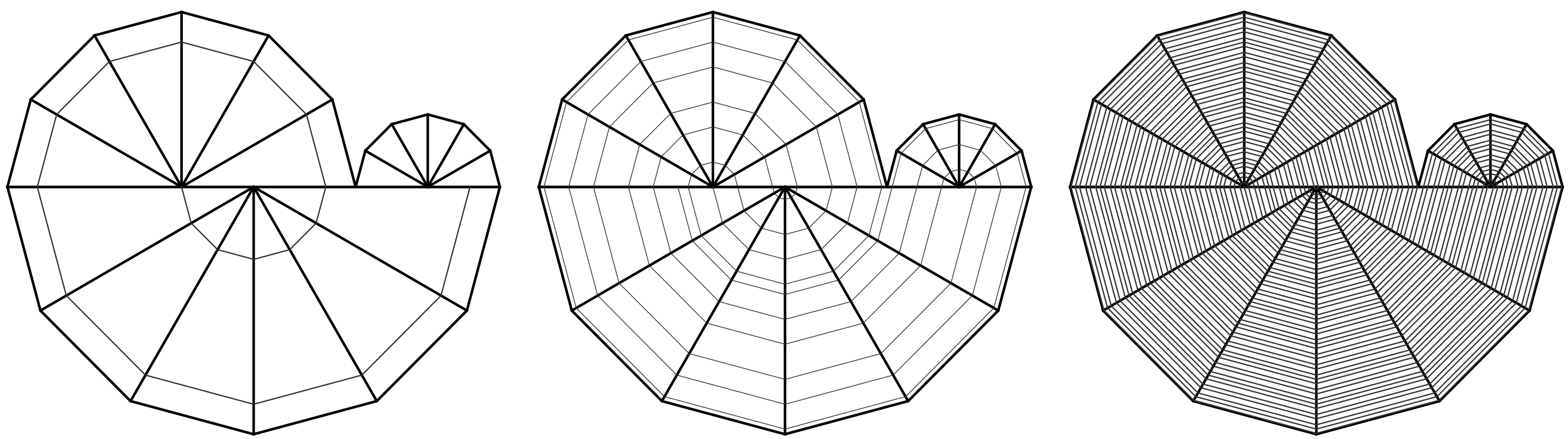
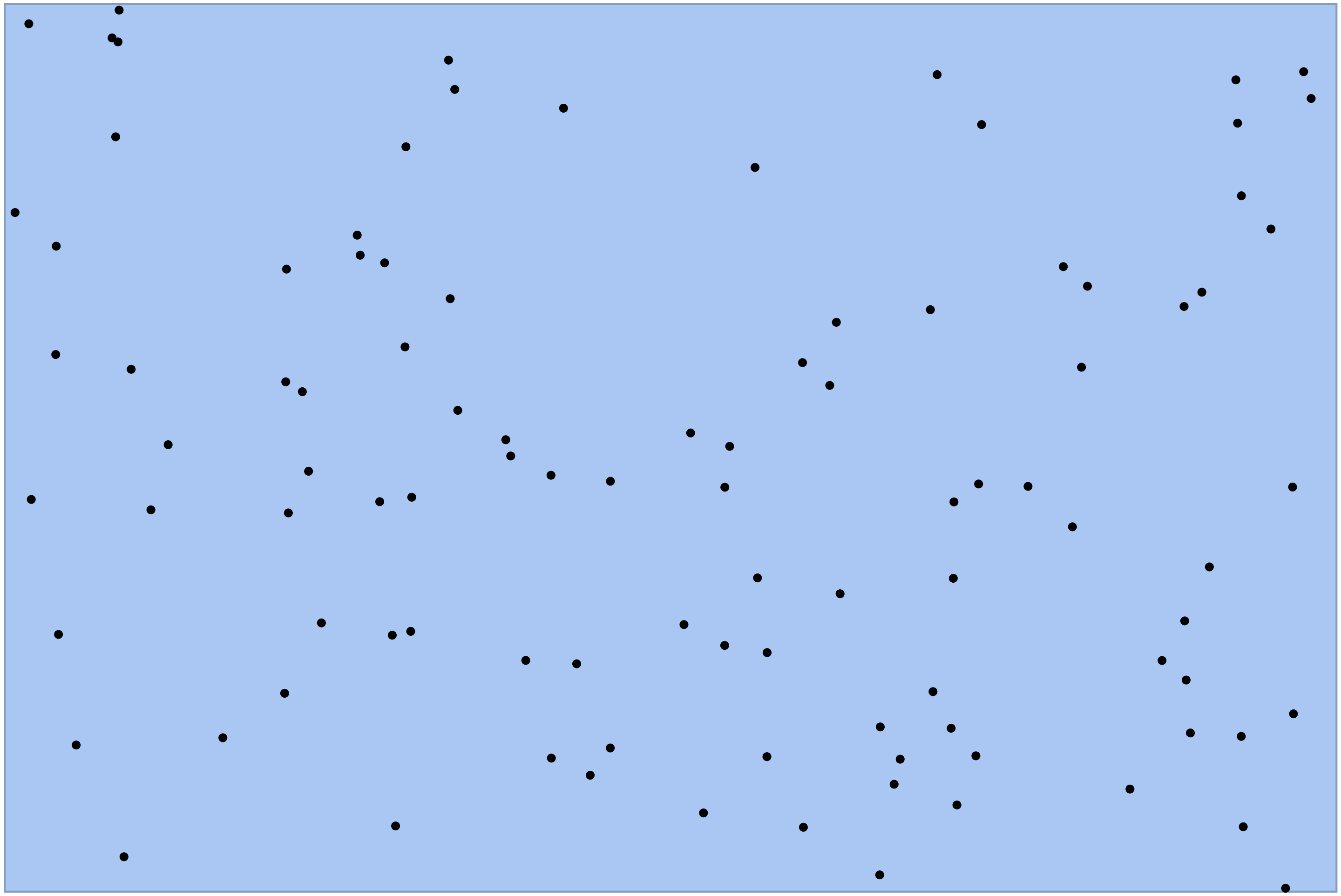
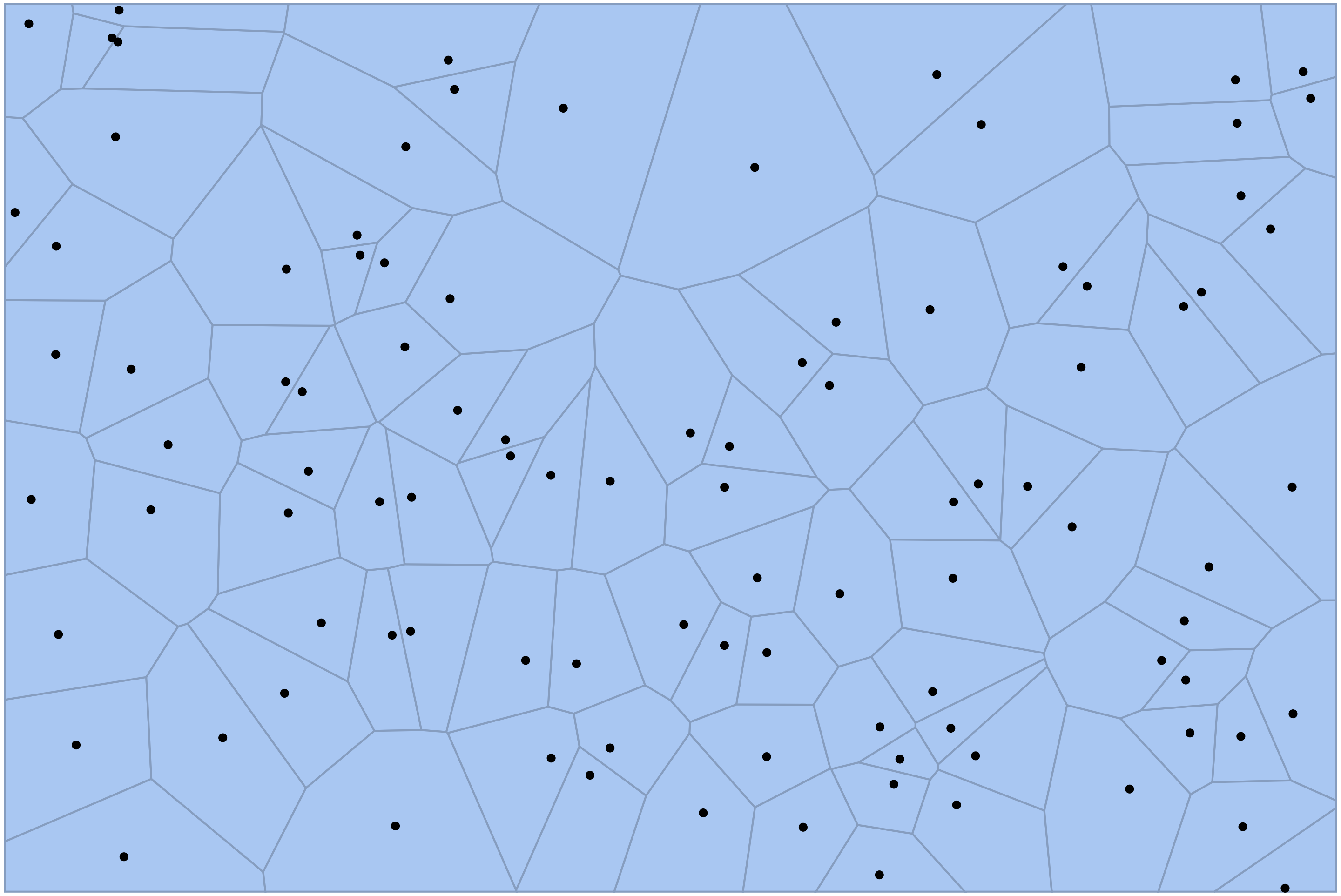
From fractals to phones: hyperbolic ideas in Euclidean geometry

Christopher J. Bishop
Stony Brook University

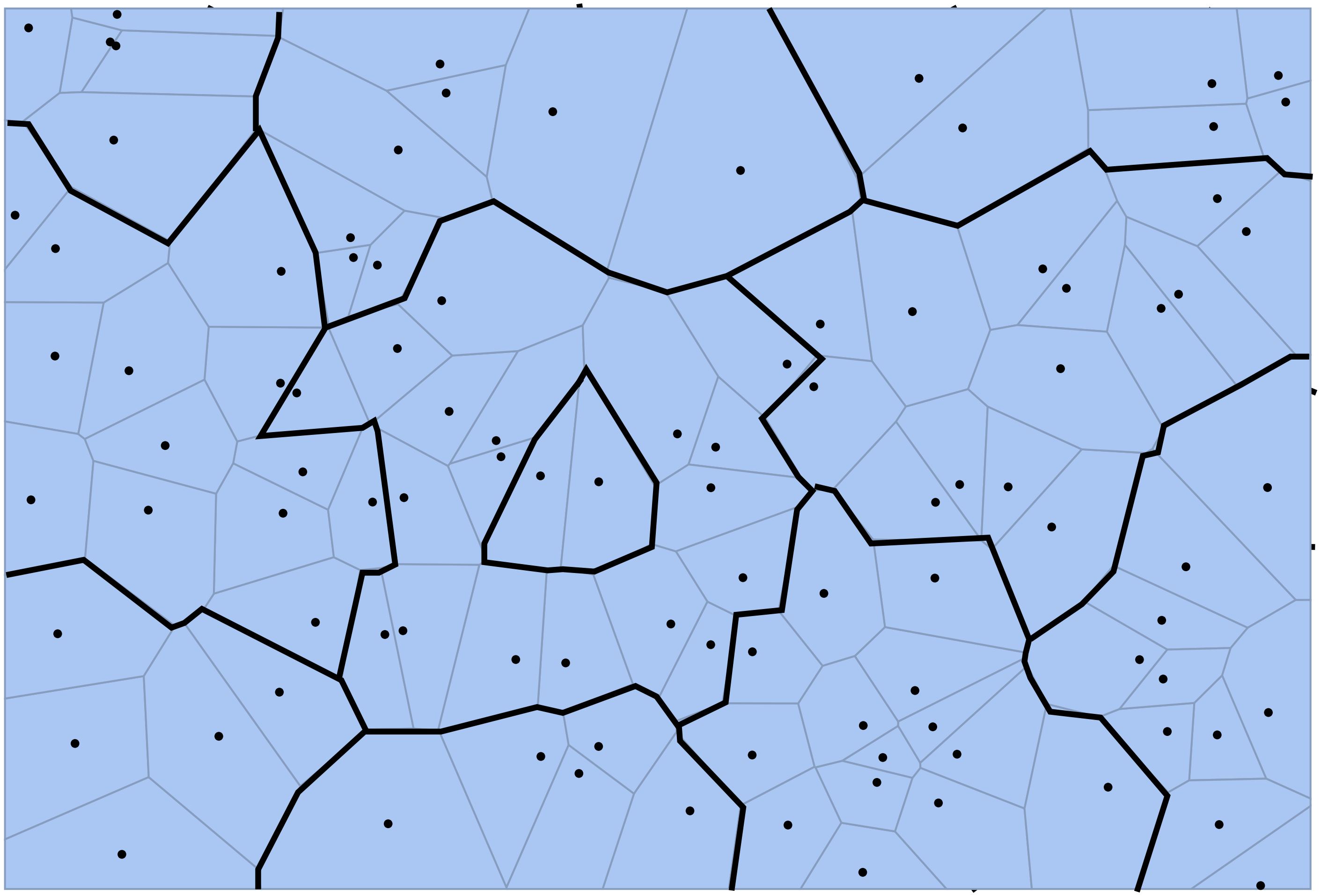
Dartmouth Geometry Seminar
Feb. 28, 2018



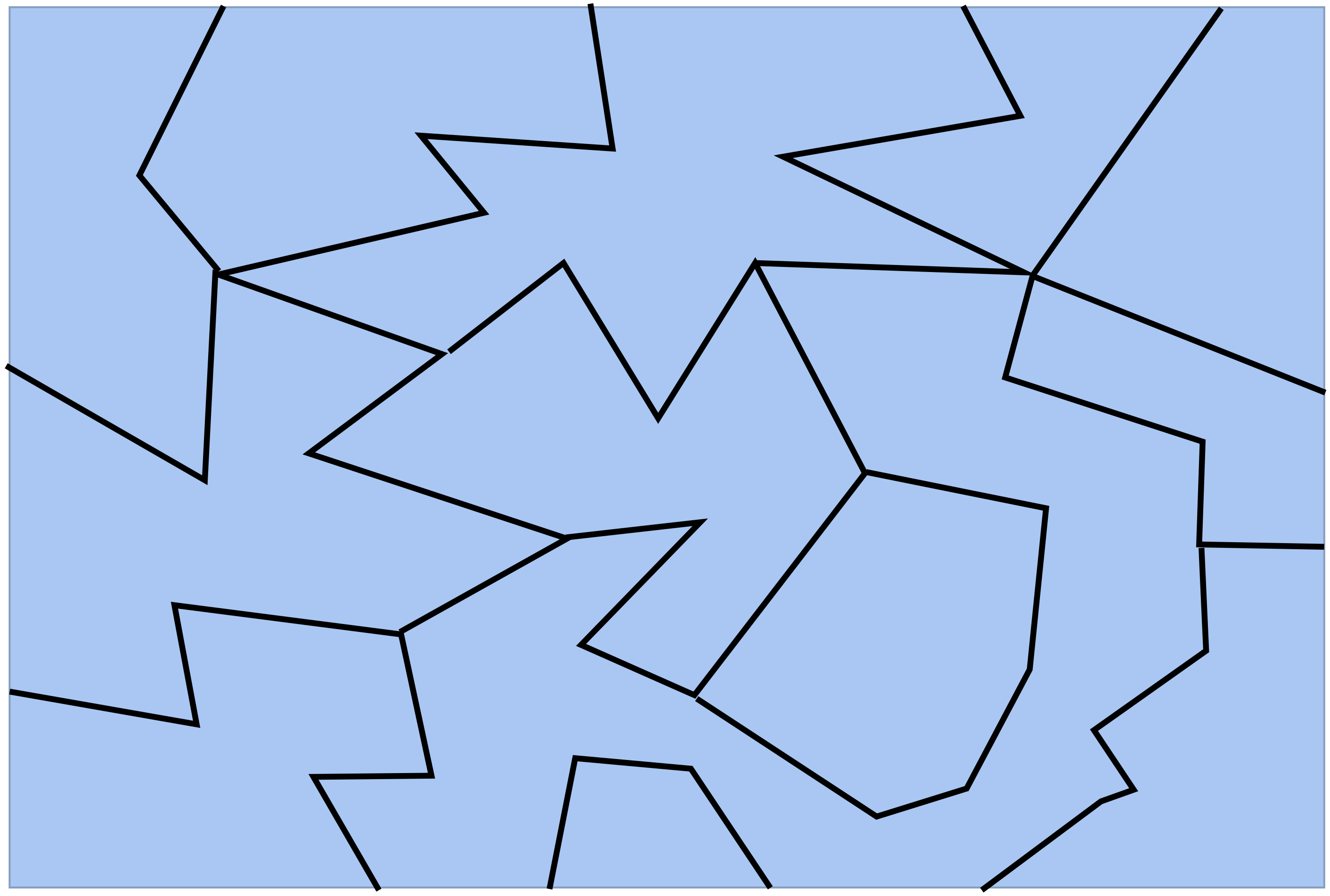




Voronoi cells (think of cell phone connecting to closest tower).



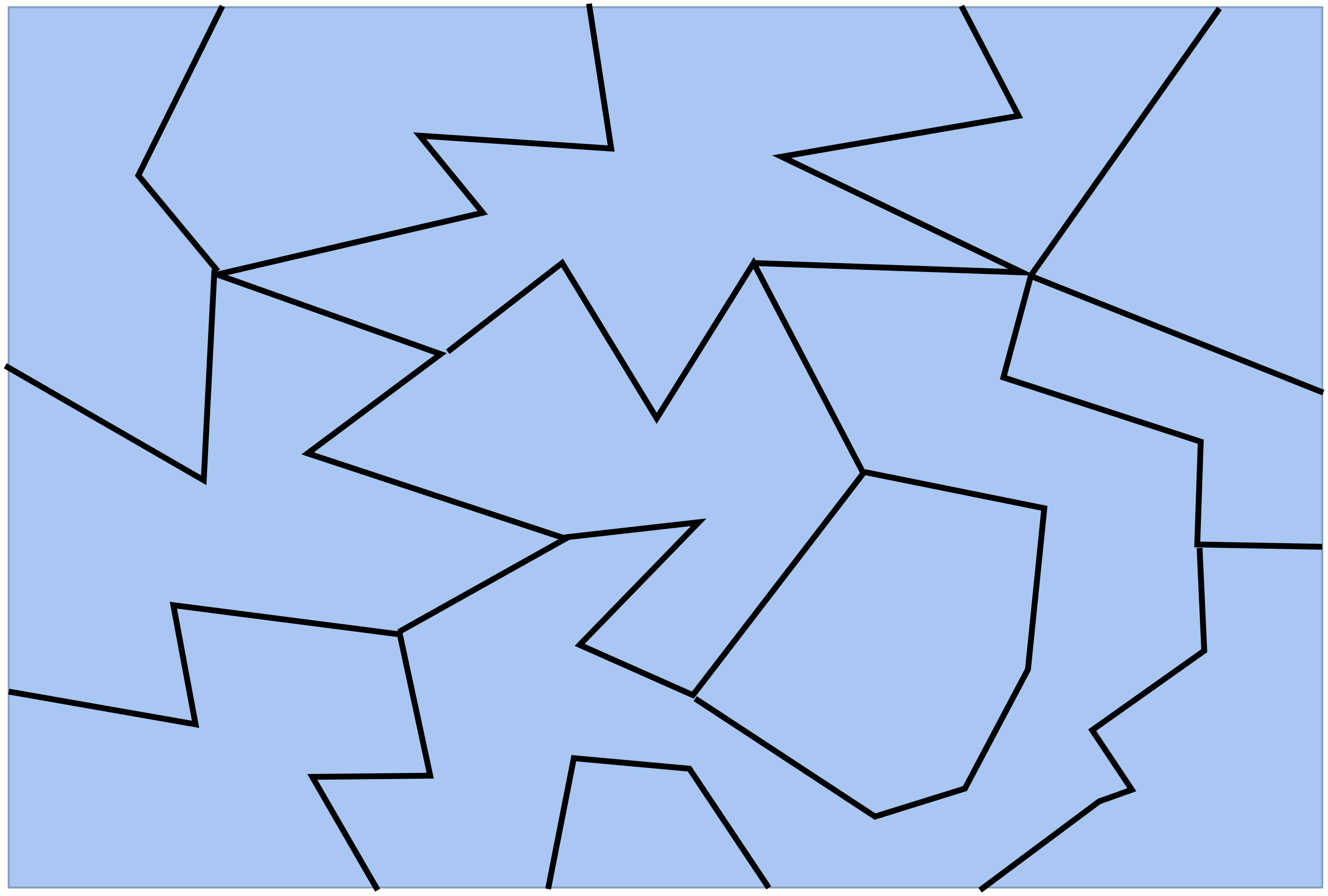
If country boundaries conform to cell boundaries, then a phone always connects to a tower in the same country.



Given countries, can we place towers so this happens?

Do a polynomial number of towers suffice?

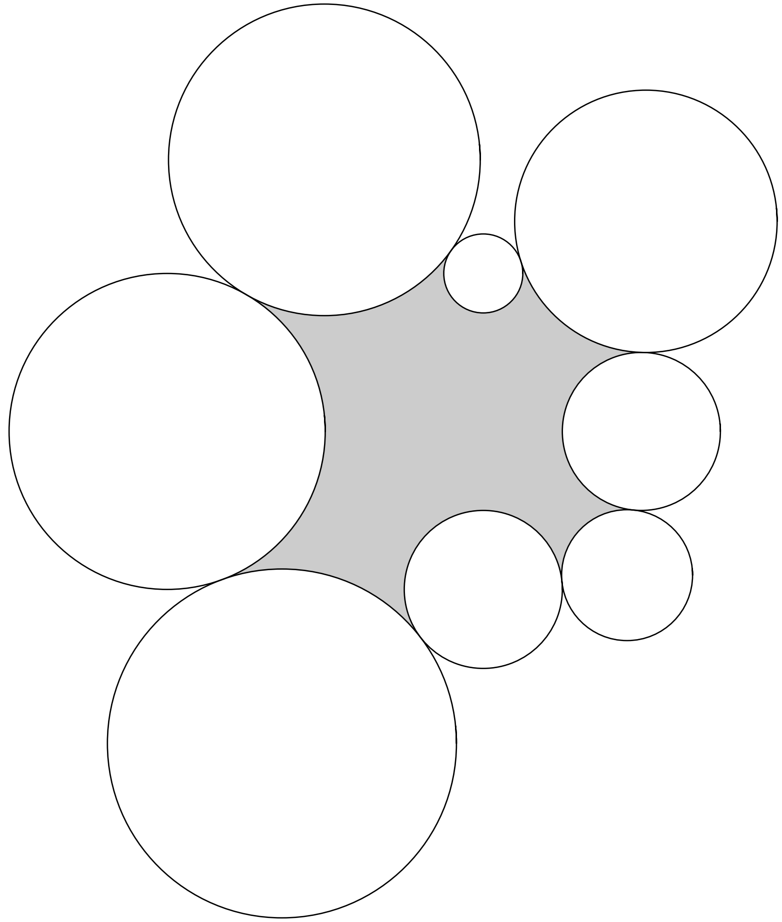
Question arose in machine learning (Salzberg, et.al, 1995)

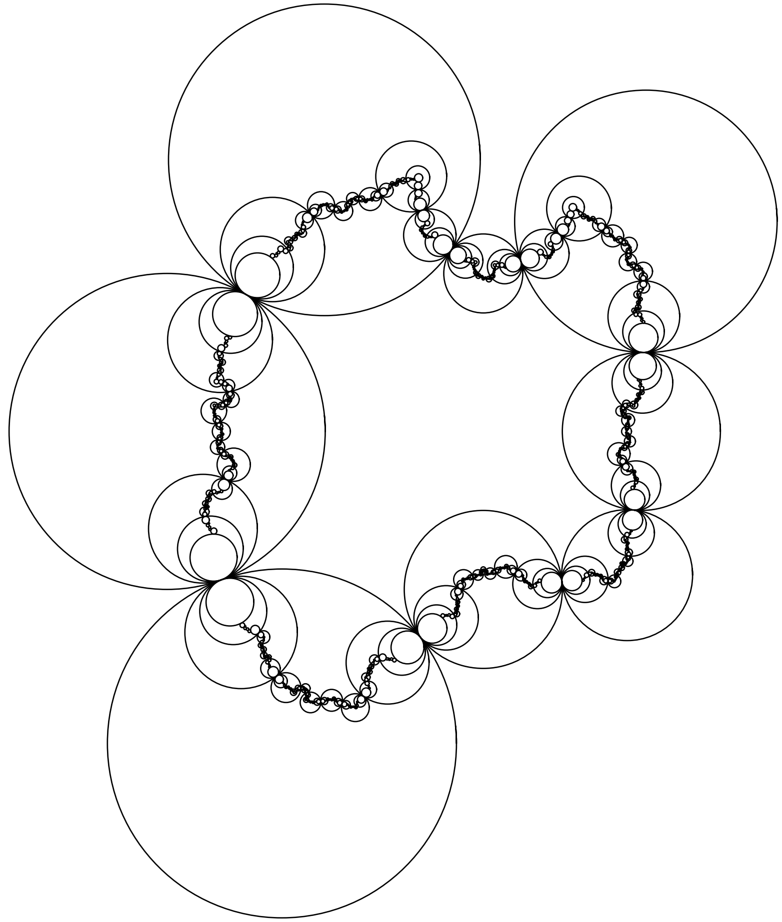


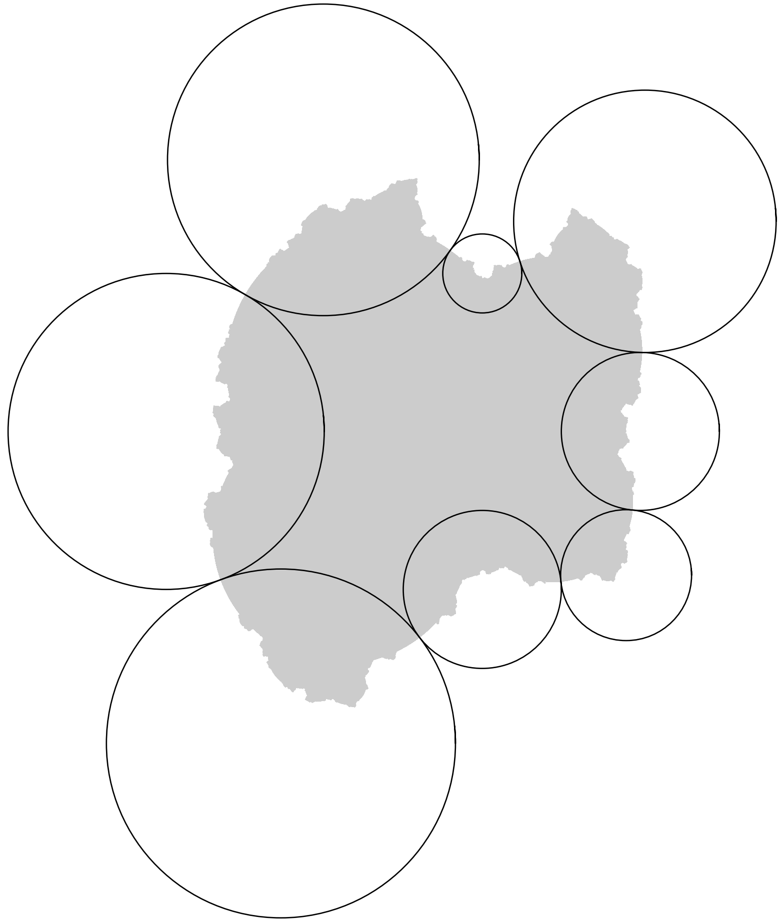
Given countries, can we place towers so this happens?

Do a polynomial number of towers suffice? Yes (B 2016)

Question arose in machine learning (Salzberg, et.al, 1995)







Bowen's Dichotomy: Limit set of a finitely generated quasi-Fuchsian group is either a circle or dim > 1.

$G =$ group of reflections

$\Lambda =$ limit set of G

$C \setminus \Lambda =$ ordinary set

$R = \Omega/G$ is a Riemann surface

Bowen's dichotomy is true for "small" surfaces.

Bowen's Dichotomy: Limit set of a finitely generated quasi-Fuchsian group is either a circle or

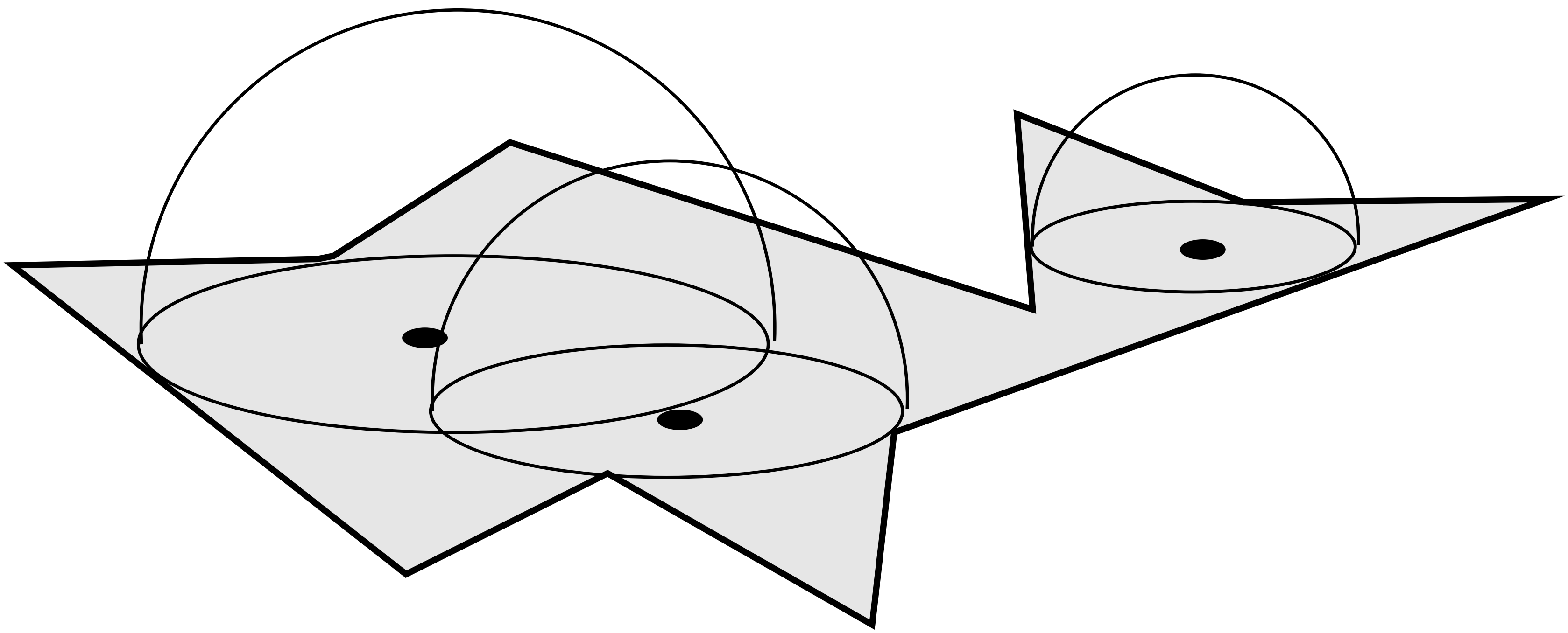
True for co-compact groups (Bowen, 1979).

True for finite area groups (Sullivan, 1984).

False if Brownian motion transient on R (Astala-Zinsmeister, 1990)

True if Brownian motion is recurrent (B, 2001)

The dome of $\Omega \subset C$ is the upper envelope of all hemispheres with base disk in Ω.

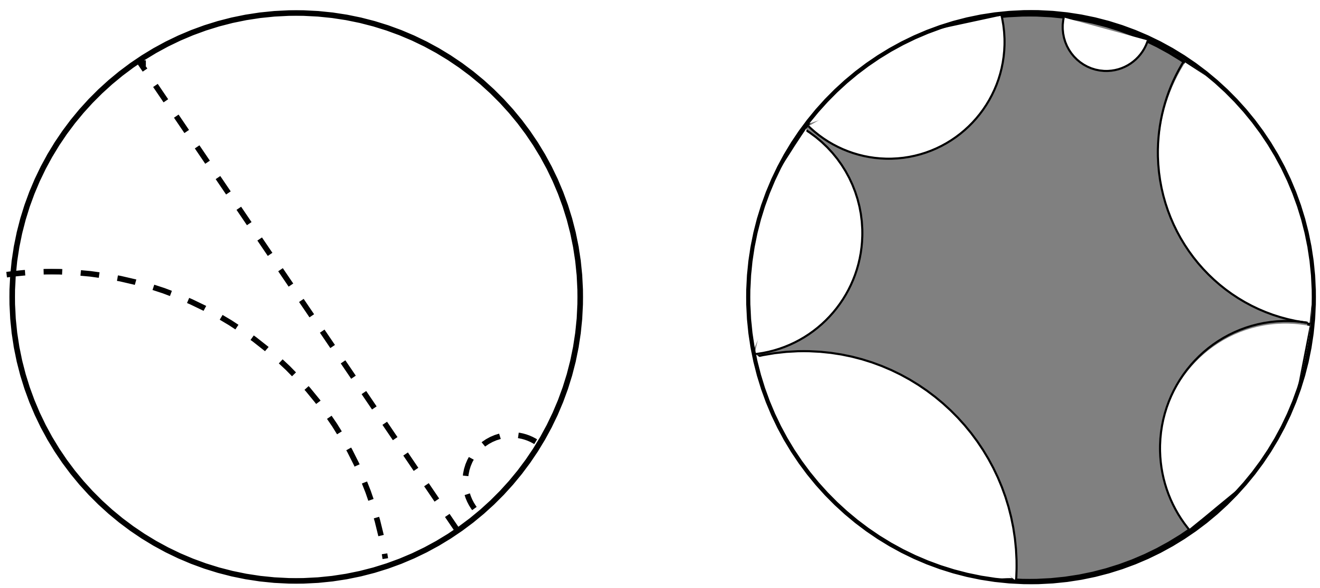


Dome = boundary of hyperbolic convex hull of $C \setminus \Omega$.

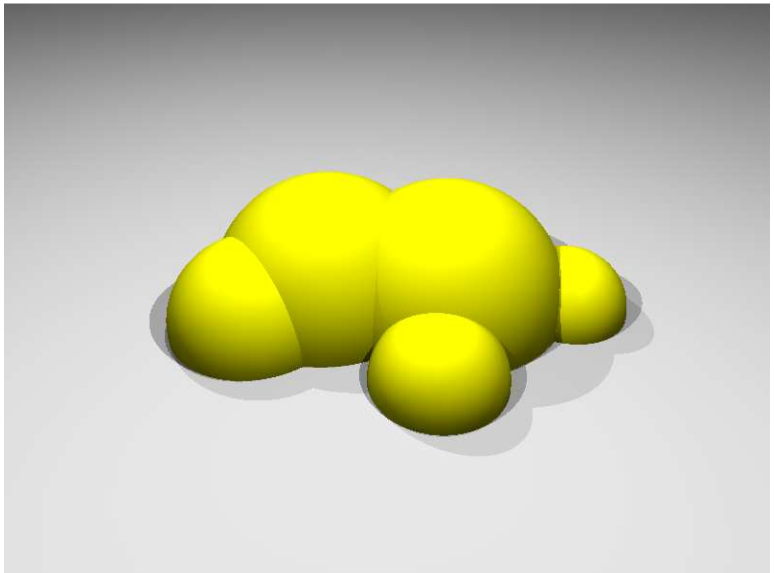
Hyperbolic metric on disk given by

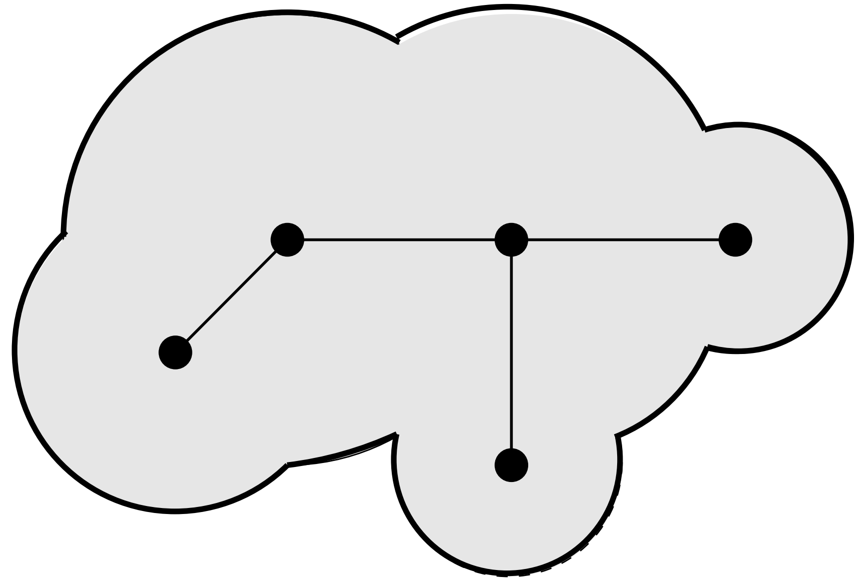
$$d\rho = \frac{ds}{1-|z|^2} \simeq \frac{ds}{\operatorname{dist}(z, \partial D)}.$$

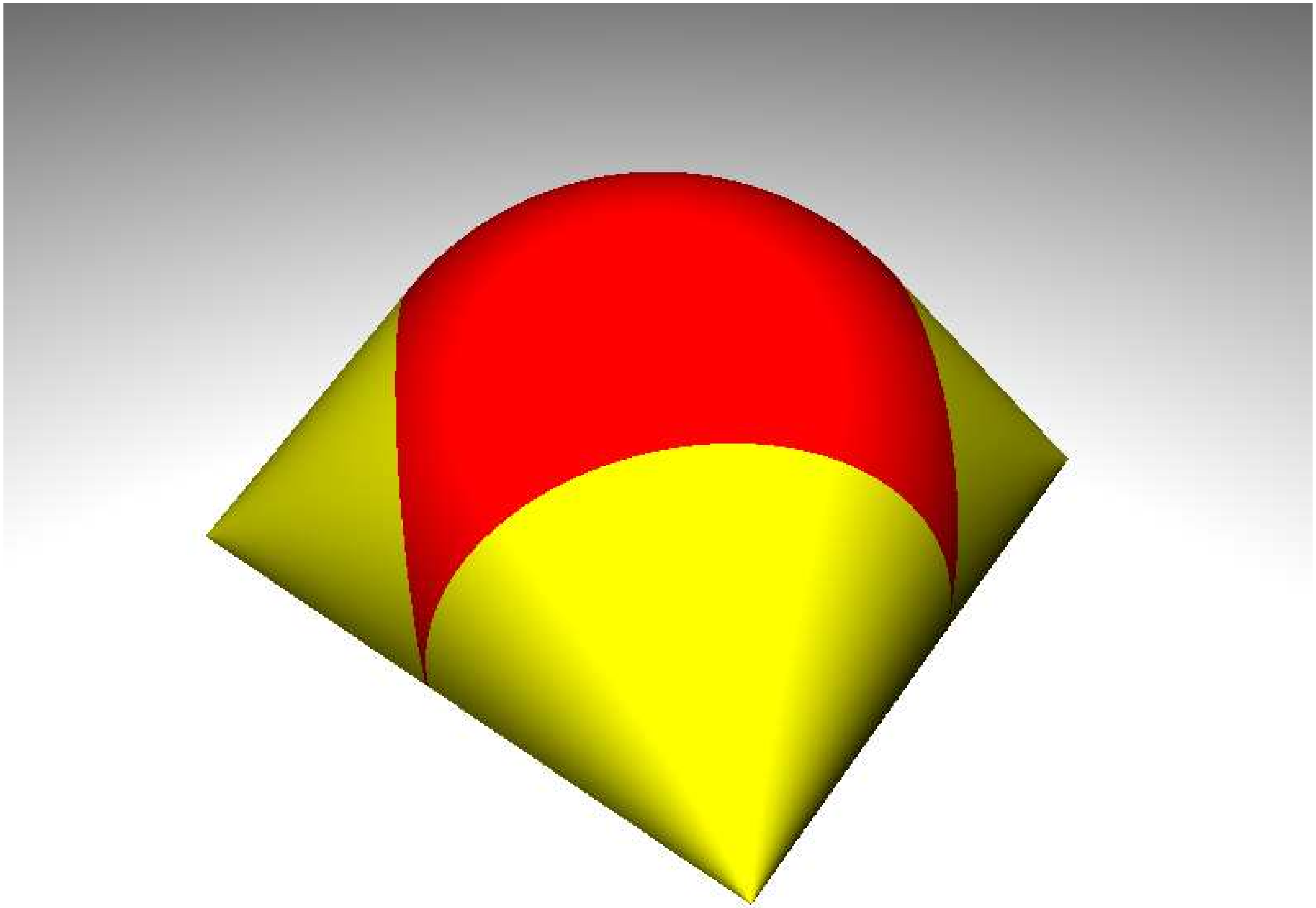
- Geodesics are circles perpendicular to boundary.
- Shaded region is hyperbolically convex.
- Metric transfers via conformal maps to other domains.



