Conformal Welding and Koebe's Theorem

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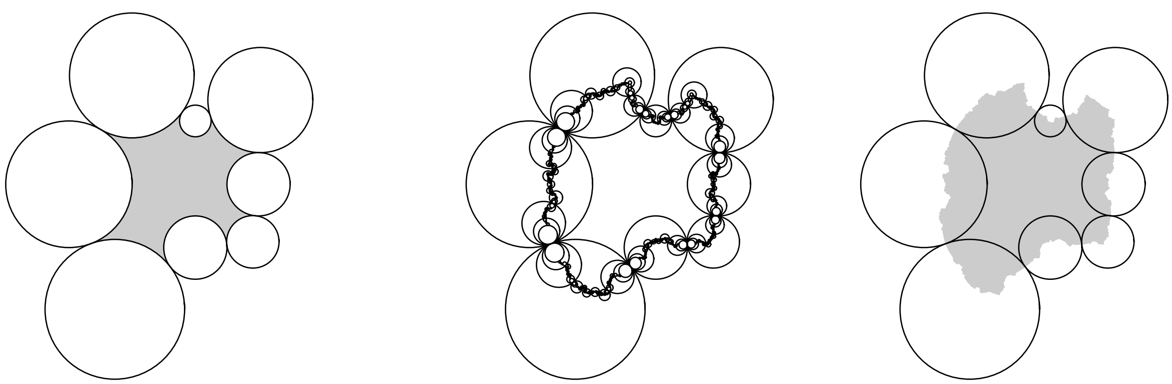
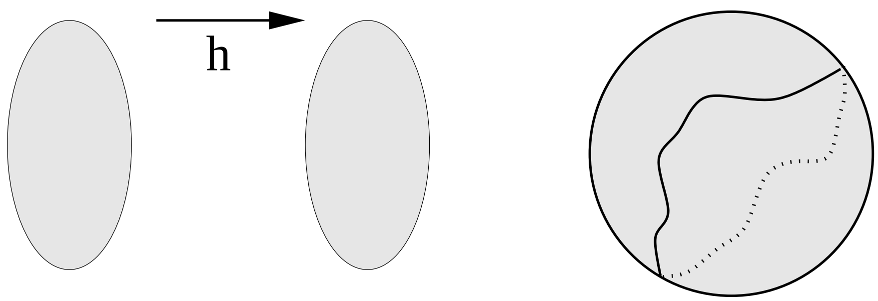


University of Maryland, May 14, 2007

www.math.sunysb.edu/~bishop/lectures

Suppose $h : \mathbb{T} \to \mathbb{T}$ is a orientation preserving homeomorphism. Attach two copies of the disk along their boundaries via h. We get a sphere S^2.



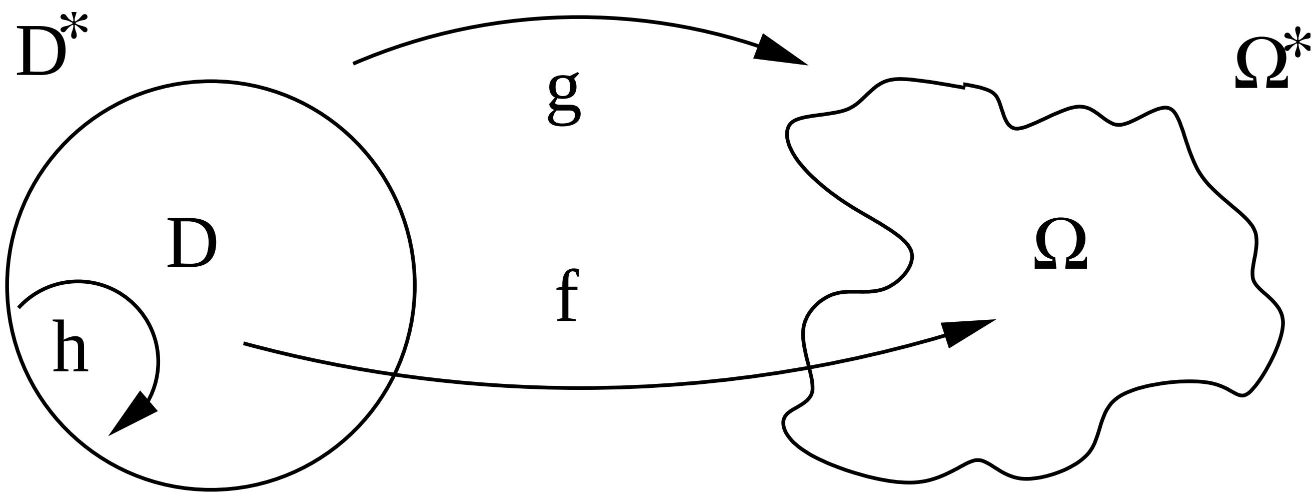
The disks and the sphere all have conformal structures. Can we make the identification respect these structures?

