True Trees

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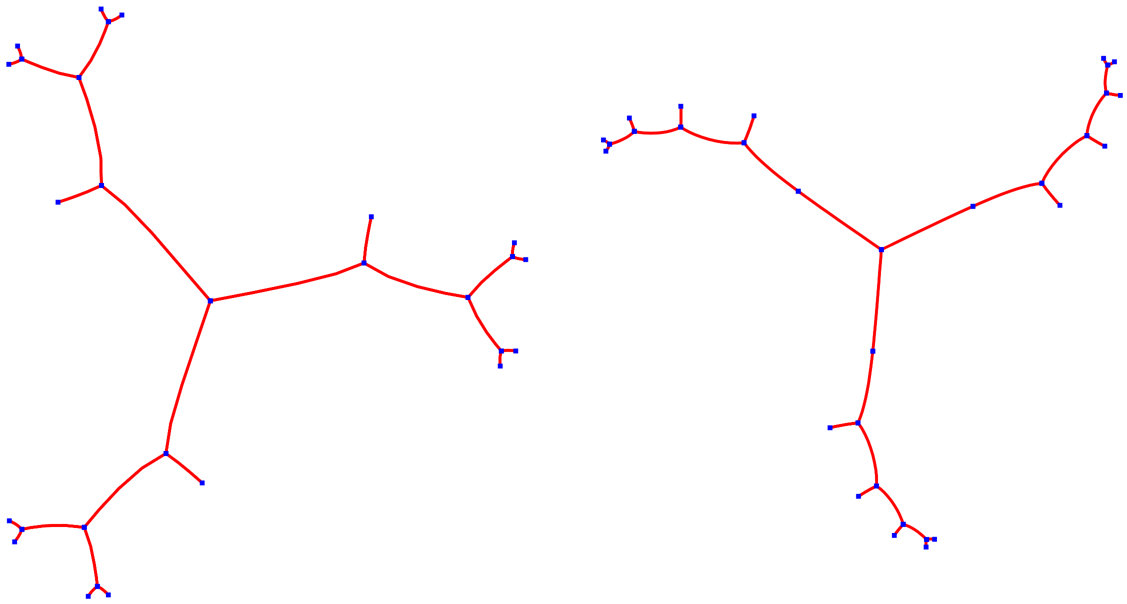
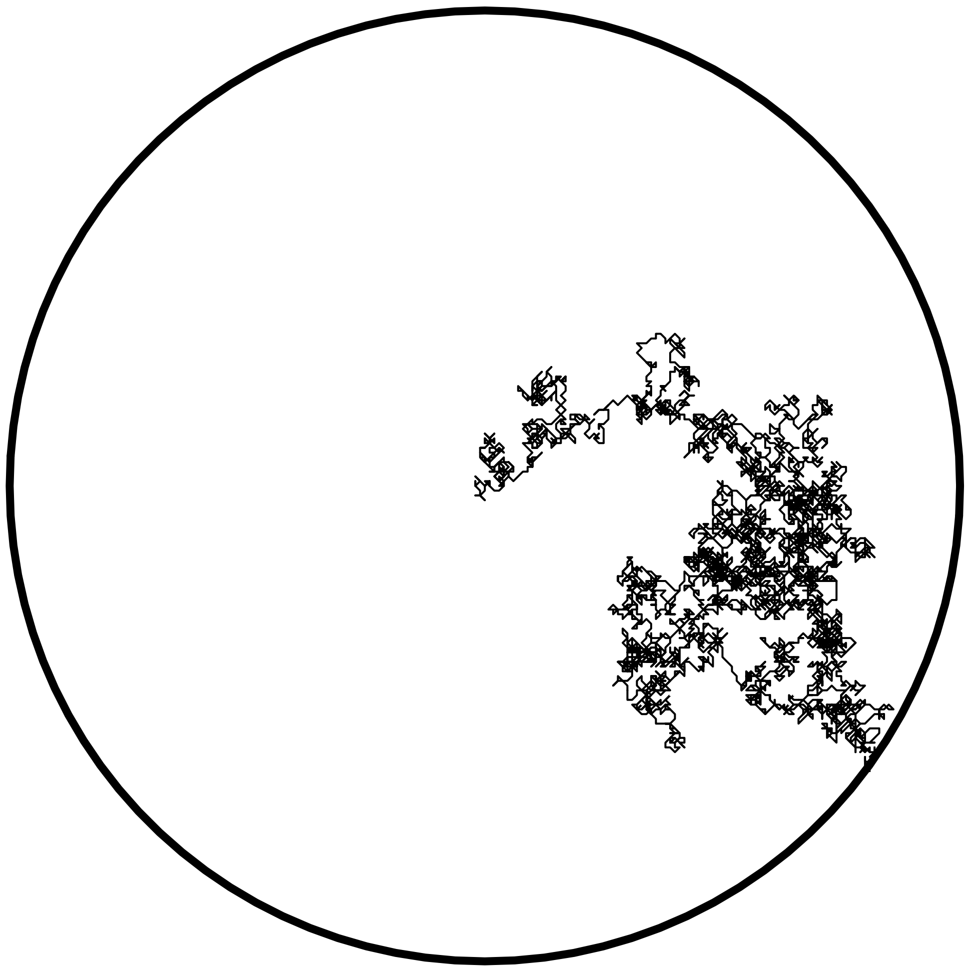
Geometry Seminar, Courant Institute, NYU, March 29, 2016



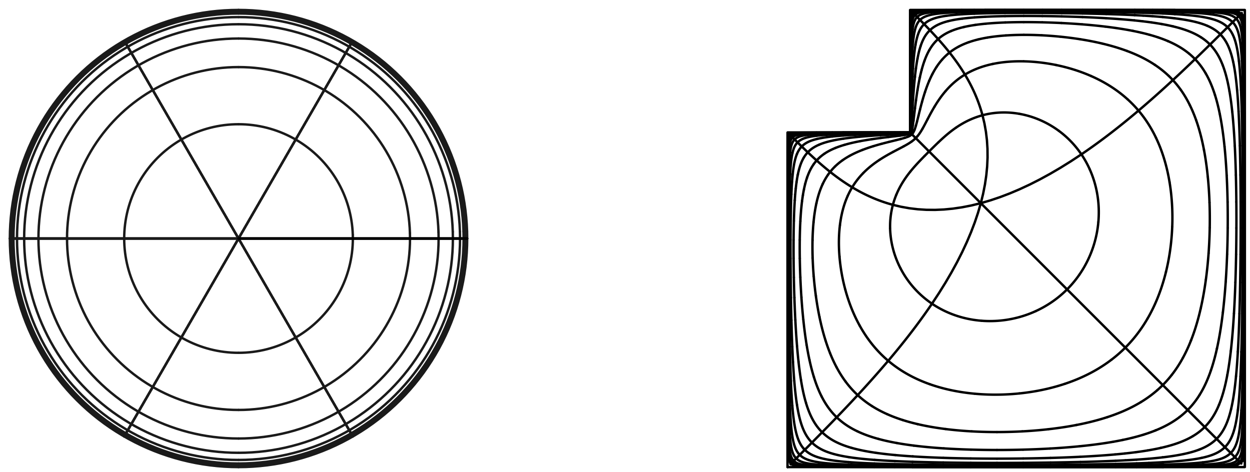
lecture slides available at

`www.math.sunysb.edu/~bishop/lectures`

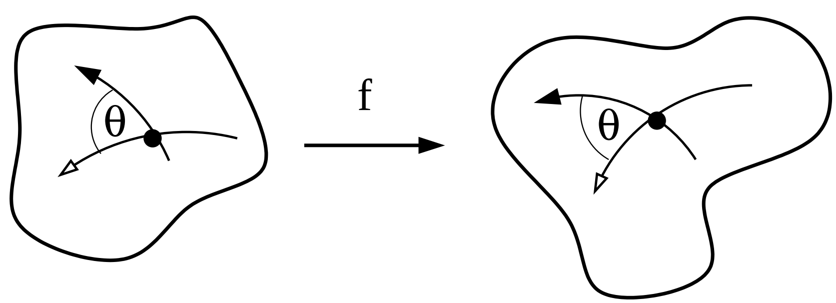


Harmonic measure = exit distribution of Brownian motion = $\omega(z, E, \Omega)$

Riemann Mapping Theorem: If $\Omega \subsetneq \mathbb{R}^2$ is simply connected, then there is a conformal map $f : \mathbb{D} \to \Omega$.



Conformal = angle preserving

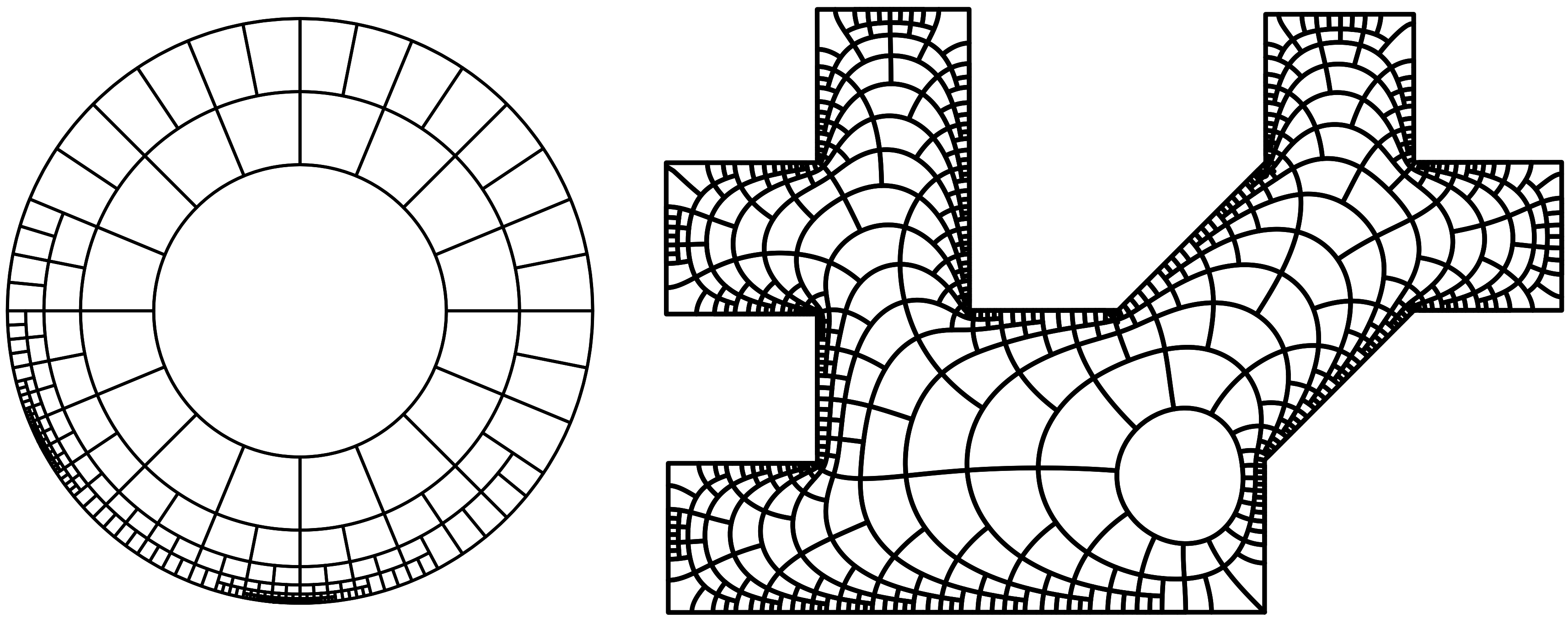


Brownian motion is conformally invariant.

If $\varphi : U \to V$ is conformal, then Brownian motion is in U is mapped to Brownian motion in V (with a time change).

Harmonic measure on U is mapped to harmonic measure on V.

By conformal invariance, harmonic measure equals image of length on unit circle under Riemann map. Powerful tool in 2 dimensions.



By symmetry $\omega_0 = \omega_\infty$ for circle.

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If $\omega_0 = \omega_\infty$ must Γ be a circle?

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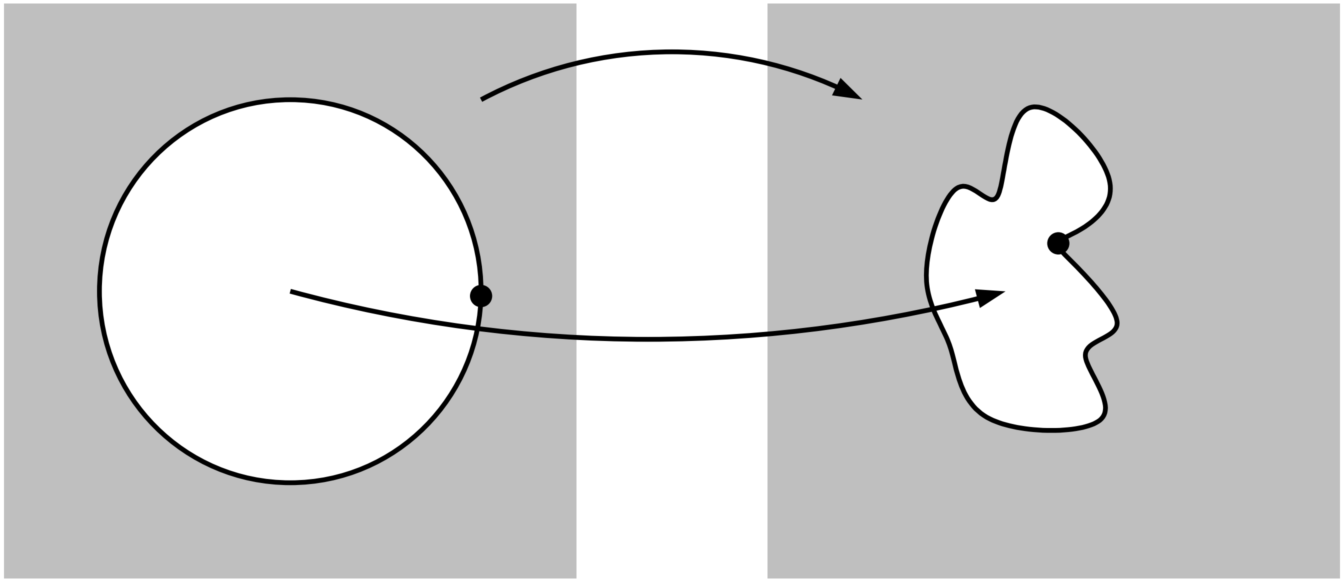
If $\omega_0 = \omega_\infty$ must Γ be a circle? **Yes**

Proof uses some complex analysis facts:

Entire function = holomorphic on whole plane

Morera's theorem: if f is continuous on whole plane and holomorphic off a line, it is entire.

Corollary of Picard's theorem: a finite-to-1 entire function is a polynomial.



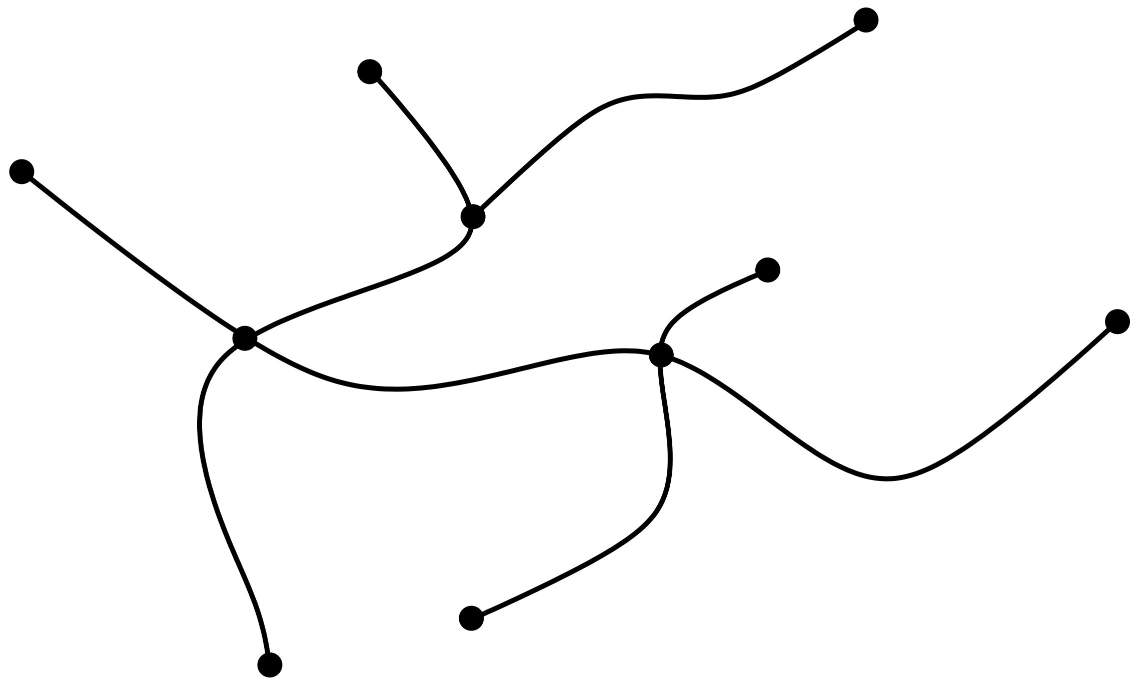
Conformally map inside to inside, outside to outside.

$\omega_0 = \omega_\infty$ means maps agree on boundary.

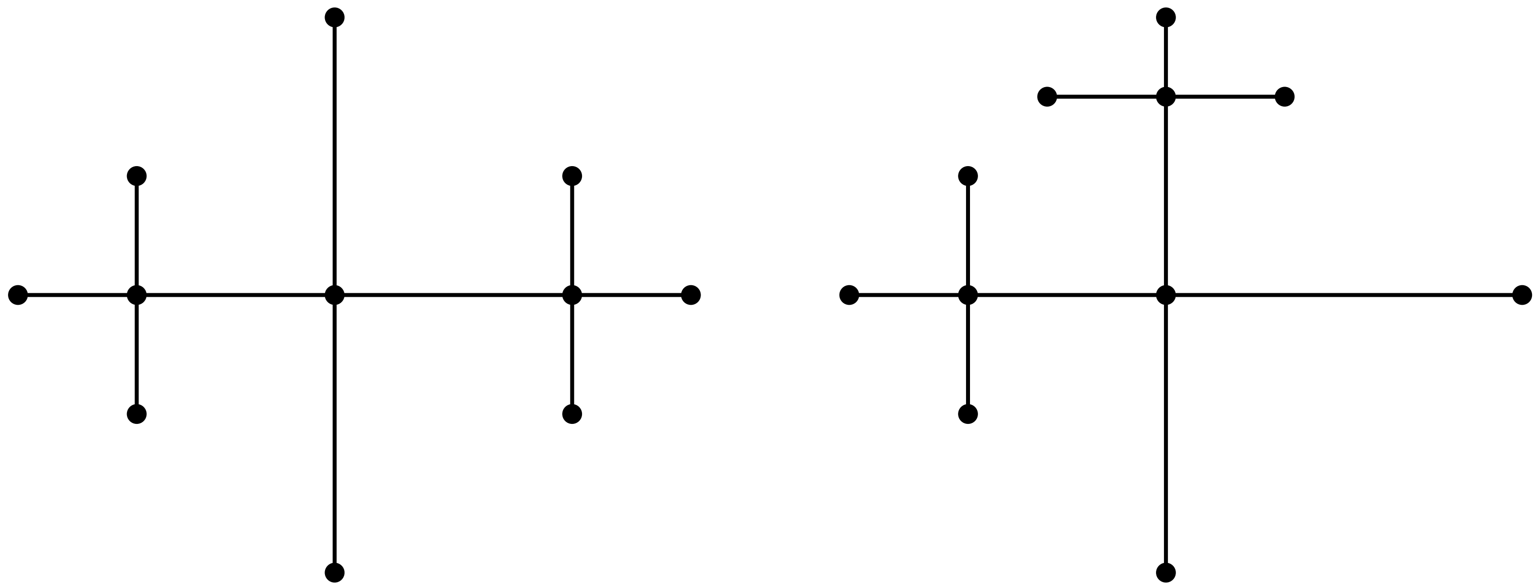
Get homeomorphism of plane holomorphic off circle.

Is entire by Morera's theorem. Plus 1-1 implies linear.

What happens if we replace closed curve by a tree?



Can we make harmonic measure ω_∞ same on "both sides"?



Same tree, different planar trees

A planar tree is **conformally balanced** if

- every edge has equal harmonic measure from ∞
- edge subsets have same measure from both sides

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A line segment is an example.