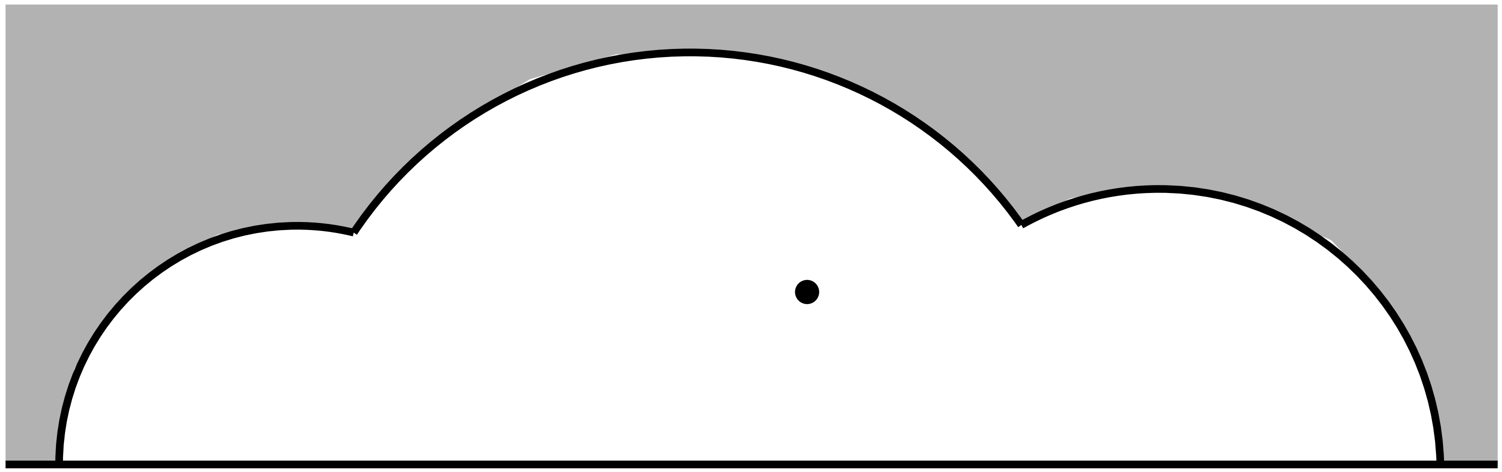
Finitely bent domain (= finite union of disks).



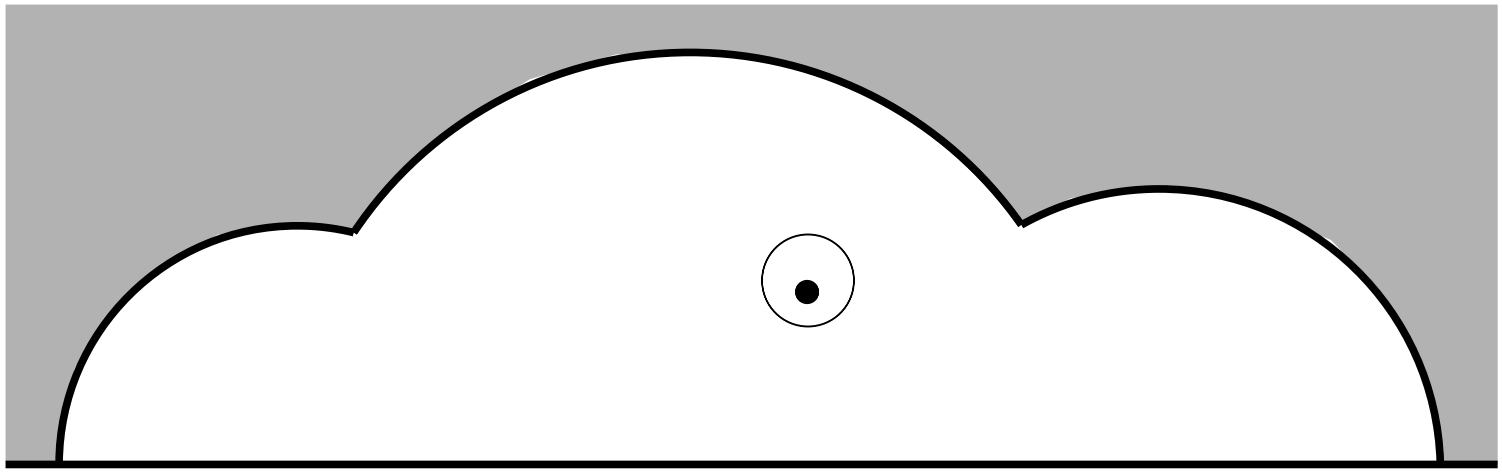




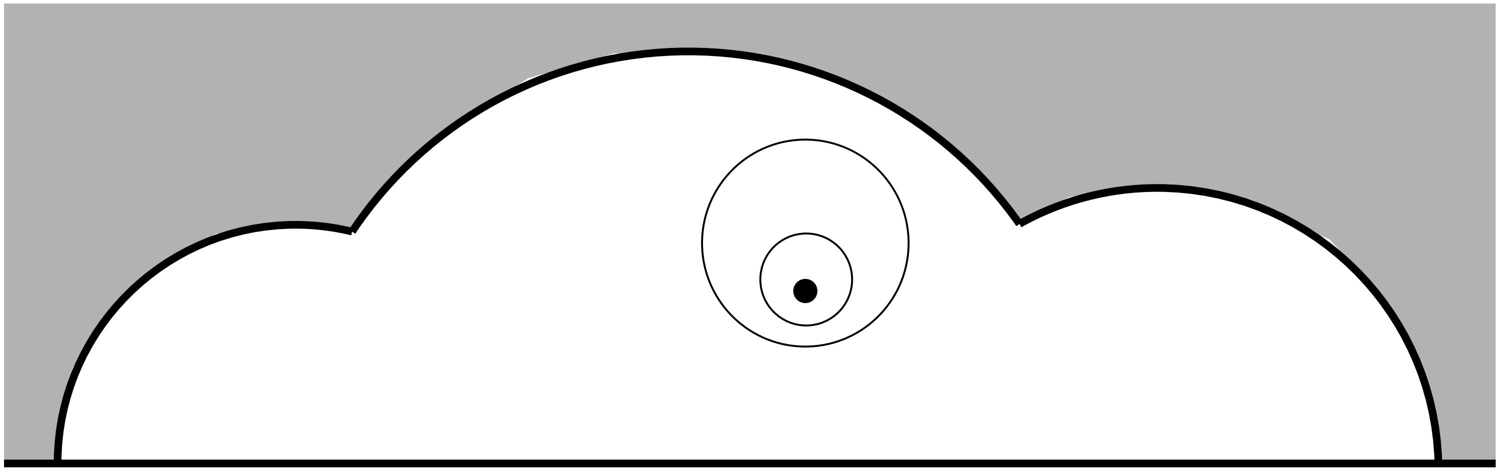
Dome of square



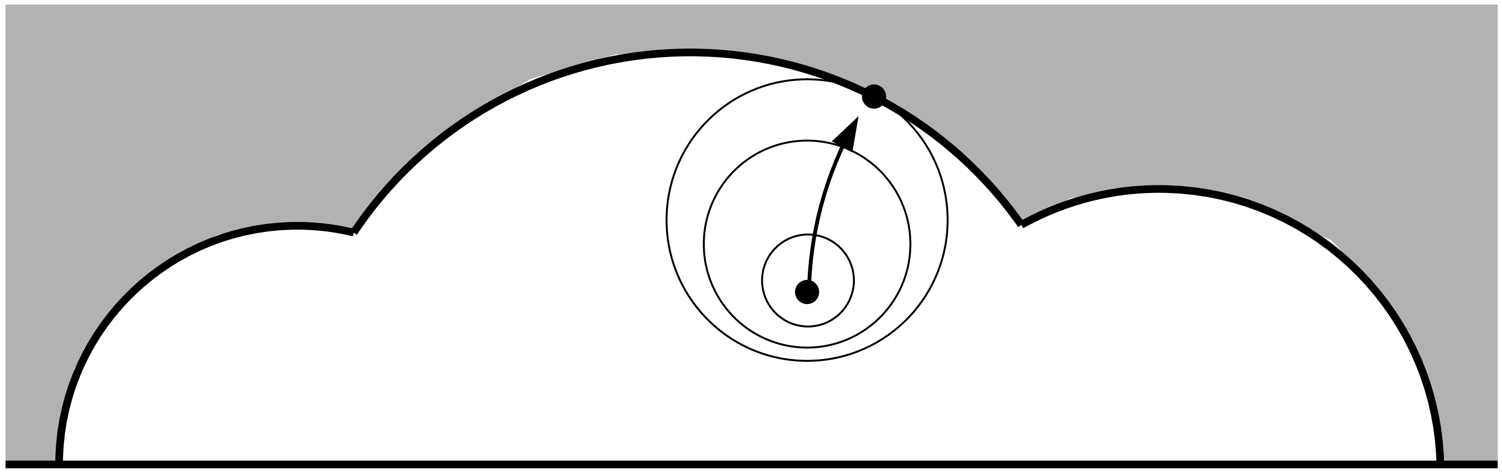
Nearest point retraction: expand ball until first contact.



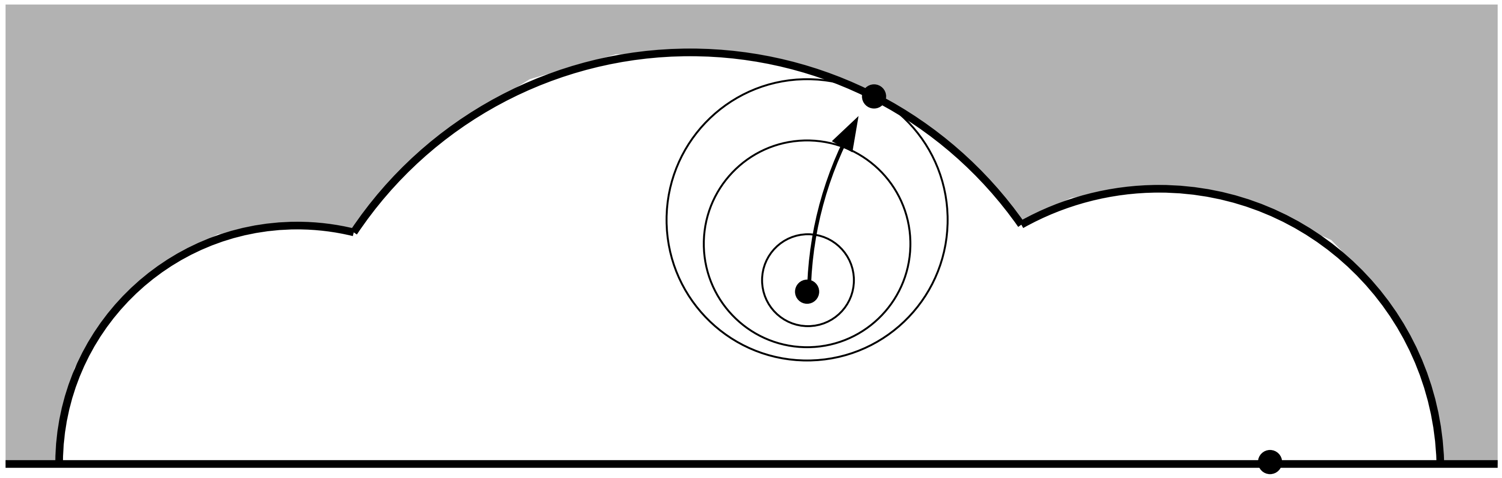
Nearest point retraction: expand ball until first contact.



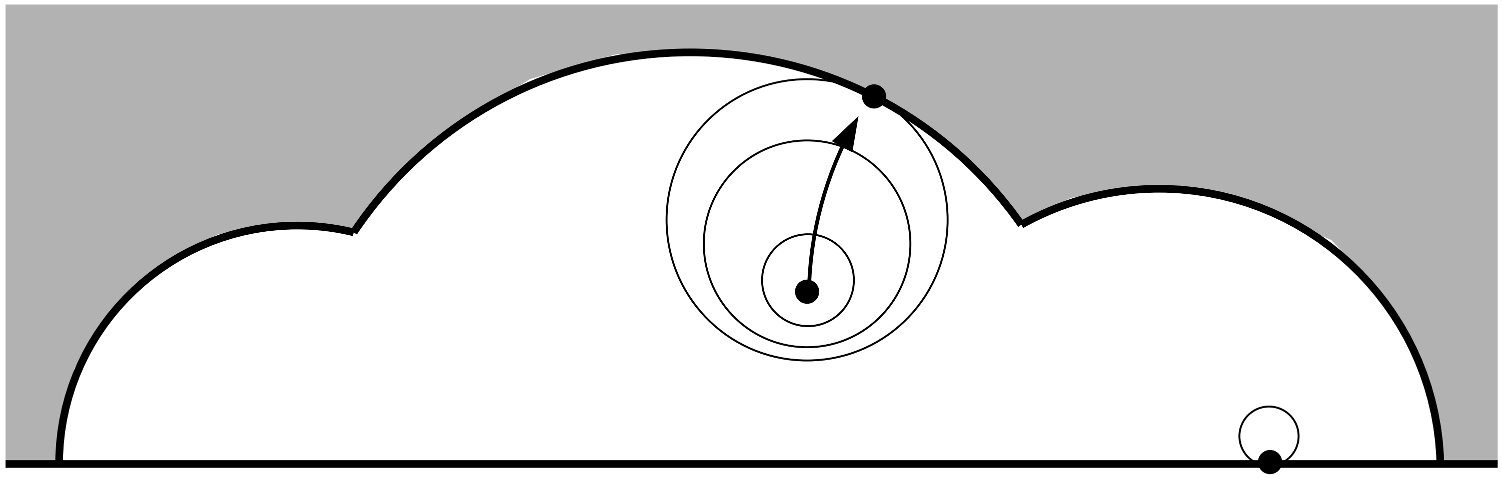
Nearest point retraction: expand ball until first contact.



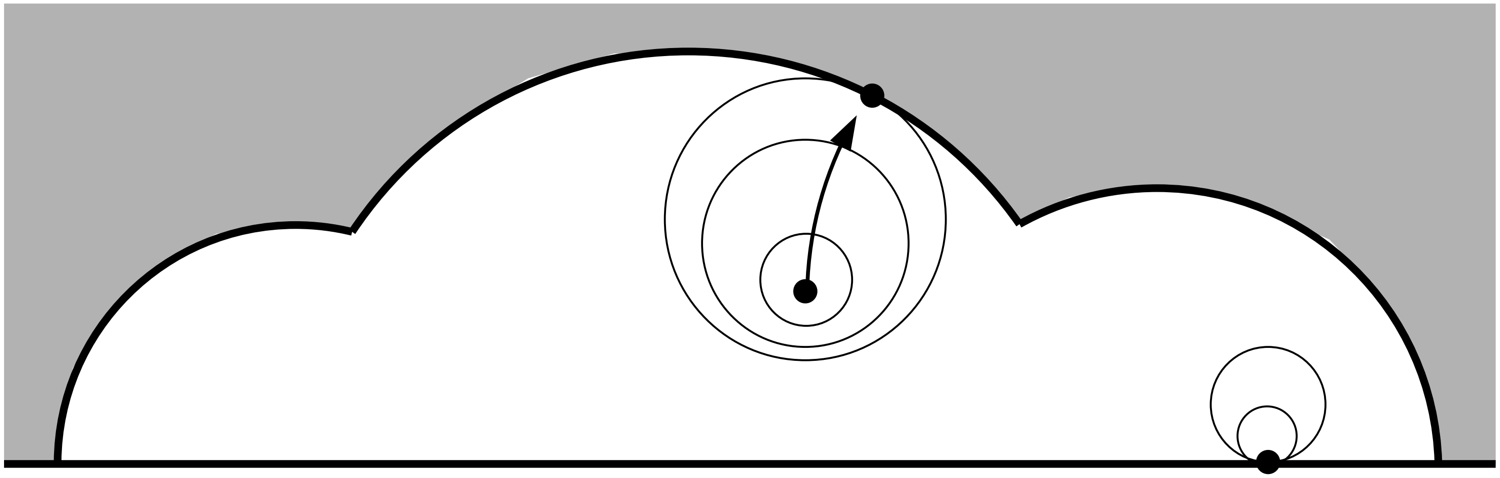
Nearest point retraction: expand ball until first contact.



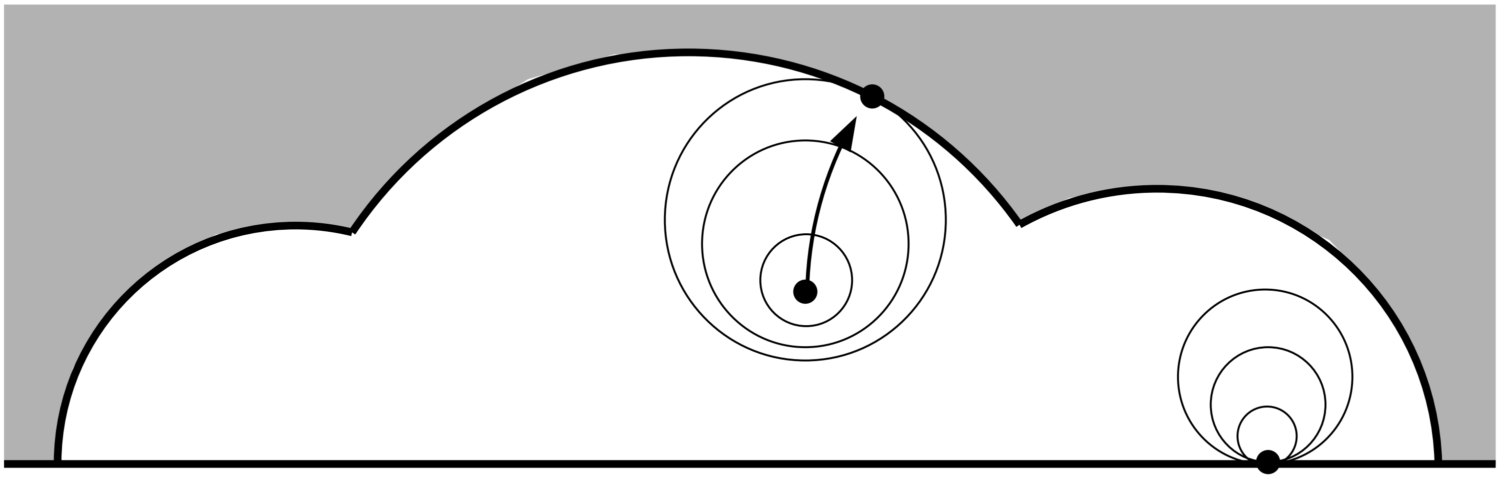
Extend to boundary by expanding horoballs.



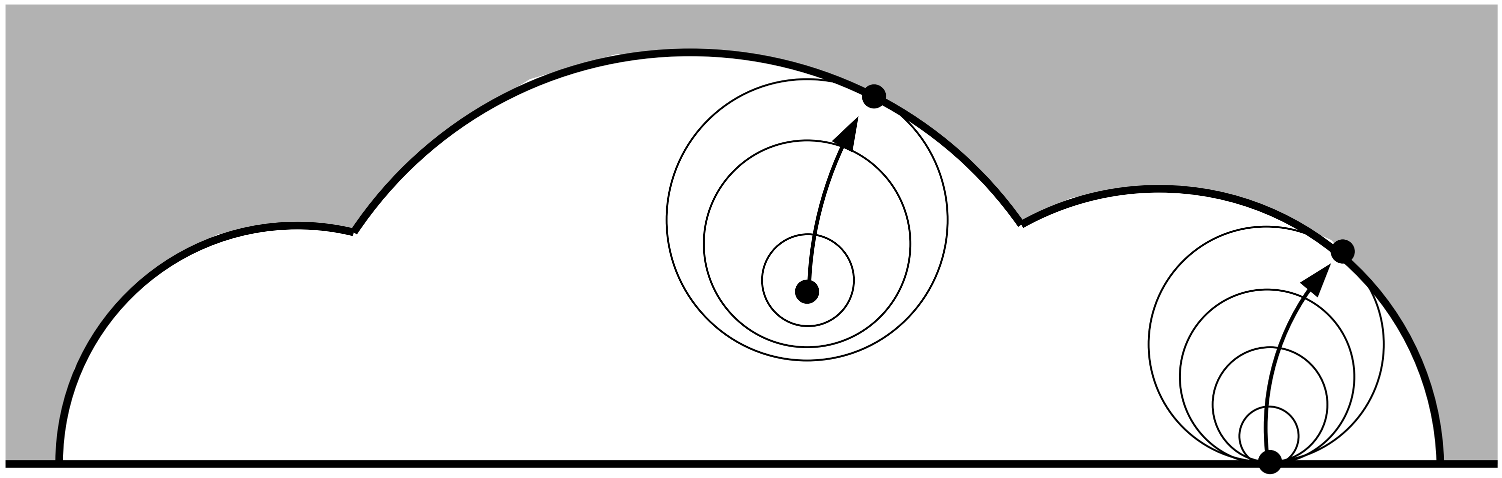
Extend to boundary by expanding horoballs.



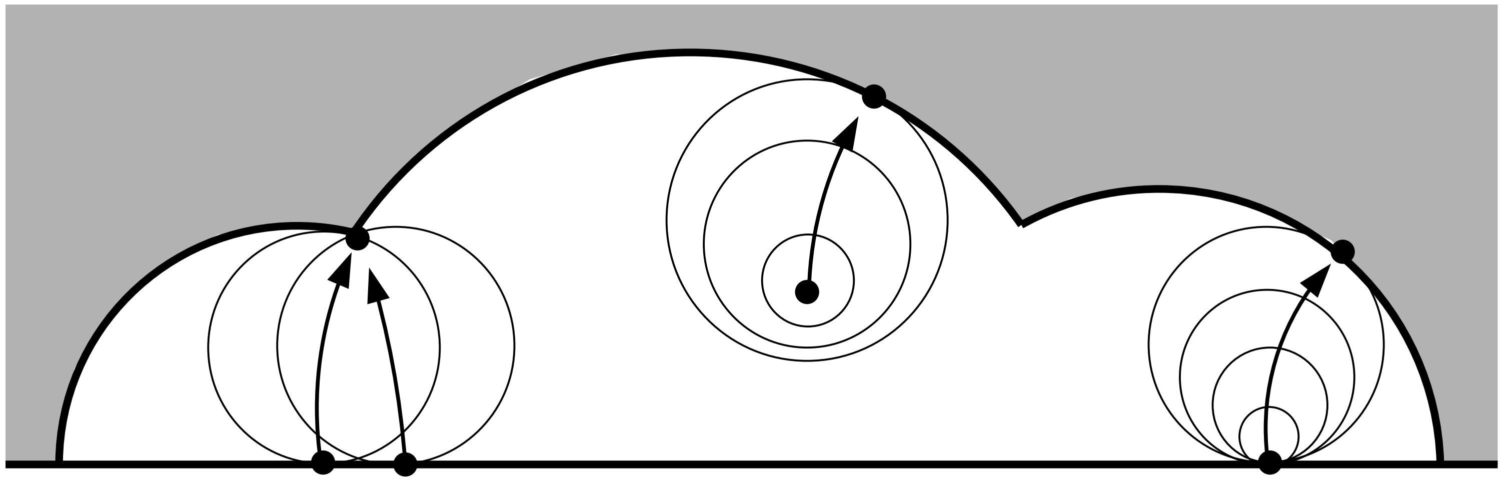
Extend to boundary by expanding horoballs.



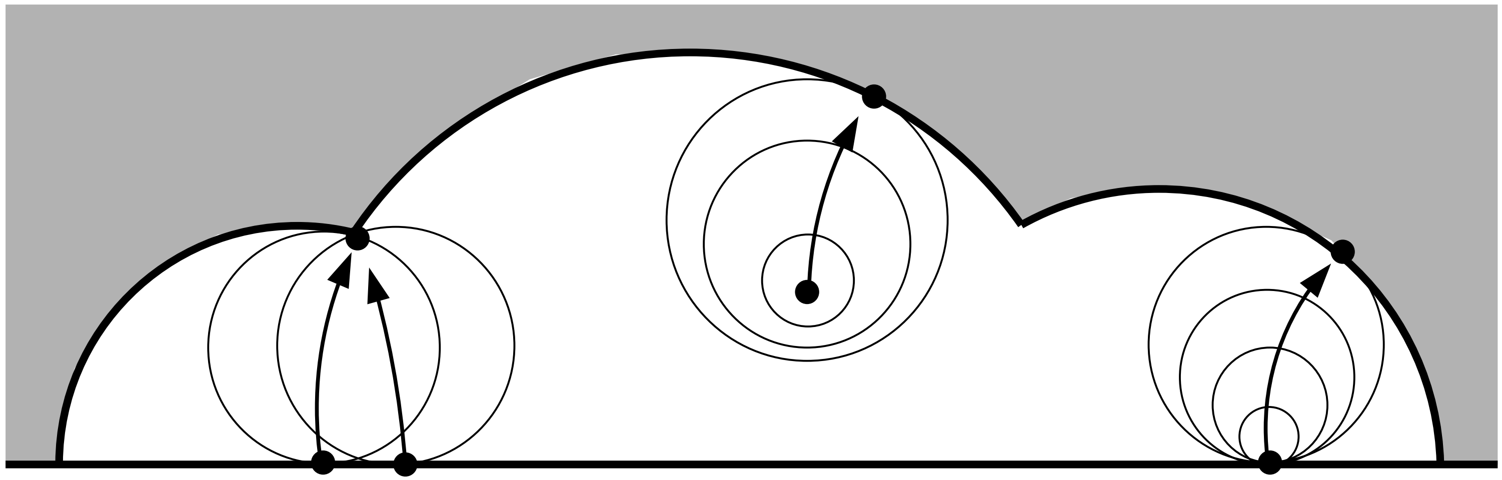
Extend to boundary by expanding horoballs.



Extend to boundary by expanding horoballs.



Not always 1-to-1.



Sullivan's Convex Hull Theorem: Nearest point retraction $R : \Omega \to S$ is a quasi-isometry: for $\rho_\Omega(x,y) > A$

$$\frac{\rho_\Omega(x,y)}{A} \le \rho_S(R(x), R(y)) \le A \cdot \rho_\Omega(x,y).$$

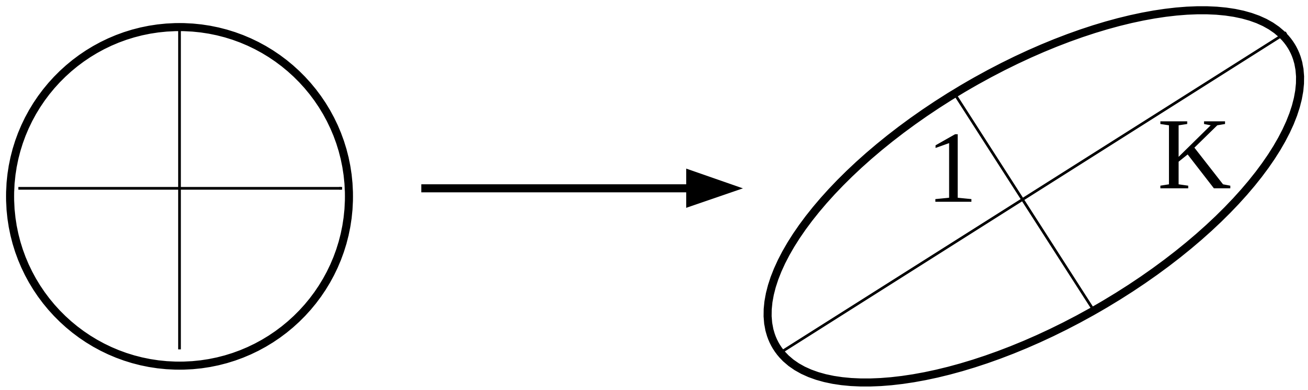
Bounds independent of Ω!

Gives biLipschitz or QC map $\Omega \to S$ fixing $\partial\Omega$.

Quasiconformal maps: bounded angle distortion

Roughly, a K-QC map multiplies angles by $\leq K$.

More precisely: tangent map sends circles to ellipses a.e.

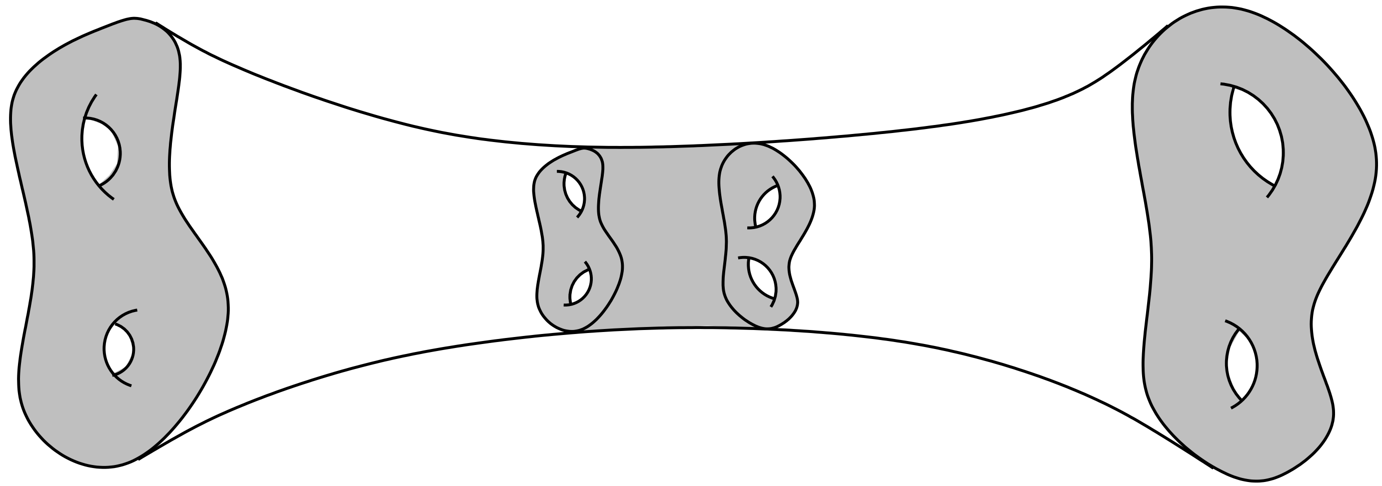


1-QC = conformal

$d_{QC}(z, w) = \inf \log K$ s.t. $\exists$ K-QC map $h(z) = w$.

QC approximation implies Euclidean approximation.

Original motivation: geometrization of 3-manifolds:



Theorem (Sullivan/Thurston): Boundary at ∞ of a hyperbolic 3-manifold is QC-equivalent to the boundary of its convex core.

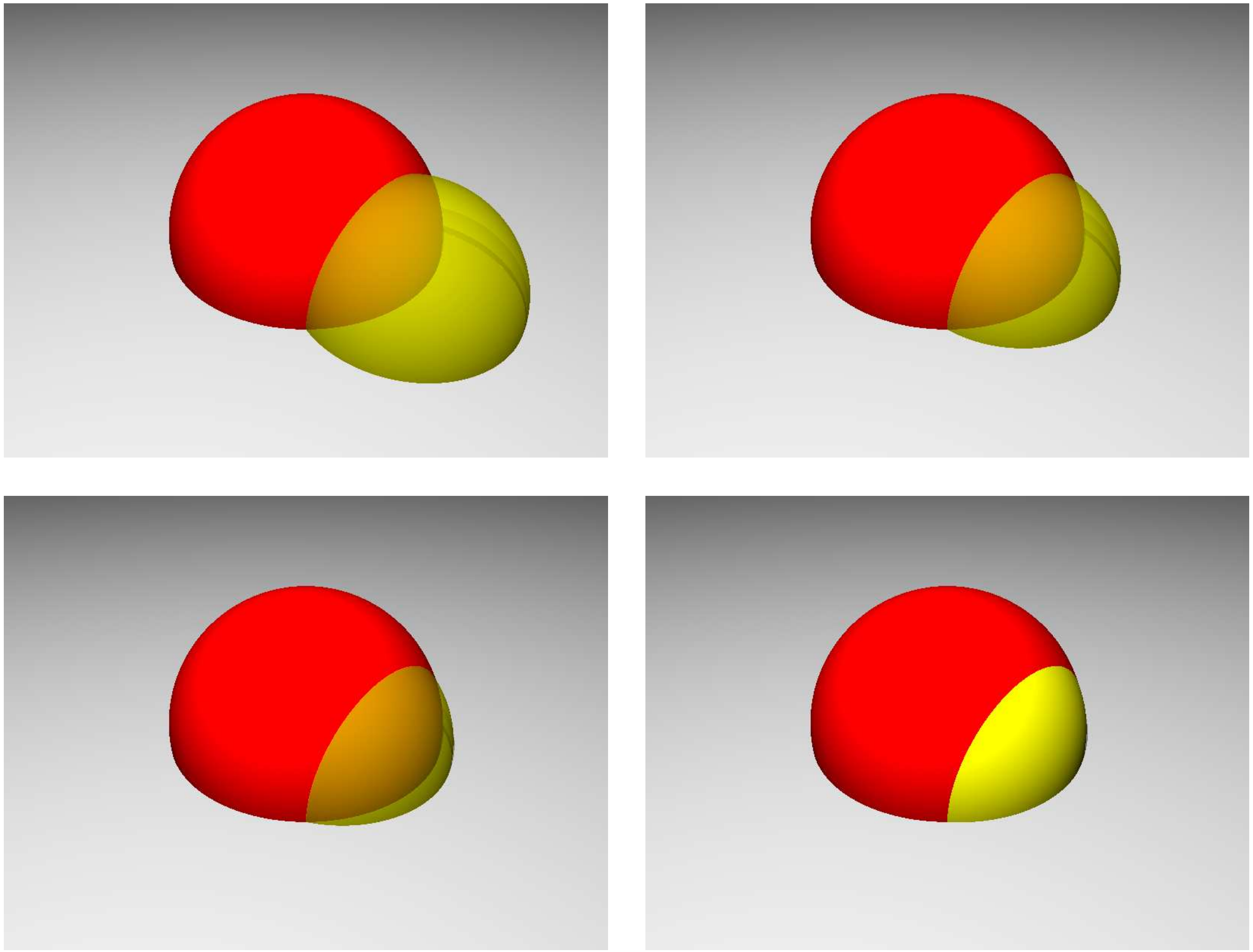
Manifold is upper half-space modulo a group of isometries.

convex core = convex hull of closed geodesics = “dome”

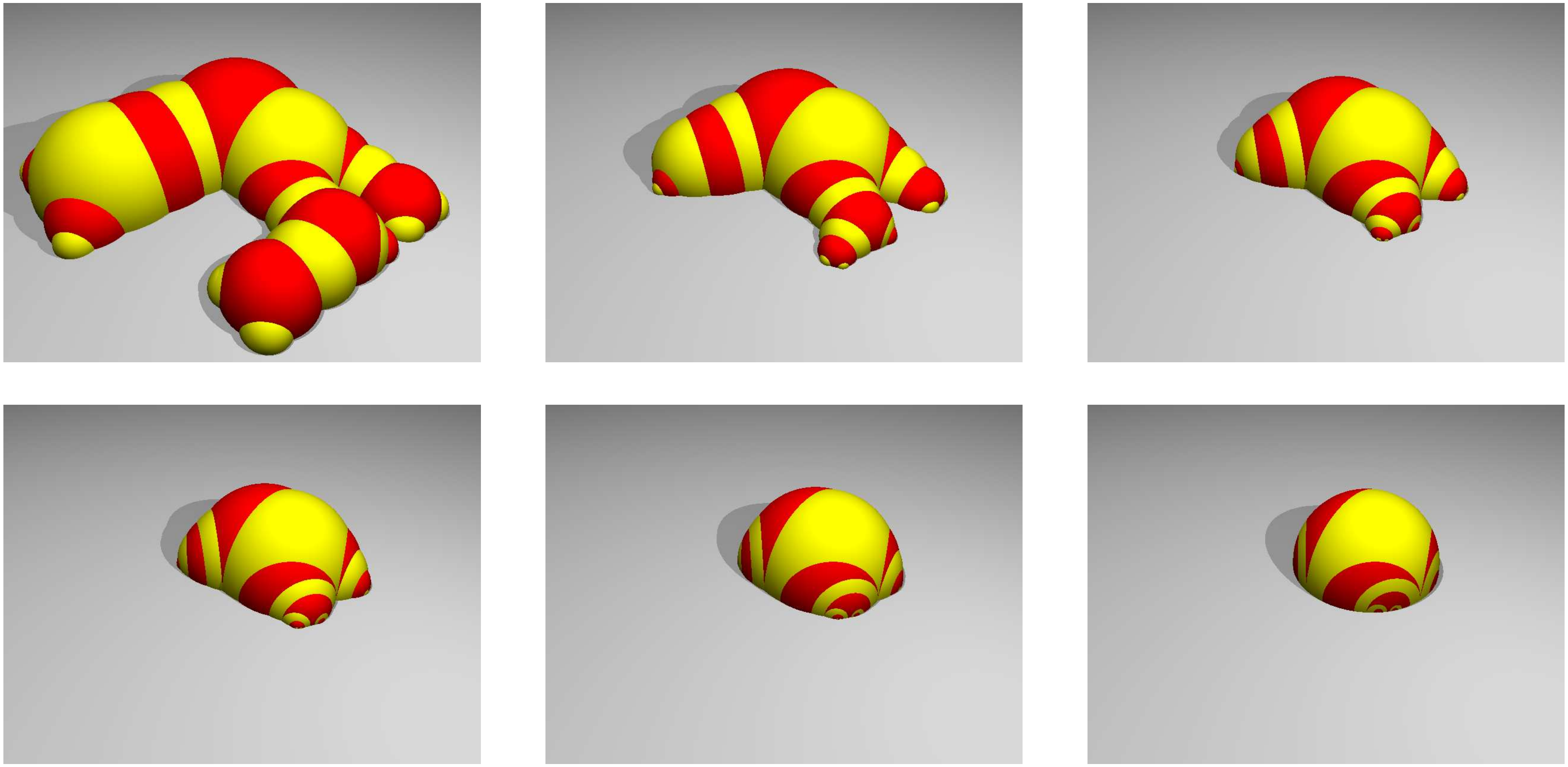
boundary at infinity = “base domain”

General domes: Epstein+Marden 1985, B. 2001.

Domes are isometric to hyperbolic disk. Isometry called Iota.



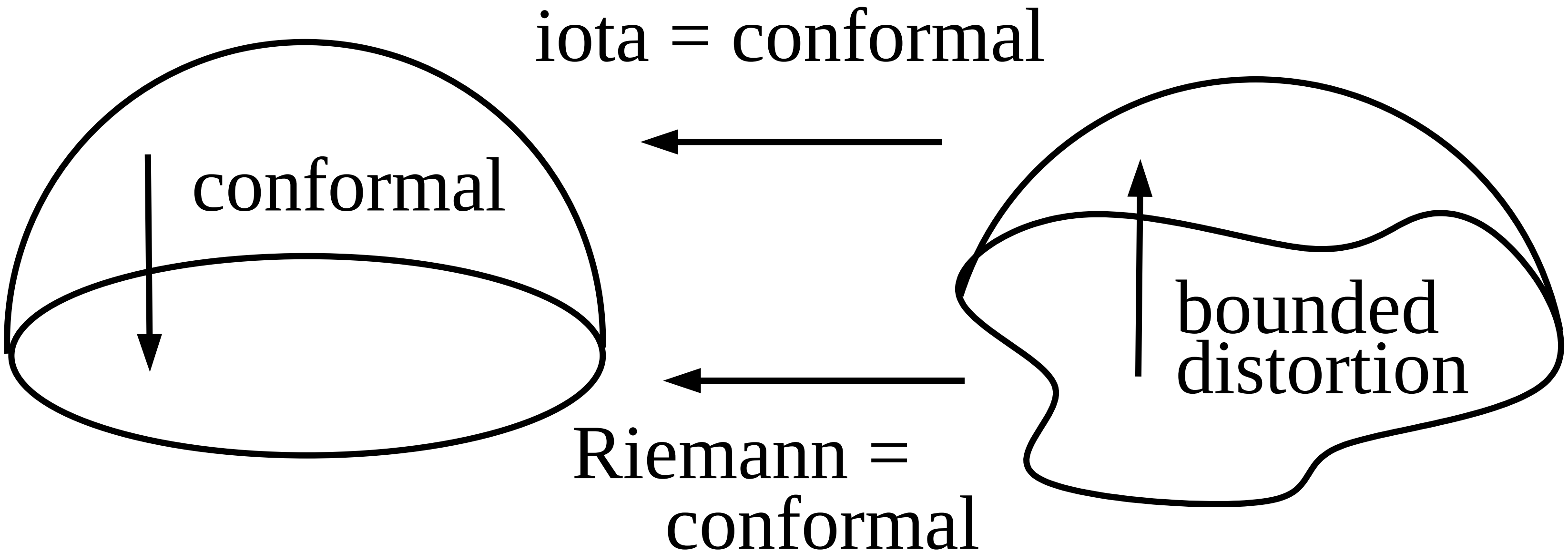
Consider union of two disks: rotating around common geodesic doesn't change path metric.



"Flattening" sends finitely bent surface to hyperbolic plane.

General dome is limit of finitely bent surfaces.

Limit of isometries is an isometry.

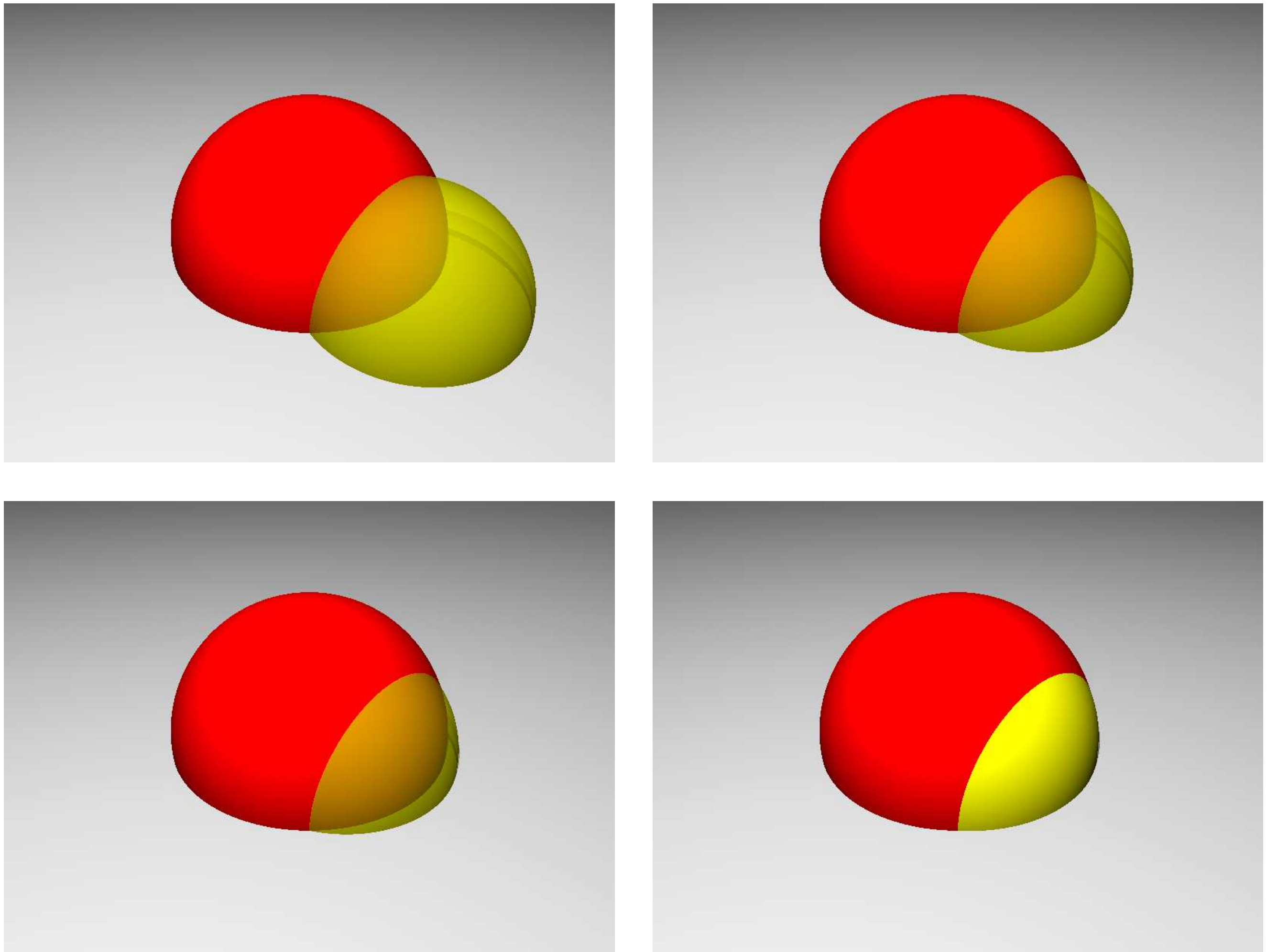


Iota on $\partial\Omega$ extends to a K-QC map $\sigma : \Omega \to D$.

