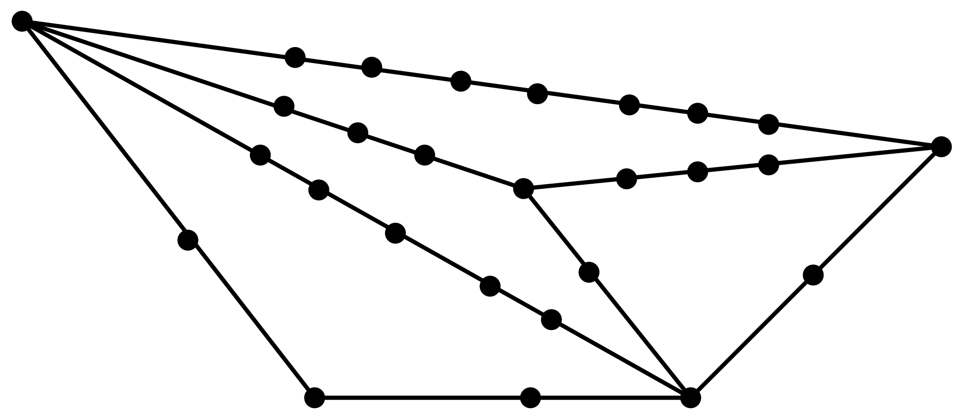
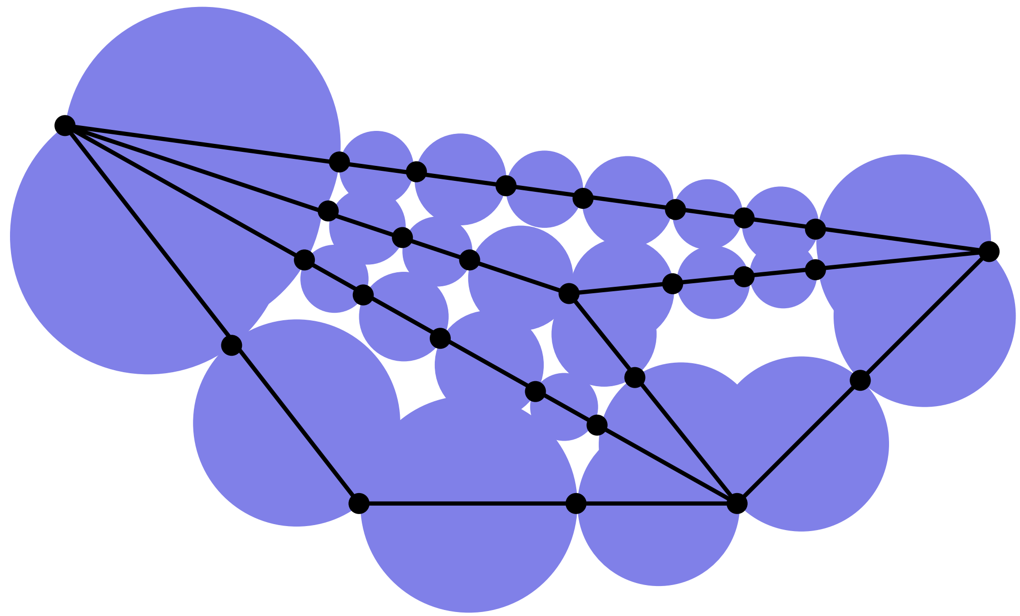
Building a NOT for a PSLG:

- Replace PSLG by triangulation of itself.
- Add vertices to make all edges Gabriel.
- Apply BMR lemma. Done.

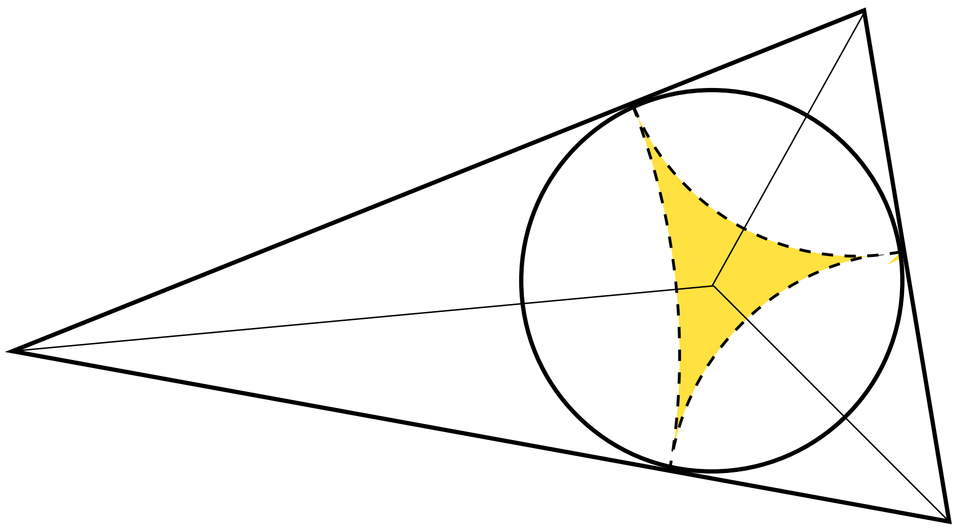








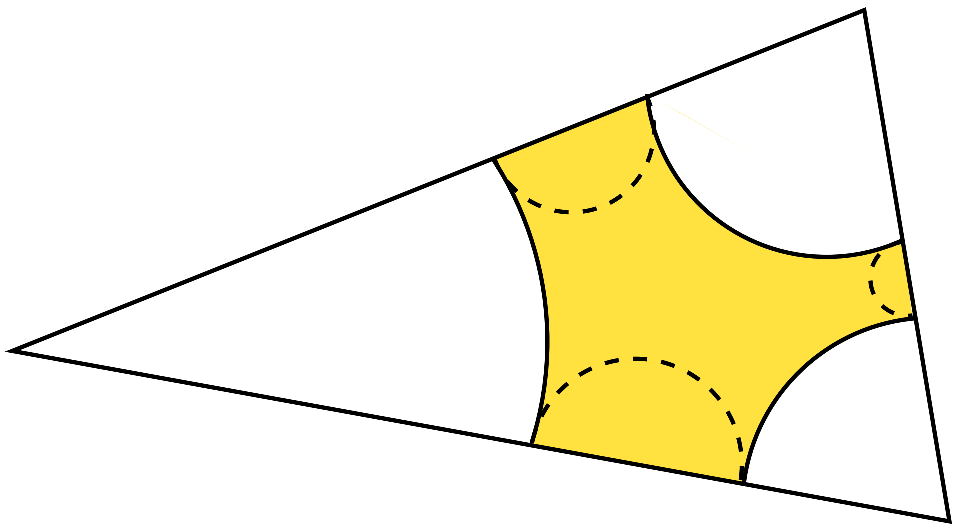
Construct Gabriel points:



Break every triangle into thick and thin parts.

Thin parts = corners,   Thick part = central region

Construct Gabriel points:

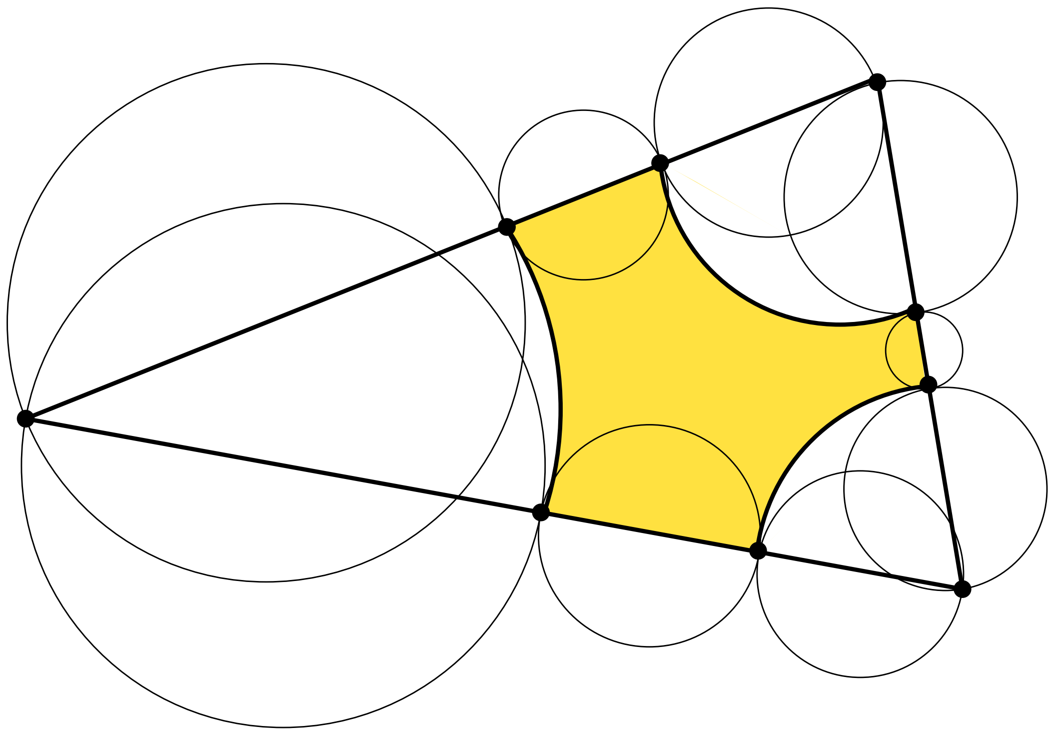


Divide triangle into thick and thin parts.

Thick sides are base of half-disk inside triangle.

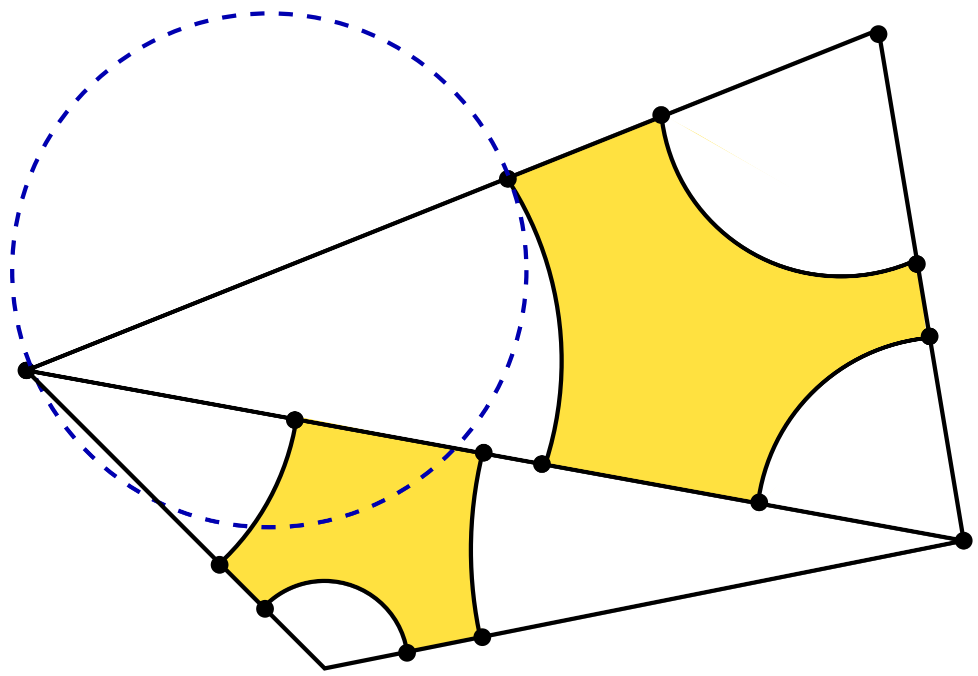
Thick version prevents infinite propagation.

