Conformal Mapping in Linear Time

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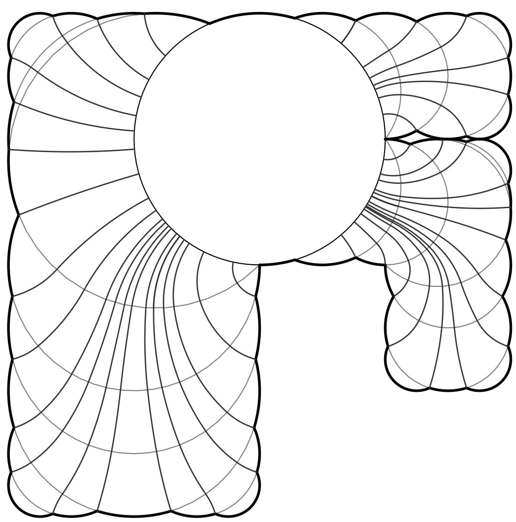
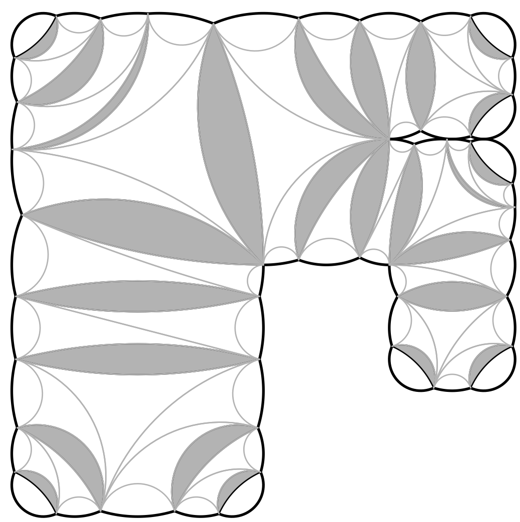
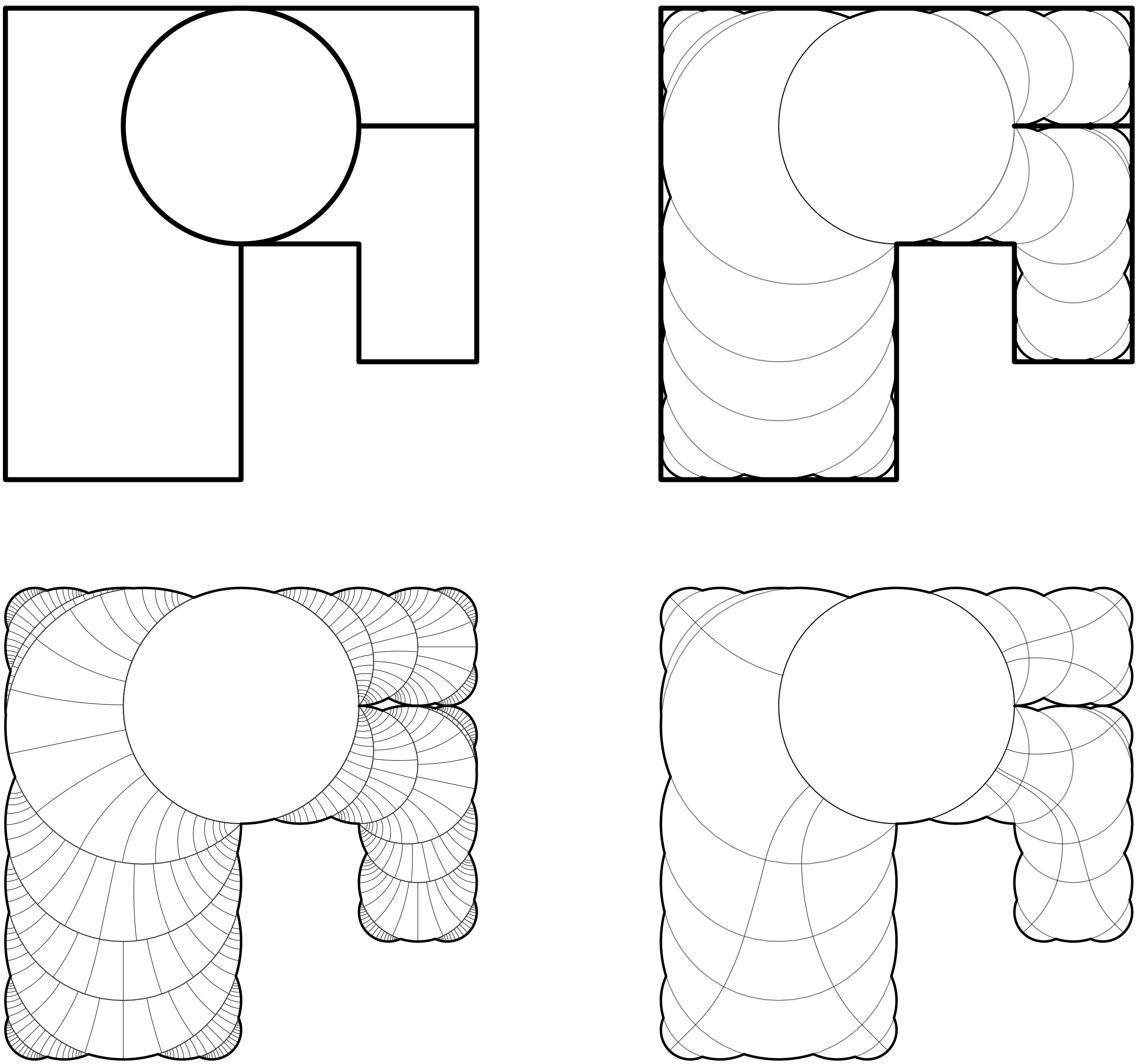




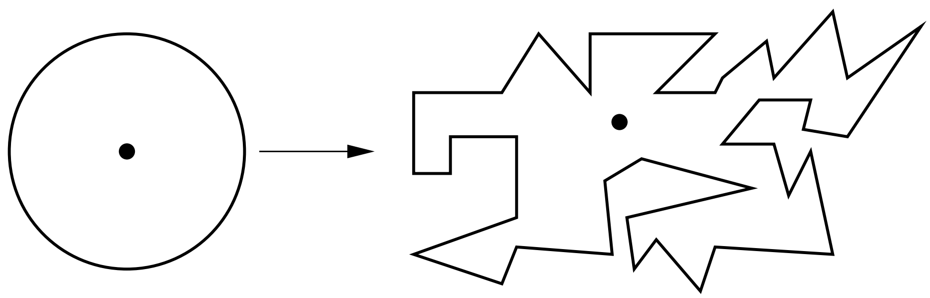
copies of lecture slides available at

`www.math.sunysb.edu/~bishop/lectures`



- Has simple geometric definition
- Only requires a “tree-of-disks” to define.
- Is stable; limit exists as disks fill in polygon.
- Fast to compute using medial axis.
- Is uniformly close to Riemann map.
- Can be used to compute Riemann map quickly.
- Definition motivated by hyperbolic 3-manifolds.
- Extends to Lipschitz map of interiors.

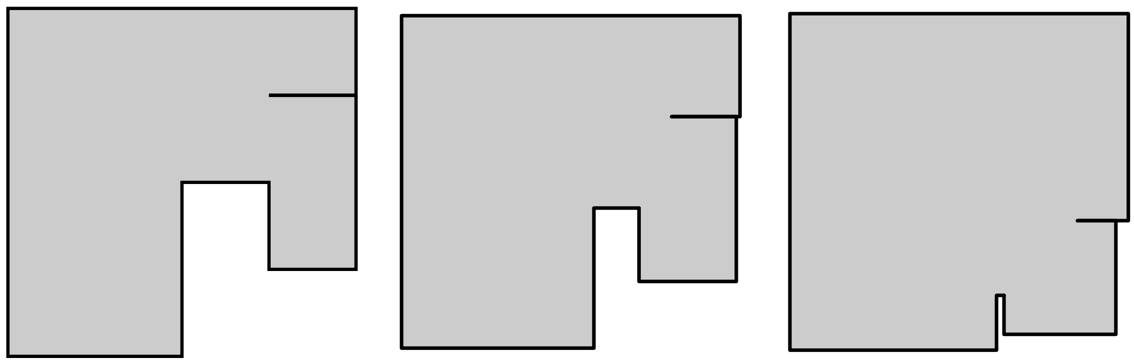
Riemann Mapping Theorem: If Ω is a simply connected, proper subdomain of the plane, then there is a conformal map $f : \Omega \to \mathbb{D}$.



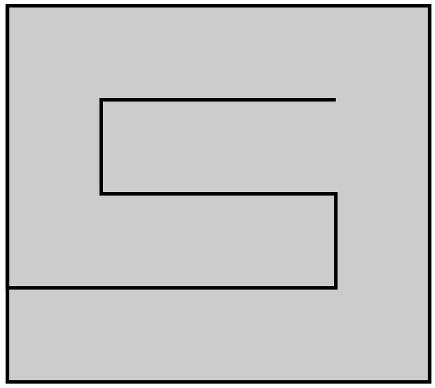
The Schwarz-Christoffel formula gives the Riemann map onto a polygonal:

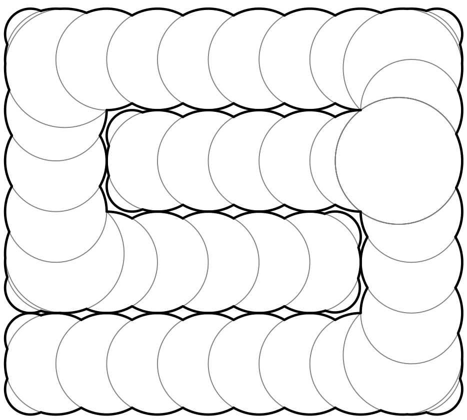
$$f(z) = A + C \int^{z} \prod_{k=1}^{n} (1 - \frac{w}{z_k})^{\alpha_k - 1} dw.$$

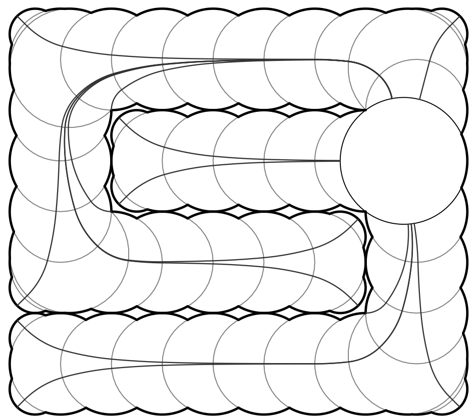
α's are known (interior angles) but z's are not (preimages of vertices).

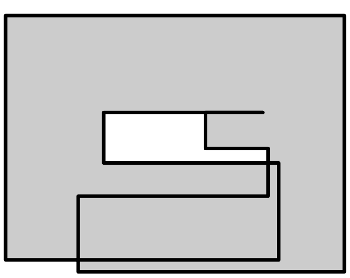


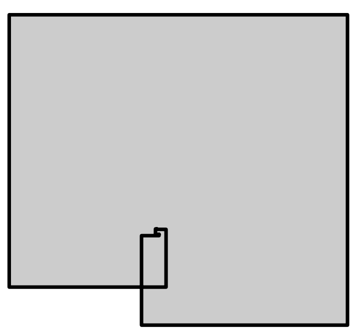
If we plug in ι-images of vertices we almost get the correct polygon (center). Using uniformly spaced points is clearly worse (right).