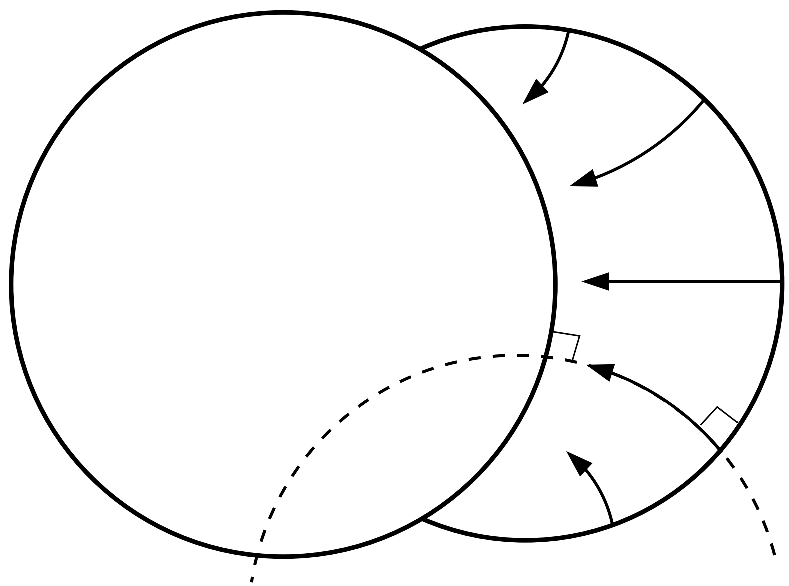
K independent of Ω!

Iota is fast to compute:

Theorem (B 2010): For an n-gon the iota images of the n-vertices on the unit circle can be computed in time $O(n)$.

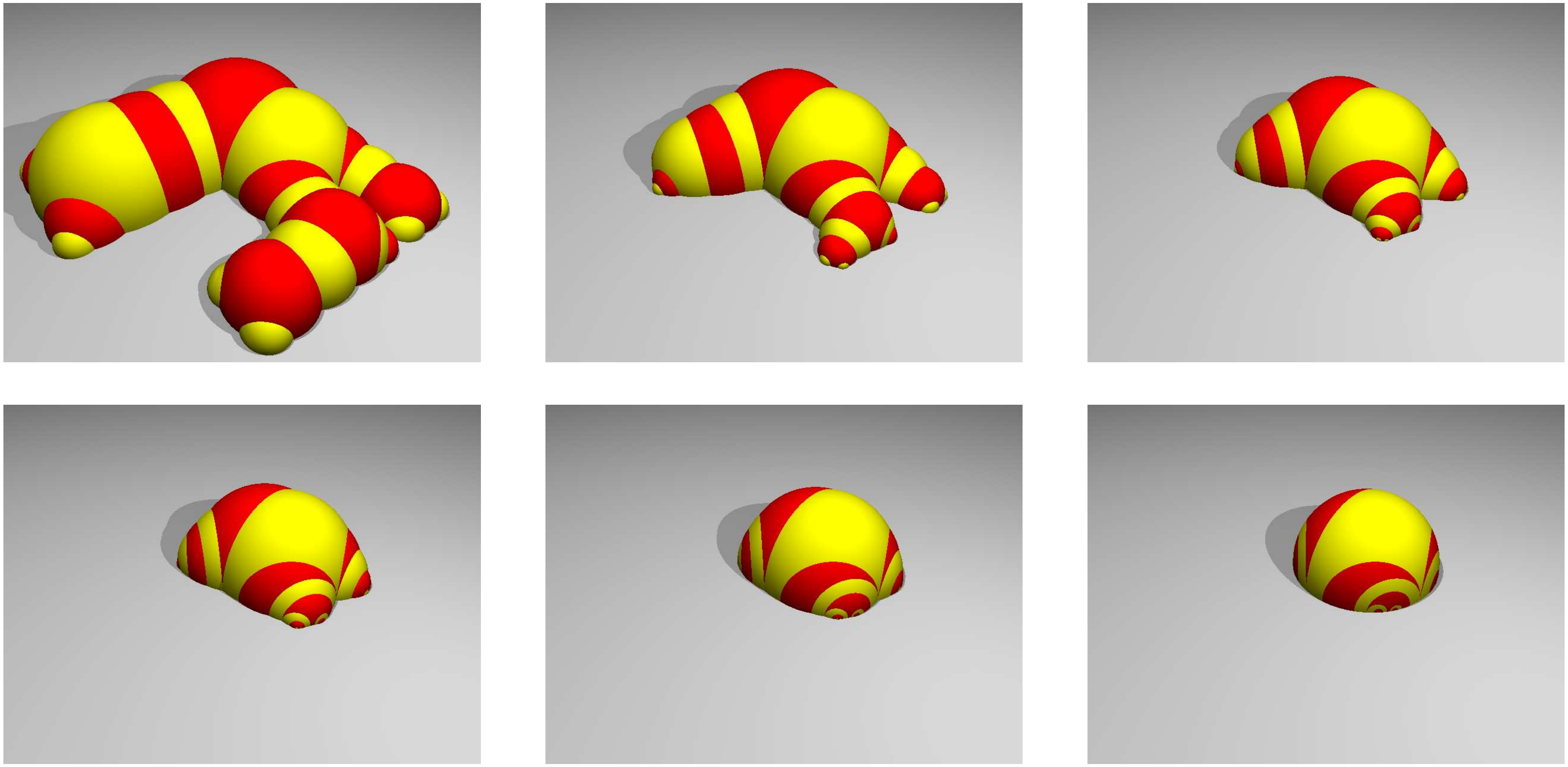


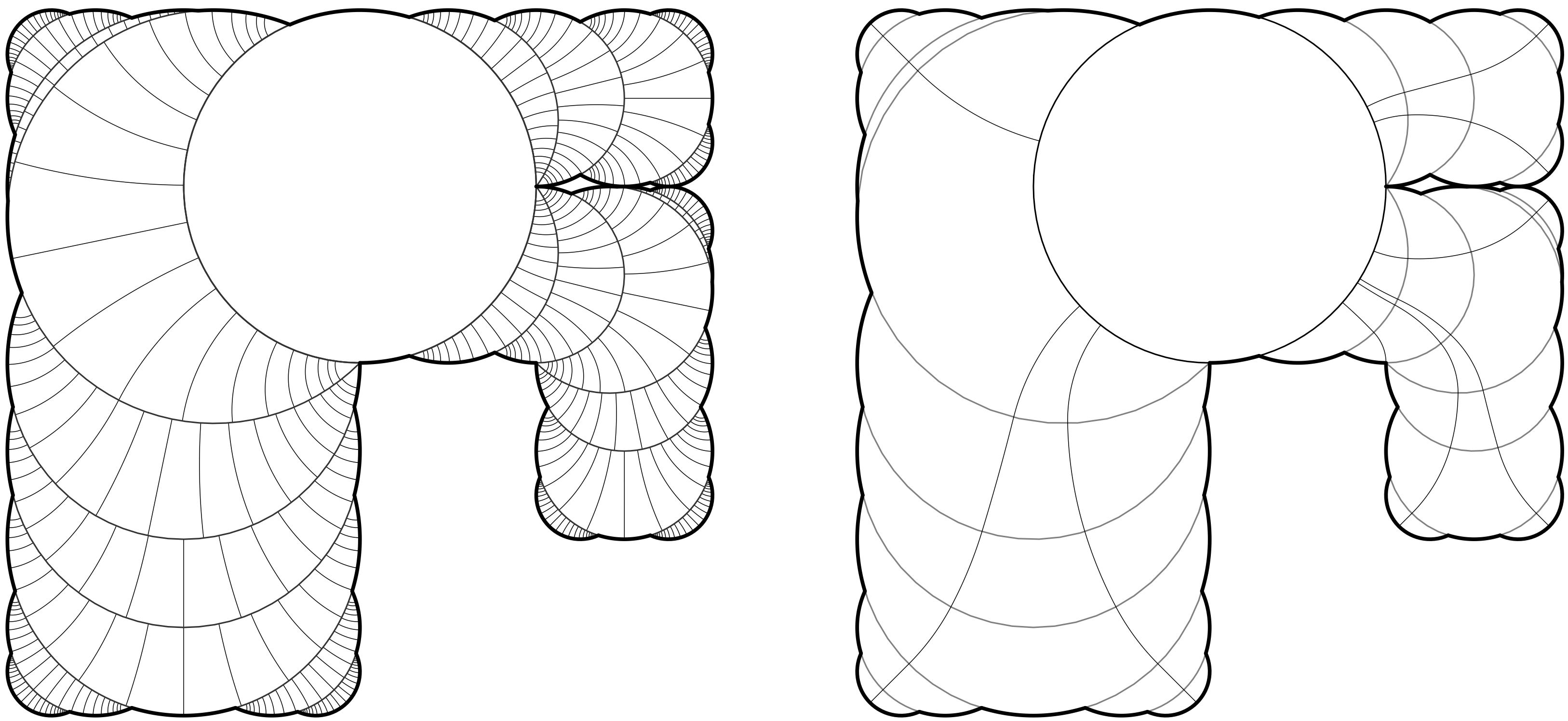
Flattening dome in 3-space, corresponds to collapsing crescents in base by collapsing orthogonal arcs.



Points move by elliptic Möbius map fixing circle intersections.

Iota is “built” from Möbius transformations.





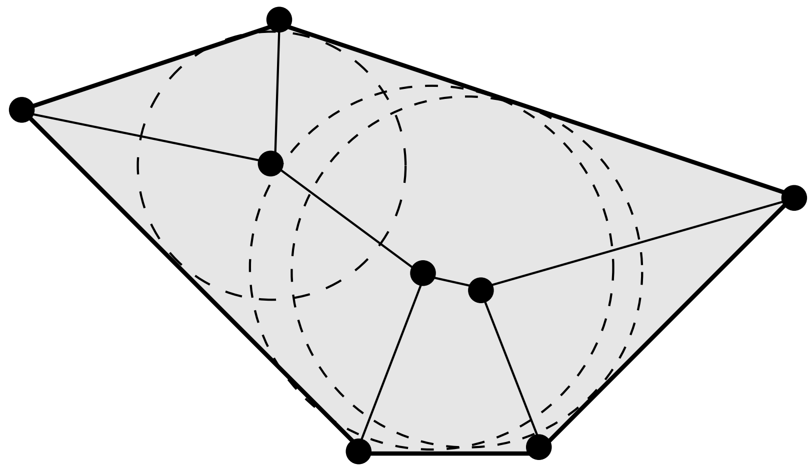
Defines Medial Axis flow from boundary to circle.

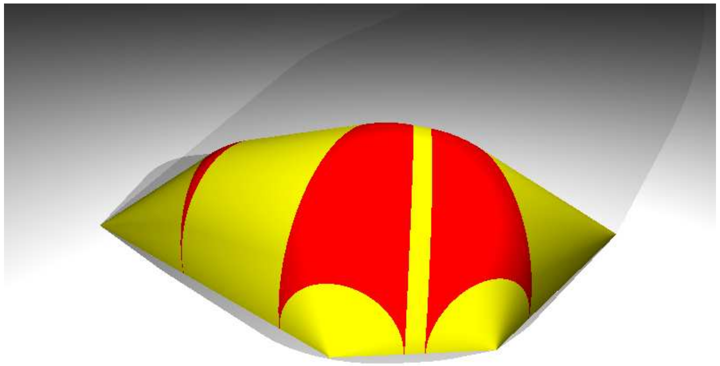
Easy to define for finite unions of disks.

Taking limits, this flow exists for any domain.

What is natural way to write domain as union of disks?

Medial axis (MA) is set of centers of subdisks of Ω that hit boundary in at least two points.

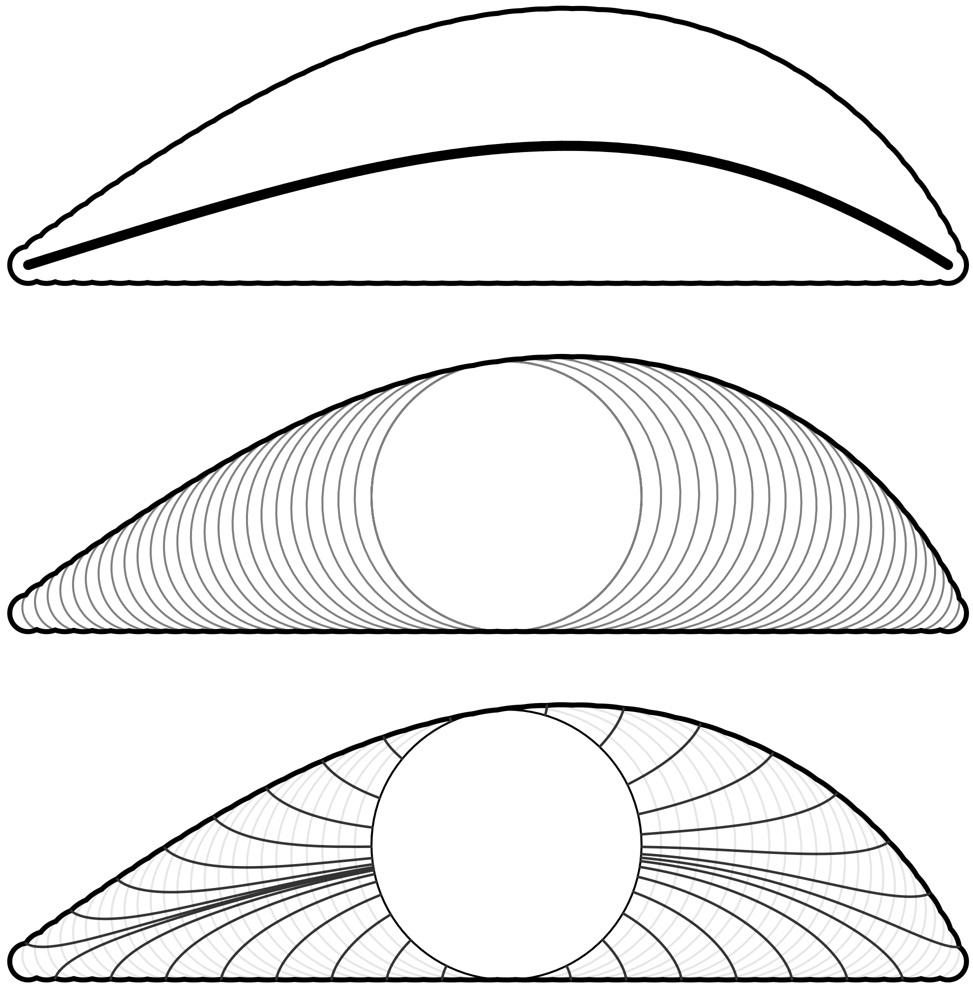




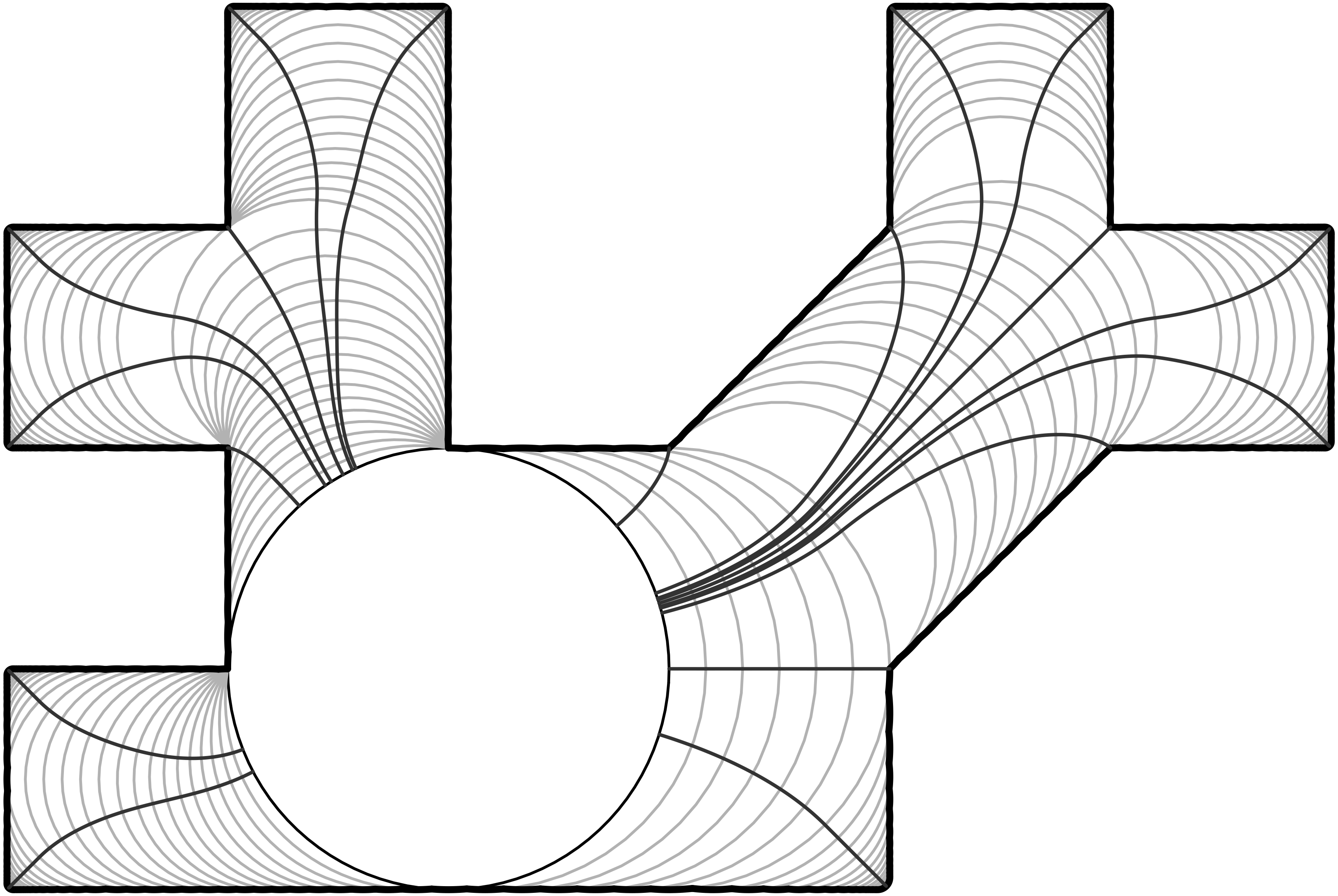
For a n-gon, MA is a finite tree, computable in $O(n)$ time.

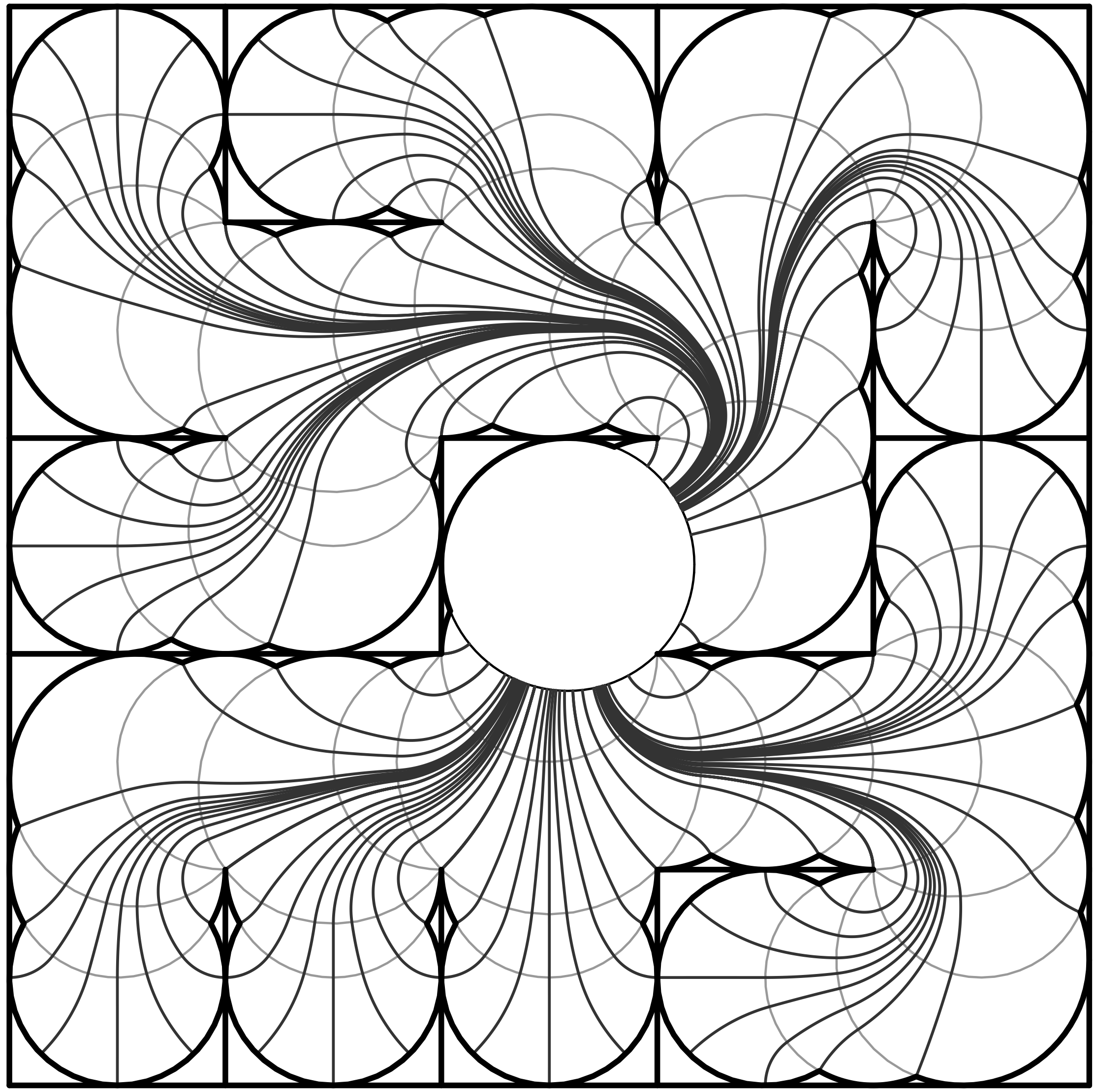
Chin, Snoeyink, Wang 1999. Based on (hard) linear triangulation of Chazelle 1991.

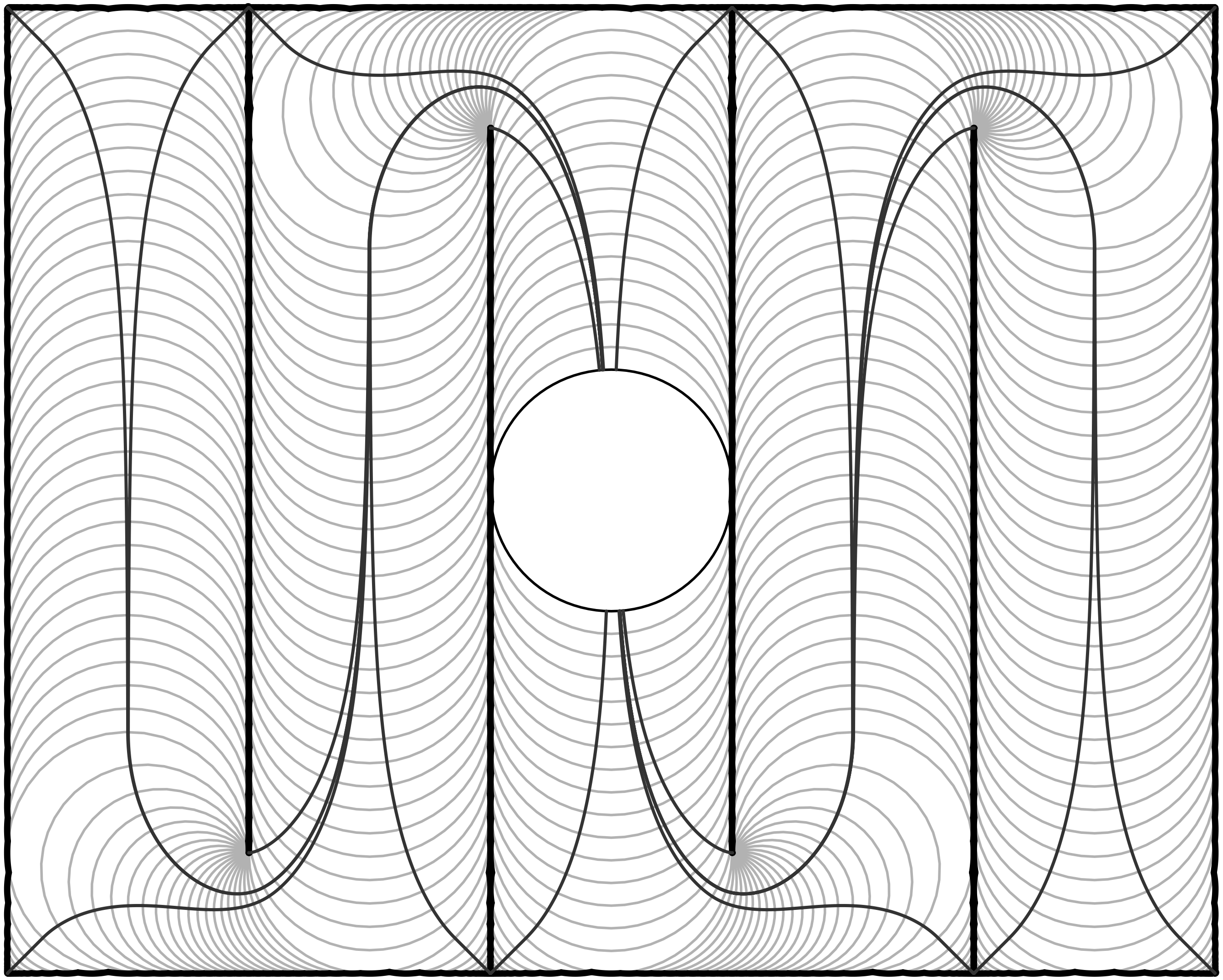
Iota-images of all n vertices can be computed in time $O(n)$ from Medial Axis (B., 2010).



Fix base disk. MA disks foliate remainder. Medial axis flow goes from domain boundary to disk, orthogonal to foliation





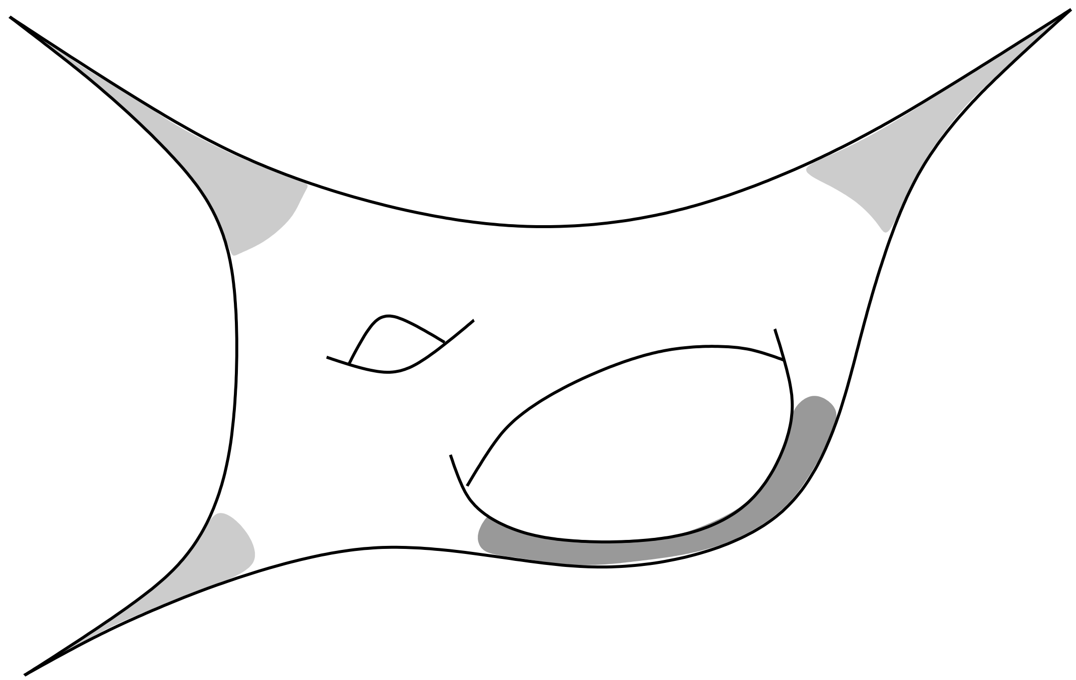


Fast QC Mapping Theorem: Given an n-gon we can compute in time $O(n)$ the images of all n vertices under some 8-QC map to the disk.

Can we take $K \to 1$? (fast conformal mapping)

First, we need another idea from hyperbolic geometry.

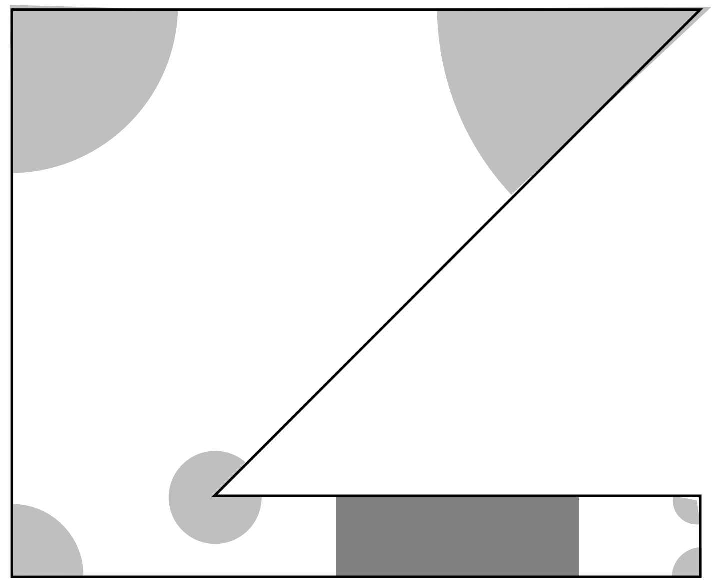
Thick and thin parts of hyperbolic surfaces



Thin part is union of non-trivial loops of length $\leq \epsilon$.

parabolic = punctures, hyperbolic = handles

Thick and thin parts of polygons

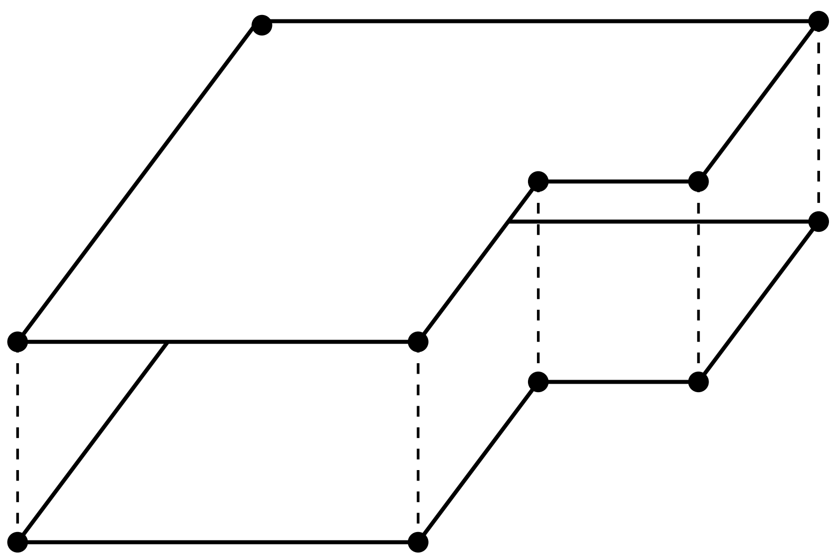


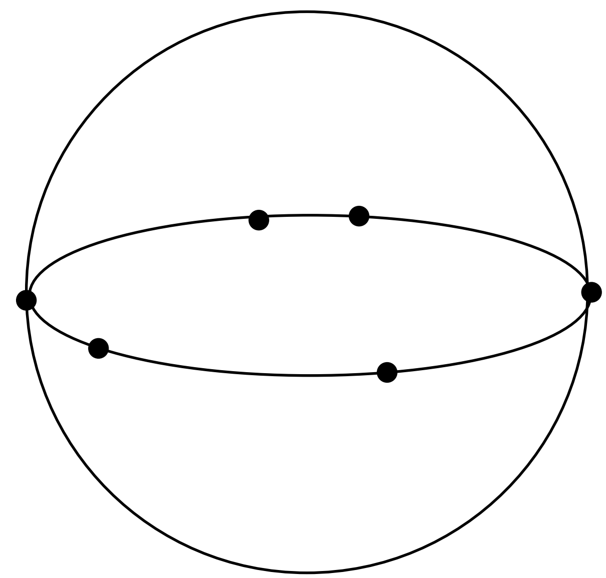
Thin parts connect sides with small extremal distance.

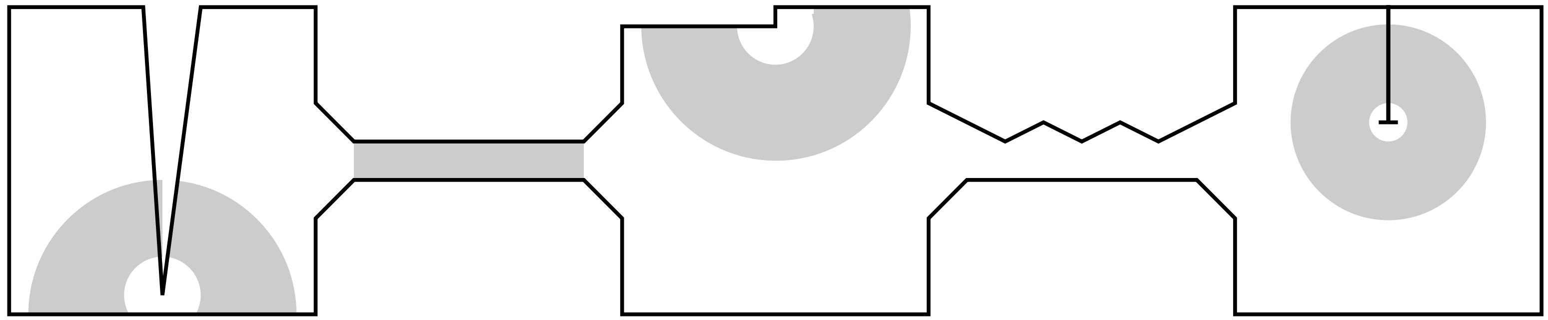
Roughly: distance between sides $\ll$ minimum side diameters

Parabolic = adjacent side ,   Hyperbolic = non-adjacent sides

Polygon thin parts correspond to thin parts of n-punctured sphere obtained by gluing 2 copies of n-gon along open edges.







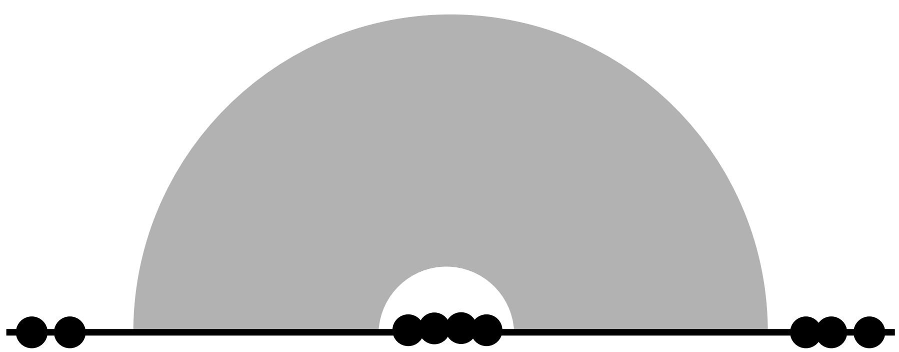
Each thin part partitions polygon vertices into two groups highly separated in extremal distance.

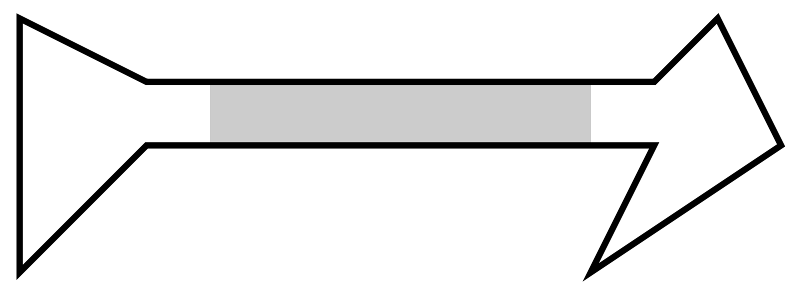
Theorem: Thin parts can be found in time $O(n)$.

Proof sketch: Compute iota images of vertices V on circle. Extremal distance preserved up to constant factor. In linear time find nested clusters S so that

$$\mathrm{diam}(S) \ll \mathrm{dist}(S, V \setminus S)$$

Thin parts correspond to these clusters.





Fast Mapping Theorem: Given $\epsilon > 0$ and an n-gon P, there is $\mathbf{w} = \{w_1, \ldots, w_n\} \subset T$ so that

1. $\mathbf{w}$ can be computed in at most Cn steps, where $C = O(1 + \log\frac{1}{\epsilon}\log\log\frac{1}{\epsilon})$,
2. $d_{QC}(\mathbf{w}, \mathbf{z}) < \epsilon$ where $\mathbf{z}$ are the conformal preimages.

Fast Mapping Theorem: Given $\epsilon > 0$ and an n-gon P, there is $\mathbf{w} = \{w_1, \dots, w_n\} \subset T$ so that

1. $\mathbf{w}$ can be computed in at most Cn steps, where $C = O(1 + \log \frac{1}{\epsilon} \log\log \frac{1}{\epsilon})$,
2. $d_{QC}(\mathbf{w}, \mathbf{z}) < \epsilon$ where $\mathbf{z}$ are the conformal preimages.

• On thin parts, conformal map has formula.

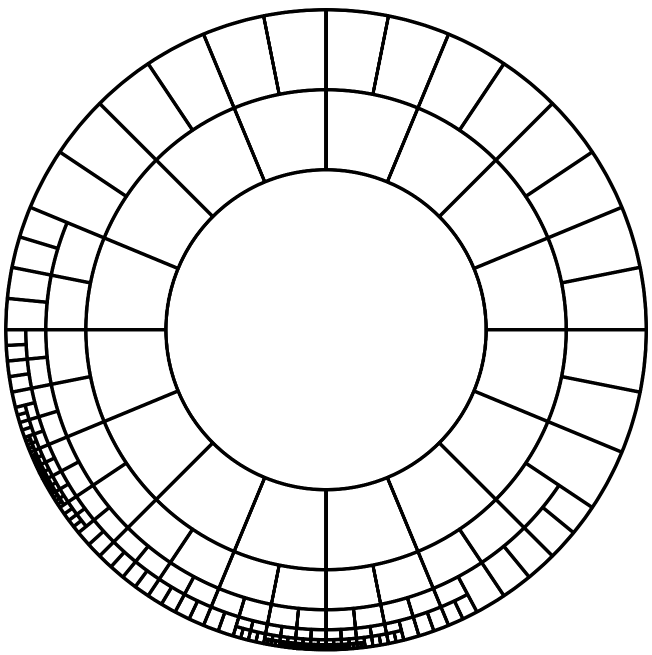
• On thick parts, Newton type iteration starting with iota map quadratically improves dilatation at each step. (fast multipole, structured linear algebra, approximate Beltrami equation,...)

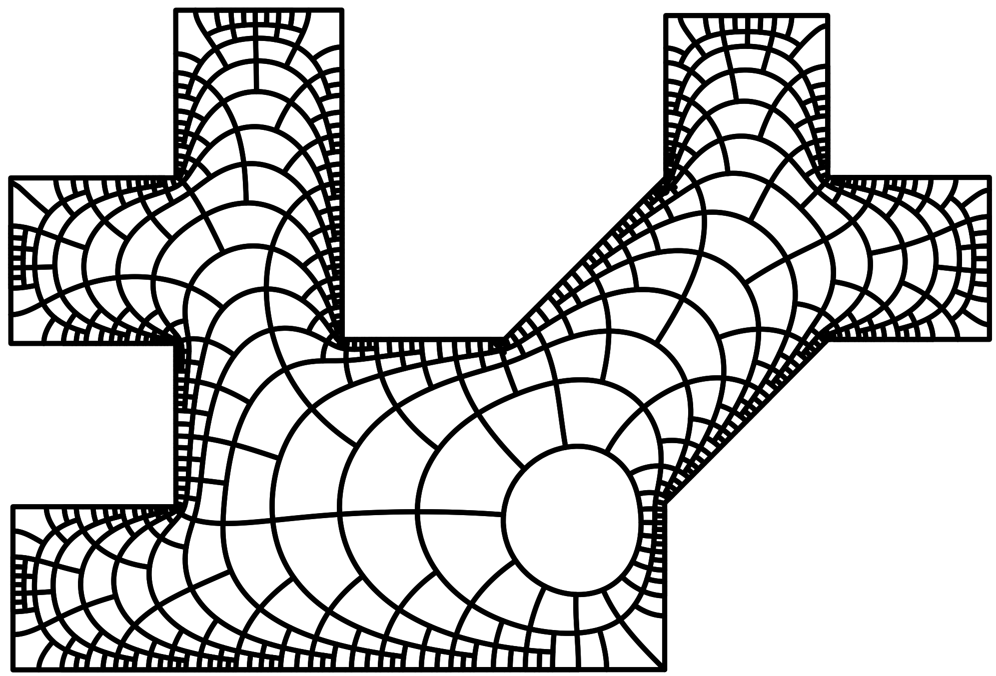
• Combine results with partition of unity.