Construct Gabriel points:



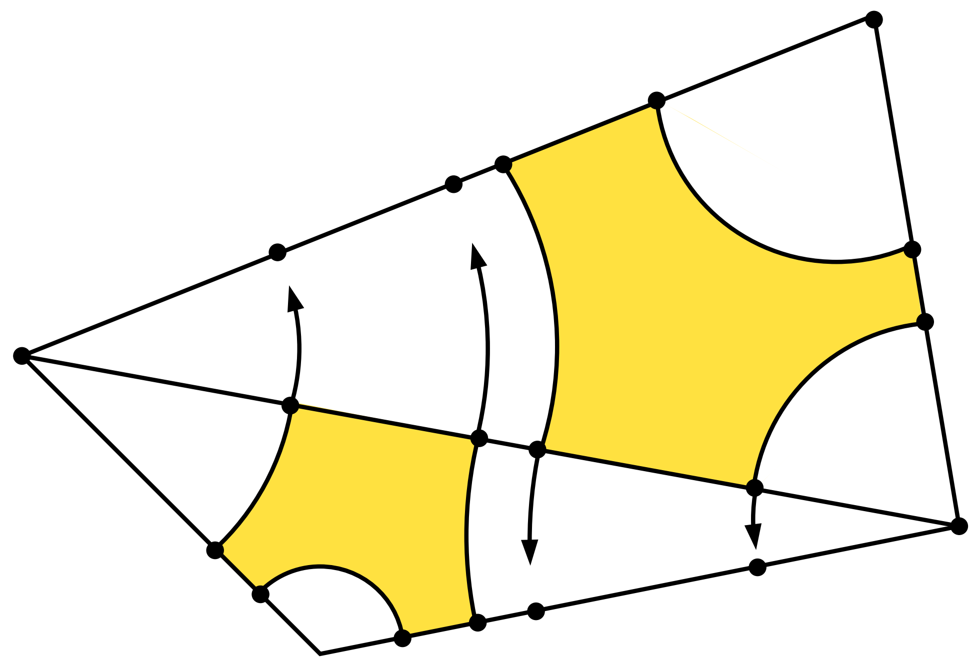
Easy to check that vertices of thick part give Gabriel edges.

Construct Gabriel points:



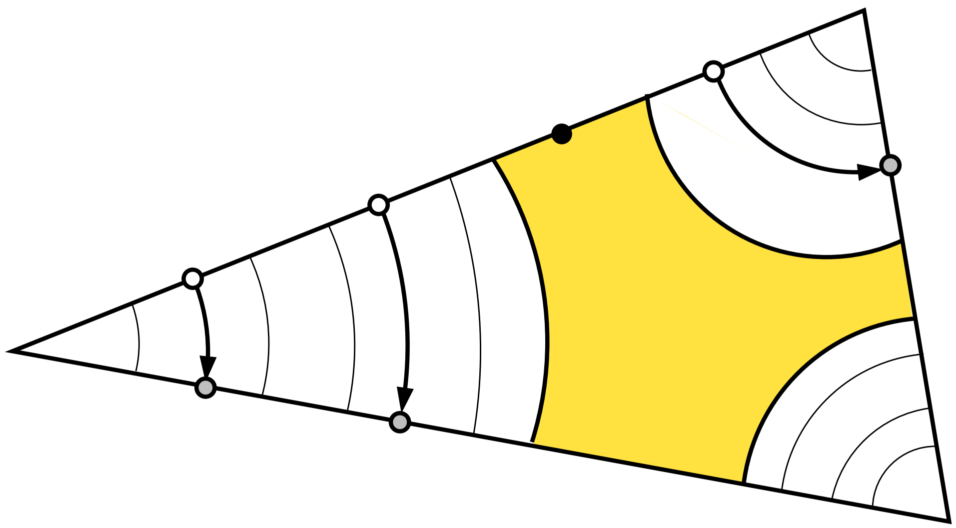
But, adjacent triangle can make Gabriel condition fail.

Construct Gabriel points:



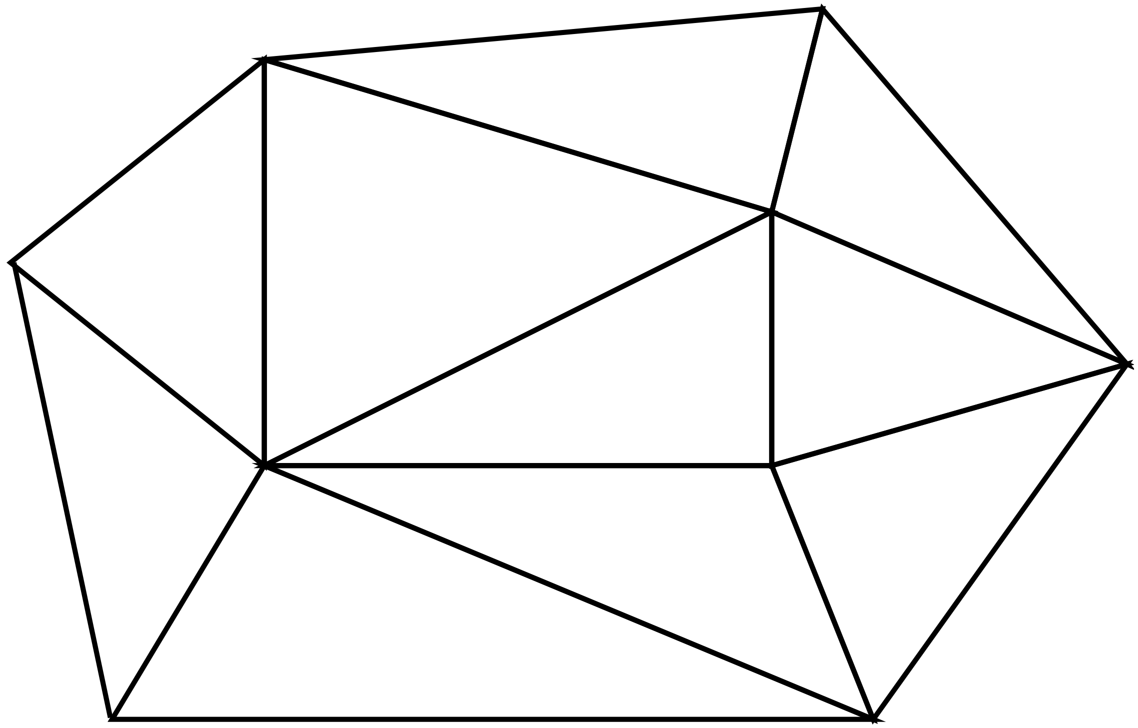
Idea: "Push" vertices across the thin parts.

Construct Gabriel points:

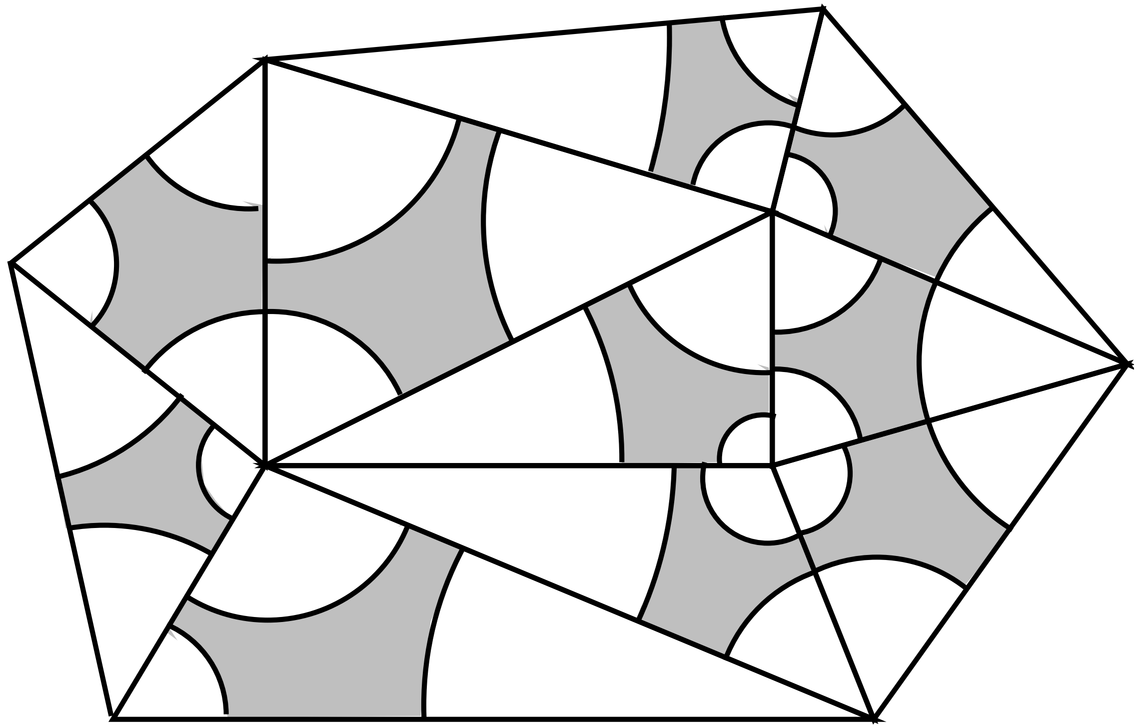


Thin parts foliated by circles centered at vertices.

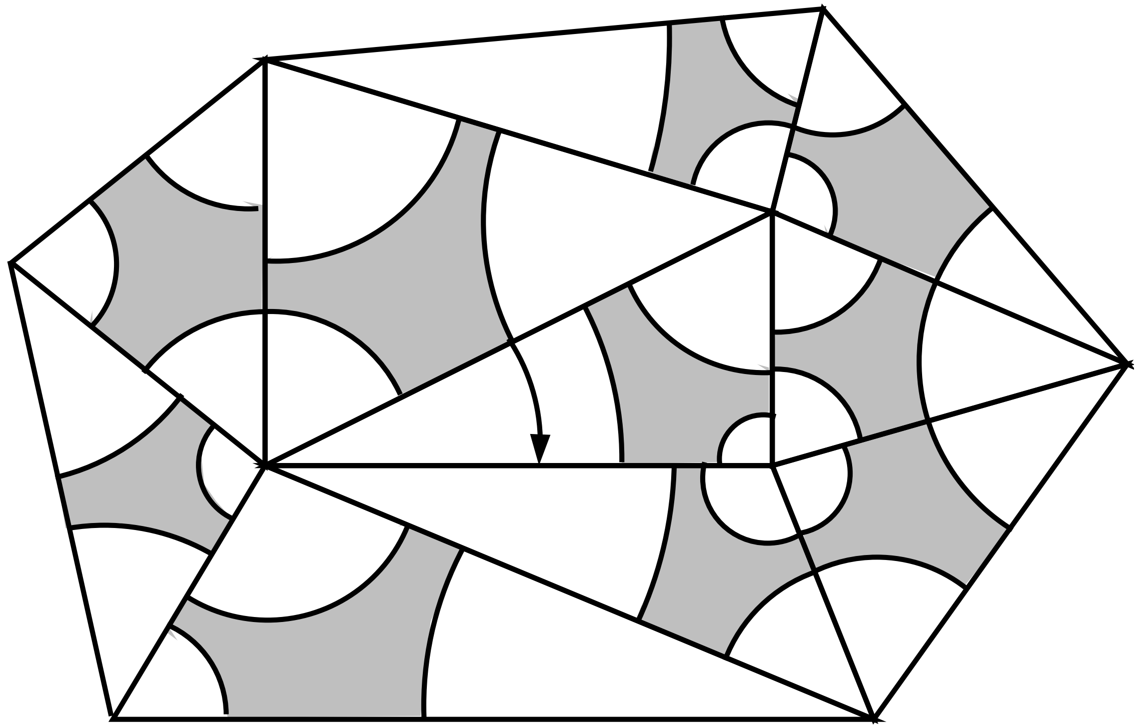
Push vertices along foliation paths.