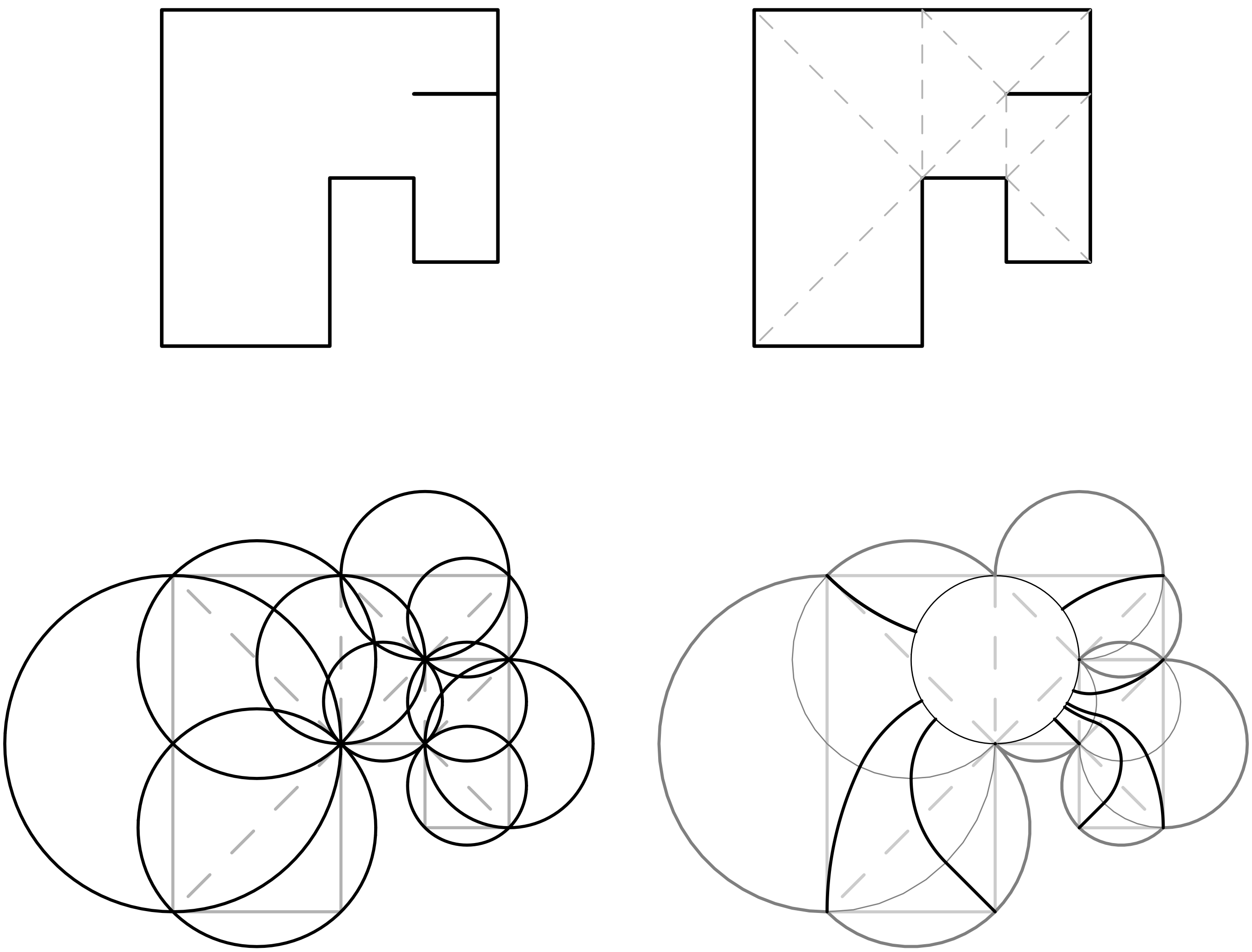


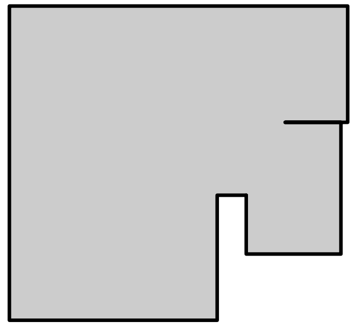






"Numerical conformal mapping using cross ratios and Delaunay triangulation" by Driscoll and Vavasis (1998).





Theorem: If $\partial\Omega$ is an n-gon we can compute a ϵ-approximation of the conformal map between Ω and $\mathbb{D}$ in time $C(\epsilon)n$.

Theorem: Suppose $\partial\Omega$ is an n-gon. We can construct points $\mathbf{w} = \{w_1, \ldots, w_n\} \subset \mathbb{T}$ so that:

1. requires at most $C(\epsilon)n$ steps.
2. $|\mathbf{w} - \mathbf{z}| < \epsilon$, $\mathbf{z} =$ true conformal prevertices.

$C(\epsilon) = \log^2 \frac{1}{\epsilon} \log\log \frac{1}{\epsilon}$.

But, what metric are we using?

Hyperbolic half-plane: Metric on $\mathbb{R}^2_+$,

$$d\rho = |dz|/\mathrm{dist}(z, \mathbb{R}^2).$$

Geodesics are circles or lines orthogonal to $\mathbb{R}$.
Hyperbolic disk: Metric on $\mathbb{D}$,

$$d\rho = |dz|/1 - |z|^2.$$

Geodesics are circles or lines orthogonal to $\partial\mathbb{D}$.

The hyperbolic metric on a simply connected domain plane Ω is defined by transferring the metric on the disk by the Riemann map.

Important Fact: $\rho \simeq \tilde{\rho}$ where

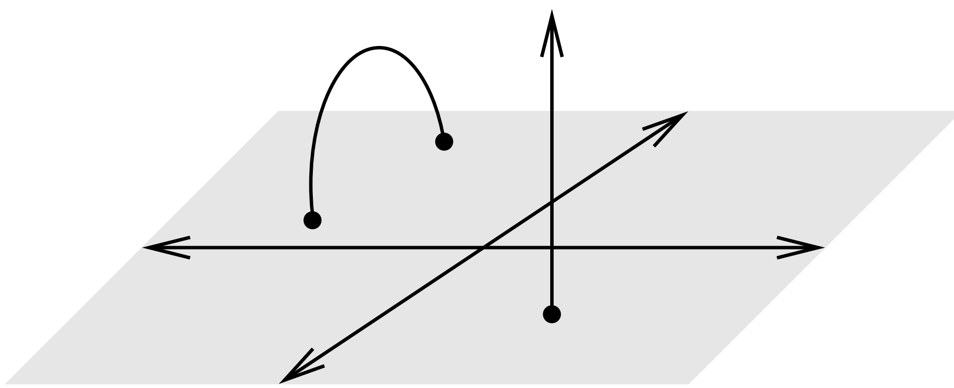
$$d\tilde{\rho} = \frac{|dz|}{\mathrm{dist}(z, \partial\Omega)},$$

is pseudo-hyperbolic metric.

Hyperbolic space: Metric on $\mathbb{R}^3_+$,

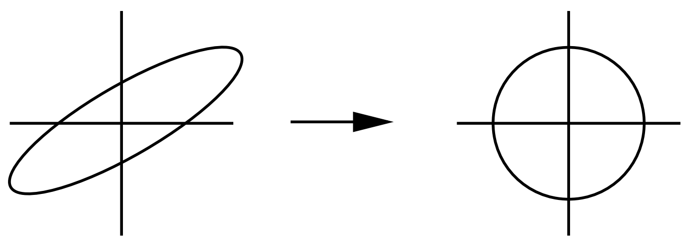
$$d\rho = |dz|/\mathrm{dist}(z, \mathbb{R}^2).$$

Geodesics are circles or lines orthogonal to $\mathbb{R}^2$.



A mapping is K-quasiconformal if either:

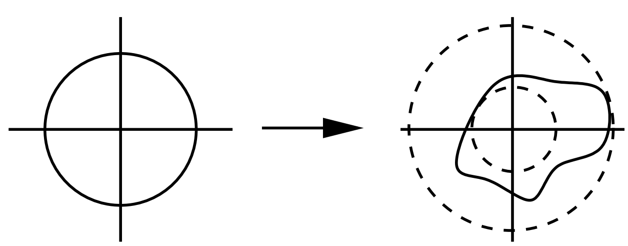
Analytic definition: $|f_{\bar{z}}| \leq \frac{K-1}{K+1}|f_z|$



$$f_z = \frac{1}{2}(f_x - i f_y),\ f_{\bar{z}} = \frac{1}{2}(f_x + i f_y).$$

Metric definition: For every $x \in \Omega$, $\epsilon > 0$ and small enough $r > 0$, there is $s > 0$ so that

$$D(f(x), s) \subset f(D(x, r)) \subset D(f(x), s(K+\epsilon)).$$



Notation for today: ϵ-conformal $= e^{\epsilon}$-quasiconformal.

- The map is determined (up to Möbius maps) by
$$\mu_f = f_{\bar{z}}/f_z,$$
For μ with $\|\mu\|_\infty < 1$, there is a f with $\mu_f = \mu$.

- $\|u\|_\infty \leq k$, $k = (K-1)/(K+1)$ iff f is K-QC.

- $\mu = 0$ iff f is conformal.

- K-QC maps form a compact family.

- f is a **bi-Lipschitz** if

$$\frac{1}{A}\rho(x,y) \le \rho(f(x), f(y)) \le A\rho(x,y).$$

- f is a **quasi-isometry** if

$$\frac{1}{A}\rho(x,y) - B \le \rho(f(x), f(y)) \le A\rho(x,y) + B.$$

- QI=BL at "large scales".

- On hyperbolic disk, BL $\Rightarrow$ QC $\Rightarrow$ QI.

Theorem: $f : \mathbb{T} \to \mathbb{T}$ has a QC-extension to interior iff it has QI-extension (hyperbolic metric) iff it has a BL-extension.

Theorem: If $\partial\Omega$ is an n-gon we can compute a $(1+\epsilon)$-quasiconformal map between Ω and $\mathbb{D}$ in time $O(n \log^2 \frac{1}{\epsilon} \log\log \frac{1}{\epsilon})$.

Theorem: Suppose $\partial\Omega$ is an n-gon. We can construct points $\mathbf{w} = \{w_1, \ldots, w_n\} \subset \mathbb{T}$ so that:

1. requires at most $C(\epsilon)n$ steps.
2. $d_{QC}(\mathbf{w}, \mathbf{z}) < \epsilon$.

$\mathbf{z} = f^{-1}(\mathbf{v})$ are conformal prevertices.

$d_{QC}(\mathbf{w}, \mathbf{z}) = \inf\{\log K : \exists h \in \mathrm{QC_K}, \mathrm{h}(\mathbf{w}) = \mathbf{z}\}$.

$\mathrm{QC_K} = K$-quasiconformal maps.

$C(\epsilon) = C + C \log^2 \frac{1}{\epsilon} \log\log \frac{1}{\epsilon}$

Proof of theorem is in two steps:

Step 1: Given $\epsilon < \epsilon_0$ and ϵ-QC $f_n : \Omega \to \mathbb{D}$ construct $C\epsilon^2$-QC map $f_{n+1} : \Omega \to \mathbb{D}$. Construction takes time $C(\epsilon) = C + C \log^2 \frac{1}{\epsilon} \log\log \frac{1}{\epsilon}$.

Step 2: Build domains and finite boundary sets

$$(\Omega_0, V_0), \ldots, (\Omega_N, V_N)$$

so that

- $\Omega_0 = \mathbb{D}$,
- $\Omega_N = \Omega$, $V_N = V$,
- δ-QC maps $g_k : \Omega_k \to \Omega_{k+1}$, $V_k \to V_{k+1}$.

If $\delta < \epsilon_0/2$ then find conformal maps by induction (use previous map as starting point of Step 1 to find next map).

Amazing Fact 1: Can take ϵ_0 independent of Ω and n.

Amazing Fact 2: Can take N independent of Ω and n.