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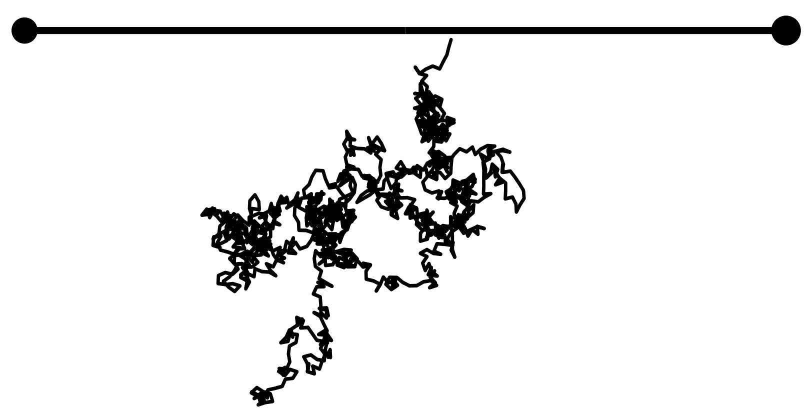
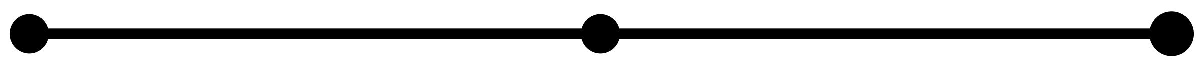
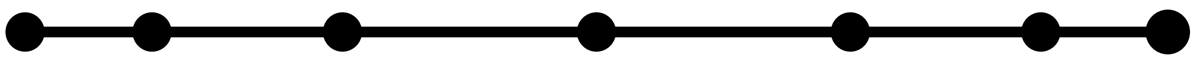
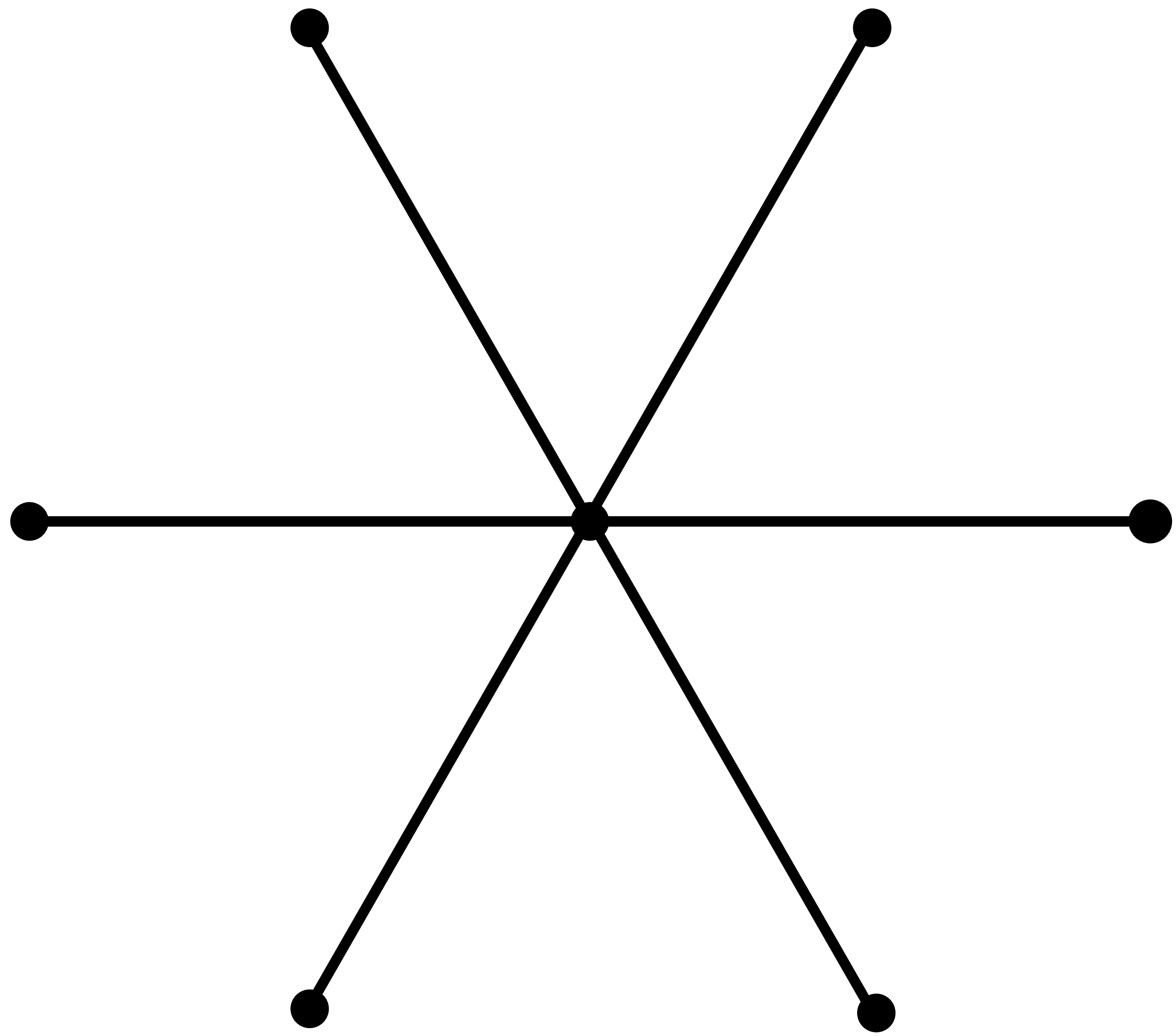
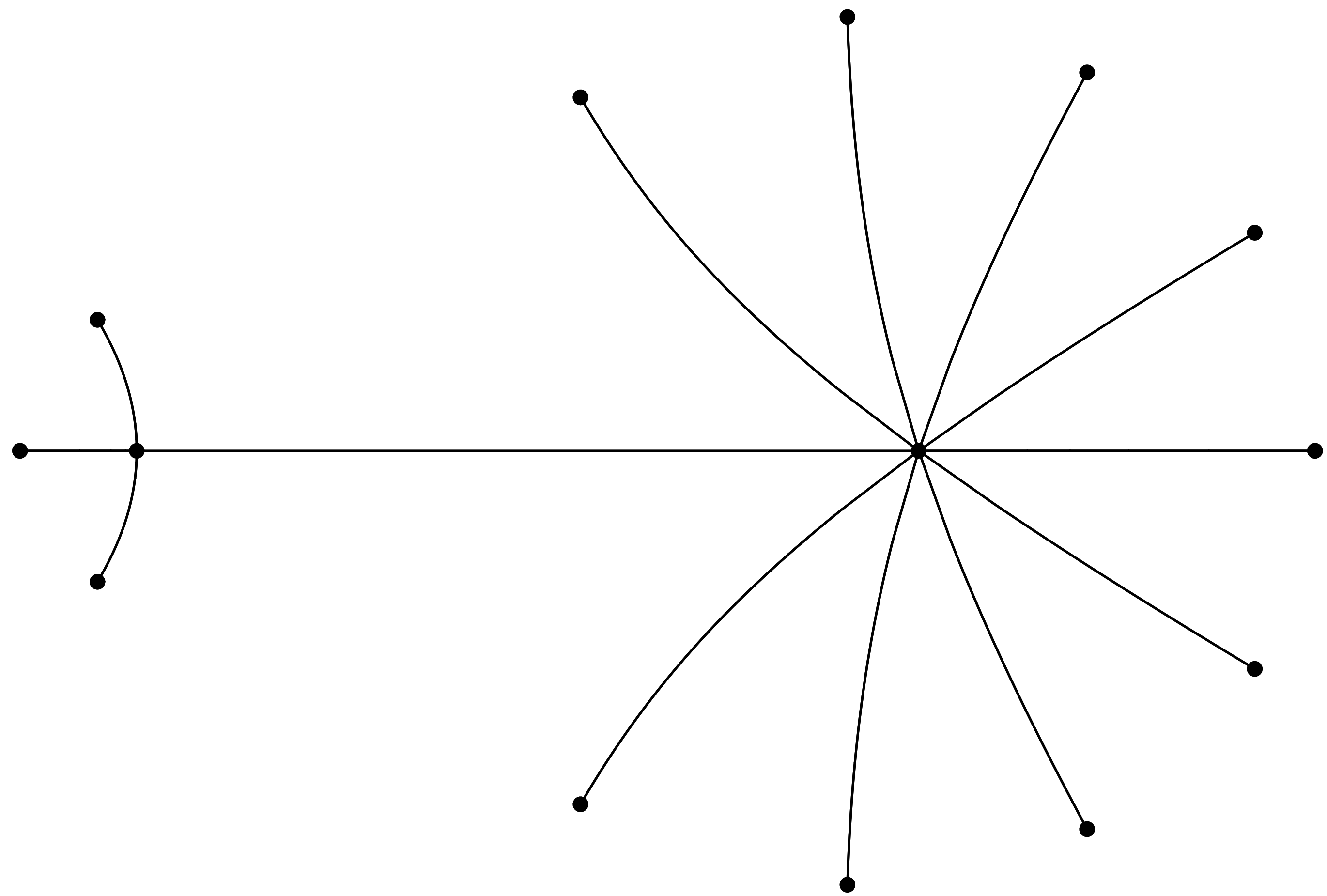
A line segment is an example. Are there others?

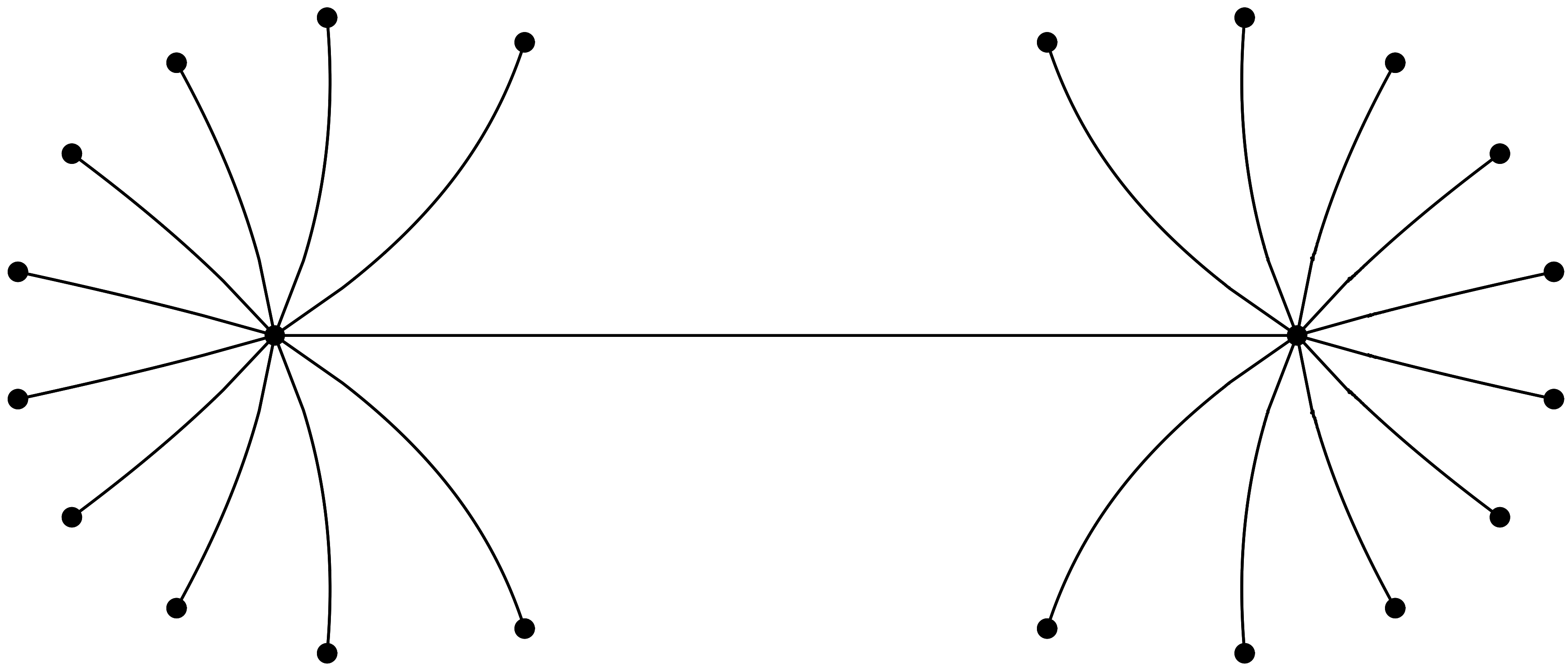












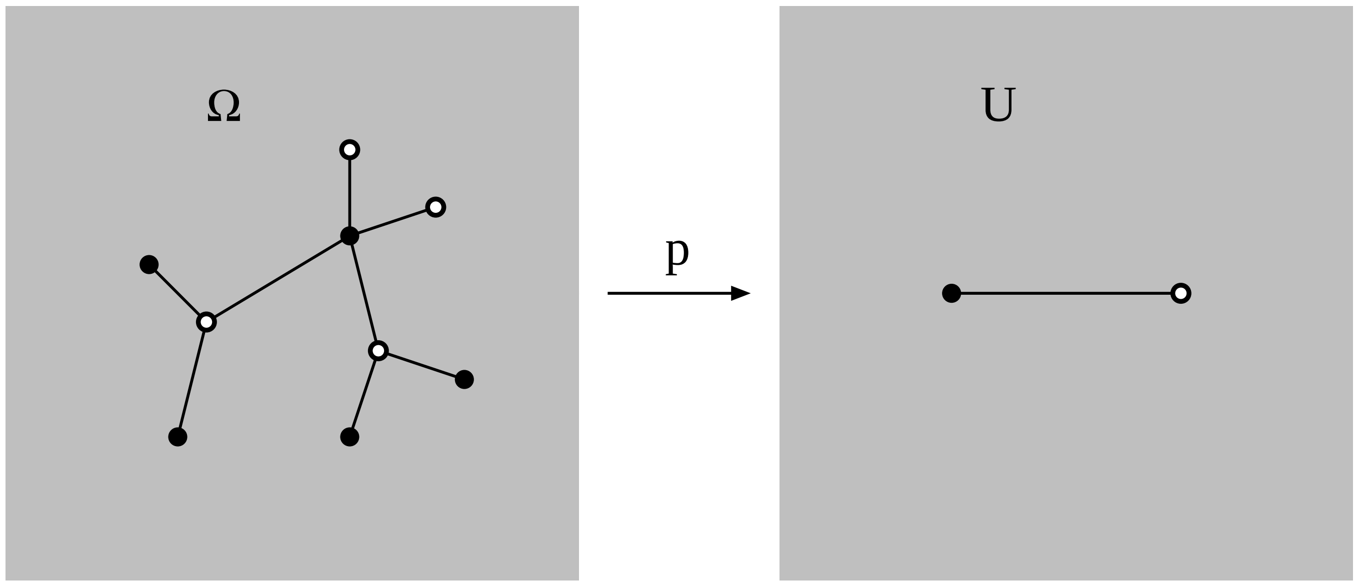
$p =$ polynomial

$\mathrm{CV}(p) = \{p(z) : p'(z) = 0\} =$ critical values

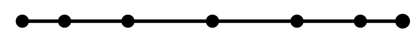
If $\mathrm{CV}(p) = \pm 1$, p is called **generalized Chebyshev** or **Shabat**.

Balanced trees $\leftrightarrow$ Shabat polynomials

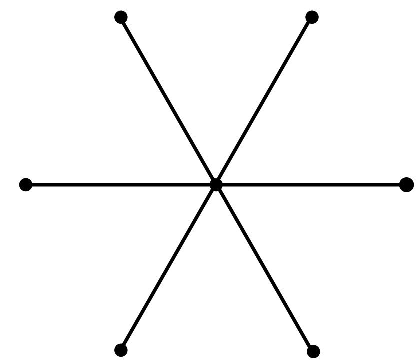
Fact: T is balanced iff $T = p^{-1}([-1,1])$, $p =$ Shabat.



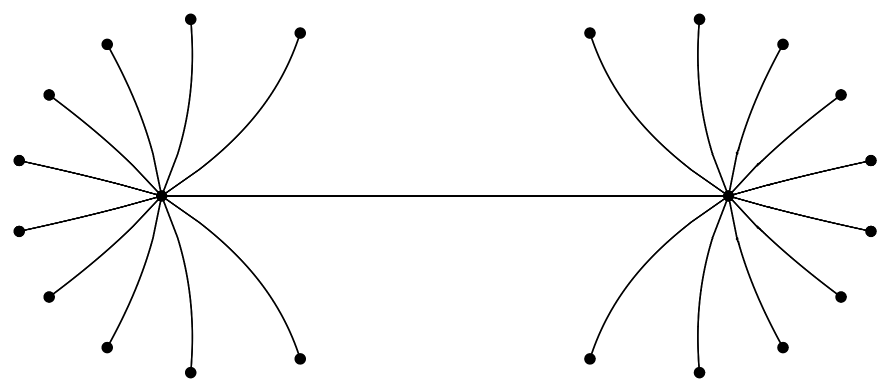
$\Omega = \mathbb{C} \setminus T$ $\qquad$ $U = \mathbb{C} \setminus [-1,1]$



