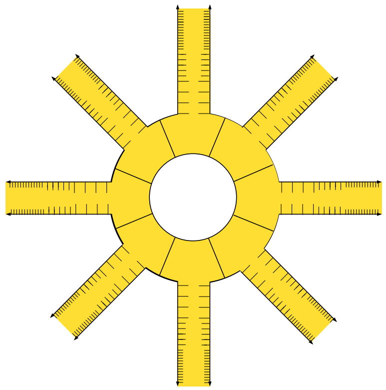
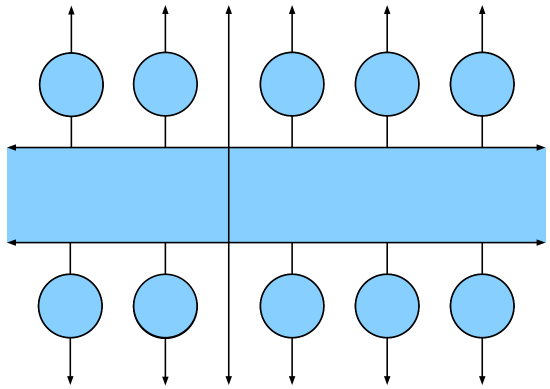
TREES, TRIANGLES AND TRACTS II

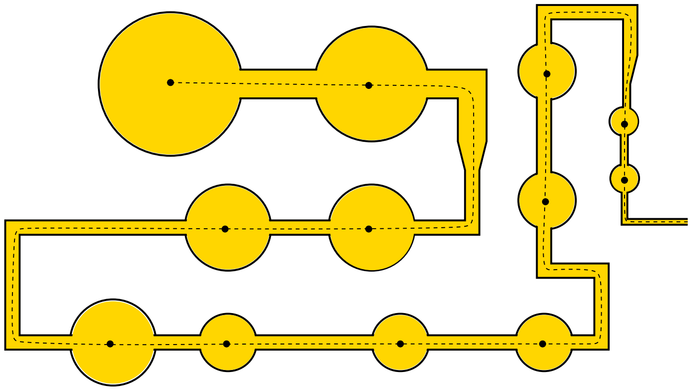
Christopher Bishop, Stony Brook

TOPICS IN COMPLEX DYNAMICS 2021
Transcendental dynamics and beyond, April 19-23, 2021

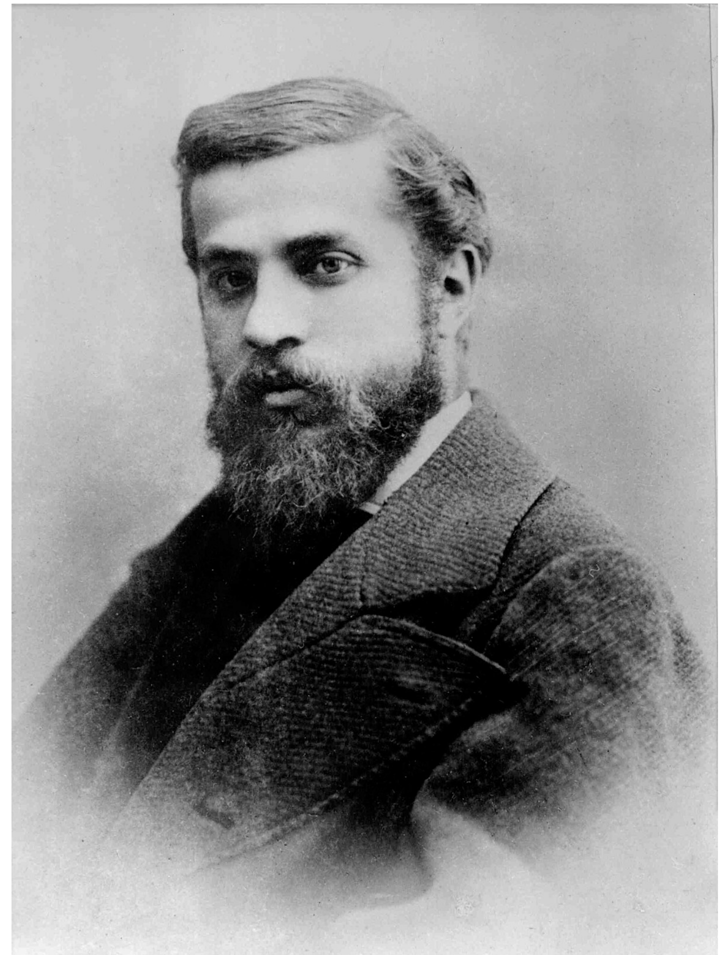
www.math.sunysb.edu/~bishop/lectures





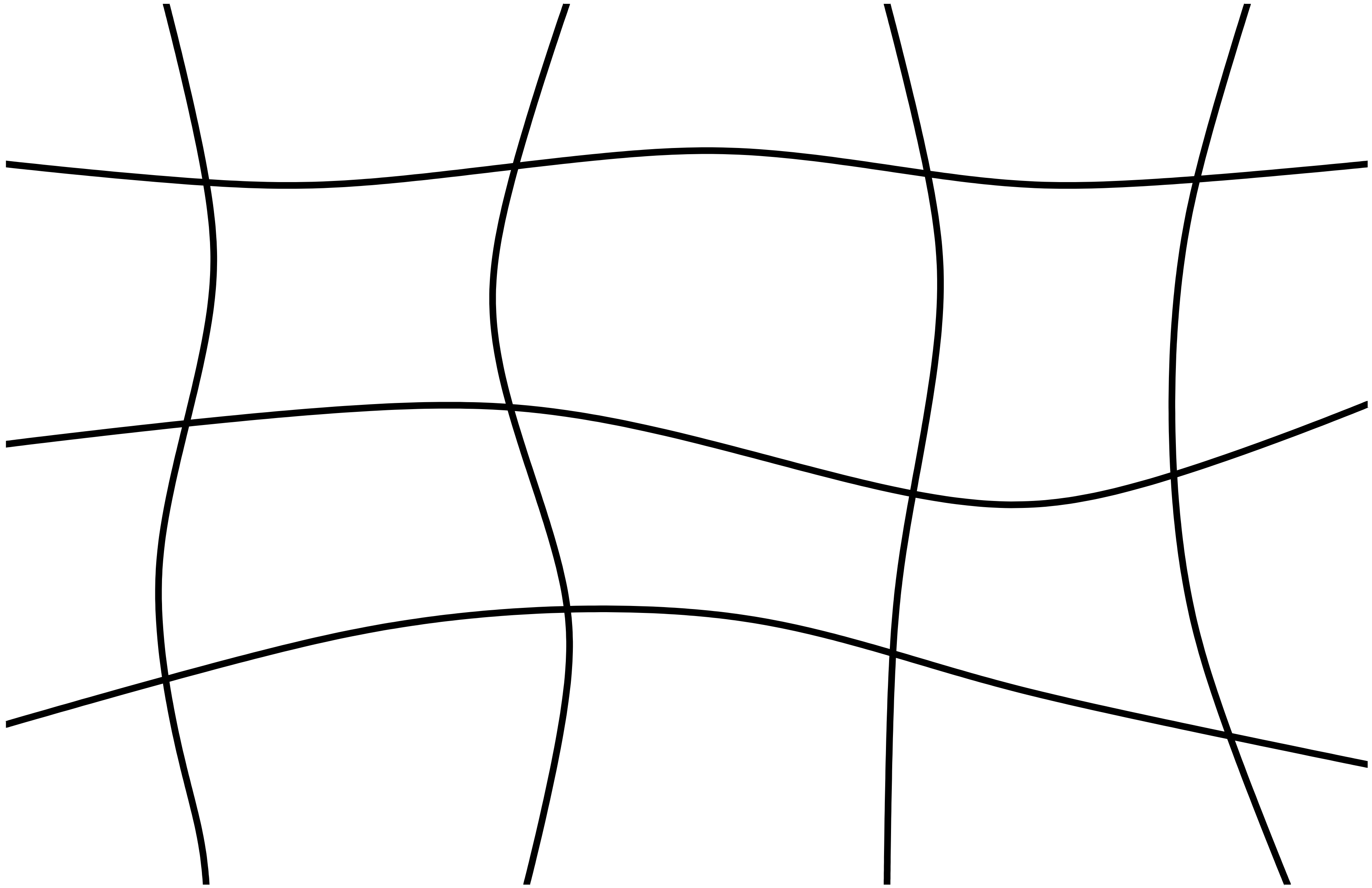


Basílica de la Sagrada Família
1882–present

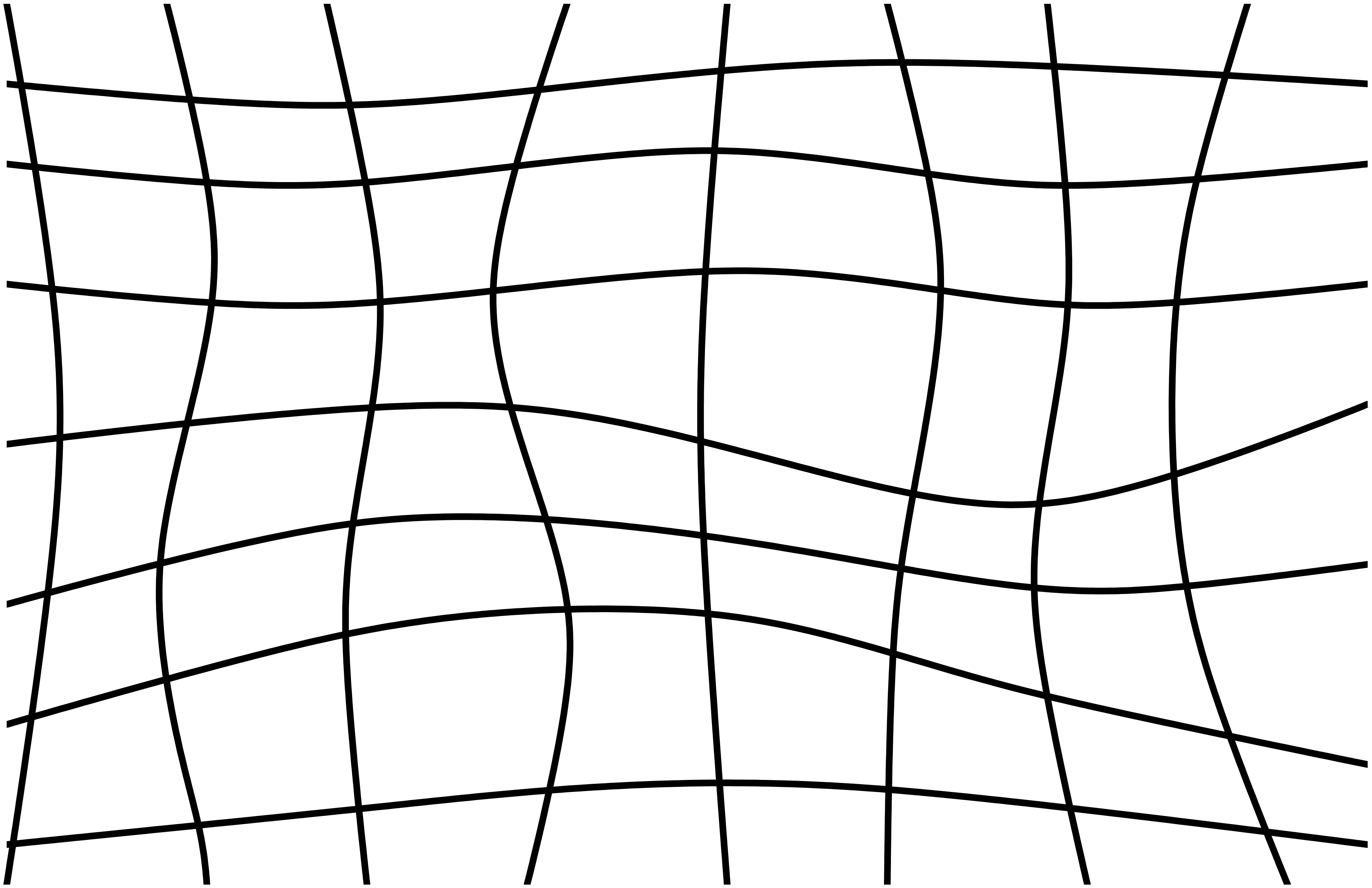


Antoni Gaudí i Cornet
1852–1926

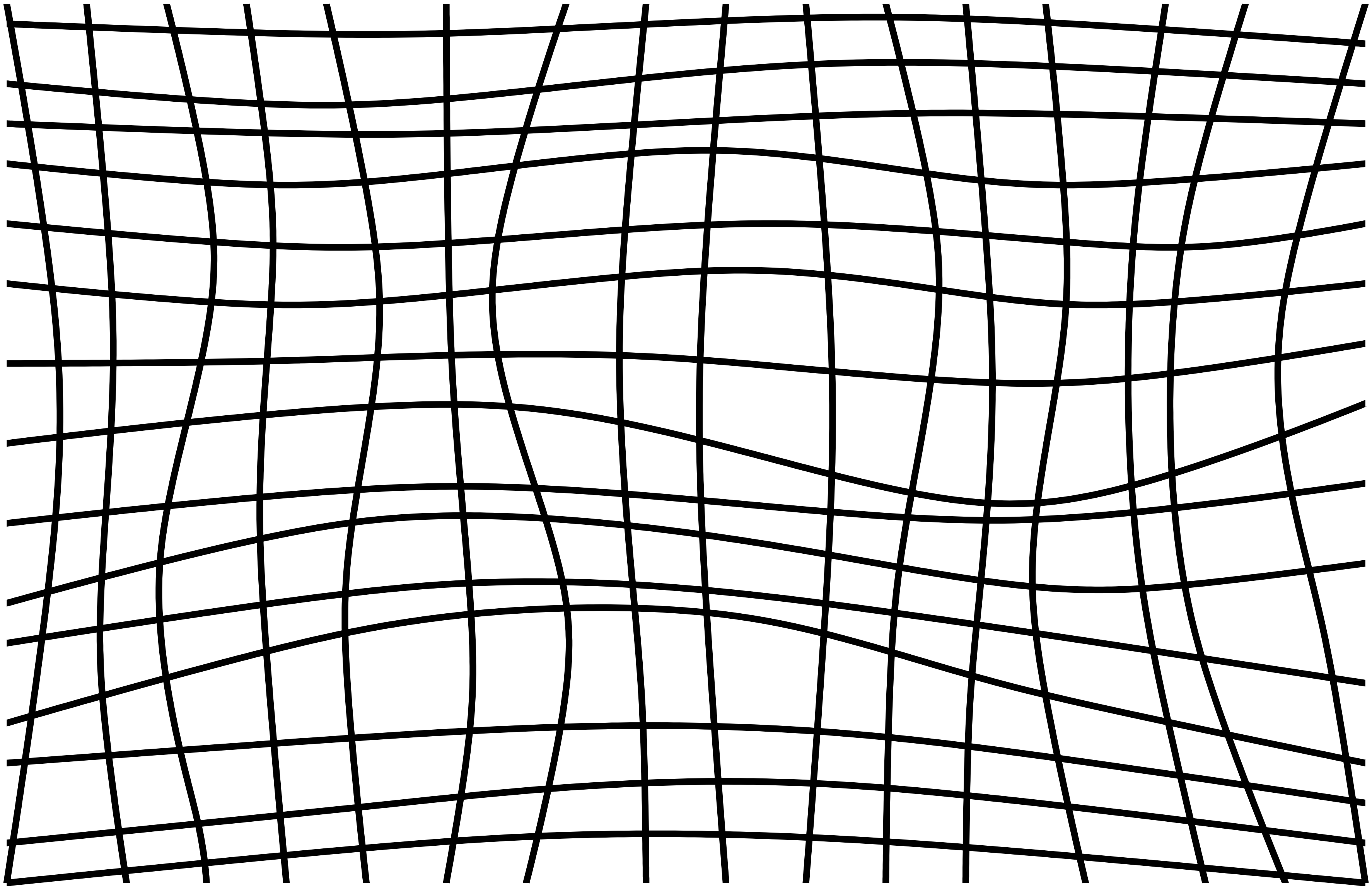
"The tiles made me want to talk about them .... so had I to find something mathematical to say related to them." D. Sullivan