Consequence: Get ϵ_0 approximation in time $O(n)$ (independent of Ω). Then just repeat Step 1 until get desired accuracy :

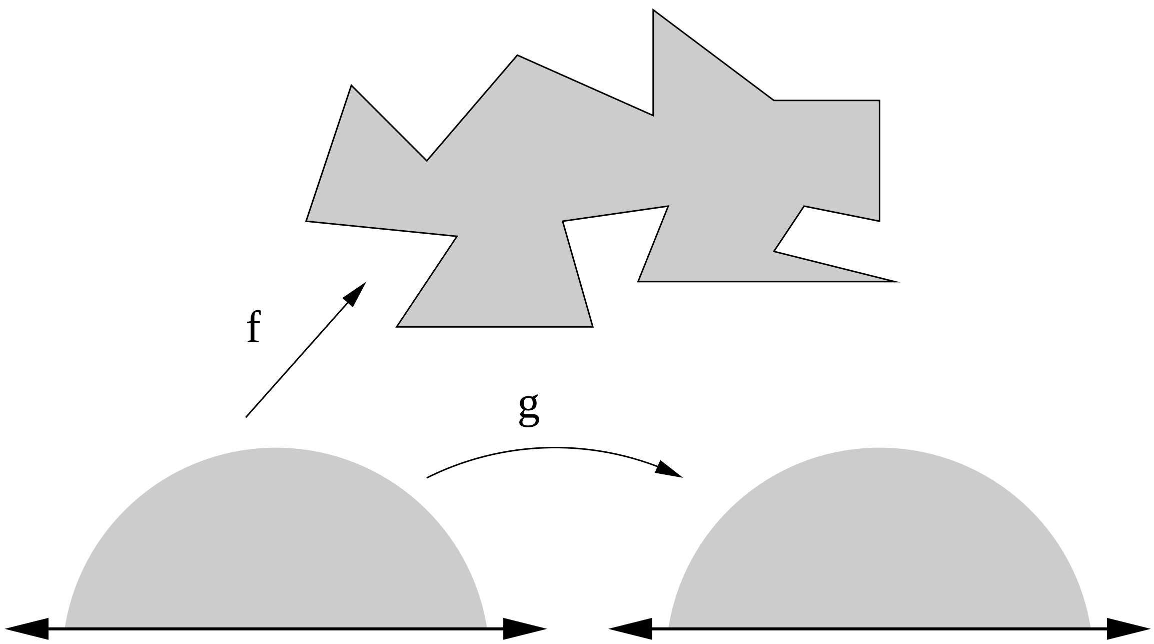
$$\epsilon_0, C\epsilon_0^2, \ldots C^k\epsilon_0^{2^k}.$$

About $\log\log\epsilon$ iterations suffice and time for kth iteration is $O(k2^{2k})$, so work dominated by final step.

Idea for Step 1: Suppose

$$f : \mathbb{H} \to \Omega, \qquad g : \mathbb{H} \to \mathbb{H}, \qquad \mu_f = \mu_g.$$

Then $f \circ g^{-1} : \mathbb{H} \to \Omega$ is conformal.

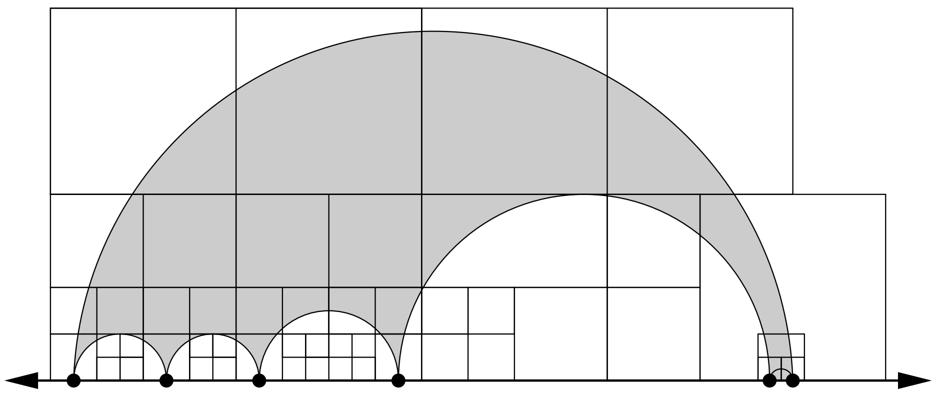


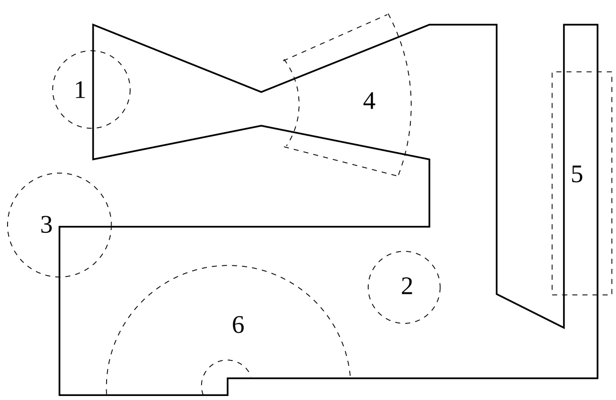
Can't solve Beltrami equation $g_{\bar{z}} = \mu g_z$ exactly in finite time, but can quickly solve

$$g_{\bar{z}} = (\mu + O(\|\mu\|^2))g_z.$$

Then $f \circ g^{-1}$ is $(1 + C\|\mu\|^2)$-QC.

Cut $\mathbb{H}$ into $O(n)$ pieces on which f, f^α or $\log f$ has nice series representation. Need $p = O(|\log \epsilon|)$ terms on each piece to get ϵ accuracy.





Our approximation is of the form

$$\sum_i \varphi_i(z) g_i\left(\sum_{k=-p}^{p} a_{i,k}(z-z_i)^k\right),$$

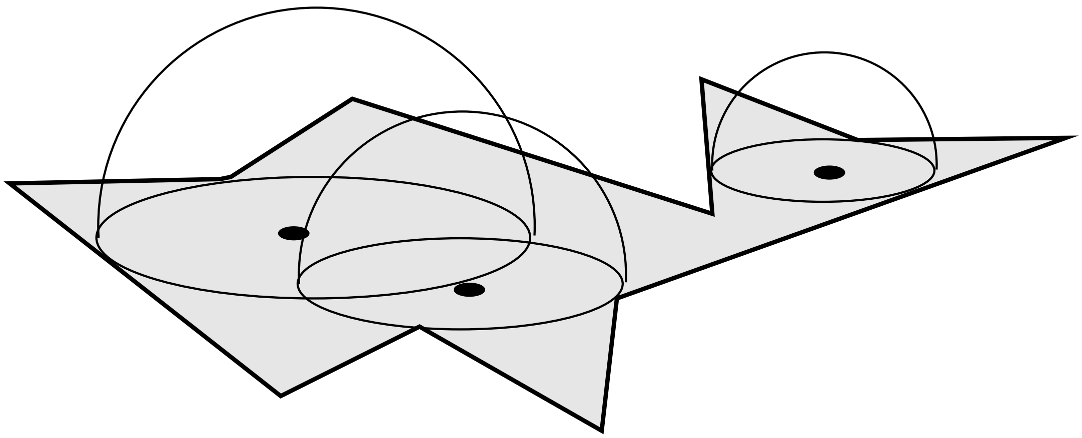
• $p = O(|\log \epsilon|)$,
• g is a known elementary function, z, z^α, $\log z$, or SC-integral (only for arches)
• $\{\varphi\}$ is a piecewise polynomial partition of unity for the decomposition; is non-constant only near boundaries of pieces
• $\{z_i\}$ are centers of the pieces
• sums are truncation of series that converge uniformly on big neighborhood of each piece (negative terms only occur for arches).
• QC-constant can be bounded above by checking agreement of terms on overlaps.

Can compute μ explicitly. Use fast multipole method to approximately solve in time $O(n)$. Finds new decomposition, and new series expansions. Bottleneck is time to compute p terms of series expansion of a product and composition of p linear fractional transformations. Takes time $p^2 \log p$.

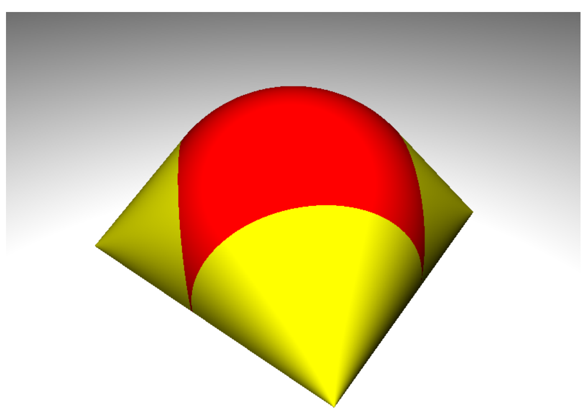
That completes sketch of Step 1.

Idea for step 2 is to consider the "iota" map from beginning of talk. Show how to factor it into composition of maps with small constant. Images of these are elements of our chain. First review "domes" and "medial axis".

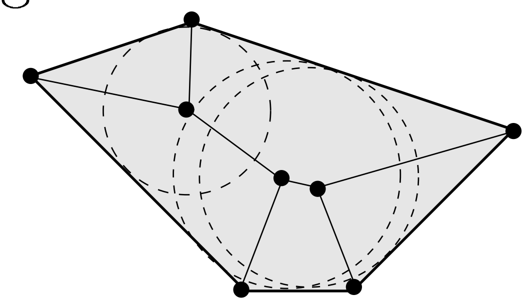
The **dome** of Ω is boundary of union of all hemispheres with bases contained in Ω.

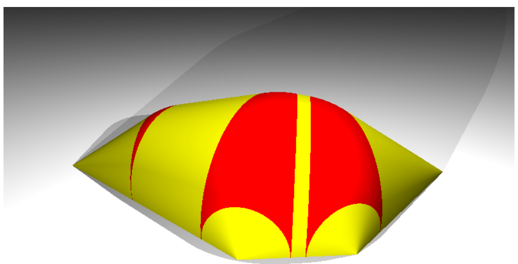


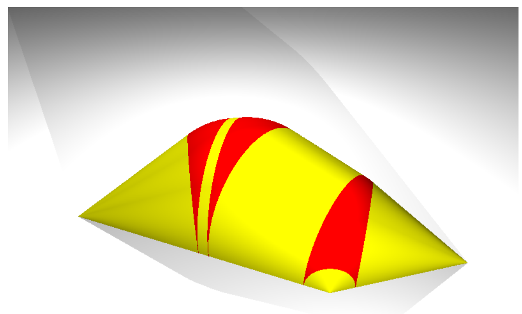
Equals boundary of hyperbolic convex hull of Ω^c. Similar to Euclidean space where complement of closed convex set is a union of half-spaces.



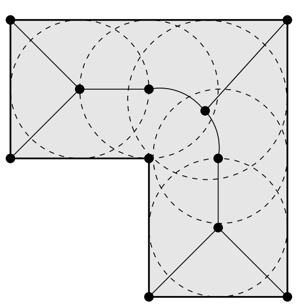
A convex polygon:

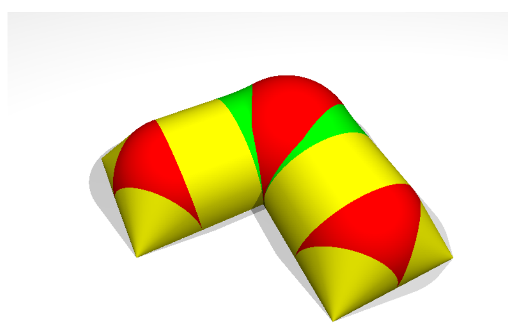






A non-convex polygon:





Each point on Dome(Ω) is on dome of a maximal disk D in Ω. Must have $|\partial D \cap \partial\Omega| \geq 2$. The centers of these disks form the **medial axis**.