Conformal mapping in linear time, DCG 2010, 330–428

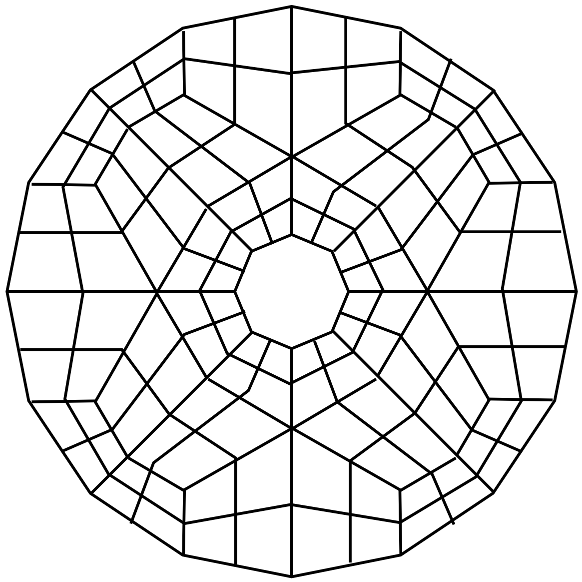
Fast Mapping Theorem: Given $\epsilon > 0$ and an n-gon P, there is $\mathbf{w} = \{w_1, \ldots, w_n\} \subset T$ so that

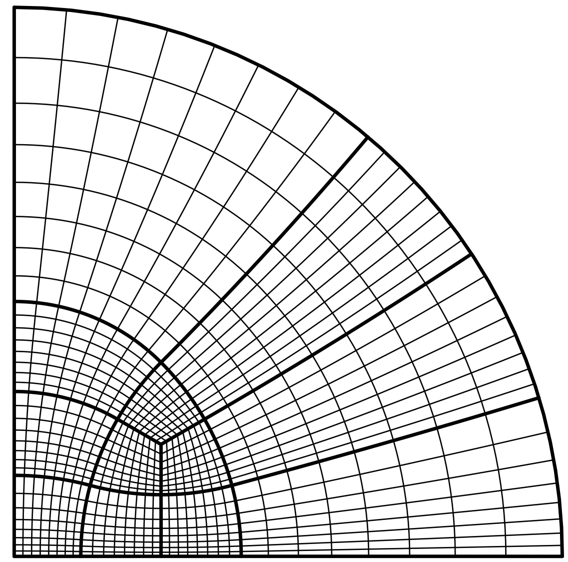
1. $\mathbf{w}$ can be computed in at most Cn steps, where $C = O(1 + \log\frac{1}{\epsilon}\log\log\frac{1}{\epsilon})$,
2. $d_{QC}(\mathbf{w}, \mathbf{z}) < \epsilon$ where $\mathbf{z}$ are the conformal preimages.

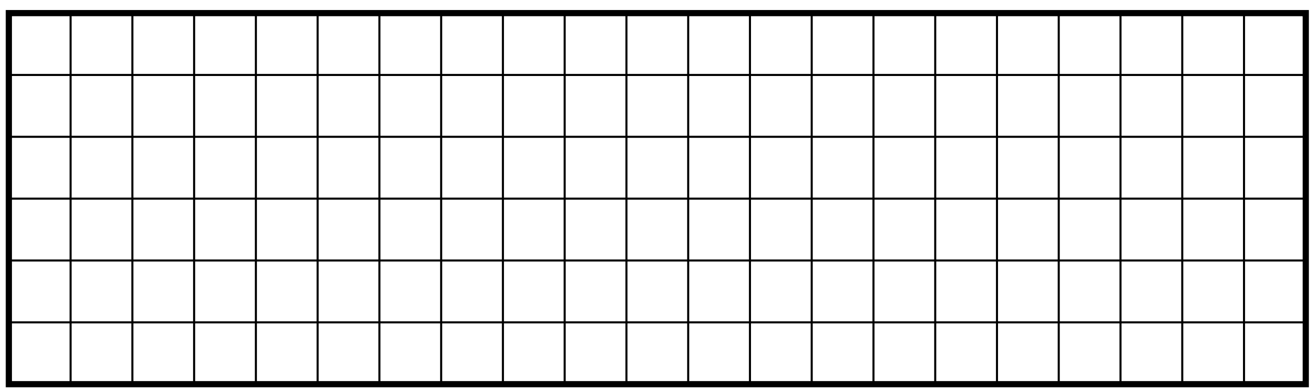




Quadrilateral meshes:







Theorem (Bern-Eppstein, 2000): Every simply n-gon has a quadrilateral mesh with every angle $\leq 120°$. The mesh can be computed in time $O(n \log n)$ time.

Angle bound is sharp.

Bern asked if lower angle bound is possible?

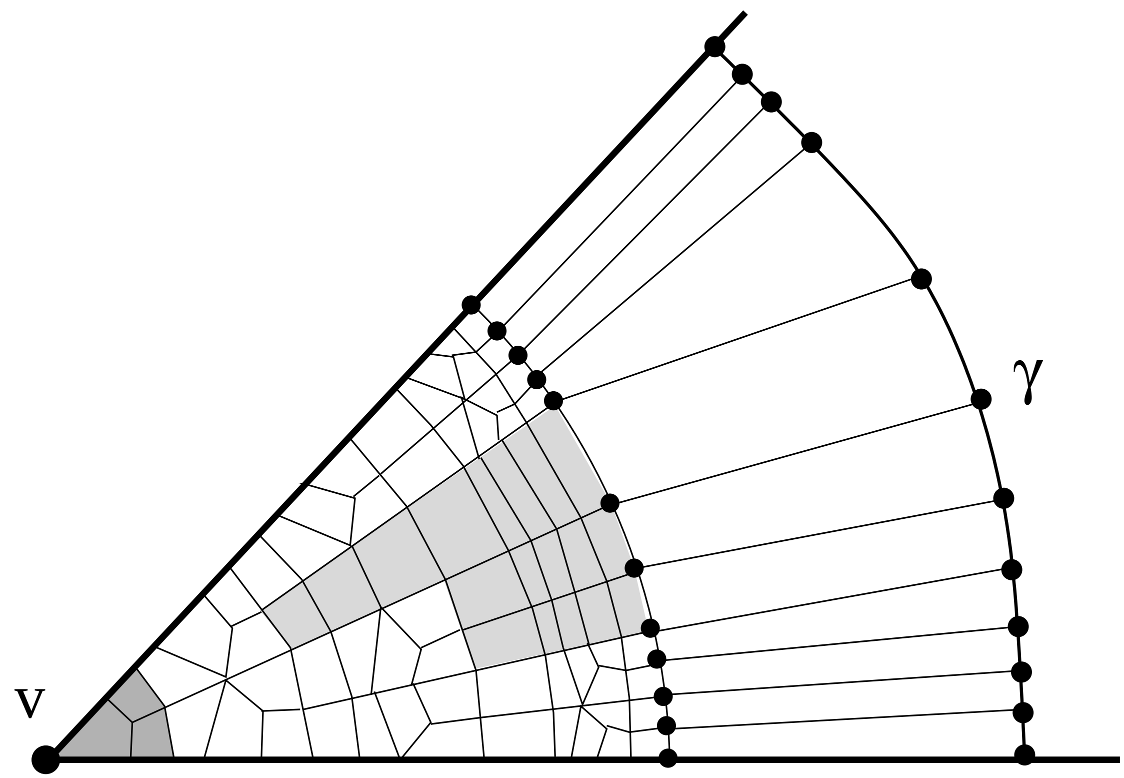
Quad-Mesh Theorem (B 2010): Every simply n-gon has a quadrilateral mesh with every angle $\leq 120°$ and all new angles $\geq 60°$. It has $O(n)$ elements and can be found in $O(n)$ time.

Lower angle bound and time are both sharp.

Existing angles of polygon remain in mesh, so we only ask for new angles to be bounded away from zero.

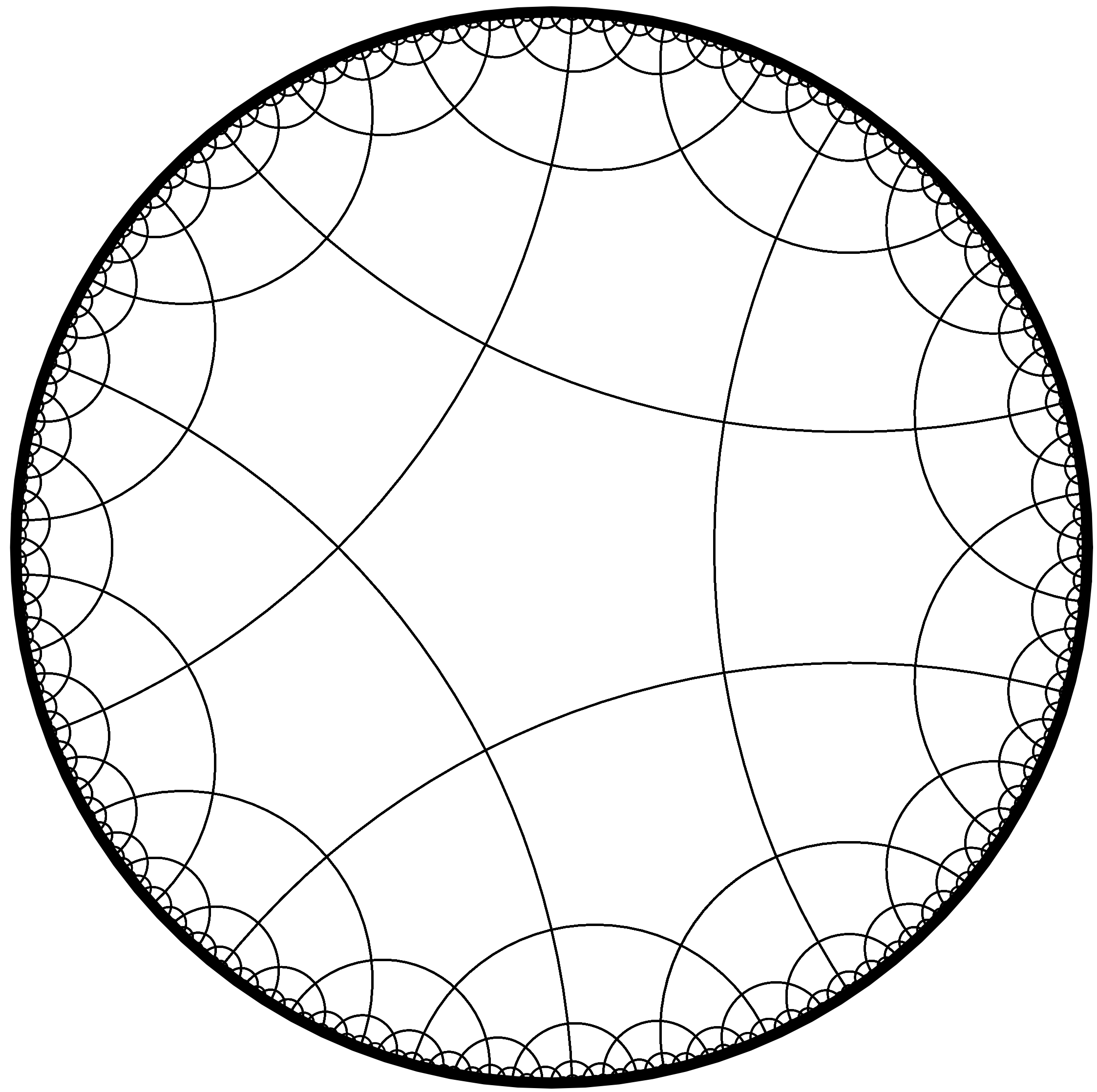
Proof uses thick/thin decomposition and FMT.

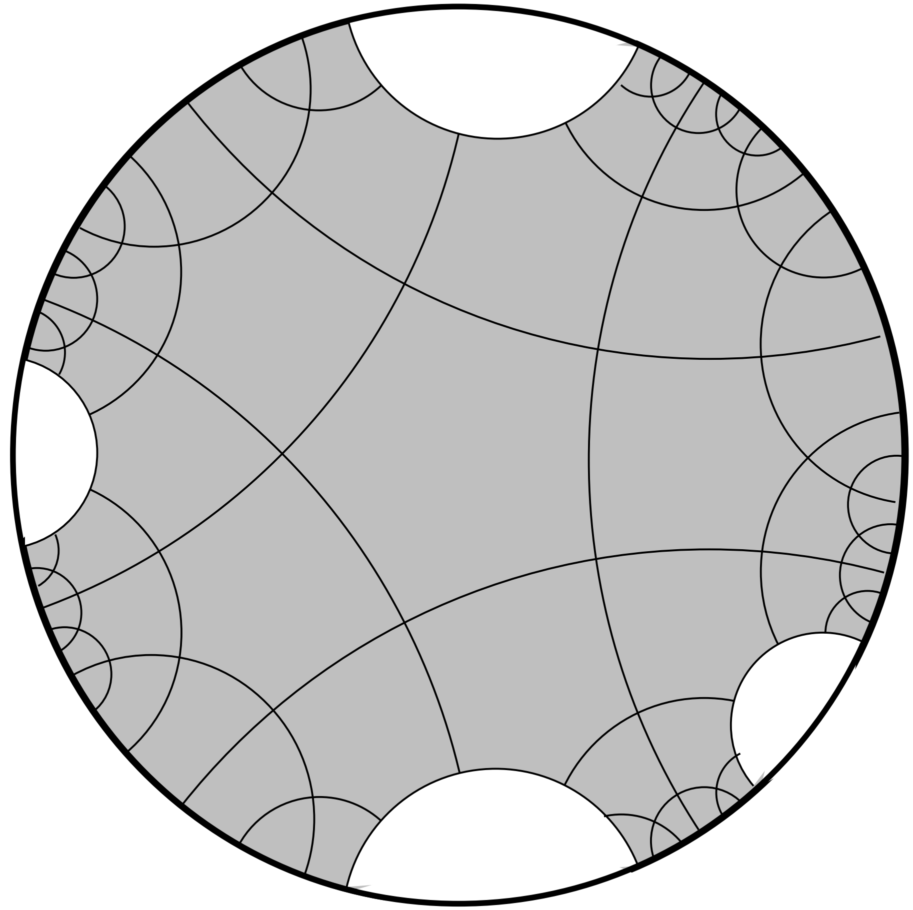
Thin parts have few possible shapes. Use explicit meshes.



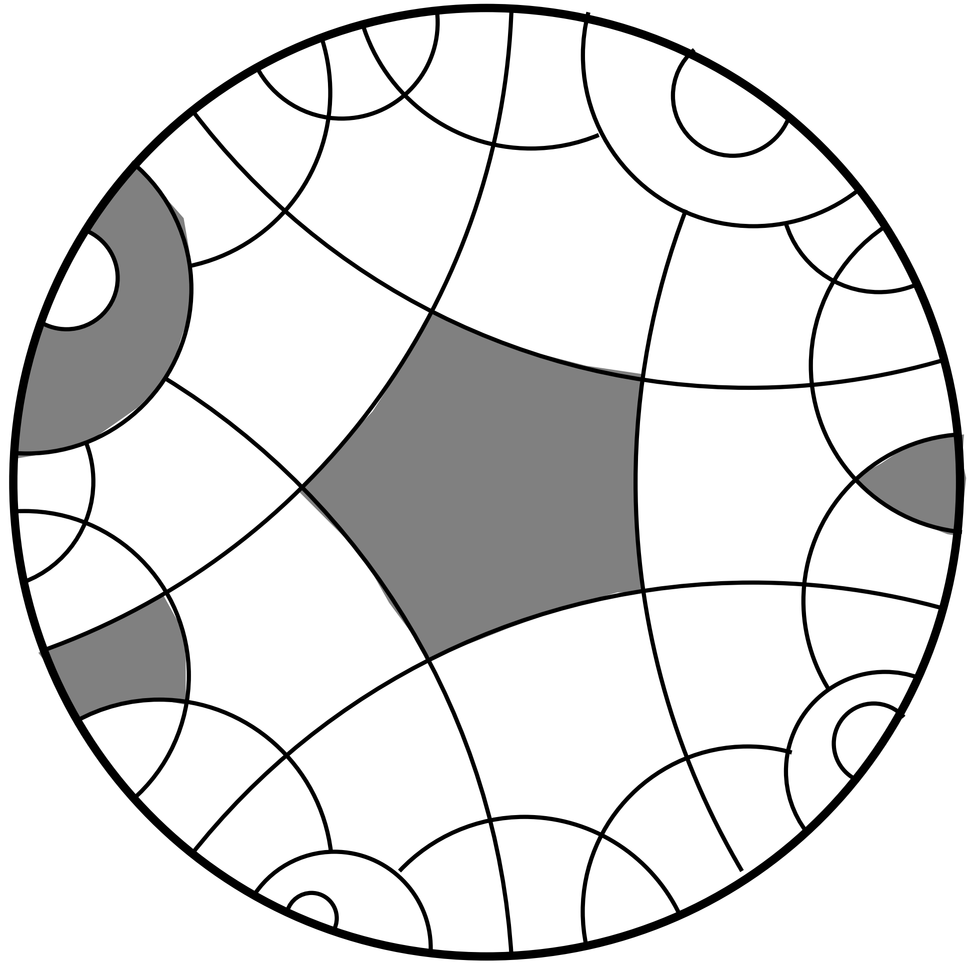
On thick parts we transfer mesh from disk.

Start with hyperbolic pentagonal tesselation of disk.

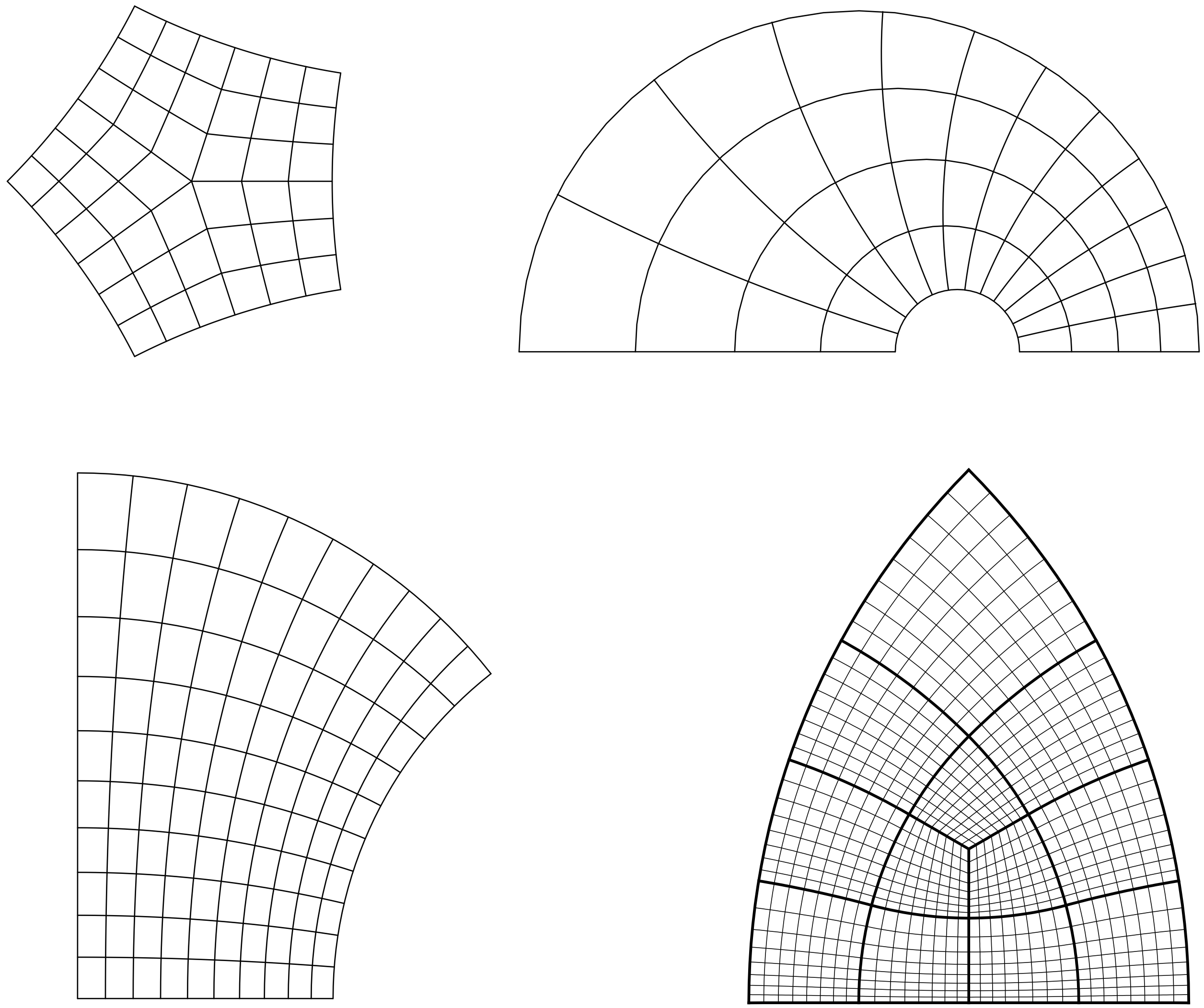




Conformal image of thick part covered by **linear number** of "Whitney pentagons" and "Carleson boxes".



There are four types of boxes that occur.

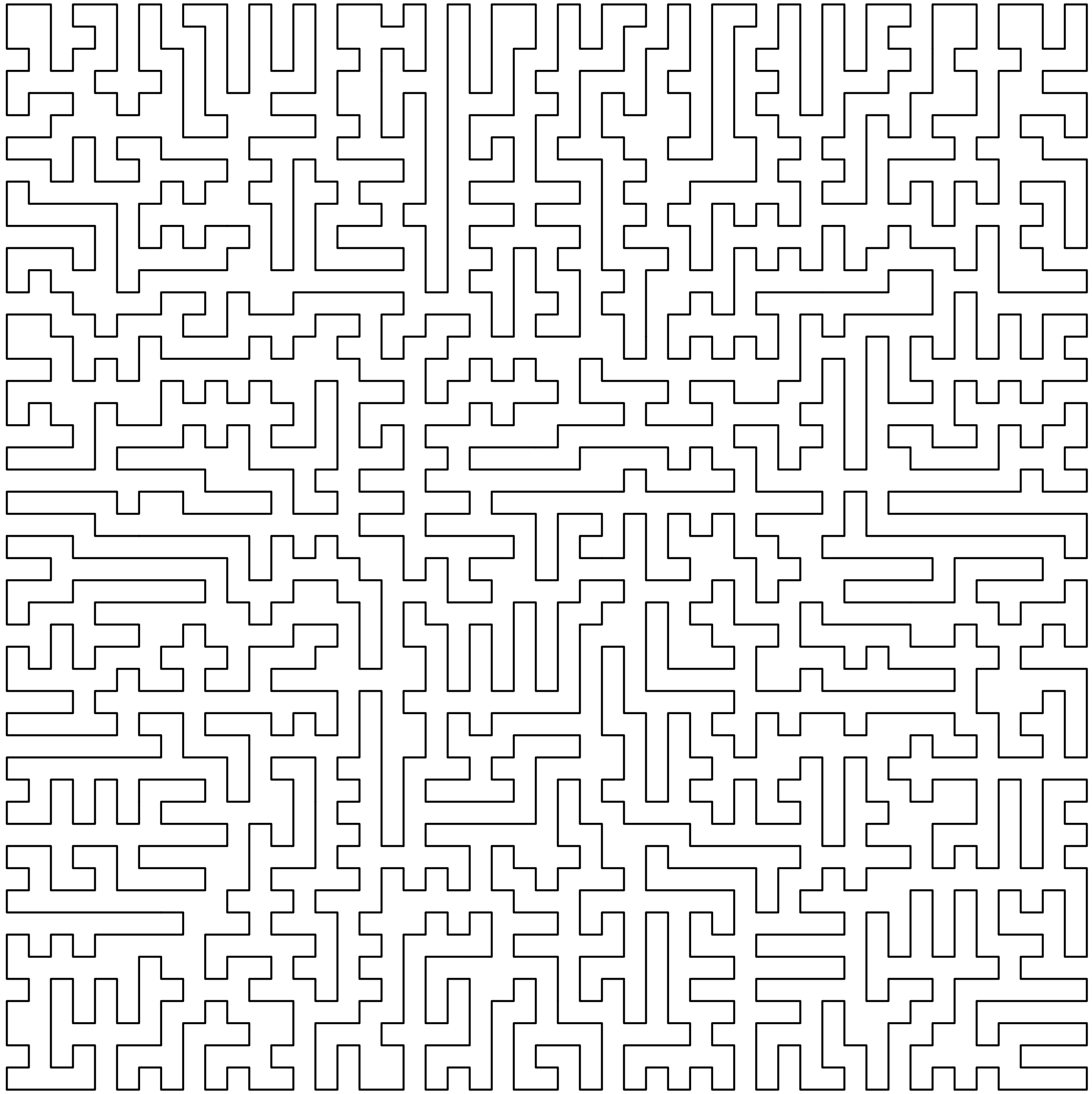


Meshes designed to match along common edges.
Transferred to polygon by (almost) conformal map.

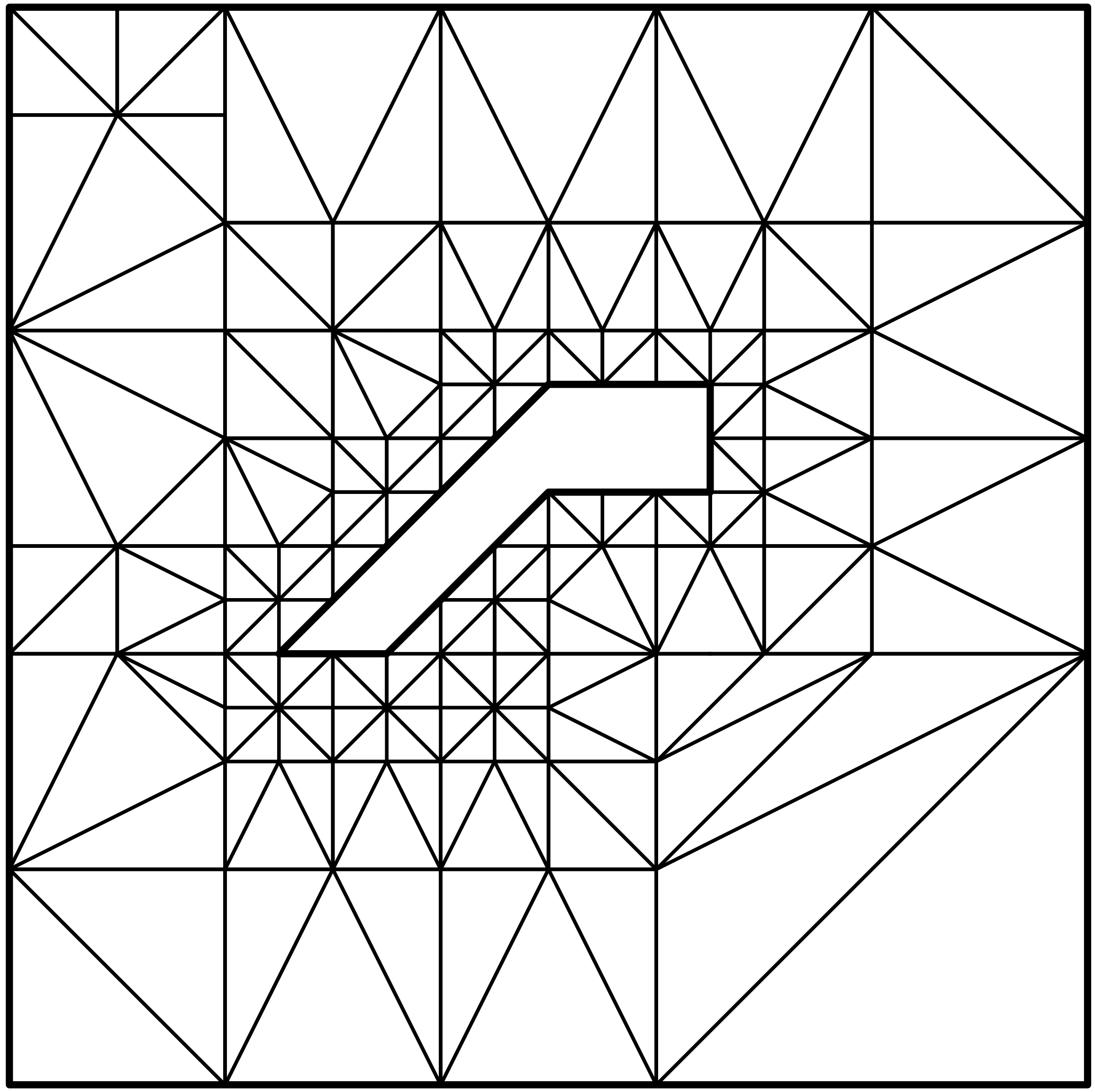
Harder version of quad-meshing problem: PSLGs

A planar straight line graph (PSLG) is the union of a finite number segments and points.

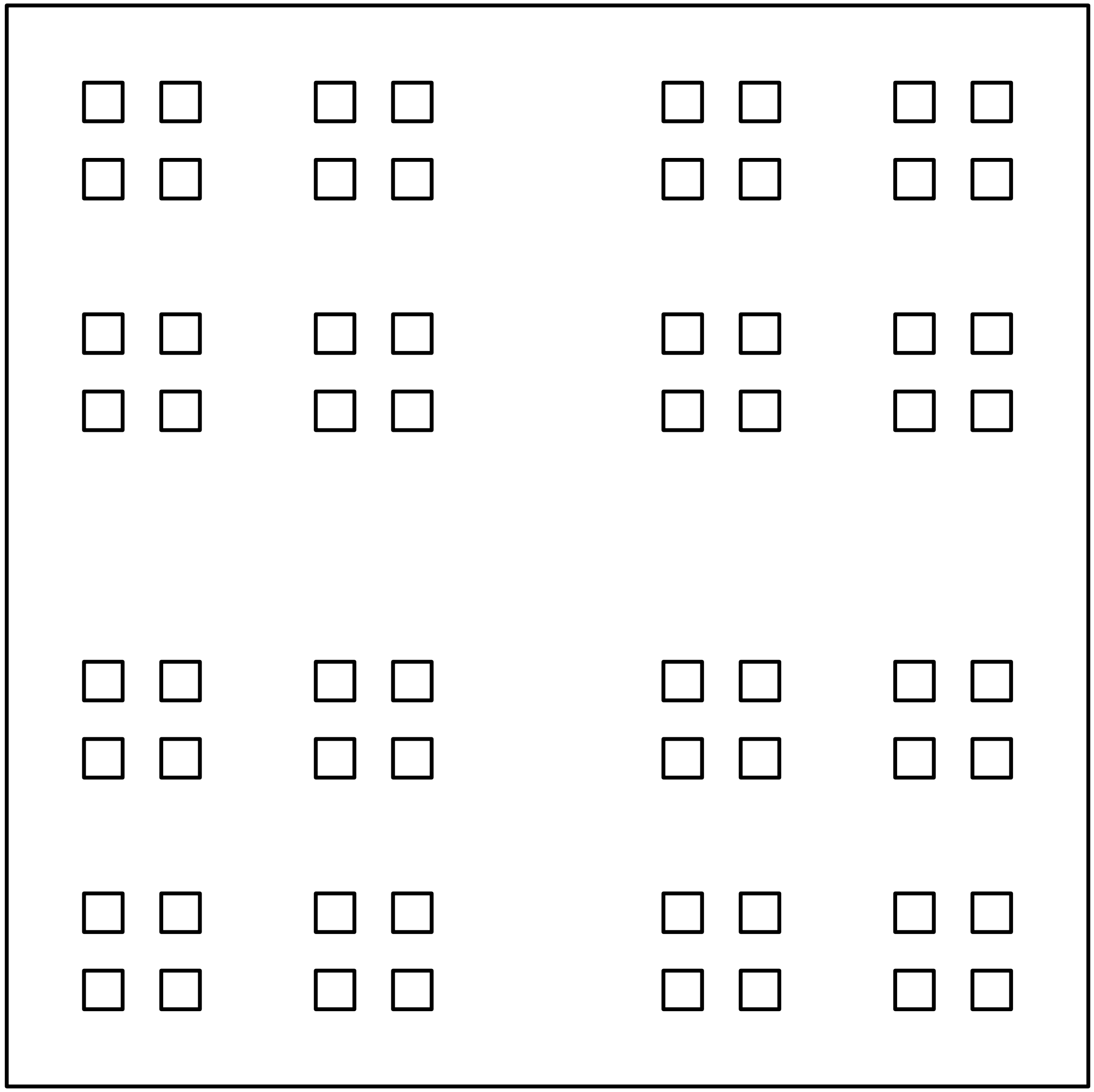
A simple polygon is a special case.



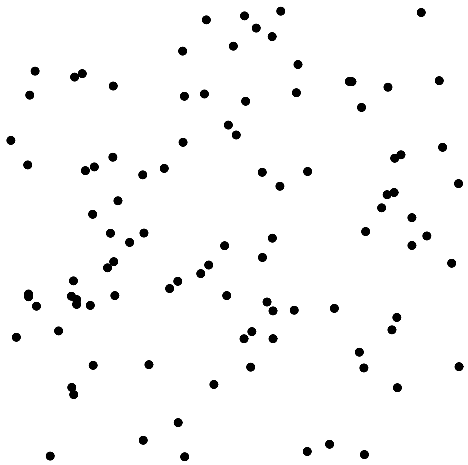
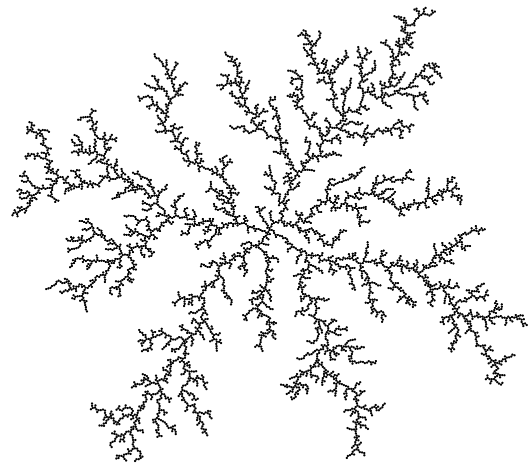
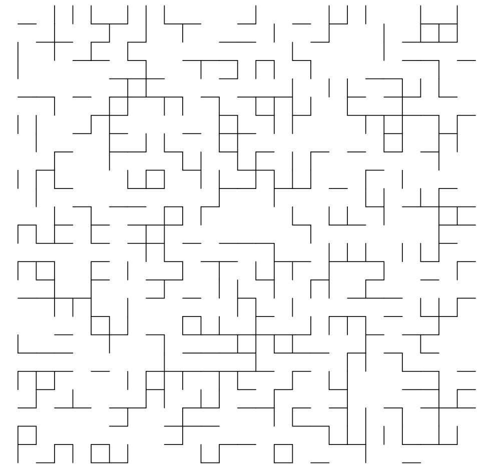
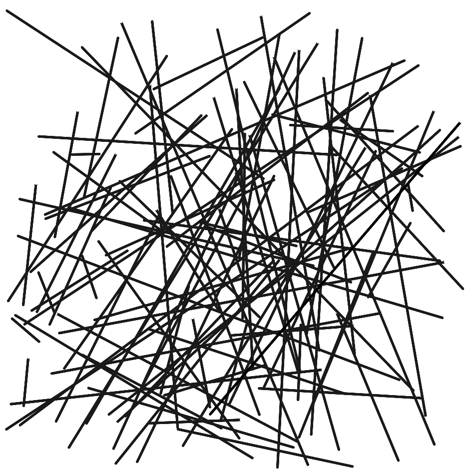
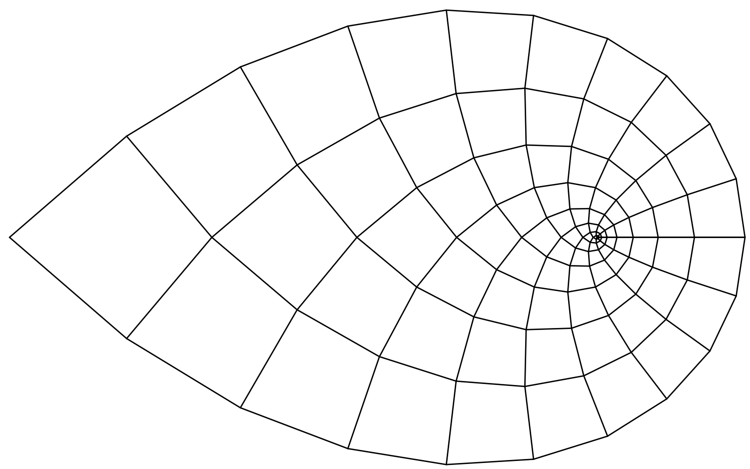
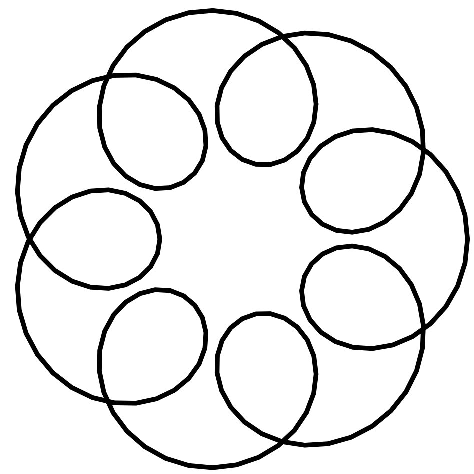
Simple polygon



Triangulation



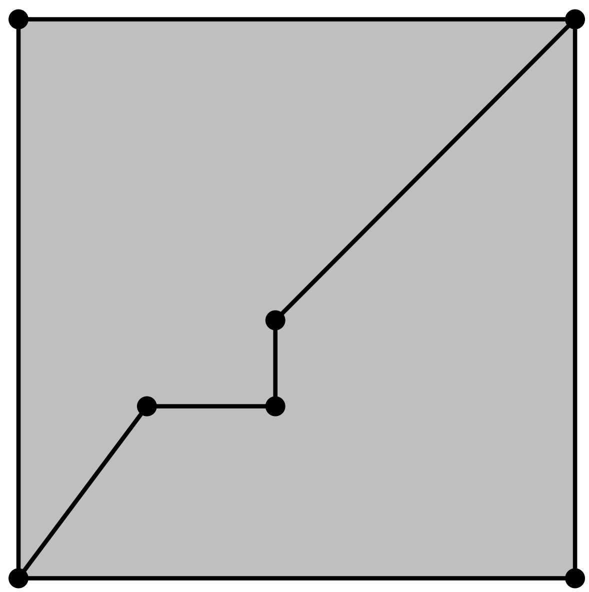
Polygon with holes

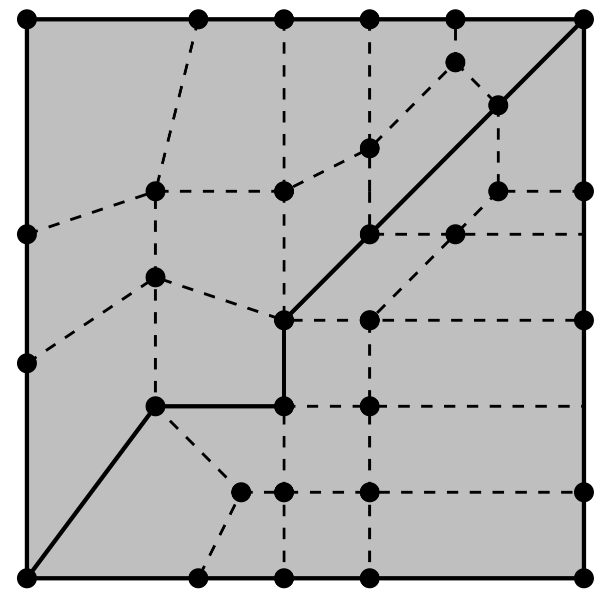
A mesh of a PSLG Γ is another PSLG Γ' obtained by adding vertices and edges, (and subdividing edges) so that every face of Γ' is a simple polygon.

Triangulation = all faces are triangles

Quad-mesh = all faces are quadrilaterals.



PSLG with two faces



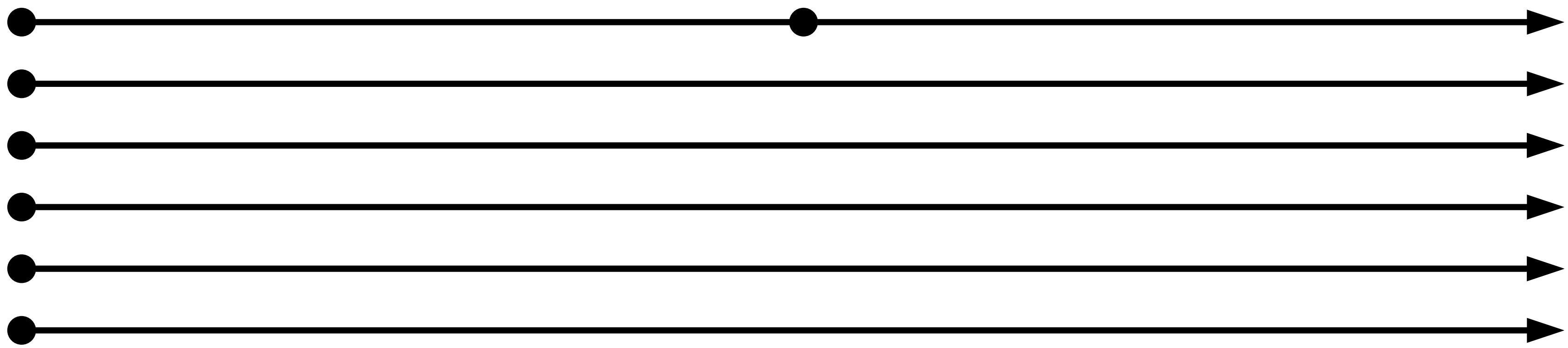
conforming

Theorem (B, 2016): Every PSLG has $O(n^2)$ quad-mesh with all angles $\leq 120°$ and all new angles $\geq 60°$.