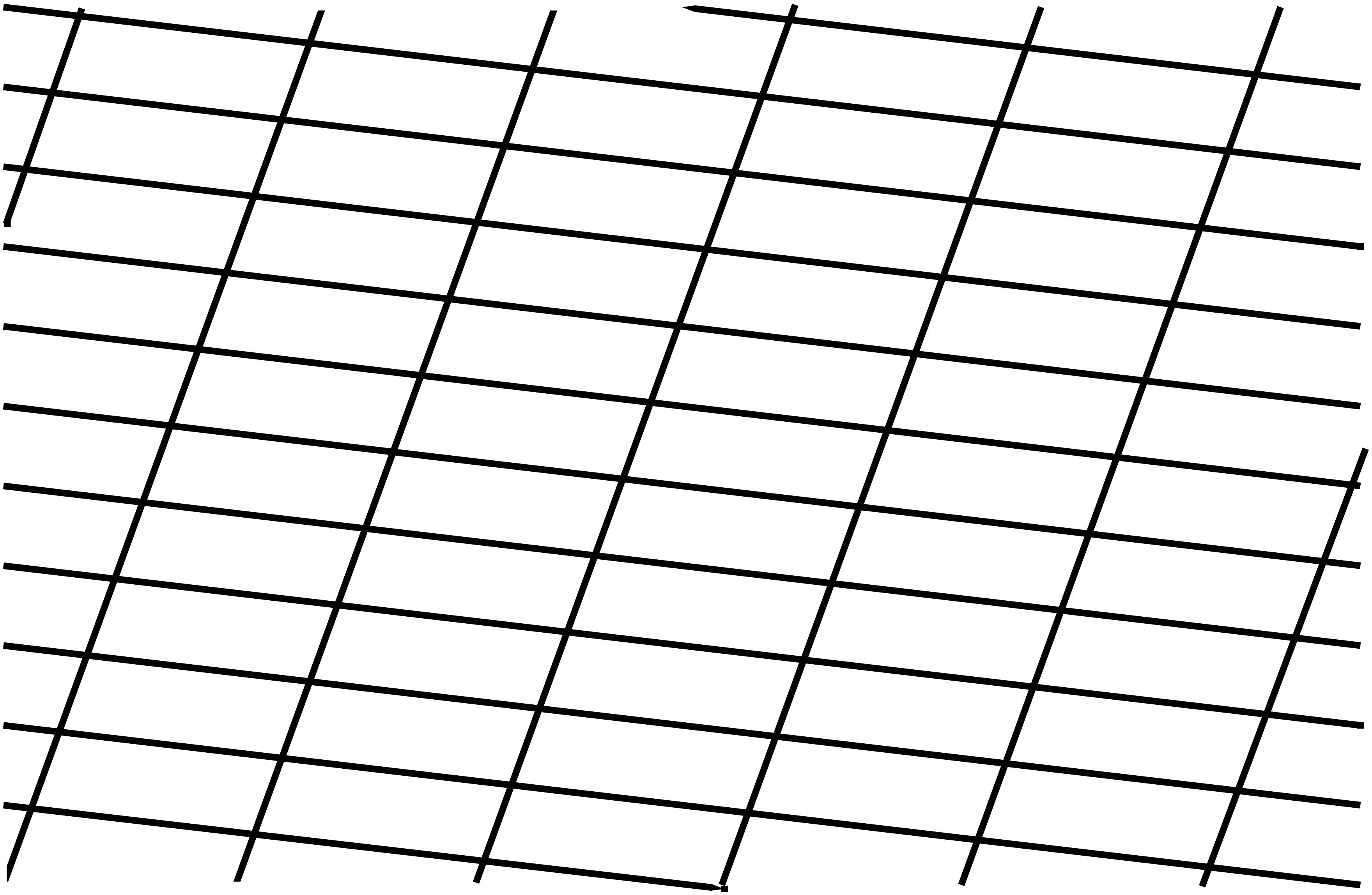
"The tiles made me want to talk about them .... so had I to find something mathematical to say related to them." D. Sullivan



Consider sequence of nested quadrilateral meshes.
All combinatorially $= \mathbb{Z} \times \mathbb{Z}$. Bounded geometry tiles.



Choose a random point z in plane and rescale around that point.



“Gaudi’s Thm”: Rescaled meshes converge to lattice for a.e. z.

Dennis Sullivan, 1941–present, 1000 tiles

Dennis Sullivan, 1941–present, 2000 tiles

Dennis Sullivan, 1941–present, 4000 tiles

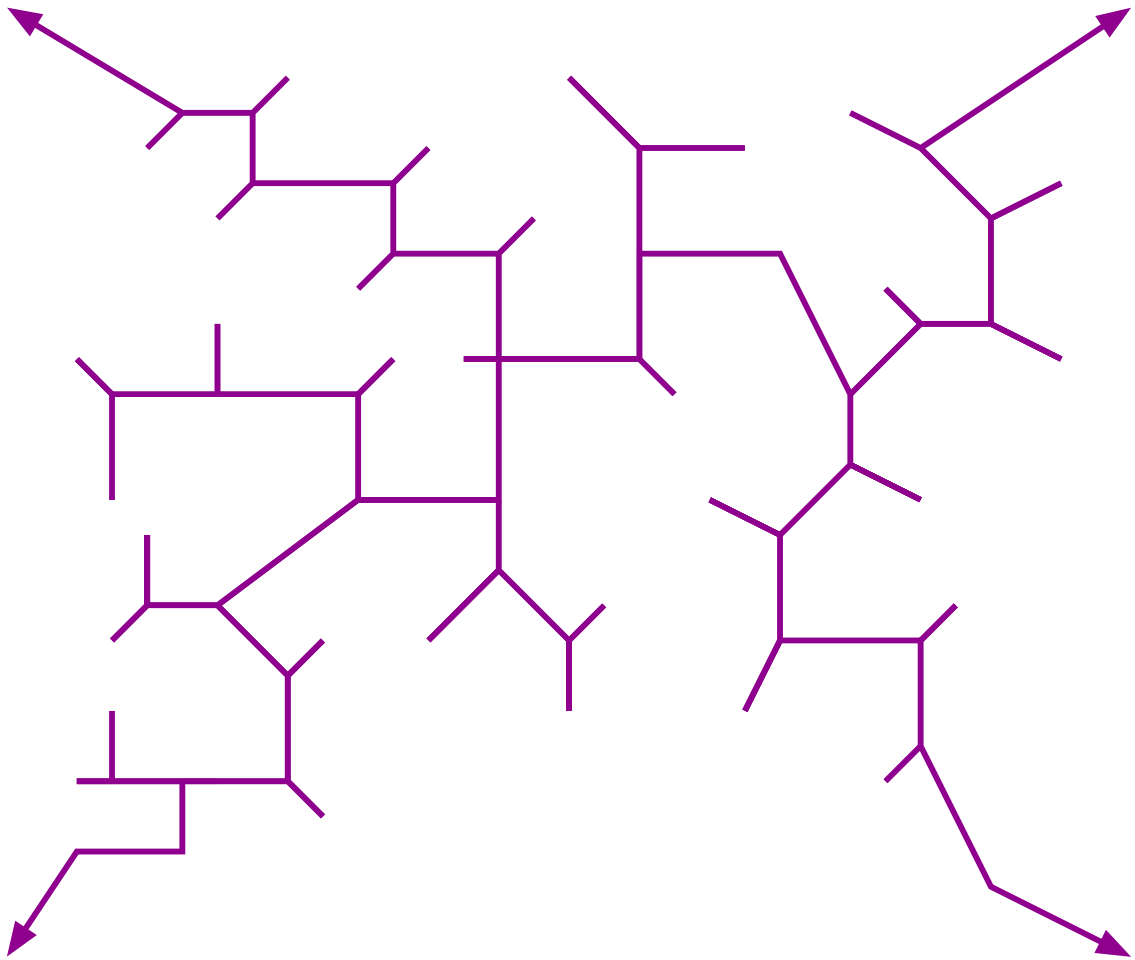
Dennis Sullivan, 1941–present, 8000 tiles

In the first talk we saw that

- Every finite planar tree has a true form.
- Every tree can be approximated in Hausdorff metric by true trees.

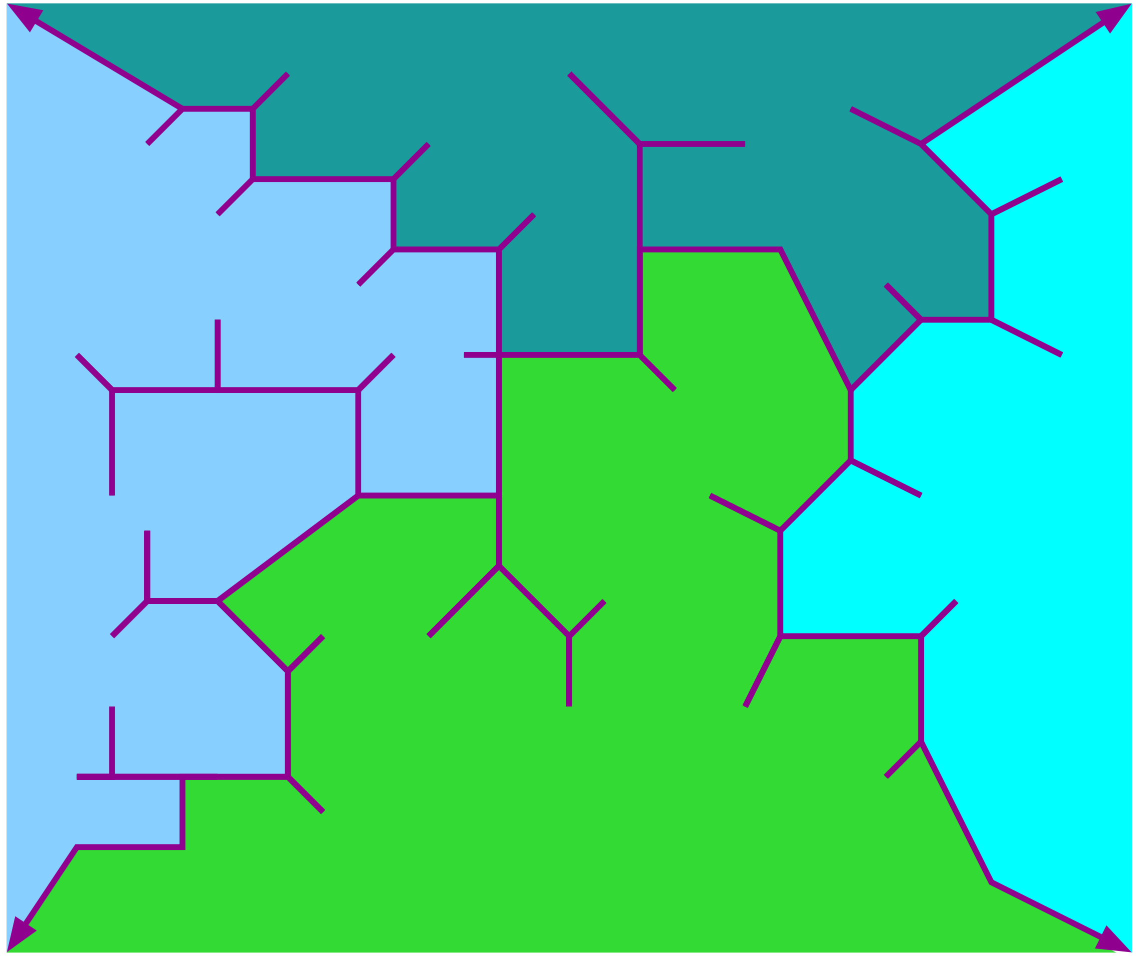
What about infinite trees?

- What is the true form on an infinite planar tree?
- What trees can we approximate by infinite true trees?
- What is this good for?



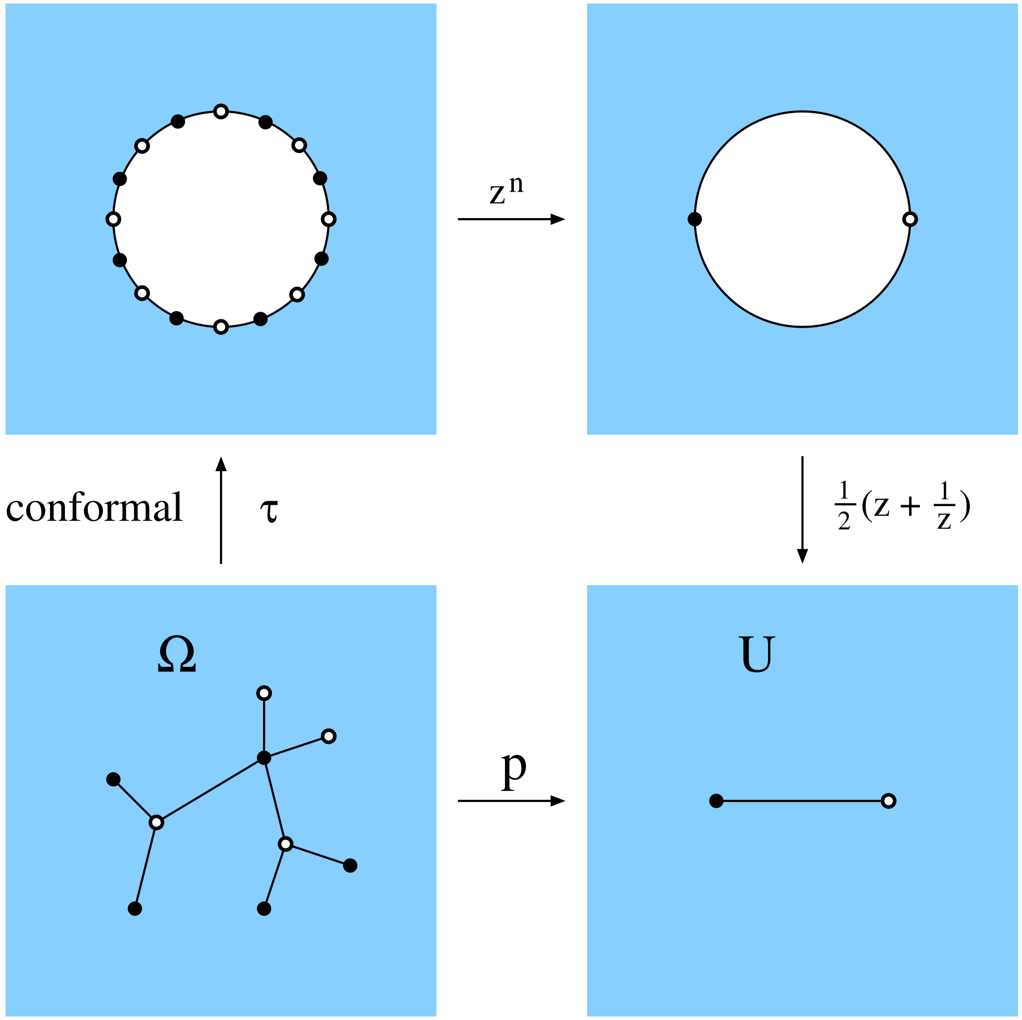
Finite trees correspond to polynomials with 2 critical values.