For polygons is a finite tree with 3 types of edges:

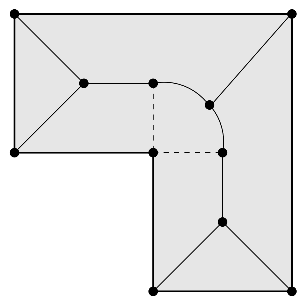
- point-point bisectors (straight)
- edge-edge bisectors (straight)
- point-edge bisector (parabolic arc)

For applications see:
www.ics.uci.edu/ eppstein/gina/medial.html+

In CS is attributed to Blum (1967), but Erdös proved dim(MA) = 1 in 1945.

Goggle("medial axis")= 26,300
Goggle("hyperbolic convex hull")= 71

Medial axis is boundary of Voronoi cells:

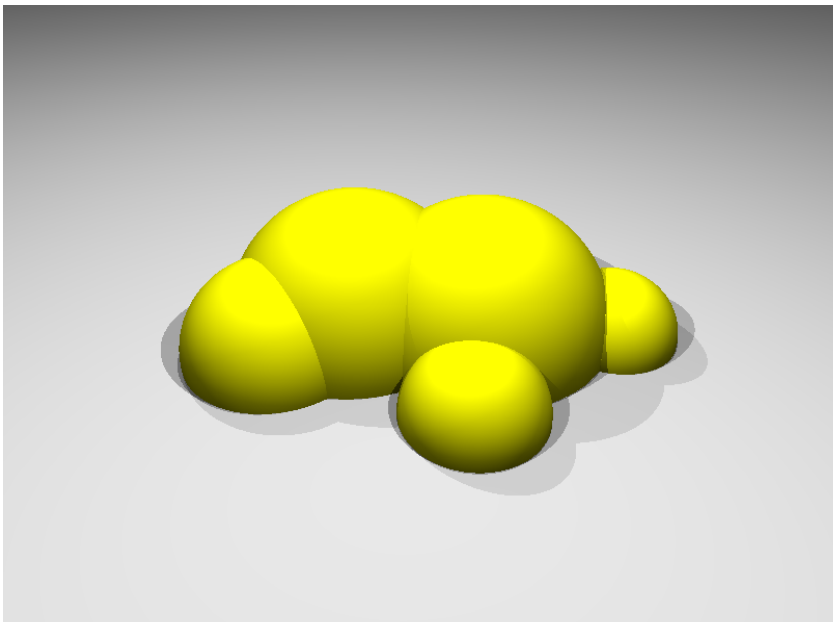


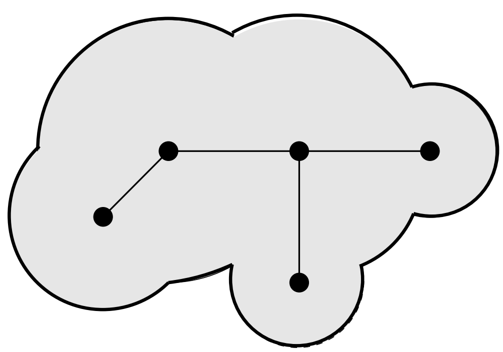
Chin-Snoeyink-Wang (1998) gave $O(n)$ algorithm. Uses Chazelle' theorem (1991): an n-gon can be triangulated in $O(n)$ time.

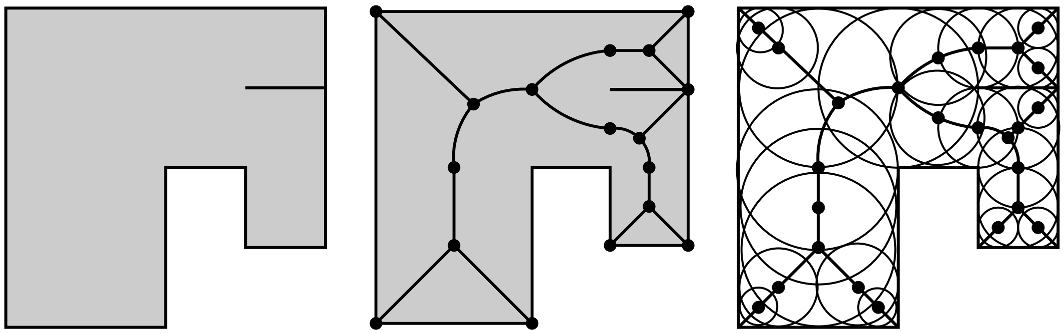
They use this to divide polygon into almost convex regions ("monotone histograms"); compute for each piece (Aggarwal-Guibas-Saxe-Shor, 1989) and merge results.

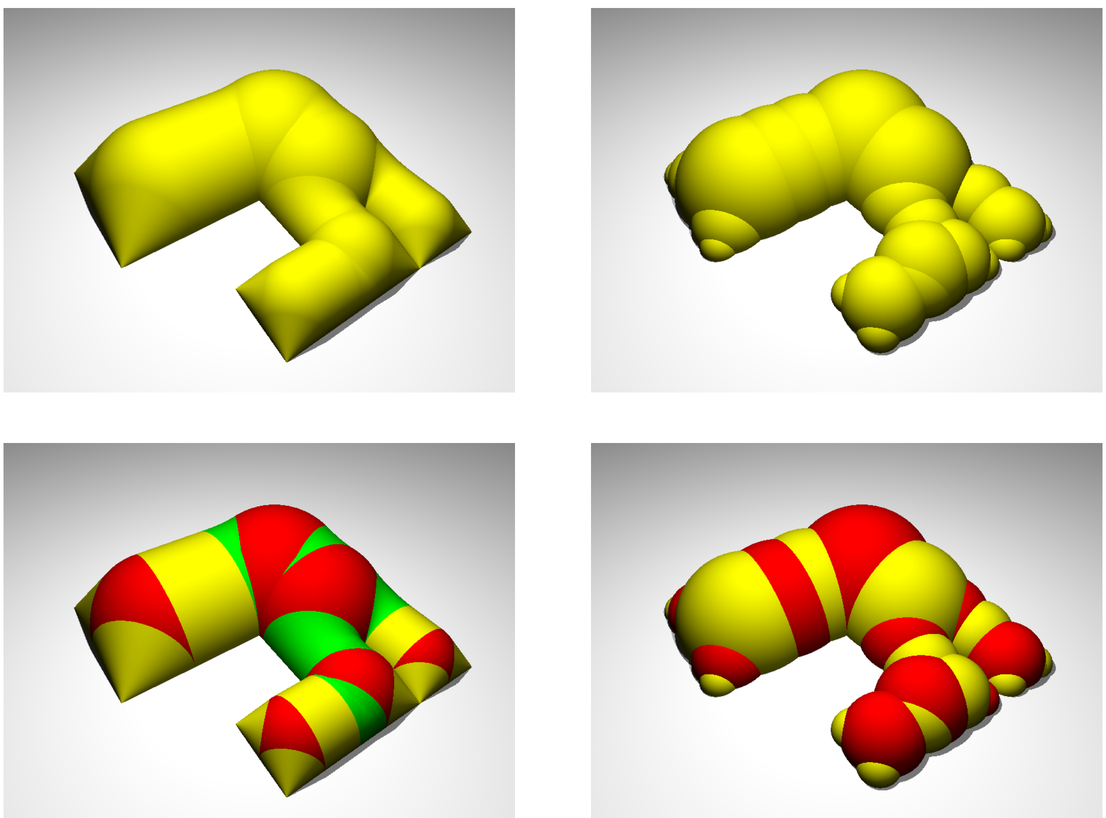
Merge Lemma: Suppose n sites $S = S_1 \cup S_2$ are divided by a line. Then diagram for S can be built from diagrams for S_1, S_2 in time $O(n)$.

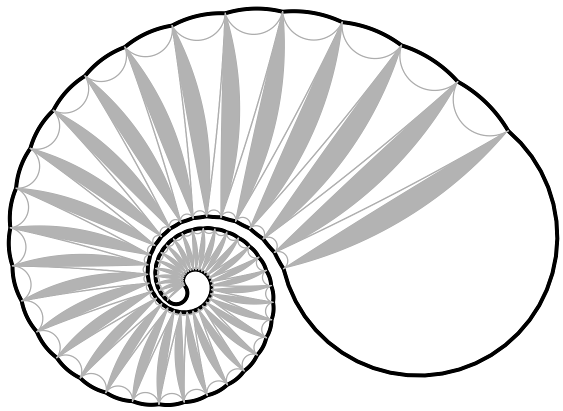
Finitely bent domain (= finite union of disks).

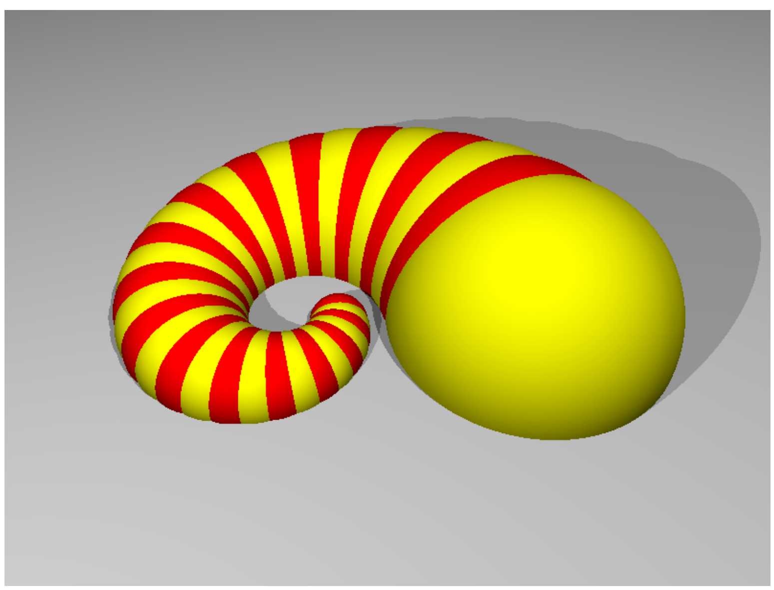








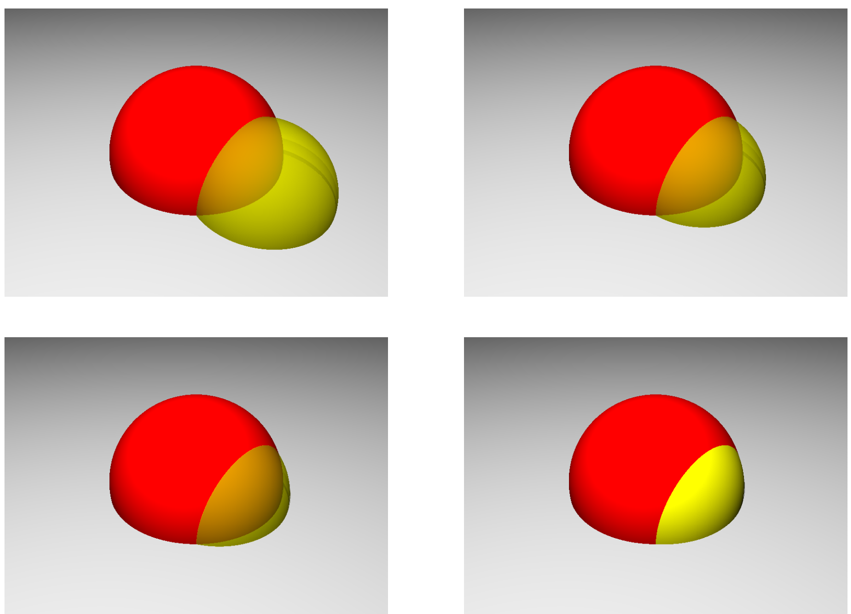




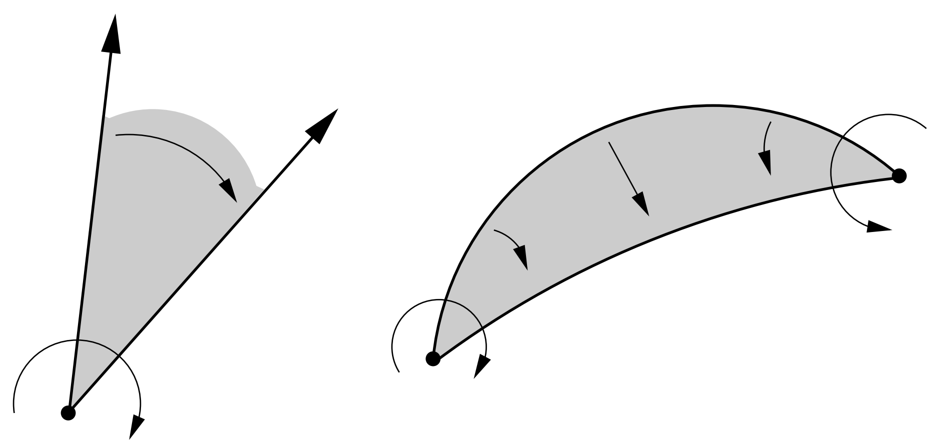
Let ρ_S be the hyperbolic path metric on S.