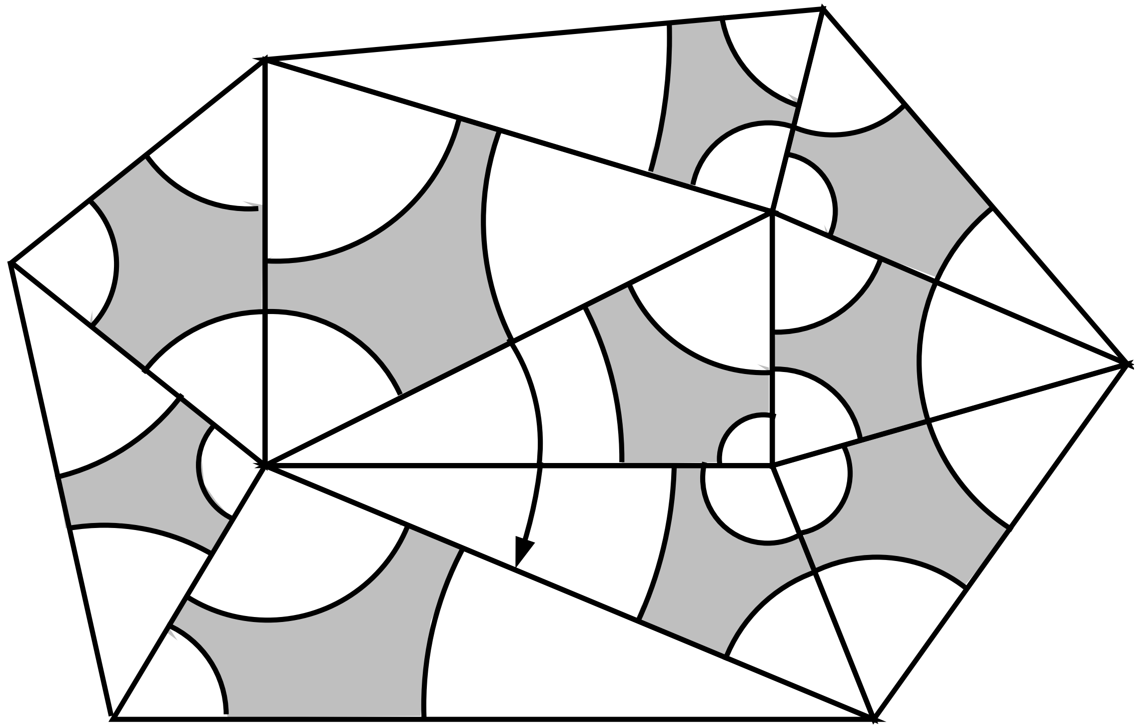
- Start with any triangulation.



- Start with any triangulation.
- Make thick/thin parts.



- Start with any triangulation.
- Make thick/thin parts.
- Propagate vertices until they leave thin parts.



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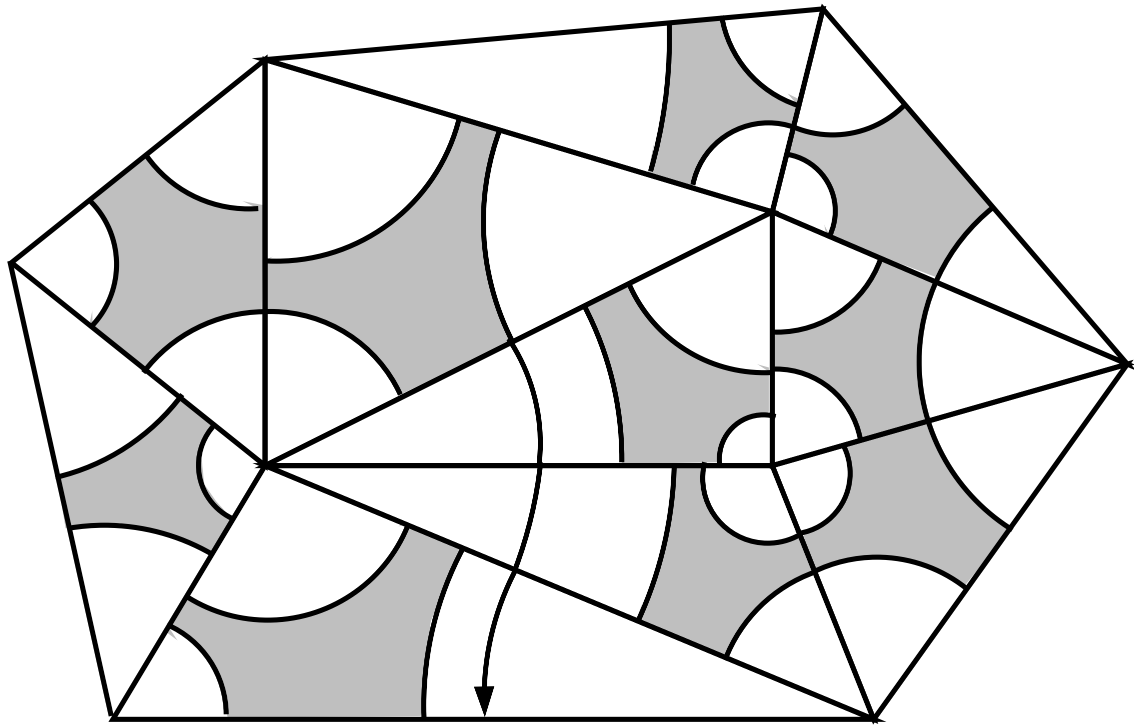

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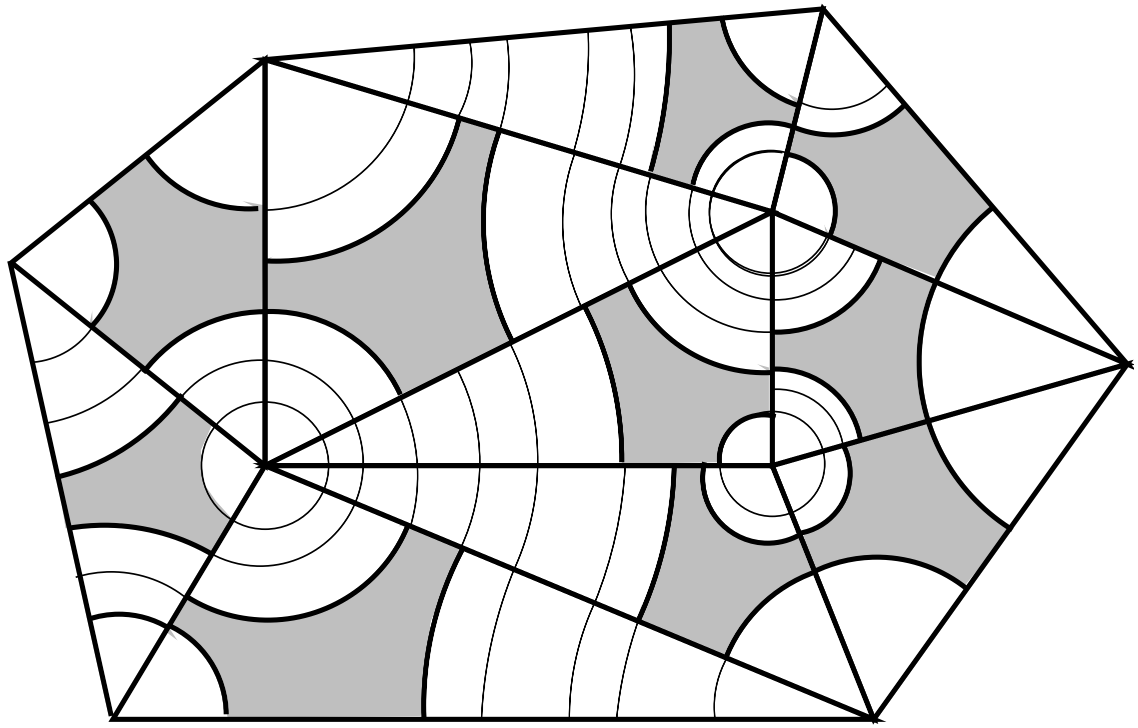