Do infinite trees correspond to entire functions with 2 critical values?



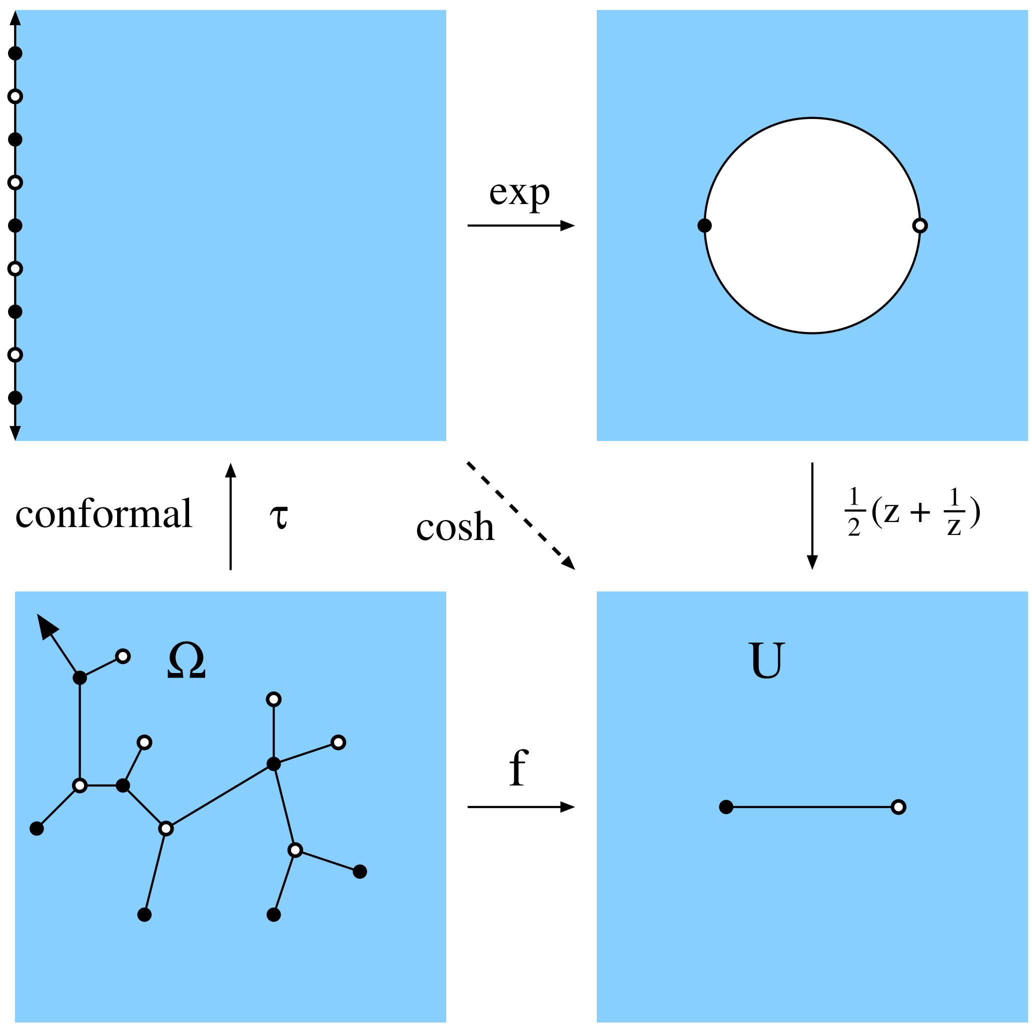
Main difference:
$\mathbb{C}\setminus$ finite tree = one topological annulus
$\mathbb{C}\setminus$ infinite tree = many simply connected components

Recall finite case

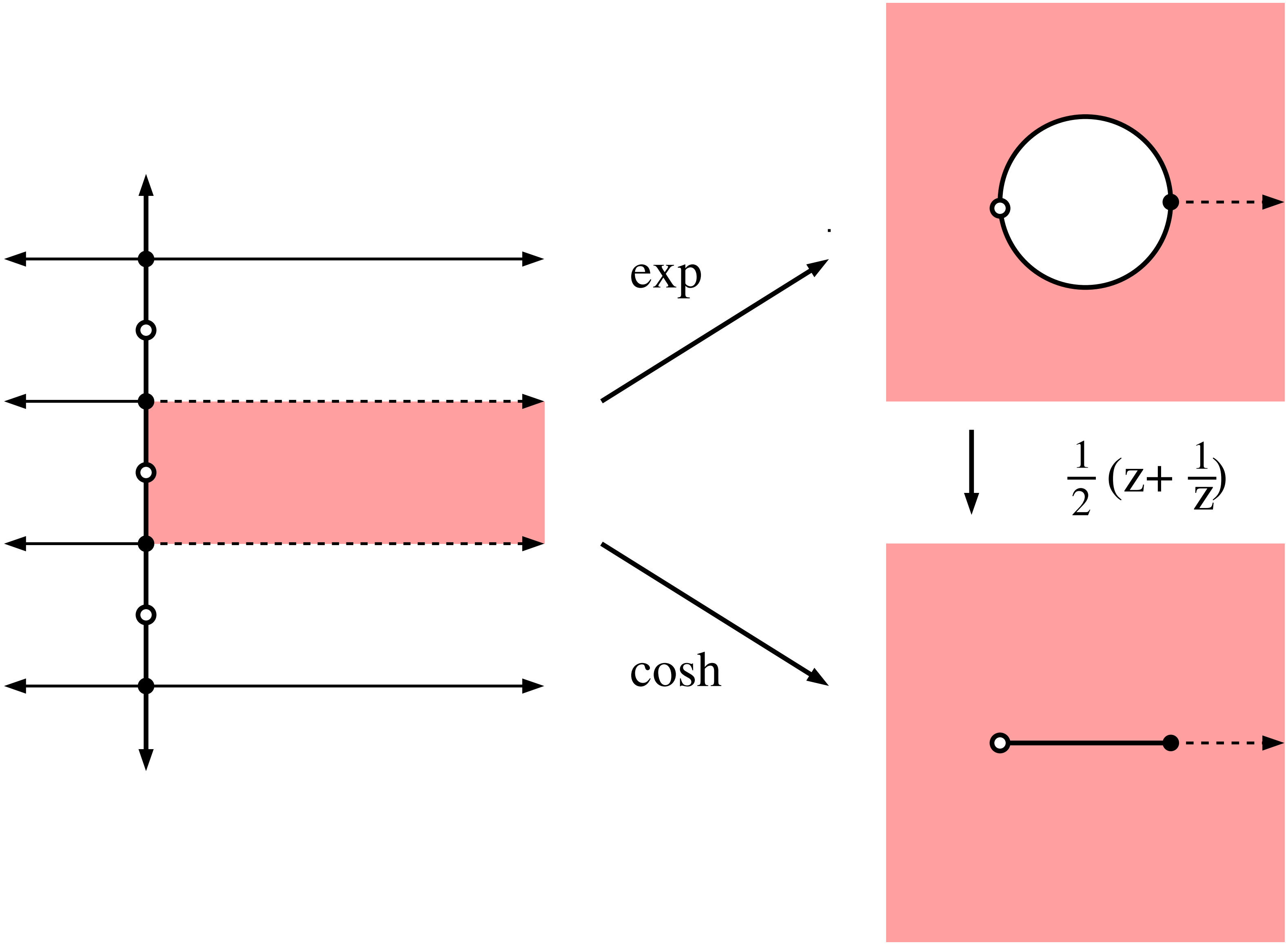


T is true tree $\Leftrightarrow p = \frac{1}{2}(\tau^n + 1/\tau^n)$ is continuous across T.

Infinite case



Infinite balanced tree $\Leftrightarrow$ $f = \cosh \circ \tau$ is continuous across T.



Definitions of exp and cosh.

We need three definitions before stating our result.

These adapt “obvious” properties of finite trees to infinite case.

(1) Tree neighborhoods: replaces Hausdorff metric ϵ-neighborhoods.

(2) Bounded geometry: nearby edges have comparable sizes.

(3) τ-lower bound: lower bound for measure of edges.

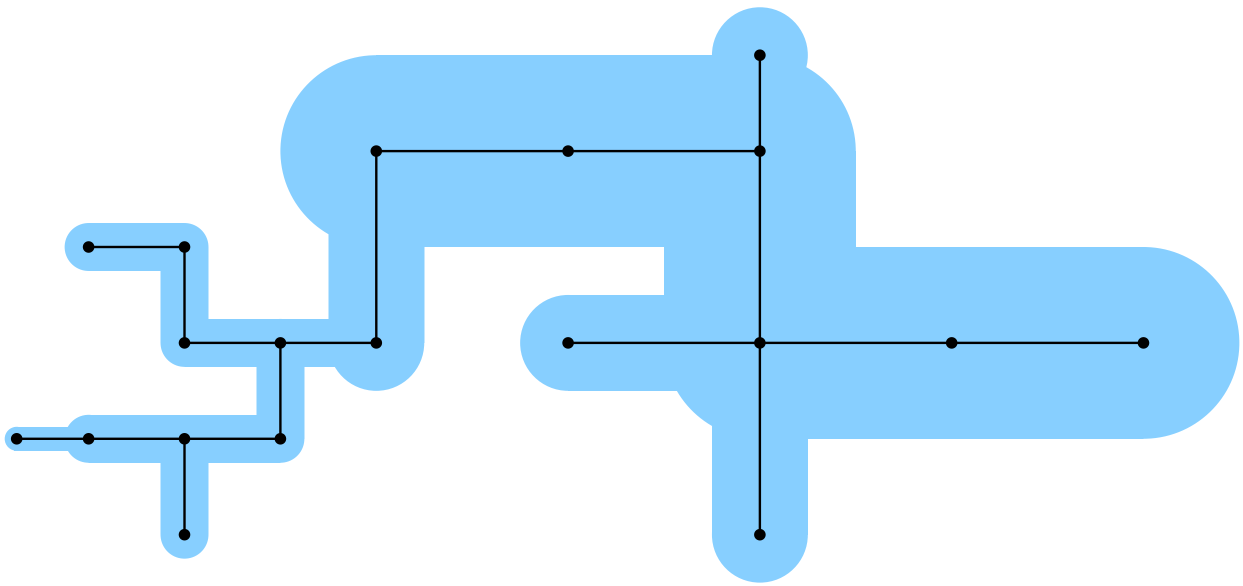
If e is an edge of T and $r > 0$ let

$$e(r) = \{z : \mathrm{dist}(z, e) \leq r \cdot \mathrm{diam}(e)\}$$

If e is an edge of T and $r > 0$ let

$$e(r) = \{z : \mathrm{dist}(z, e) \leq r \cdot \mathrm{diam}(e)\}$$

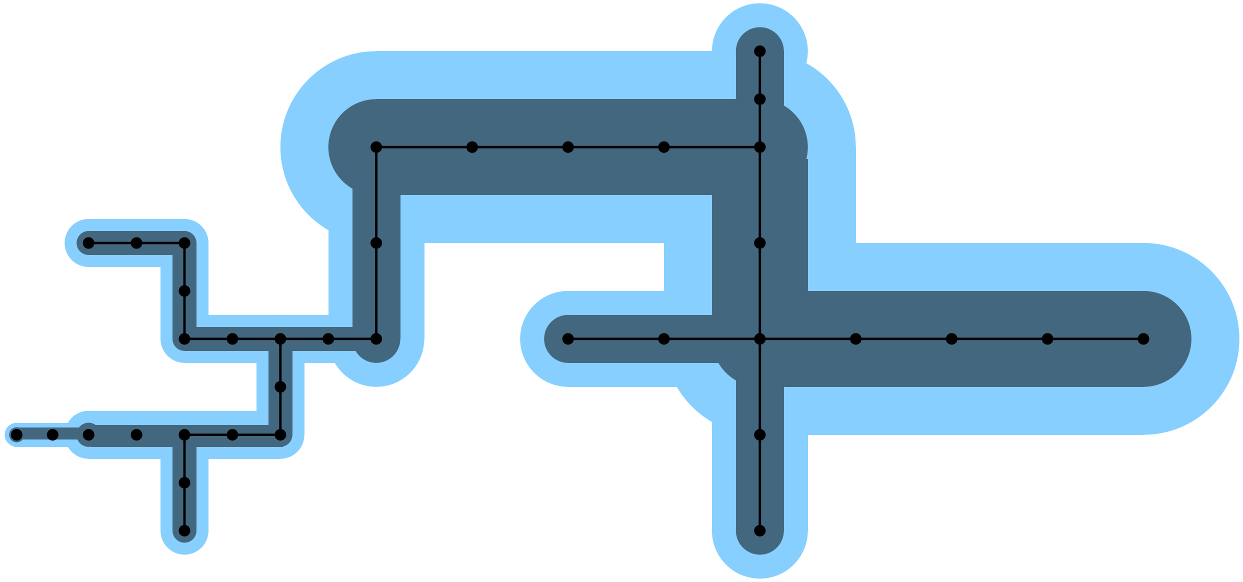
Define neighborhood of T: $T(r) = \cup\{e(r) : e \in T\}$.



If e is an edge of T and $r > 0$ let

$$e(r) = \{z : \mathrm{dist}(z, e) \leq r \cdot \mathrm{diam}(e)\}$$

Define neighborhood of T: $T(r) = \cup\{e(r) : e \in T\}$.