Theorem (Thurston): There is an isometry ι from (S, ρ_S) to the hyperbolic disk.

For finitely bent domains rotate around each bending geodesic by an isometry to remove the bending (more obvious if vertices are 0 and ∞).

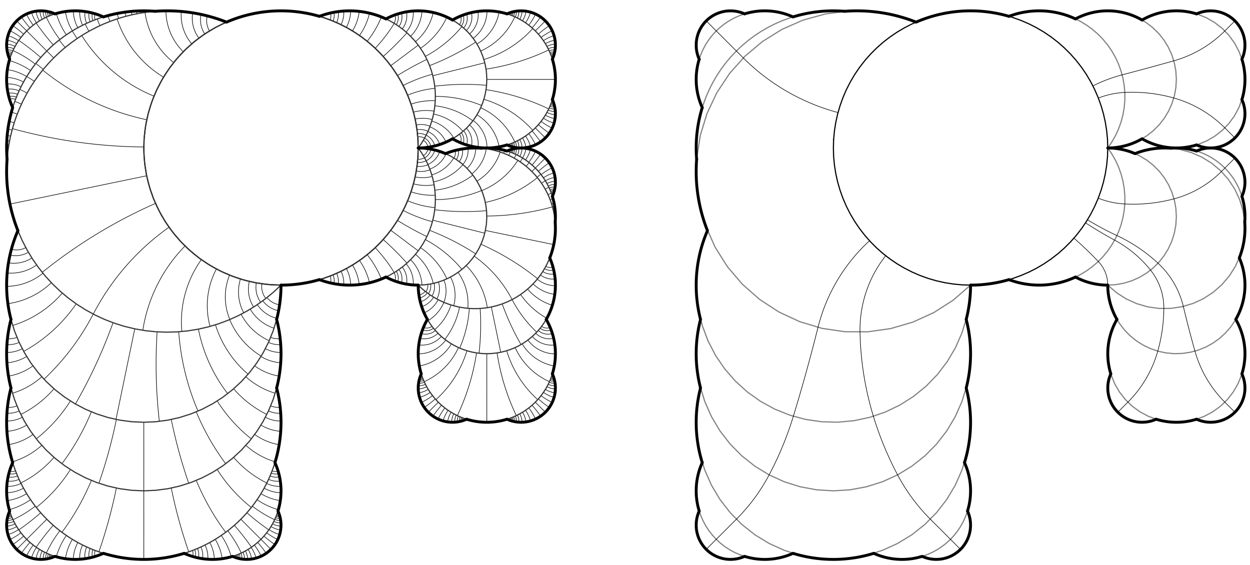


Elliptic Möbius transformation is conjugate to a rotation.



Elliptic transformation determined by fixed points and angle of rotation θ. It identifies sides of a crescent of angle θ: think of flow along circles orthogonal to boundary arcs.

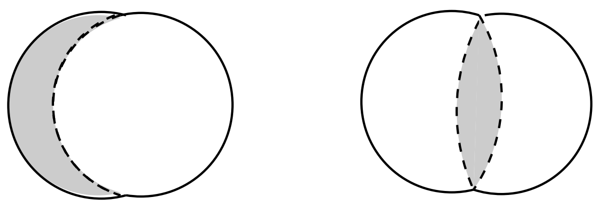
Visualize ι as a flow: Write finitely bent Ω as a disk D and a union of crescents. Foliate crescents by orthogonal circles. Following leaves of foliation in $\Omega \setminus D$ gives $\iota : \partial\Omega \to \partial D$.



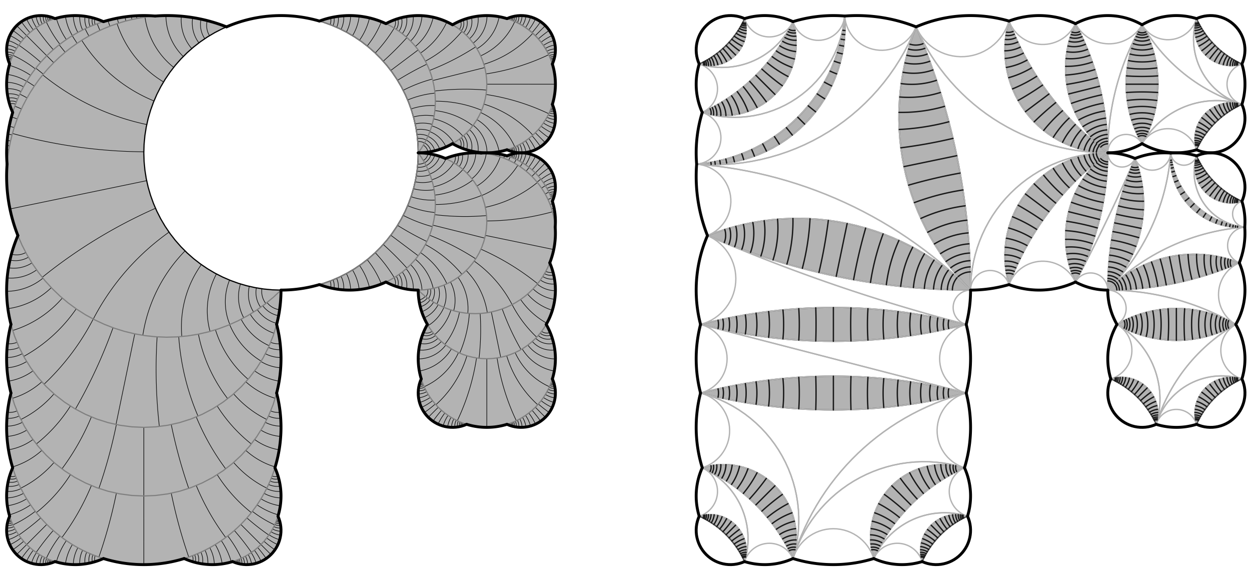
Has continuous extension to interior: identity on disk and collapses orthogonal arcs to points.

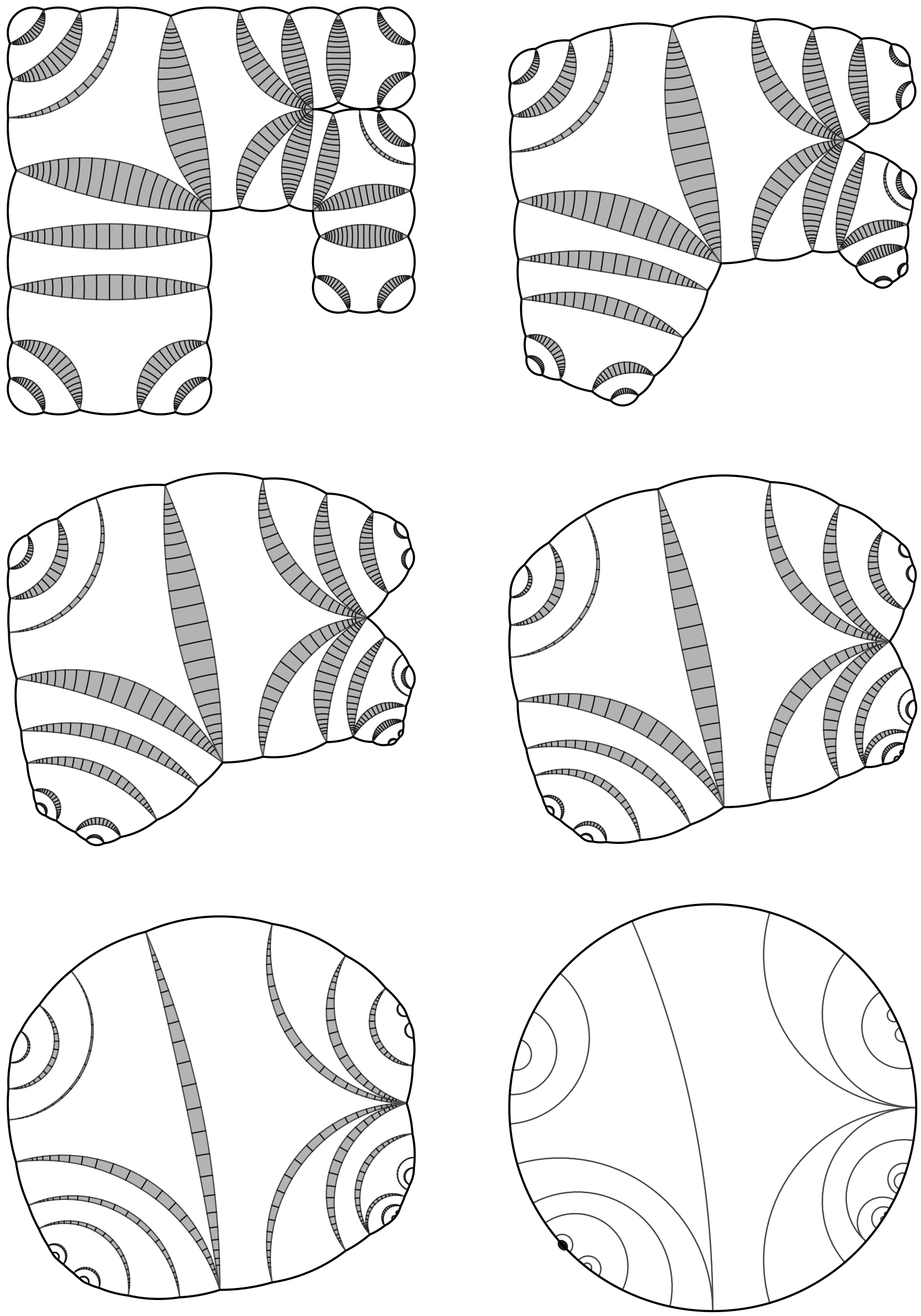
- ι has K-QC extension to interior.
- ι can be evaluated at n points in time $O(n)$.