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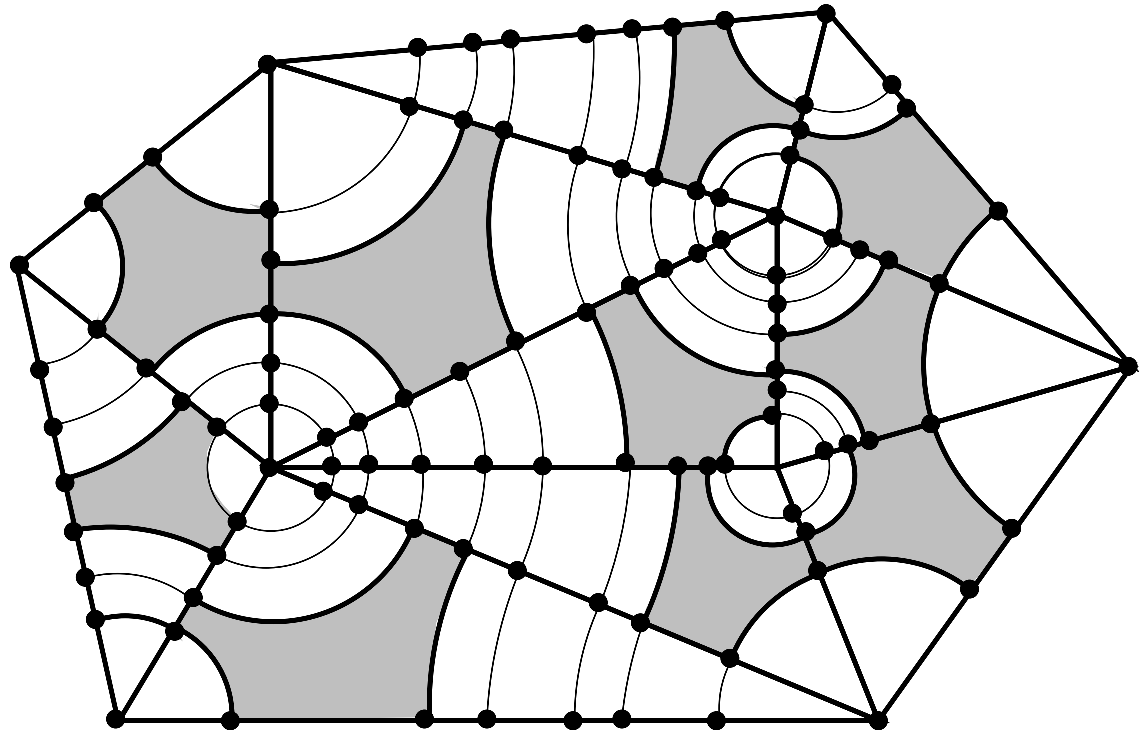
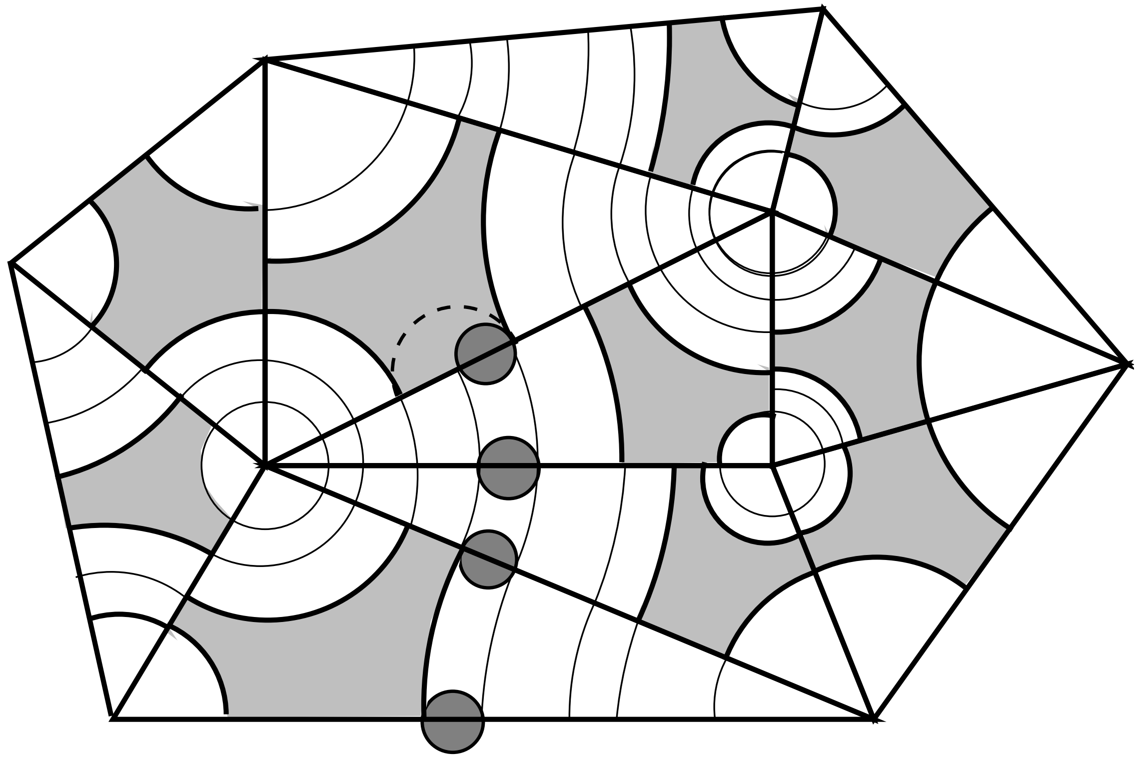
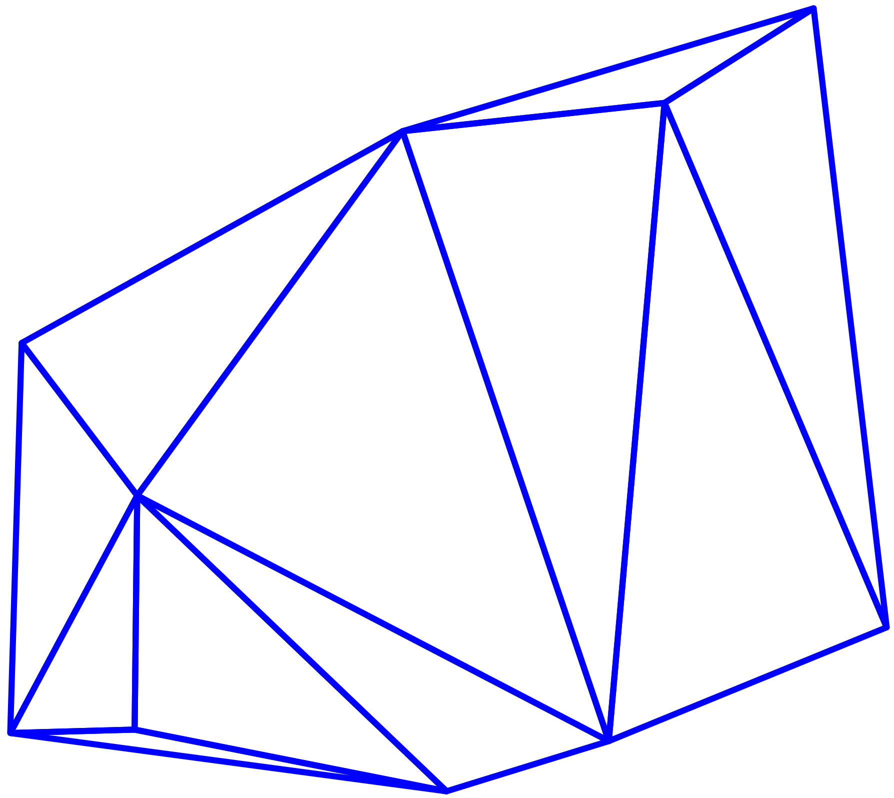


- Start with any triangulation.
- Make thick/thin parts.
- Propagate vertices until they leave thin parts.
- Intersections satisfy Gabriel condition. Why?

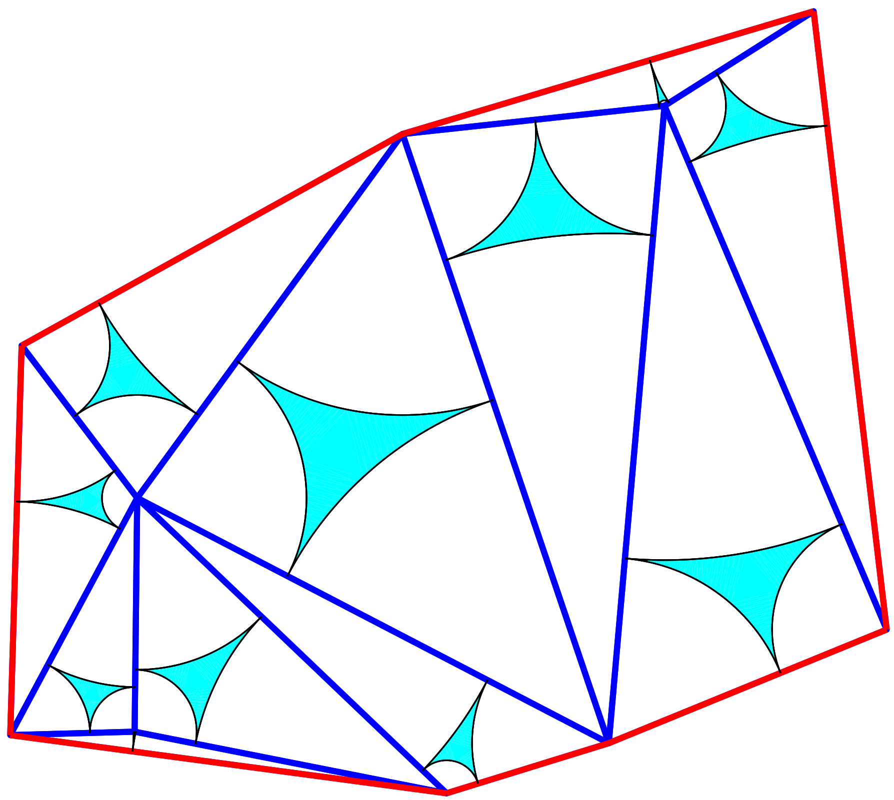


Tube is “swept out” by fixed diameter disk.

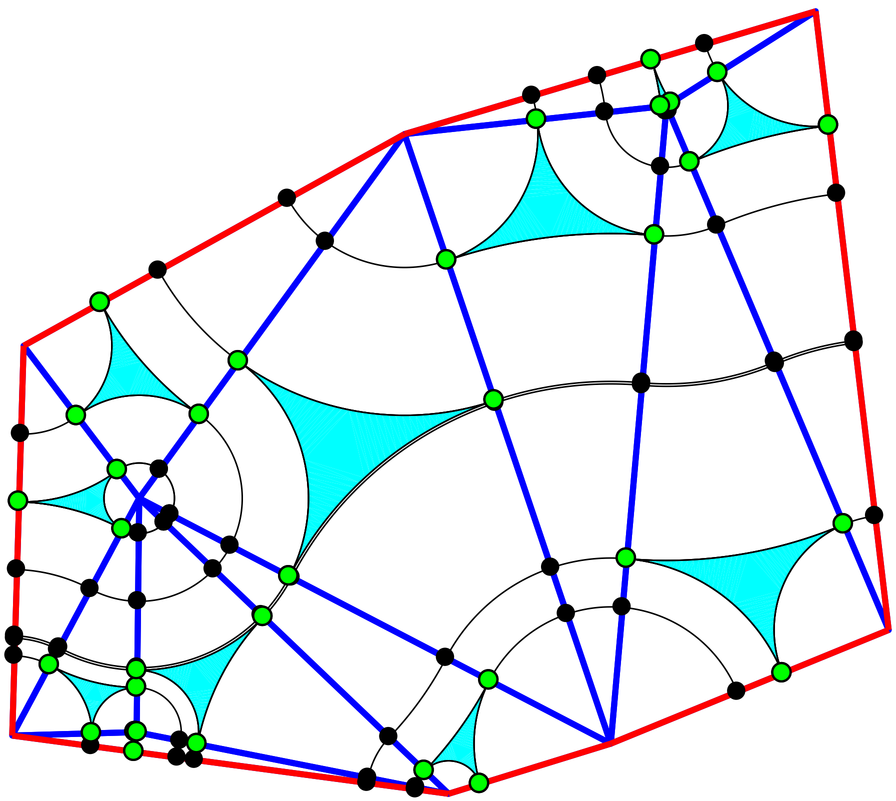
Disk lies inside tube or thick part or outside convex hull.