$$\Gamma = \text{ closed Jordan curve}$$

$$S^2 \setminus \Gamma = \Omega \cup \Omega^*.$$

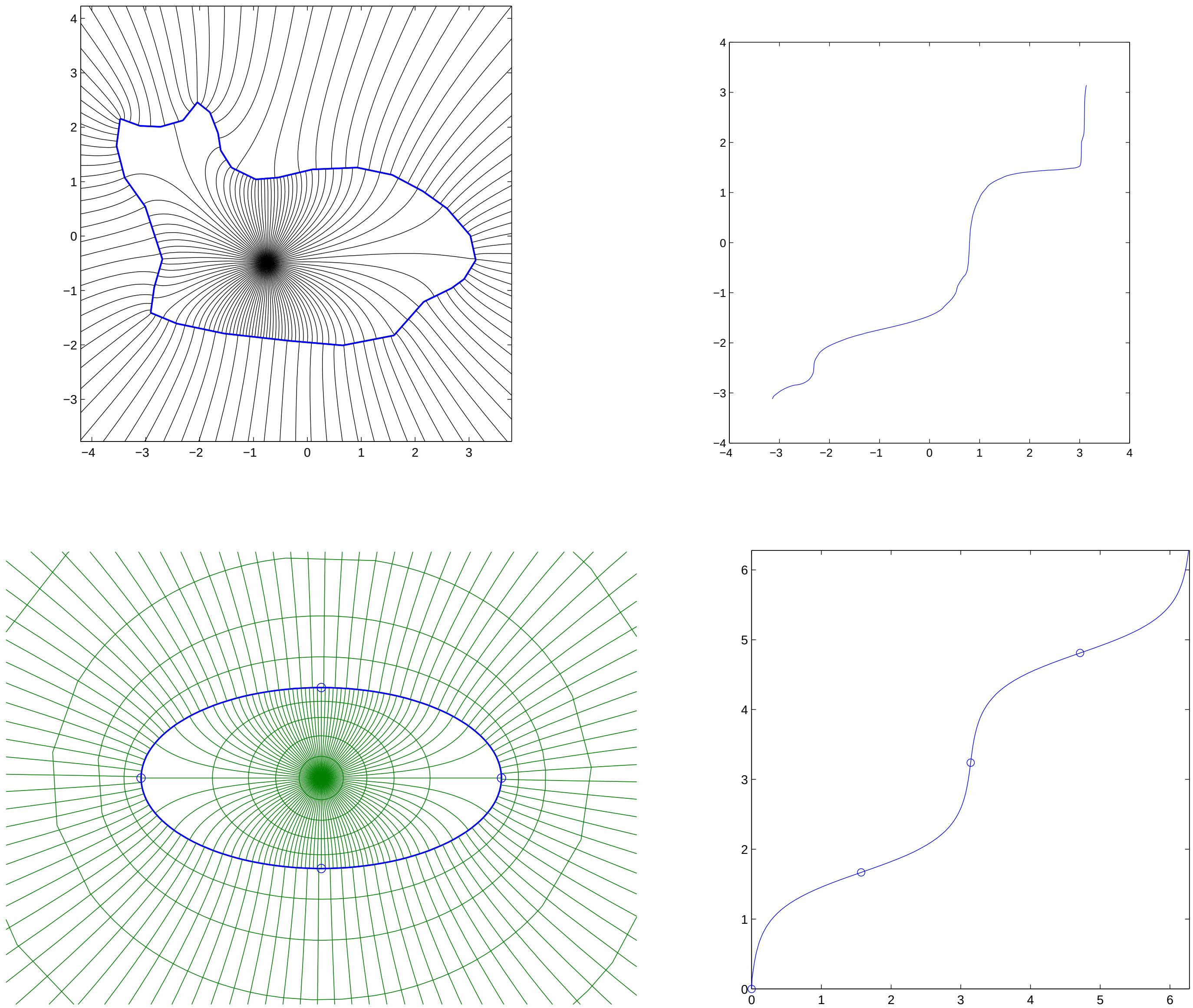
$$f : \mathbb{D} \to \Omega, \quad g : \mathbb{D}^* \to \Omega^* \quad \text{conformal}$$

$$h = g^{-1} \circ f : \mathbb{T} \to \mathbb{T} \quad \text{conformal welding}$$

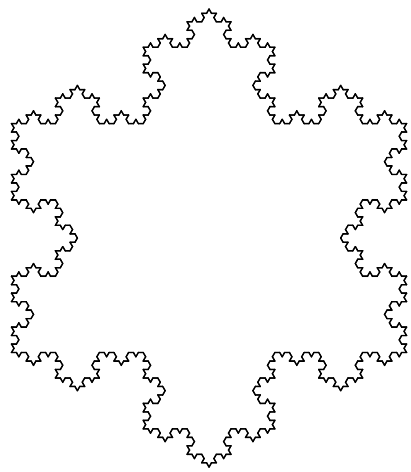


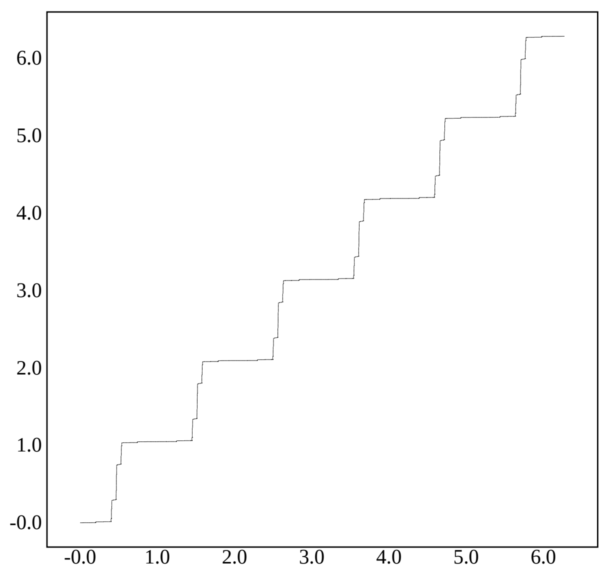
smooth $\Gamma \leftrightarrow$ smooth h is 1-to-1, onto (up to Möbius equivalence).

For general curves, neither 1-to-1 nor onto.



From paper of Mumford and Sharon. Application of conformal welding to pattern recognition.