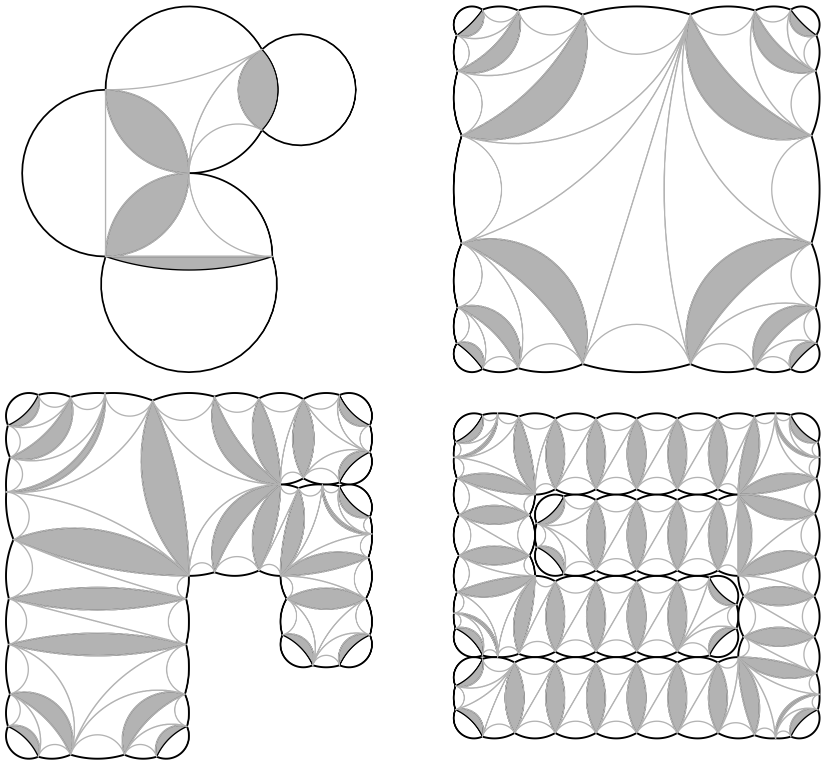
There are at least two ways to decompose a finite union of disks using crescents (with same angles and vertices in both cases).

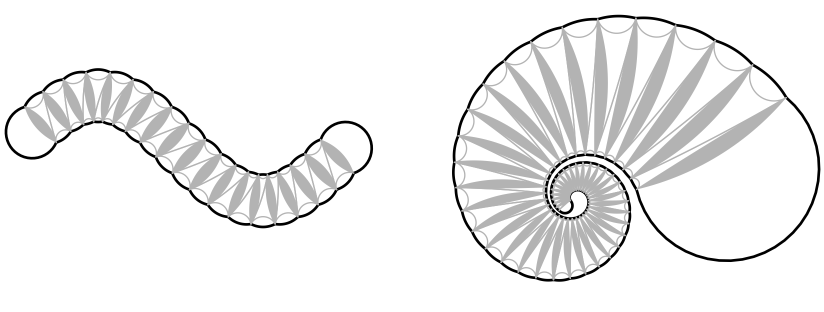


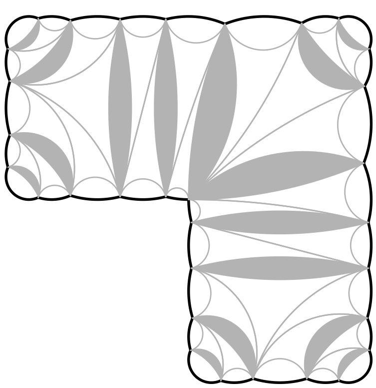
We call these **tangential** and **normal** crescents. A finitely bent domain can be decomposed with either kind of crescent.