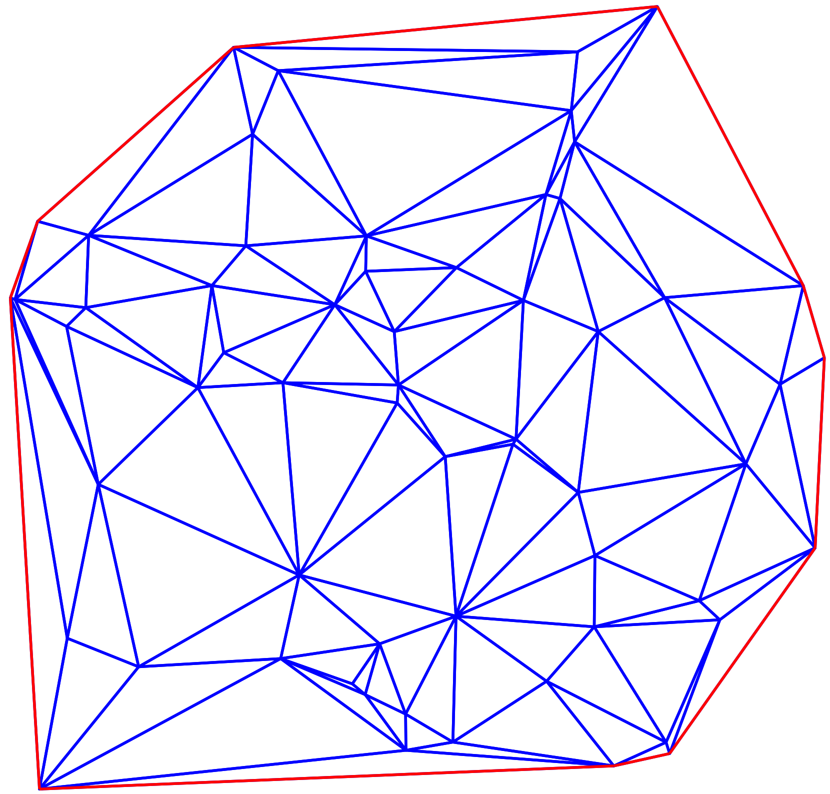
Delaunay triangulation of 10 random points,



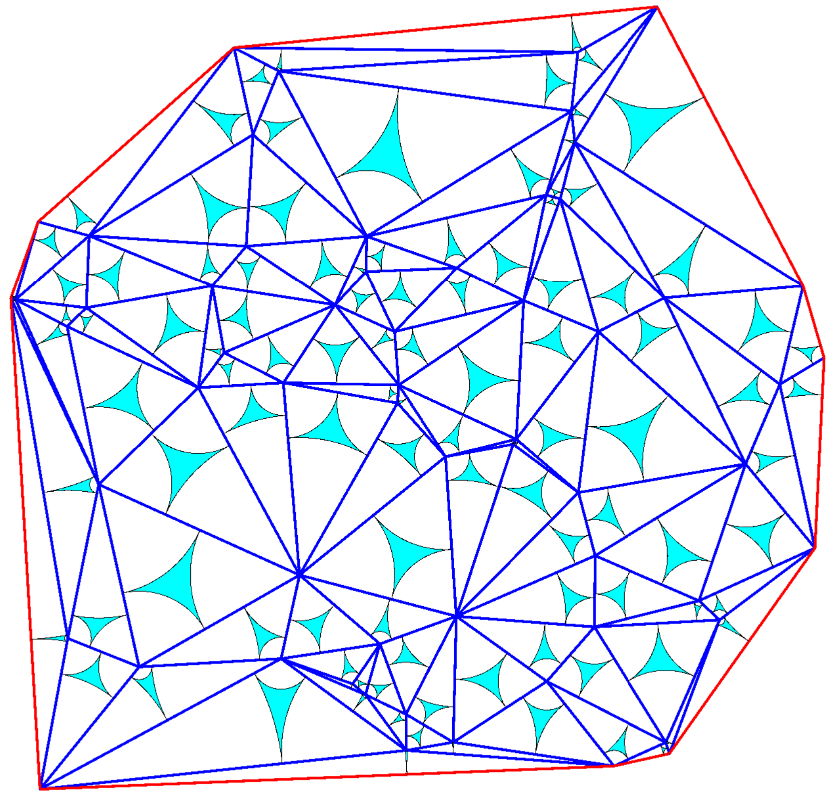
The central regions.



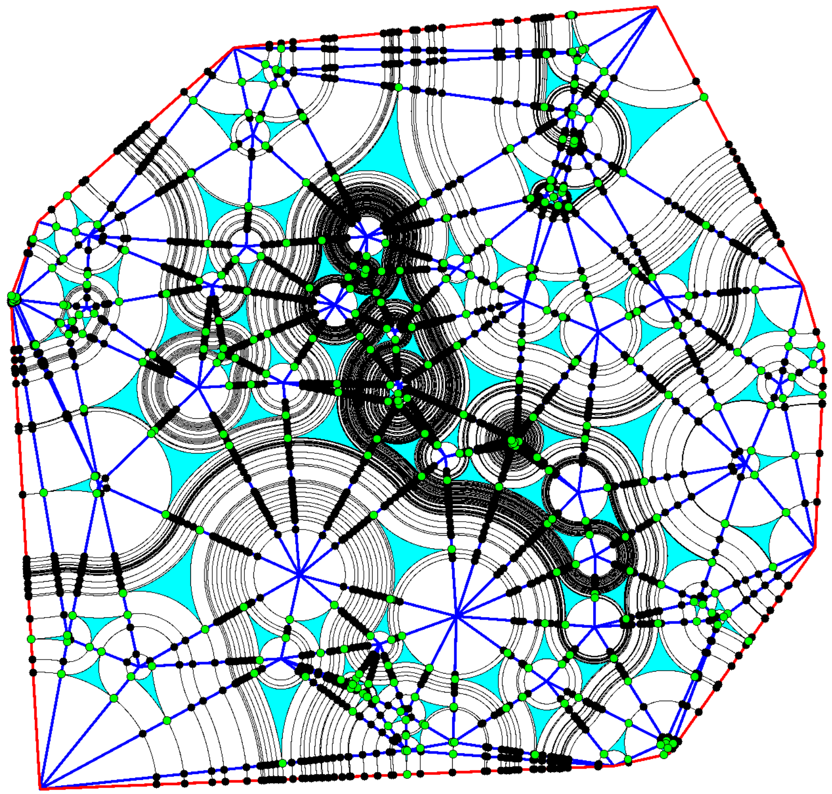
Propagation lines starting at all cusp points.



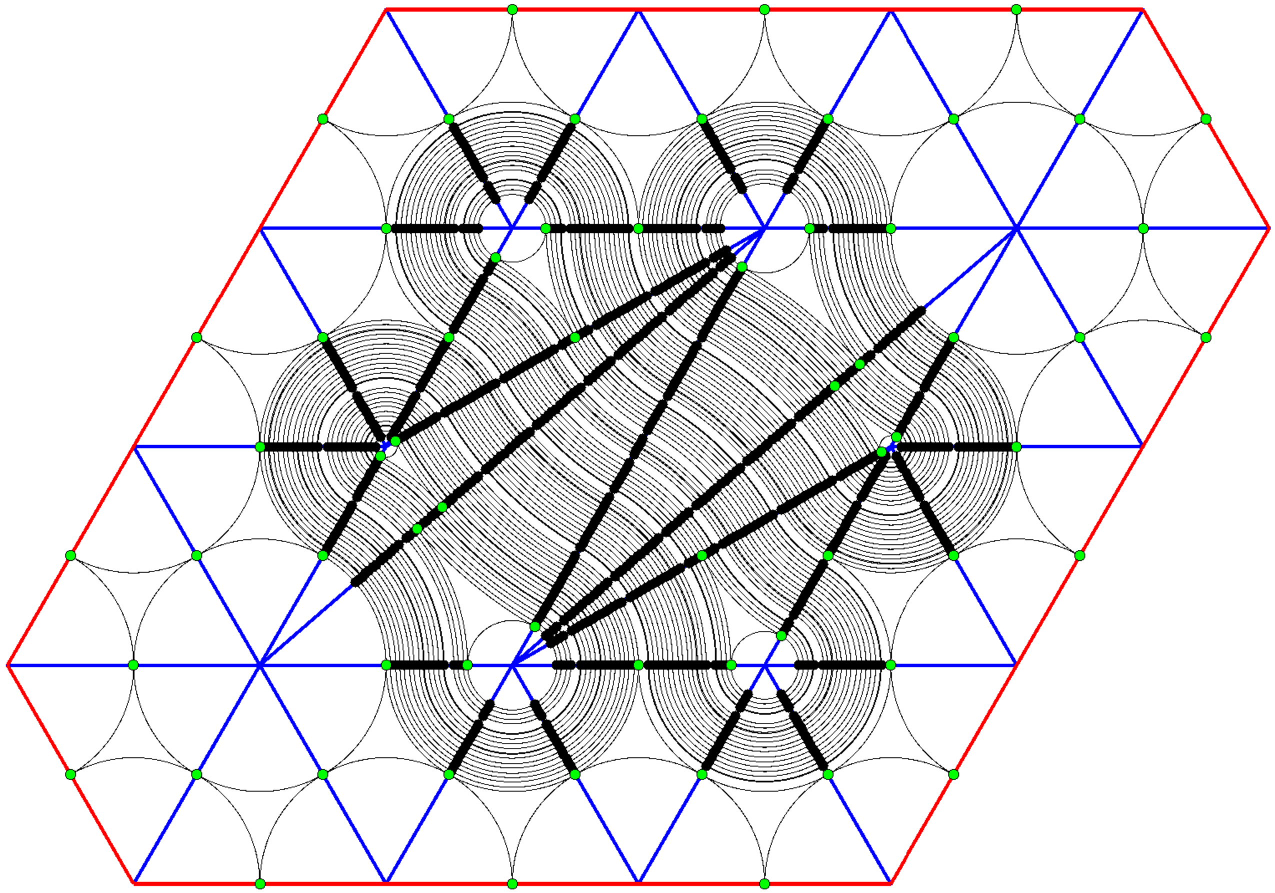
60 points



The central regions.



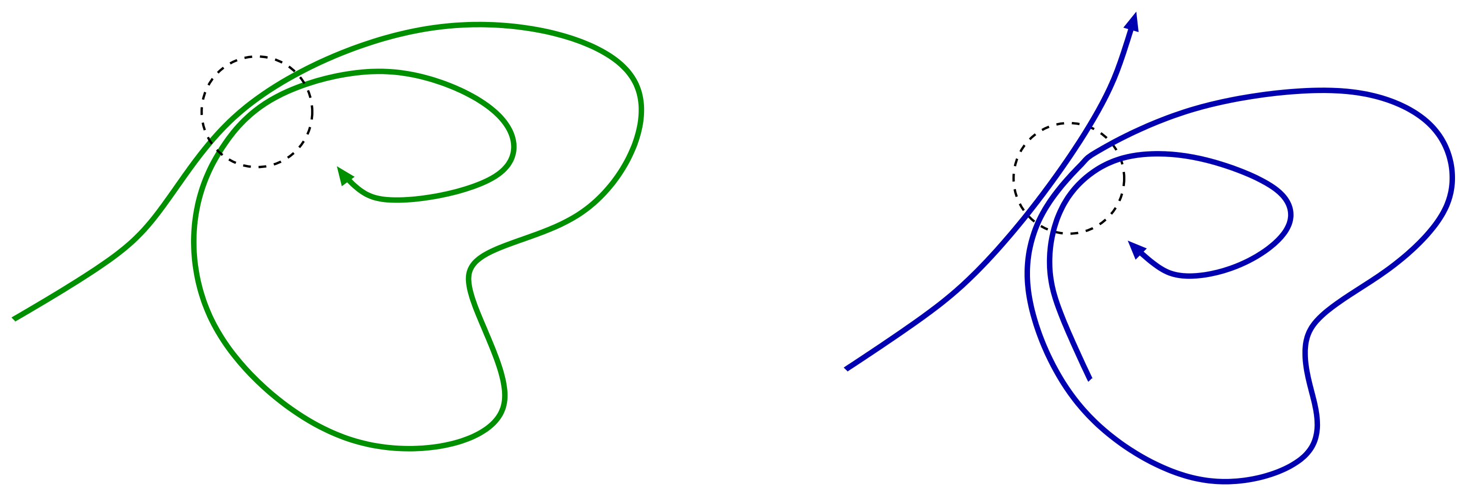
Propagation lines starting at all cusp points.
In random case, each flow line hits $\sim n^{1.5}$ triangles on average.