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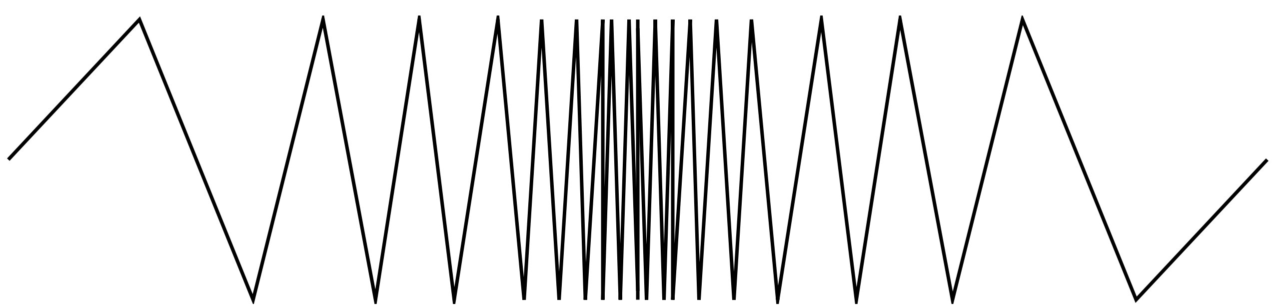
Let K be the closure of the graph of $\sin(1/x)$.

Let F, G be conformal maps from regions above and below K to $\mathbb{D}, \mathbb{D}^*$.

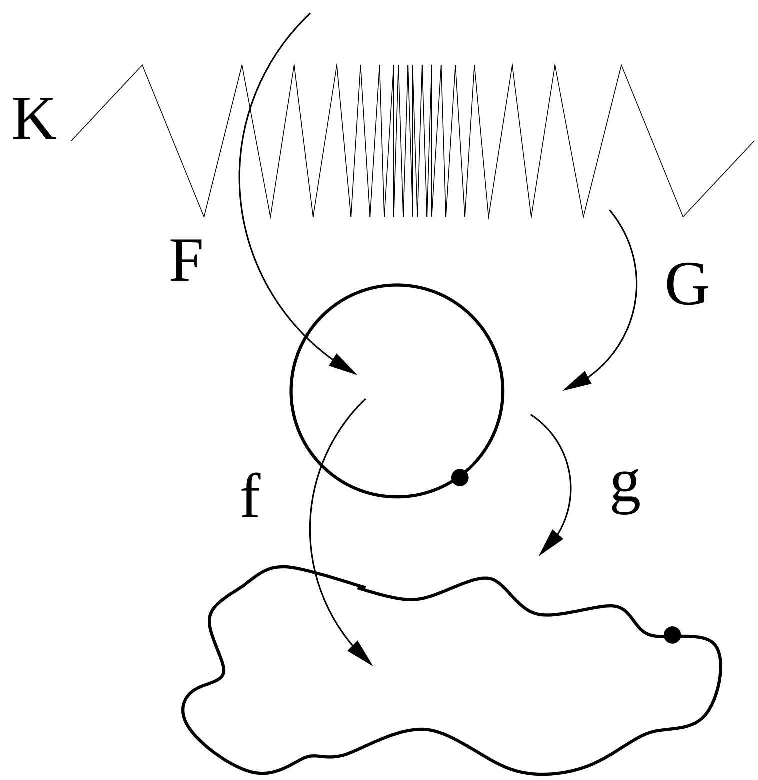
$h = G \circ F^{-1}$ is well defined, continuous and 1-1 except at one point. Thus it extends to a homeomorphism h of circle.



Claim: h is not a conformal welding.

Suppose $h = g^{-1} \circ f$ for some curve Γ. Then $f \circ F$ and $g \circ G$ would be conformal off K and continuous except on the segment $I = [i, -i]$.

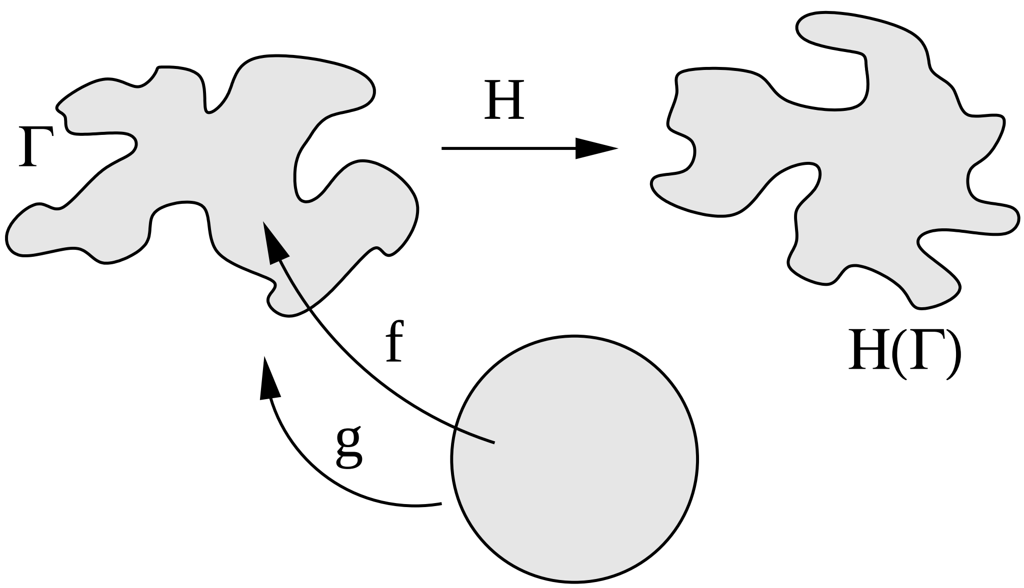
By Morera's theorem they extend to conformal map from complement of I to complement of a point. This contradicts Liouville's theorem.



To see that the map is not 1-to-1 is more difficult.

There are curves Γ and homeomorphisms of the sphere H which are conformal off Γ but which are not Möbius.