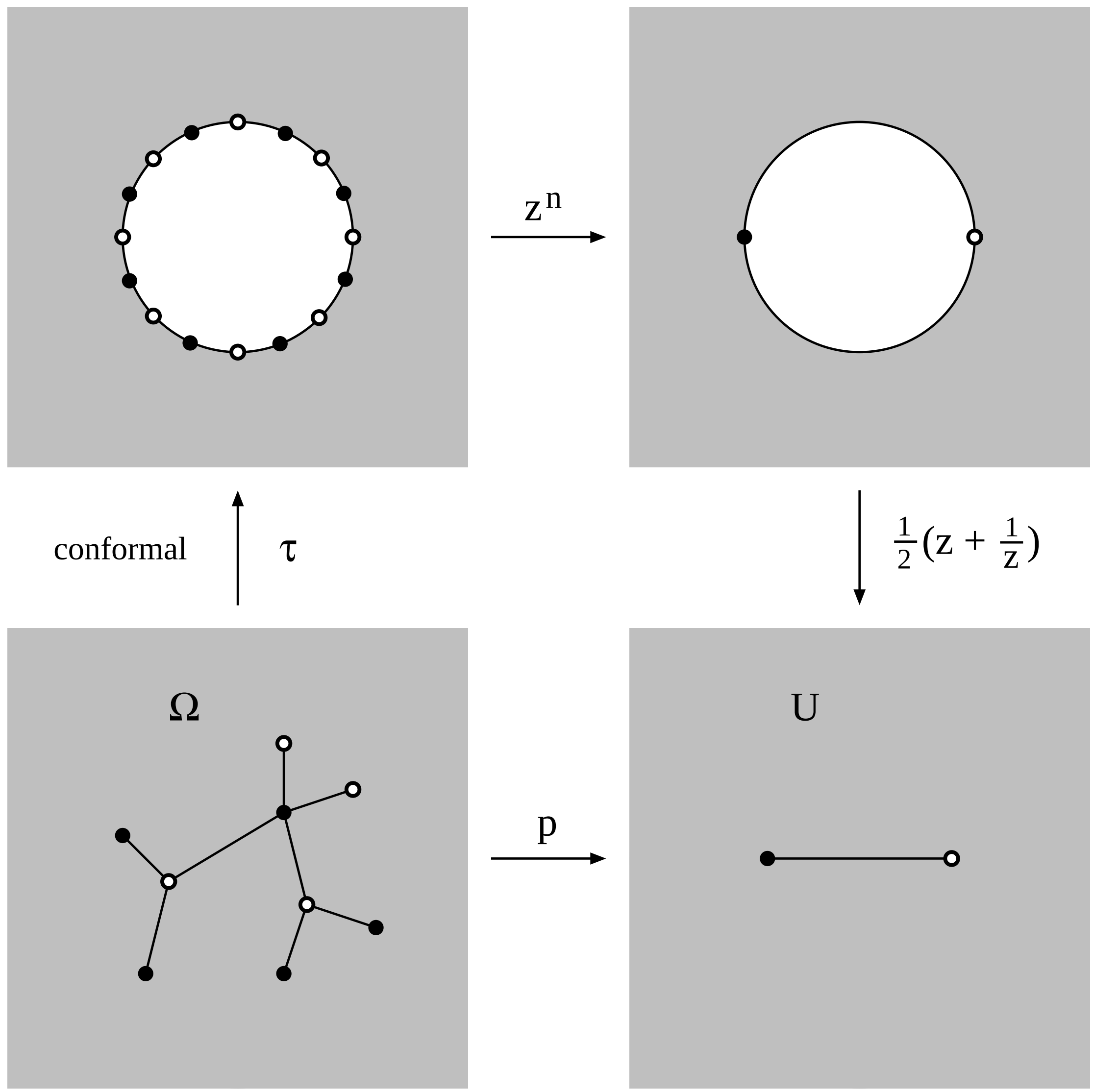
p(z) = 1st type Chebyshev



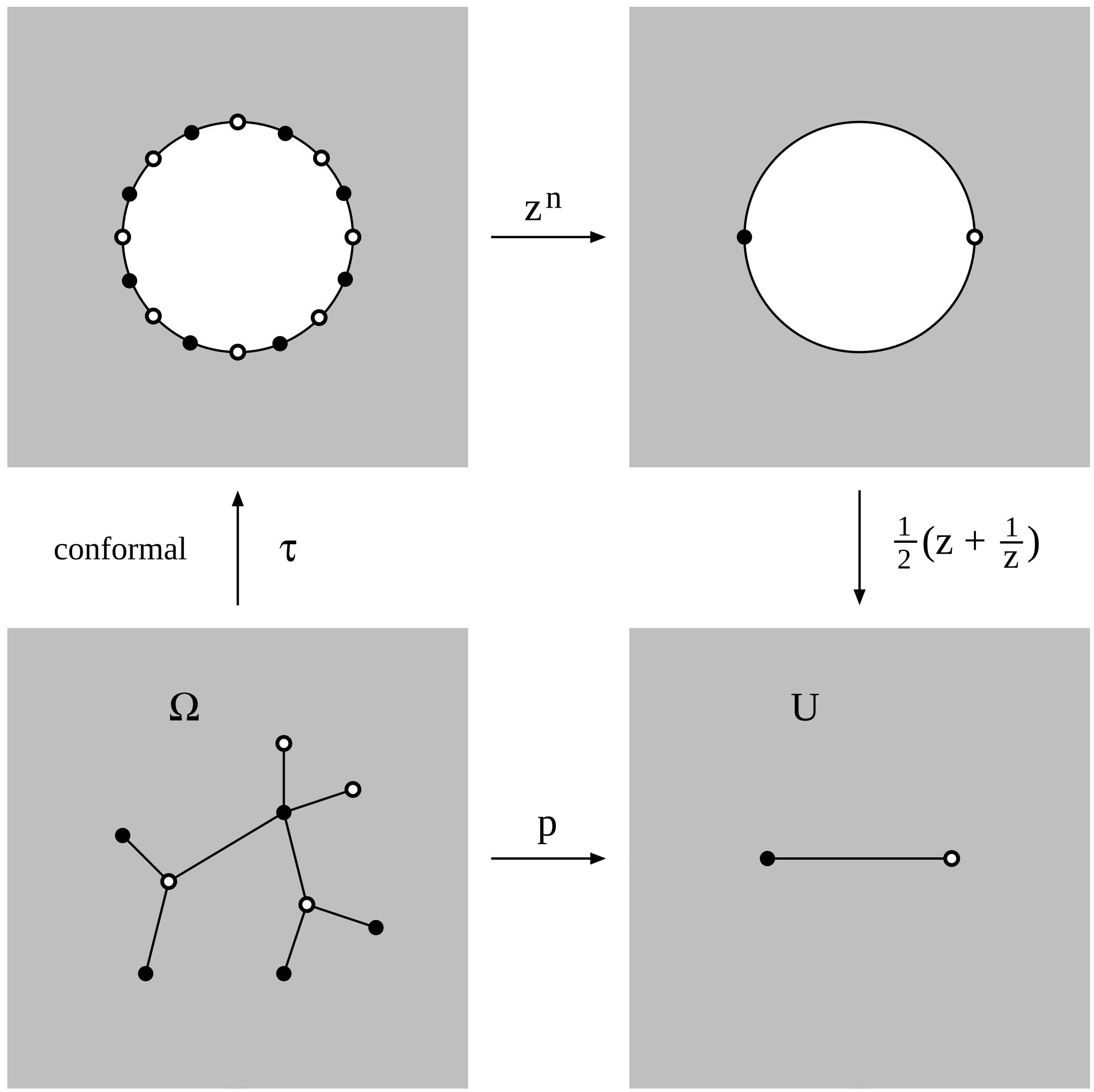
$p(z) = 2z^n - 1$



$p'(z) = c(z+1)^a(z-1)^b$



$$\tau = \text{conformal } \Omega \to \mathbb{D}^* = \{|z| > 1\}.$$



p is entire and n-to-1. $\Rightarrow p =$ polynomial

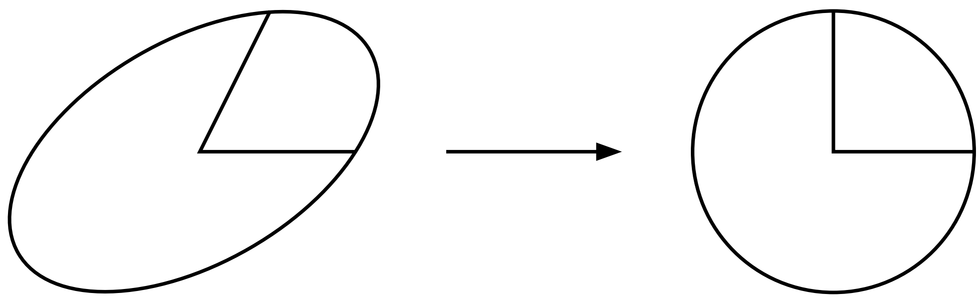
What if the tree is not balanced?

What if the tree is not balanced?

Replace conformal map by quasiconformal map.

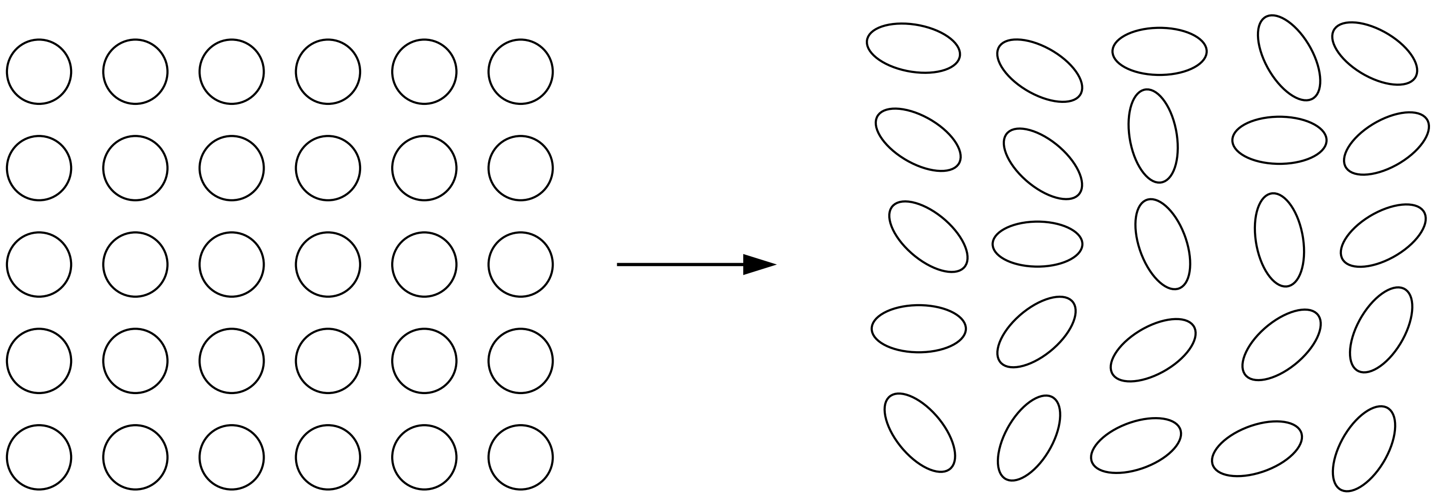
dilatation $= \mu_f = f_{\overline{z}}/f_z =$ measure of non-conformality

$K = (\mu+1)/(\mu-1) =$ ratio of singular values of tangent map



quasiconformal = homeomorphism with bounded angle distortion.

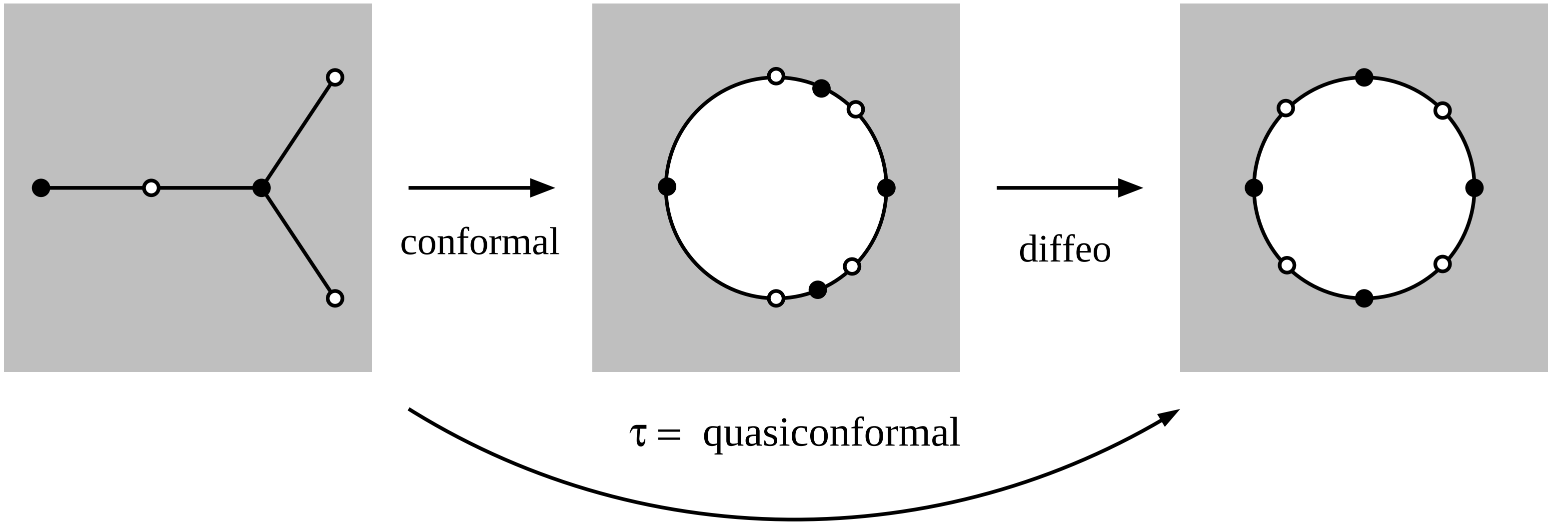
= homeomorphism with $\sup|\mu_f| \leq k < 1$.



quasiregular = function with $\sup |\mu_f| \leq k < 1$.

Theorem: If f is quasi-regular and finite-to-one then $f = g \circ \phi$ where g is polynomial and ϕ is a QC homeomorphism.

This is corollary of **Measurable Riemann Mapping Theorem**.

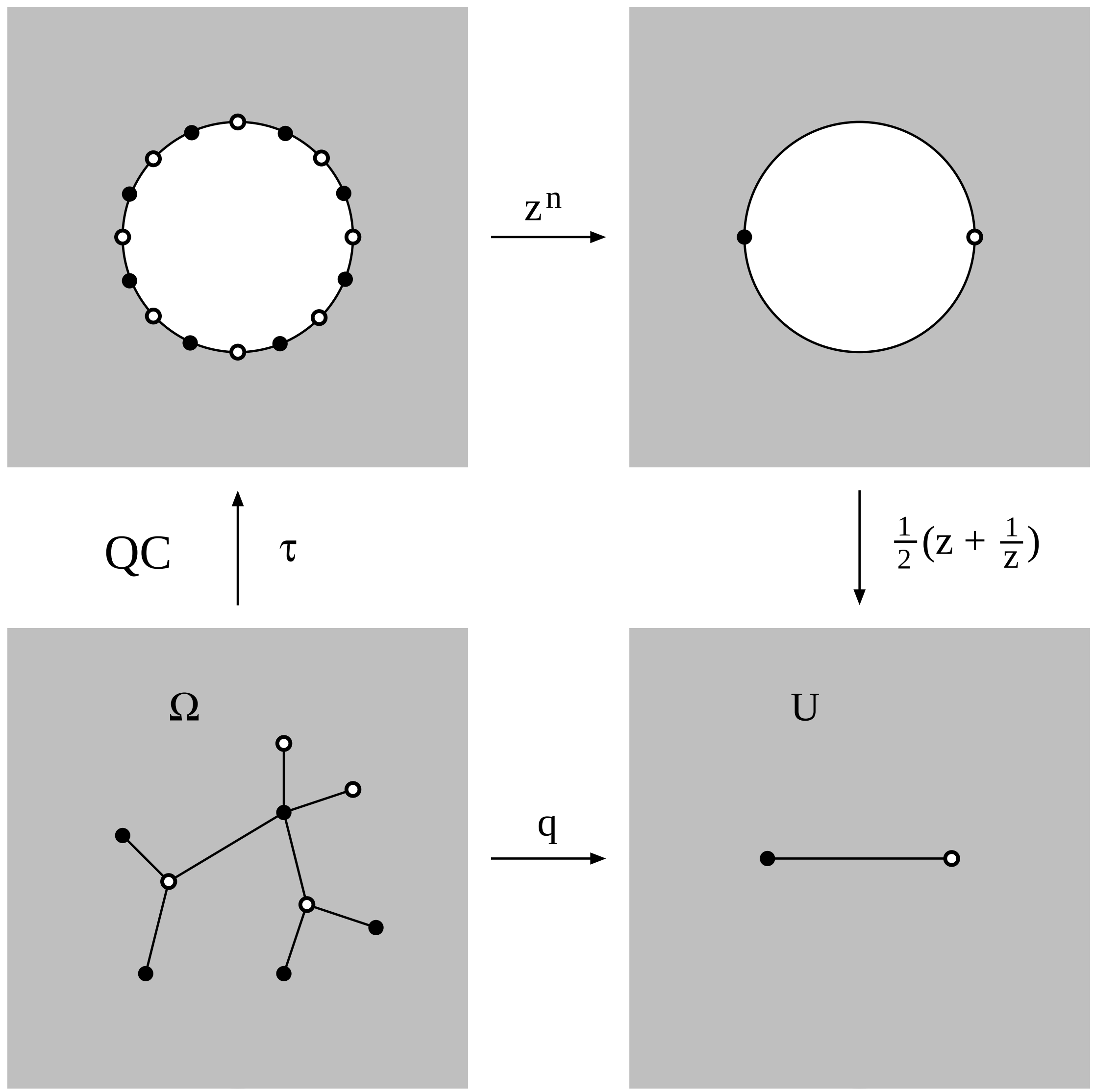


Map $\Omega \to \{|z| > 1\}$ conformally, then "even out" by diffeomorphism.

If tree is "nice" (smooth edges, equal angles at each vertex), then

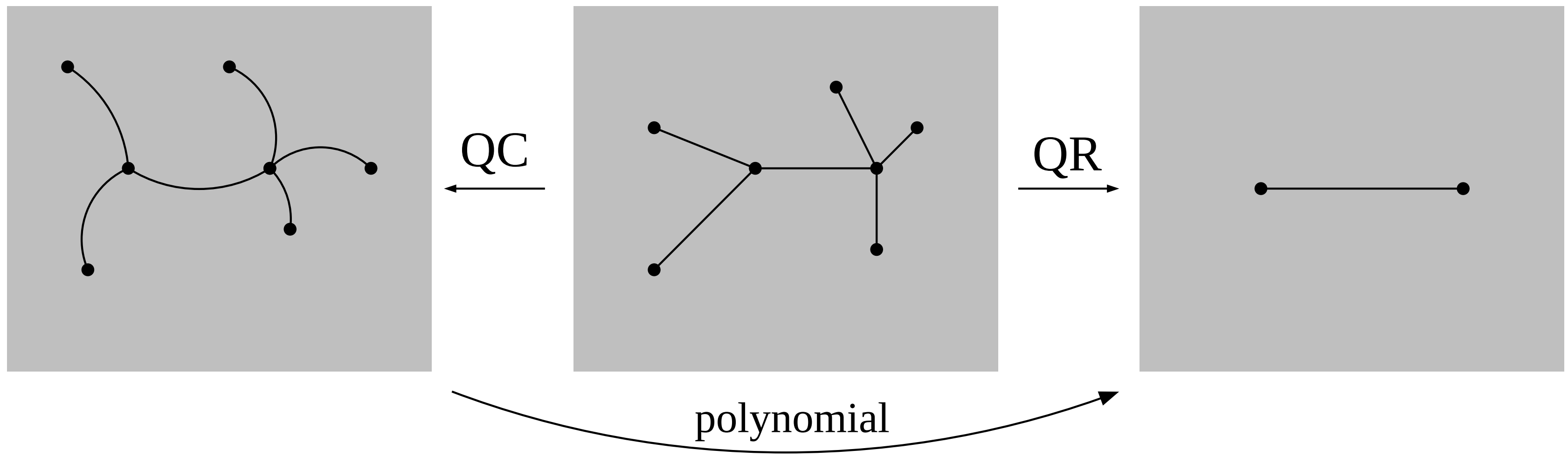
- composition is QC
- any subset of an edge has two images of equal length

QC constant depends on tree. Is finite since tree is finite.



τ is quasiconformal on Ω. p is quasi-regular on plane.

Factor the QR map into a QC map and polynomial:



Cor: Every planar tree has a true form.

In Grothendieck's theory of *dessins d'enfants*, a finite graph of a topological surface determines a conformal structure and a Belyi function (a meromorphic function to sphere branched over 3 values).

Shabat polynomial is special case of Belyi function.
(branch points $= -1, 1, \infty$).

Polynomial has coefficients in a number field (finite extension of $\mathbb{Q}$). Universal Galois group acts on these polynomials, hence has an action on finite trees. Determining orbits is major open problem.

"conformally balanced tree" = "true form of a tree".

Is the polynomial computable from the tree?

Kochetkov, *Planar trees with nine edges: a catalogue*, 2007.
"The complete study of trees with 10 edges is a difficult work, and probably no one will do it in the foreseeable future".

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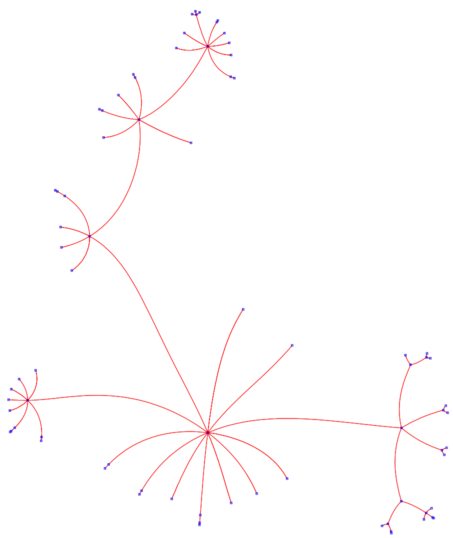
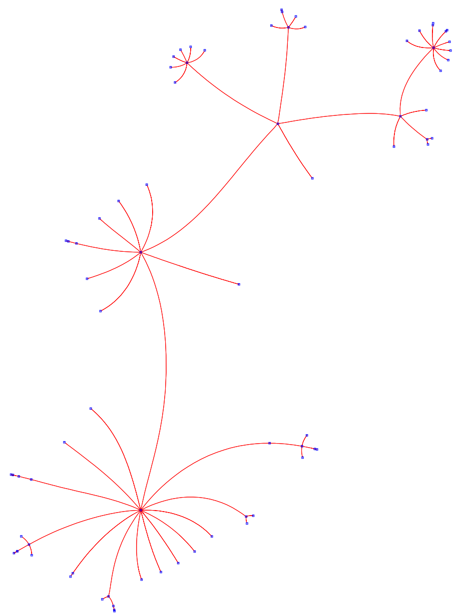
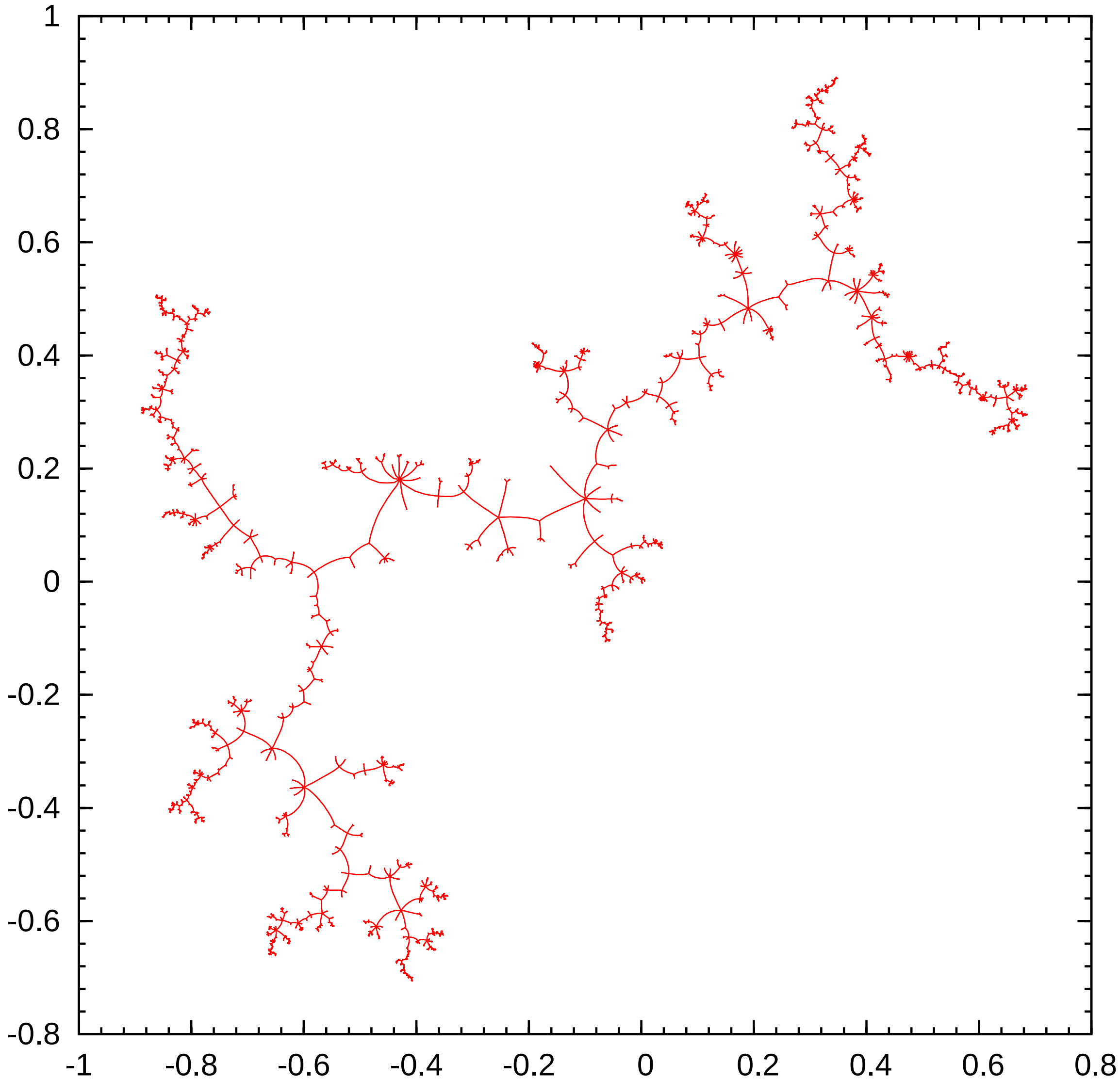
Kochetkov gave "short catalog" of 10-edge trees in 2014.