In special cases, flows never stop.

All paths end if central regions have sides, not cusps.

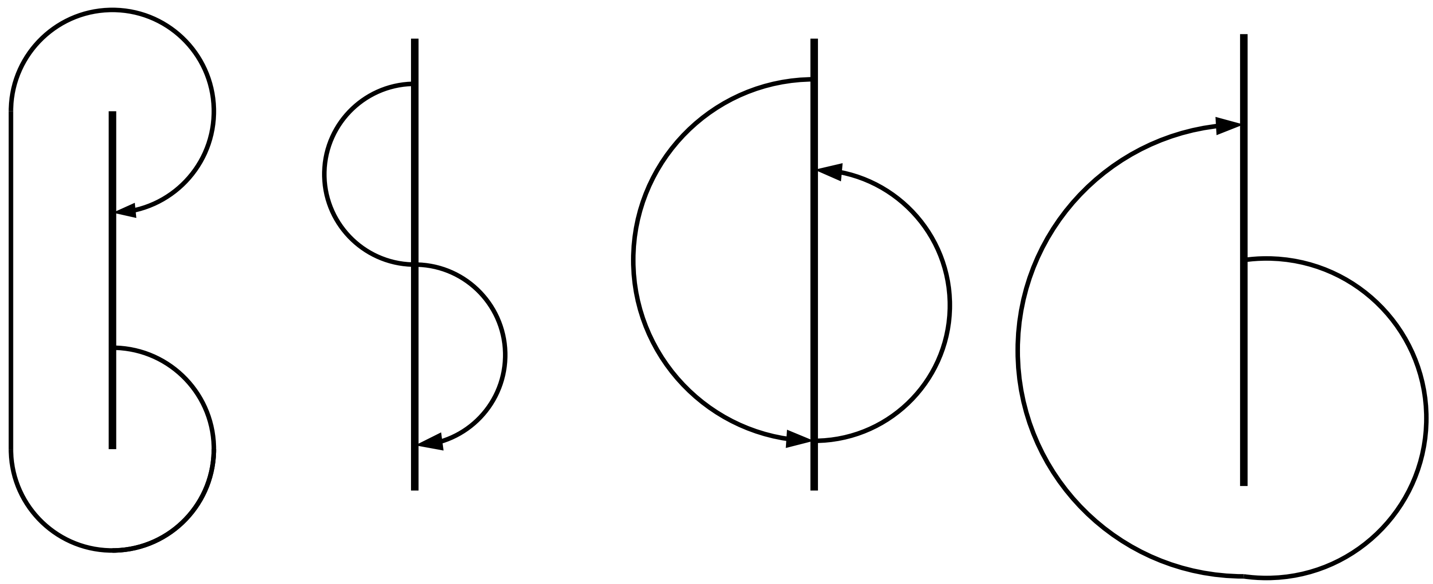
Closing lemma (Pugh, 1967): given vector field on a surface, and a flow line that returns arbitrarily close to itself, we can make a C^1 perturbation to create a closed loop.



This remains open for C^2 perturbations.

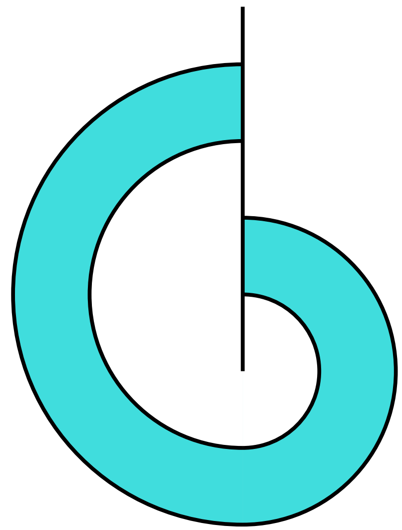
Discrete version: perturb so each flow line hits only $O(n)$ triangles, but still gives Gabriel points.

If a path returns to same thin edge at least 3 times it has a sub-path that looks like one of these:



C-curve, S-curve, G-curves

Return region consists of paths “parallel” to one of these.



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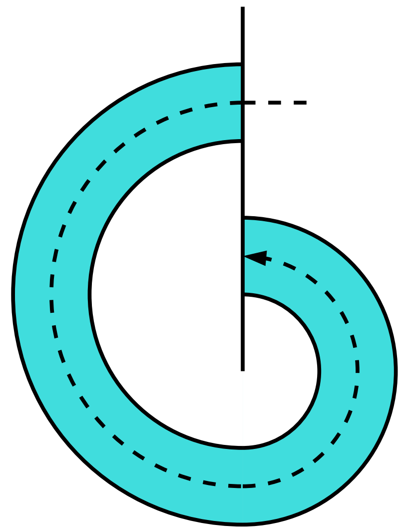
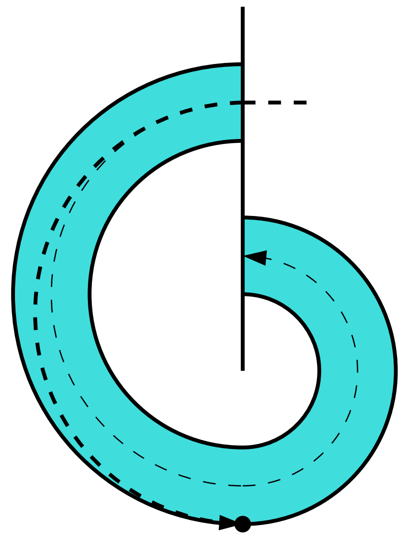


There are $O(n)$ return regions and every propagation path enters one after crossing at most $O(n)$ thin parts.

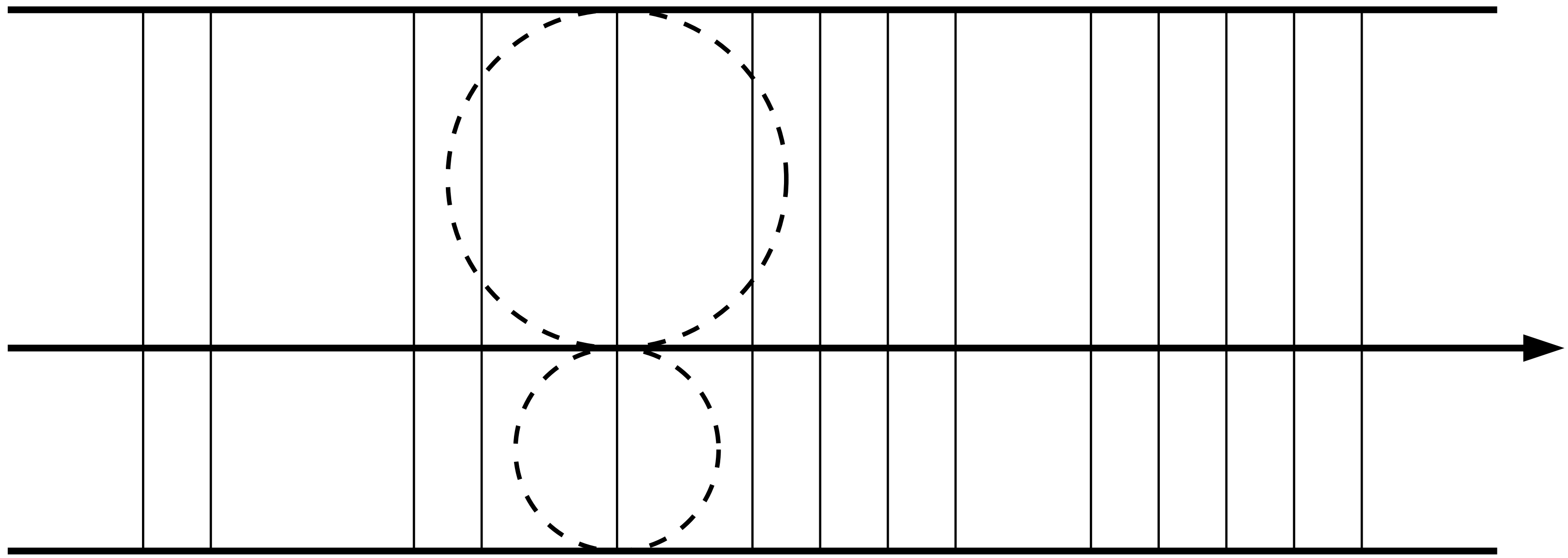
Return region consists of paths "parallel" to one of these.

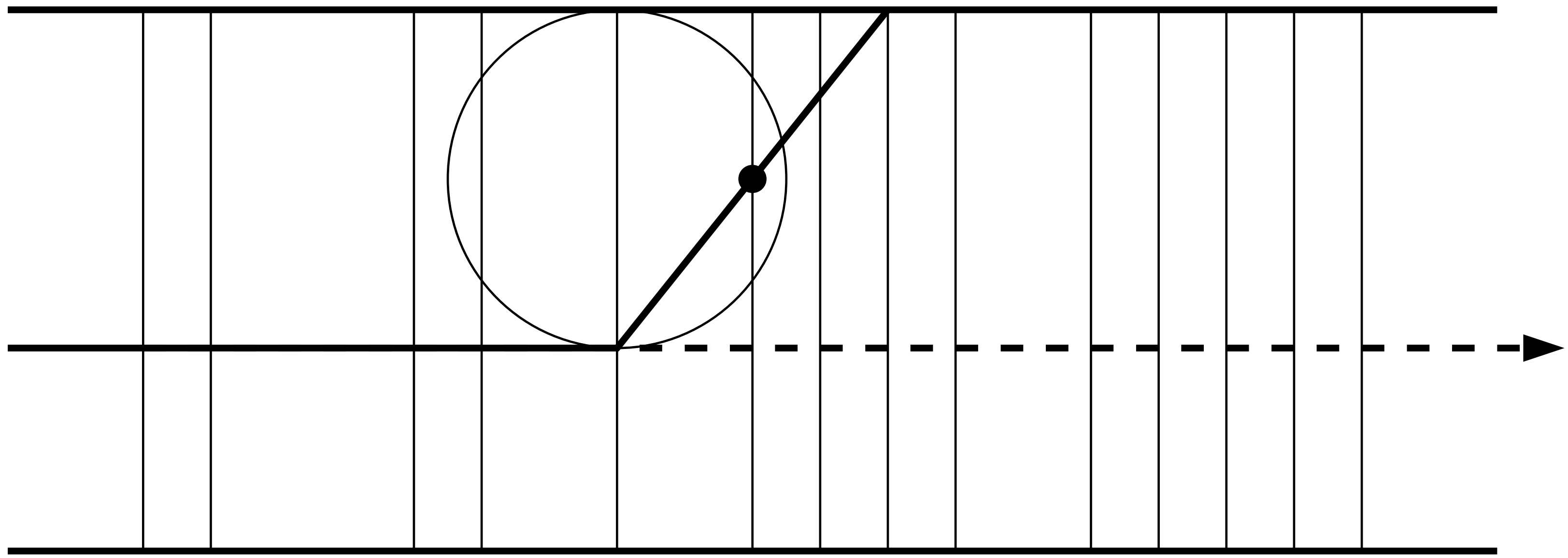


There are $O(n)$ return regions and every propagation path enters one after crossing at most $O(n)$ thin parts.