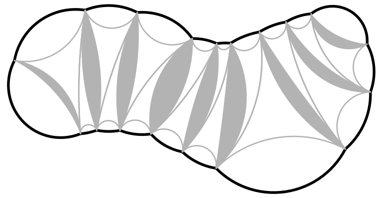


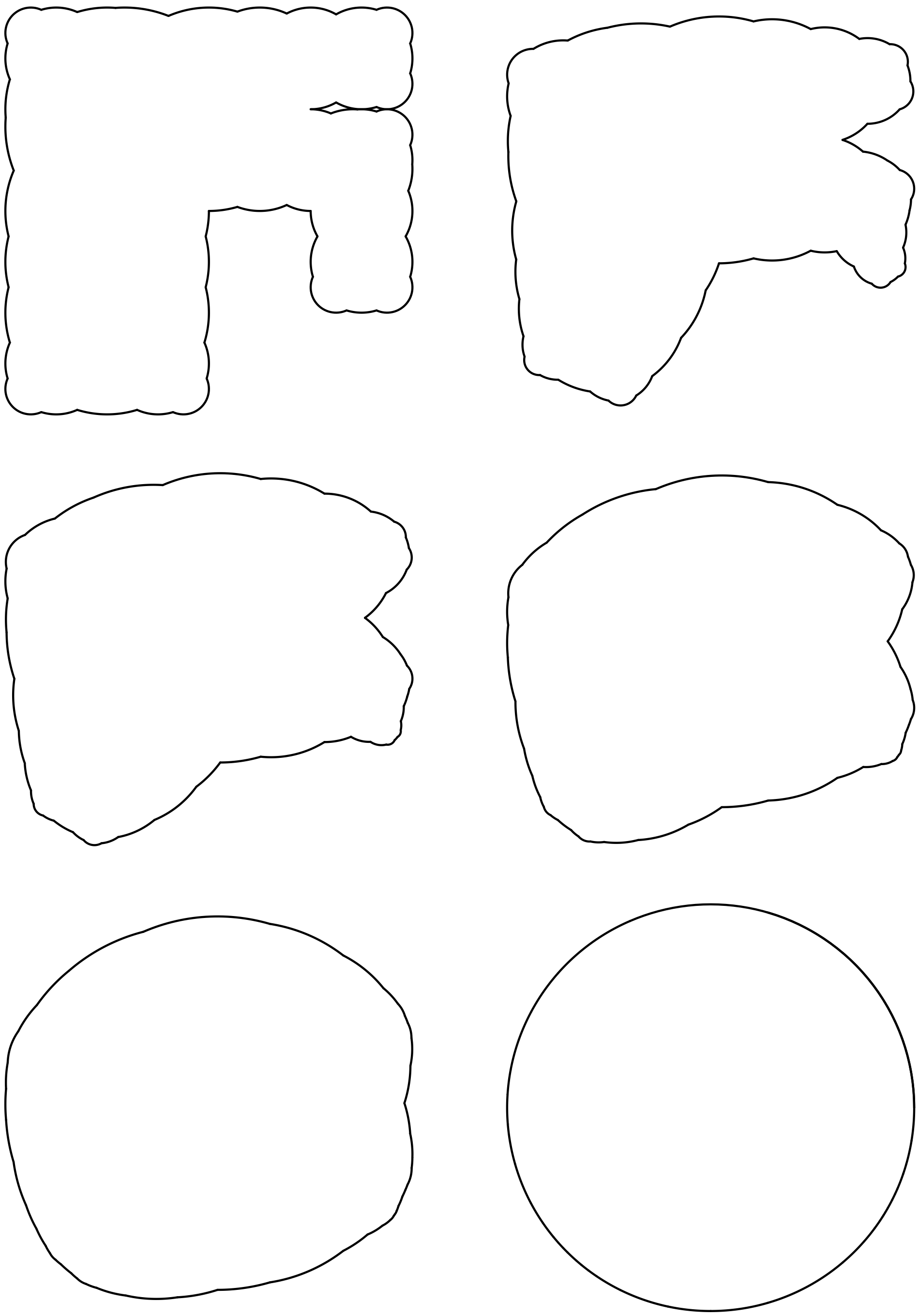


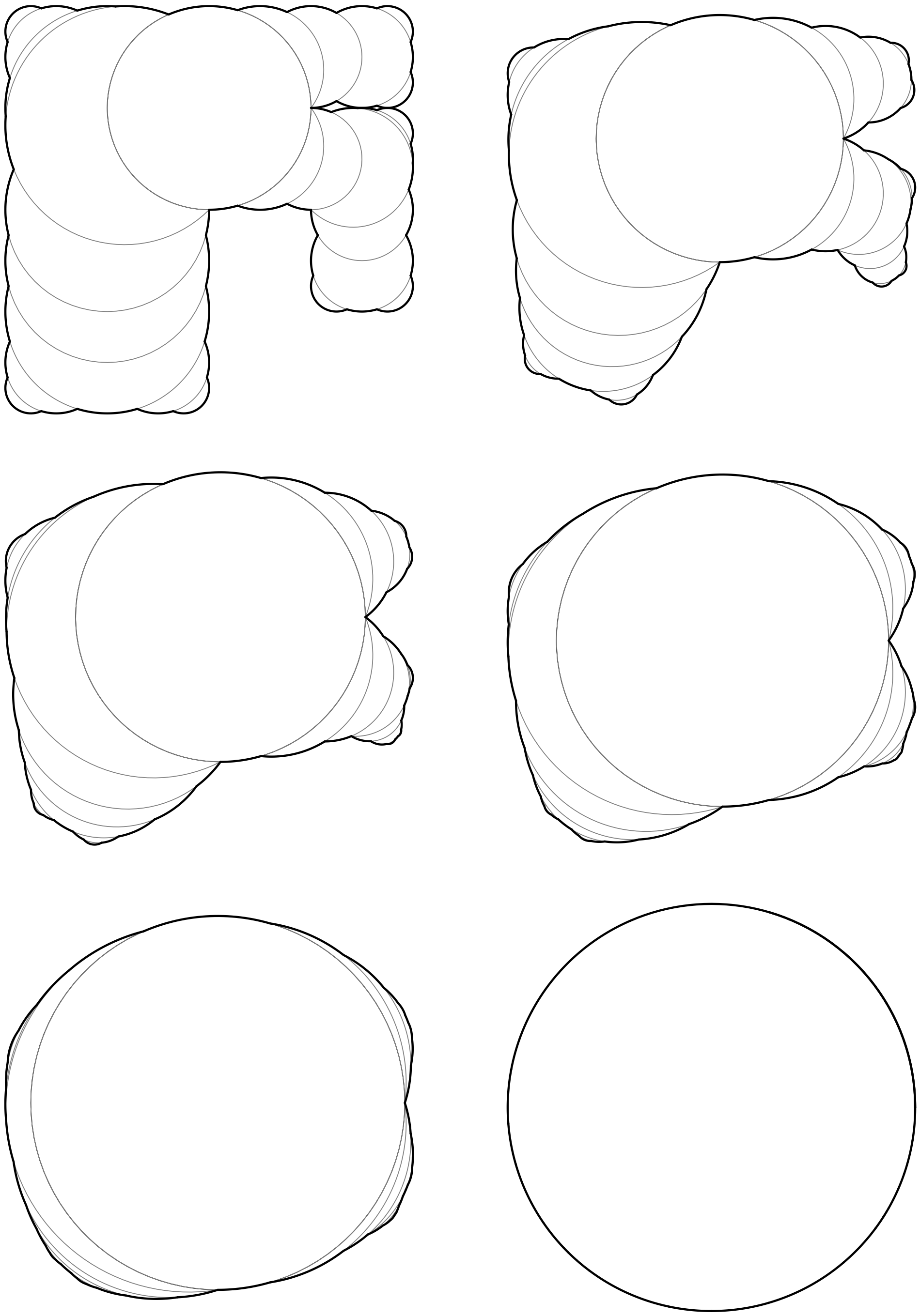


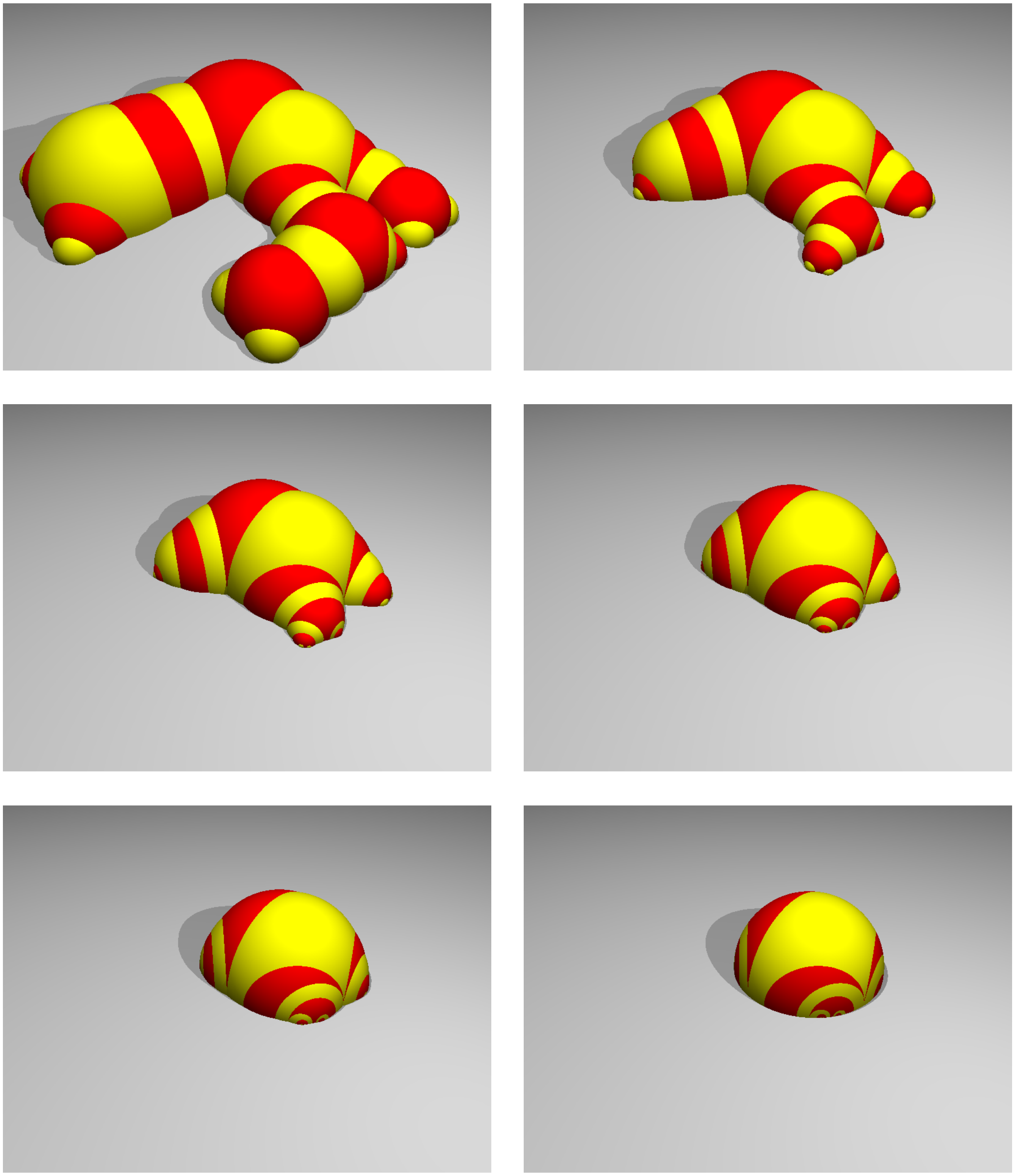


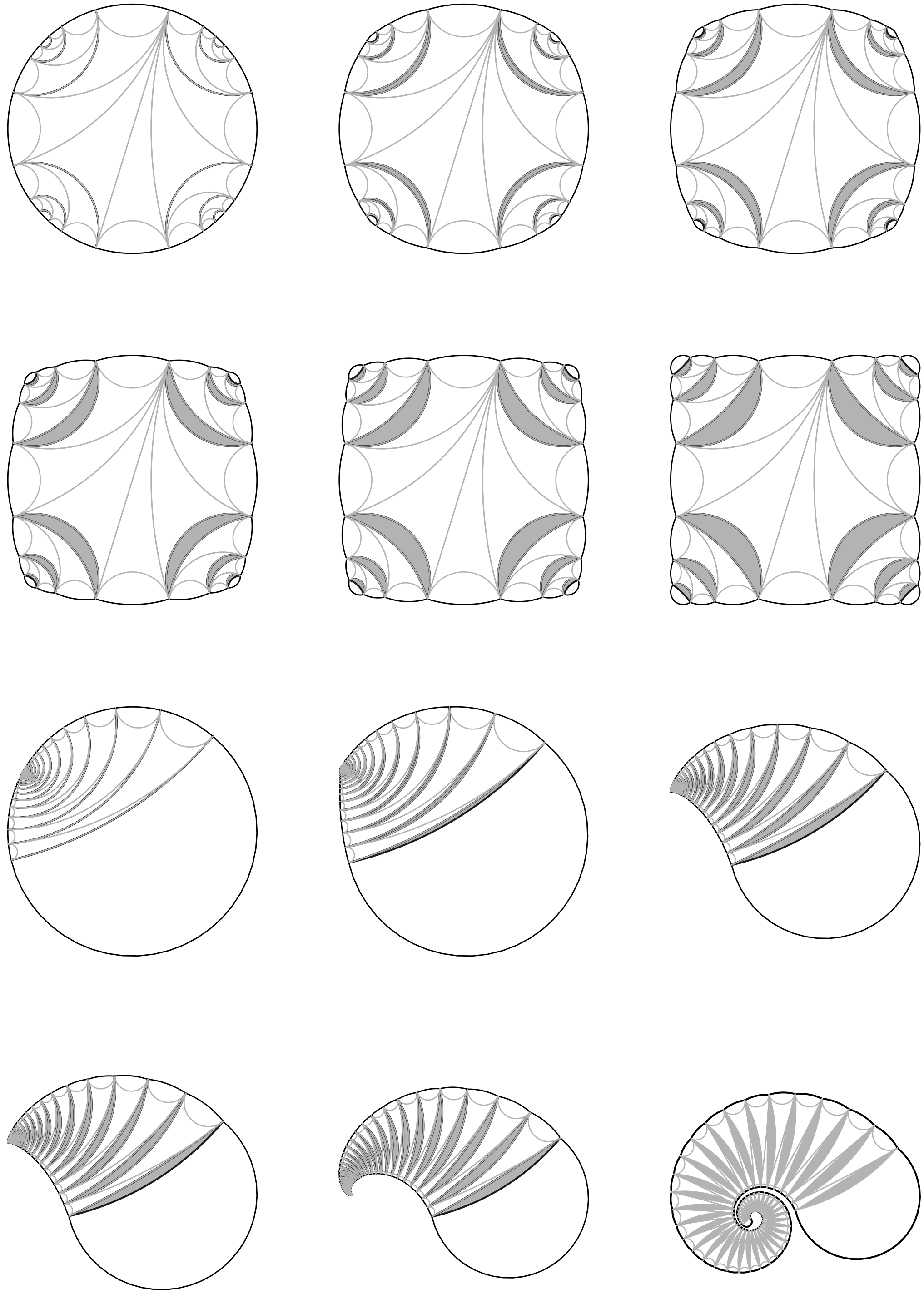


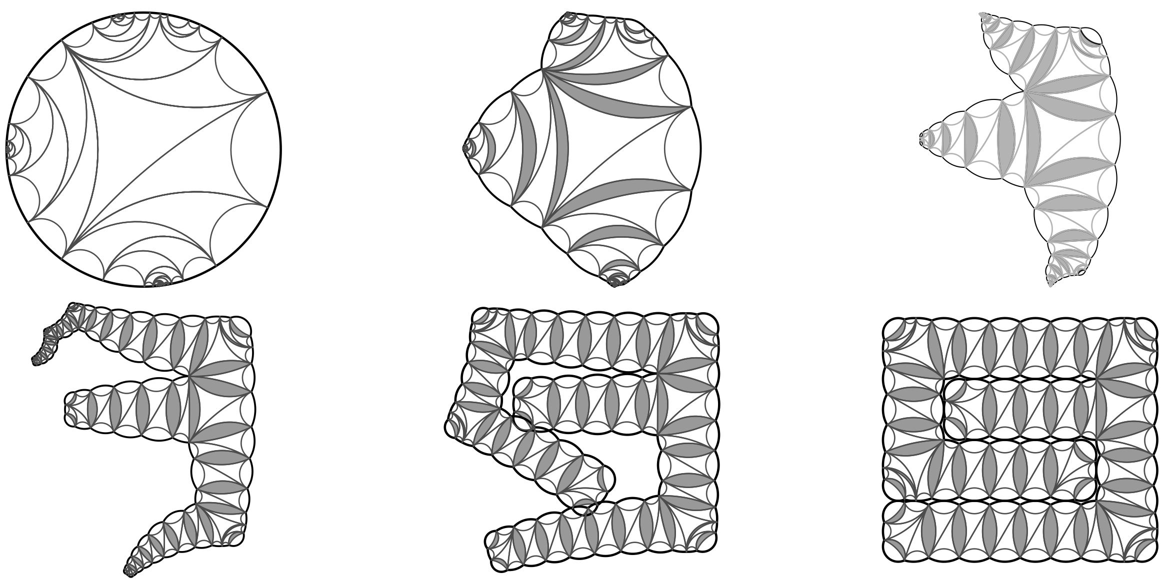












Theorem: Collapsing normal crescents gives hyperbolic quasi-isometry $R : \Omega \to \mathbb{D}$.

Corollary: ι has a K-QC extension to interior.

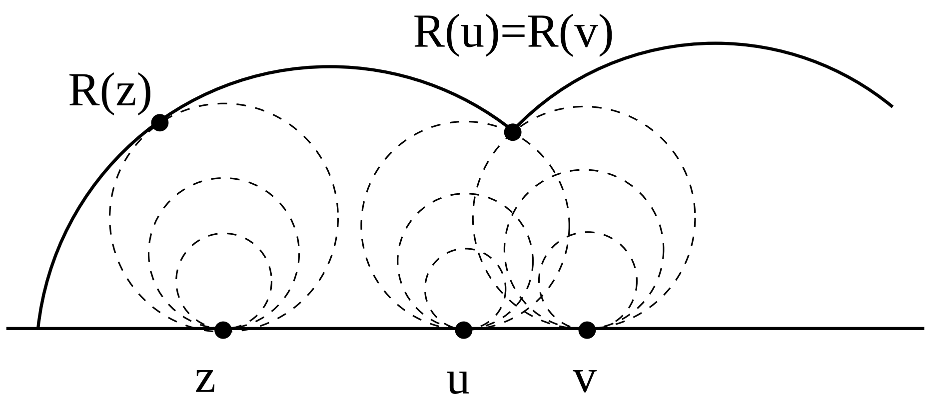
Corollary (Sullivan, Epstein-Marden): There is a K-QC map $\sigma : \Omega \to S_\Omega$ so that $\sigma = \mathrm{Id}$ on $\partial\Omega = \partial S$.

Result comes from hyperbolic 3-manifolds. If Ω is invariant under Möbius group G, $M = \mathbb{R}^3_+/G$ is hyperbolic manifold,

$$\partial_\infty M = \Omega/G, \quad \partial C(M) = \mathrm{Dome}(\Omega)/G.$$

Thurston conjectured $K = 2$ is possible. Best known upper bound is $K < 7.82$.

Nearest point retraction $R : \Omega \to \mathrm{Dome}(\Omega)$**:**
Expand ball tangent at $z \in \Omega$ until it hits a point $R(z)$ of the dome.