Then Γ and $\Gamma' = H(\Gamma)$ are not equivalent, but they do give the same homeomorphism h.



$$h = g^{-1} \circ f = (H \circ g)^{-1} \circ (H \circ f)$$

For example, every curve of positive area (define QC map with non-zero dilatation on Γ).

Non-uniqueness can fail more dramatically. There are h's with corresponding Γ's dense in all closed curves (in Hausdorff metric).

We call these **flexible curves**.

Such example can have $\dim = 1$.

Fundamental thm of conformal welding: quasisymmetric $\Rightarrow$ conformal welding.

h is quasisymmetric (QS) if there is an $M < \infty$.

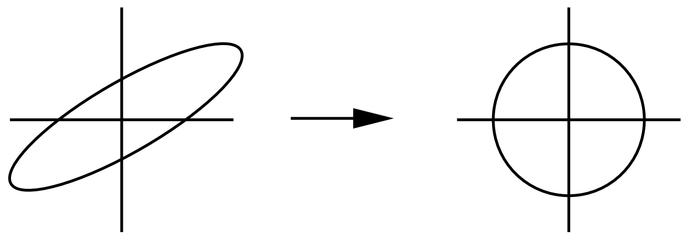
$$\frac{1}{M} \leq \frac{|h(I)|}{|h(J)|} \leq M,$$

for any adjacent intervals I, J of equal length.

By a famous theorem of Ahlfors and Beurling, these are same as boundary values of quasiconformal self mappings of the disk.

First proof of fund thm was by Pfluger (1960) using measurable Riemann mapping theorem.

A map is K-quasiconformal if preimages of small disks are ellipses of eccentricity $\leq K$.



$$f_z = \frac{1}{2}(f_x - i f_y), \quad f_{\bar{z}} = \frac{1}{2}(f_x + i f_y)$$
$$|\mu_f| = |\frac{f_{\bar{z}}}{f_z}| \leq \frac{K-1}{K+1} < 1$$

Measureable Riemann mapping theorem:
If $\|\mu\|_\infty < 1$ then there is a QC f so $\mu_f = \mu$.

- normalized K-QC maps are equicontinuous.
- $f : \mathbb{H} \to \mathbb{H}$ is QC, reflection gives QC map.

Standard proof of fund thm:

$h : \mathbb{T} \to \mathbb{T}$ extends to QC-map $H : \mathbb{R}^2 \to \mathbb{R}^2$.

Let F be QC map with dilatation equal to H's in $\mathbb{D}$, equal to 0 outside disk. Let $\Gamma = F(\mathbb{T})$.

Let $g = F|_{\mathbb{D}^*}$ and $f = F \circ H^{-1}|_{\mathbb{D}}$. THen

$$f : \mathbb{D} \to \Omega, \quad \text{conformal}$$
$$g : \mathbb{D}^* \to \Omega^*, \quad \text{conformal}$$
$$g^{-1} \circ f = H \circ F^{-1} \circ F = H = h \quad \text{on} \quad \mathbb{T}$$

QS map include all diffeomorphisms and include many non-smooth maps ($h' = 0$ a.e. is possible).

Let X be space of all homeomorphisms $\mathbb{T} \to \mathbb{T}$

$$d(f,g) = |\{f \neq g\}| + |\{f^{-1} \neq g^{-1}\}|.$$

QS maps are nowhere dense in this space.

Theorem 1: Weldings are dense in (X, d).

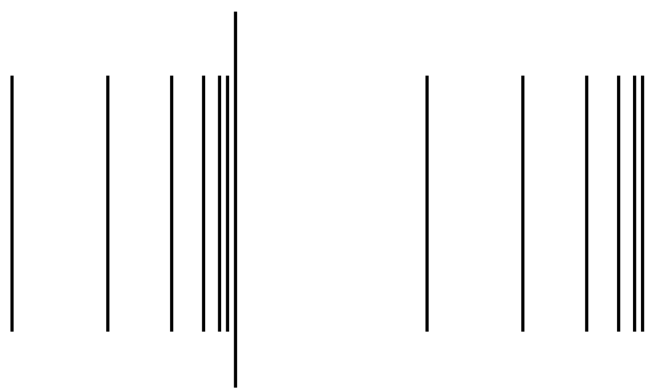
Every circle homeomorphism agrees with a conformal welding except on a set of length $< \epsilon$.

There are homeomorphisms which do not agree with any QS map on positive measure.

(Amusing fact: X is path connected but has no finite length paths.)

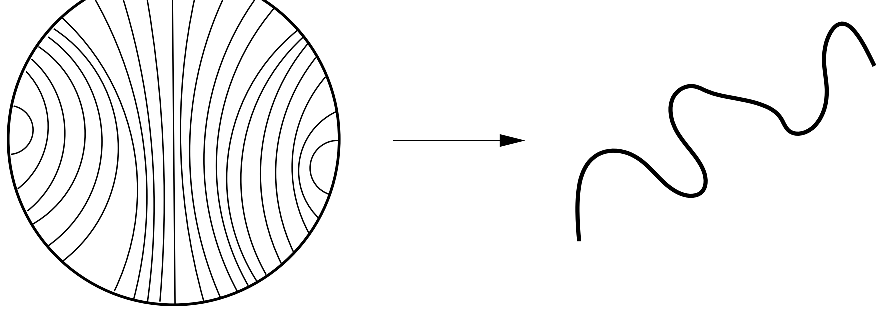
A **decomposition** of a compact set K is a collection of disjoint closed sets whose union is K.

A collection $\mathcal{C}$ of closed sets in the plane is called **upper semi-continuous** if a collection of elements which converges in the Hausdorff metric must converge to a subset of another element.