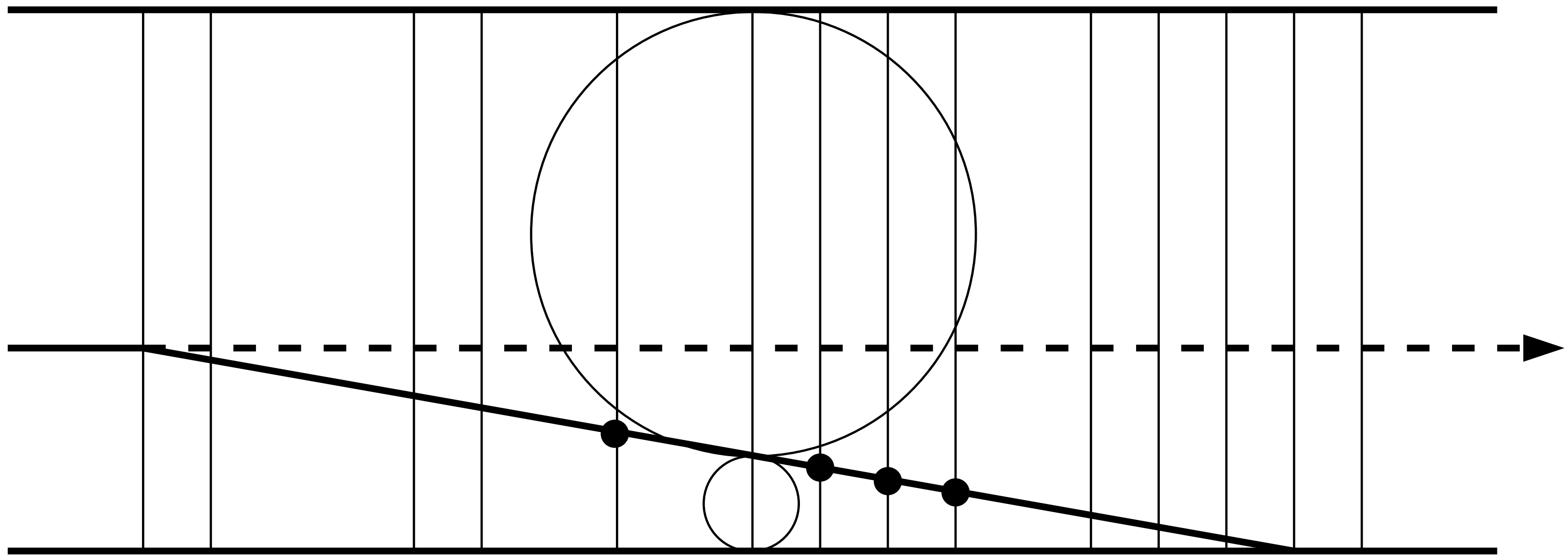
IDEA: bend paths to terminate before they exit.

Gives $O(n^2)$ if it works.

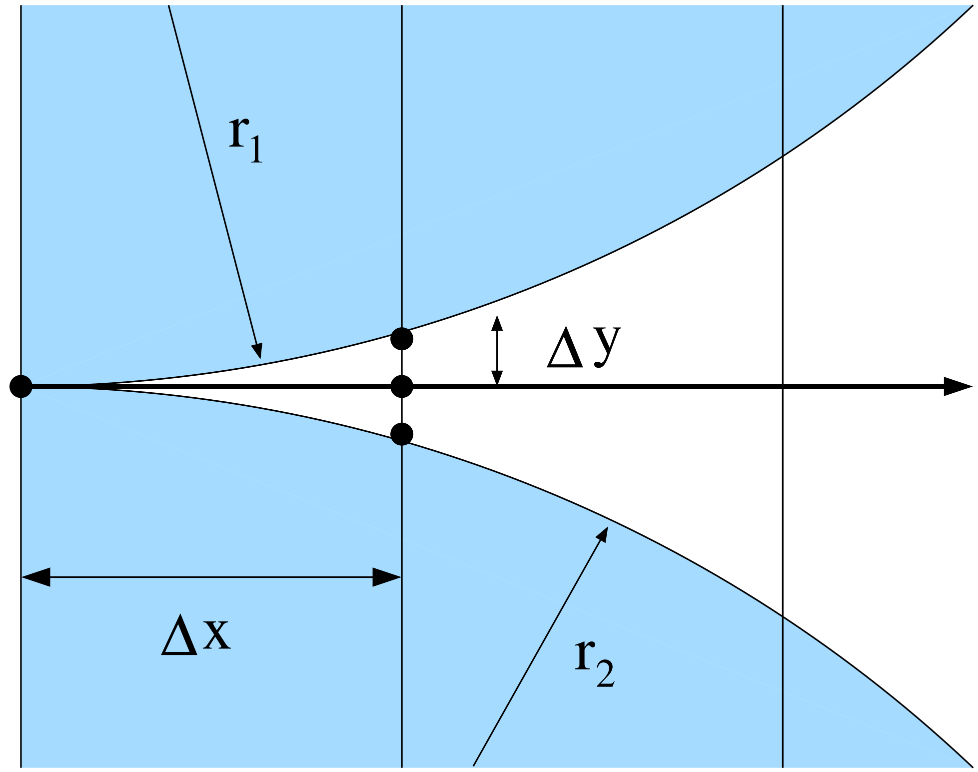




If path bends too fast, Gabriel condition can fail.

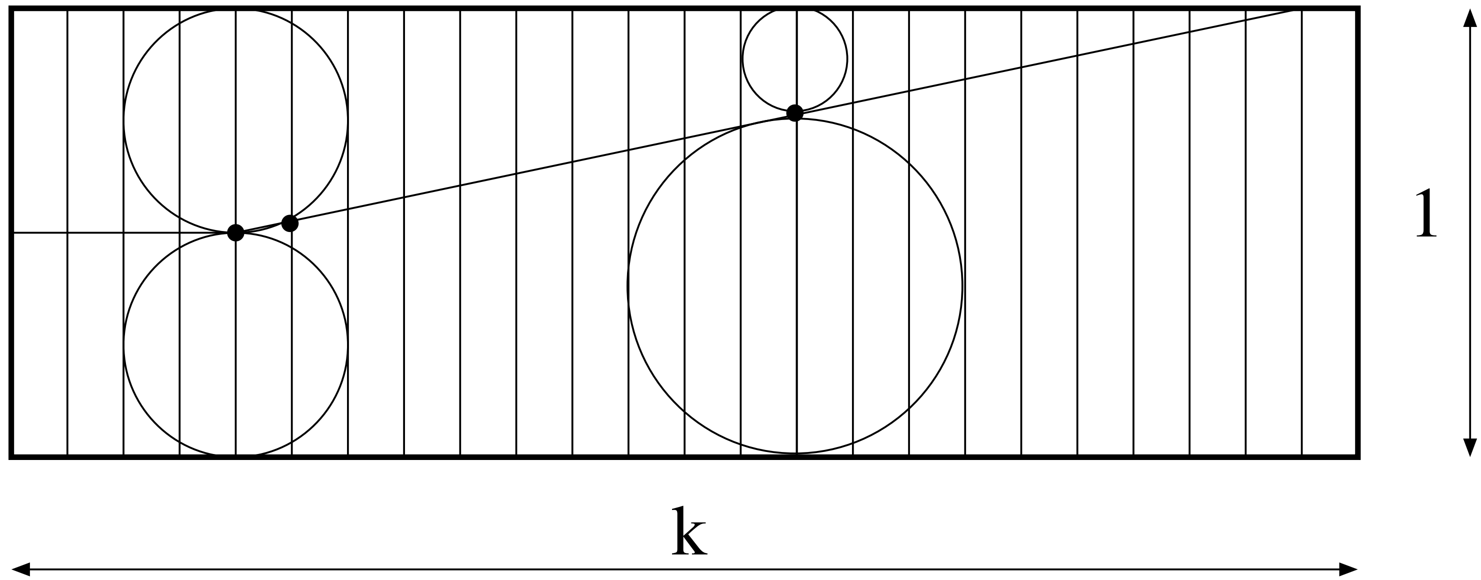


Bend slowly enough to satisfy Gabriel condition.



$$\Delta y \approx (\Delta x / r)^2 r = (\Delta x)^2 / r.$$

$$r = \max(r_1, r_2).$$



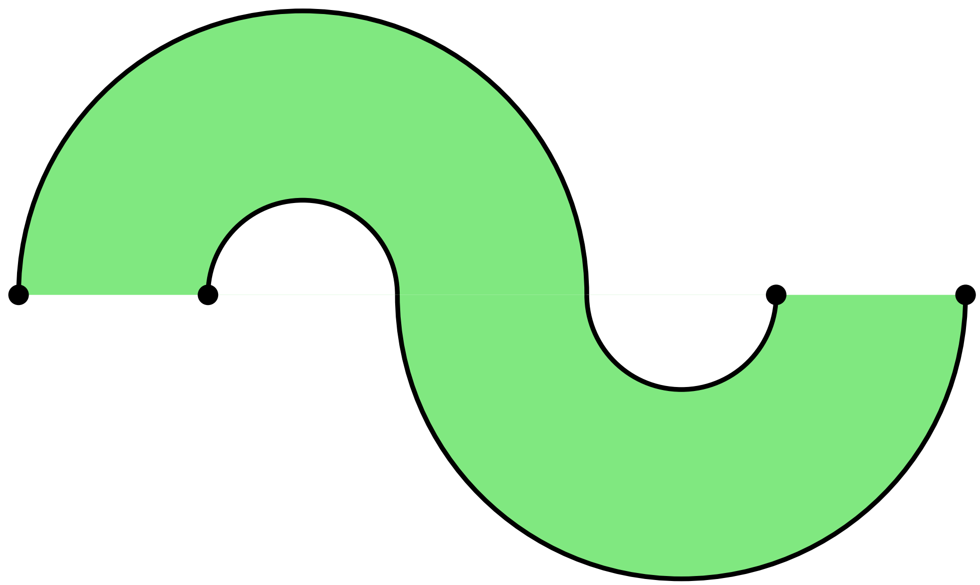
$k \times 1$ region crossing n (equally spaced) thin parts,

$r \approx 1, \quad \Delta x \approx k/n, \quad \Rightarrow \quad \Delta y \approx k^2/n^2$

Need $1 \leq \sum \Delta y = n\Delta y = k^2/n$.

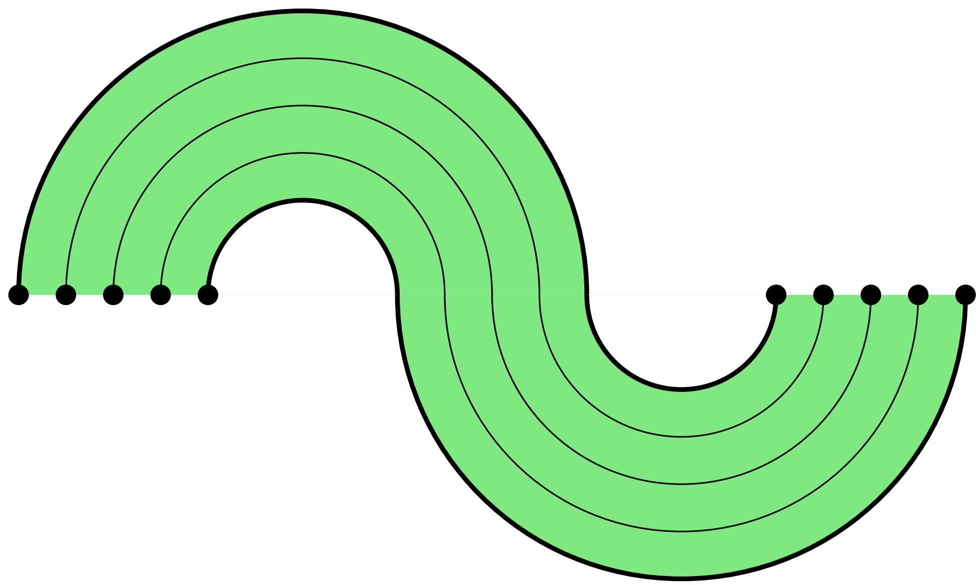
Bent path hits side of region if $k \gg \sqrt{n}$.