Marshall and Rohde approximated all 95,640 true trees with 14 edges.

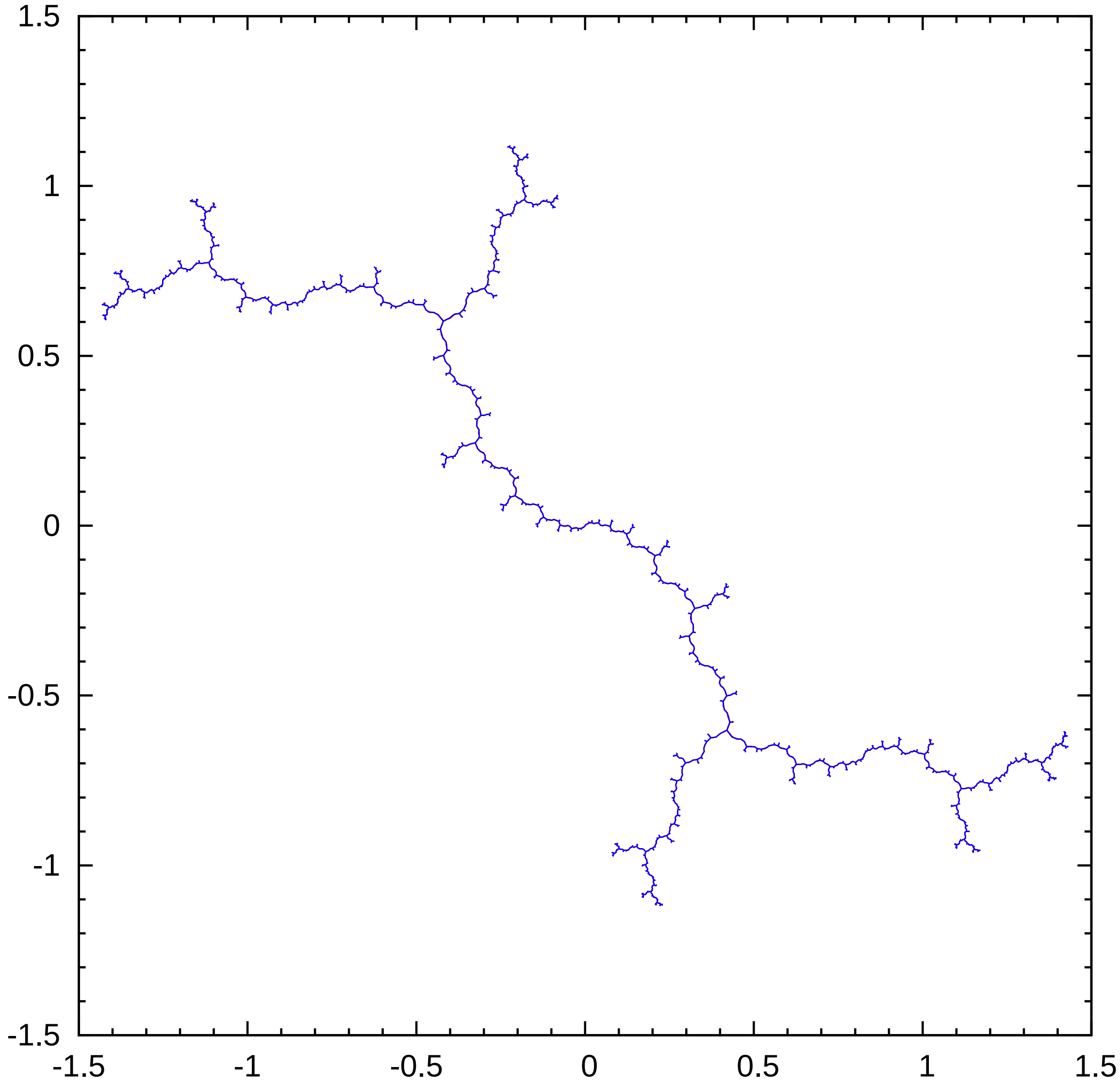




They can get 1000's of digits of accuracy. Can such approximations and lattice reduction (e.g., PSLQ) give the exact algebraic coefficients?



Random true tree with 10,000 edges



Tree with dynamical combinatorics

Every planar tree has a true form.

In other words, all possible **combinatorics** occur.

What about all possible **shapes**?

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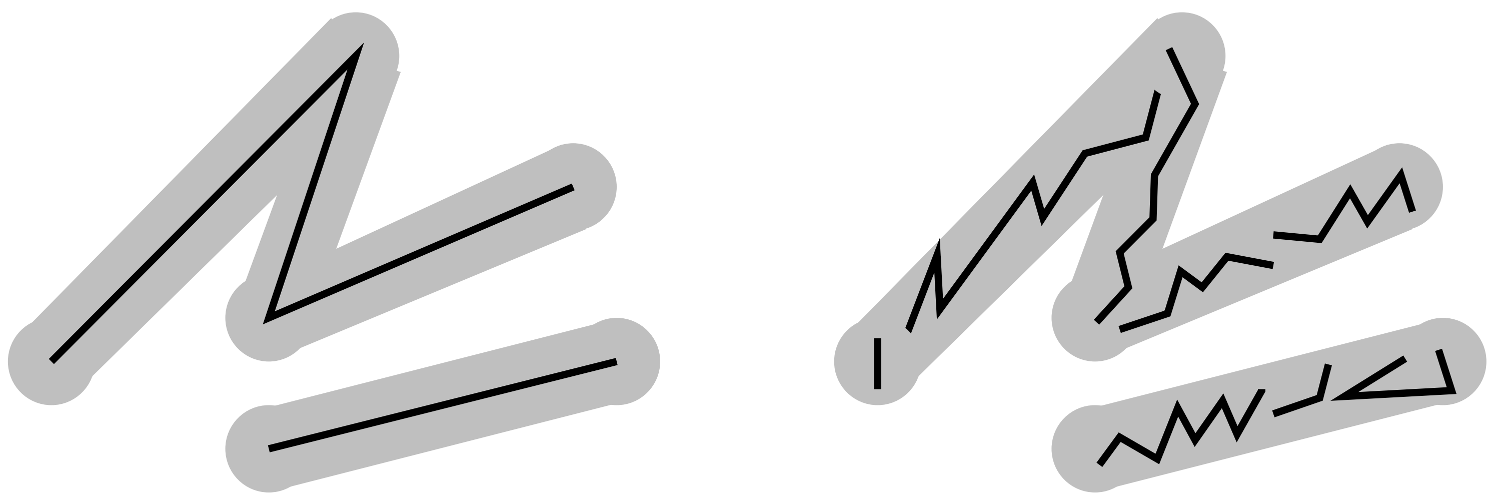
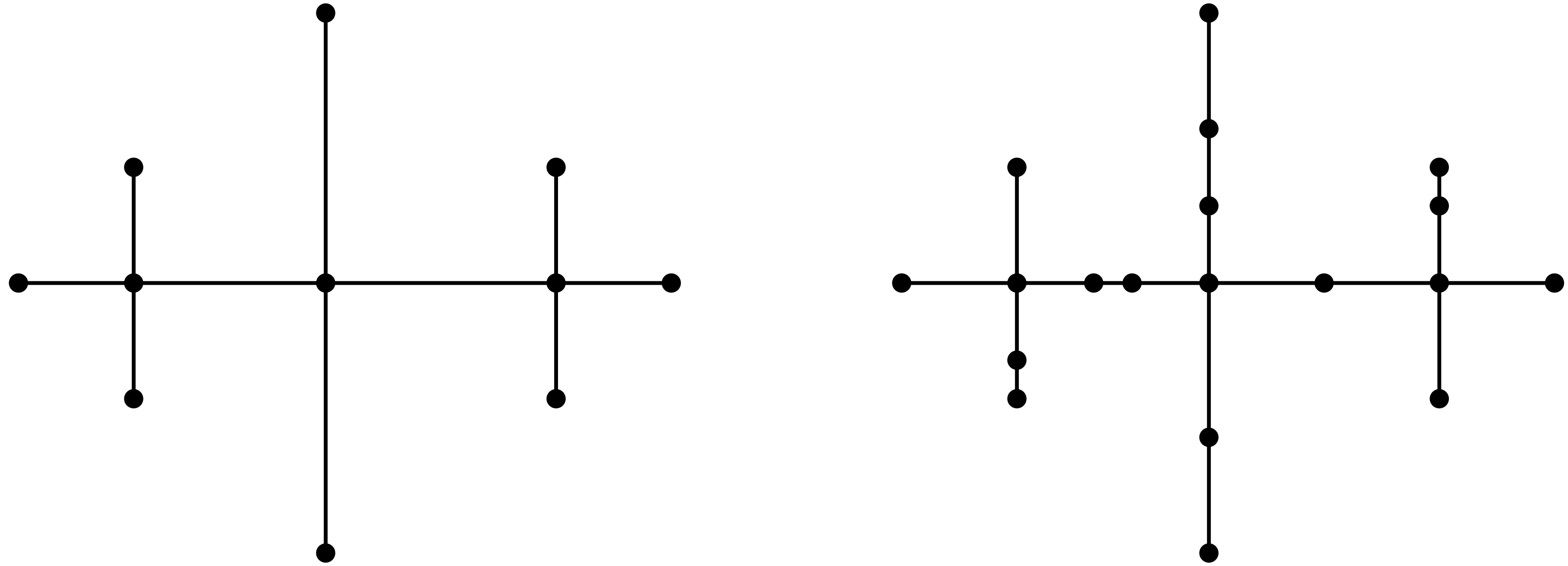
What about all possible **shapes**?

Hausdorff metric: if E is compact,

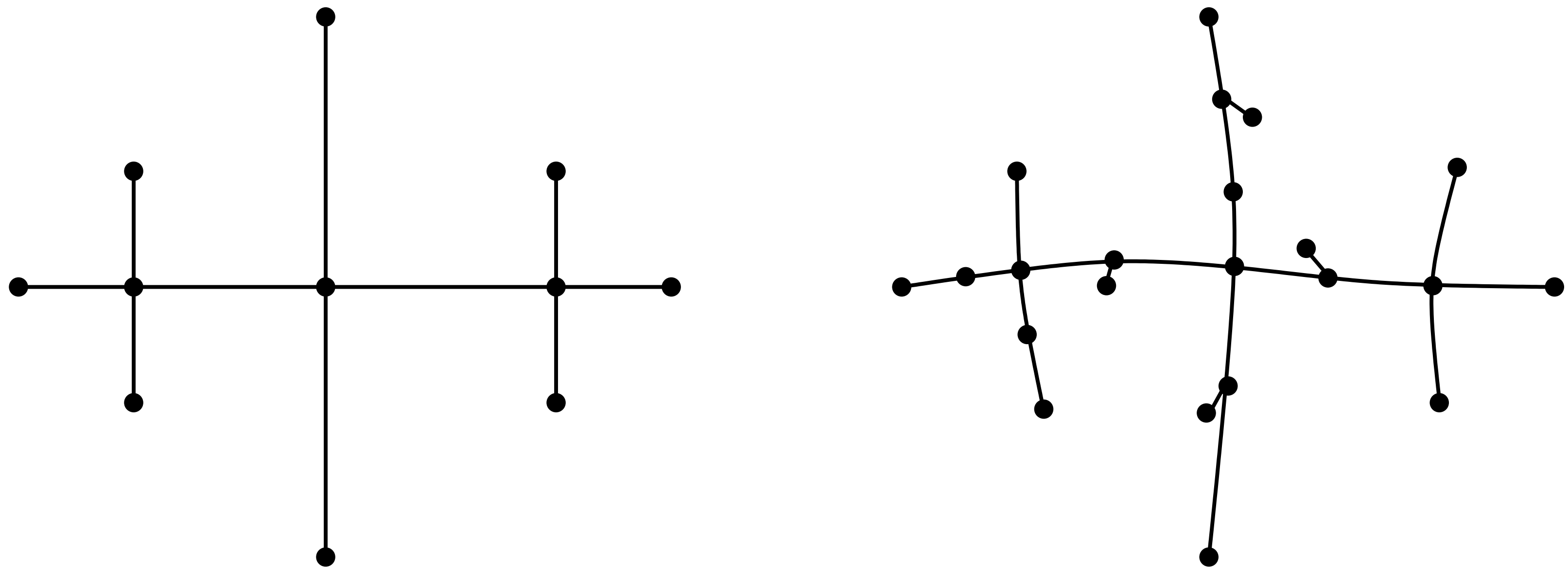
$$E_\epsilon = \{z : \mathrm{dist}(z, E) < \epsilon\}.$$

$$\mathrm{dist}(E, F) = \inf\{\epsilon : E \subset F_\epsilon, F \subset E_\epsilon\}.$$





Different combinatorics, same shape



Different trees, similar shapes

Close in Hausdorff metric

Theorem: Every continuum is a limit of true trees.

True trees are dense, Inventiones Mat., 197(2014), 433-452.

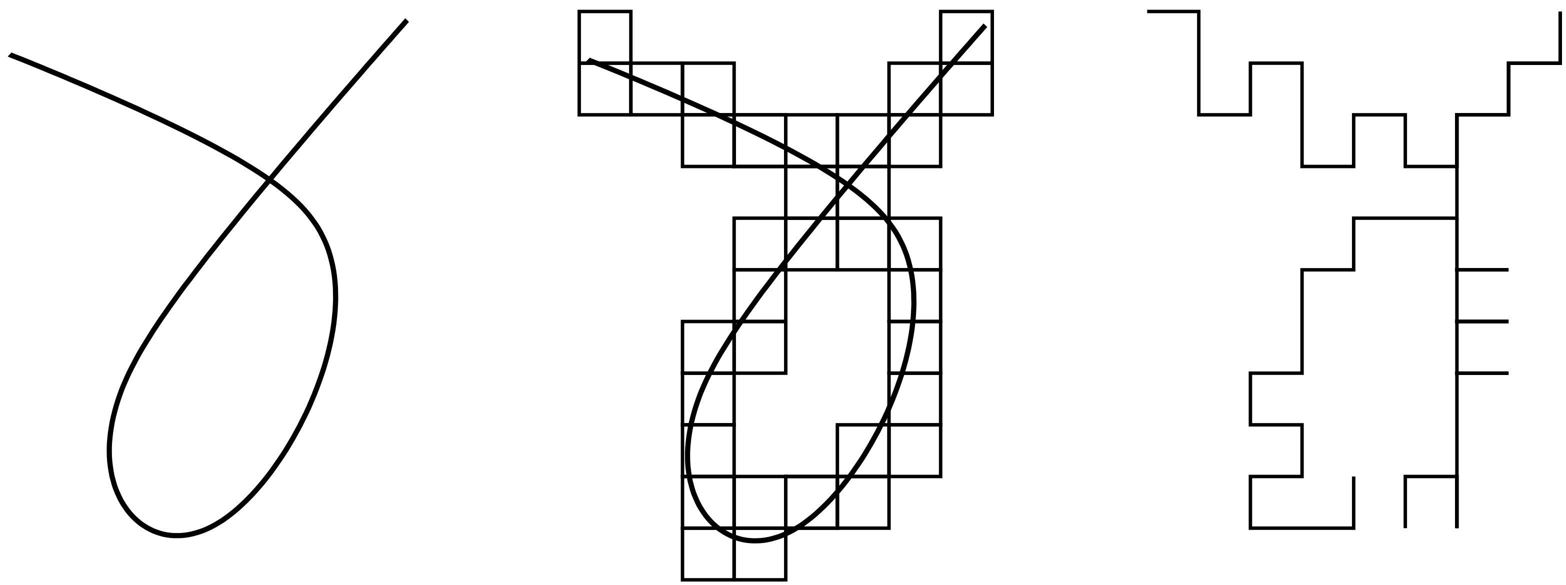
Limit in Hausdorff metric.

continuum = compact, connected set

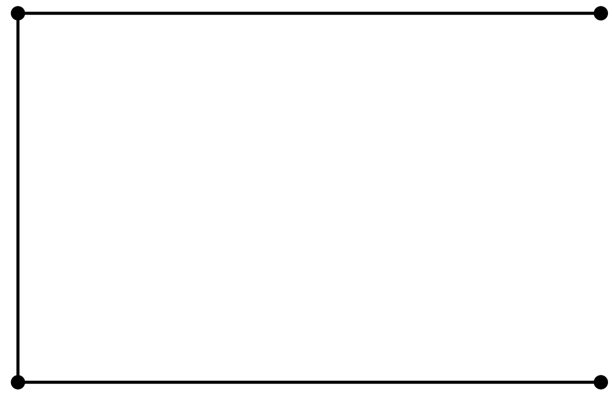
Theorem: Every continuum is a limit of true trees.

Answers question of Alex Eremenko.

Enough to approximate certain finite trees by true trees.

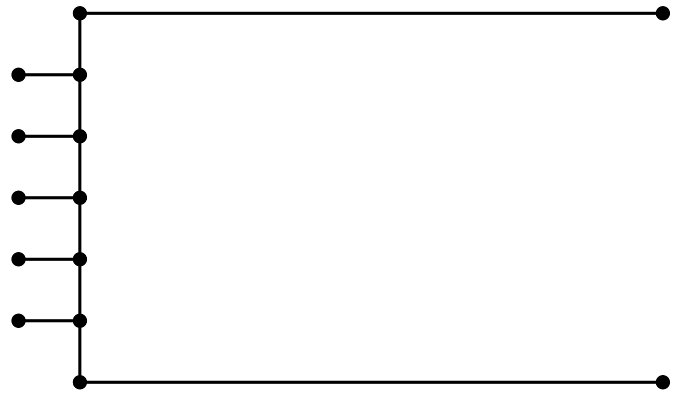


Idea: reduce harmonic measure ratio by adding edges.



Vertical side has much larger harmonic measure from left.

Idea: reduce harmonic measure ratio by adding edges.



"Left" harmonic measure is reduced (roughly 3-to-1).

New edges are approximately balanced (universal constant).

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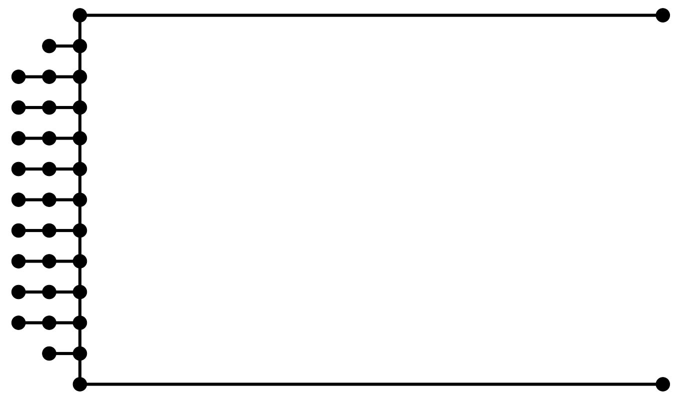

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Longer spikes mean more reduction. Spikes can be very short

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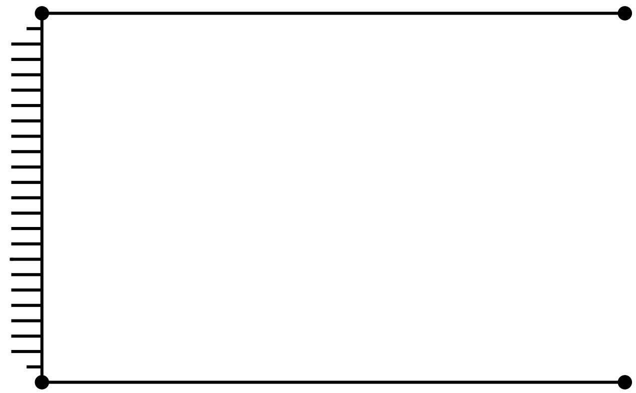


"Left" harmonic measure is reduced (roughly 3-to-1).

New edges are approximately balanced (universal constant).