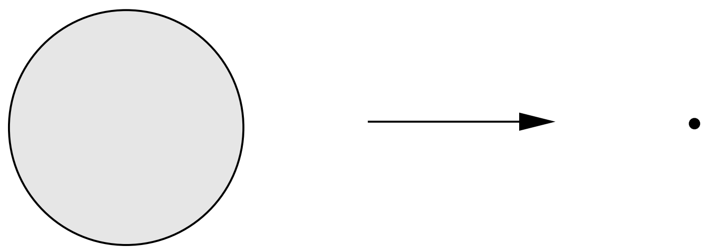
A **Moore decomposition** of the plane is an upper semi-continuous decomposition by compact, connected sets not seperating the plane.

Theorem (R.L. Moore, 1925): The quotient space formed by identifying each element of a Moore decomposition to a point is homeomorphic to $\mathbb{R}^2$.

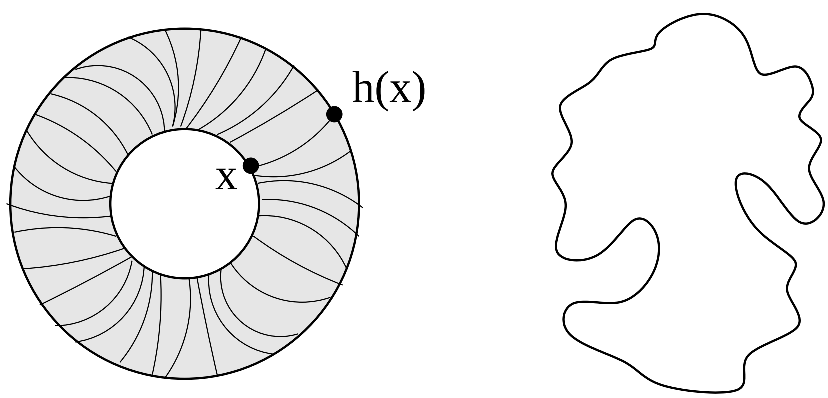


We will call a Moore decomposition **conformal** if the quotient map can be taken to be conformal on $\Omega(\mathcal{C})$, the interior of the set of singletons.

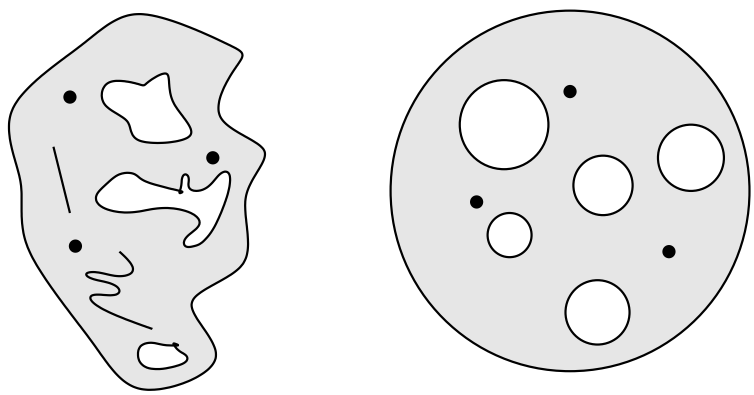
Not every Moore decomposition is conformal, e.g., suppose $\mathcal{C}$ consists of a single closed disk and singletons. If this were conformal we would contradicts Liouvilles's theorem.



If every Moore decomposition was conformal, then every cirle homeomorphsim would be a conformal welding.



Koebe's circle domain theorem (1908): Every finitely connected domain can be conformally mapped to a Koebe domain (components of Ω^c are disks and points).



Koebe's Conjecture: Every planar domain can be mapped to a Koebe domain.

Koebe's conjecture is known for some special cases. He and Schramm proved it for countably many boundary components (1993).

A **Koebe decomposition** is a Moore decomposition using only disks and points.

General Koebe Conjecture: Every Moore decomposition is conformally equivalent to a Koebe decomposition.

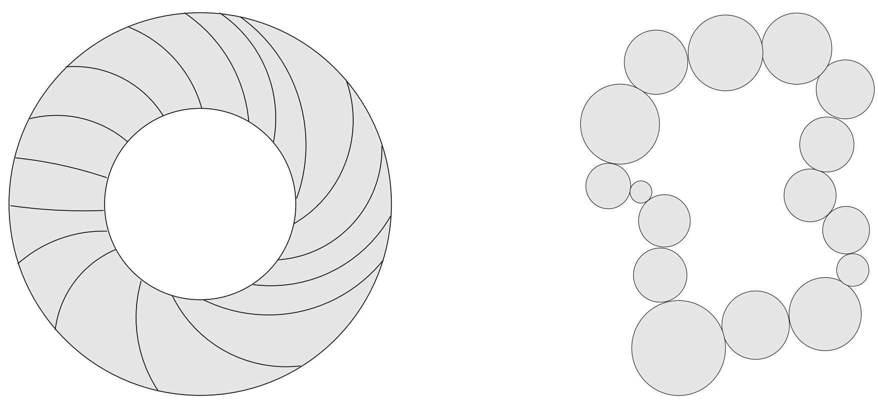
All but countable many components map to points.

This contains Koebe's conjecture. If Ω is connected then decomposition of $S^2 \setminus \Omega$ into its connected components is a Moore decom. (Moore 1925).

It would also be nice to have conditions which imply all elements collapse to points.

Koebe's thm gives conformal weldings:

- Connect points $\{x_k\}_1^n \subset \mathbb{T}$ by disjoint curves $\{\gamma_n\}$ to the points $2h(x_k) \in \{|z| = 2\}$.
- Let Ω be $\mathbb{D} \cup 2\mathbb{D}^*$ plus ϵ-nbhd of each γ_n.
- Ω can be mapped to a circle domain.
- As $\epsilon \to 0$, get chain of tangent circles.