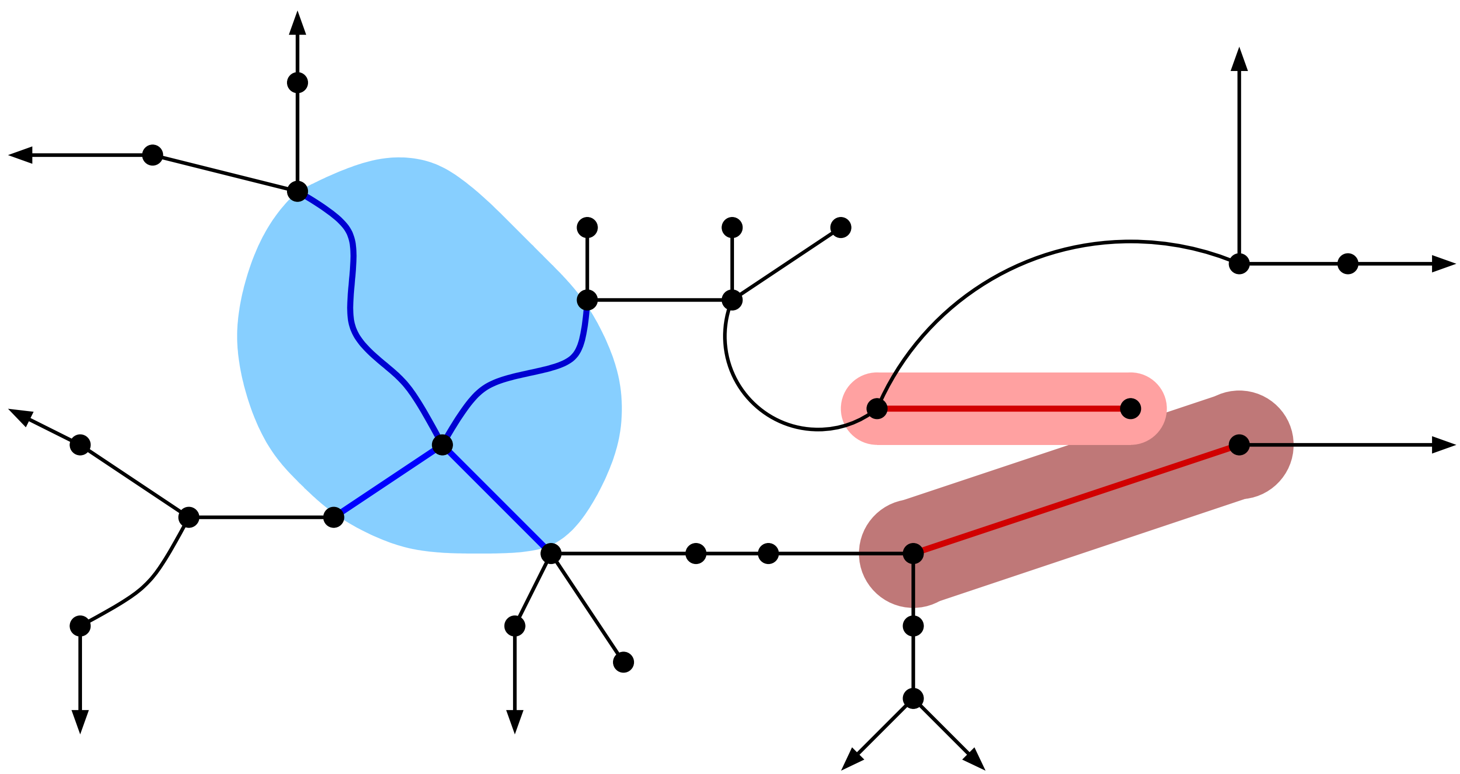


Adding vertices reduces $T(r)$. Useful scaling property.

Bounded Geometry (local condition; easy to verify):

- edges are uniformly smooth.
- adjacent edges form bi-Lipschitz image of a star $= \{z^n \in [0, r]\}$
- non-adjacent edges are well separated,

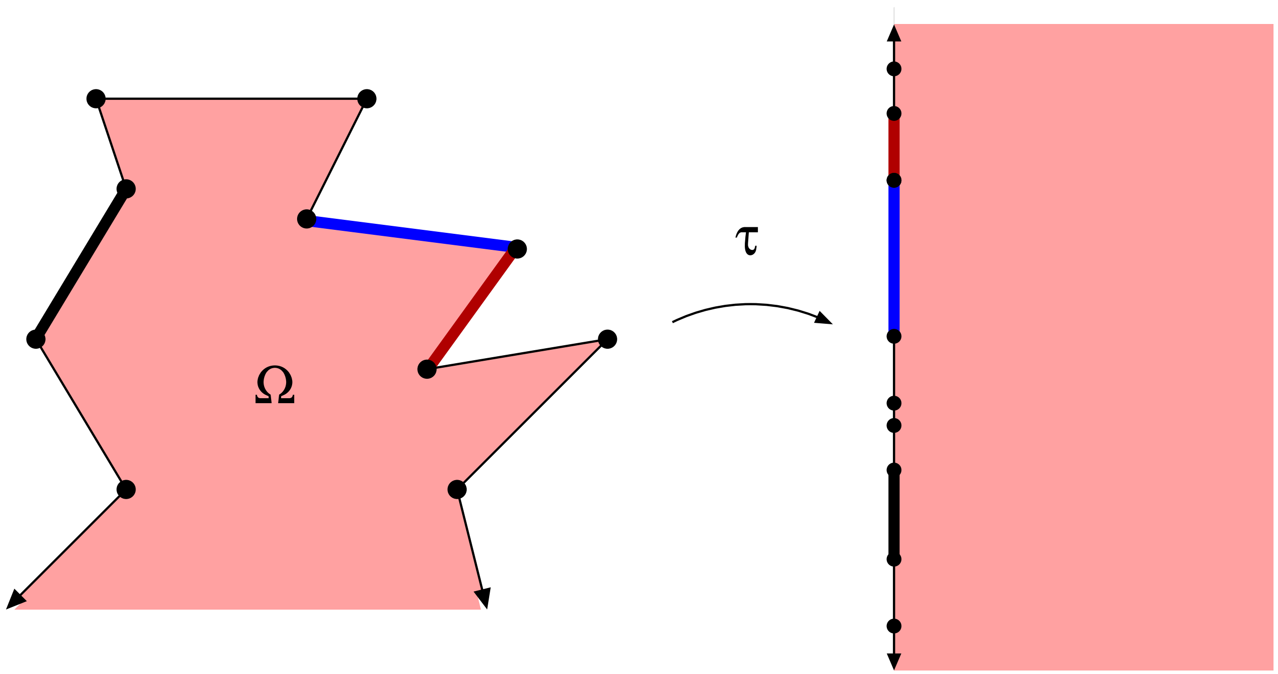
$$\mathrm{dist}(e, f) \geq \epsilon \cdot \min(\mathrm{diam}(e), \mathrm{diam}(f)).$$



τ-Lower Bound (global condition; harder to check):

Complementary components of tree are simply connected.

Each can be conformally mapped to right half-plane. Call map τ.

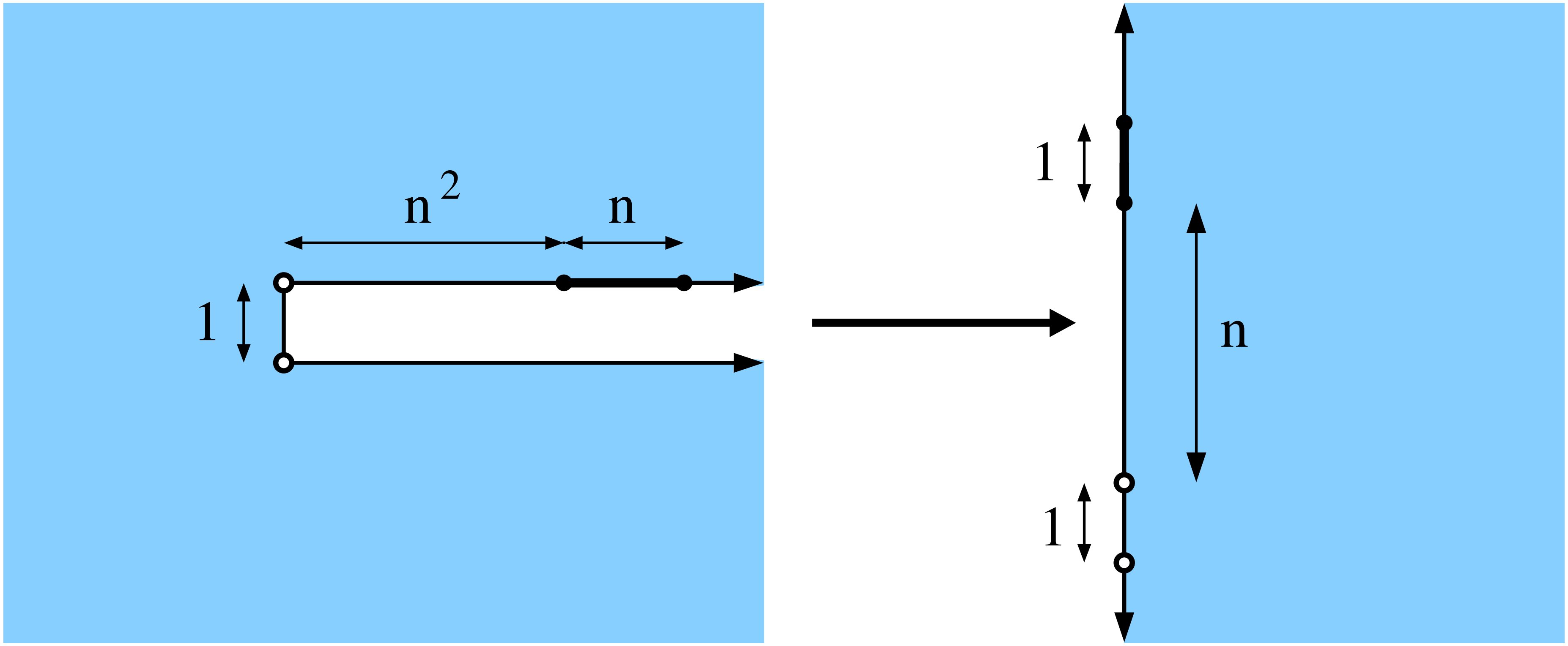


We assume all images have length $\geq \pi$.

Need positive lower bound; actual value usually not important.

Components are "thinner" than half-plane near ∞.

Non-example: half-strip. “Inside” is OK, but ...

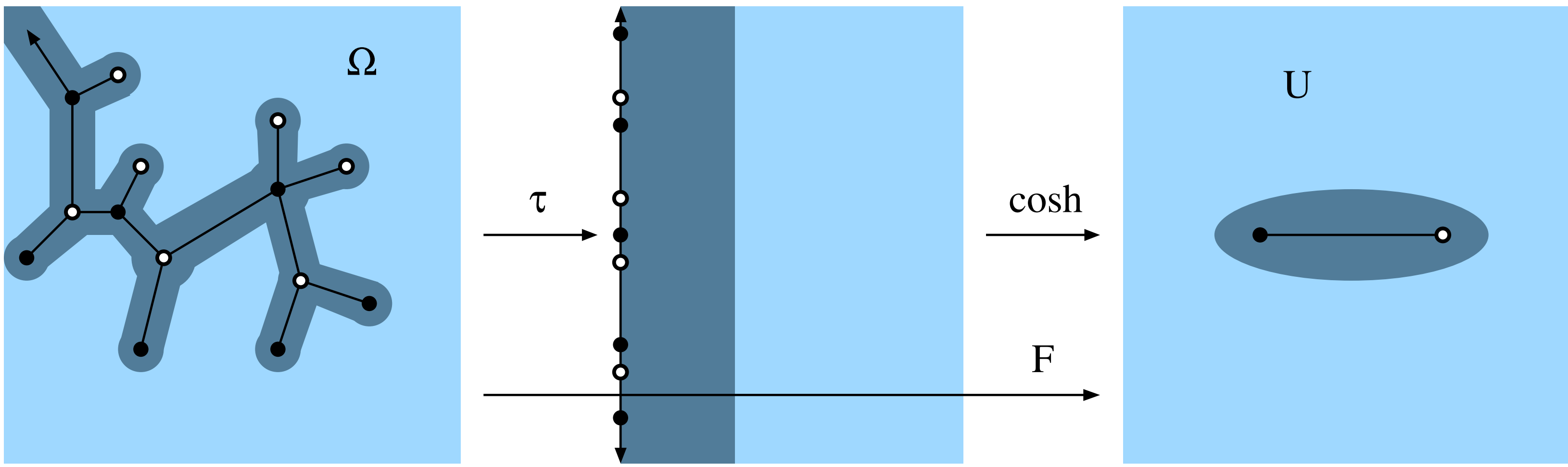


Conformal map of outside to half-plane is $\tau(z) \approx \sqrt{z}$.

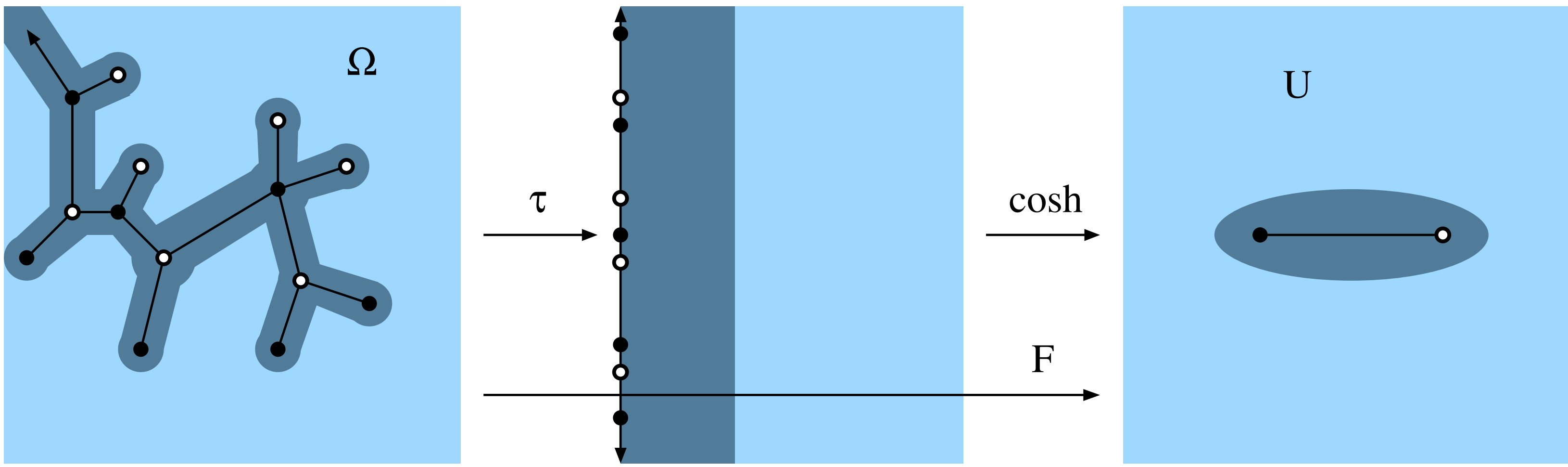
Unit intervals on half-plane have pre-images $\simeq n$.

$\Rightarrow$ Bounded geometry and τ-condition can't both hold.

QC-Folding Theorem: Suppose T has bounded geometry and the τ-lower bound. Let $F = \cosh \circ \tau$. There is a K-quasiregular g and $r > 0$ such that $g = F$ off $T(r)$ (shaded) and $\mathrm{CV}(g) = \pm 1$.



QC-Folding Theorem: Suppose T has bounded geometry and the τ-lower bound. Let $F = \cosh \circ \tau$. There is a K-quasiregular g and $r > 0$ such that $g = F$ off $T(r)$ (shaded) and $\mathrm{CV}(g) = \pm 1$.

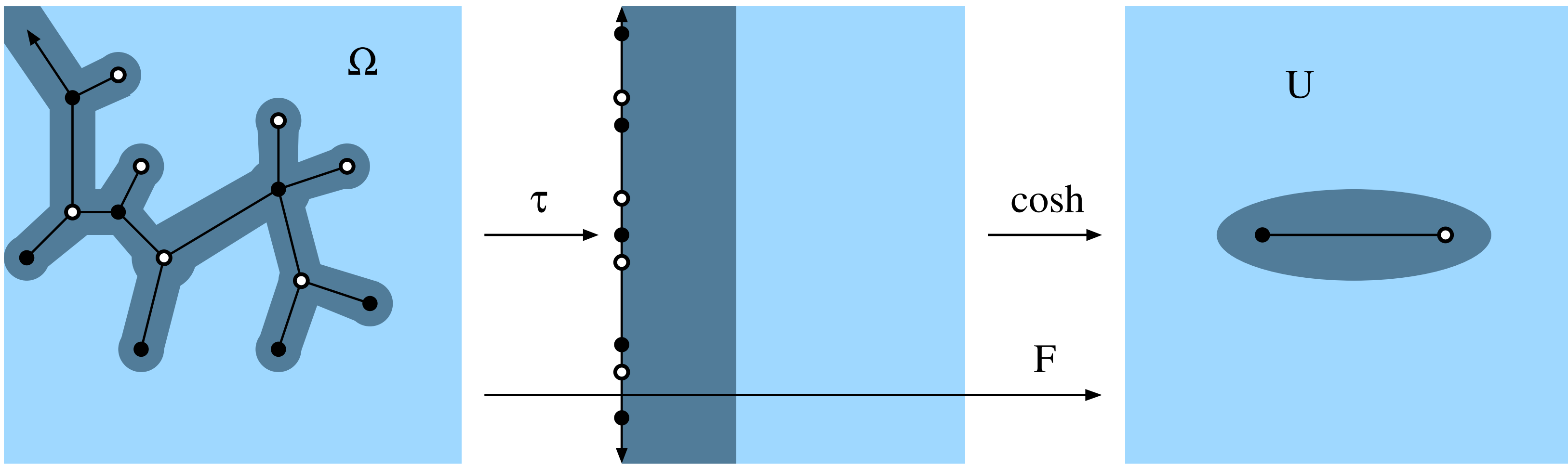


K and r only depend on the bounded geometry constants.