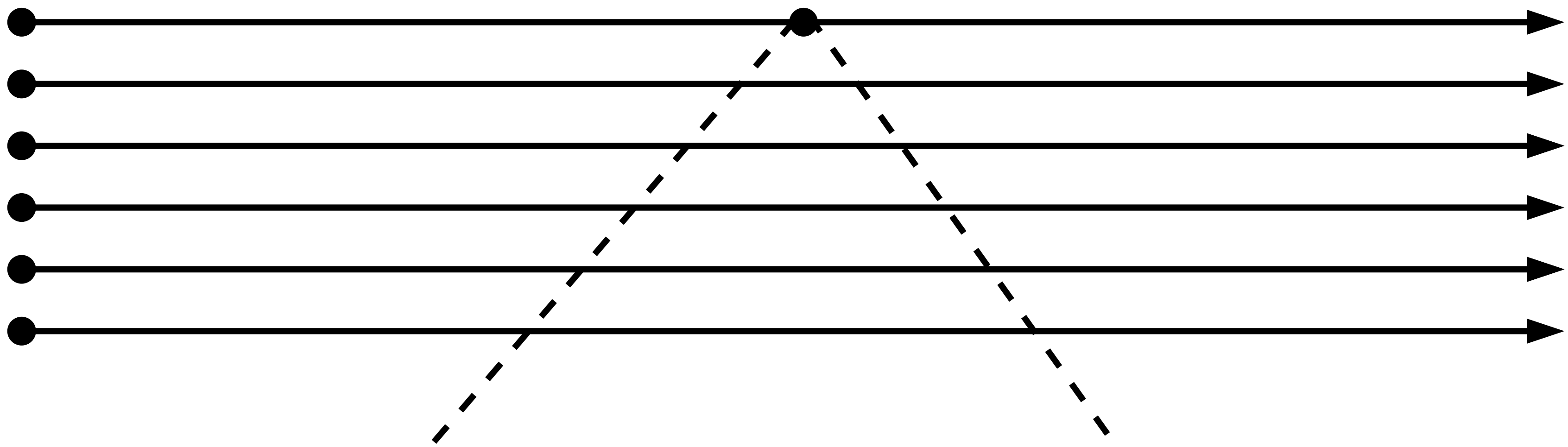
Angle and complexity bounds all sharp.

The $O(n^2)$ is sharp because any mesh with maximum angle $\leq \theta < 180^\circ$ sometimes requires n^2 elements.

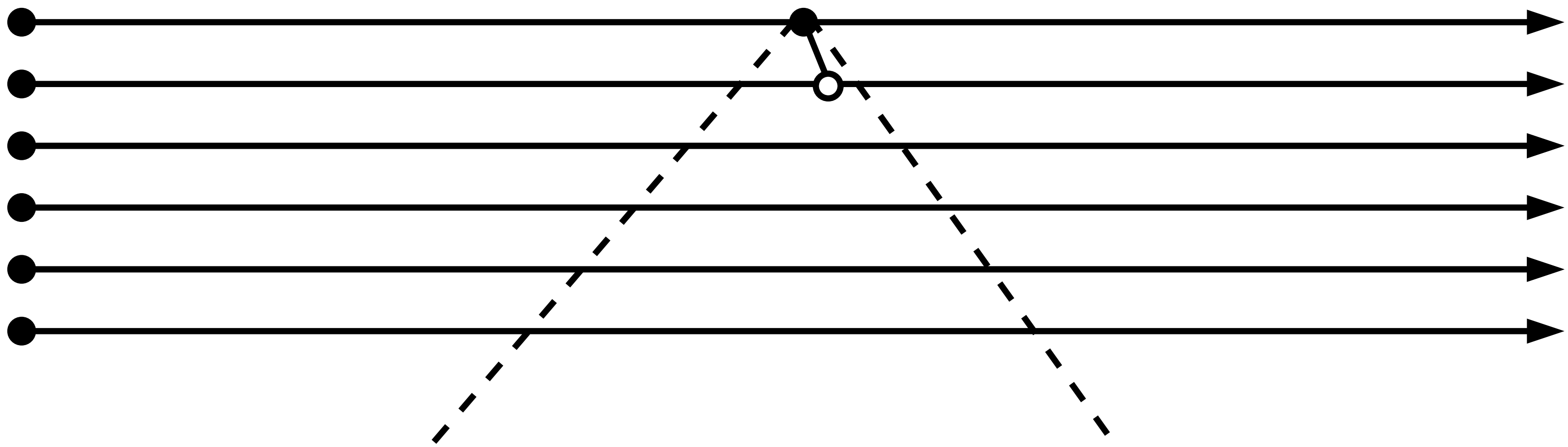
The $O(n^2)$ is sharp because any mesh with maximum angle $\leq \theta < 180^\circ$ sometimes requires n^2 elements.



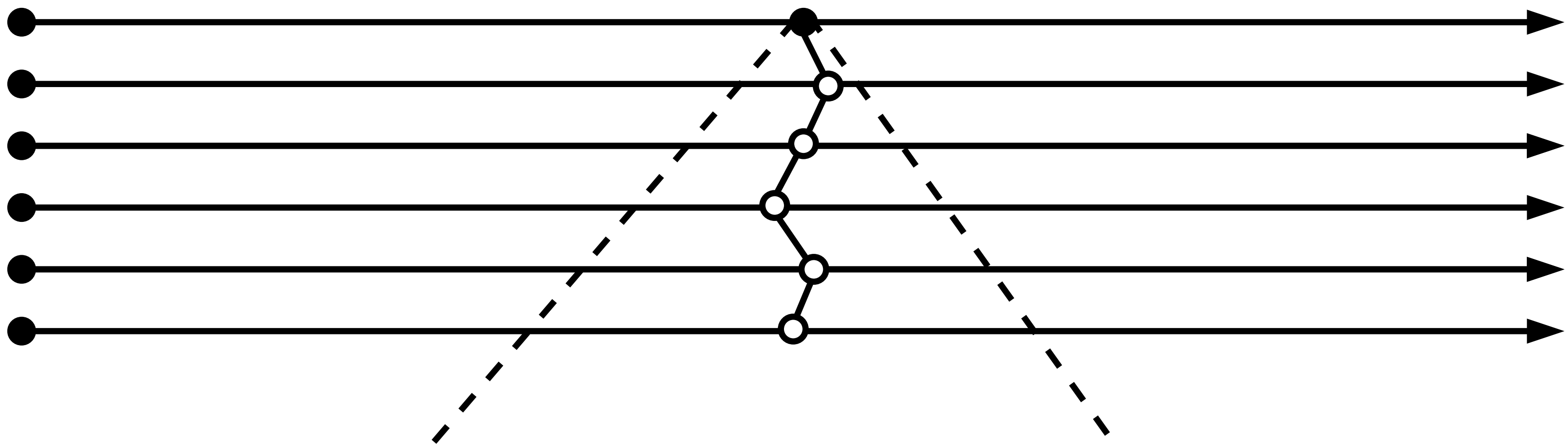
The $O(n^2)$ is sharp because any mesh with maximum angle $\leq \theta < 180^\circ$ sometimes requires n^2 elements.



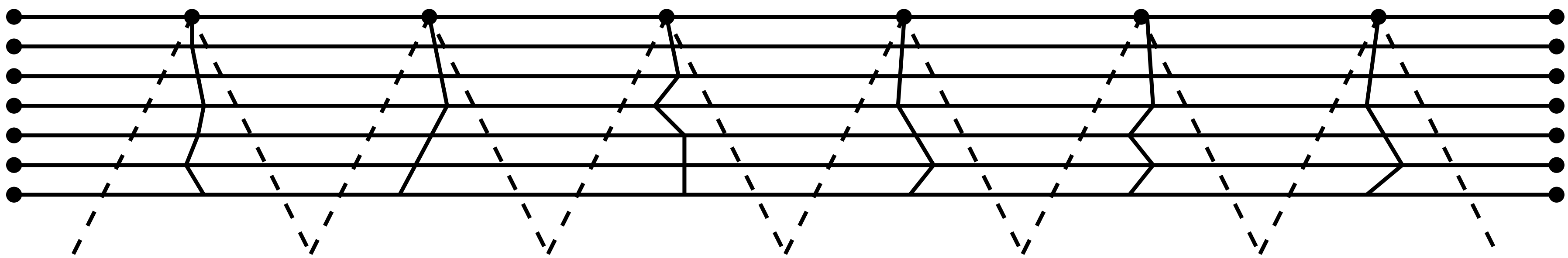
The $O(n^2)$ is sharp because any mesh with maximum angle $\leq \theta < 180^\circ$ sometimes requires n^2 elements.



The $O(n^2)$ is sharp because any mesh with maximum angle $\leq \theta < 180^\circ$ sometimes requires n^2 elements.

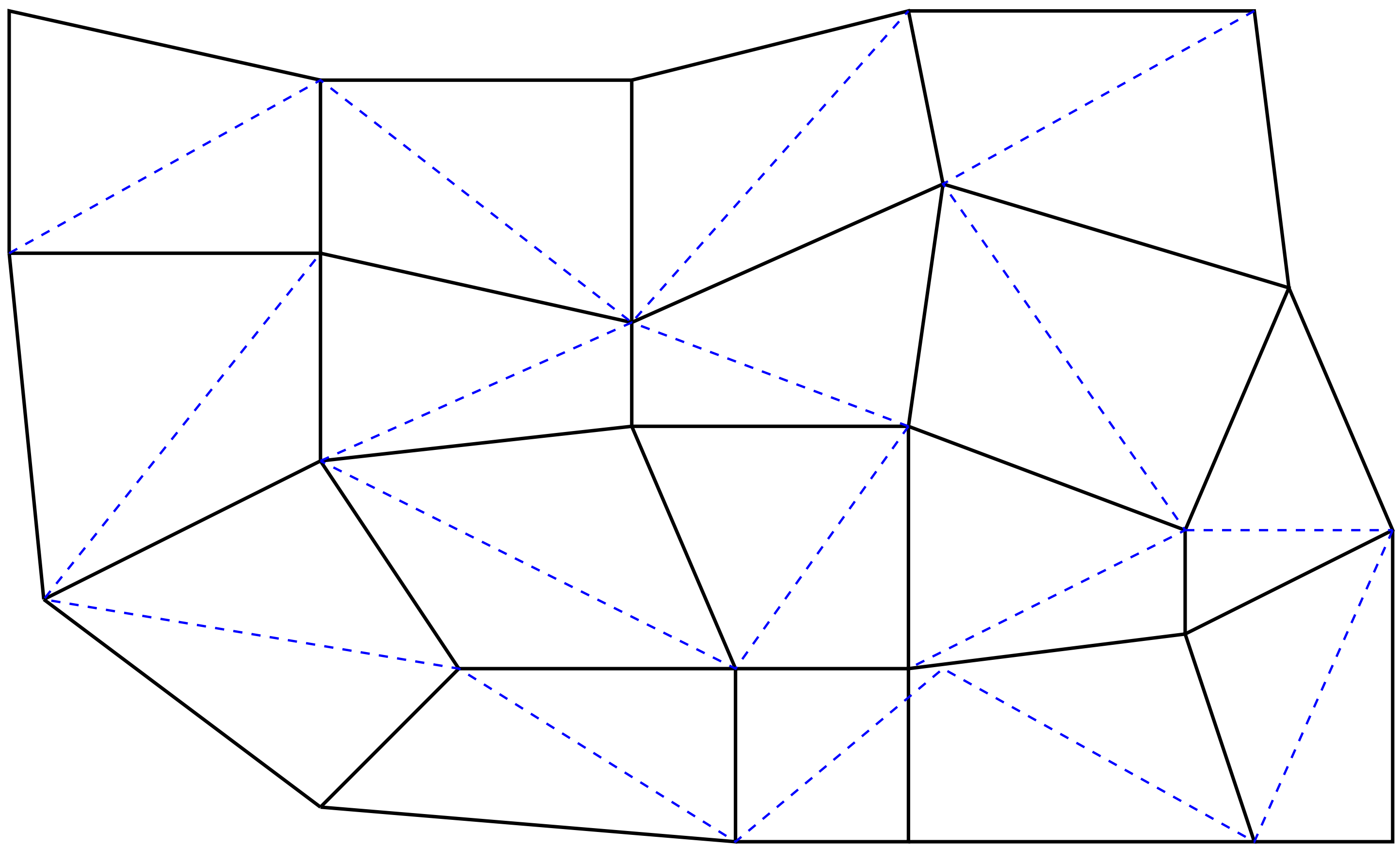


The $O(n^2)$ is sharp because any mesh with maximum angle $\leq \theta < 180^\circ$ sometimes requires n^2 elements.



If we add diagonals to each quadrilateral we get:

Cor: Every PSLG has $O(n^2)$ triangulation with angles $\leq 120°$.



If we add diagonals to each quadrilateral we get:

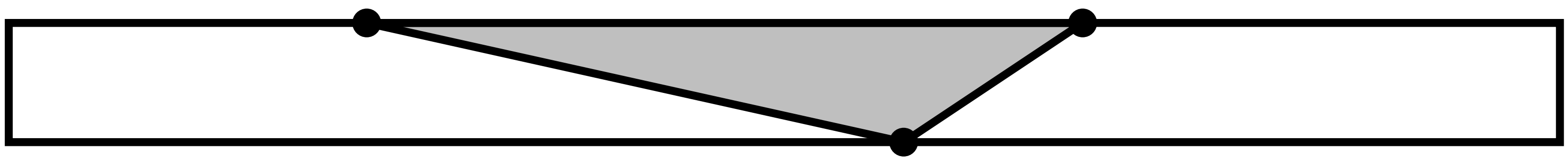
Cor: Every PSLG has $O(n^2)$ triangulation with angles $\leq 120°$.

Previously: 157.5° (Mitchell, 1993) and 132° (Tan, 1996).

- $O(n^2)$ complexity is sharp.
- Can we get better upper angle bound?
- Can we get any lower angle bound?

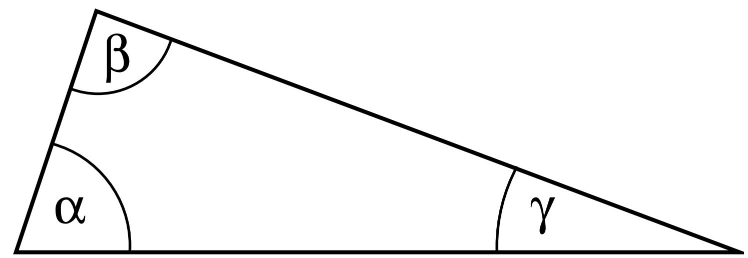
No lower angle bound. For $1 \times R$ rectangle

$$\text{number of triangles} \gtrsim R \times (\text{smallest angle})$$



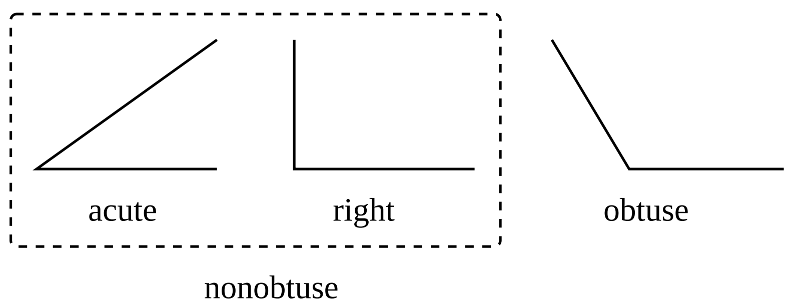
So, bounded complexity $\Rightarrow$ no lower angle bound.

Upper bound $< 90^\circ$ **implies lower bound** > 0**:**



$$\alpha, \beta < (90^\circ - \epsilon) \Rightarrow \gamma = 180^\circ - \alpha - \beta \geq 2\epsilon.$$

So 90° is best uniform upper bound we can hope for.



Many practical applications of non-obtuse triangulations.

For example, Vavasis studies discrete solutions of

$$\nabla \cdot (c\nabla u) = 0$$

and bounds condition number of matrices arising in the finite element method.

General estimate is exponential in the number of mesh elements, but for nonobtuse triangulations it is linear.

Brief history of NOTs:

- Always possible: Burago, Zalgaller 1960.
- Rediscovered: Baker, Grosse, Rafferty, 1988.
- $O(n)$ for points sets: Bern, Eppstein, Gilbert 1990
- $O(n^2)$ for polygons: Bern, Eppstein, 1991
- $O(n)$ for polygons: Bern, Mitchell, Ruppert, 1994
- PSLG's: exist, n^2 lower bound

Is there a polynomial bound for PSLGs?

The NOT Theorem (B 2016): Every PSLG has a conforming non-obtuse triangulation of size $O(n^{2.5})$.

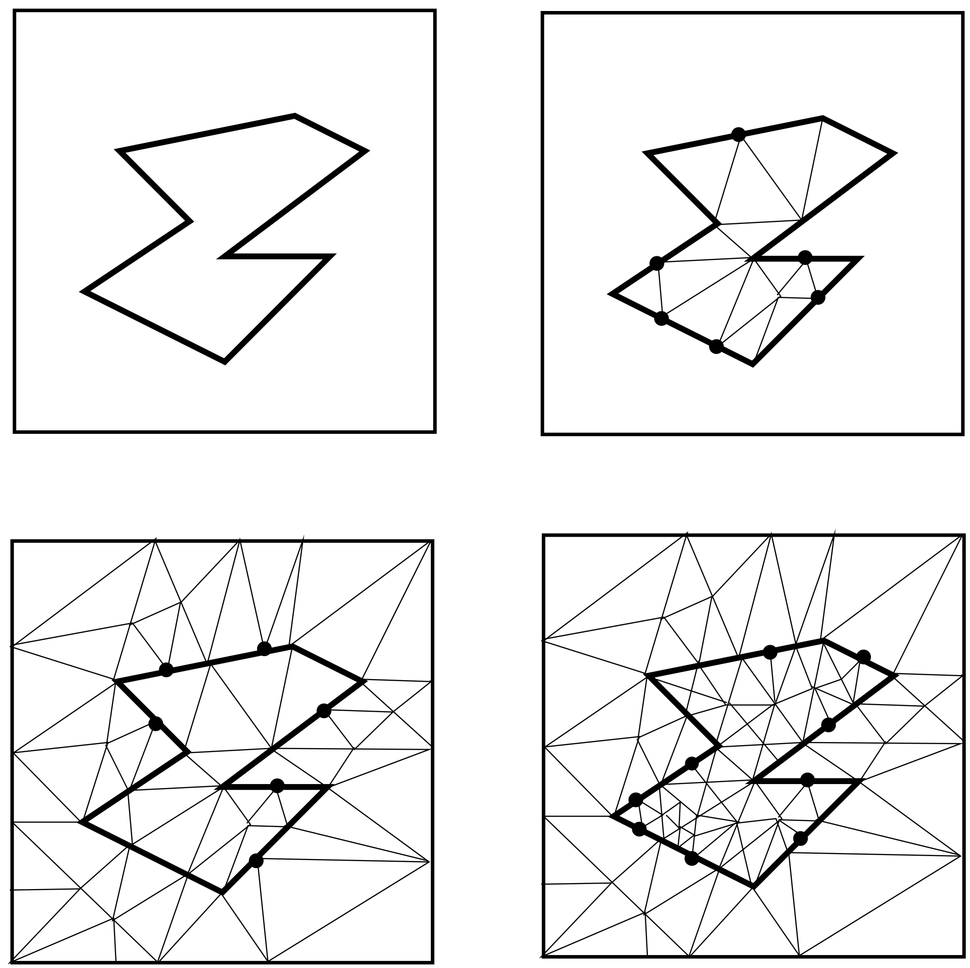
Mesh can be all right triangles, or all strictly acute.

The NOT Theorem (B 2016): Every PSLG has a conforming non-obtuse triangulation of size $O(n^{2.5})$.

Corollary: Every PSLG has a conforming Delaunay triangulation of size $O(n^{2.5})$.

Previous best was $O(n^3)$ (Edelsbrunner and Tan, 1993).

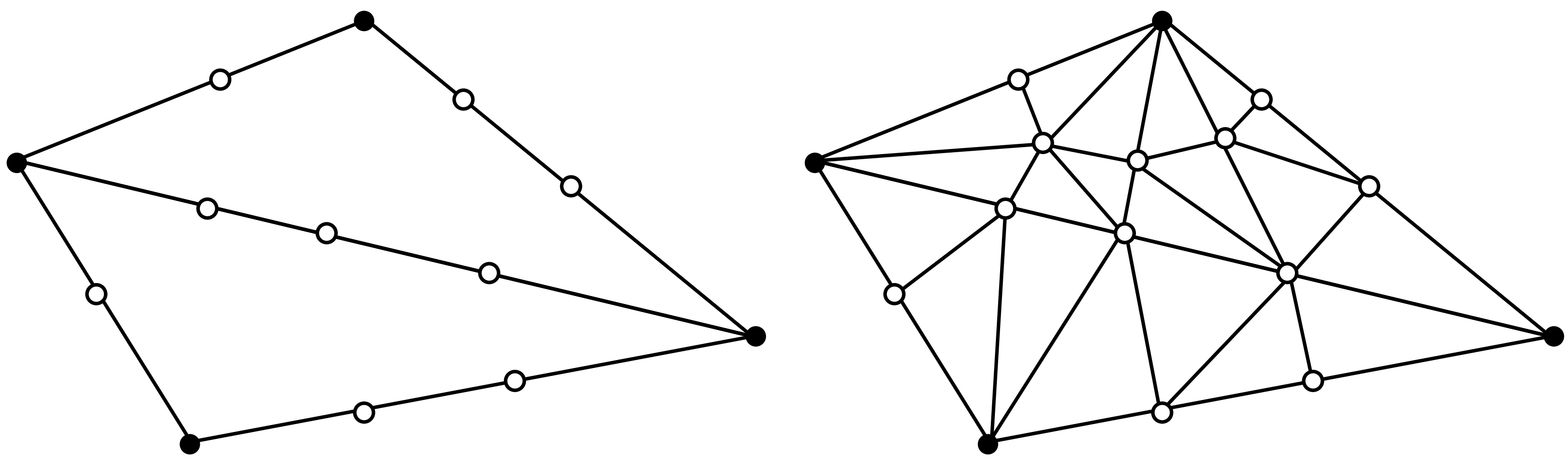
Corollary: Every PSLG has a triangulation with all angles $\leq 90° + \epsilon$ and complexity $O(n^2/\epsilon^2)$.



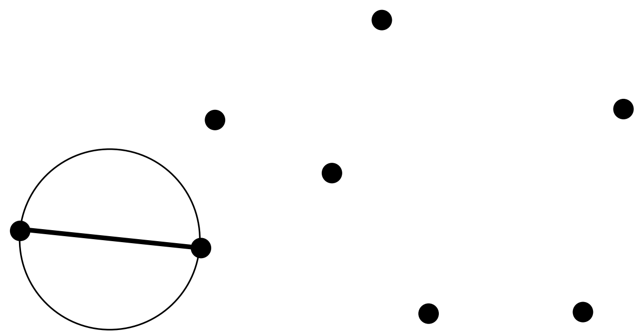
We can find NOTs for each face separately, but these create new boundary vertices, that must be incorporated into mesh on "other side". Does this ping-pong process terminate?

Idea of proof:

- Reduce to case PSLG = triangulation of a point set (easy).
- Choose special points on edges of triangles.
- Find non-obtuse mesh in each triangle using only those points.

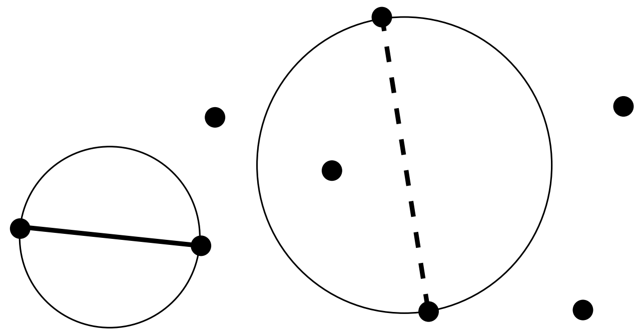


The segment $[v, w]$ is a **Gabriel** edge of a point set V if it is the diameter of an open disk missing V.



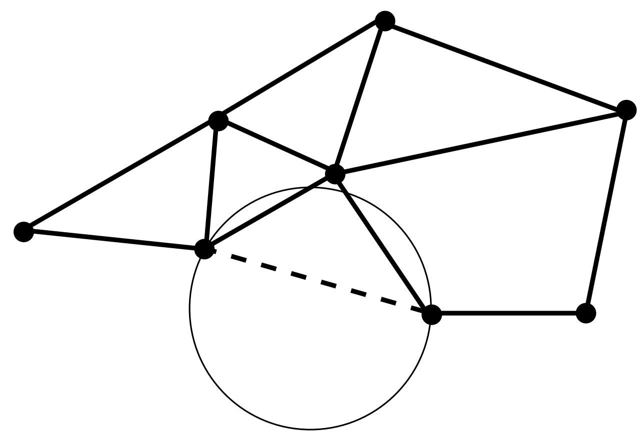
Gabriel edge.

The segment $[v, w]$ is a **Gabriel** edge of a point set V if it is the diameter of an open disk missing V.



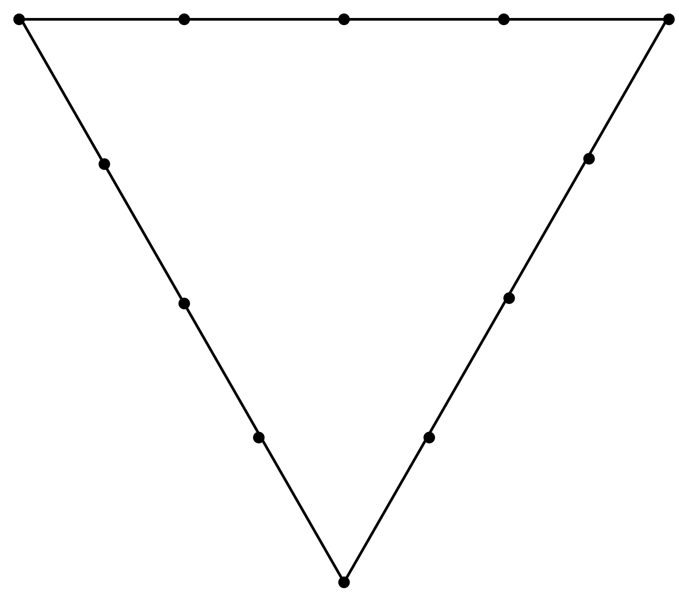
Not a Gabriel edge.

Gabriel edge is a special case of a **Delaunay** edge: $[v, w]$ is a **chord** of an open disk not hitting V.



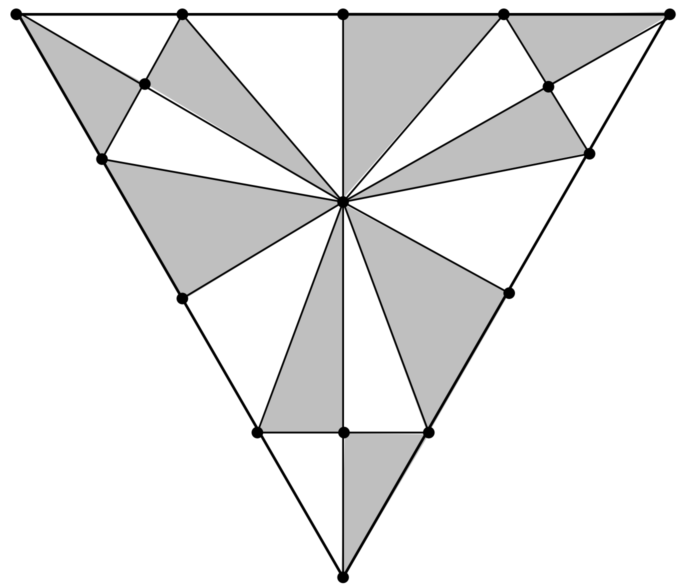
Bern-Mitchell-Ruppert (1994)

Lemma: Add k vertices to sides of triangle, so all edges become Gabriel, then add all midpoints. Resulting polygon has a $O(k)$ NOT, using **no additional** boundary vertices.



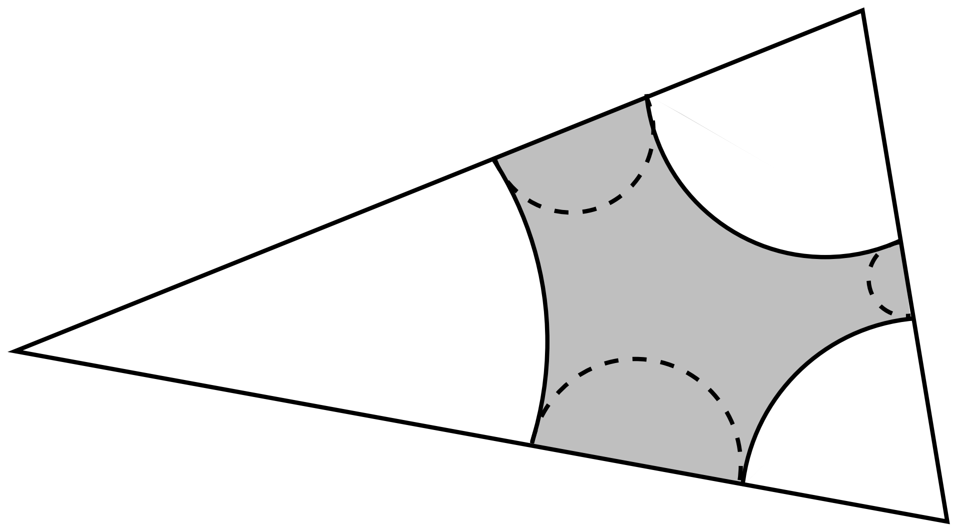
Bern-Mitchell-Ruppert (1994)

Lemma: Add k vertices to sides of triangle, so all edges become Gabriel, then add all midpoints. Resulting polygon has a $O(k)$ NOT, using **no additional** boundary vertices.



Proof sketched in Appendix C.

Construct Gabriel points:

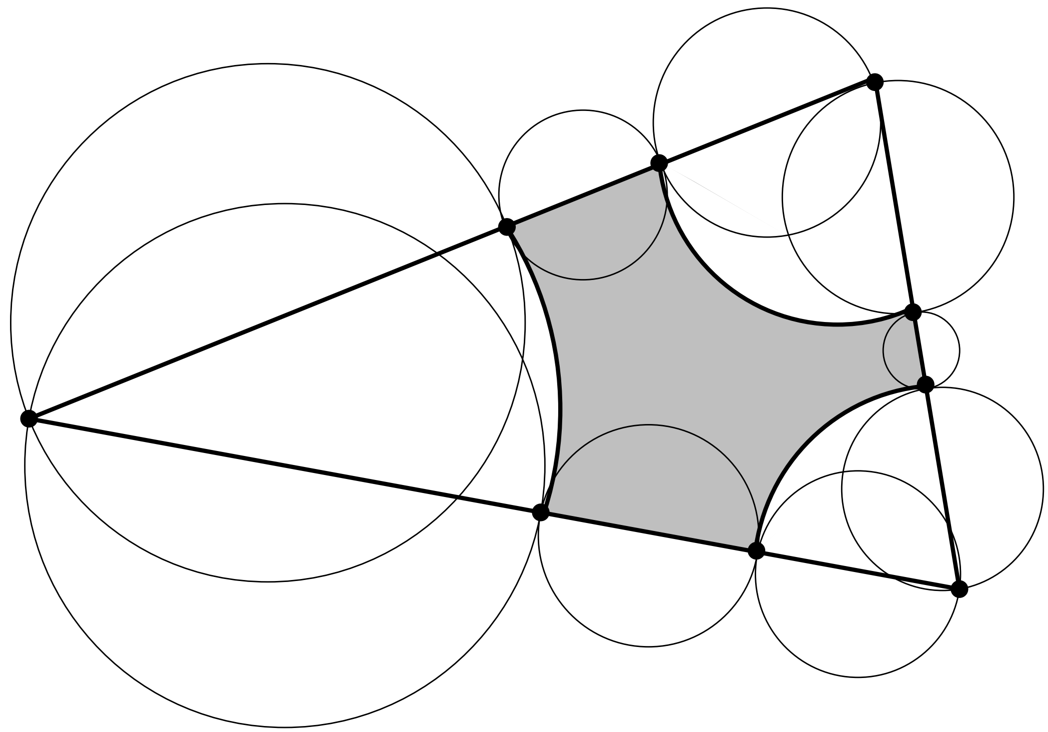


Break every triangle into thick and thin parts.

Thin parts = corners, Thick part = central region

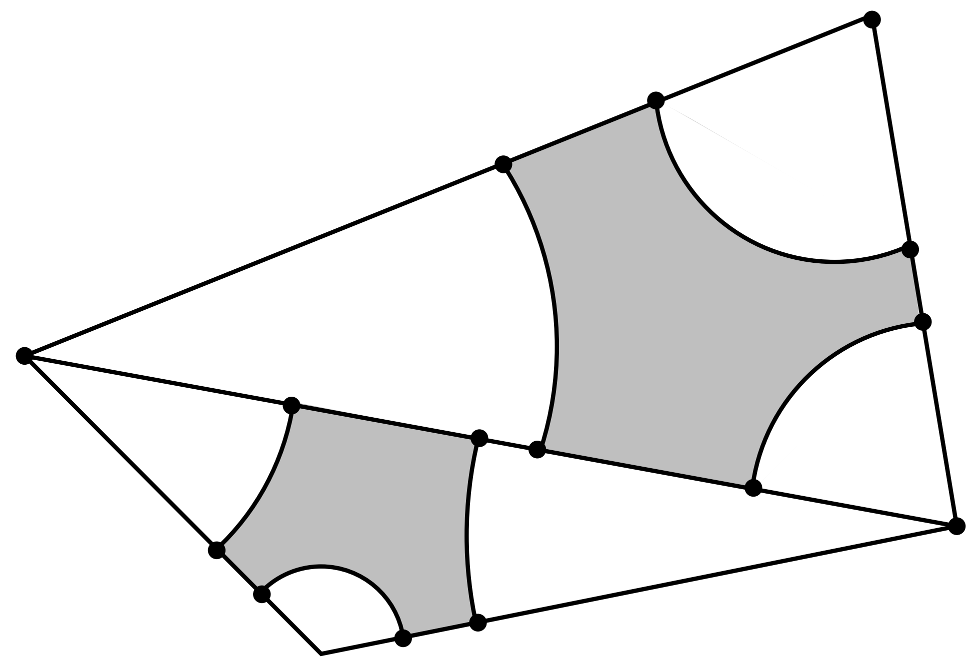
Thick sides are bases of disjoint half-disks inside triangle.

Construct Gabriel points:



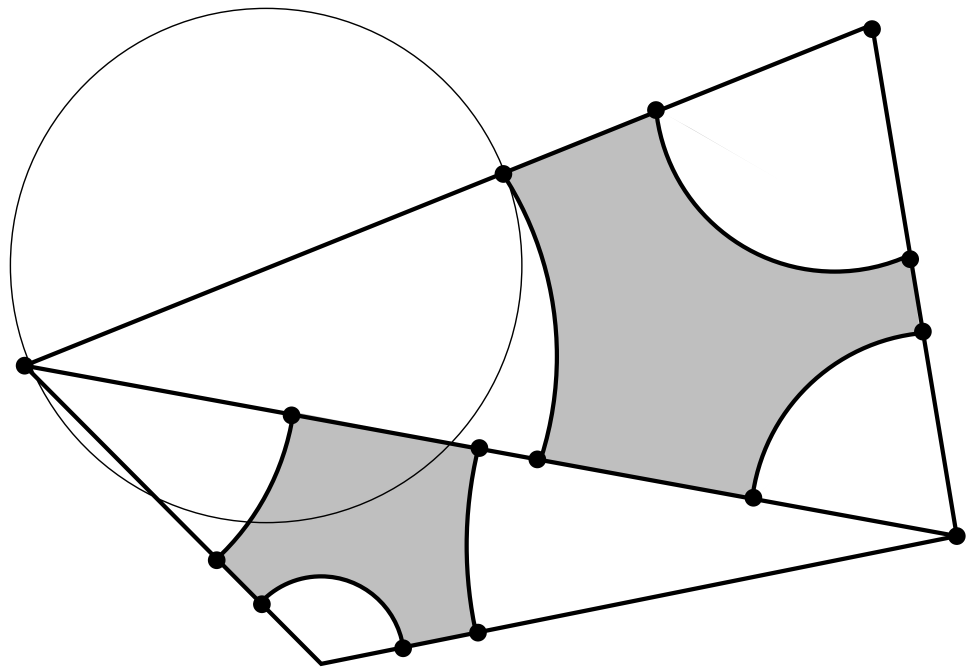
Then vertices of thick part give Gabriel edges.

Construct Gabriel points:



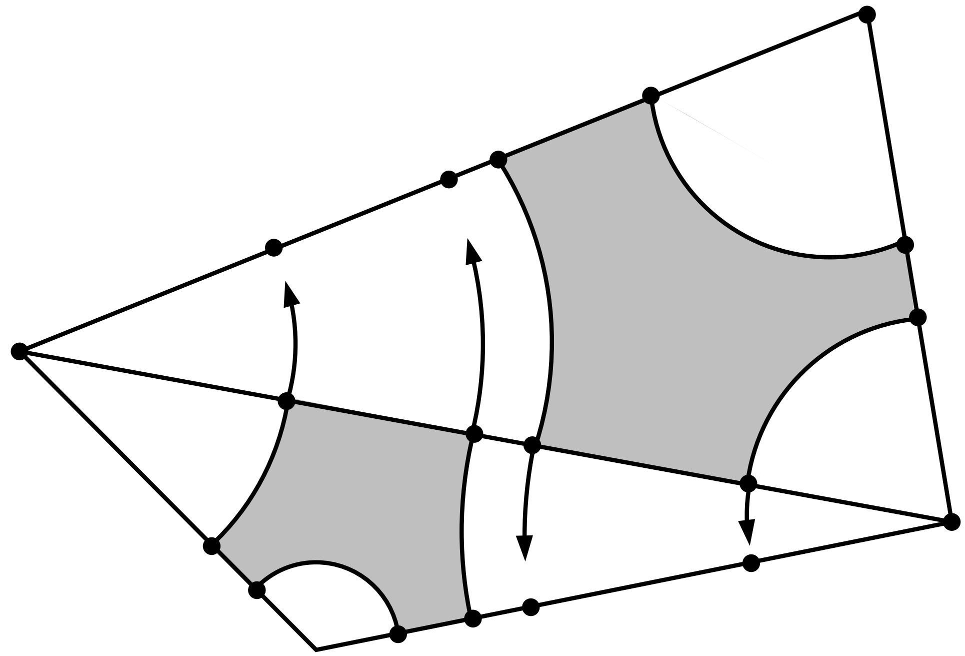
But vertices for adjacent triangles need not match up.

Construct Gabriel points:



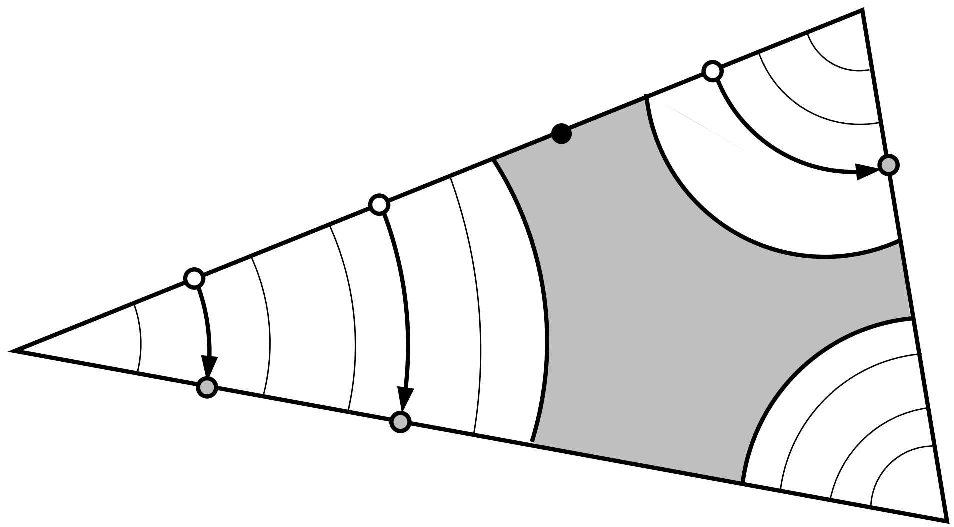
Adjacent triangle can make Gabriel condition fail.

Construct Gabriel points:

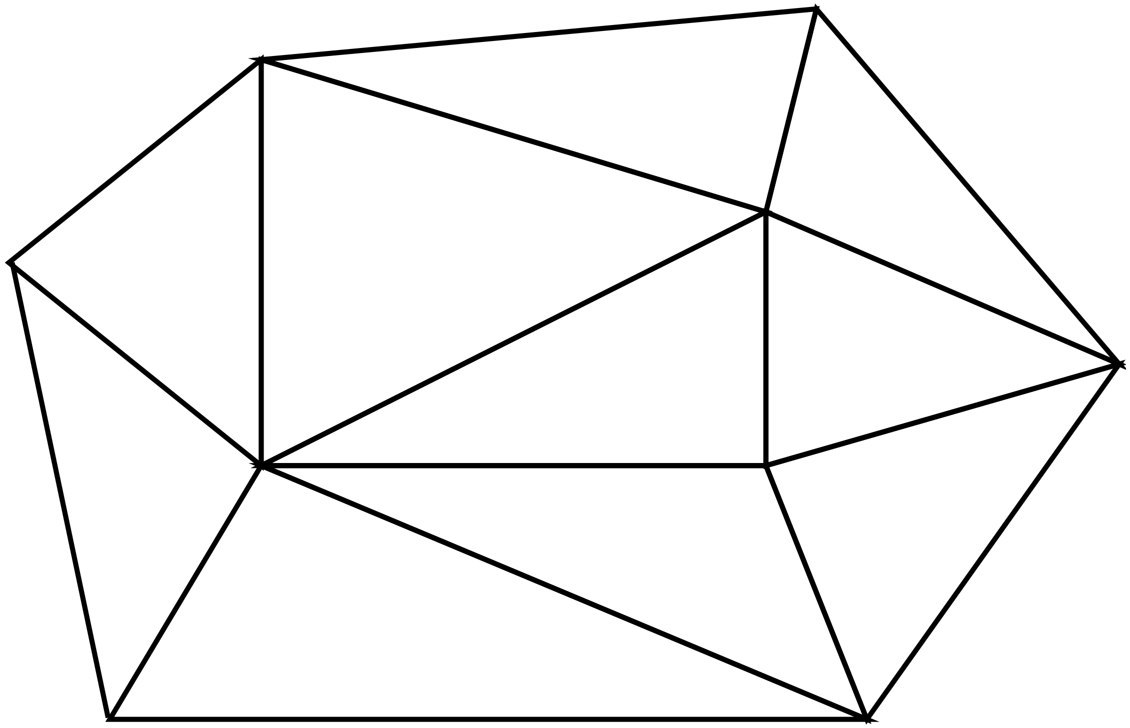


Idea: "Push" vertices across the thin parts.