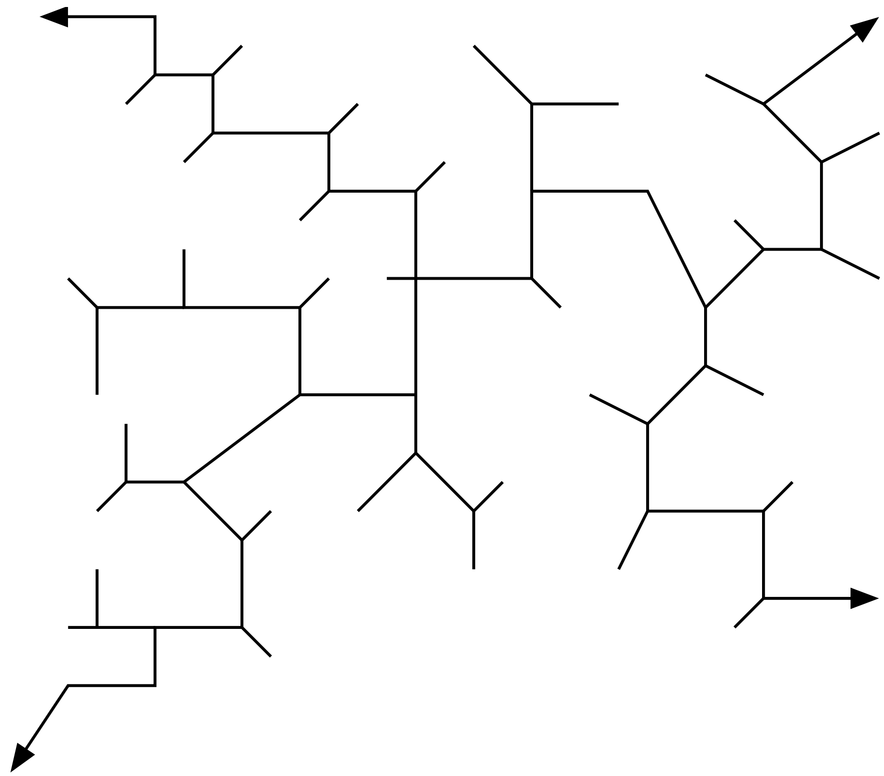
Longer spikes mean more reduction. Spikes can be very short.

Approximately balanced $\Rightarrow$ exactly balanced by MRMT.

QC-constant uniformly bounded. Only non-conformal very near tree.

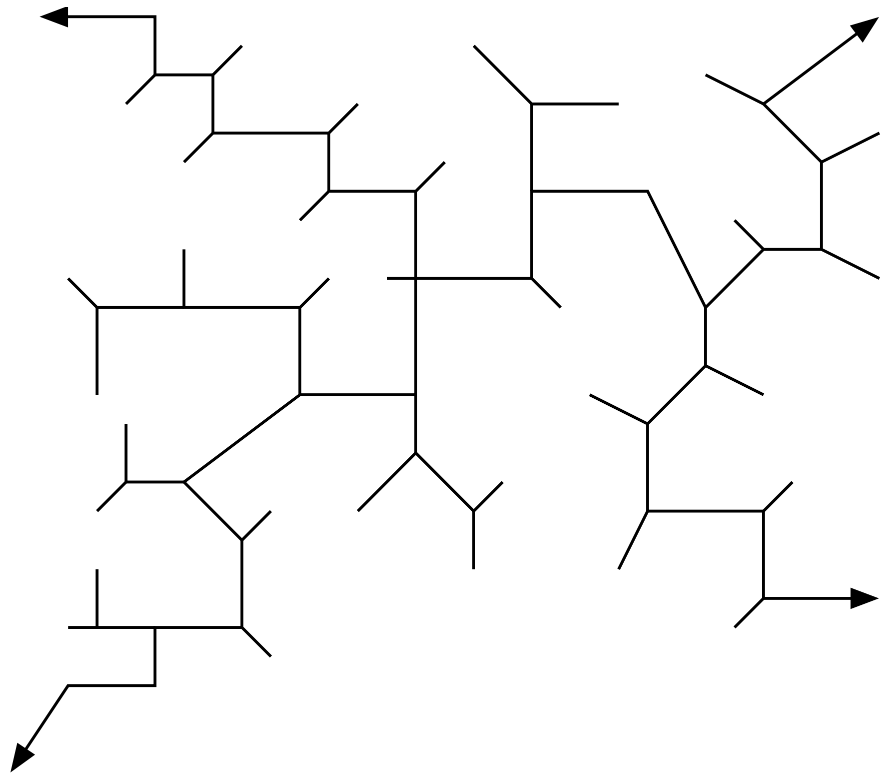
Implies correction map close to identity. This proves theorem.

What about infinite trees?



Is there a theory of *dessins d'adolescents* that relates infinite trees to entire functions with two critical values?

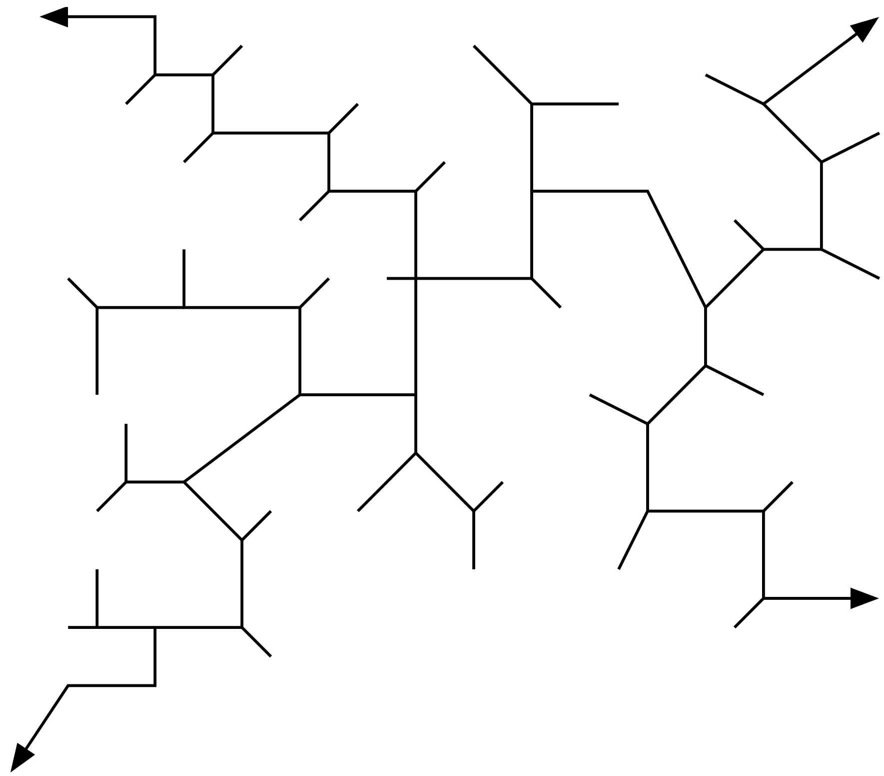
What about infinite trees?



What does “balanced” mean now?

Harmonic measure from ∞ doesn't make sense.

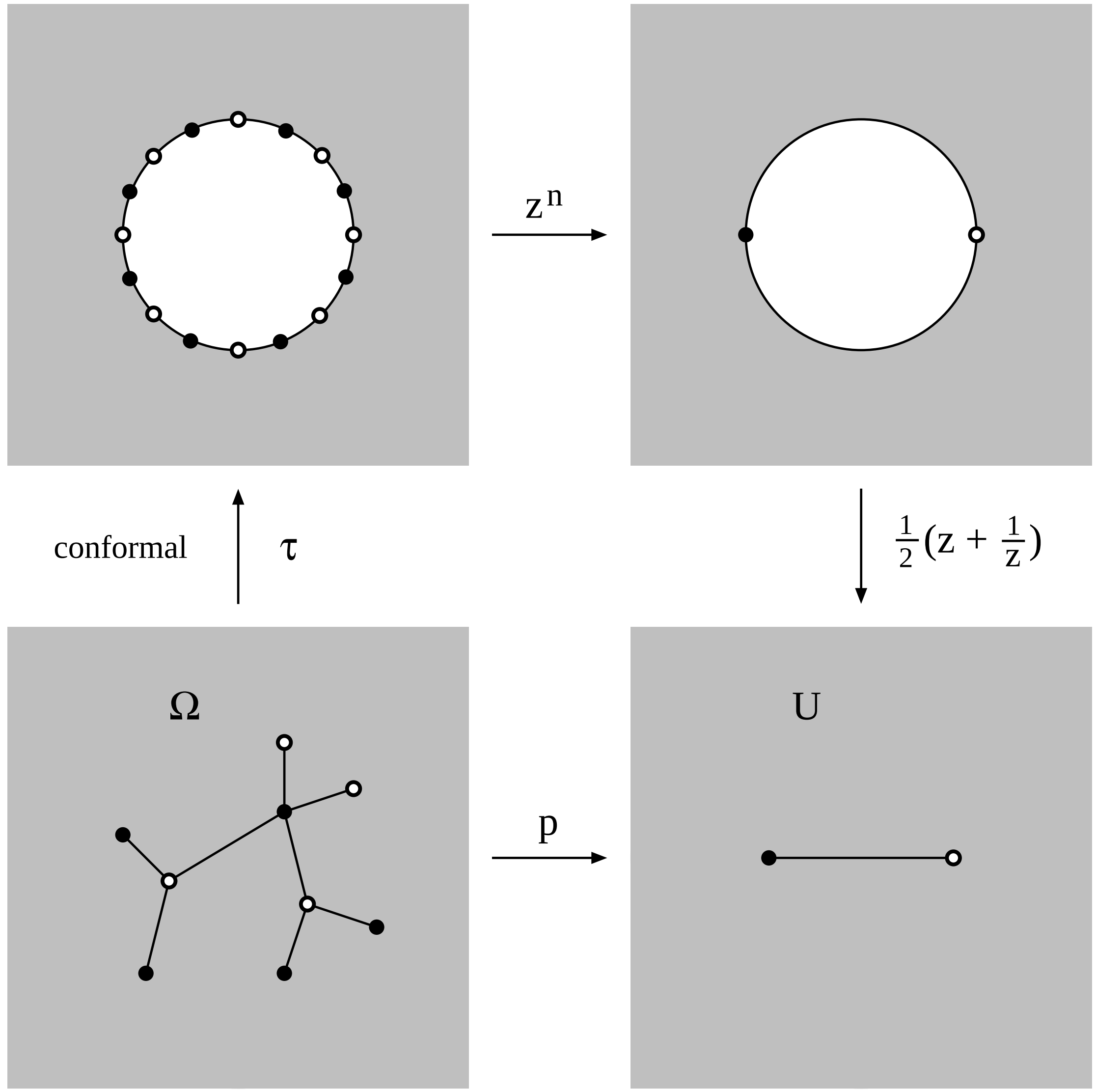
What about infinite trees?



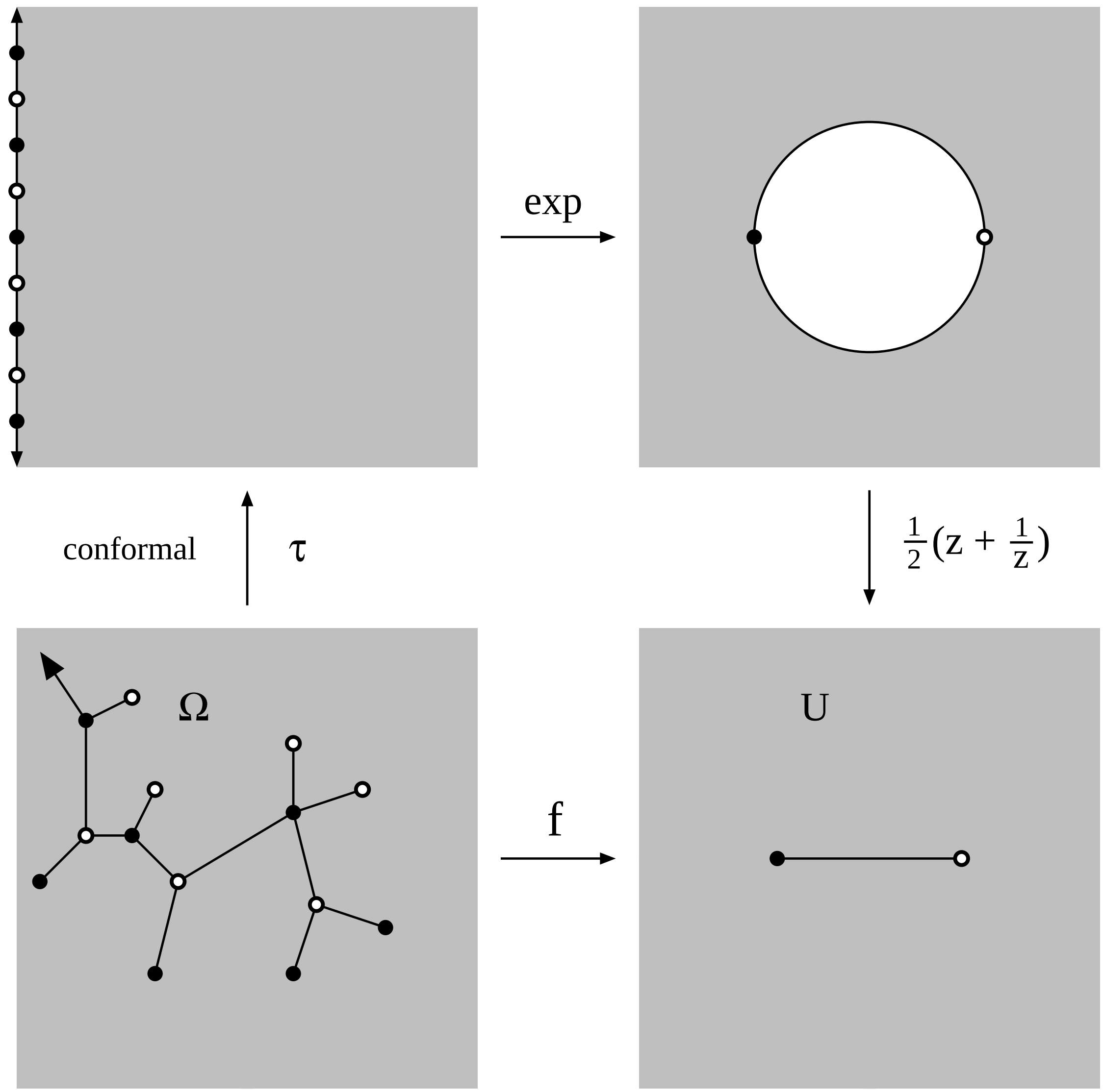
Main difference:

$\mathbb{C}\setminus$ finite tree = one annulus

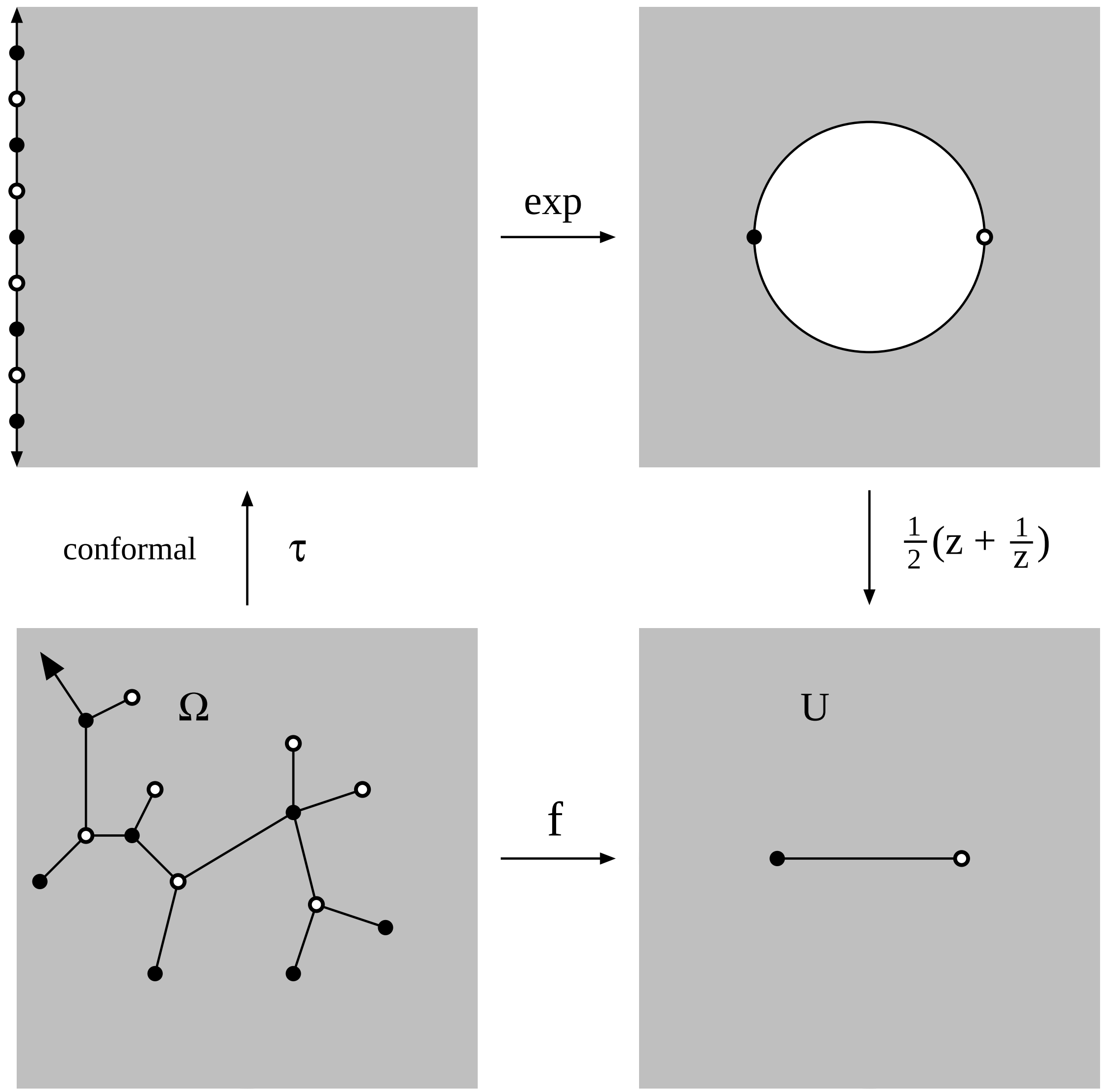
$\mathbb{C}\setminus$ infinite tree = many simply connected components



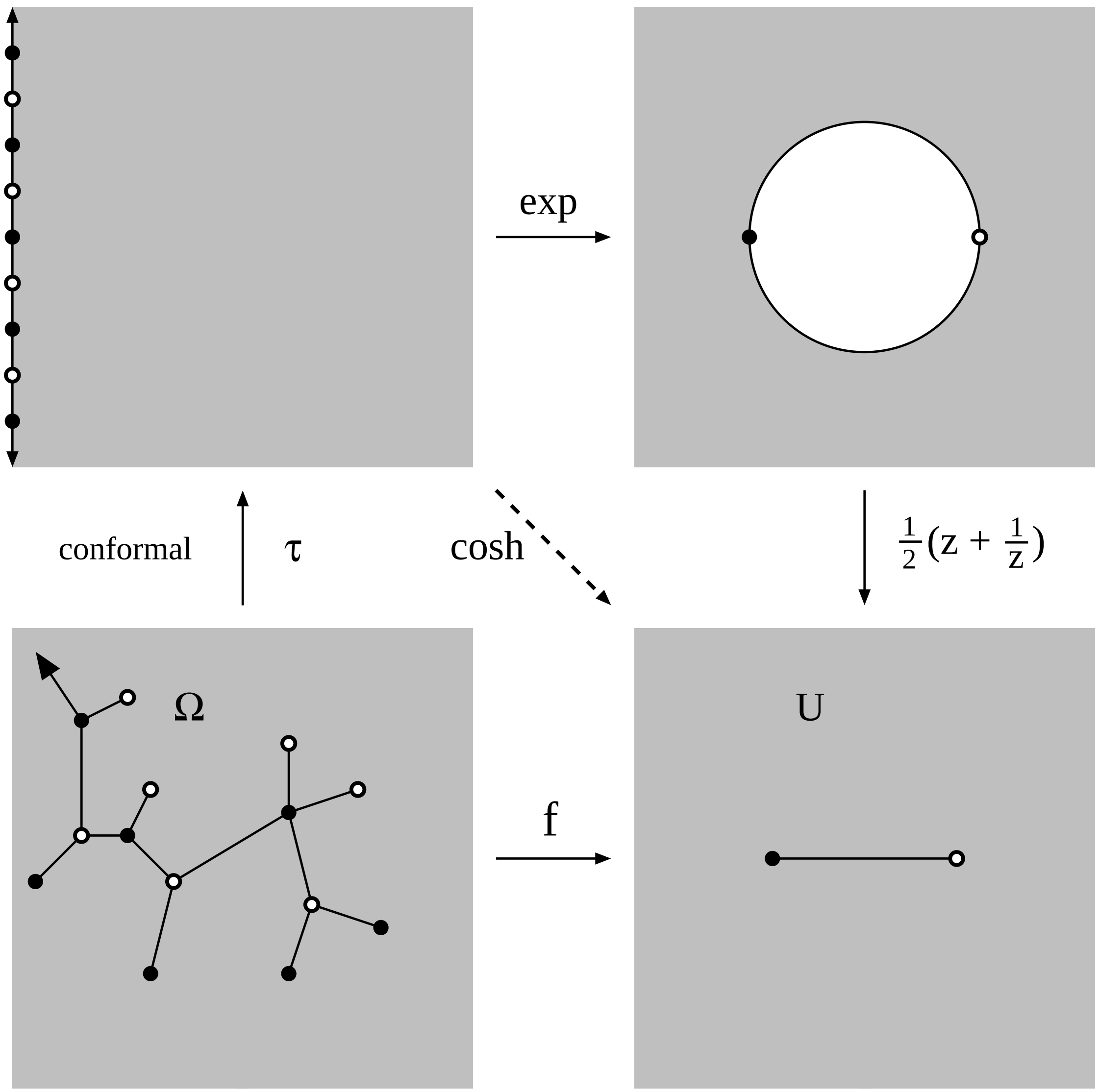
Recall finite case. Infinite case is very similar.