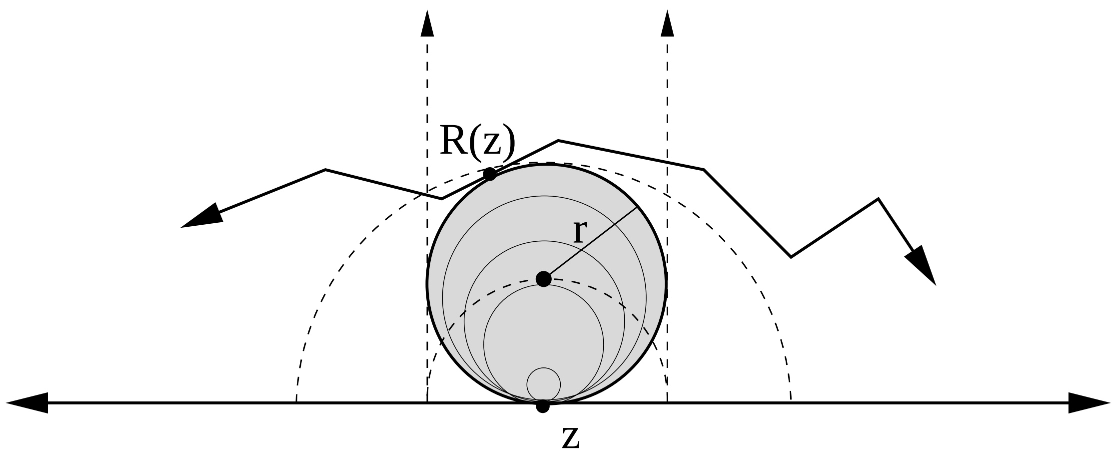
$$\text{normal crescents} = R^{-1}(\text{bending lines})$$
$$\text{gaps} = R^{-1}(\text{faces})$$

collapsing crescents = nearest point retraction

Suffices to show nearest point retraction is a quasi-isometry. This follows from three easy facts.

Fact 1: If $z \in \Omega$, $\infty \notin \Omega$,

$$r \simeq \mathrm{dist}(z, \partial\Omega) \simeq \mathrm{dist}(R(z), \mathbb{R}^2) \simeq |z - R(z)|.$$



Fact 2: R is Lipschitz.

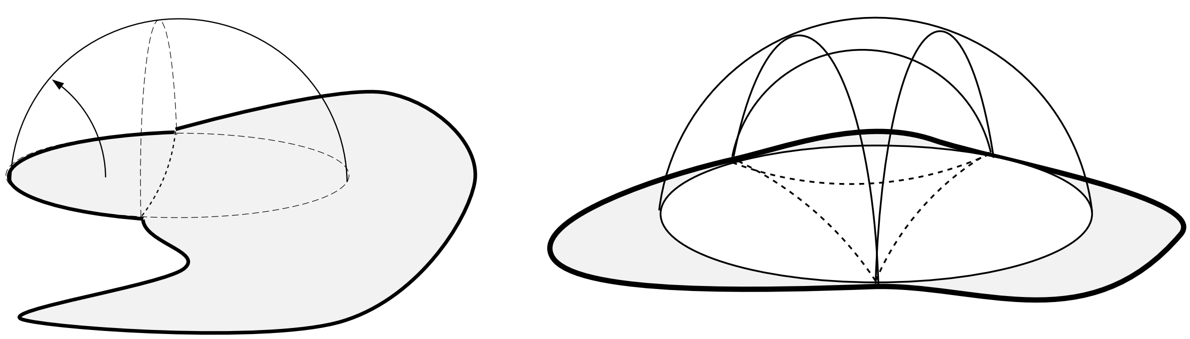
Ω simply connected $\Rightarrow$

$$d\rho \simeq \frac{|dz|}{\operatorname{dist}(z, \partial\Omega)}.$$

$z \in D \subset \Omega$ and $R(z) \in \text{Dome(D)} \Rightarrow$

$$\operatorname{dist}(z, \partial\Omega)/\sqrt{2} \leq \operatorname{dist}(z, \partial D) \leq \operatorname{dist}(z, \partial\Omega)$$

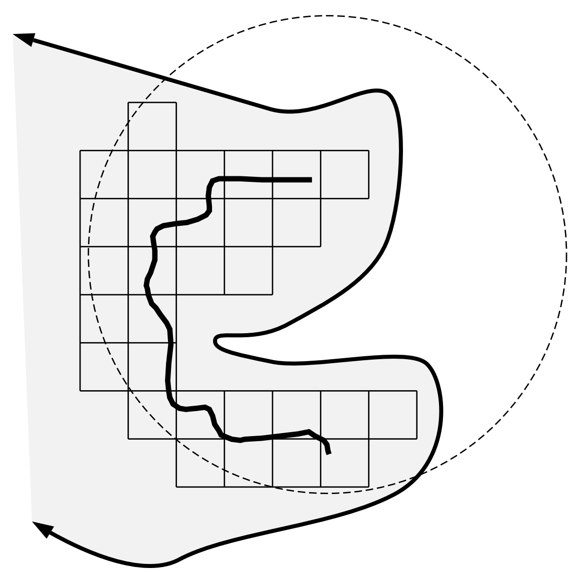
$$\Rightarrow \quad \rho_\Omega(z) \simeq \rho_D(z) = \rho_{\text{Dome}}(R(z)).$$



Fact 3: $\rho_S(R(z), R(w)) \leq 1 \Rightarrow \rho_\Omega(z, w) \leq C$.

Suppose $\text{dist}(R(z), \mathbb{R}^2) = r$ and γ is geodesic from z to w.

$$
\begin{aligned}
\Rightarrow \quad & \text{dist}(\gamma, \mathbb{R}^2) \simeq r \\
\Rightarrow \quad & \text{dist}(R^{-1}(\gamma), \partial\Omega) \simeq r, \\
& \quad R^{-1}(\gamma) \subset D(z, Cr) \\
\Rightarrow \quad & \rho_\Omega(z, w) \leq C
\end{aligned}
$$

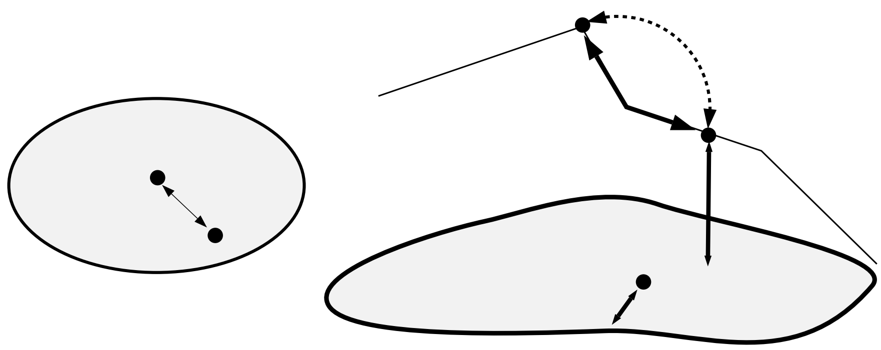


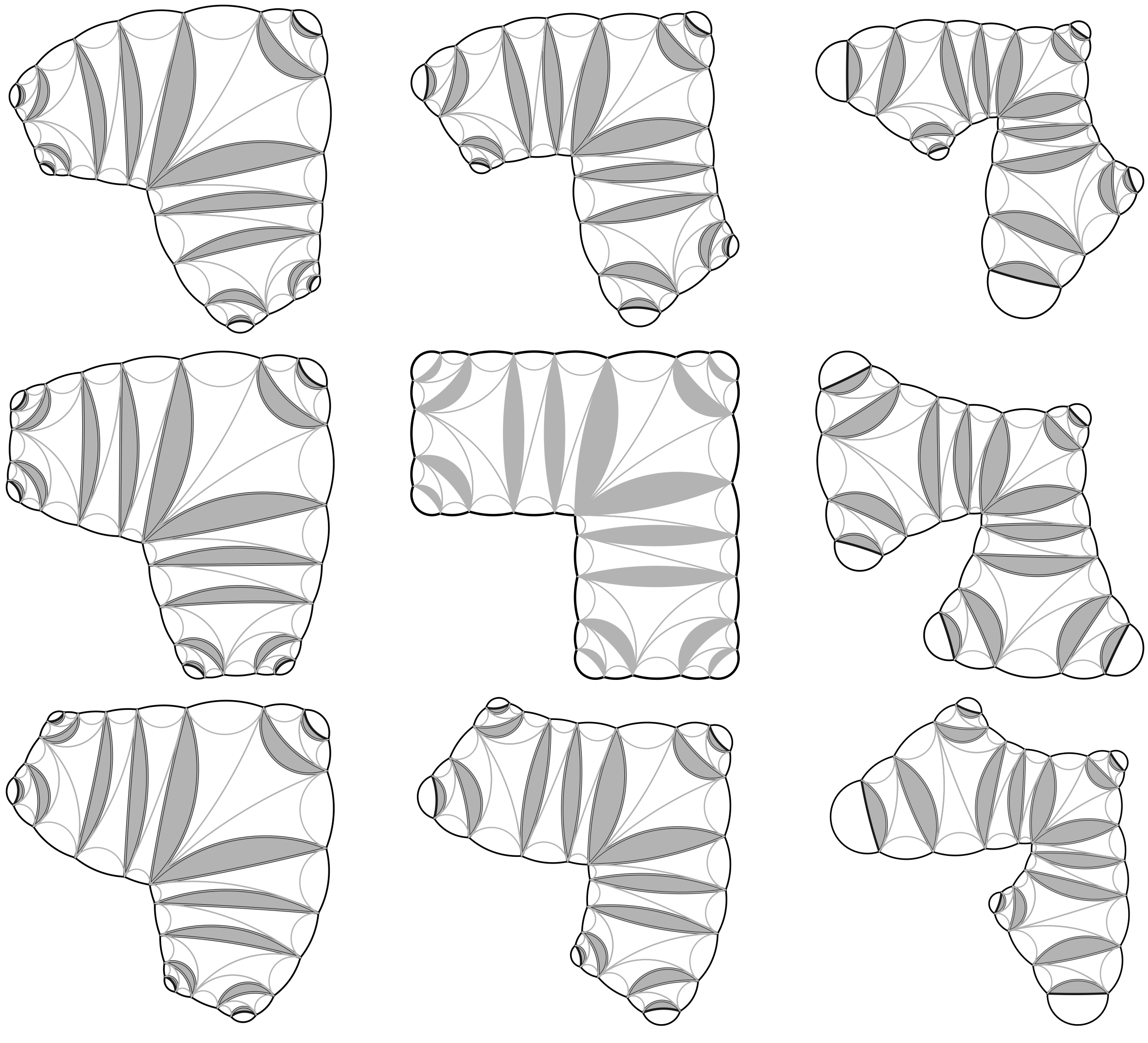
Moreover, $g = \iota \circ \sigma : \Omega \to \mathbb{D}$ is locally Lipschitz. Standard estimates show

$$|g'(z)| \simeq \frac{\operatorname{dist}(g(z), \partial\mathbb{D})}{\operatorname{dist}(z, \partial\Omega)}.$$

Use Fact 1

$$\begin{aligned}
\operatorname{dist}(z, \partial\Omega) &\simeq \operatorname{dist}(\sigma(z), \mathbb{R}^2) \\
&\simeq \exp(-\rho_{\mathbb{R}^3_+}(\sigma(z), z_0)) \\
&\gtrsim \exp(-\rho_S(\sigma(z), z_0)) \\
&= \exp(-\rho_D(g(z), 0)) \\
&\simeq \operatorname{dist}(g(z), \partial D)
\end{aligned}$$





If you understand the figures, you understand the book.

John Garnett,
Bounded Analytic Functions, 1981

"Ah!" replied Pooh. He'd found that pretending a thing was understood was sometimes very close to actually understanding it. Then it could easily be forgotton with no one the wiser...

Winnie-the-Pooh

I wouldn't even think of playing music if I was born in these times... I'd probably turn to something like mathematics. That would interest me.

Bob Dylan, 2005