$g = F$ on light blue.

F may be discontinuous across T. g is continuous everywhere.

QC-Folding Theorem: Suppose T has bounded geometry and the τ-lower bound. Let $F = \cosh \circ \tau$. There is a K-quasiregular g and $r > 0$ such that $g = F$ off $T(r)$ (shaded) and $\mathrm{CV}(g) = \pm 1$.

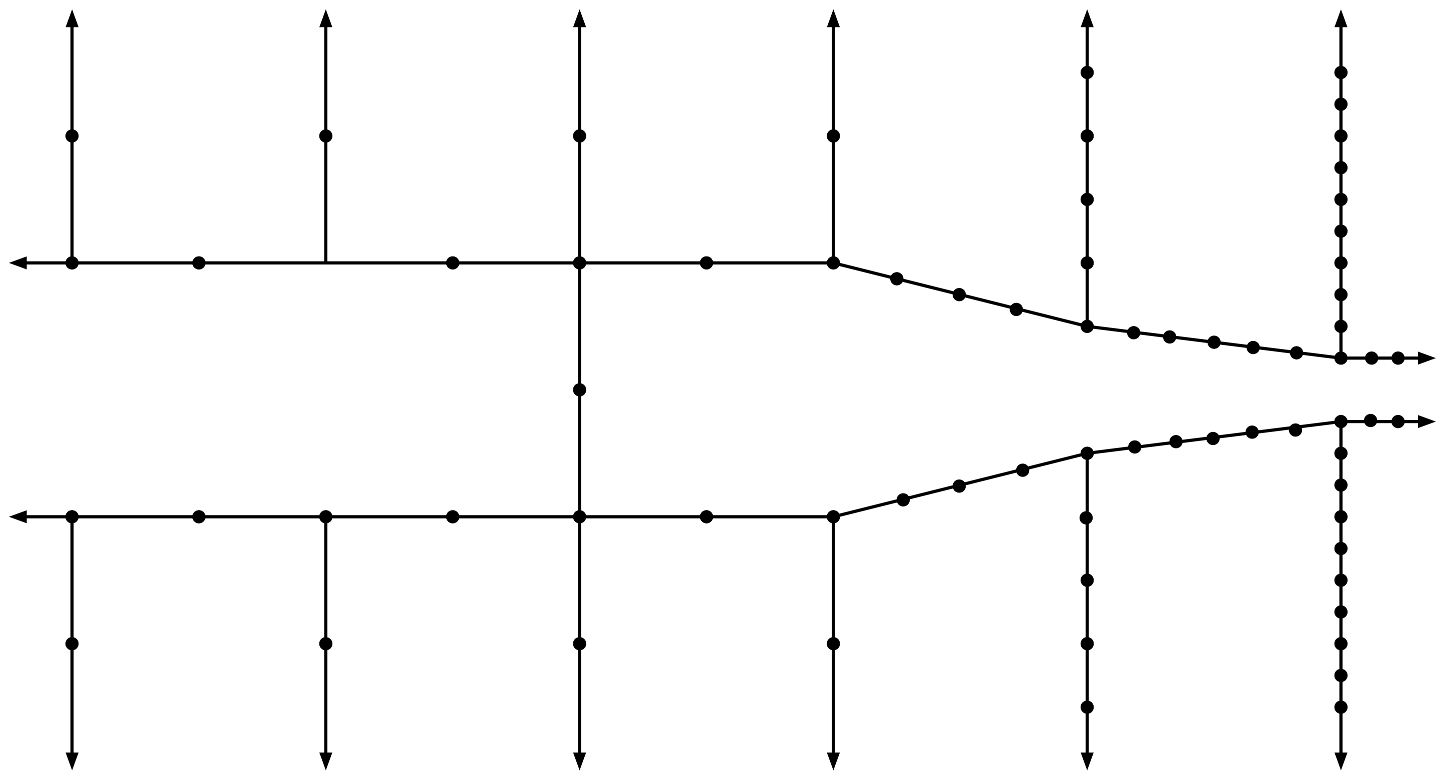


K and r only depend on the bounded geometry constants.

Cor: Any T as above is approximated by $\varphi(T')$ where $T' = f^{-1}([-1, 1])$, f is entire with $\mathrm{CV}(f) = \pm 1$, φ is QC and conformal off $T(r)$.

In many applications, φ is close to identity, so $T \approx T'$.

Example:

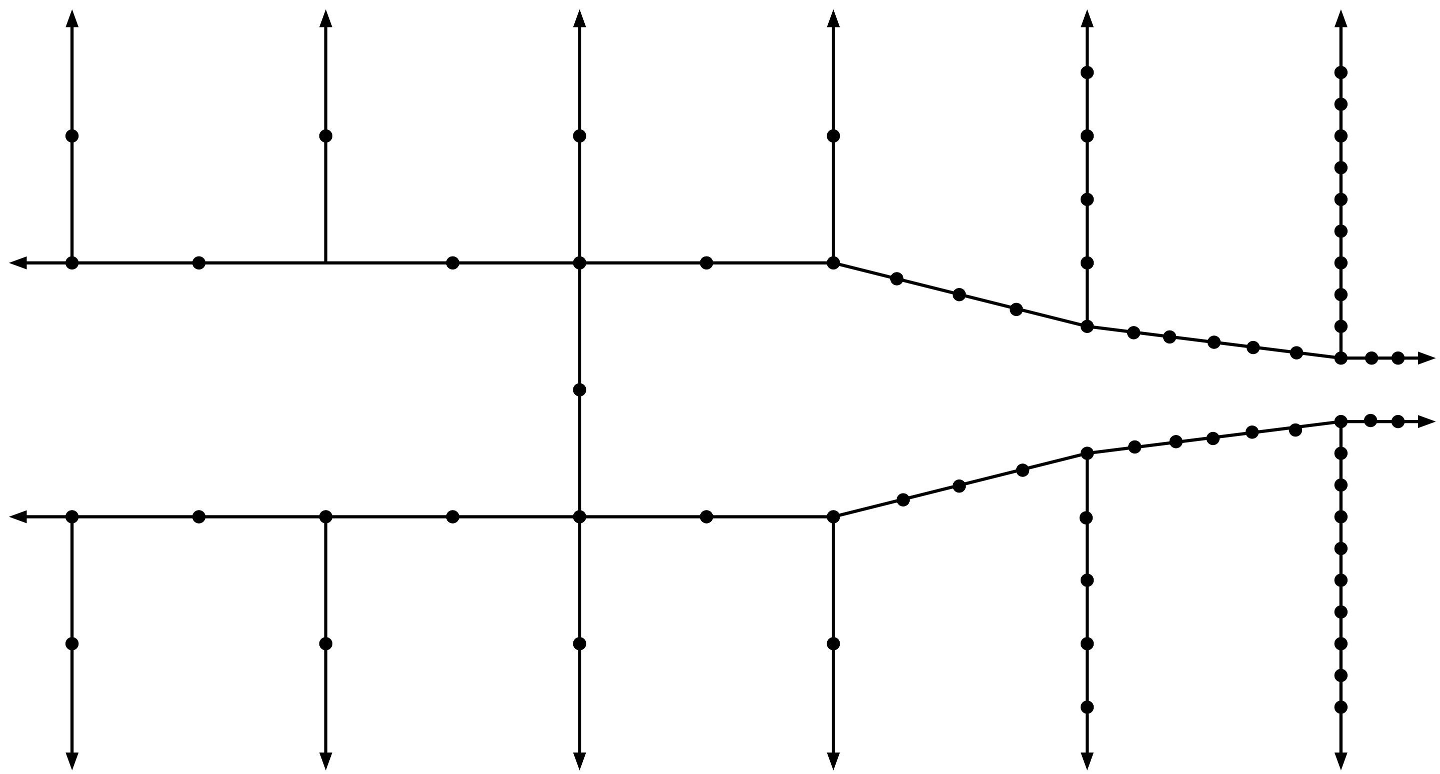


Check that this tree has:

(1) bounded geometry,

(2) the τ-lower bound.

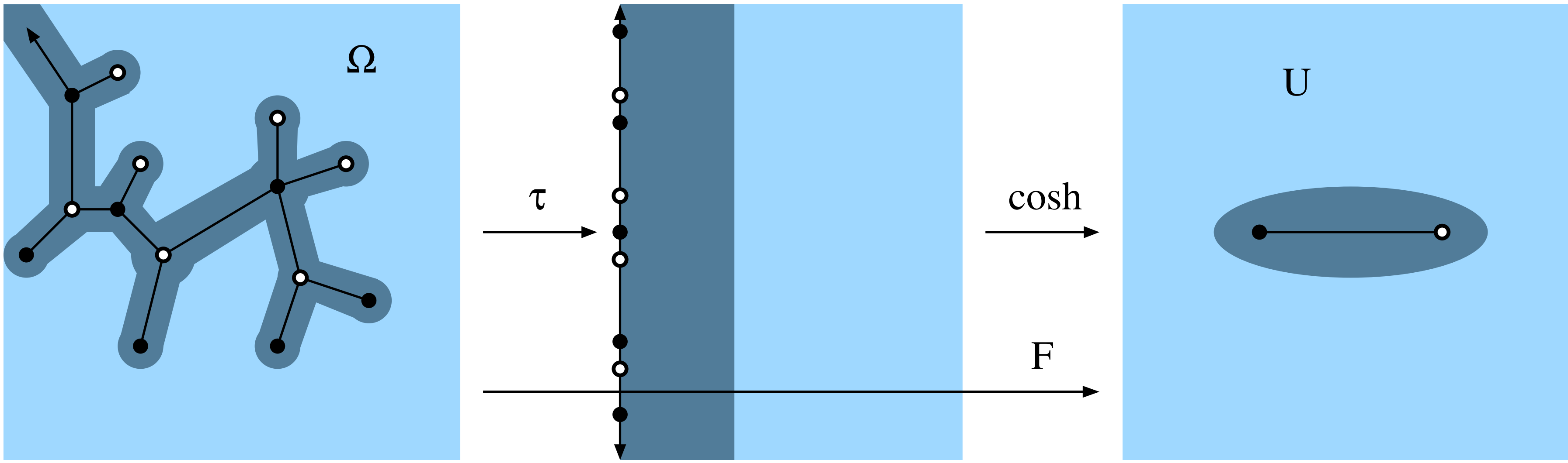
Rapid increase in Speiser class



We get f with 2 singular values, $f(x) \nearrow \infty$ as fast as we wish.

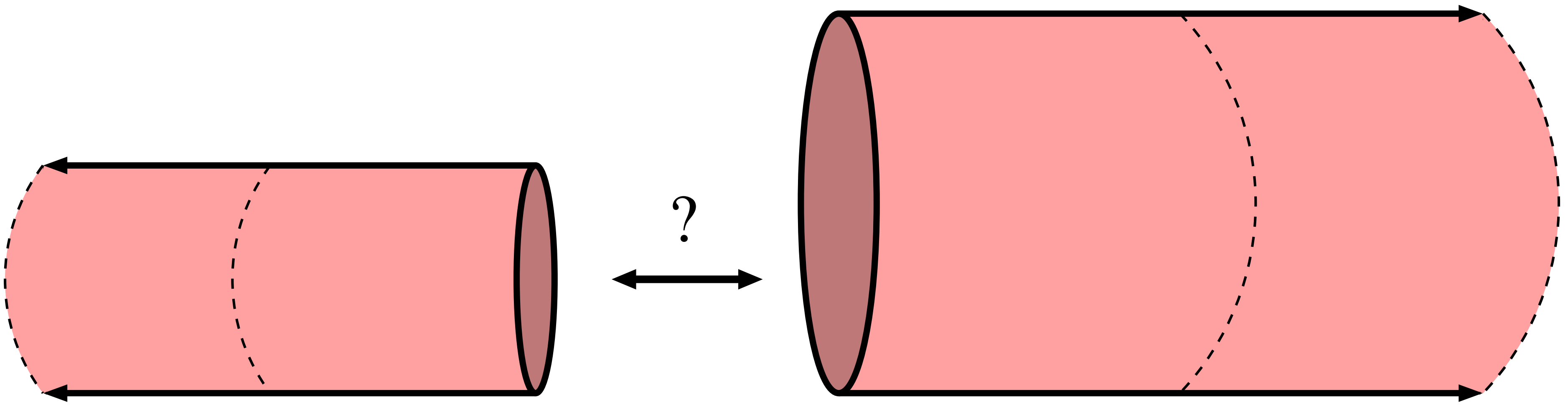
Correction map φ is Hölder, only slows growth a little.

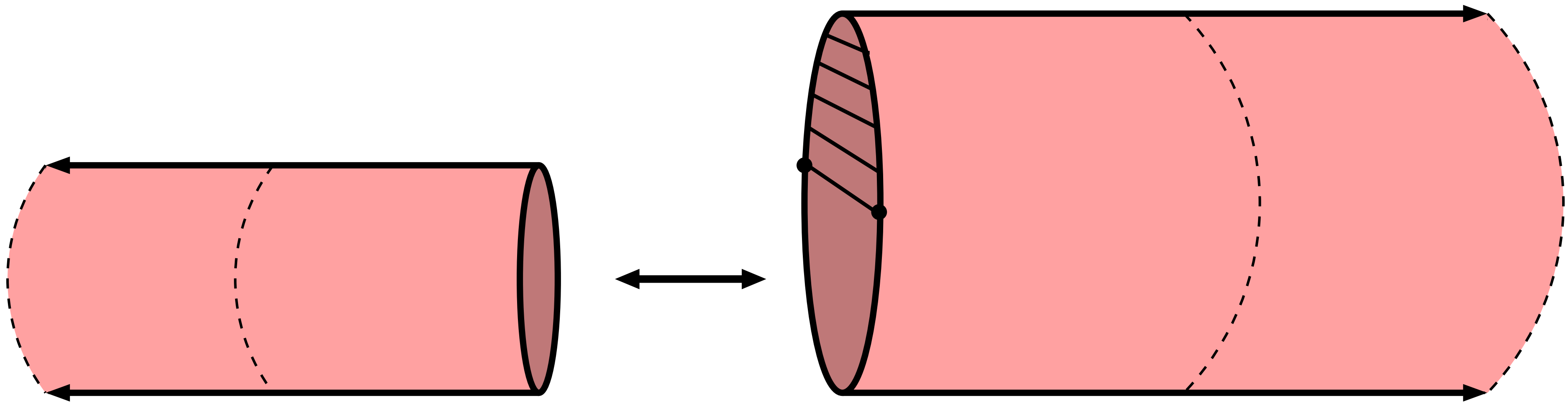
Similar examples due to Sergei Merenkov (2008) (3 singular values).