τ maps components of $\Omega = \mathbb{C} \setminus T$ to right half-plane.



Pullback length to tree. Every side gets τ-length π.

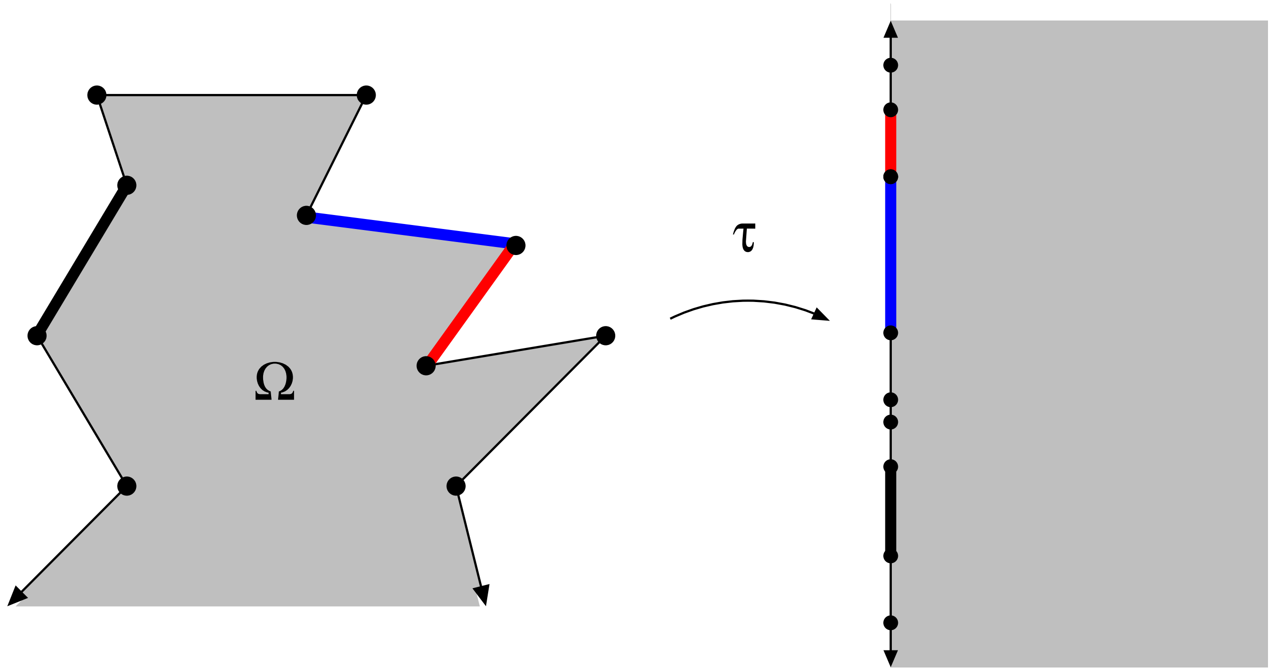


Balanced tree $\Leftrightarrow f = \cosh \circ \tau$ is entire, $\mathrm{CV(f)} = \pm 1$.

For general tree T, define:

τ is conformal from components of $\Omega = \mathbb{C} \setminus T$ to RHP=$\{x > 0\}$.

τ-length is pull-back if length on imaginary axis to sides of T.



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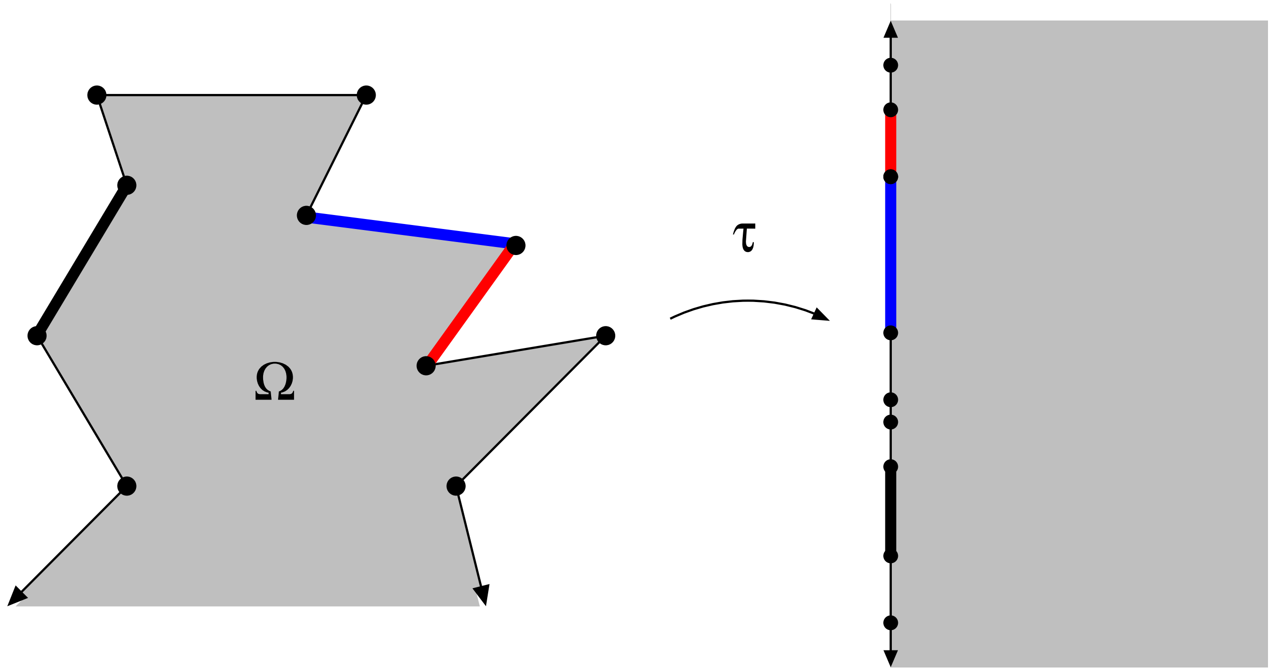


We make two assumptions about components of Ω.

1. Adjacent sides have comparable τ-length (local, bounded geometry)

2. All τ-lengths are $\geq \pi$ (global, smaller than half-plane)

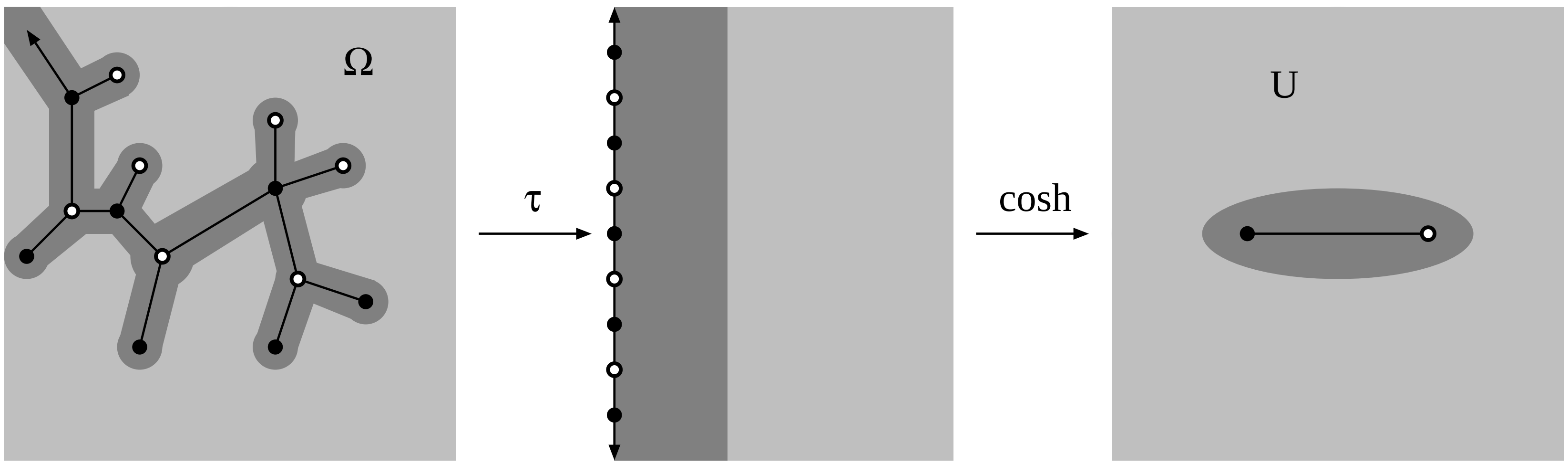
"Adjacent sides have comparable τ-length" follows if T has **bounded geometry**:

- edges are uniformly C^2
- angles are bounded away from 0
- adjacent edges have comparable lengths
- non-adjacent edges satisfy $\mathrm{diam}(e) \leq C \mathrm{dist}(e, f)$.

This is usually **easy** to check.

"τ lower bound" less obvious, but is "standard" problem in complex analysis (extremal length, hyperbolic metric, harmonic measure).

QC Folding Thm: If (1) and (2) hold, then there is a quasi-regular g such that $g = \cosh \circ \tau$ off $T(r)$ and $\mathrm{CV}(g) = \pm 1$.



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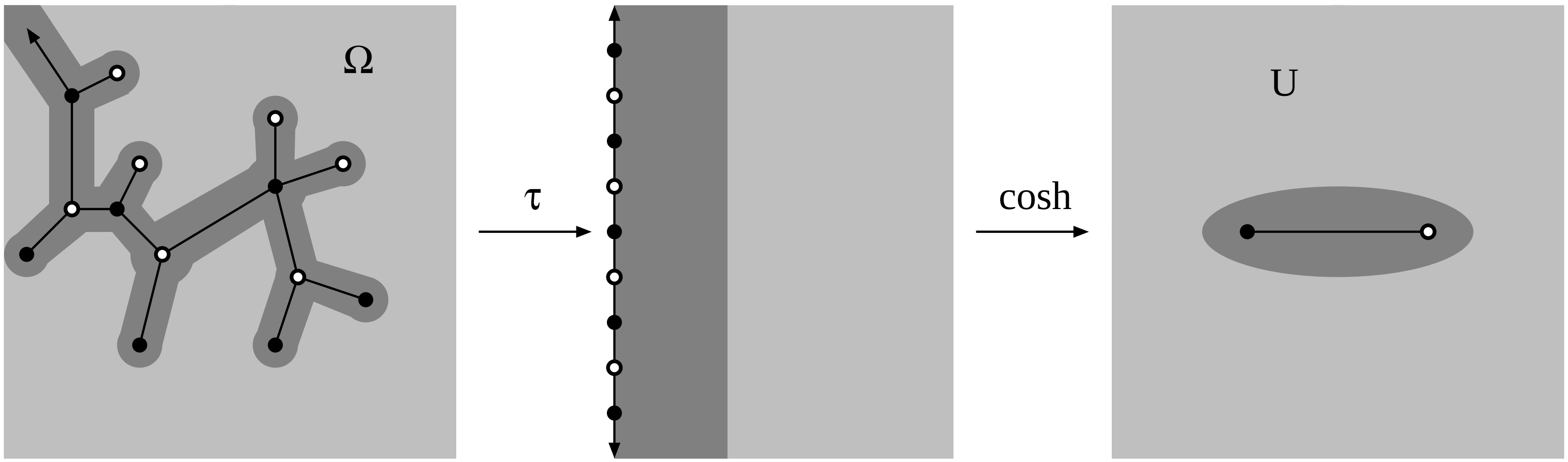

$T(r)$ is a "small" neighborhhod of the tree T.

QR constant depends only on constants in (1).

Cor: There is an entire function $f = g \circ \varphi$ with $\mathrm{CV}(f) = \pm 1$ so that $f^{-1}([-1, 1])$ approximates the shape of T.

What is $T(r)$? If e is an edge of T and $r > 0$ let

$$e(r) = \{z : \operatorname{dist}(z, e) \le r \cdot \operatorname{diam}(e)\}$$

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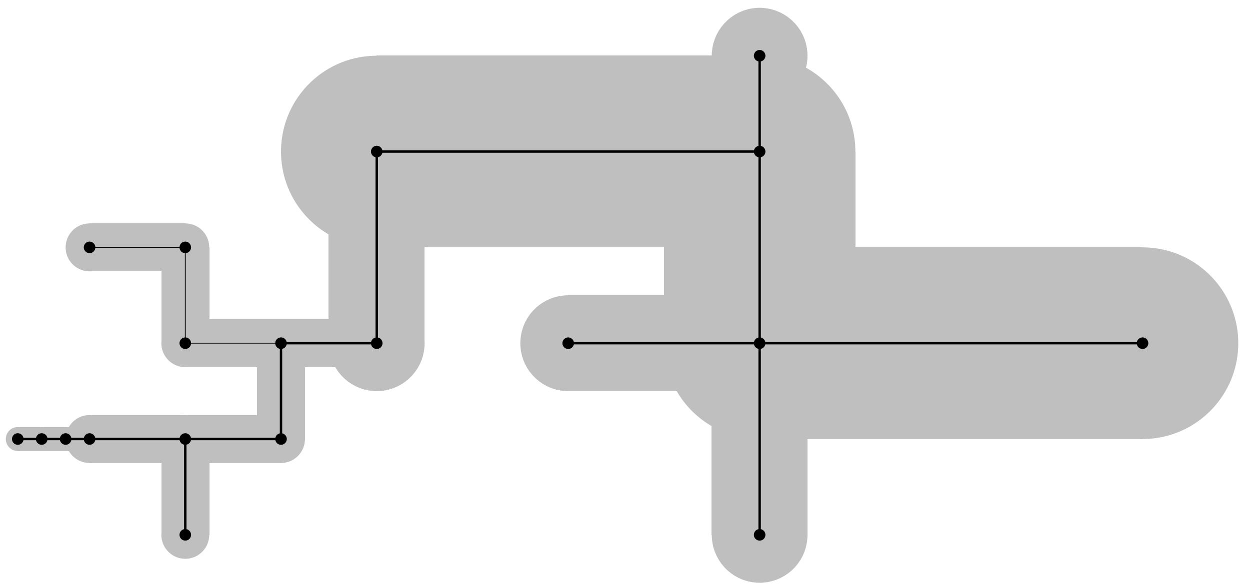
Define neighborhood of T: $T(r) = \cup\{e(r) : e \in T\}$.



$T(r)$ for infinite tree replaces Hausdorff metric in finite case.

What is $T(r)$? If e is an edge of T and $r > 0$ let

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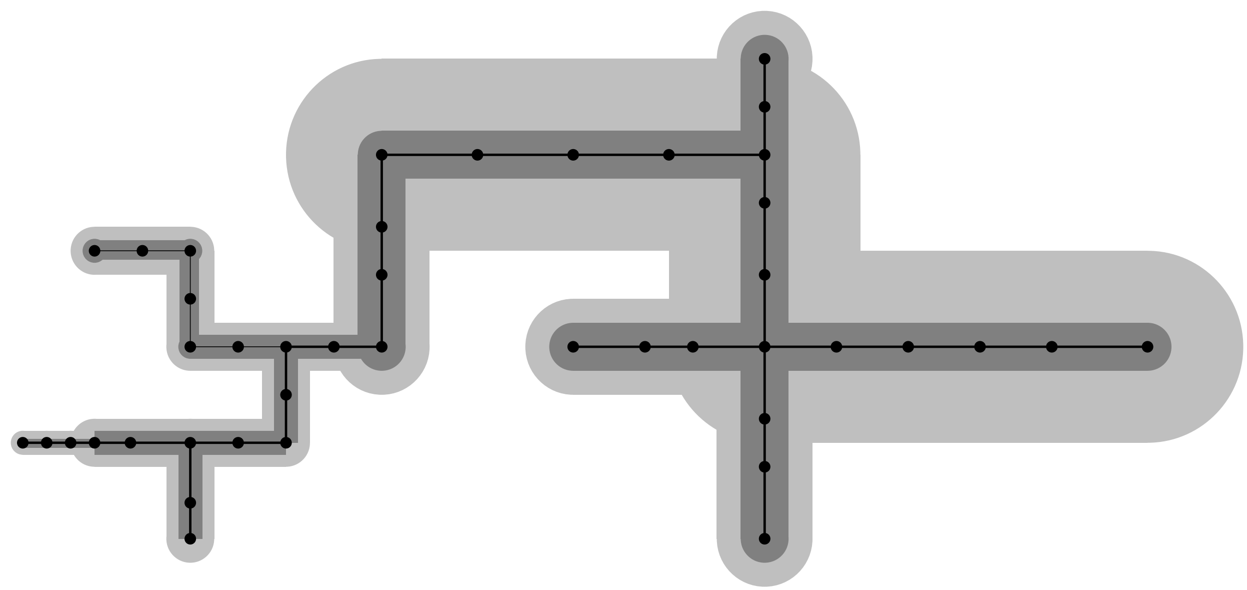
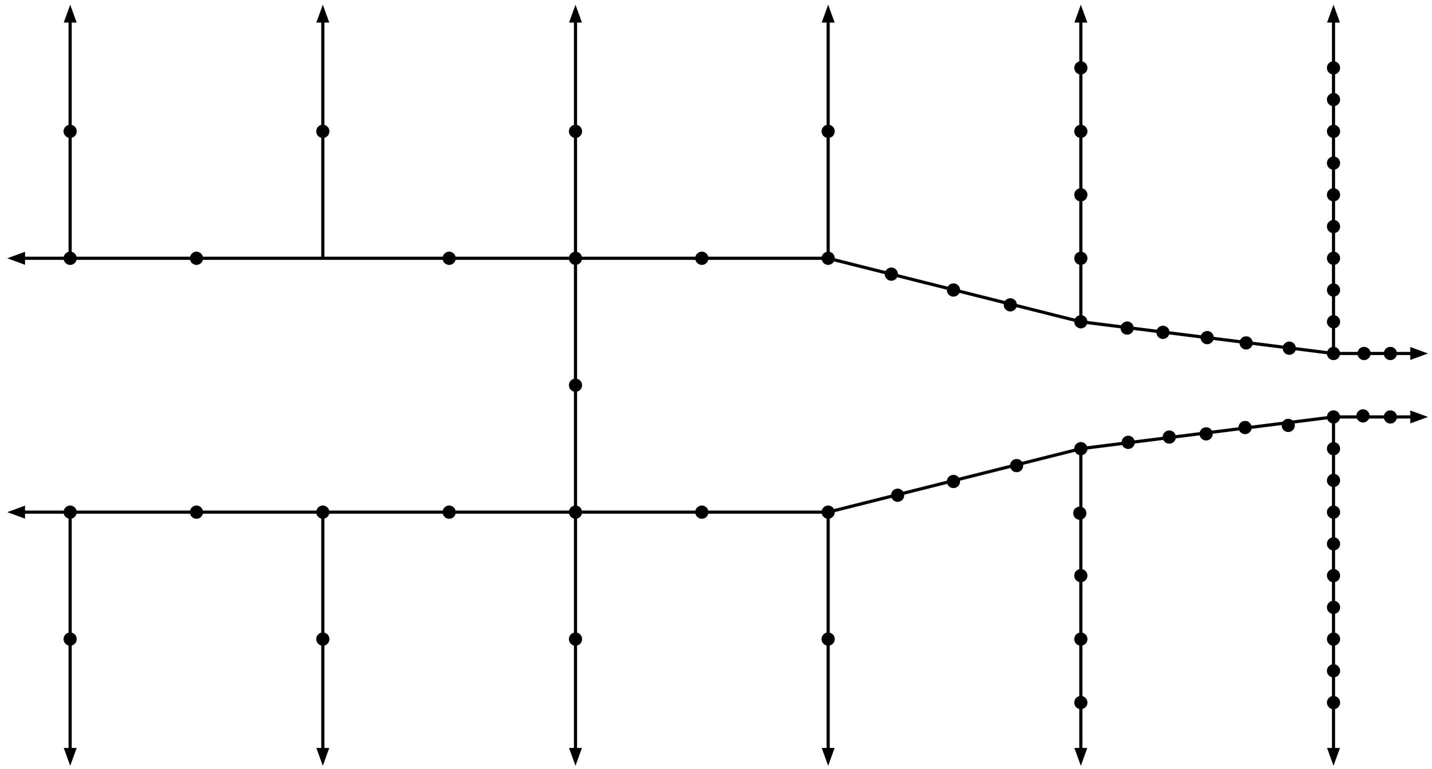


Adding vertices reduces size of $T(r)$.