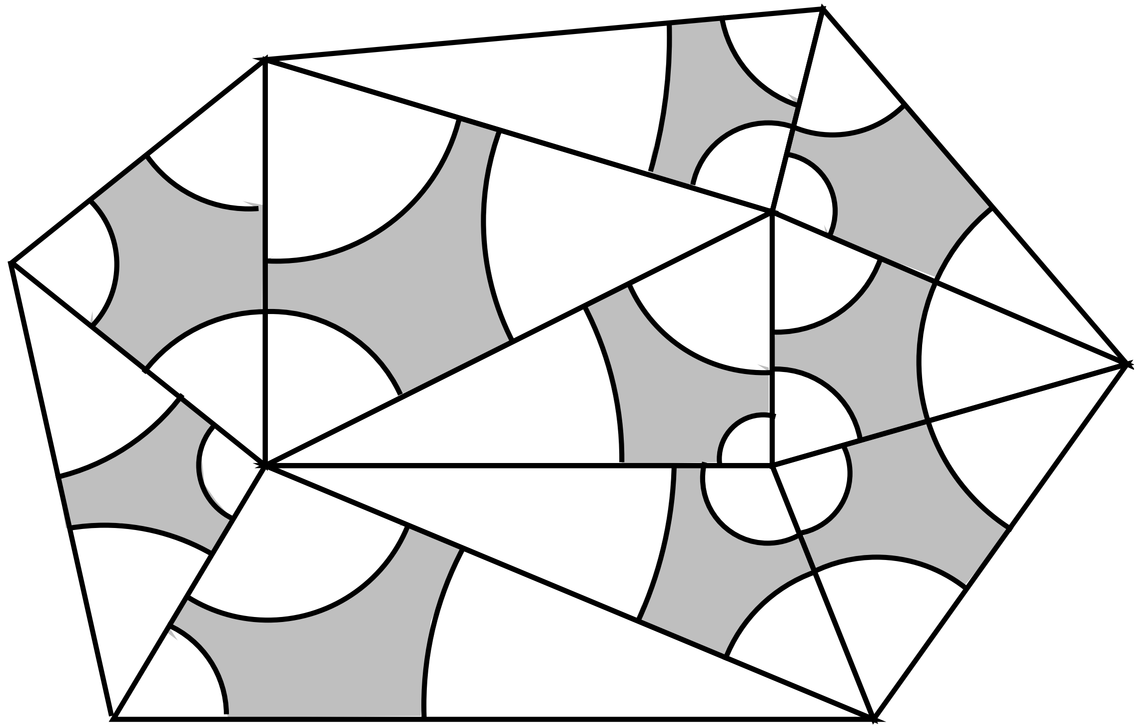
Construct Gabriel points:



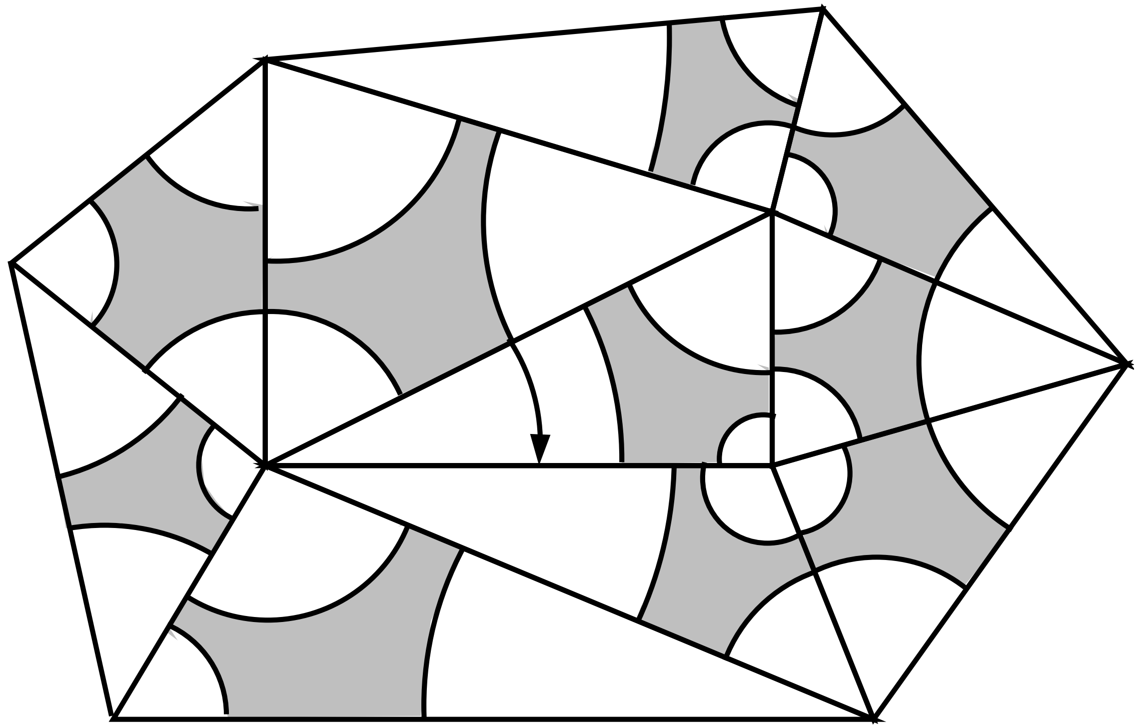
Thin parts foliated by circles centered at vertices.



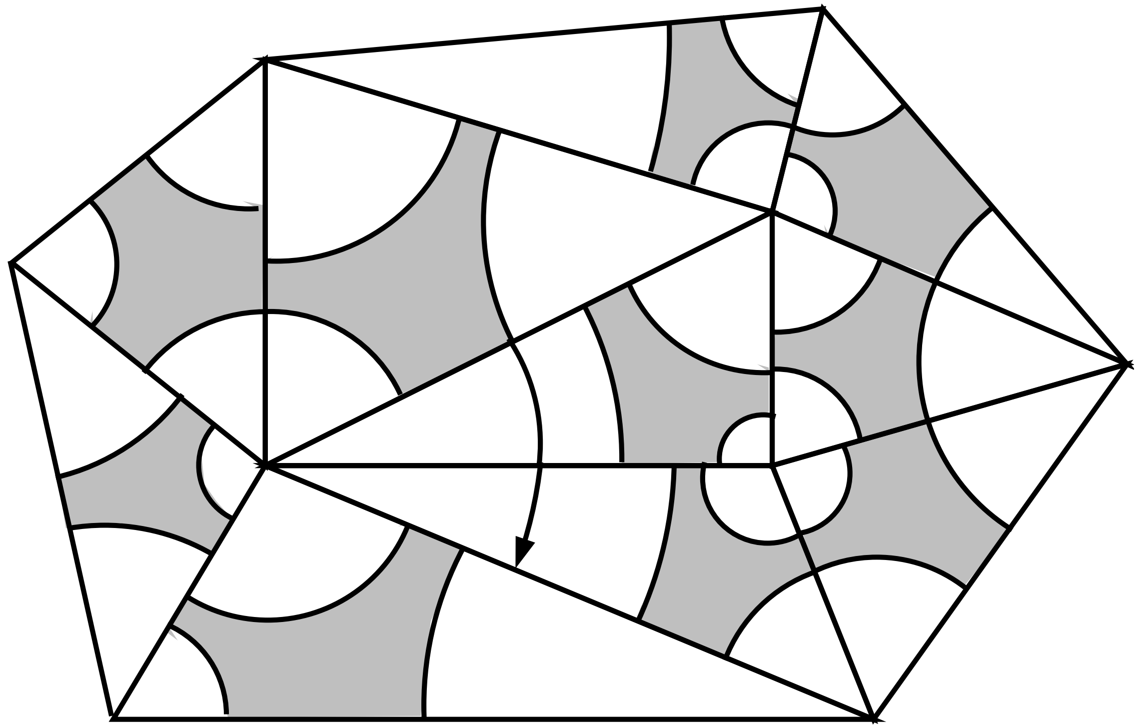
- Start with any triangulation.



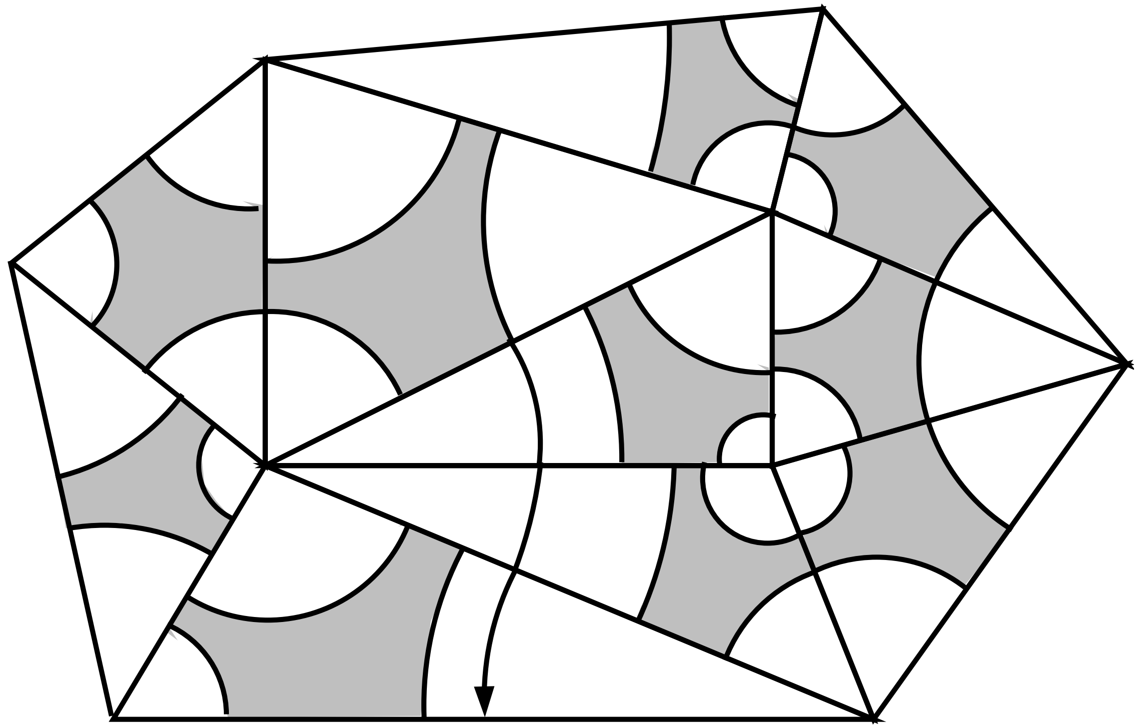
- Start with any triangulation.
- Make thick/thin parts.



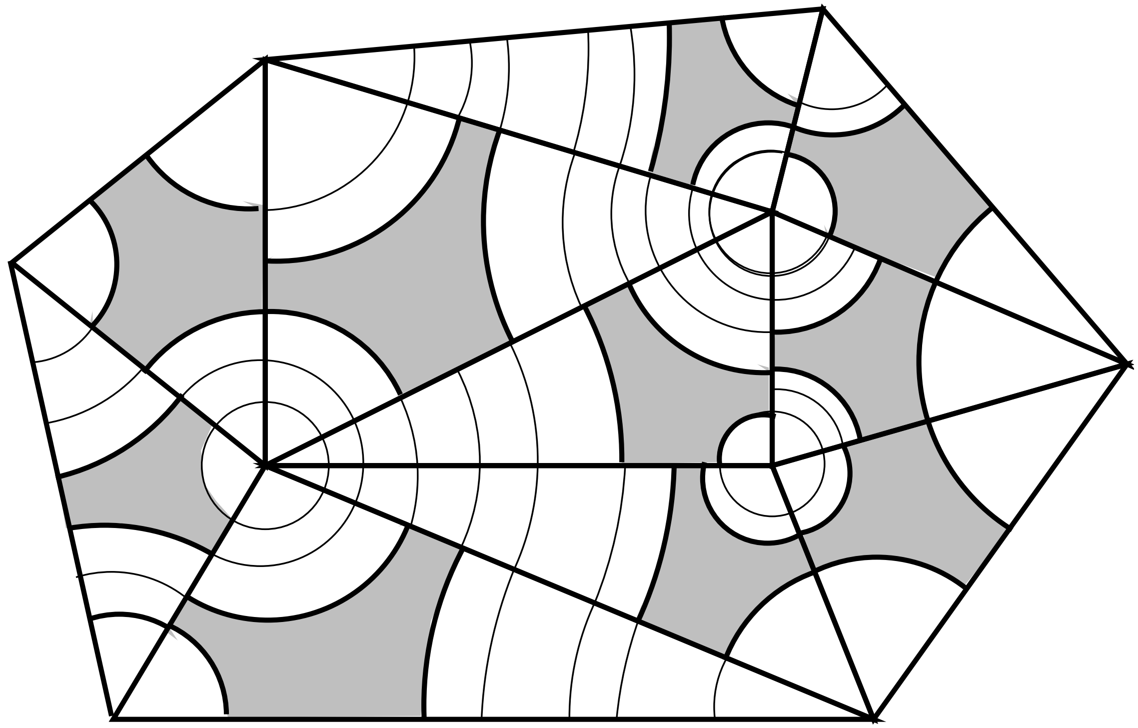
- Start with any triangulation.
- Make thick/thin parts.
- Propagate vertices until they leave thin parts.



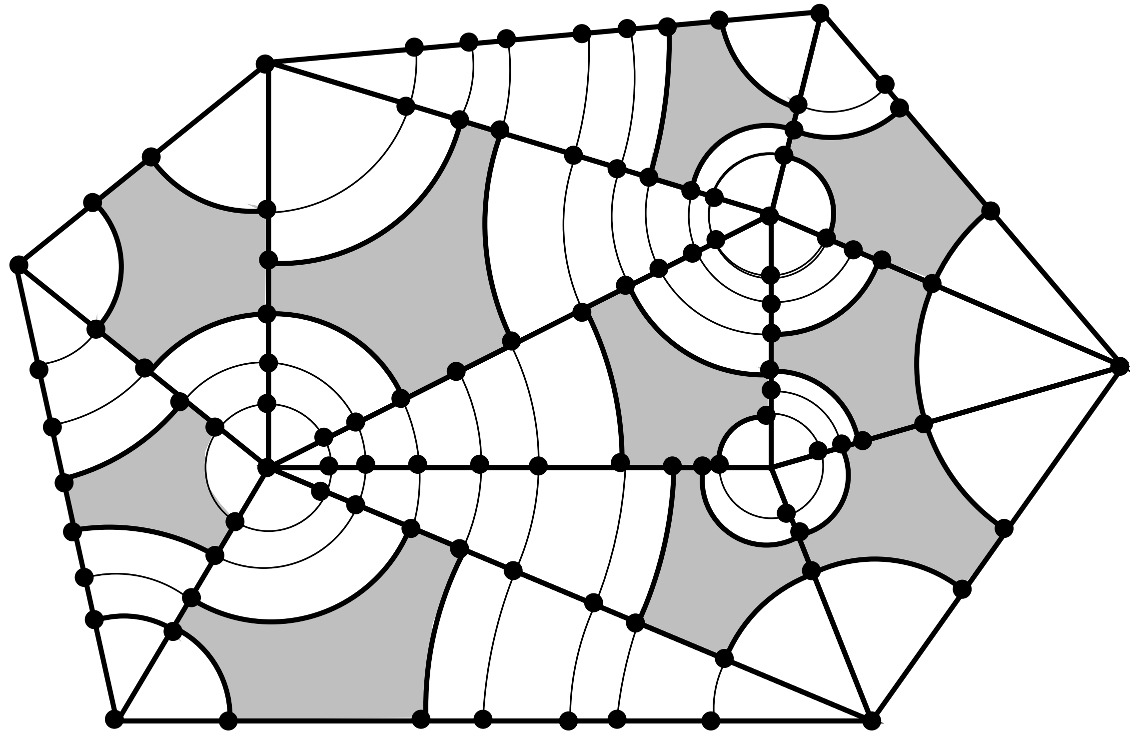
- Start with any triangulation.
- Make thick/thin parts.
- Propagate vertices until they leave thin parts.



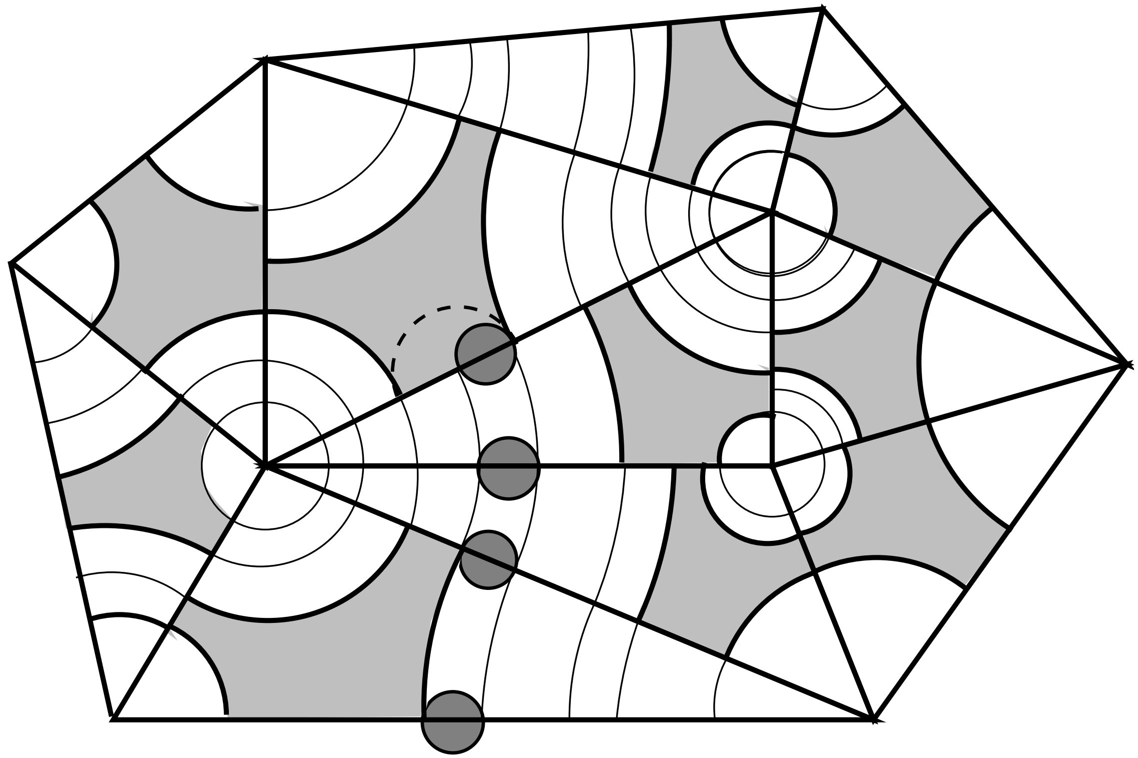
- Start with any triangulation.
- Make thick/thin parts.
- Propagate vertices until they leave thin parts.



- Start with any triangulation.
- Make thick/thin parts.
- Propagate vertices until they leave thin parts.



- Start with any triangulation.
- Make thick/thin parts.
- Propagate vertices until they leave thin parts.
- Intersections satisfy Gabriel condition. Why?

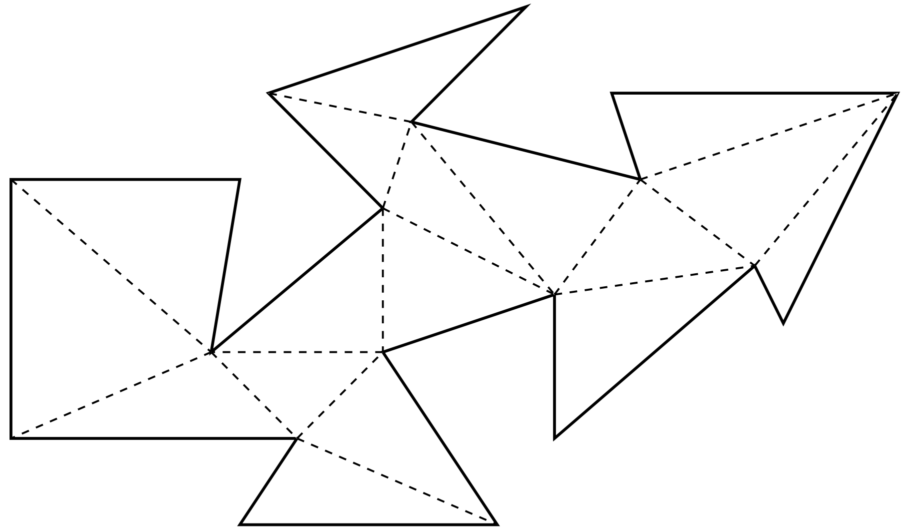


Tube is “swept out” by fixed diameter disk. Verifies Gabriel.

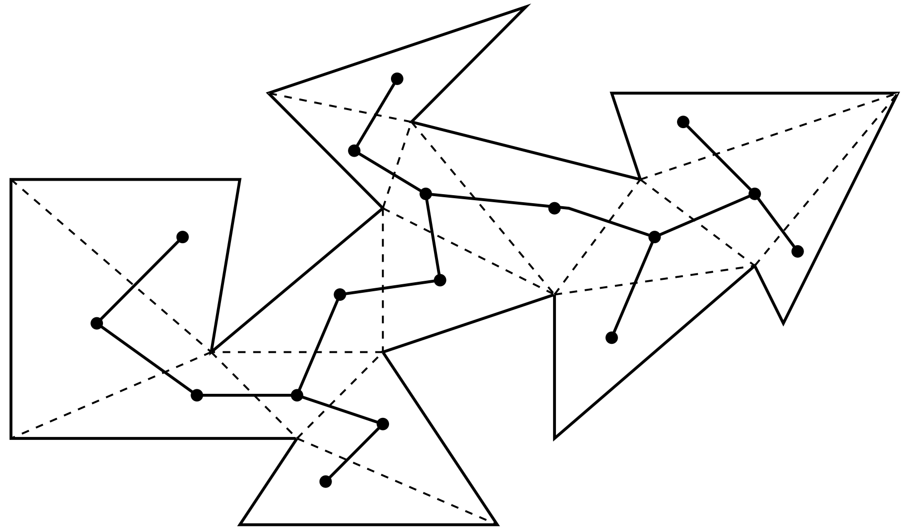
Disk lies inside tube or thick part or outside convex hull.

How many points have we generated?

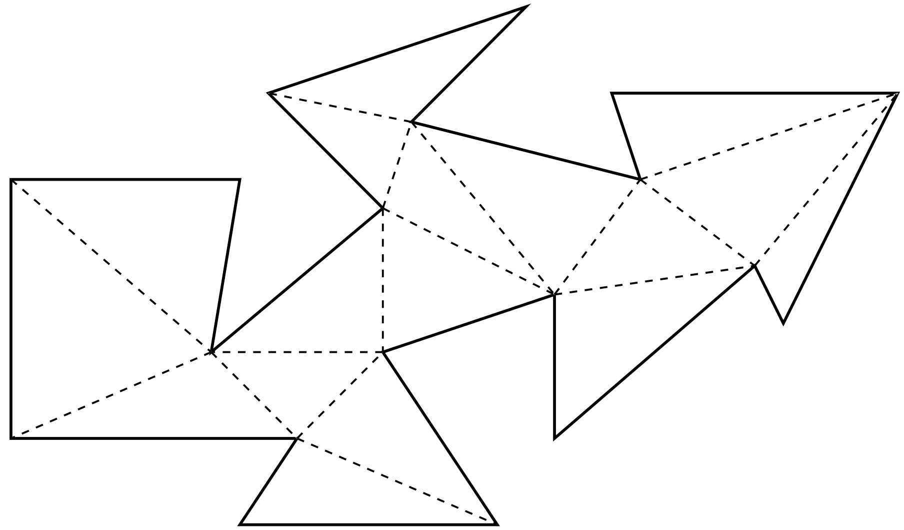
Special case: triangulation of an n-gon by diagonals.



Special case: in triangulation of a n-gon, triangles form tree.



Special case: in triangulation of a n-gon, triangles form tree.

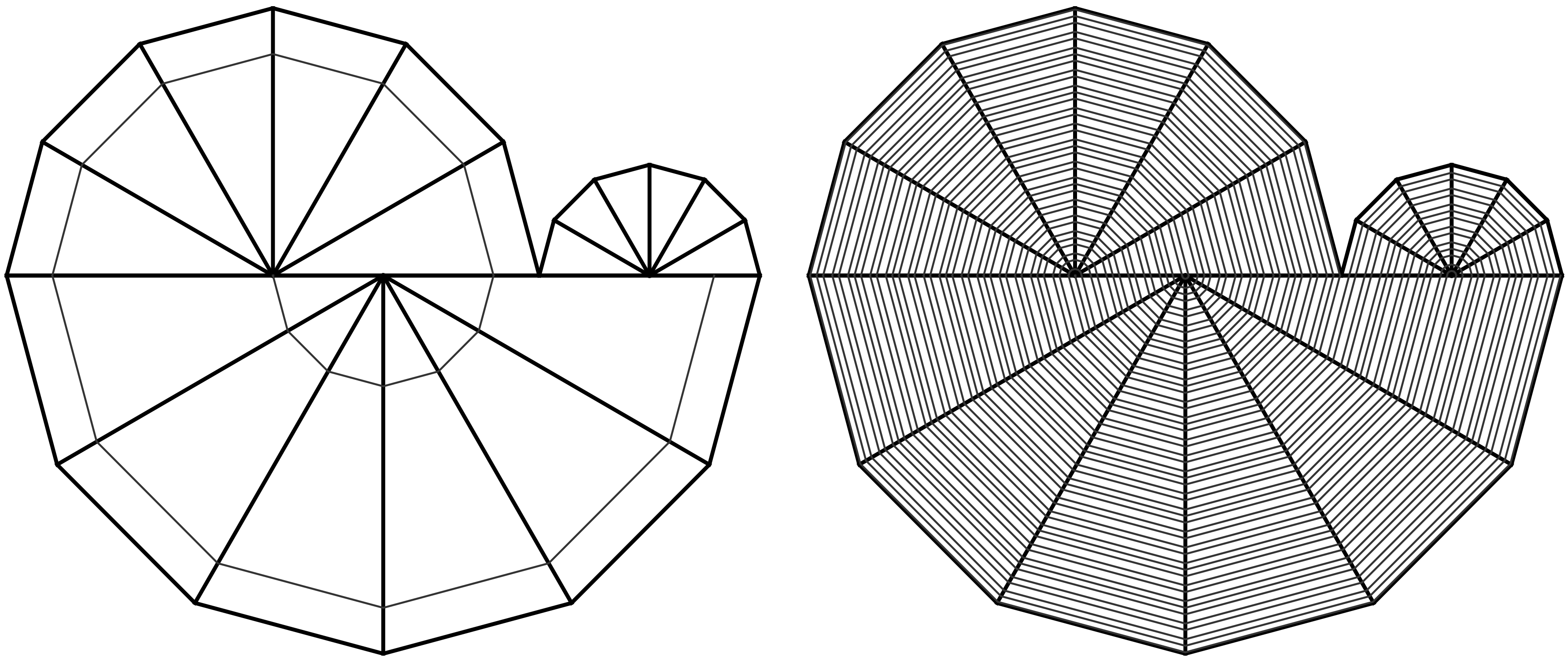


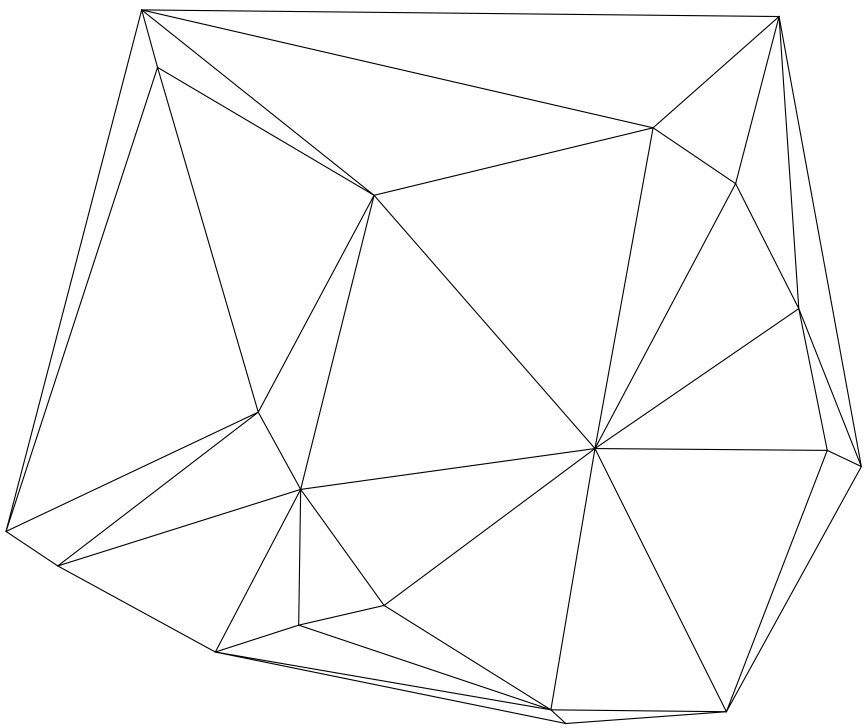
Hence propagating paths never revisit a triangle.
$6n$ starting points, so $O(n^2)$ points are created.

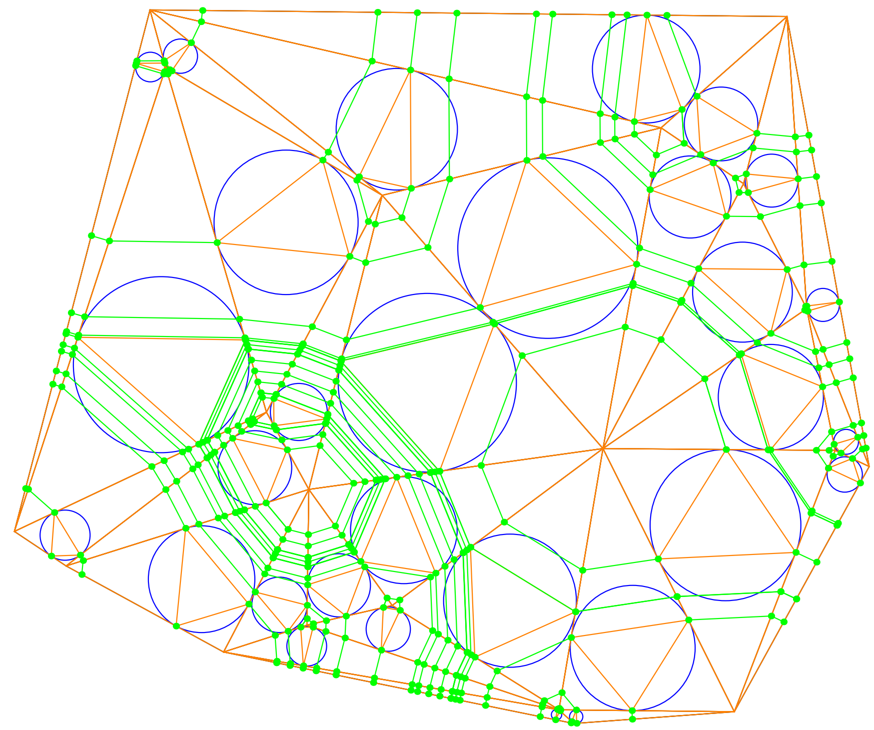
Theorem: Any triangulation of an n-gon by diagonals has a refinement into $O(n^2)$ non-obtuse triangles.

Improves $O(n^4)$ bound by Bern and Eppstein (1992).

In general case, path can hit same thin part many times.







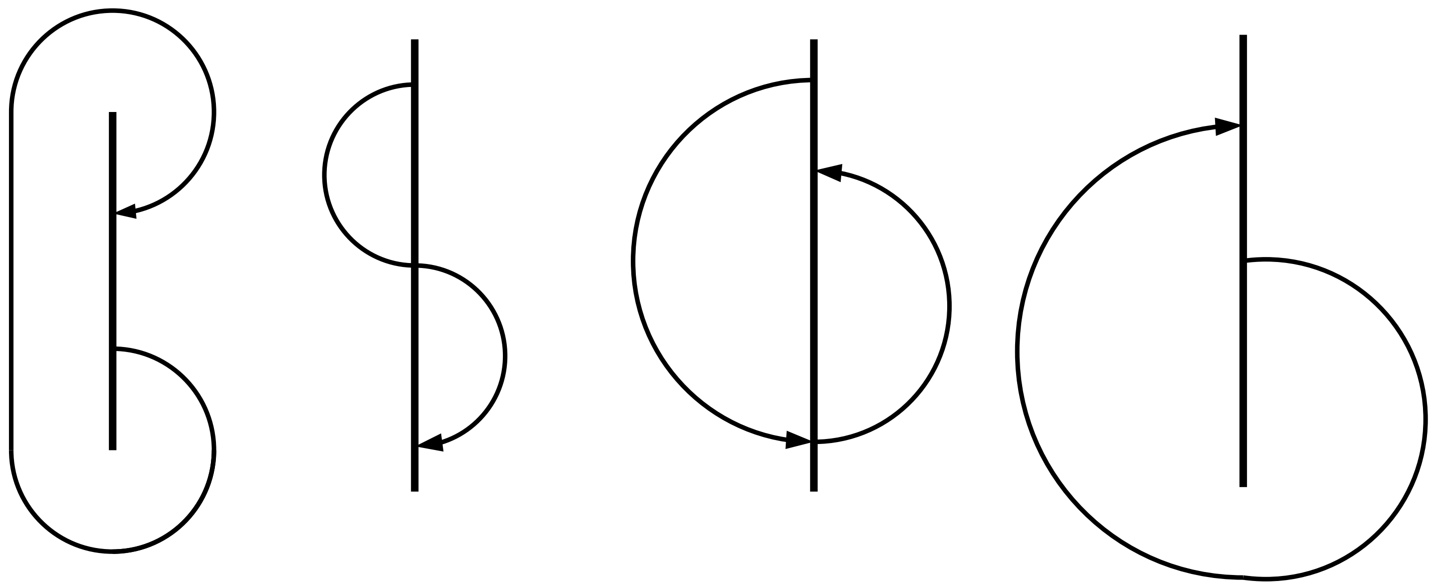
To get complexity bounds in general case, we must prevent propagation paths from returning to same triangle too often.

Basic strategy:

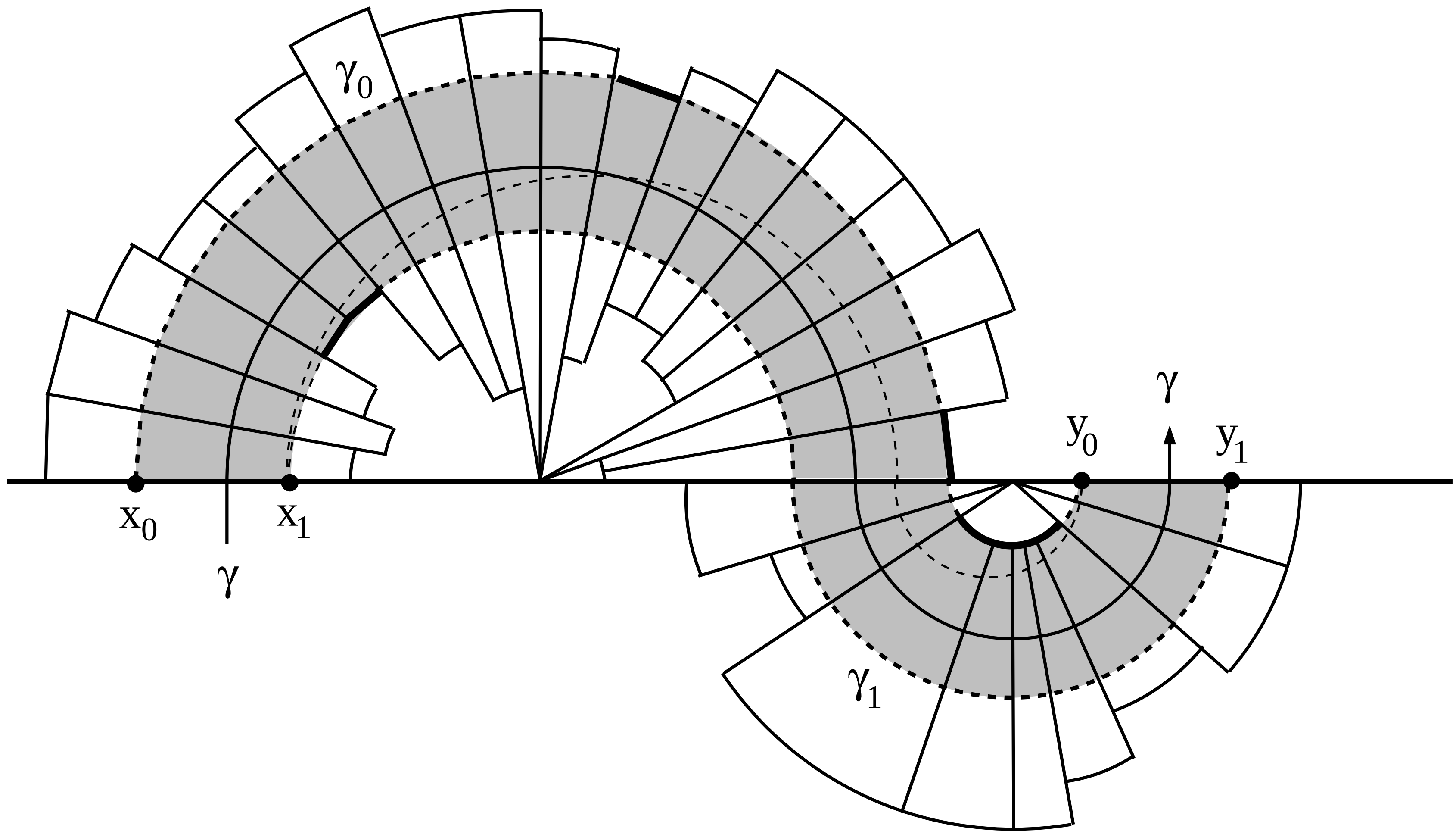
- Identify "return regions" that all paths must enter within $O(n)$ steps.

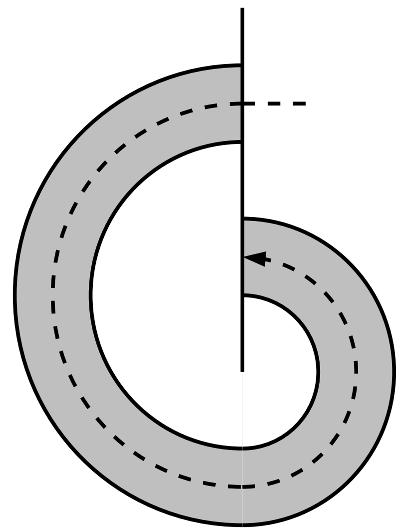
- "Bend" paths within return regions so that they terminate (collide with other paths). Maintain the Gabriel condition.

If a path hits an edge 3 times, then some subpath looks like:

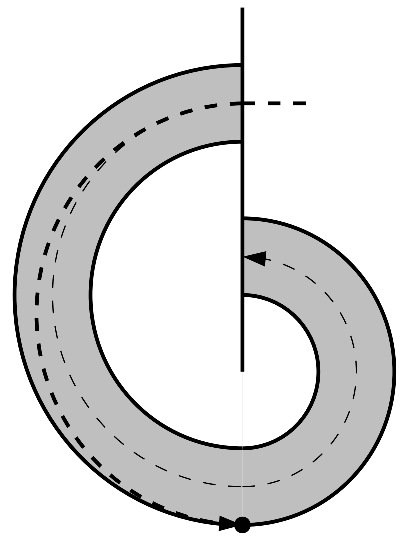


Return region is maximal width union of "parallel" paths:





There are $O(n)$ return regions and every propagation path enters one after crossing at most $O(n)$ thin parts.



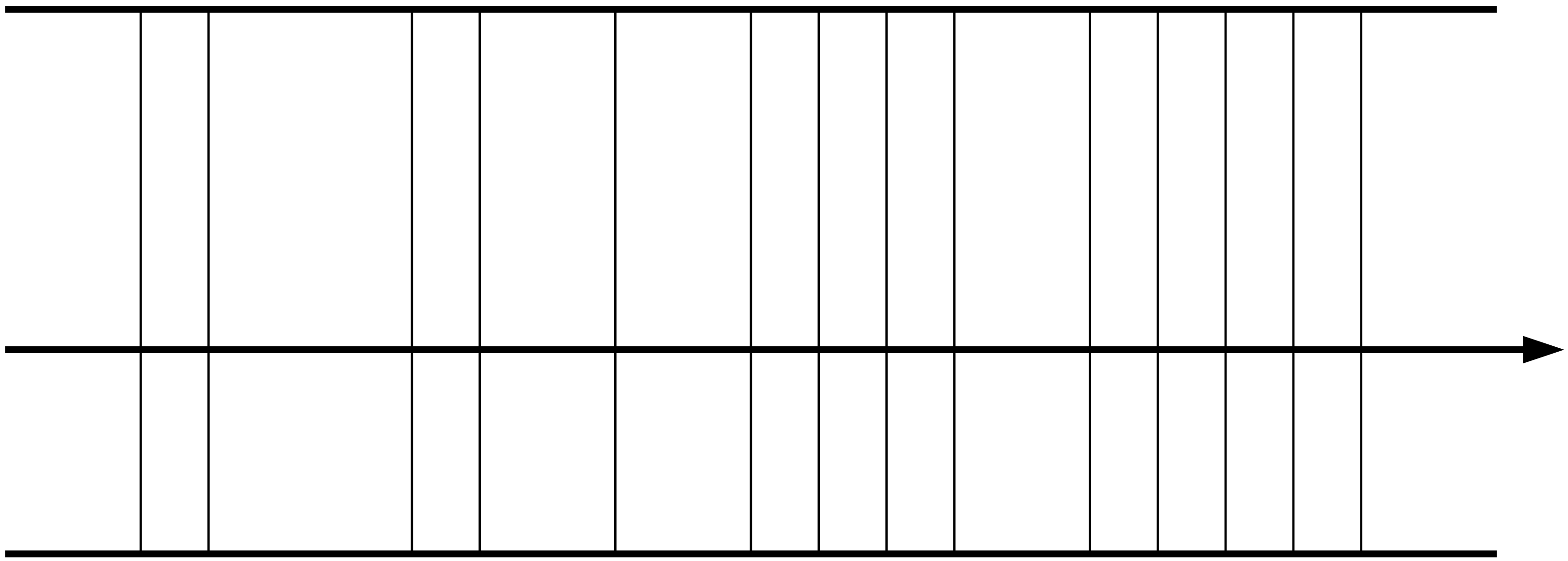
There are $O(n)$ return regions and every propagation path enters one after crossing at most $O(n)$ thin parts.