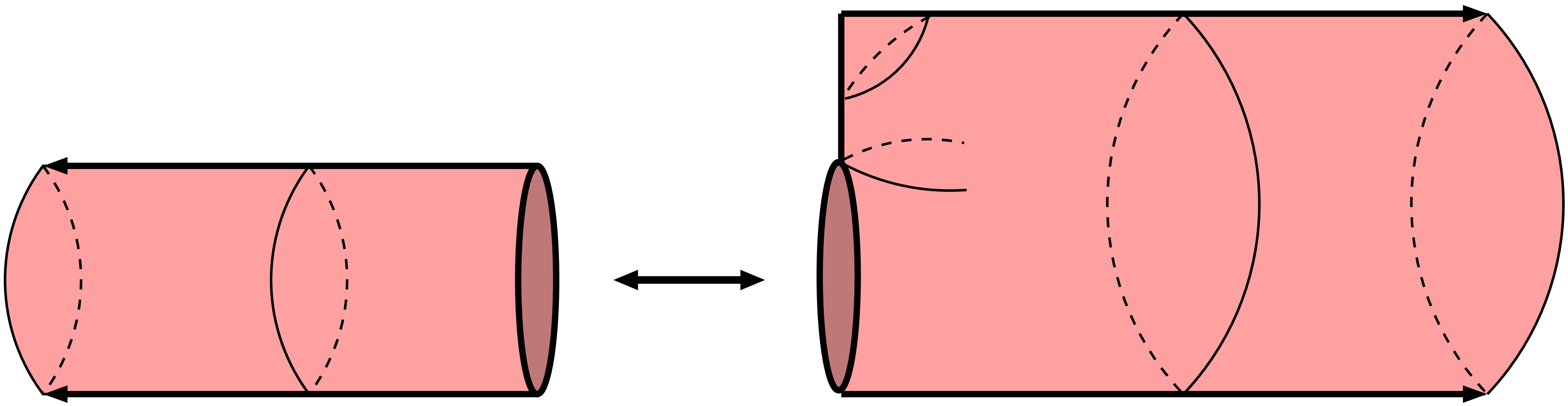


Idea of proof of folding theorem:

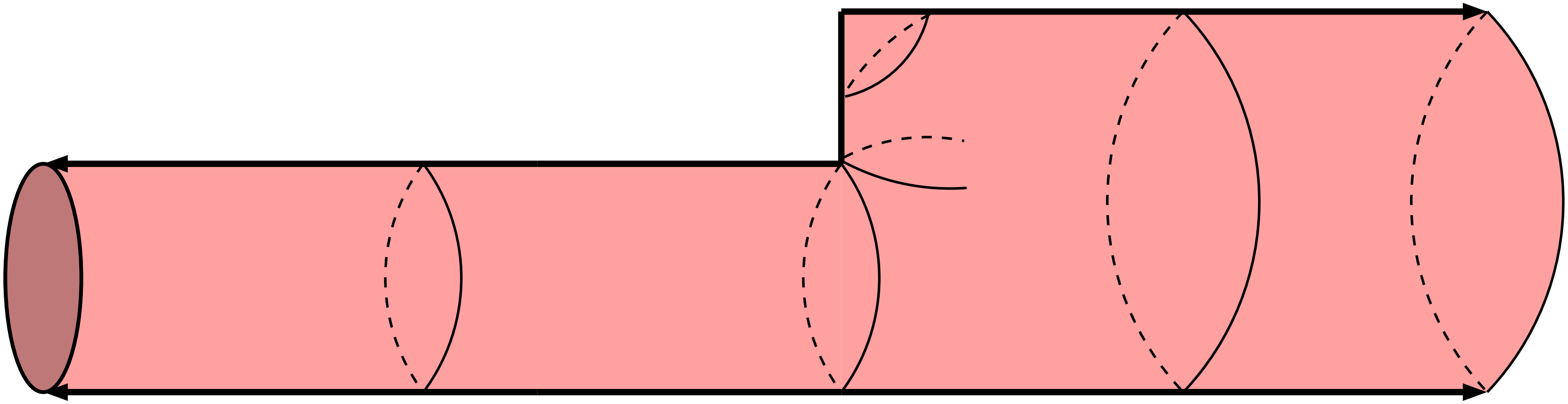
- τ maps both sides of an edge into $[-1, 1]$.
- One side might cover $[-1, 1]$ m times, the other n times, $n \neq m$.
- Must match m critical points to n critical points?
- Consider an analogy ...



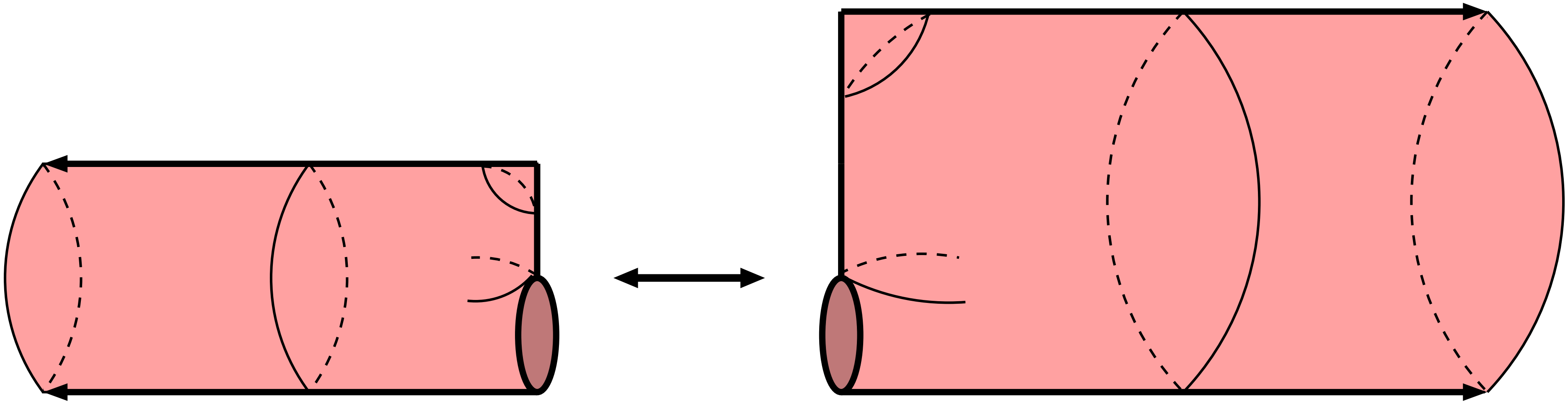




folding = conformal sewing with pleats



To attach different “sizes”, introduce critical values.
Remove excess length, then attach the remaining pieces.

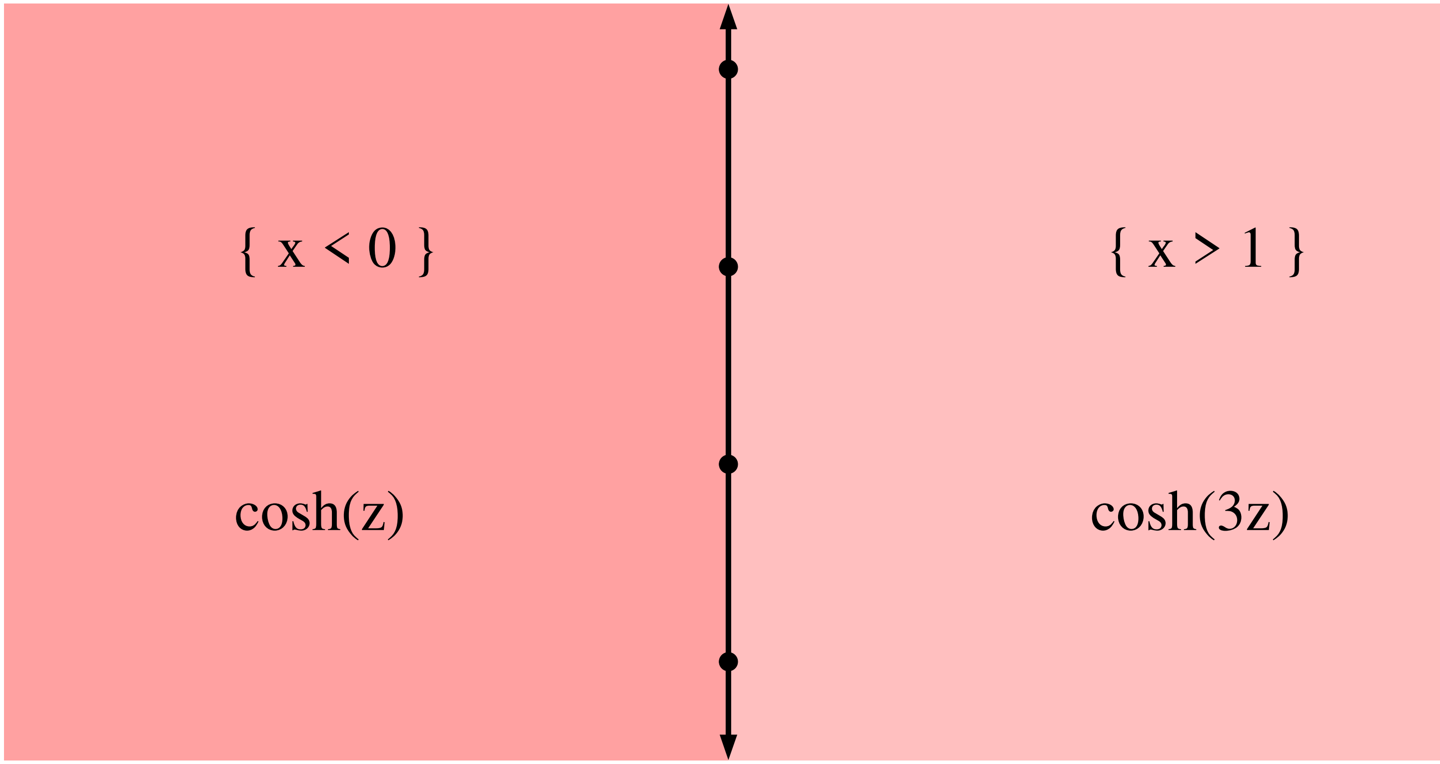


Often convenient to shorten both sides to a common length.
With multiple connections, shorten everything to minimal length.

Next, the idea of the proof of folding in a simple case.

The tree is a vertical line with even spaced vertices.

The τ map are different linear maps on each side.

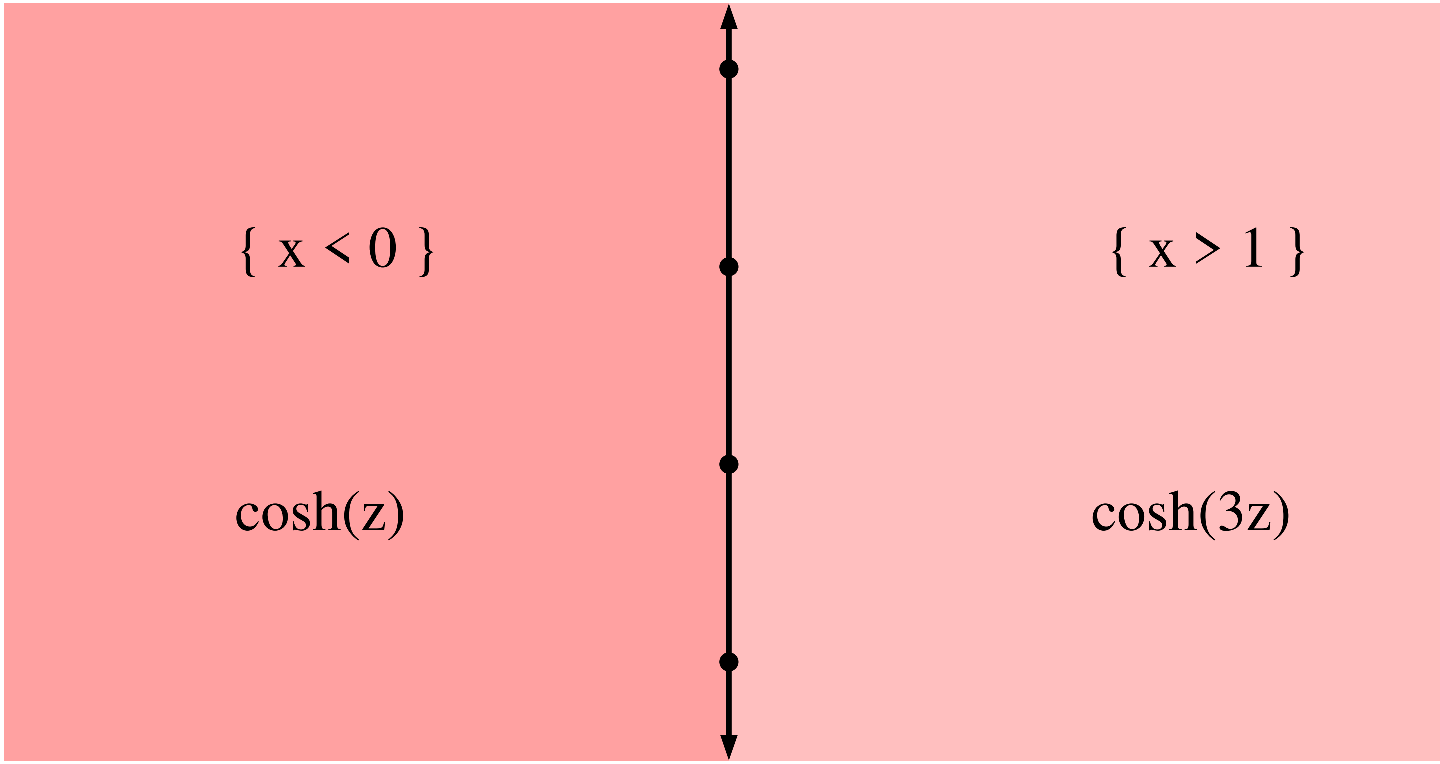


Construct an entire function F so that

$$F(z) = \cosh(z) \text{ on left half-plane } = \mathbb{H}_l = \{z = x + iy : x < 0\}$$

$$F(z) = \cosh(3z) \text{ on right half-plane } = \mathbb{H}_r = \{z = x + iy : x > 0\}$$

$\tau(z) = z$ on left, $\tau(z) = 3z$ on right.