If chains stay inside $\{|z| < R\}$, then there are $\leq (R/\epsilon)^2$ disks of size $> \epsilon$, independent of n.

Requires mild condition on h ($\text{cap}(E) = 0 \Rightarrow \text{cap}(h(E^c)) > 0$, i.e., h not log-singular).

Theorem 1: For any o.p. homeomorphism h there are non-degenerate sequences

$$f_n : \mathbb{D} \to \Omega_n, \quad g_n : \mathbb{D}^* \to \Omega_n^*$$

of conformal maps onto disjoint domains with

$$|f_n(x) - g_n(h(x))| \to 0,$$

for all $x \in \mathbb{T}$ minus a countable set.

Conjecture 2: Suppose $h : \mathbb{T} \to \mathbb{T}$ is any o.p. homeomorphism. Then there are conformal maps f, g onto disjoint domains such that $f = g \circ h$ on $\mathbb{T}$ minus a countable set.

Follows from the generalized Koebe Conjecture.

The problem is passing to the limit. We can have

$$\lim_n f_n = f, \lim_n g_n = g, \quad \text{unif. on compacta}$$

$$f_n(x) = g_n(h(x)) \quad \forall n,$$

but still $f(x) \neq g(x)$ some $x \in \mathbb{T}$.

Moreover, we only know f and g have radial limits off a set of zero logarithmic capacity.

If $f_n \to f$ uniformly on compacta, then there is subsequence so $f_n(x) \to f(x)$ for all $x \in \mathbb{T}$ except a set of zero K-capacity, any K with $\log \frac{1}{t} = o(K(t))$. False for log-capacity. (Hamilton, Lundberg).

Capacity: For $\mu > 0$ define

$$I_K(\mu) = \iint K(|z-w|)d\mu(z)d\mu(w).$$

$$\text{cap}_K(E) = \sup\{I(\mu)^{-1} : \mu \text{ prob. meas. on} E\}.$$

$K = \log \frac{1}{t}$ gives log-capacity. For $E \subset \mathbb{T}$, cap $>$ 0, monotone and countably subadditive.

$E \subset \mathbb{T}$ has zero logarithmic capacity iff

(i) Planar Brownian motion never hits E a.s.

(ii) Extremal length from $D(0, 1/2)$ to E is ∞.

(iii) E is the zero set of some conformal map f.

(iv) $\exists$ conformal f with no radial limits on E.

cap$(E) = 0$ implies $\dim(E) = 0$ (very small!).

Suppose

$$f : \mathbb{D} \to \Omega$$
$$g : \mathbb{D}^* \to \Omega^*$$

map onto disjoint simply connected domains.

If $E \subset \mathbb{T}$ and $h = g^{-1} \circ f$ on E (radial limits), h is a generalized conformal welding on E.

David Hamilton (1991) introduced this idea.

Try to pass to limit in Theorem 2. Lose control of boundary values on sets of log capacity zero.

Theorem 3: Every o.p. homeomorphism h is a generalized conformal welding on $\mathbb{T} \setminus (F_1 \cup F_2)$, where F_1 and $h(F_2)$ have zero log capacity.

Corollary: If h preserves sets of zero capacity then it is a generalized welding except on a set of zero capacity.

Corollary: If h is log-regular then h is a generalized welding except on a set of Lebesgue measure zero. ($\text{cap}(E) = 0$ implies $|h(E)| = 0$.)

Theorem 3 gives no information if h is log singular, i.e., $\mathbb{T} = F_1 \cup F_2$, with $\mathrm{cap}(F_1) = \mathrm{cap}(h(F_2)) = 0$. But amazingly in this case we have

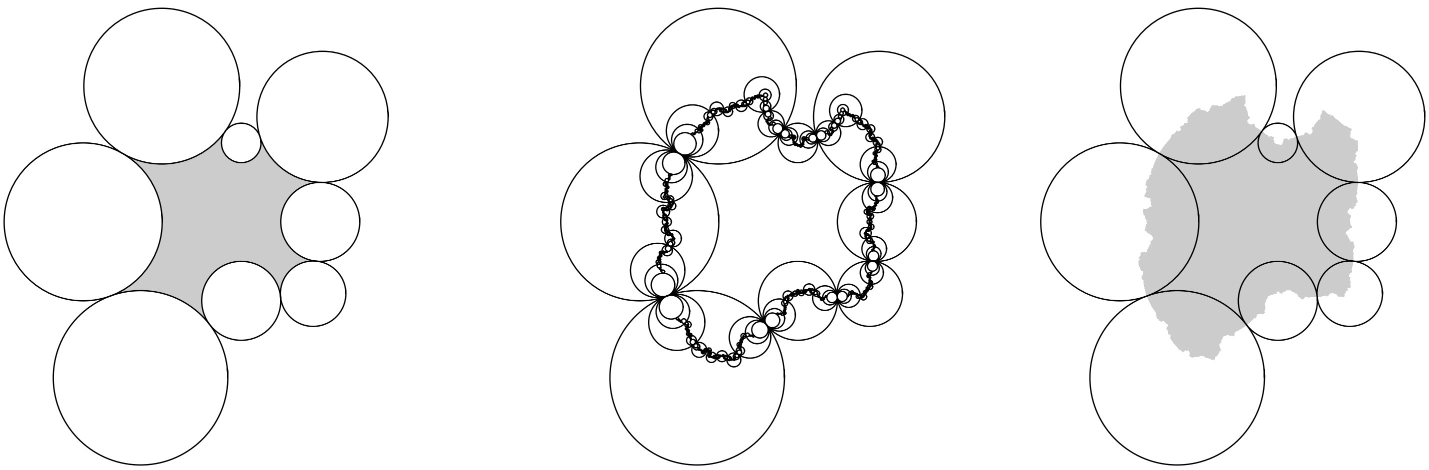
Theorem 4: h is the welding of a flexible curve iff h is log singular.