IDEA: bend paths to hit side before they exit.

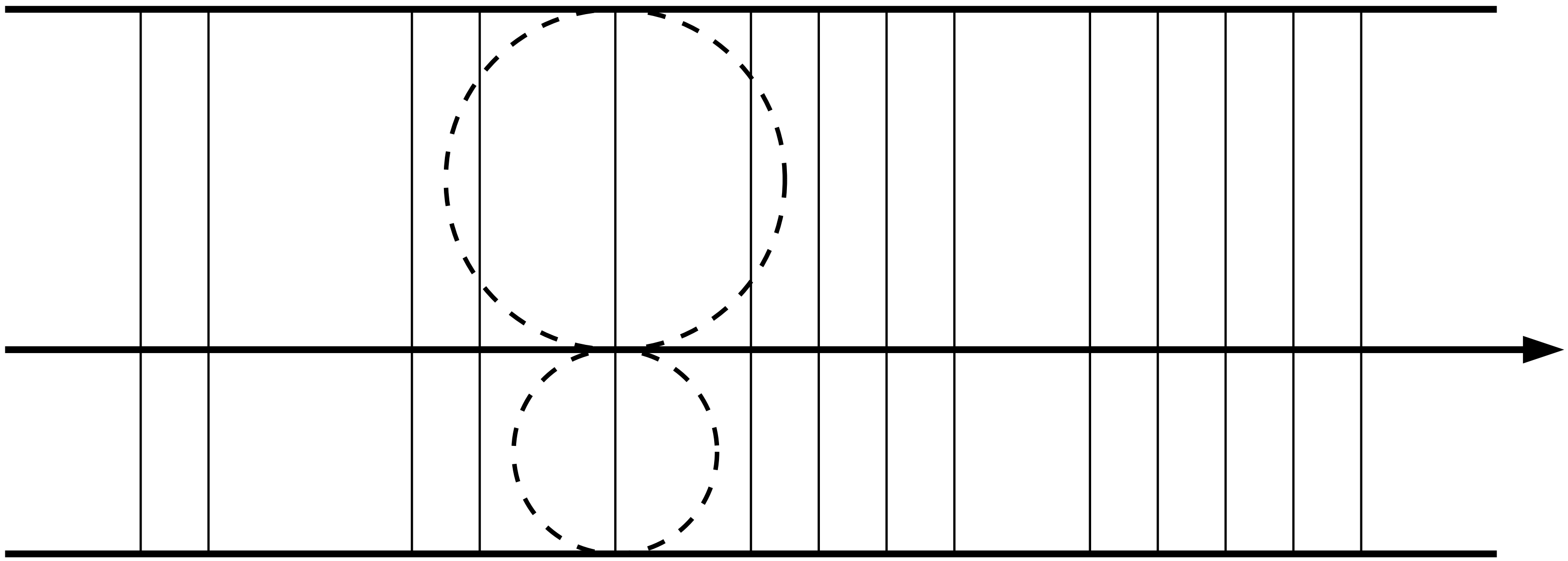
Still need Gabriel condition.

Gives $O(n^2)$ if it works.

For simplicity, “straighten” region to rectangle.



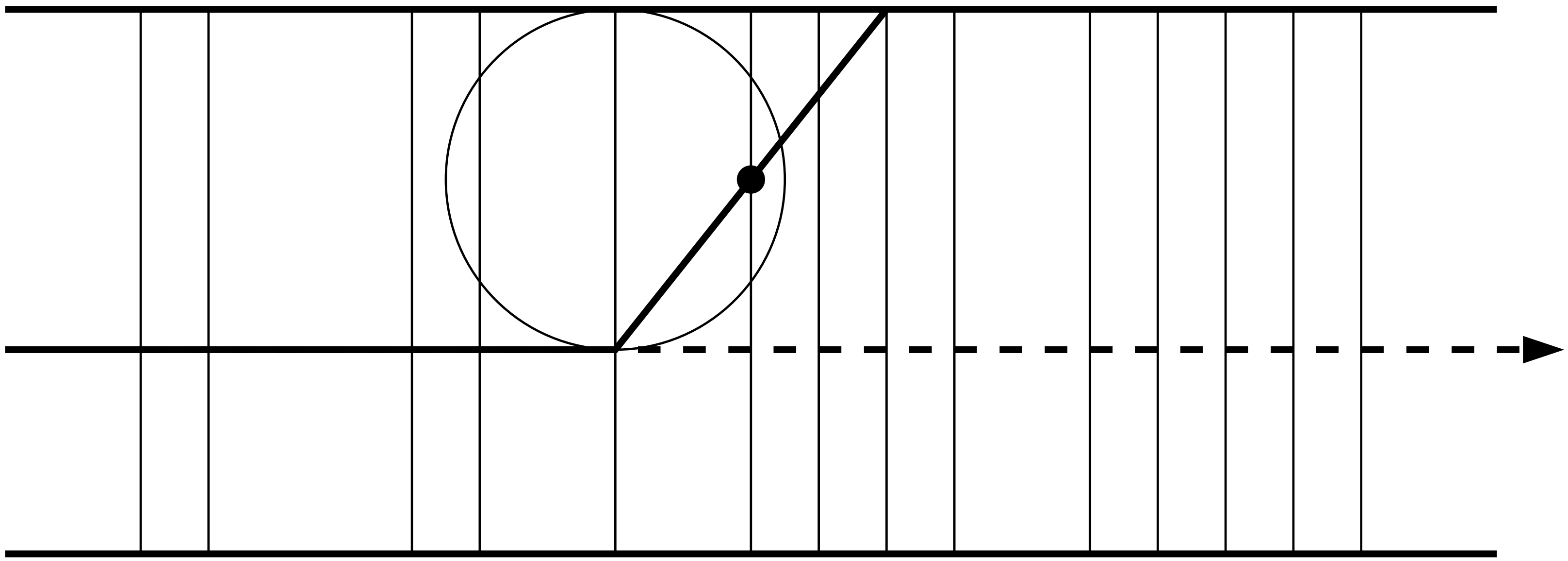
For simplicity, “straighten” region to rectangle.



Gabriel condition is satisfied if path follows foliation.

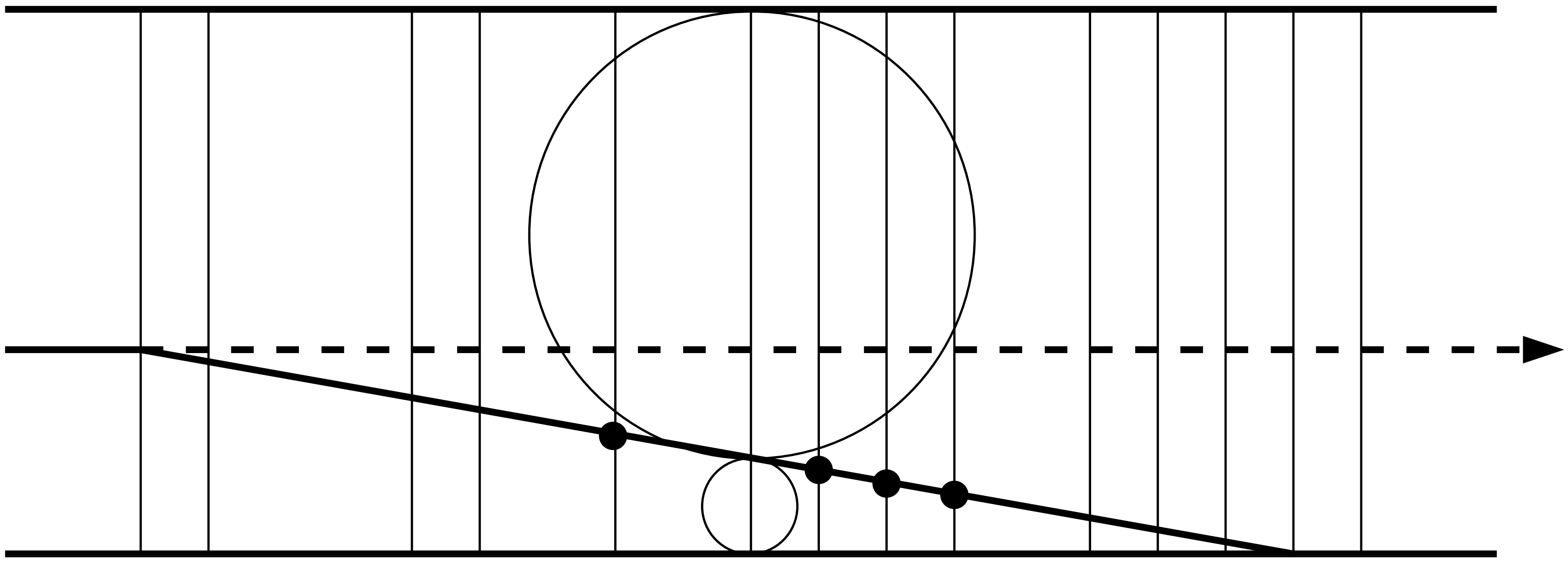
What if we bend the path?

For simplicity, “straighten” region to rectangle.



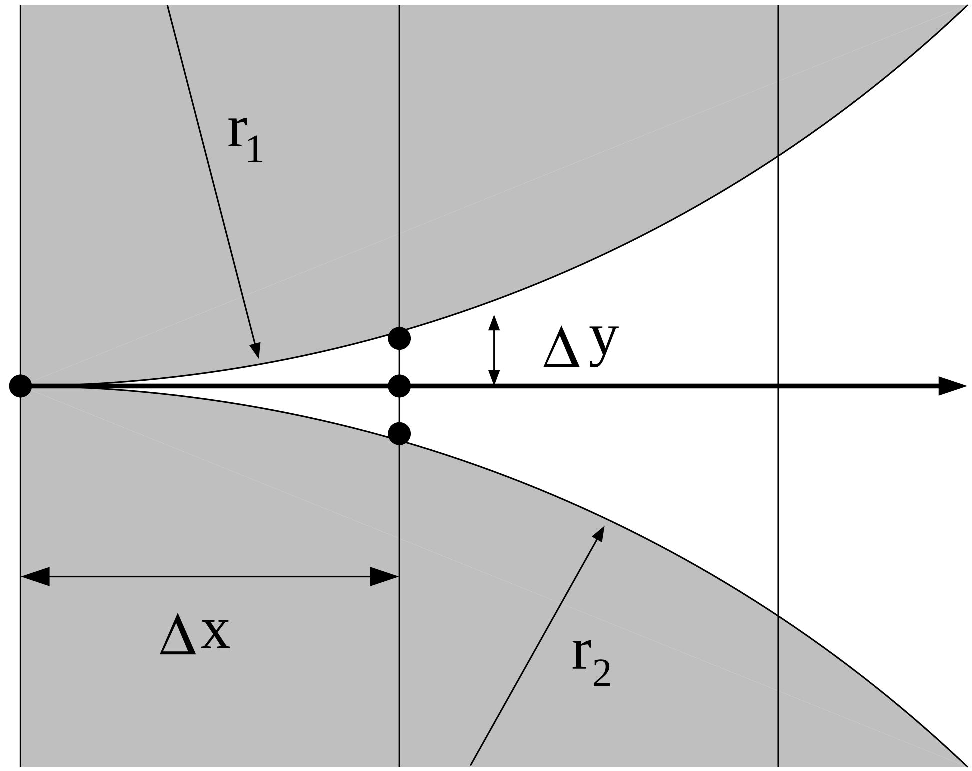
If path bends too fast, Gabriel condition can fail.

For simplicity, "straighten" region to rectangle.



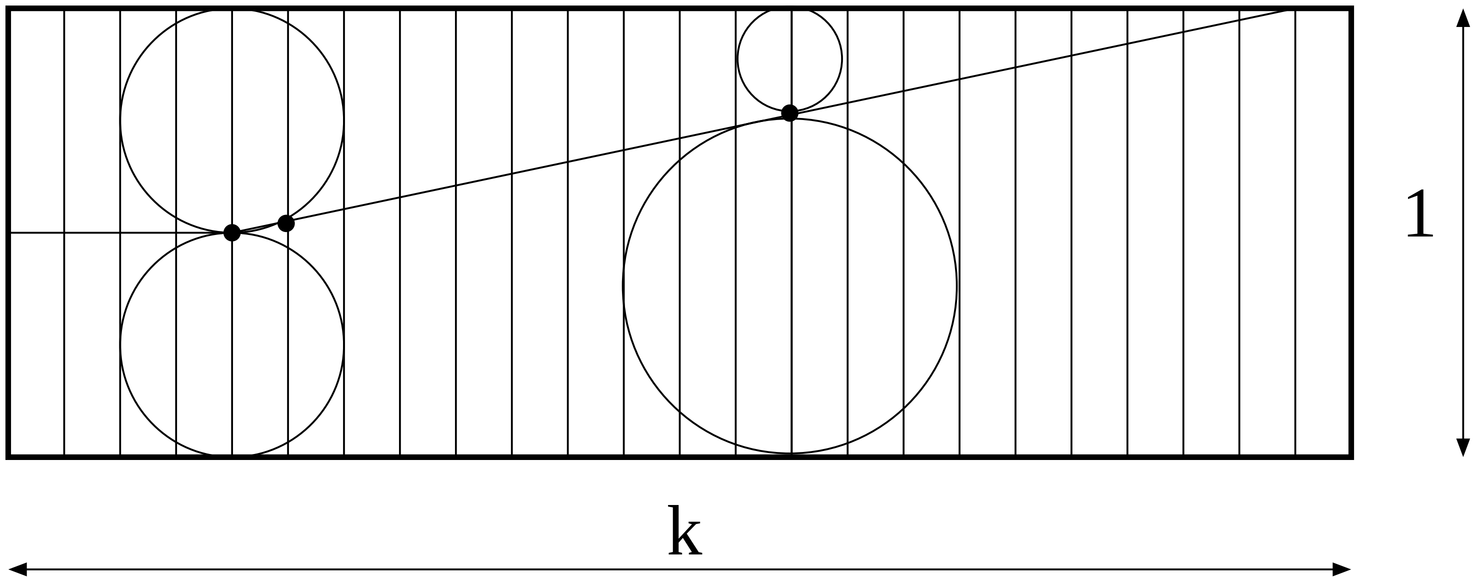
Bend slowly enough to satisfy Gabriel condition.

How far can we bend?



Answer: $\Delta y \approx (\Delta x/r)^2 r = (\Delta x)^2/r.$

$$r = \max(r_1, r_2).$$



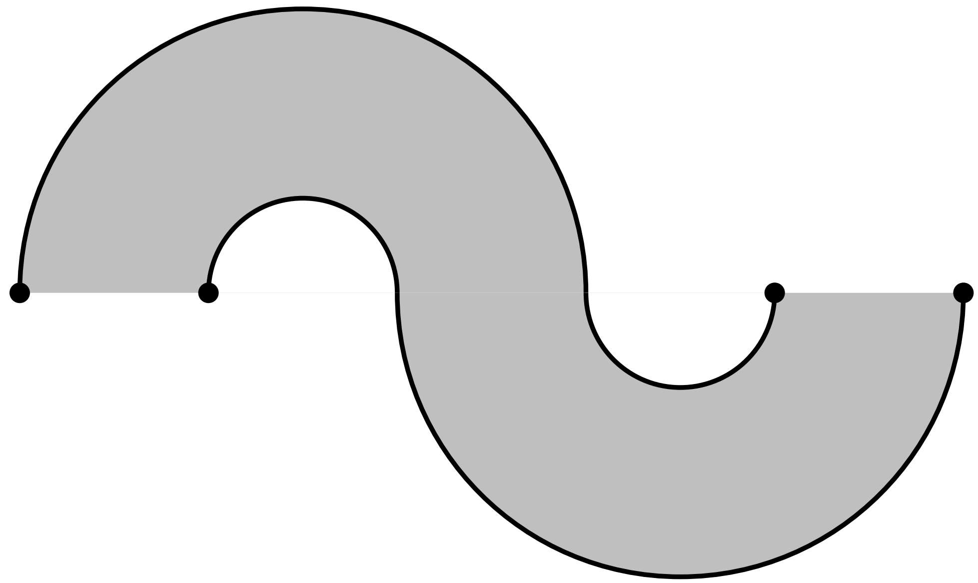
$k \times 1$ region crossing n (equally spaced) thin parts,

$r \approx 1, \quad \Delta x \approx k/n, \quad \Rightarrow \quad \Delta y \approx k^2/n^2$

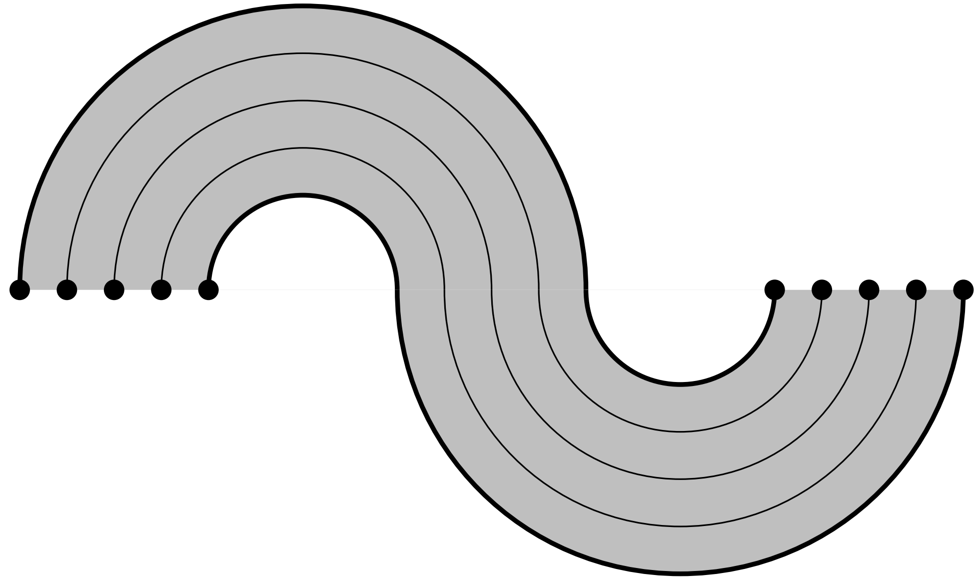
We need: $n\Delta y \geq 1$ or $k \geq \sqrt{n}$.

Return regions should have length $> \sqrt{n} \cdot$ width.

Easy case: return region length > width.

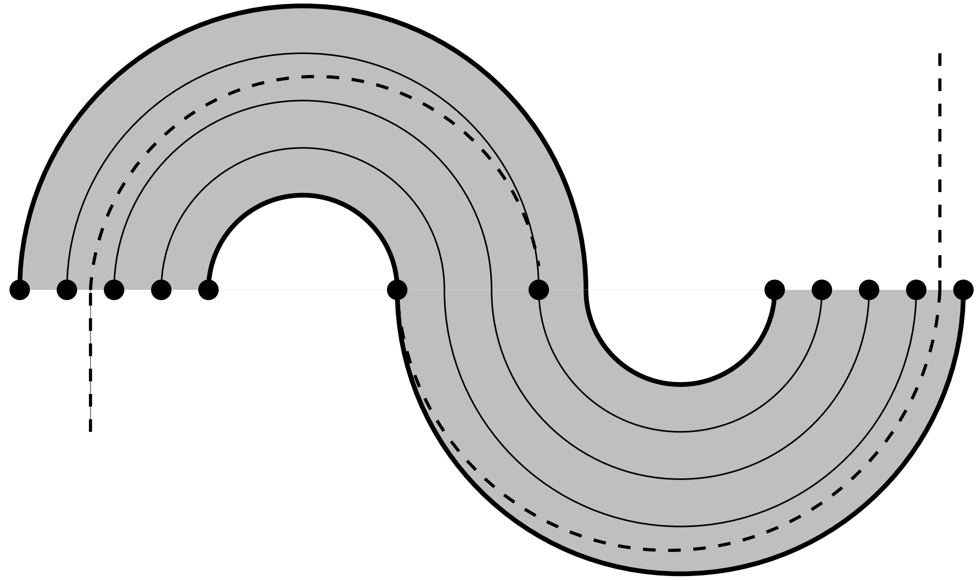


Easy case: return region length > width.



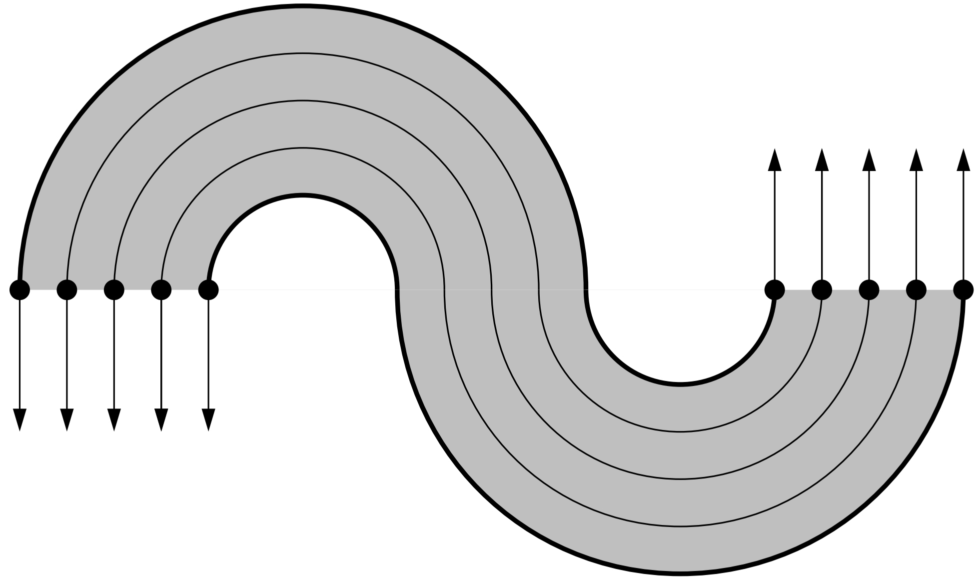
- Divide each region into $O(\sqrt{n})$ long parallel tubes.

Easy case: return region length > width.



- Divide each region into $O(\sqrt{n})$ long parallel tubes.
- Entering paths can be bent and terminated.
  Total vertices created $= O(n^2)$, but $\ldots$

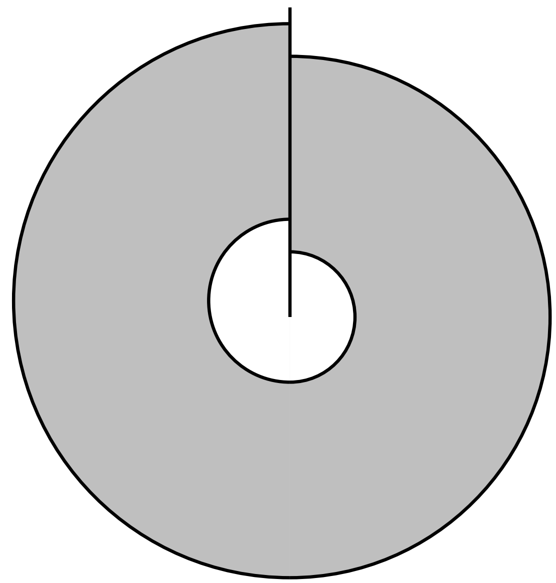
Easy case: return region length > width.



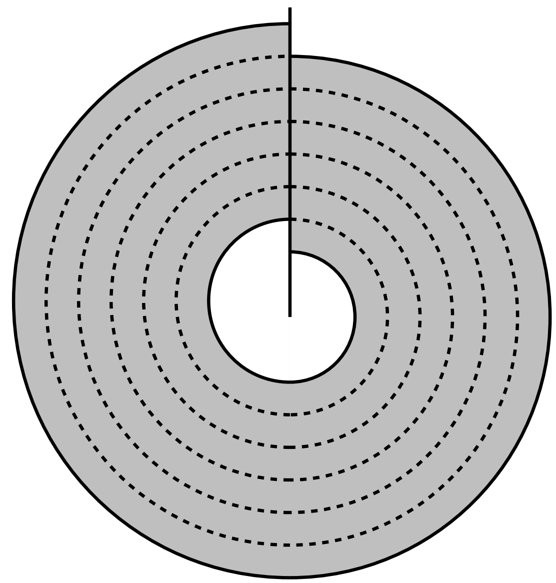
- Divide each region into $O(\sqrt{n})$ long parallel tubes.
- Entering paths can be bent and terminated.
  Total vertices created $= O(n^2)$, but ...
- Each region has $O(\sqrt{n})$ new vertices to propagate.
  $O(\sqrt{n} \cdot n \cdot n) = O(n^{2.5})$ are created.

This is where "2.5" in NOT theorem comes from.

Hard case is spirals:



Hard case is spirals:



Curves may spiral arbitrarily often (independent of n).

No curve can be allowed to pass through entire spiral.

We stop them in a 5-phase construction.

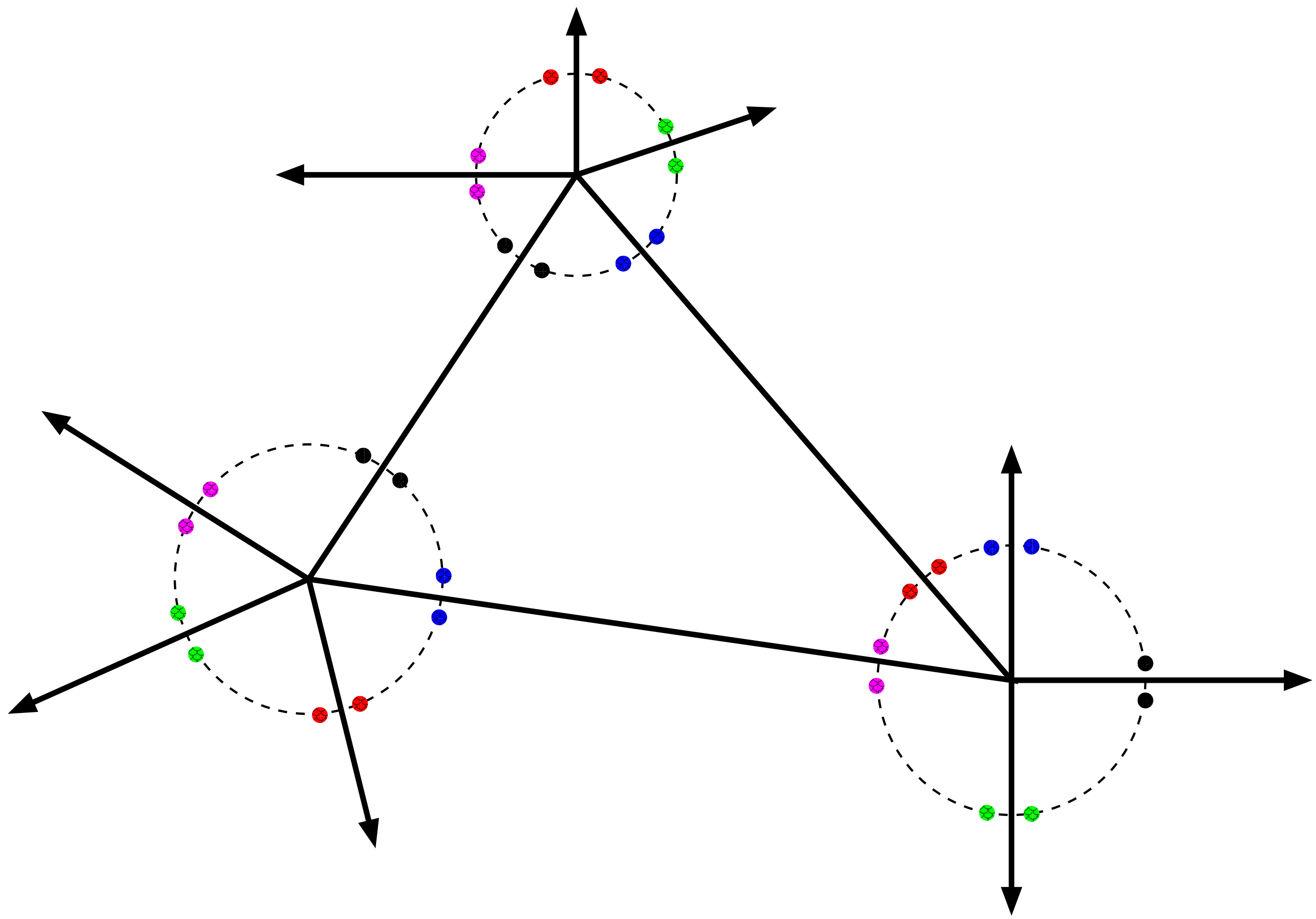
I won't discuss details here (see Appendix D).

Spiral case finishes proof of NOT theorem.

Remember the cell tower problem?

Remember the cell tower problem?

It's easy if countries are acute triangles: add points near corners, in symmetric pairs across sides. NOT thm $\Rightarrow O(n^{5/2})$ towers.



Speculation 1: NOTs and flows

The proof of the NOT theorem is very reminiscent of:

Pugh's Closing Lemma: Let V be a C^1 vector field on a compact, smooth manifold M, and $x \in M$ a point whose orbit accumulates on itself. Then V can be C^1 approximated by vector fields so that x is periodic.

Unknown for C^2 perturbations; some partial results.

Is there a connection between NOTs and closing lemmas?

Flows in NOT proof induce interval exchange maps. Exploitable?

Could dynamics get rid of extra $\sqrt{n}$ in NOT theorem?

Speculation 2: NOTs in 3 dimensions

Many applications are 3-D. Many heuristics, little theory.

Can we divide a polyhedron into pieces associated to the eight 3-D geometries and mesh accordingly?

3-D is hard: First acute triangulation of 3-cube not found until 2009. Uses 1370 tetrahedra, found by computer search. (VanderZee, Hirani, Zharnitsky and Guoy).

More "geometric" construction uses 2715 tetrahedra (Kopczynski, Pak and Przytycki). Impossible in 4-D.

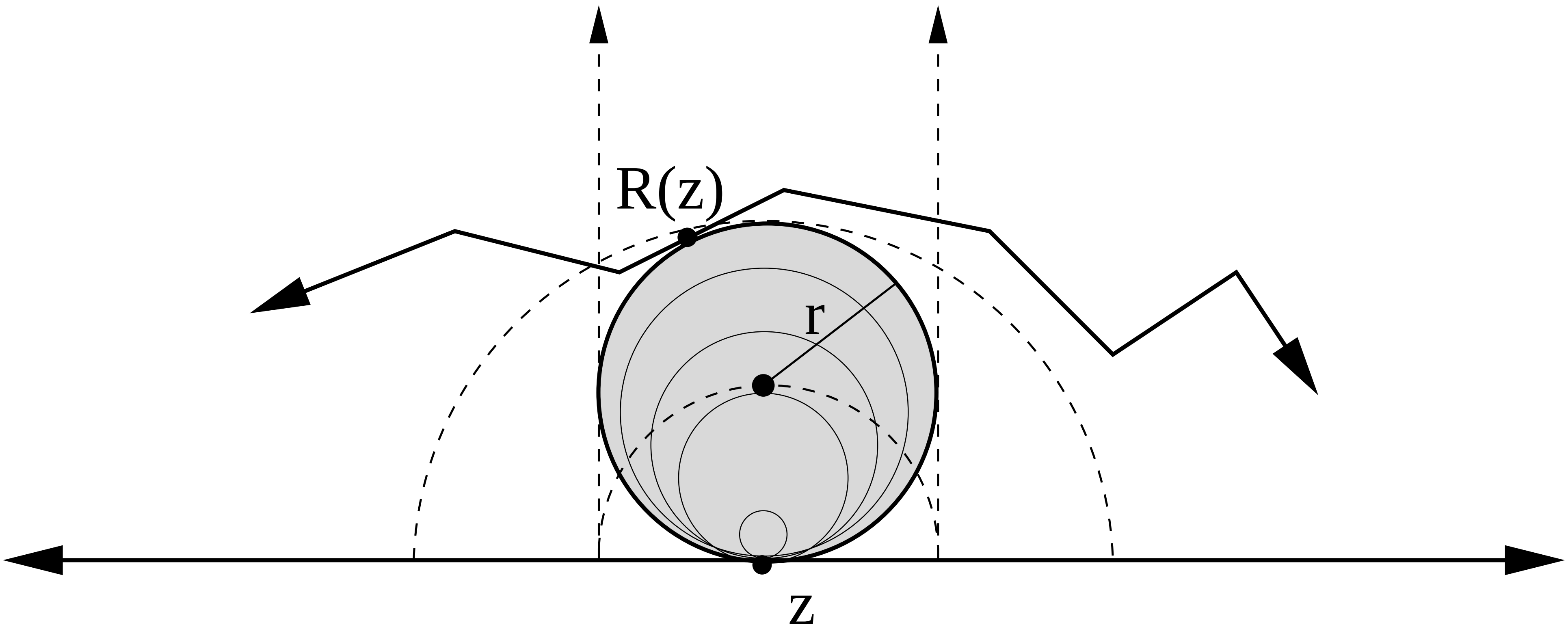
Main Conjecture: Every polyhedron in R^3 has an acute triangulation that can be computed in polynomial time.

Thanks!

APPENDIX A
PROOF OF SULLIVAN'S CONVEX HULL THEOREM

Fact 1: If $z \in \Omega$, $\infty \notin \Omega$,

$$r \simeq \mathrm{dist}(z, \partial\Omega) \simeq \mathrm{dist}(R(z), R^2) \simeq |z - R(z)|.$$



Fact 2: R is Lipschitz.

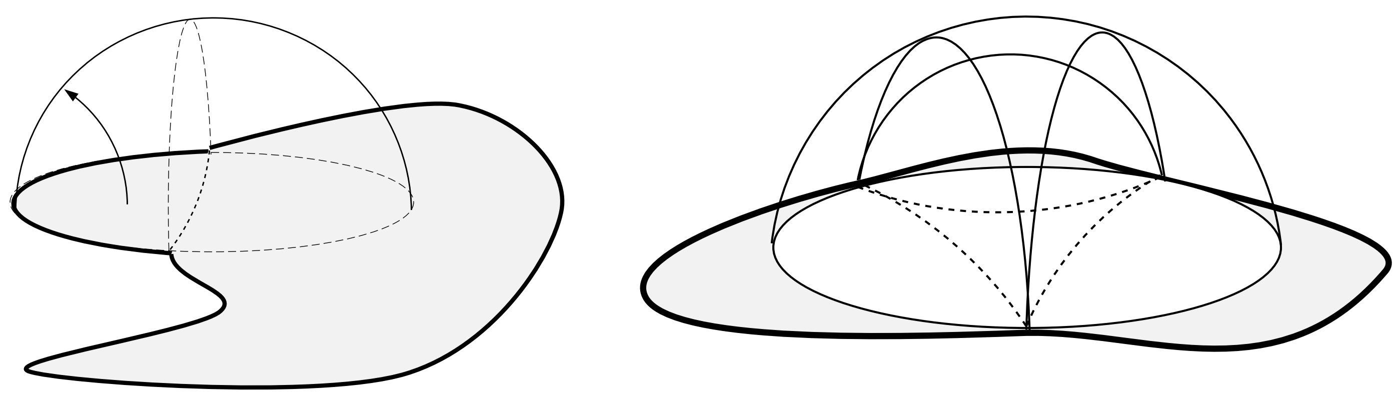
Ω simply connected $\Rightarrow$

$$d\rho \simeq \frac{|dz|}{\mathrm{dist}(z, \partial\Omega)}.$$

$z \in D \subset \Omega$ and $R(z) \in \mathrm{Dome(D)} \Rightarrow$

$$\mathrm{dist}(z, \partial\Omega)/\sqrt{2} \leq \mathrm{dist}(z, \partial D) \leq \mathrm{dist}(z, \partial\Omega)$$

$$\Rightarrow \quad \rho_\Omega(z) \simeq \rho_D(z) = \rho_{\mathrm{Dome}}(R(z)).$$



Fact 3: $\rho_S(R(z), R(w)) \leq 1 \Rightarrow \rho_\Omega(z,w) \leq C$.