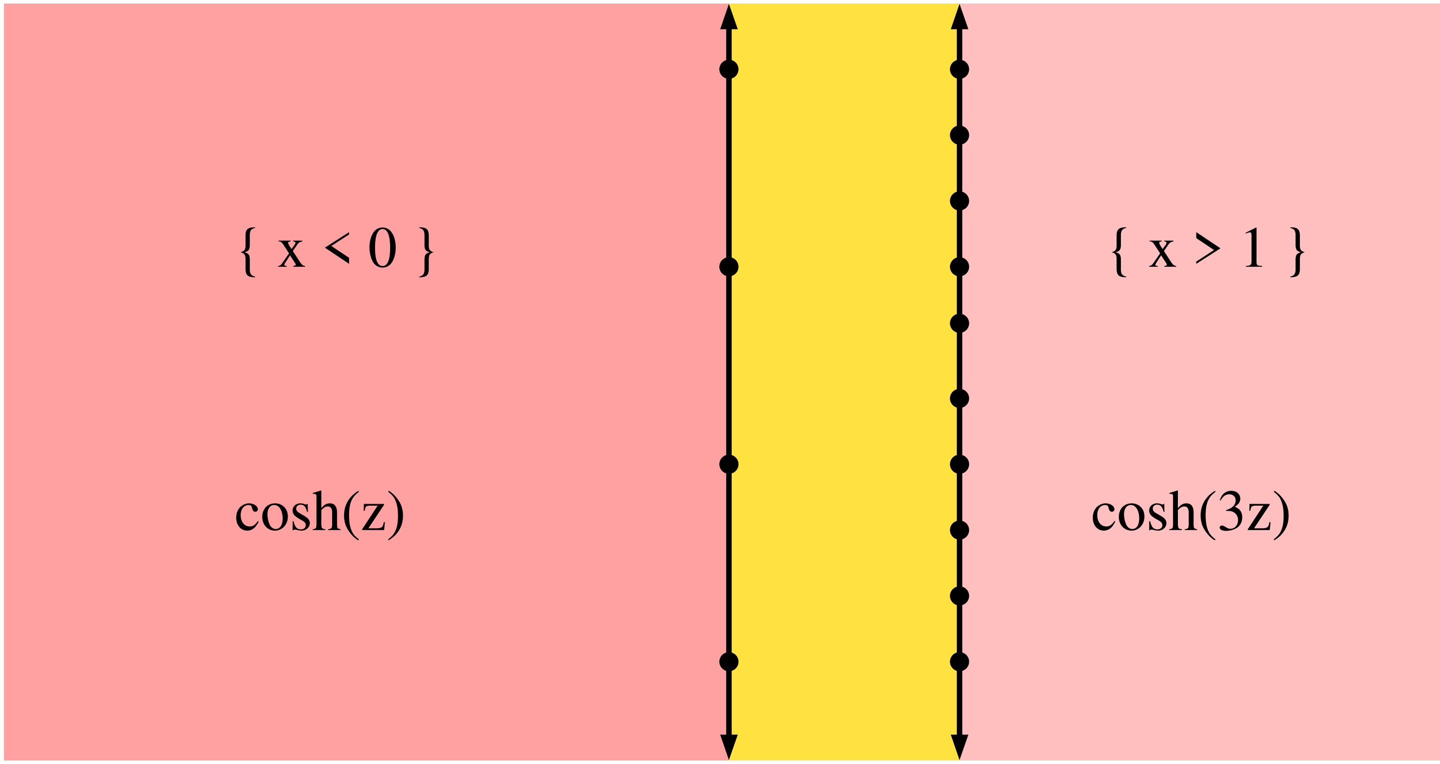


Construct an entire function F so that

$$F(z) = \cosh(z) \text{ on left half-plane } = \mathbb{H}_l = \{z = x + iy : x < 0\}$$

$$F(z) = \cosh(3z) \text{ on right half-plane } = \mathbb{H}_r = \{z = x + iy : x > 0\}$$

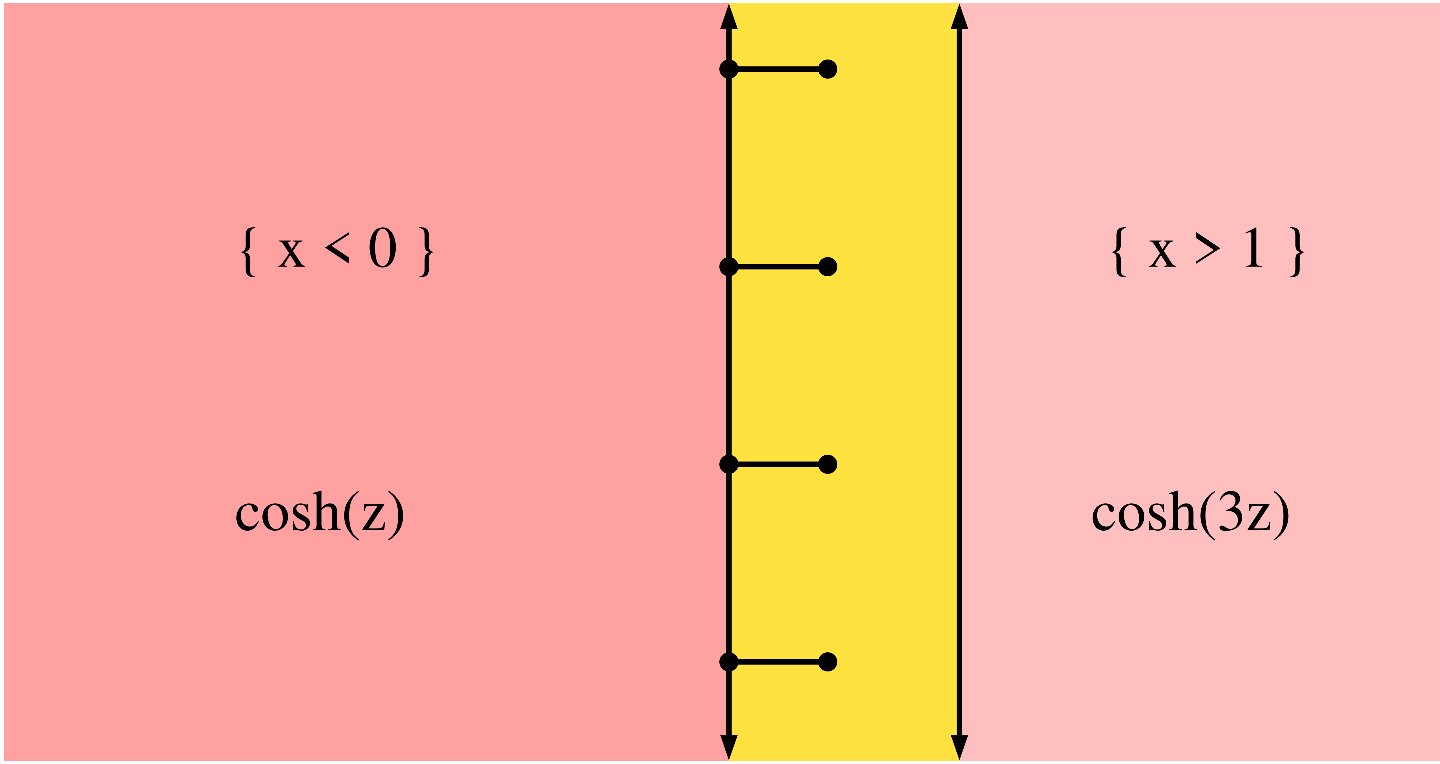
Impossible: violates unique continuation



Construct an **quasiregular** function F so that

$$F(z) = \cosh(z) \text{ on } \{x < 0\}$$

$$F(z) = \cosh(3z) \text{ on } \{x > 1\}$$

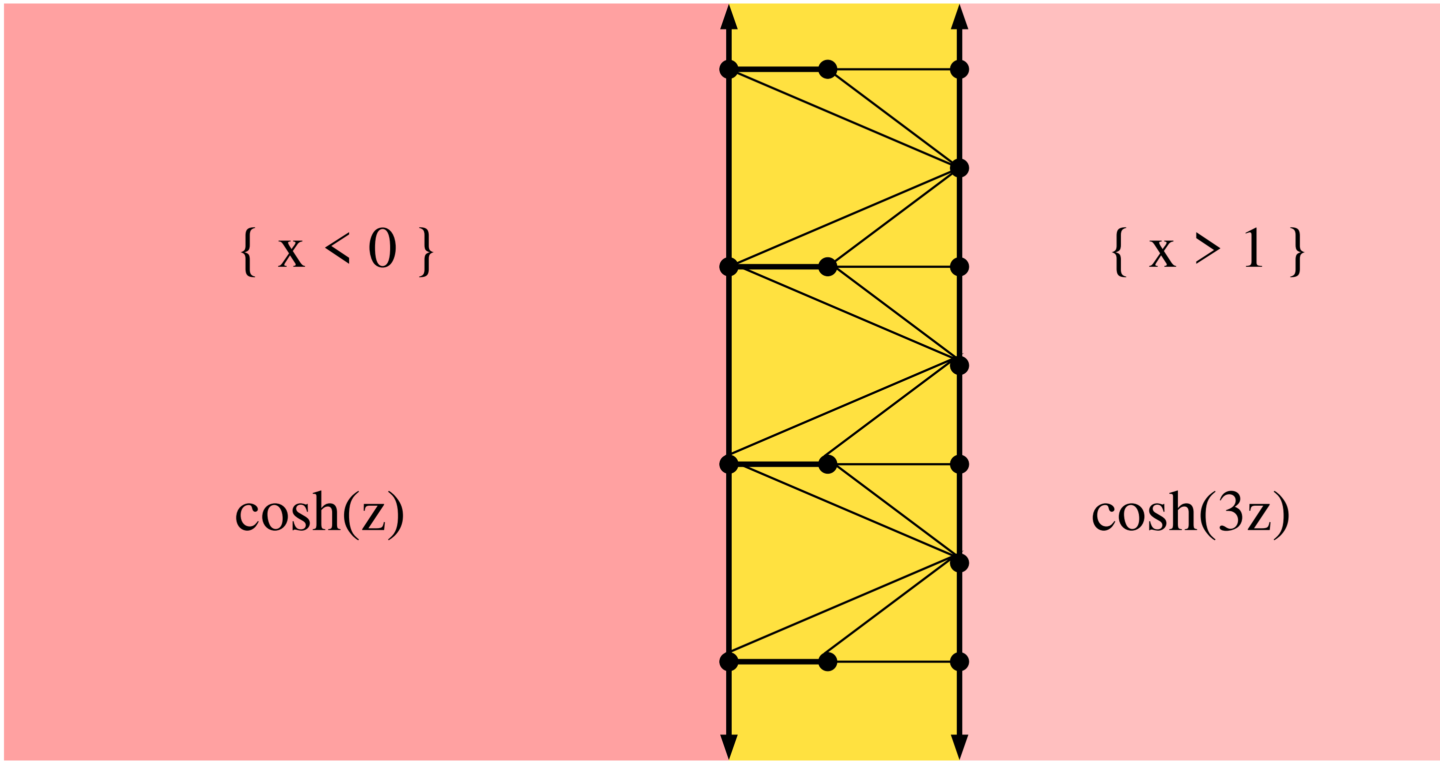


Construct an **quasiregular** function F so that

$$F(z) = \cosh(z) \text{ on } \{x < 0\}$$

$$F(z) = \cosh(3z) \text{ on } \{x > 1\}$$

This is easy. First add some spikes in the strip.

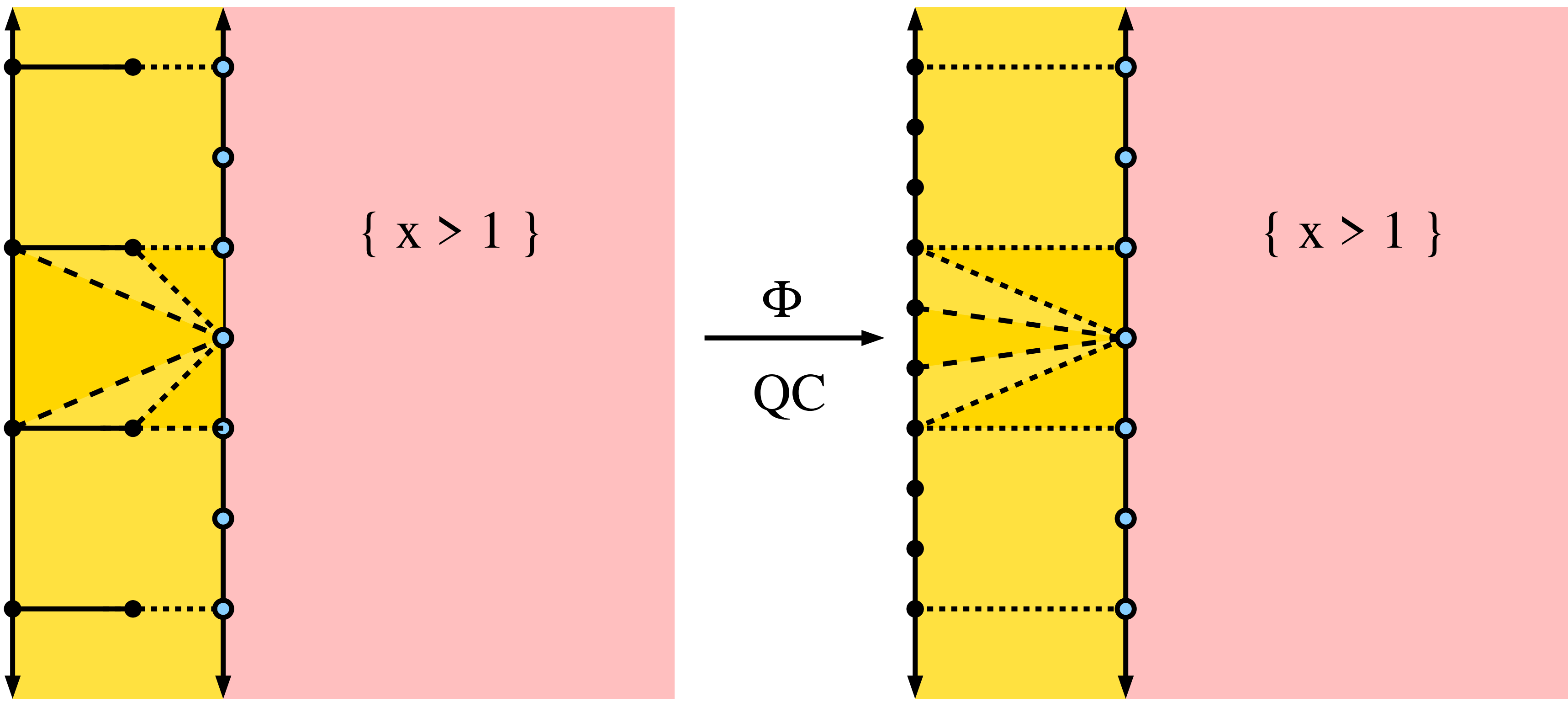


Construct an **quasiregular** function F so that

$$F(z) = \cosh(z) \text{ on } \{x < 0\}$$

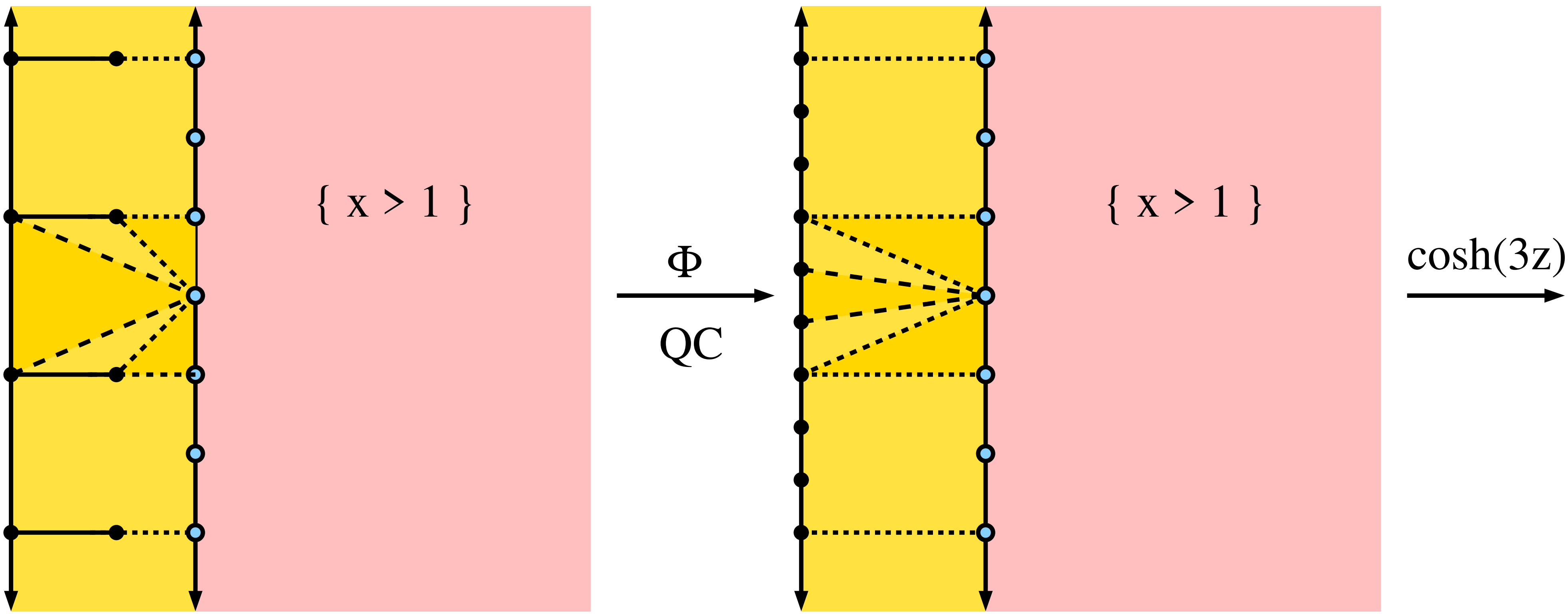
$$F(z) = \cosh(3z) \text{ on } \{x > 1\}$$

Triangulate the folded strip.