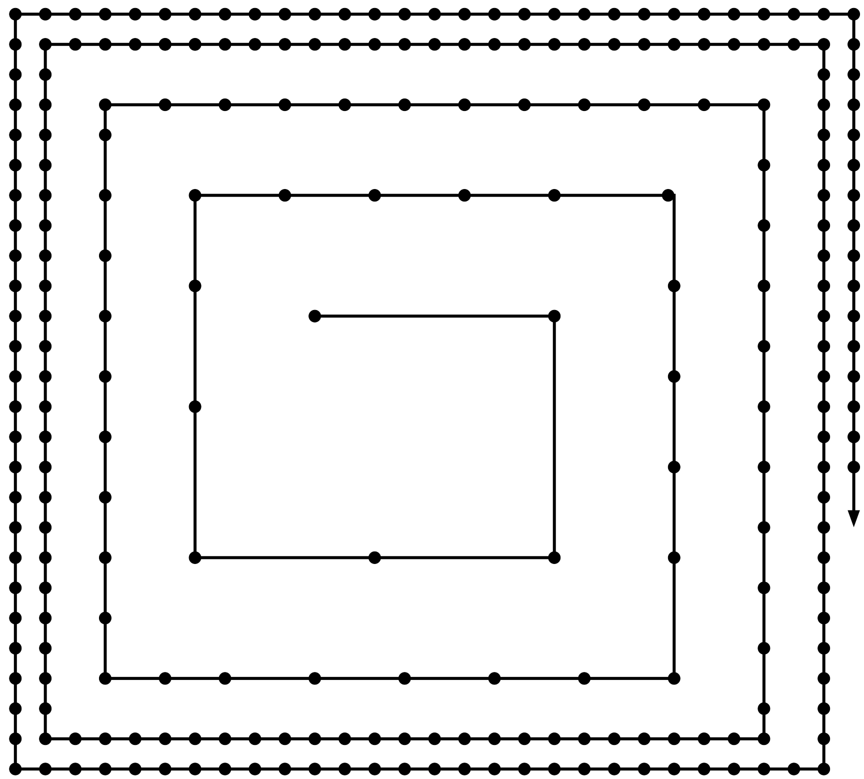
Rapid increase



f has two singular values, $f(x) \nearrow \infty$ as fast as we wish.

First such example due to Sergei Merenkov.

Fast spirals



Two singular values, the tract $\{|f| > 1\}$ spirals as fast as we wish.

Order of growth:

$$\rho(f) = \limsup_{|z|\to\infty} \frac{\log\log|f(z)|}{\log|z|}, \qquad \rho(e^{z^d}) = d.$$

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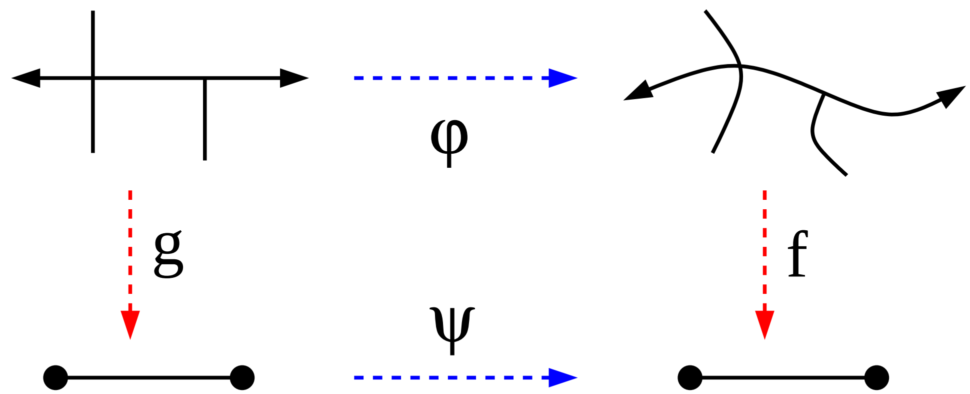
Order conjecture (A. Epstein): f, g QC-equivalent $\Rightarrow \rho(f) = \rho(g)$?

f, g are QC-equivalent if $\exists$ QC ϕ, ψ such that $f \circ \phi = \psi \circ g$.



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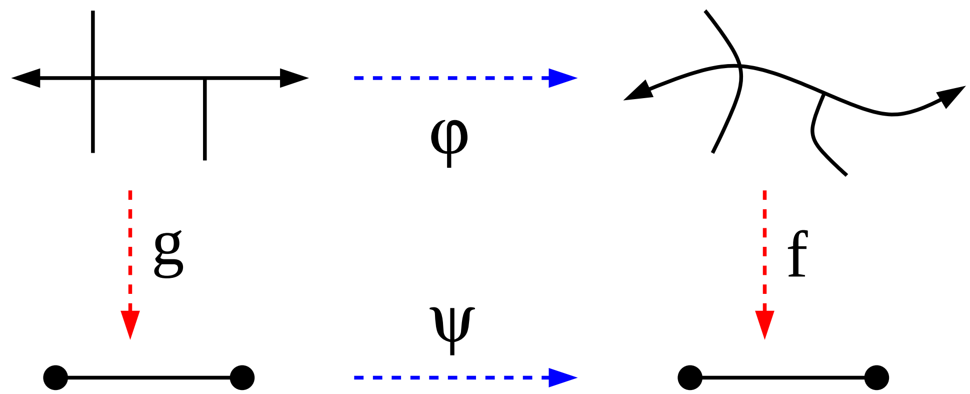
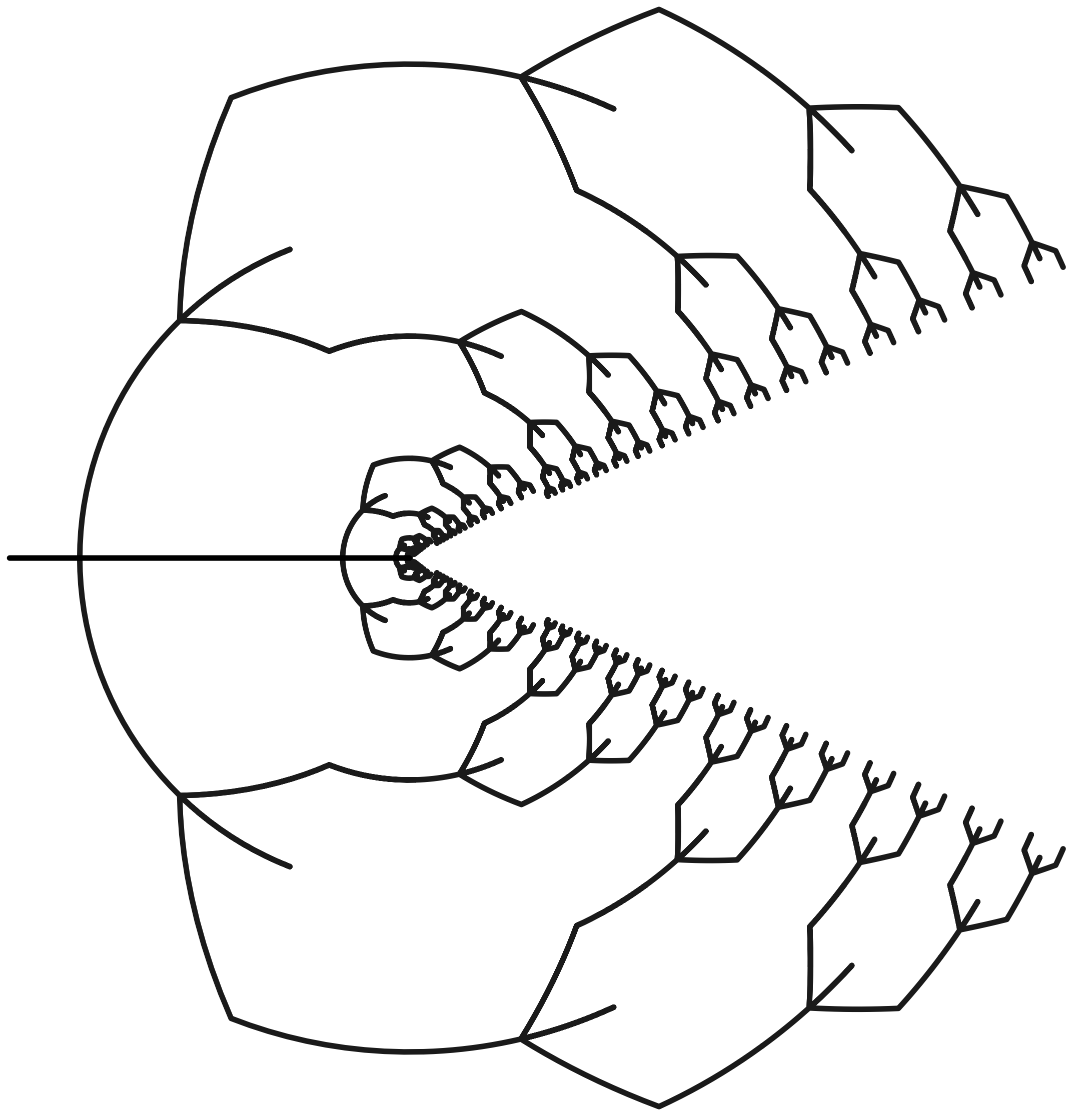


Conjecture true in some special cases.

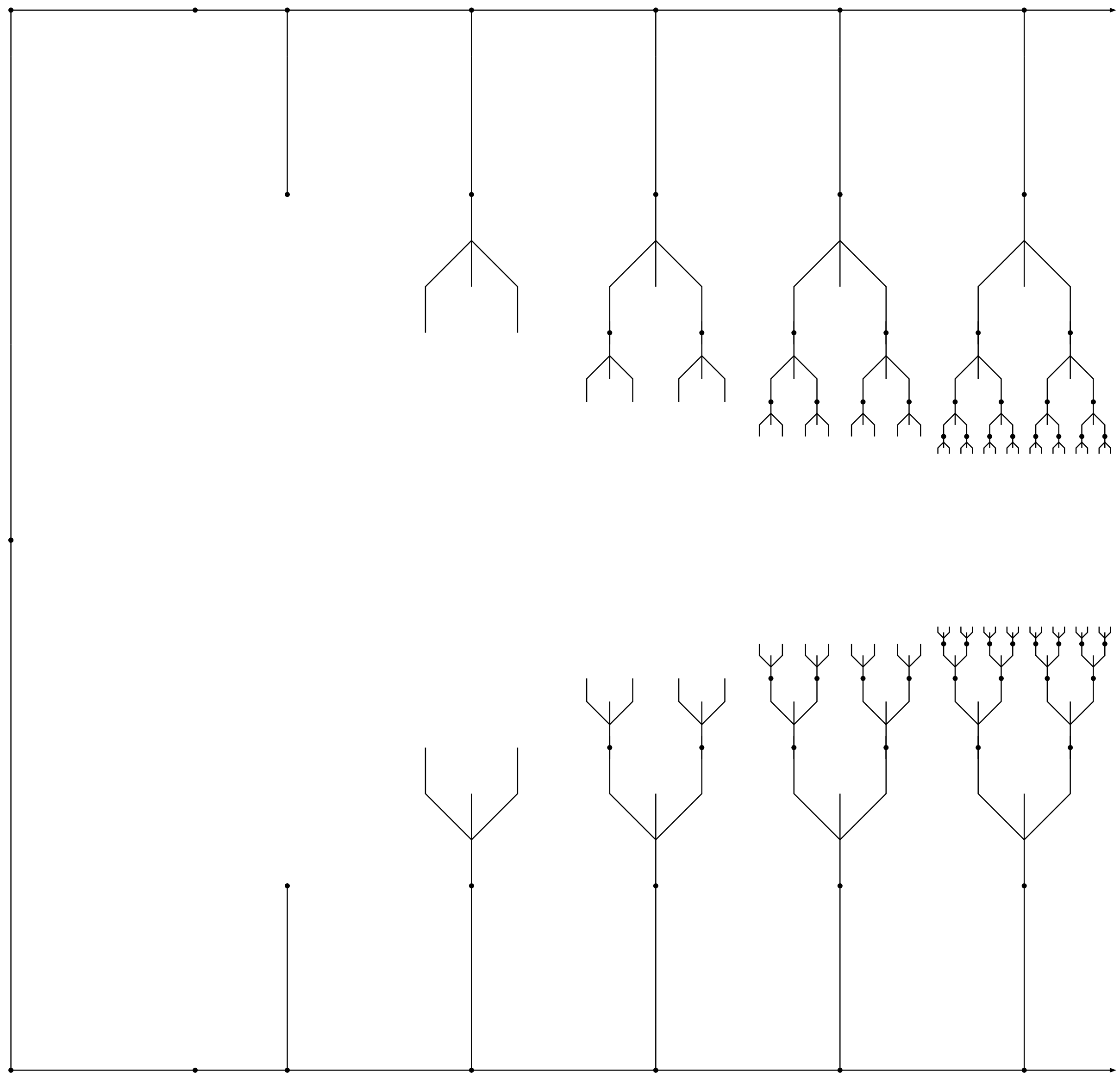
False in Eremenko-Lyubich class = bounded singular set (Epstein-Rempe)

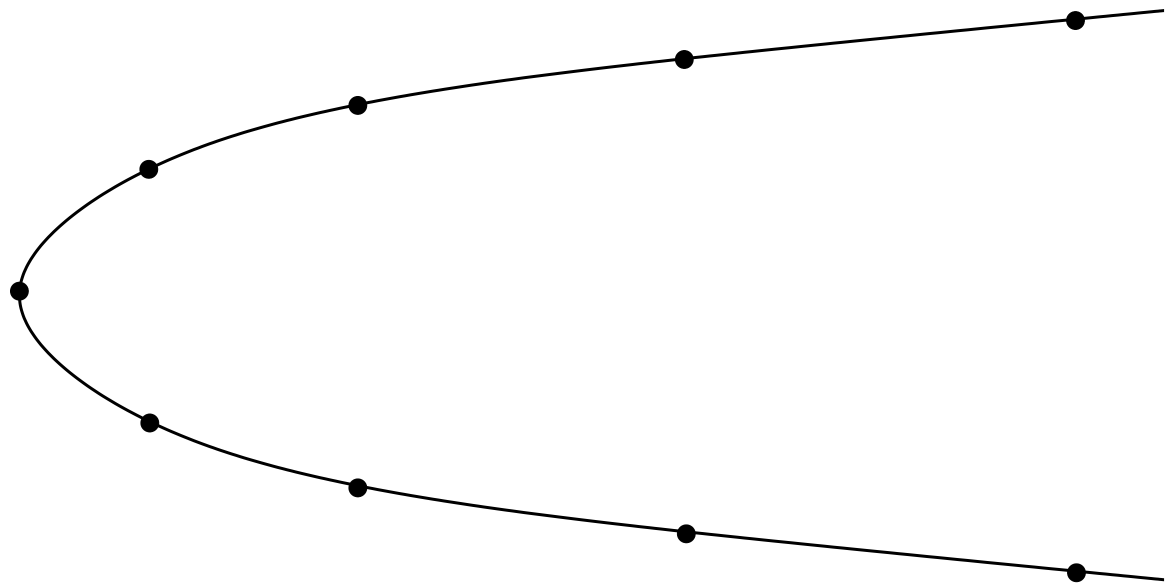
What about Speiser class = finite singular set? True for 2 singularities . . .

but counterexample with 3 singular values:

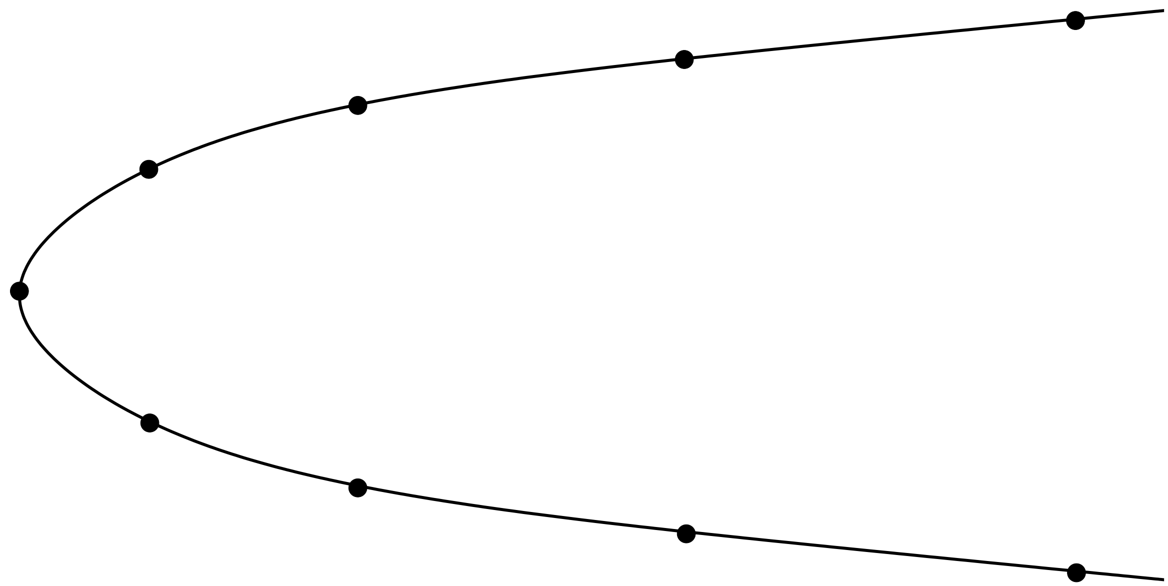


Same domain in logarithmic coordinates





Two singular values and $\dim(\mathcal{J}(f)) < 1 + \epsilon$?



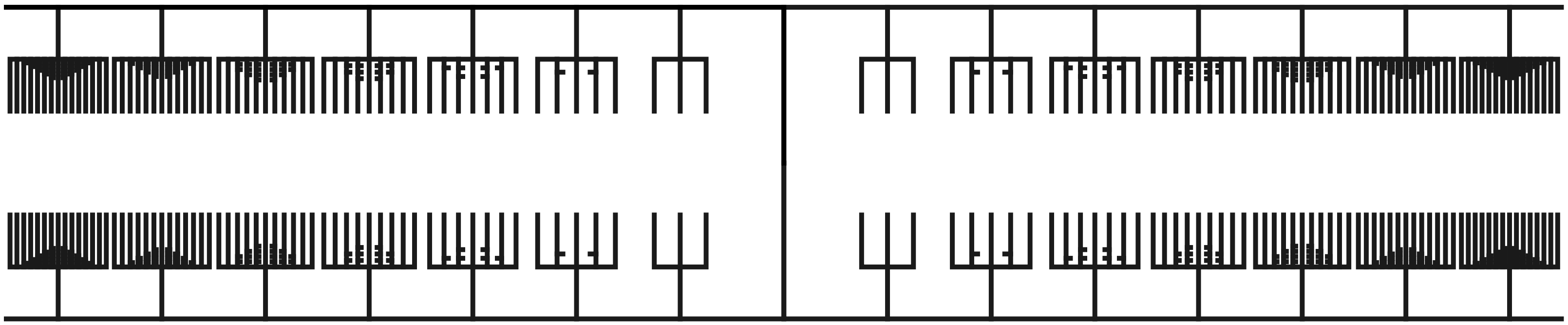
$\dim(\mathcal{J}) \geq 1$ for all transcendental entire functions (Baker).

All values in $[1, 2]$ occur (McMullen, Stallard, B).

$\dim(\mathcal{J}) > 1$ if singular set bounded; $1 + \epsilon$ occurs (Stallard).

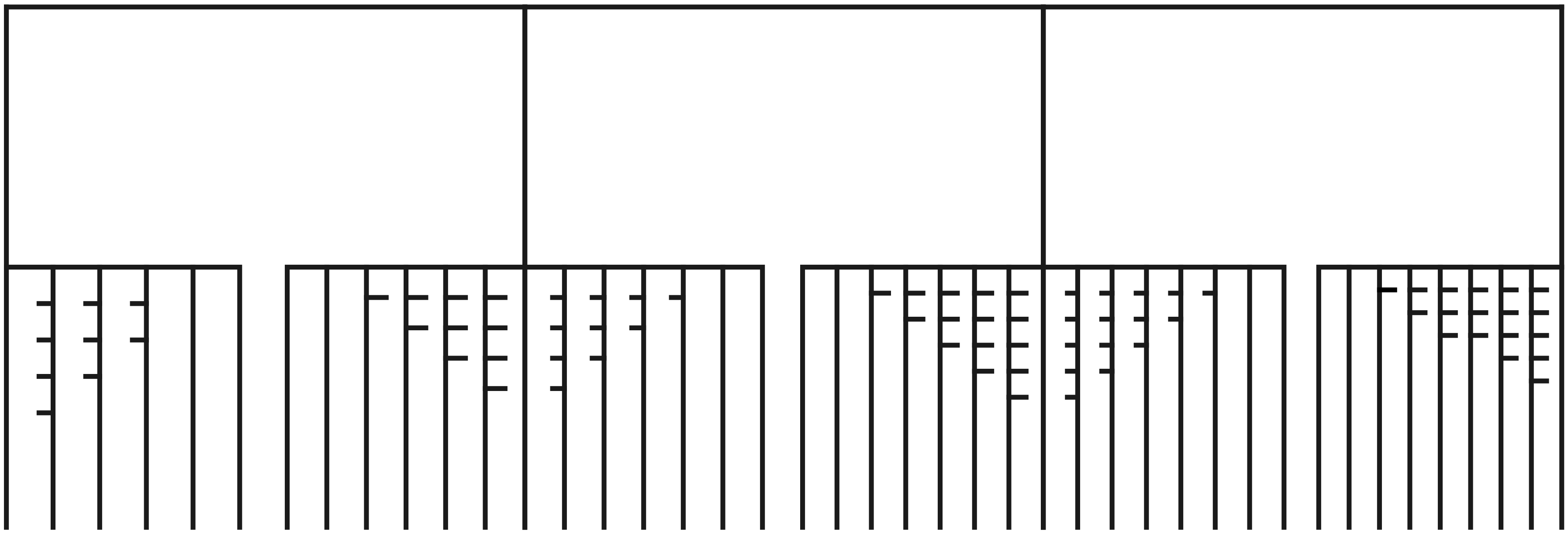
Can we get $\dim < 1 + \epsilon$ in Speiser class = finite singular set?