Suppose $\mathrm{dist}(R(z), R^2) = r$ and γ is the geodesic from z to w. Then

$$\begin{array}{cc} \Rightarrow & \mathrm{dist}(\gamma, R^2) \simeq r \\ \Rightarrow & \mathrm{dist}(R^{-1}(\gamma), \partial\Omega) \simeq r, \\ & R^{-1}(\gamma) \subset D(z, Cr) \\ \Rightarrow & \rho_\Omega(z,w) \leq C \end{array}$$

Moreover, $g = \iota \circ \sigma : \Omega \to D$ is locally Lipschitz. Standard estimates show

$$|g'(z)| \simeq \frac{\mathrm{dist}(g(z), \partial D)}{\mathrm{dist}(z, \partial\Omega)}.$$

Use Fact 1

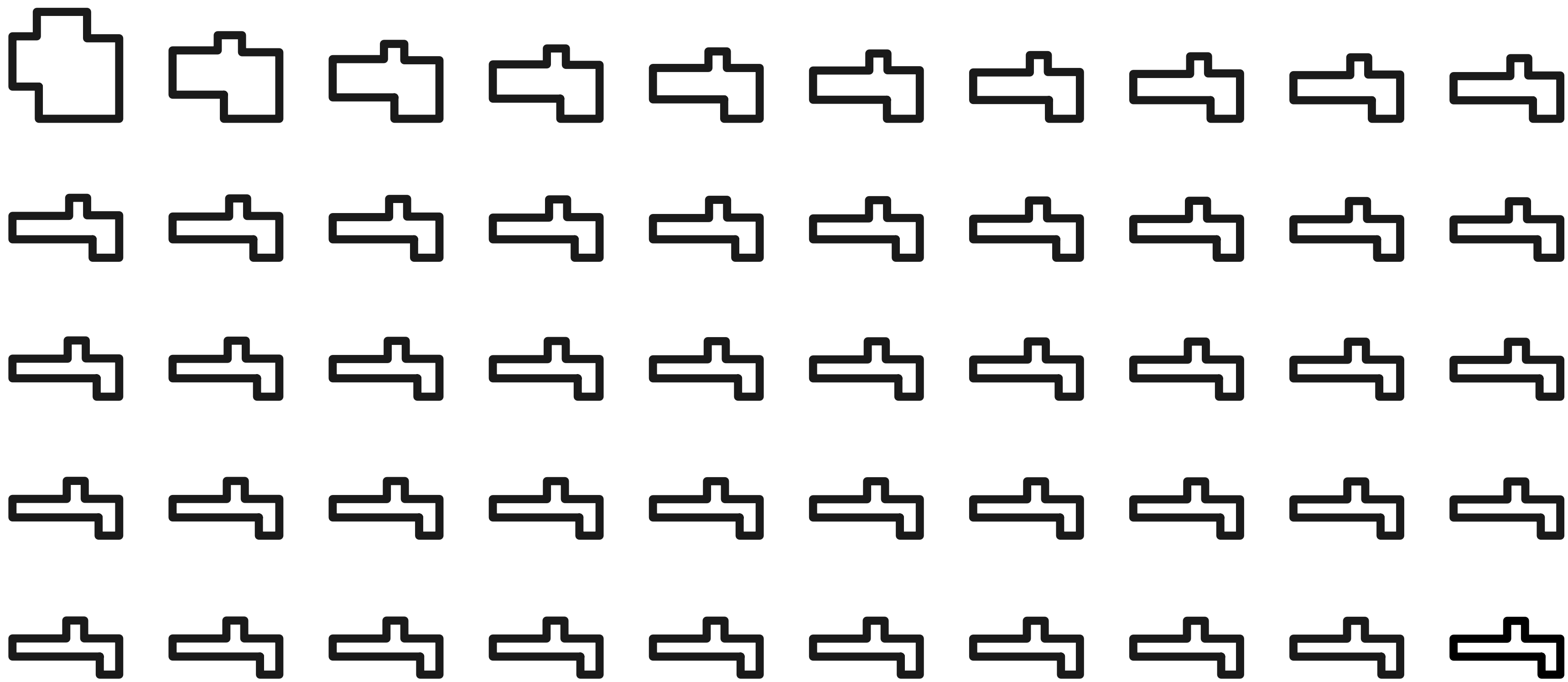
$$\begin{aligned} \mathrm{dist}(z, \partial\Omega) &\simeq \mathrm{dist}(\sigma(z), R^2) \\ &\simeq \exp(-\rho_{R^3_+}(\sigma(z), z_0)) \\ &\gtrsim \exp(-\rho_S(\sigma(z), z_0)) \\ &= \exp(-\rho_D(g(z), 0)) \\ &\simeq \mathrm{dist}(g(z), \partial D) \end{aligned}$$

APPENDIX B
DAVIS' METHOD FOR FINDING CONFORMAL PREVERTICES

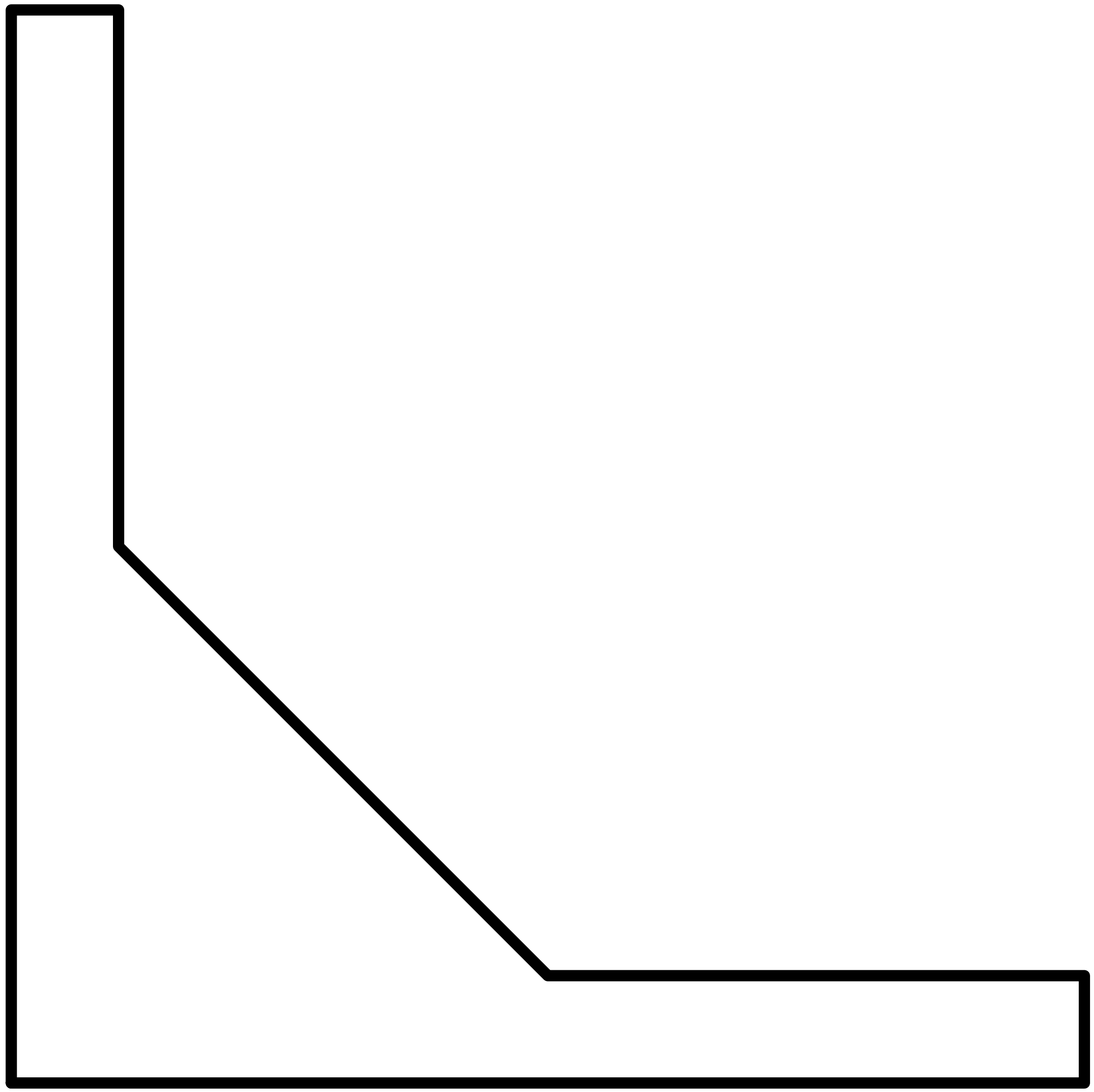
Davis' method:

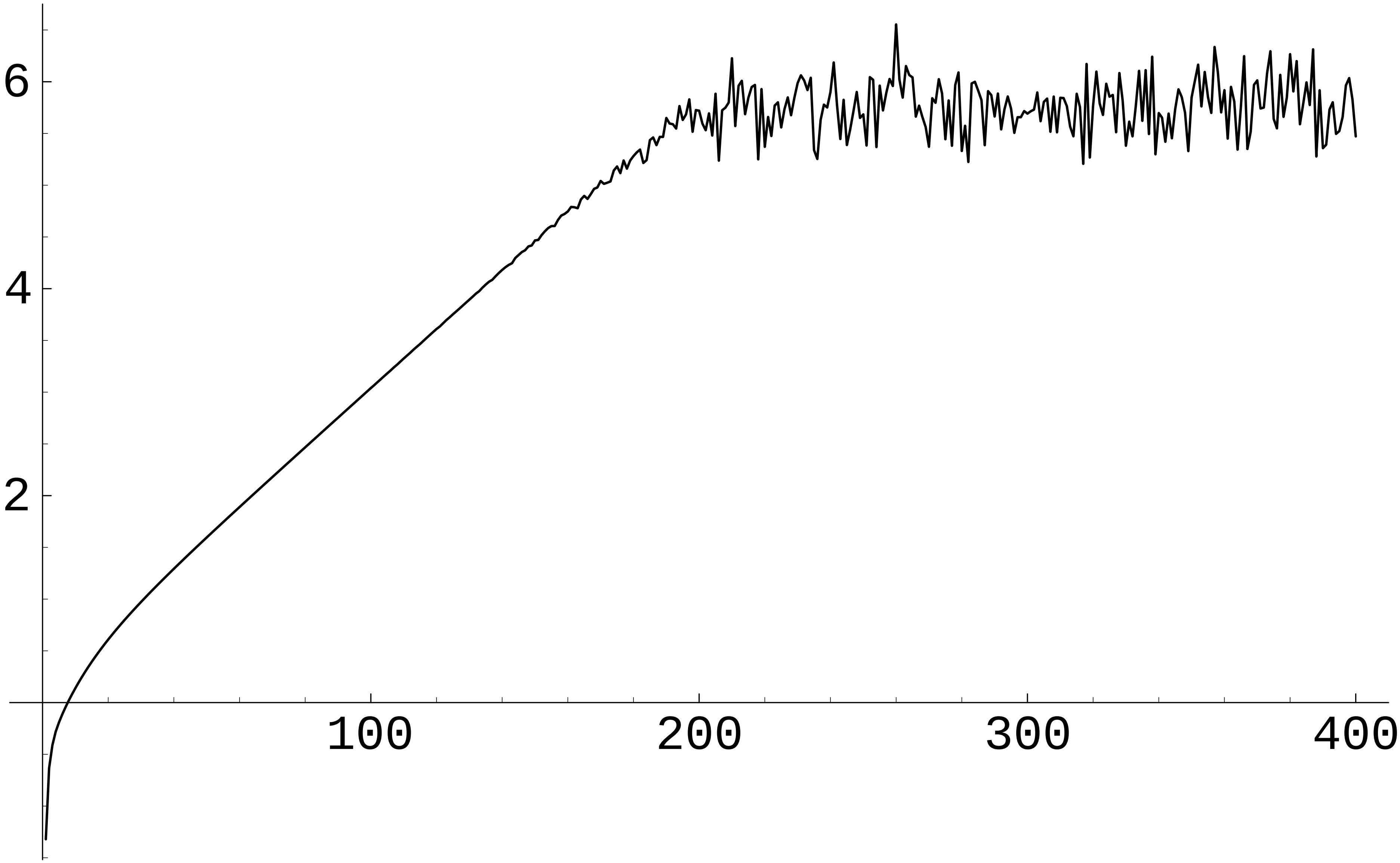
- Compare guessed polygon to target polygon.
- If an edge is too long, shorten corresponding parameter arc.
- If too long, lengthen the gap.

$$\text{new gap} = \text{old gap} \times \frac{\text{target side}}{\text{old side}} \times \text{normalizing constant}$$



50 iterations of Davis' method.

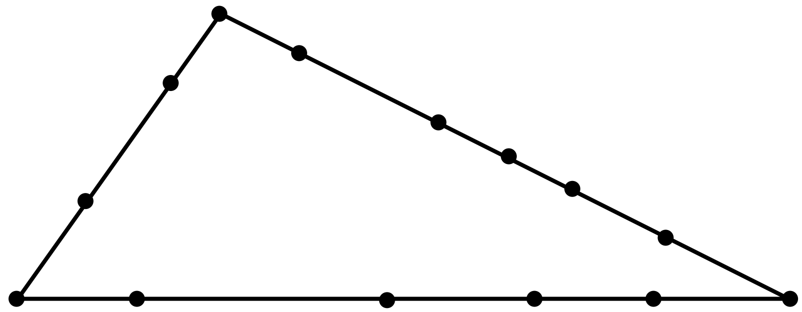




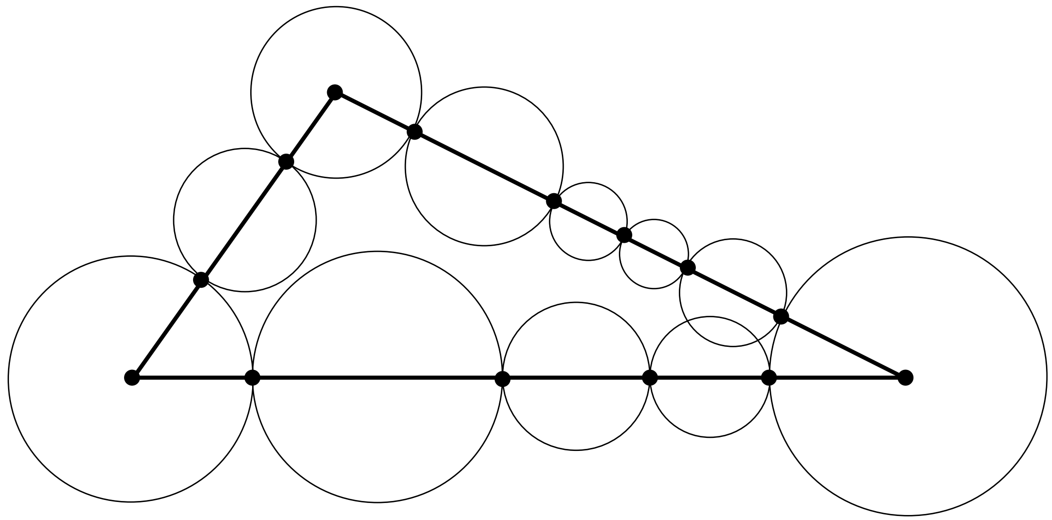
Negative log-QC error for 400 iterations of Davis' method.

APPENDIX C
SKETCH OF THE PROOF OF THE BMR LEMMA

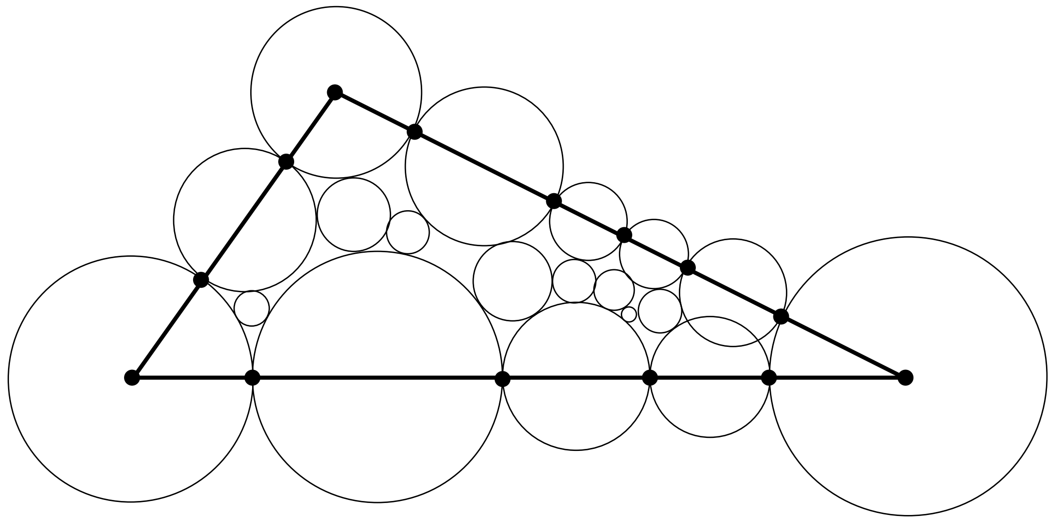
Sketch of BMR lemma:



Sketch of BMR lemma:

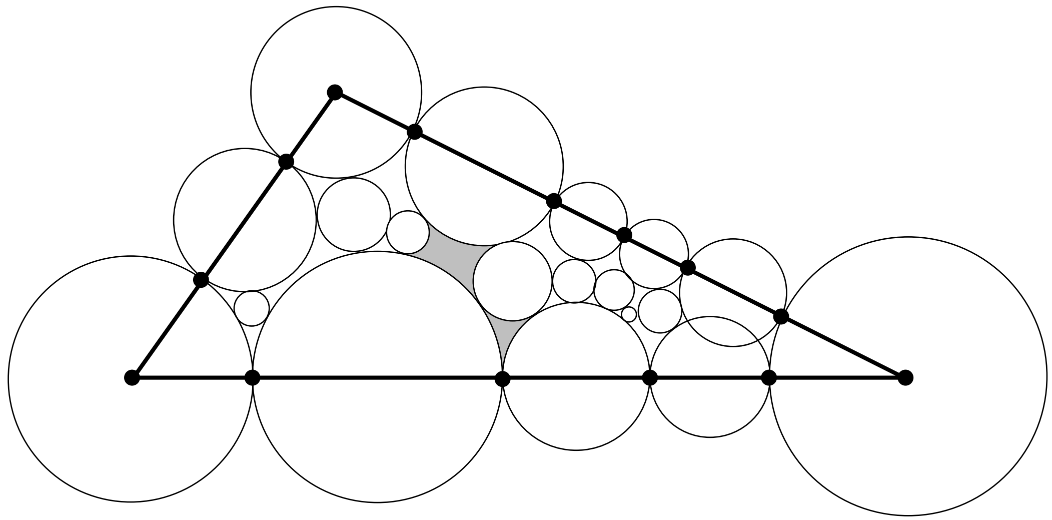


Sketch of BMR lemma:



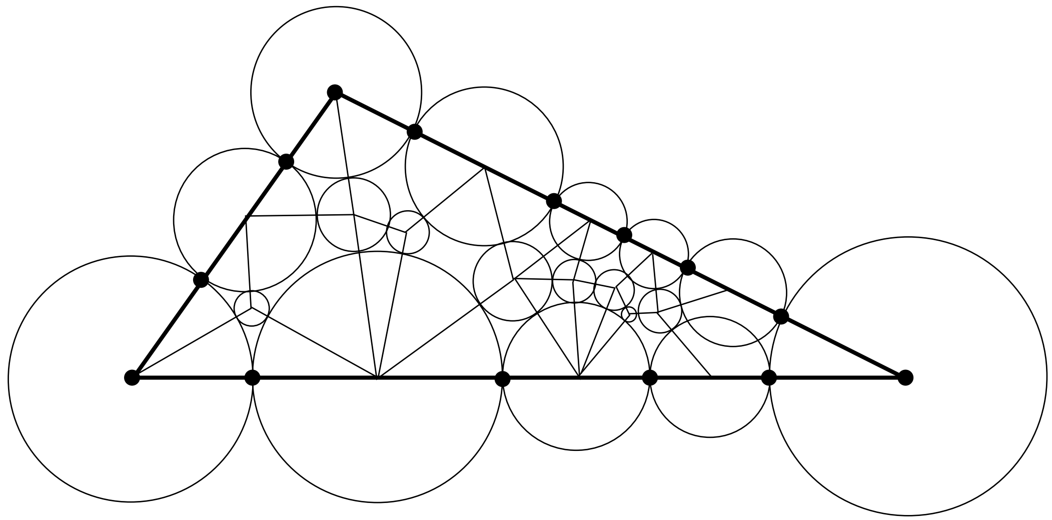
• Pack interior with disjoint disks so only 3-sided and 4-sided regions remain.

Sketch of BMR lemma:



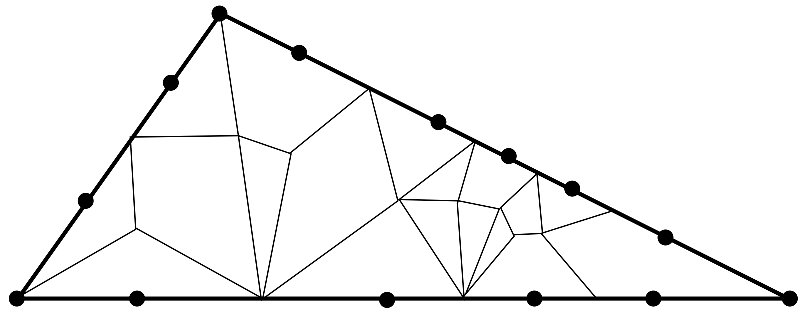
• Pack interior with disjoint disks so only 3-sided and 4-sided regions remain.

Sketch of BMR lemma:



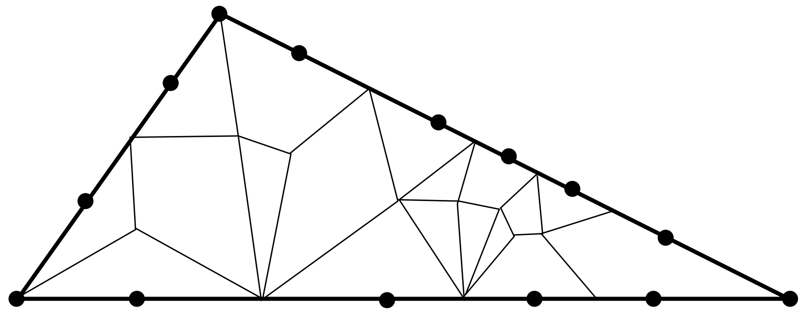
• Pack interior with disjoint disks so only 3-sided and 4-sided regions remain.
• Connect centers.

Sketch of BMR lemma:



- Pack interior with disjoint disks so only 3-sided and 4-sided regions remain.
- Connect centers.
- Divides triangle into triangles and quadrilaterals.

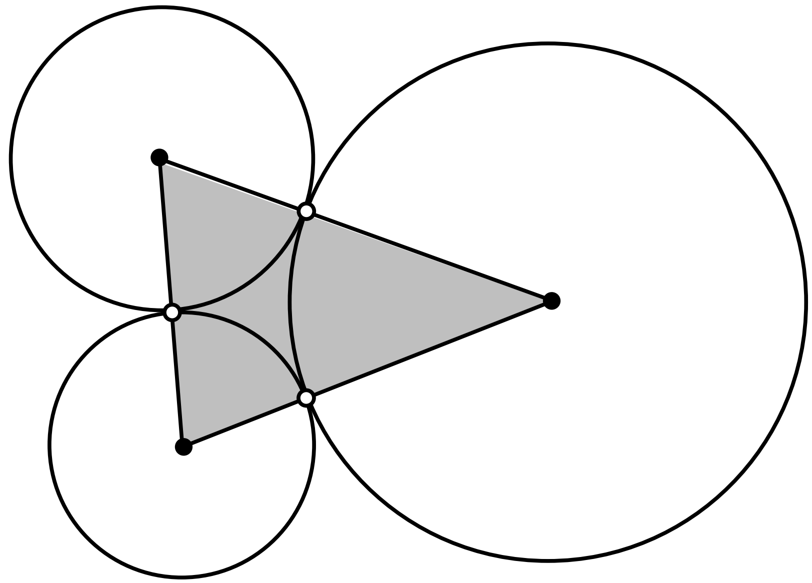
Sketch of BMR lemma:



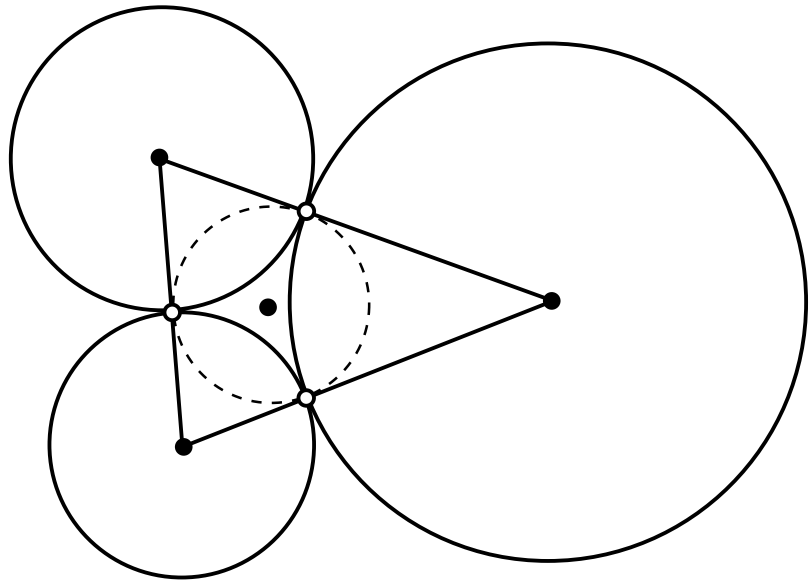
- Pack interior with disjoint disks so only 3-sided and 4-sided regions remain.
- Connect centers.
- Divides triangle into triangles and quadrilaterals.

We want to nonobtusely triangulate each region without adding new vertices along boundary. Several cases.

First case: decompose 3-region into right triangles

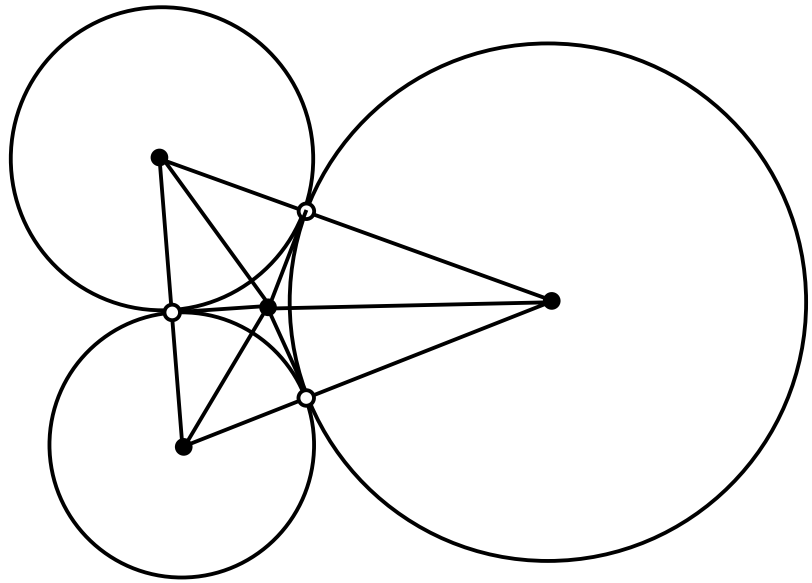


First case: decompose 3-region into right triangles



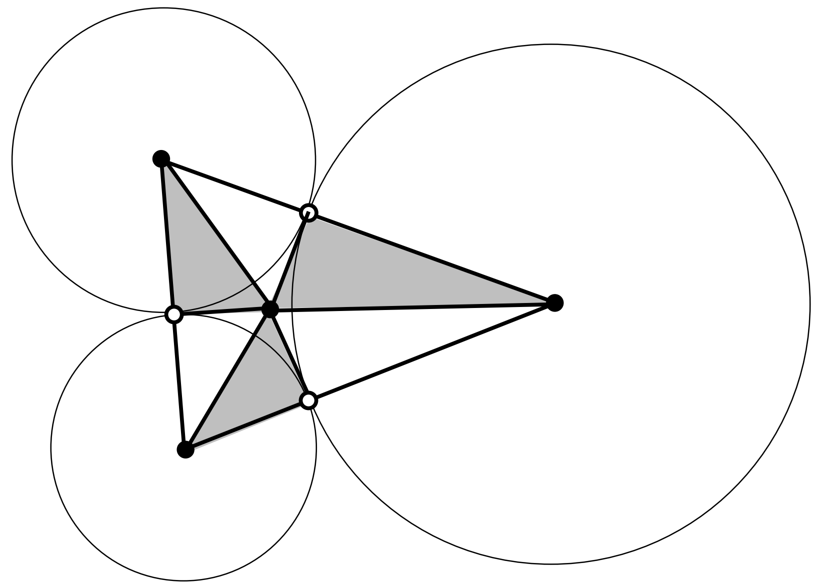
- Add center of circle through the three tangent points.

First case: decompose 3-region into right triangles



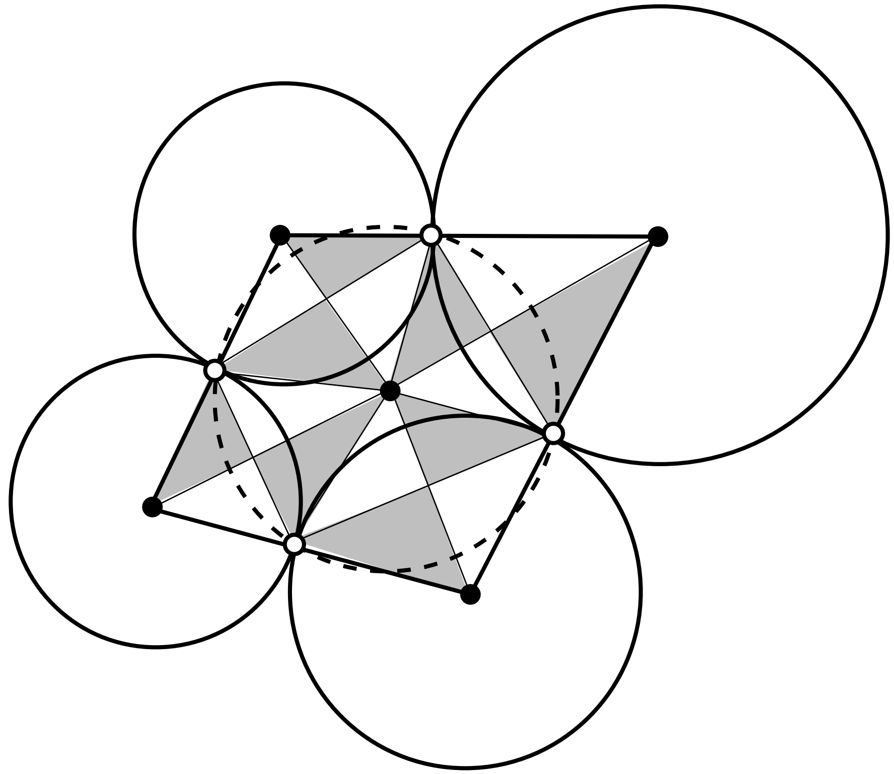
- Add center of circle through the three tangent points.
- Connect center to tangent points and centers of circles

First case: decompose 3-region into right triangles



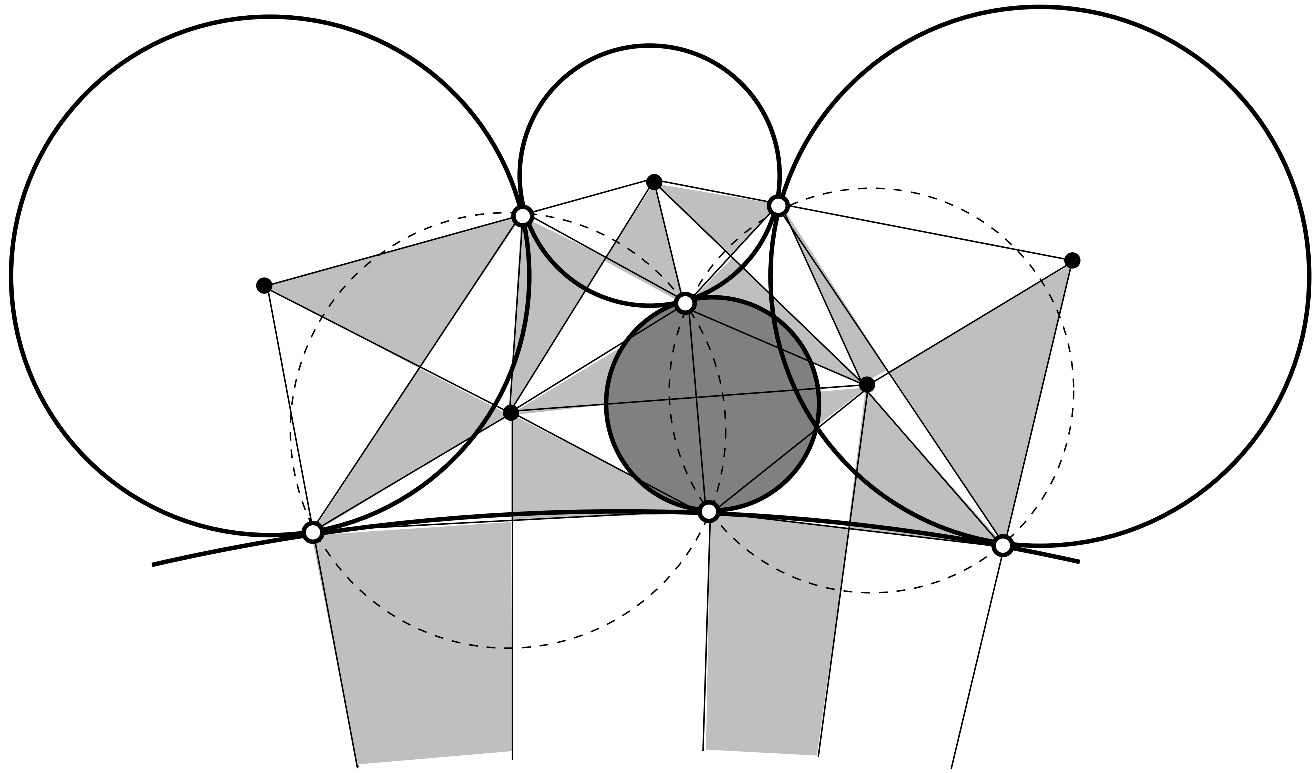
- Add center of circle through the three tangent points.
- Connect center to tangent points and centers of circles

The 4-regions are similar (but several cases arise).



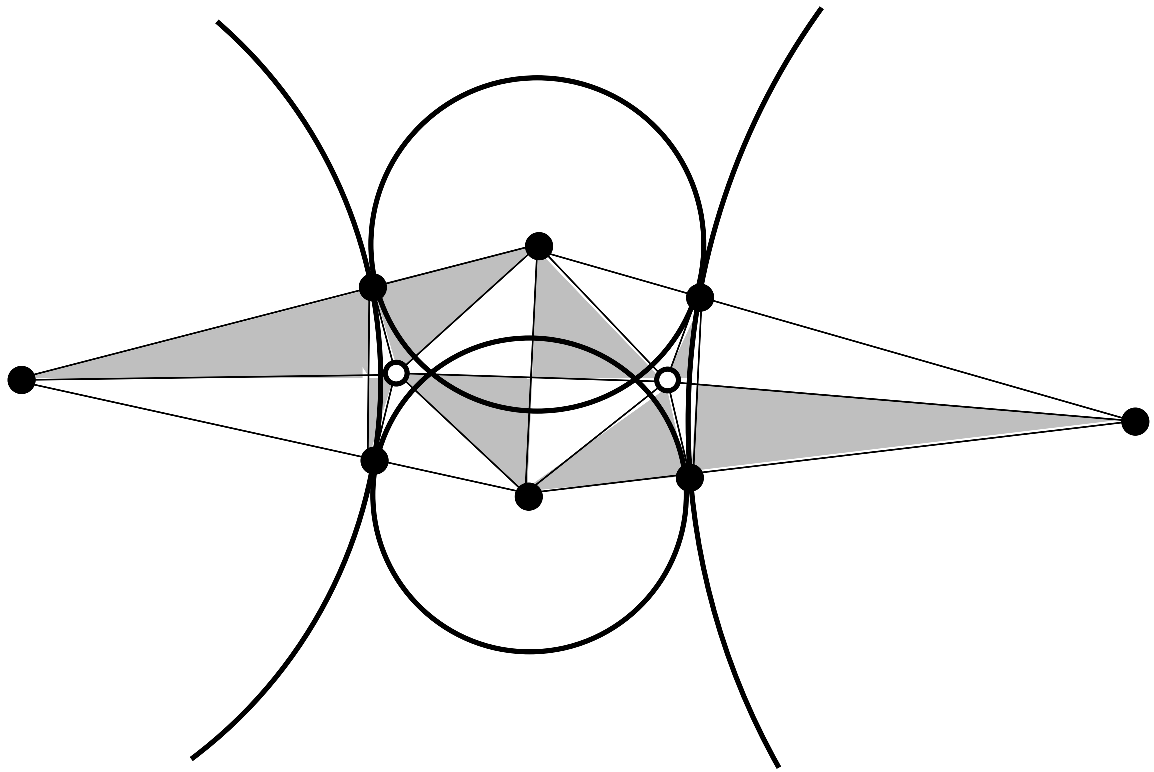
So Gabriel covering gives a nonobtuse triangulation.

The 4-regions are similar (but several cases arise).



So Gabriel covering implies a nonobtuse triangulation.

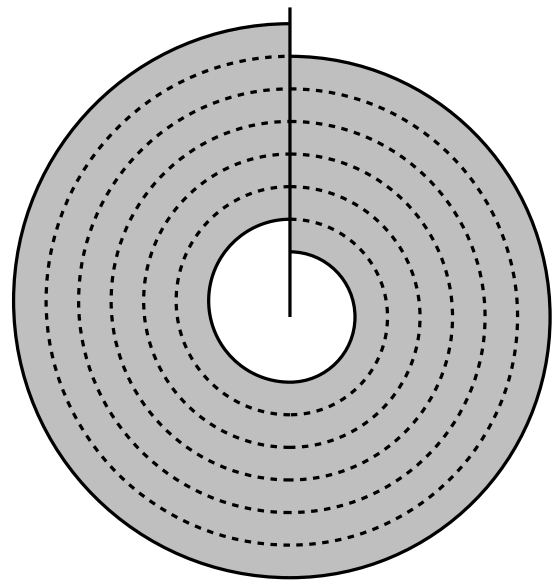
The 4-regions are similar (but several cases arise).



So Gabriel covering implies a nonobtuse triangulation.

APPENDIX D
PROOF FOR THE SPIRAL CASE IN THE NOT THEOREM

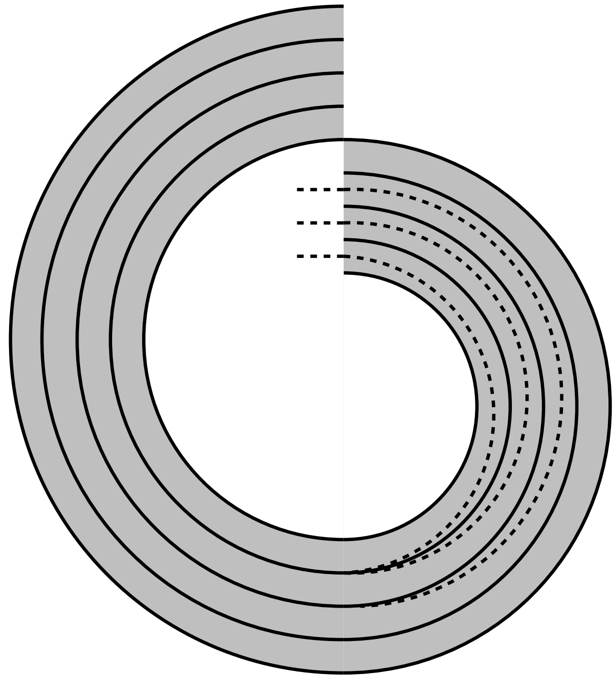
Hard case is spirals:



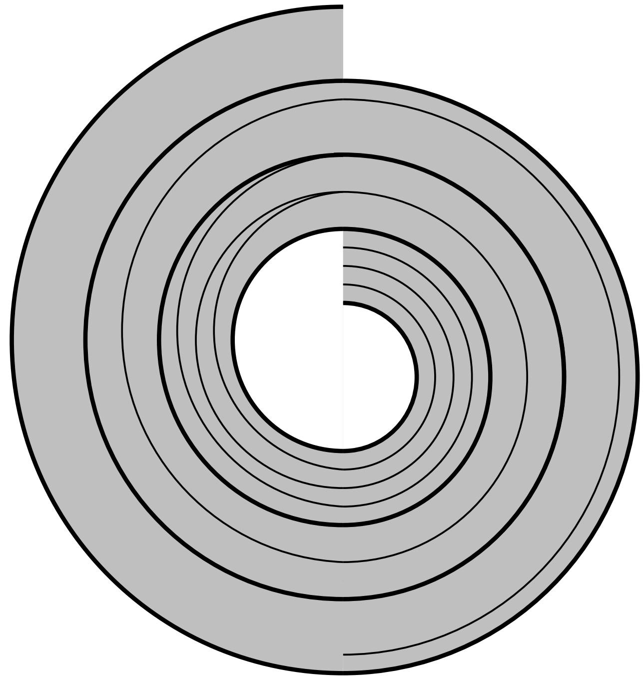
Curves may spiral arbitrarily often.

No curve can be allowed to pass all the way through the spiral. We stop them in a multi-stage construction.

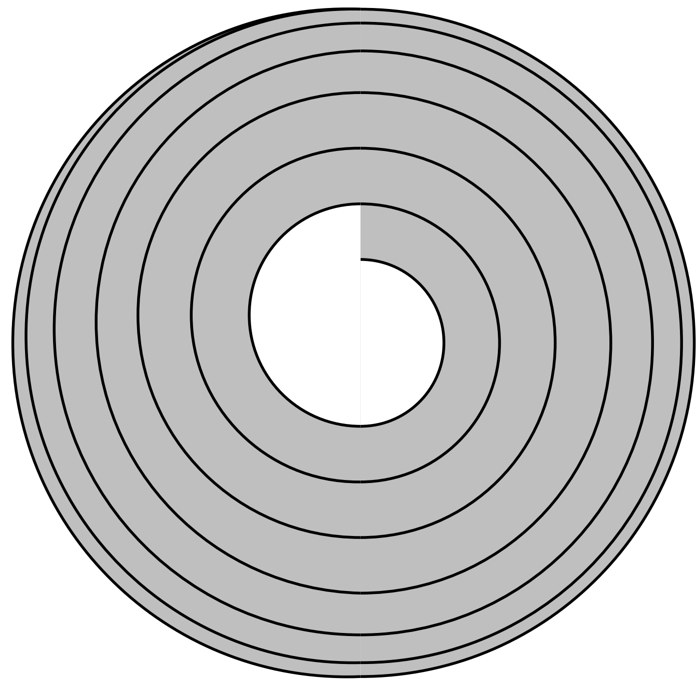
Normalize so “entrance” is unit width.



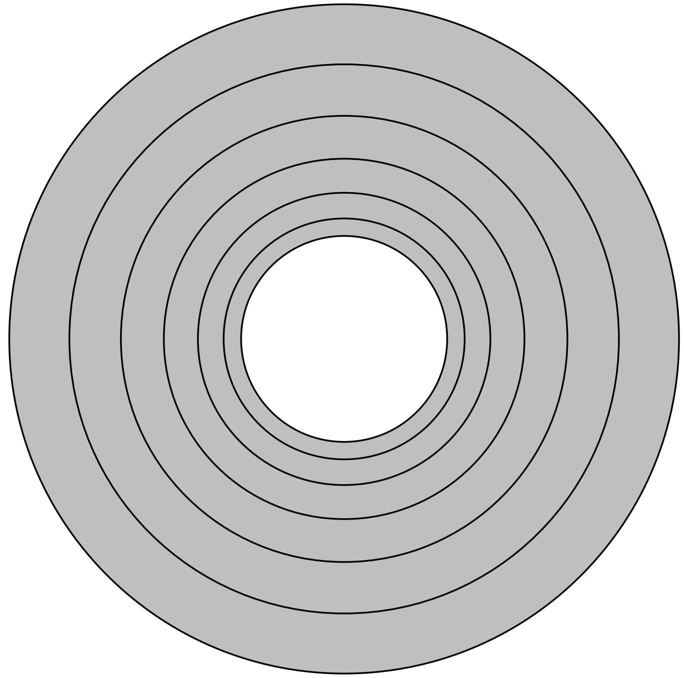
- Start with $\sqrt{n}$ parallel tubes at entrance of spiral. Terminate entering paths (1 spiral).



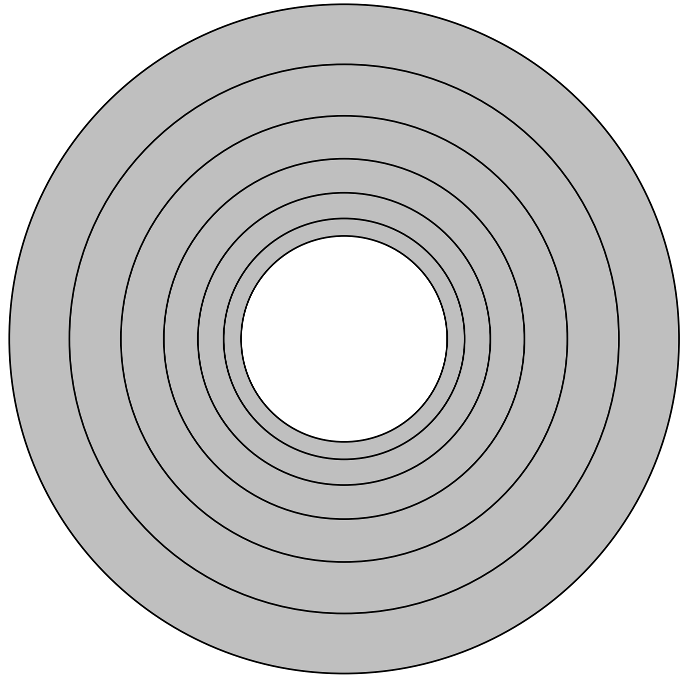
- Start with $\sqrt{n}$ parallel tubes at entrance of spiral. Terminate entering paths (1 spiral).
- Merge $\sqrt{n}$ tubes to single tube ($n^{1/3}$ spirals). (spirals get longer as we move out.)



- Start with $\sqrt{n}$ parallel tubes at entrance of spiral. Terminate entering paths (1 spiral).
- Merge $\sqrt{n}$ tubes to single tube ($n^{1/3}$ spirals). (spirals get longer as we move out.)
- Make tube edge self-intersect ($n^{1/2}$ spirals)



- Start with $\sqrt{n}$ parallel tubes at entrance of spiral. Terminate entering paths (1 spiral).
- Merge $\sqrt{n}$ tubes to single tube ($n^{1/3}$ spirals). (spirals get longer as we move out.)
- Make tube edge self-intersect ($n^{1/2}$ spirals)
- Loops with increasing gaps ($n^{1/2}$ loops, to radius n)



- Start with $\sqrt{n}$ parallel tubes at entrance of spiral. Terminate entering paths (1 spiral).
- Merge $\sqrt{n}$ tubes to single tube ($n^{1/3}$ spirals). (spirals get longer as we move out.)
- Make tube edge self-intersect ($n^{1/2}$ spirals)
- Loops with increasing gaps ($n^{1/2}$ loops, to radius n)
- Beyond radius n spiral is empty. Gives $O(n^{2.5})$.