Thus we know h is a (classical) conformal welding if it is either tame enough or wild enough!

Log-regular and log-singular maps are distance one apart in X. To show CW is dense in X, take and h and divide $\mathbb{T}$ into sets where h looks log-singular or log-regular, plus set of small length. Combine results to get CW map which agrees with h except on small measure.

Koebe's theorem ⇒ Fund. Thm. of CW:

A circle chain defines a Jordan curve Γ (limit set of corresponding reflection group).

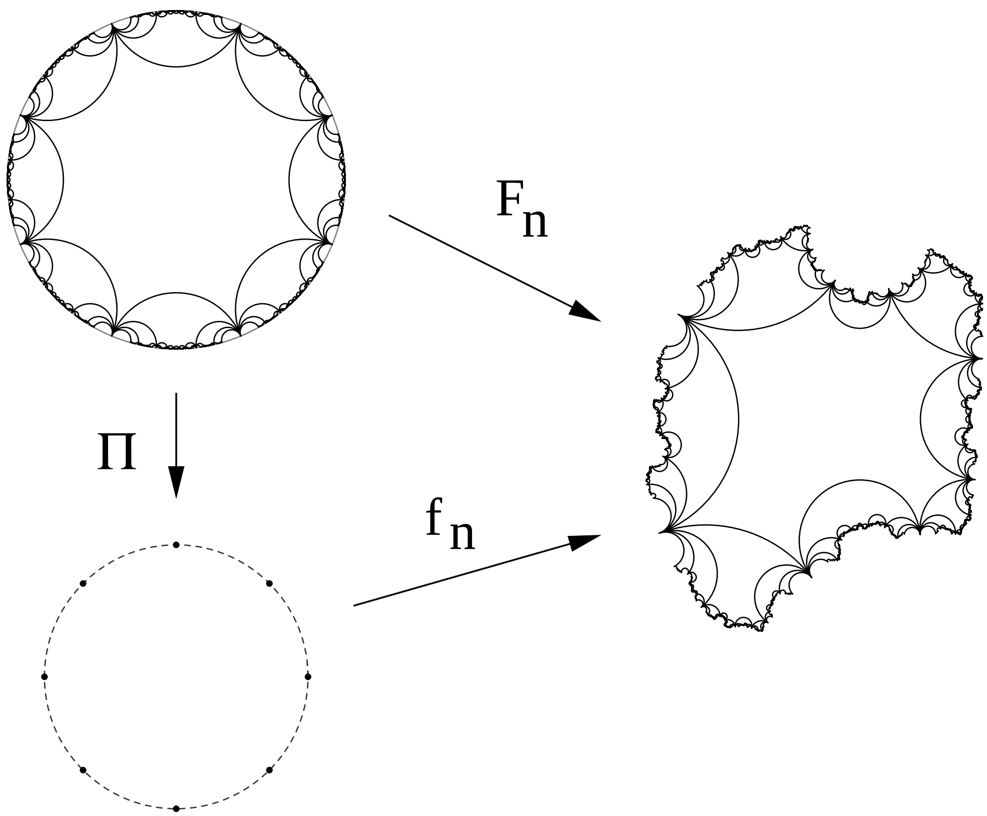


Must show:

• Conformal welding of curve agrees with h at tangents of circle chain.

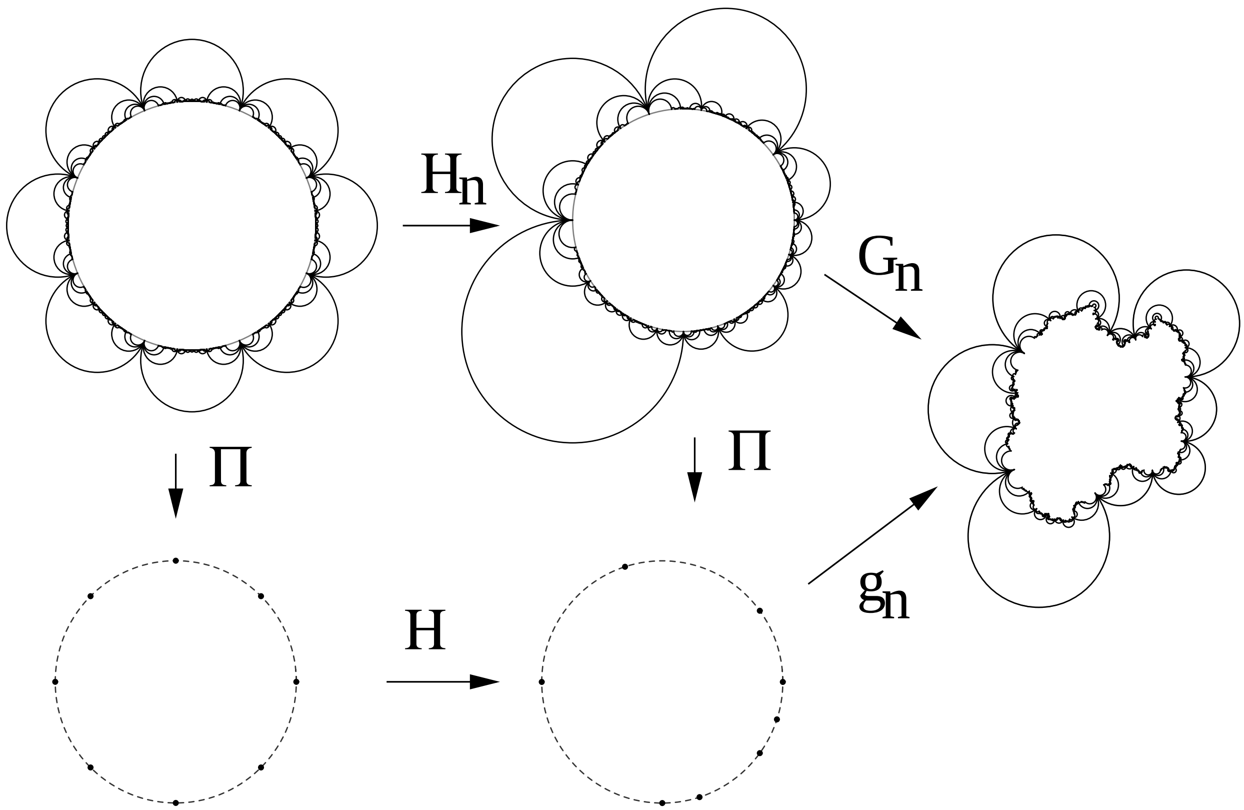
• If h has K-QC extension H, Γ is K-quasicircle