Easy case: return region length $>$ width.



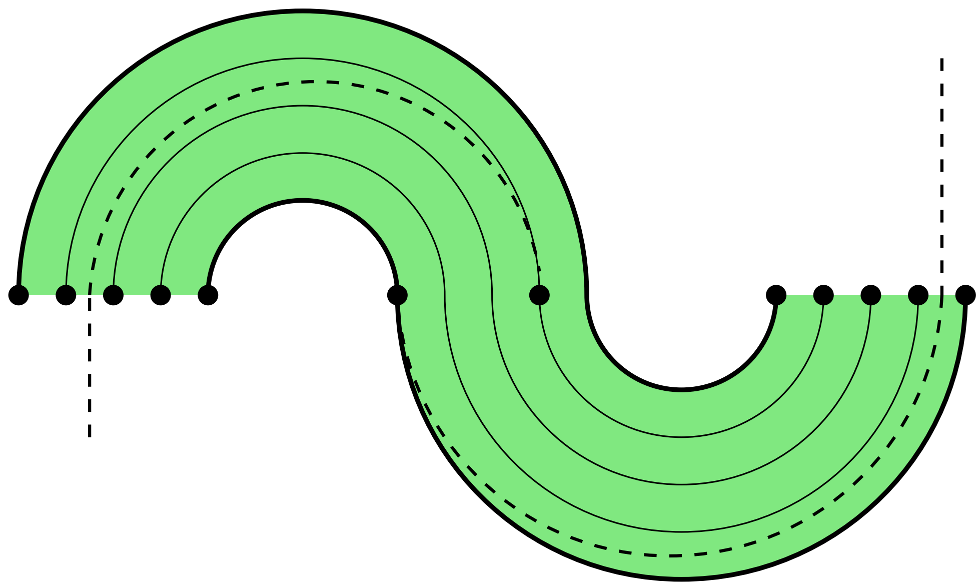
- Show there are $O(n)$ return regions.

Easy case: return region length > width.



- Show there are $O(n)$ return regions.
- Divide each region into $O(\sqrt{n})$ long parallel tubes.

Easy case: return region length > width.



- Show there are $O(n)$ return regions.
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- Entering paths can be bent and terminated.
  Total vertices created $= O(n^2)$, but ...

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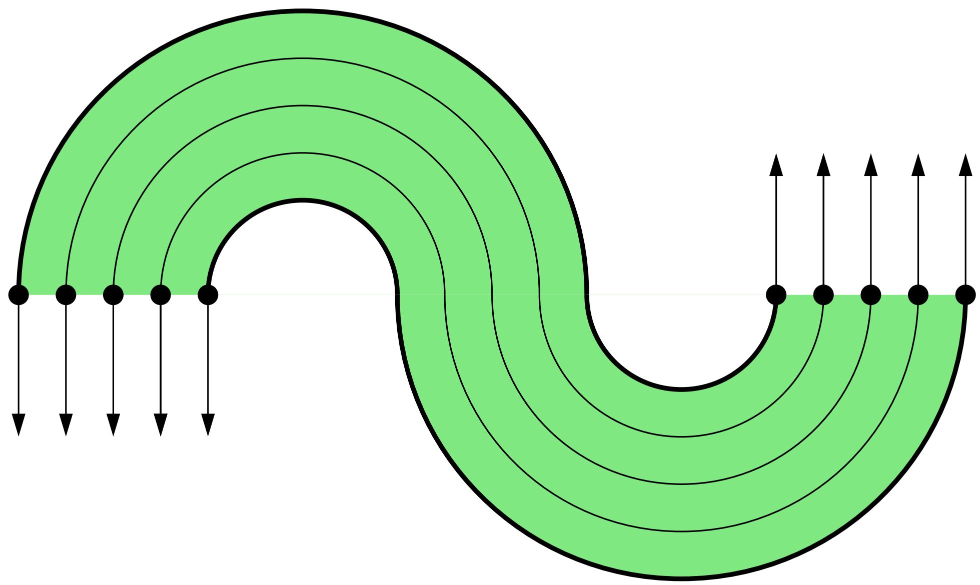
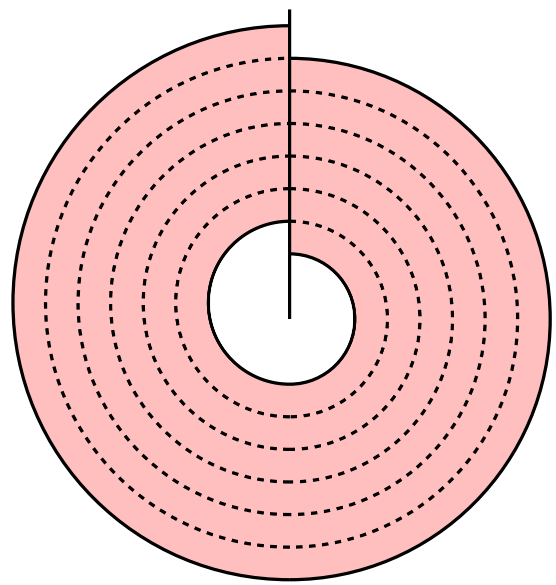


- Show there are $O(n)$ return regions.
- Divide each region into $O(\sqrt{n})$ long parallel tubes.
- Entering paths can be bent and terminated.
  Total vertices created $= O(n^2)$, but ...
- Each region has $O(\sqrt{n})$ new vertices to propagate.
  Vertices created is $O(\sqrt{n} \cdot n^2) = O(n^{2.5})$.

Hard case is spirals:



Curves may spiral arbitrarily often.

No curve can be allowed to pass all the way through the spiral.

Stop them in a multi-stage construction.

Can we do better than 90° for individual PSLGs?

Theorem: There is a $\theta_0 > 0$ so that a PSLG with all angles $\geq \theta$, has a triangulation will all angles $\leq 90^\circ - \min(\theta, \theta_0)/2$.

Cor: Any triangulation with minimal angle θ has an acute refinement with maximum angle $\leq 90^\circ - \min(\theta, \theta_0)/2$.

Corollary answers question of Florestan Brunck and Piotr Przyrycki.

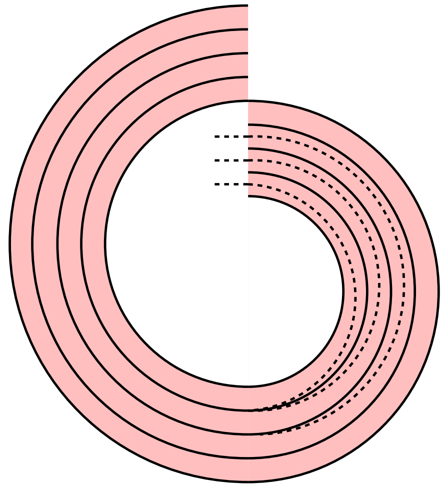
For polygons we saw the theorem holds for $\theta_0 = 36^\circ$.

Sharp θ_0 open for PSLGs.

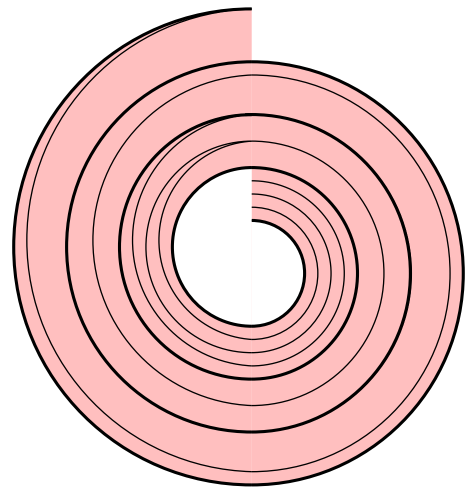
- Compute optimal angle bound for triangulating a given PSLG.
- Find optimal angle bounds for quad-mesh of a given polygon (or PSLG).
- Prove $O(n^2)$ for NOTs of PSLGs.
- Estimate minimal size of NOT for a given PSLG.
- Smallest triangulation with optimal angle bound? (PSLGs or polygons)
- Compute probability that random polygon has a $72°$-triangulation.
- Better bound for dissections of PSLGs than for triangulations?
- Faster Delaunay triangulations of PSLGs than NOTs?

DT or NOT-DT, that is the question.
Whether 'tis nobler in the mind to suffer
the slings and arrows of obtuse angles,
or take arms against a sea of paths,
and by perturbing end them?

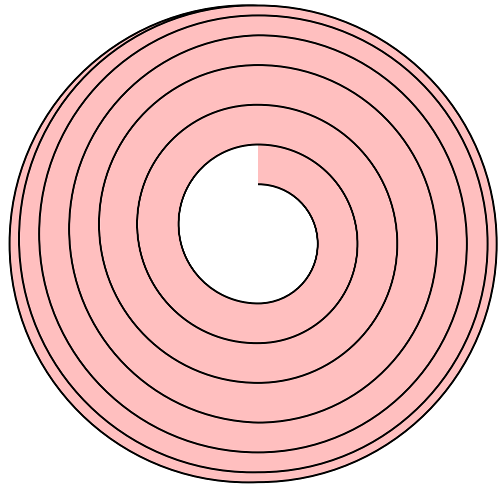
Thanks for listening. Questions?



- Start with $\sqrt{n}$ parallel tubes at entrance of spiral. Terminate entering paths (1 spiral).



- Start with $\sqrt{n}$ parallel tubes at entrance of spiral. Terminate entering paths (1 spiral).
- Merge $\sqrt{n}$ tubes to single tube ($n^{1/3}$ spirals). (get longer as we move out.)



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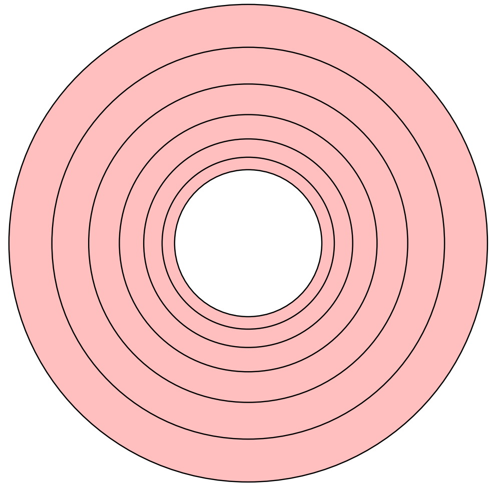
  (spirals get longer as we move out.)
- Make tube edge self-intersect ($n^{1/2}$ spirals)



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- Merge $\sqrt{n}$ tubes to single tube ($n^{1/3}$ spirals). (spirals get longer as we move out.)
- Make tube edge self-intersect ($n^{1/2}$ spirals)
- Loops with increasing gaps ($n^{1/2}$ loops, n spirals)
- Beyond radius n spiral is empty.

Careful estimates needed to get $O(n^{2.5})$ vertices.