Hard to estimate QC correction map above. Need Lipschitz bound.

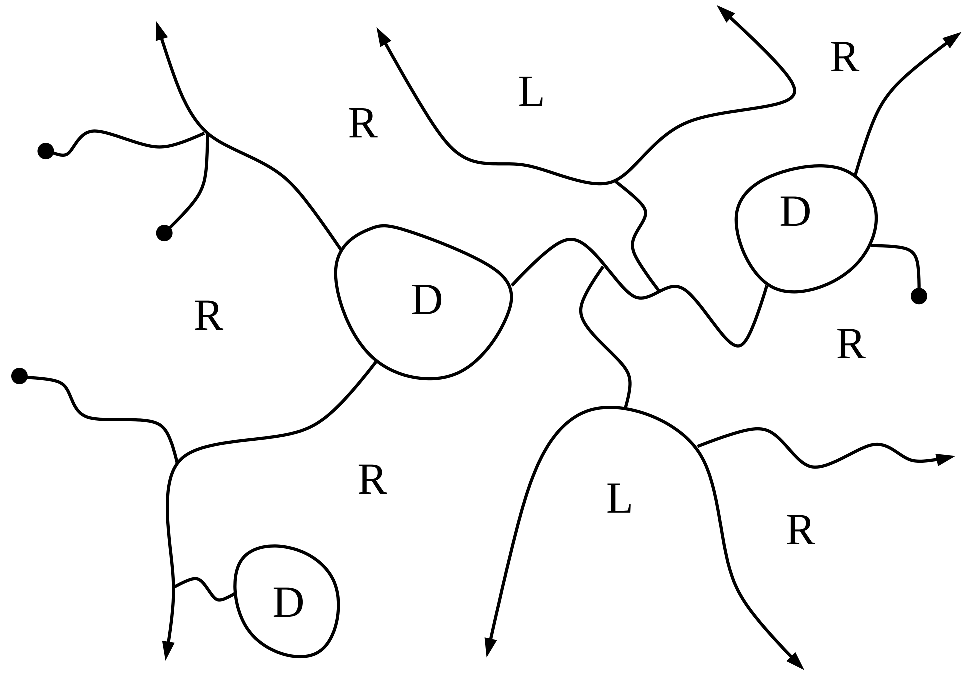


Two singular values and $\dim(\mathcal{J}(f)) < 1 + \epsilon$? (B + Albrecht)

Complicated tree, but almost τ-balanced, so folding map is "simple".



Enlargement of portion of tree.



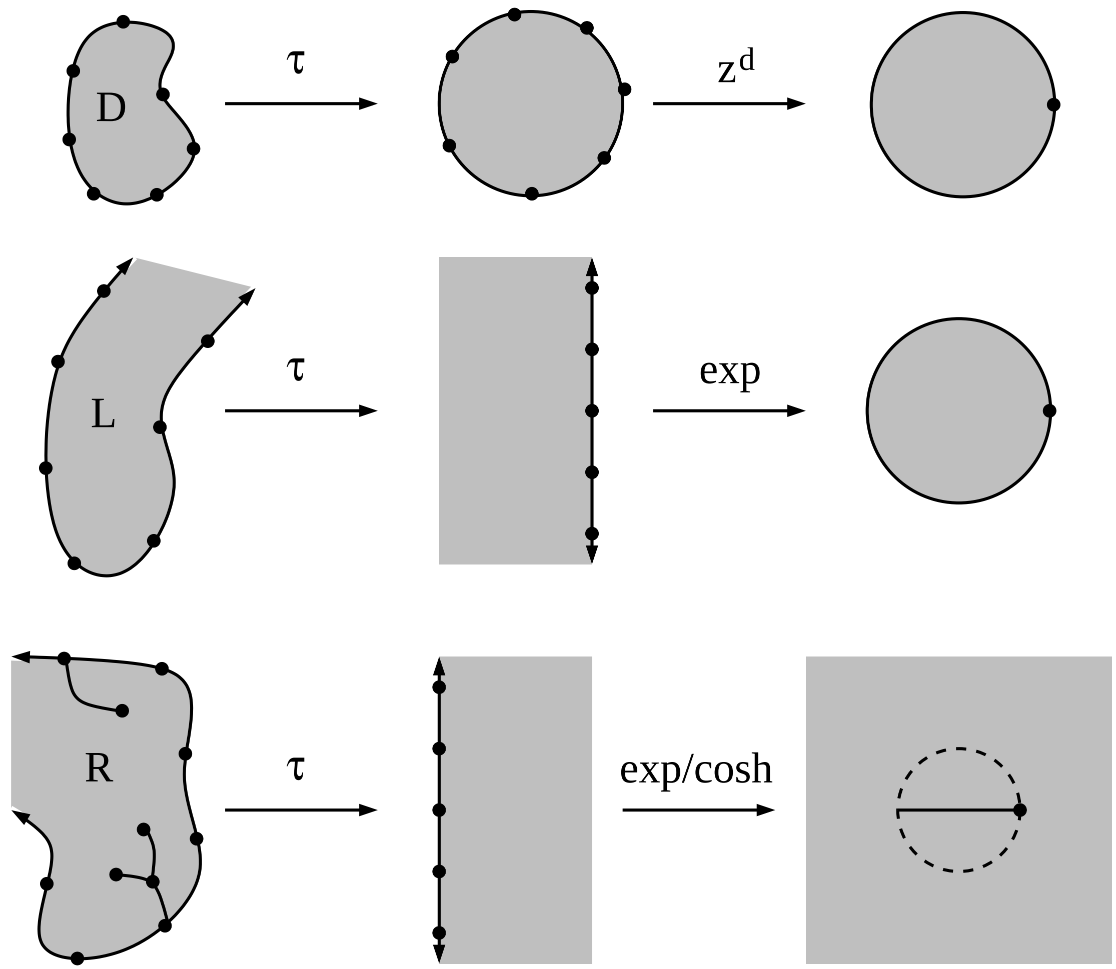
Folding theorem can be generalized from trees to graphs.

Graph faces labeled D,L,R.

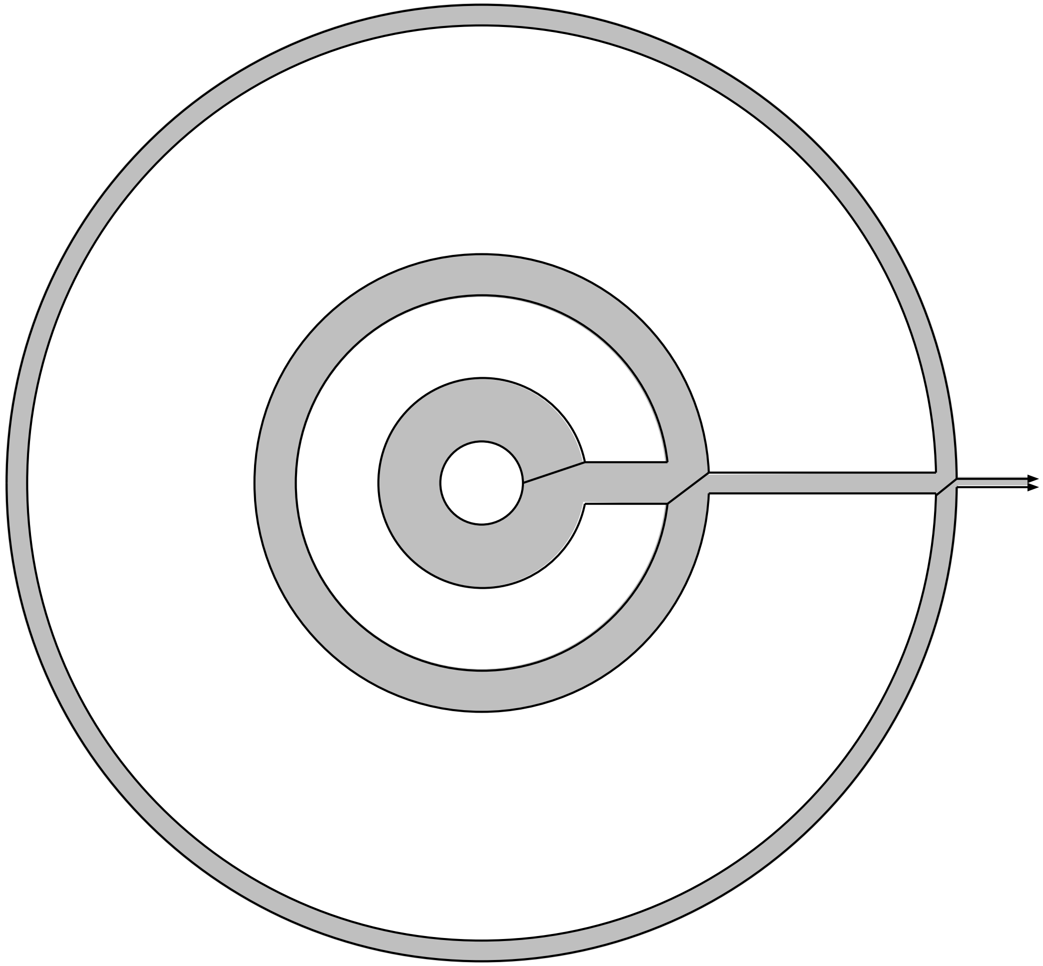
D = bounded Jordan domains (high degree critical points)

L = unbounded Jordan domains (asymptotic values)

D's and L's only share edges with R's.

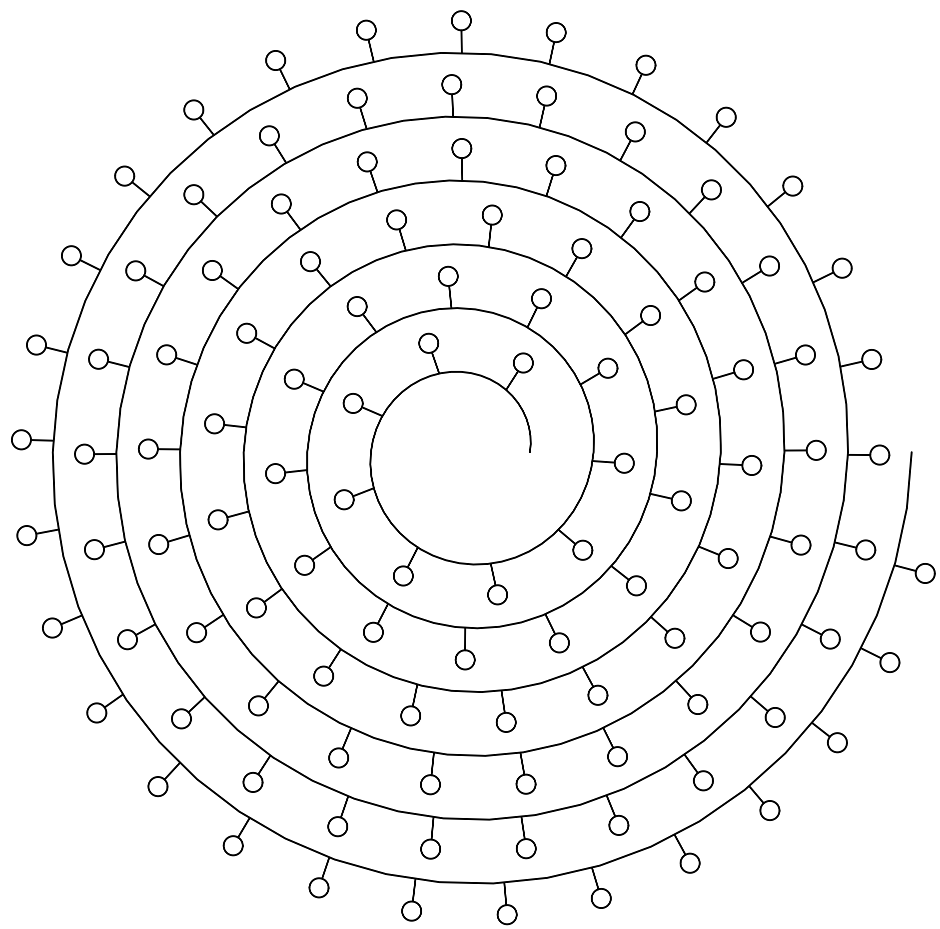


Can define holomorphic map on each component. Generalized folding modifies near graph to get quasiregular map. MRMT gives entire function.



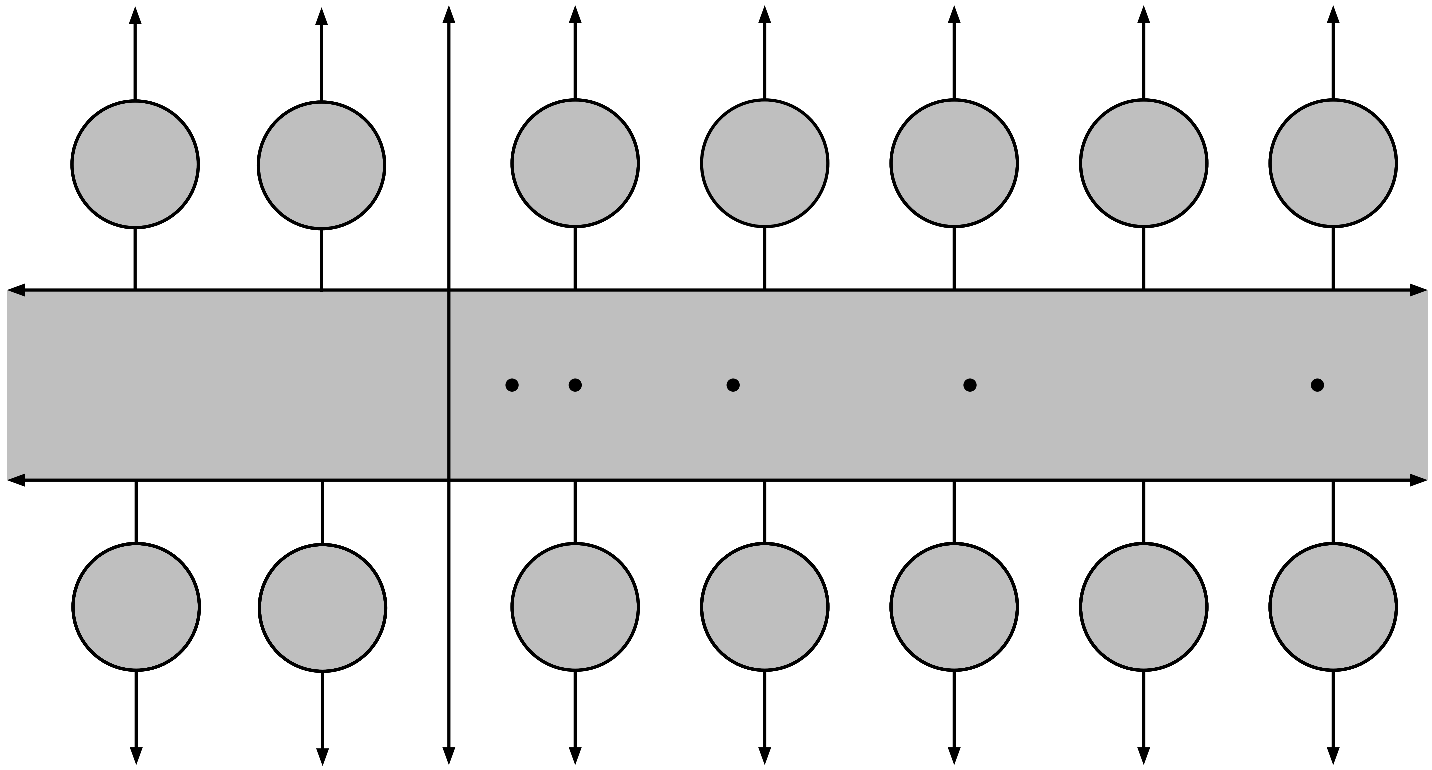
Counterexample to area conjecture (Eremenko-Lyubich)

Three critical values, area$(\{|f| > \epsilon\}) < \infty$ for all $\epsilon > 0$.



$$3 \text{ critical values and } \limsup_{r\to\infty} \frac{\log m(r,f)}{\log M(r,f)} = -\infty.$$

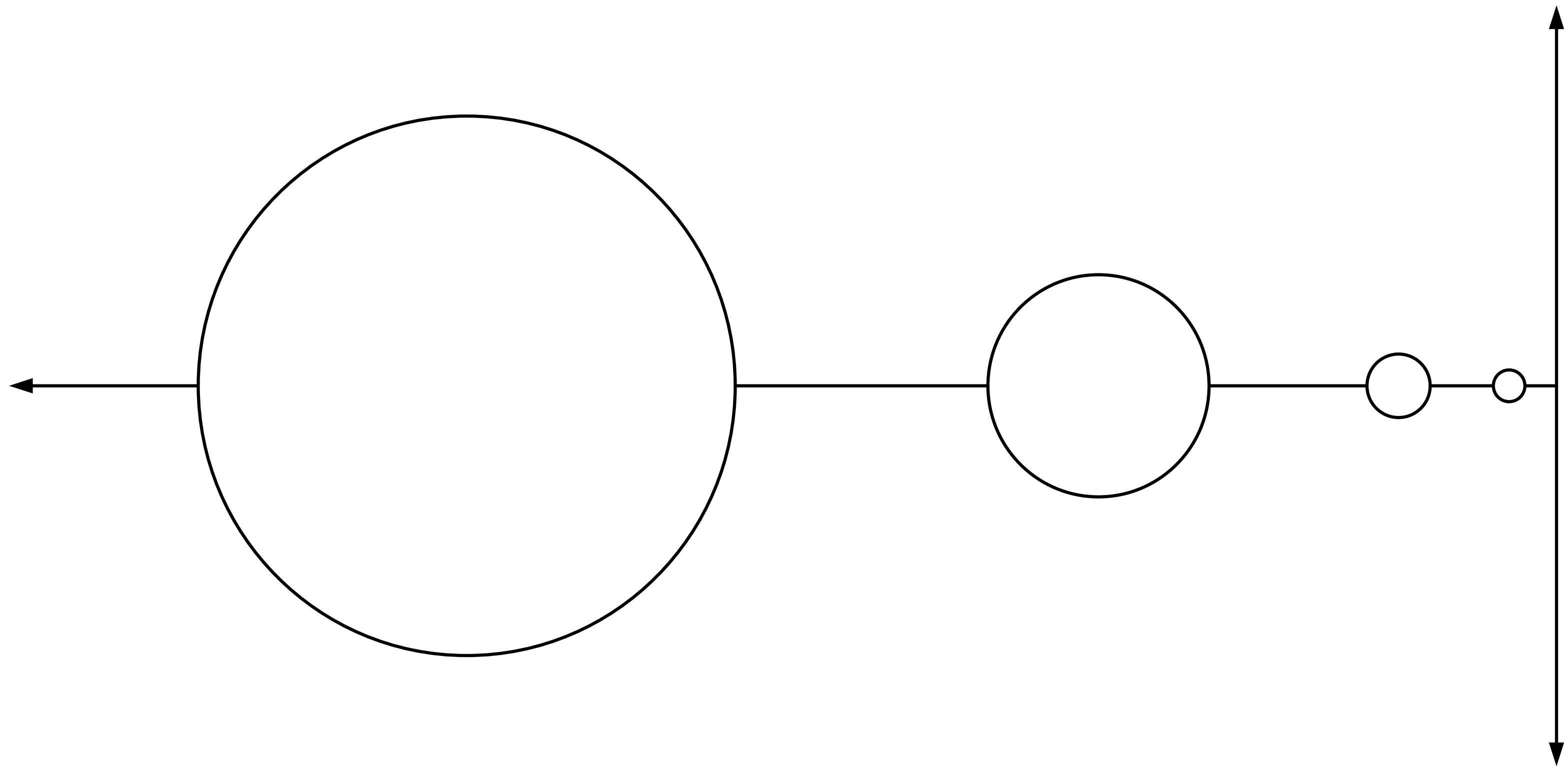
Wiman conjectured ≥ -1 (consider e^z). Disproved by Hayman.



Wandering domain in Eremenko-Lyubich class = bounded singular set

No wandering in Speiser class = finite singular set (Sullivan, Eremenko-Lyubich, Goldberg-Keen).

First wandering domains for entire functions due to Baker.



Wandering domain in Eremenko-Lyubich class with finite order of growth?

Thanks for listening. Questions?