• As $n \to \infty$ we get a K-QC limit curve.



Conformal $f_n : \mathbb{D} \to \Omega_n$ can be extended by reflection to a map of the universal cover of $W_n = S^2 \setminus \{x_1, \ldots, x_n\}$ to D_n (the inside of Γ).

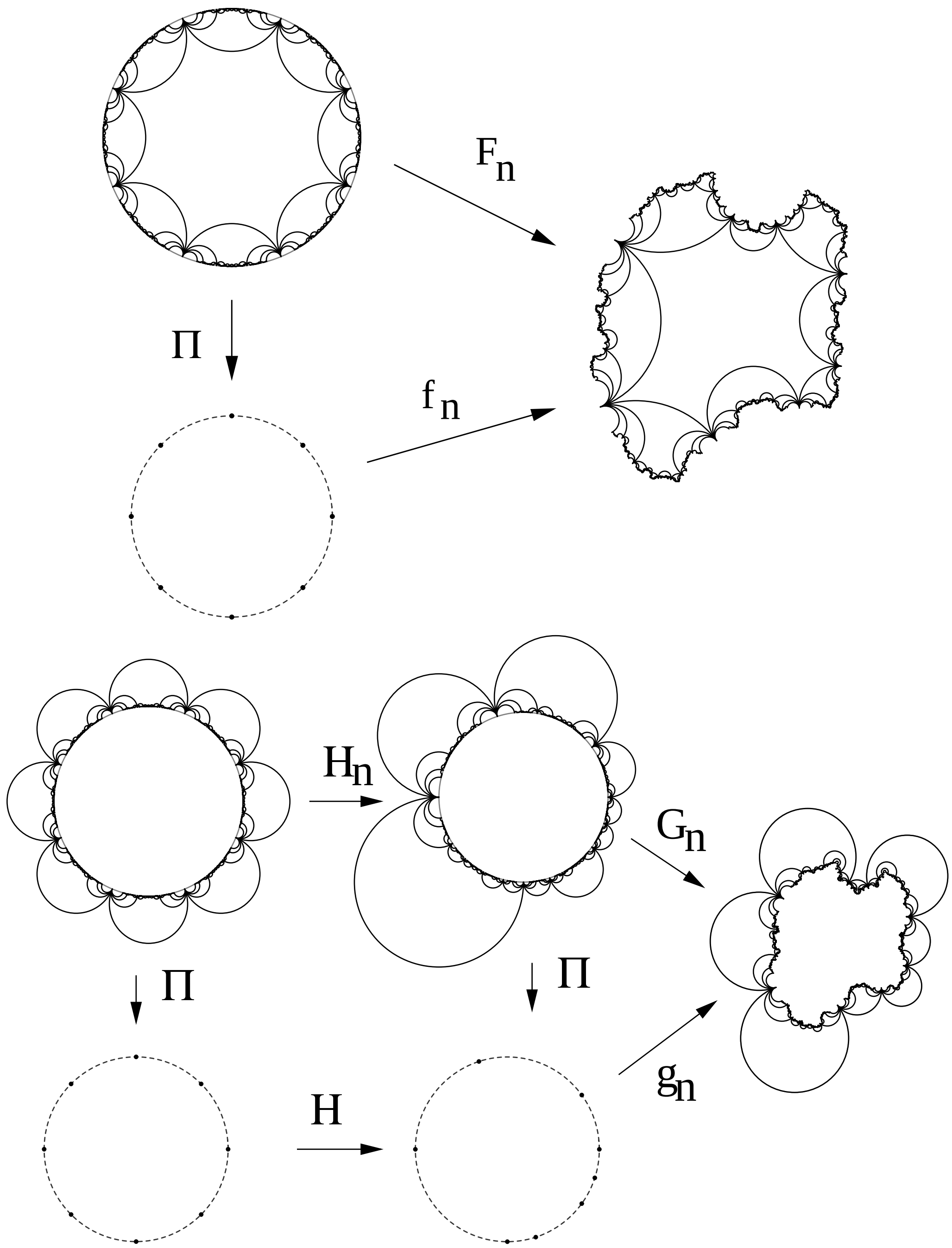
Since the universal cover can be identified with $\mathbb{D}$, we get a map of $\mathbb{D}$ to D_n.



Same construction gives a conformal map from $\mathbb{D}^*$ to D^* (the outside of Γ).

Lift H to the universal cover of $S^2 \setminus \{y_1, \ldots, y_n\}$ (use reflections). Get a K-QC map of $\mathbb{D}^* \to \mathbb{D}^*$.

Inside and outside maps agree on $\mathbb{T}$. Gives Γ as K-QC image of circle.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