Let Φ be piecewise affine map: folded strip $\to$ un-folded strip.

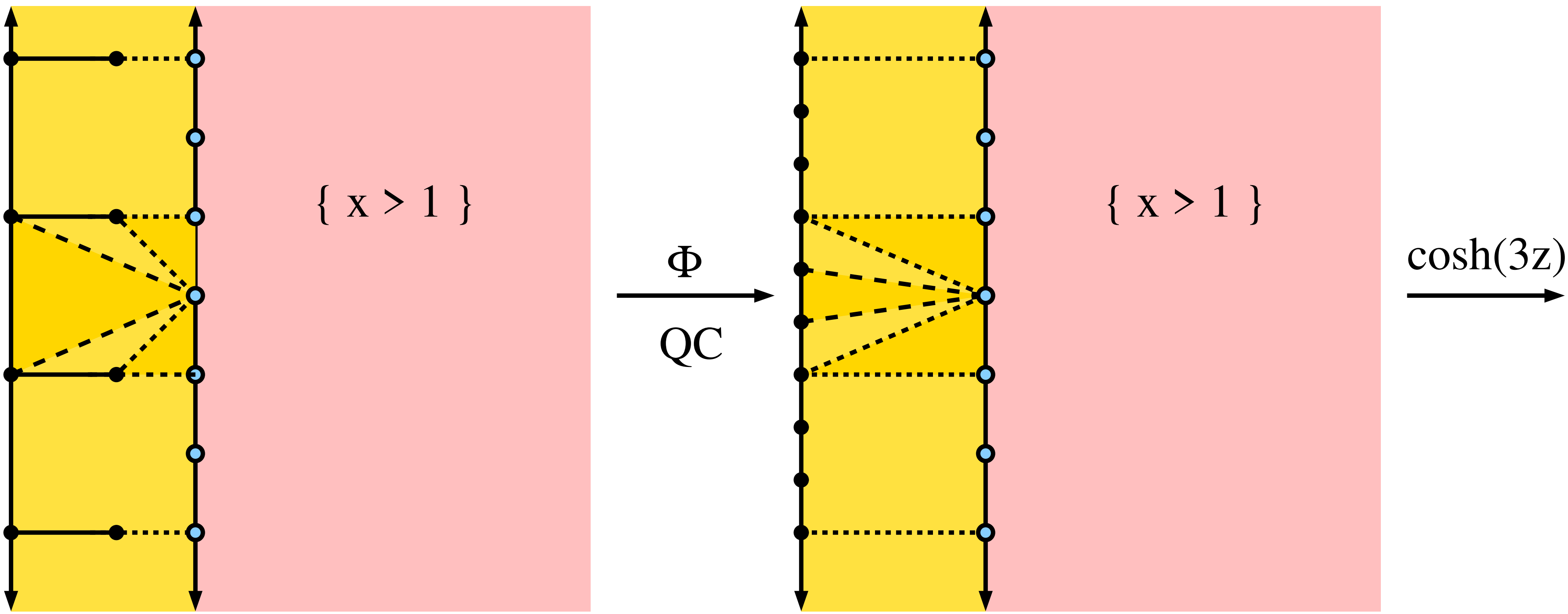
Take $\Phi =$ identity map off the strip.



$G(z) = \cosh(3 \cdot \Phi(z))$ satisfies:

- equals $\cosh 3z$ on $\{x \geq 1\}$
- equals $\cosh z$ on $\{x = 0\}$
- Is quasiregular (composition of QC and holomorphic)

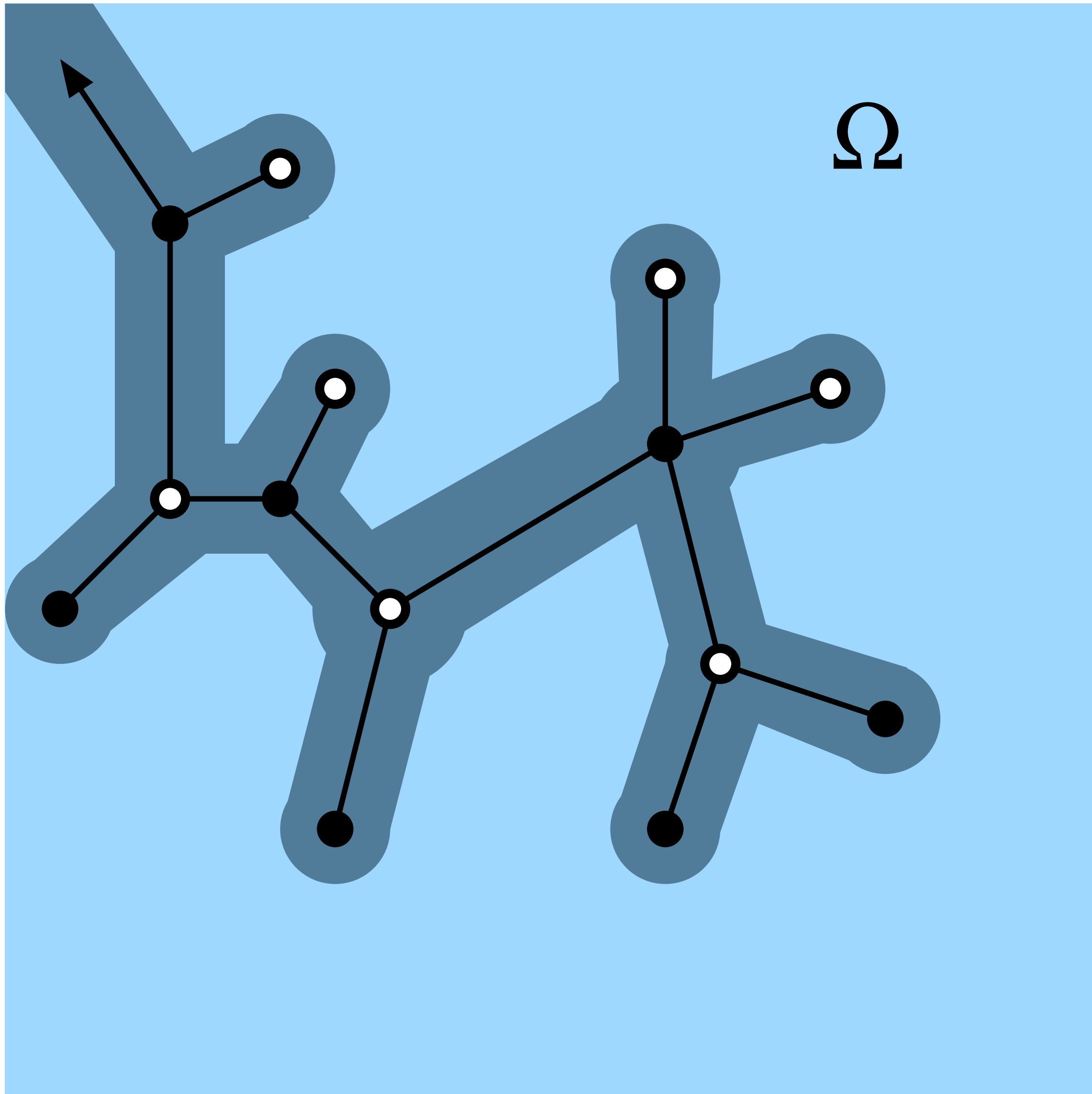
Measurable Riemann mapping theorem gives entire $F = G \circ \varphi$.



$G(z) = \cosh(3 \cdot \Phi(z))$ satisfies:

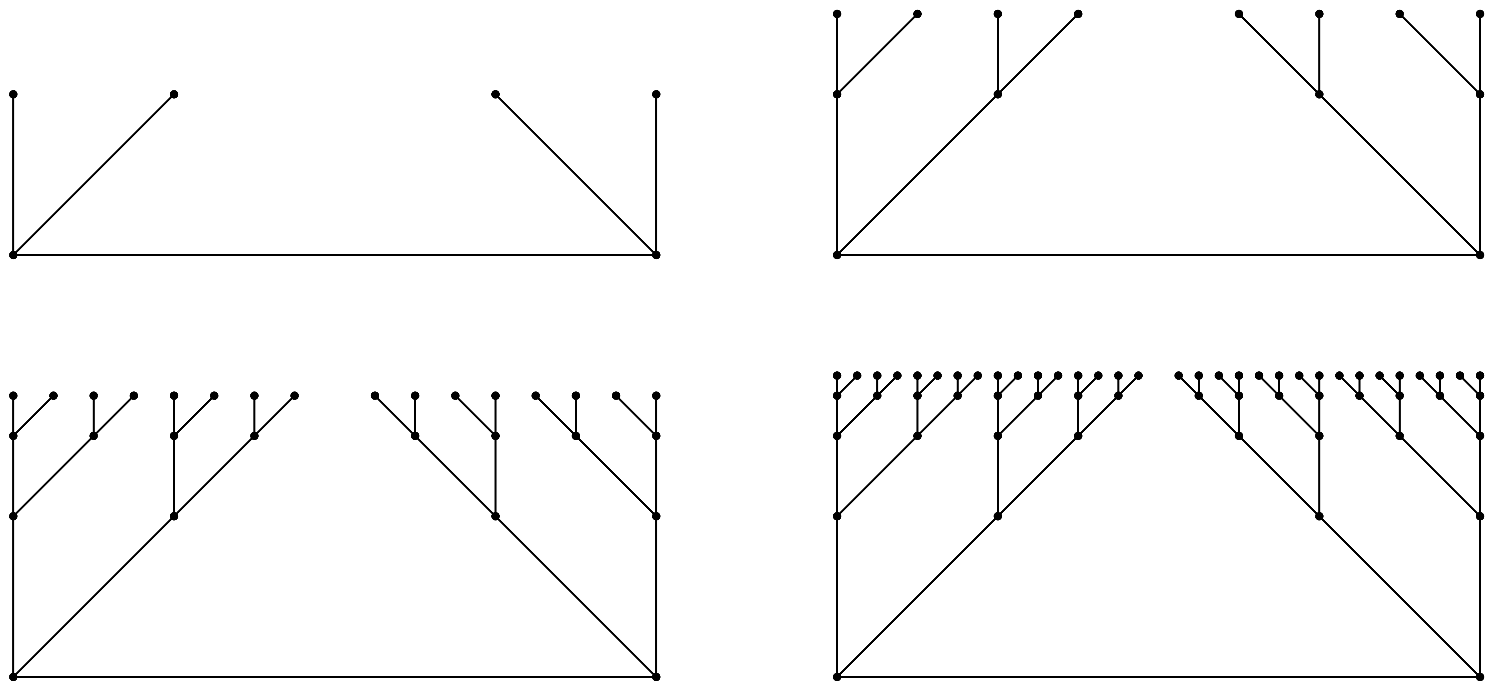
- equals $\cosh 3z$ on $\{x \geq 1\}$
- equals $\cosh z$ on $\{x = 0\}$
- Is quasiregular (composition of QC and holomorphic)

Can replace 3, by any n and keep QC constant bounded.



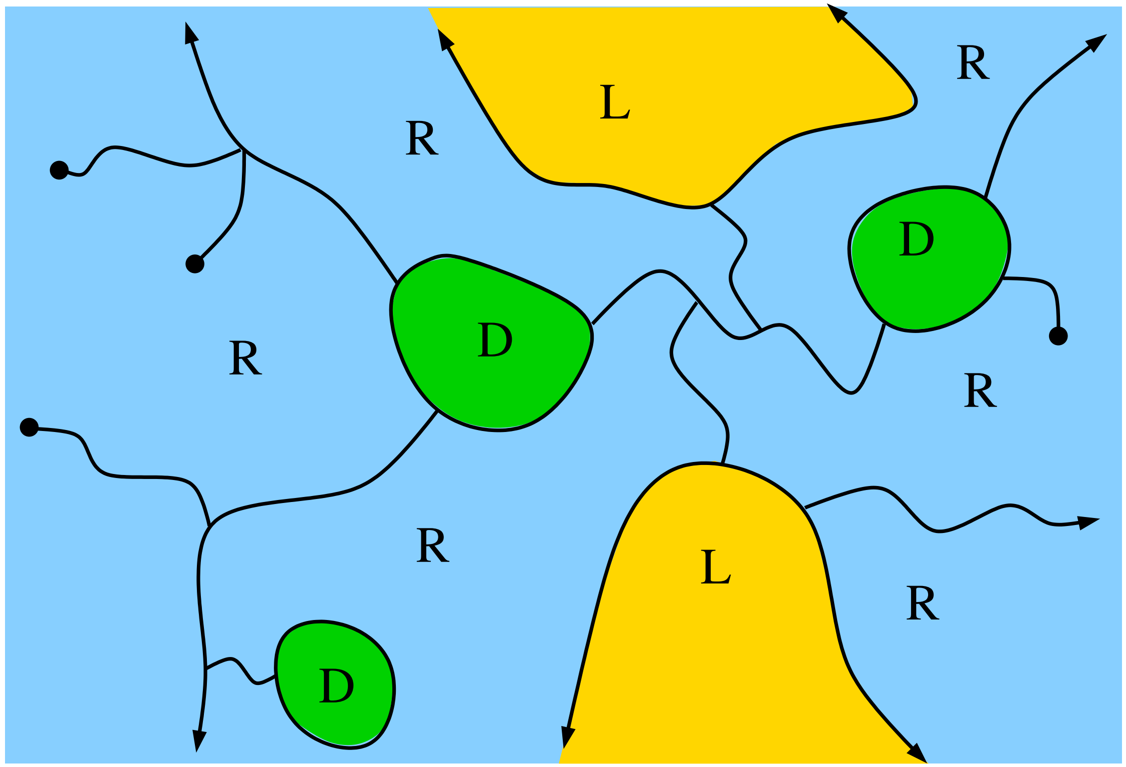
In general, add “spikes” to make all edges have same τ-length.
Construct QR maps to half-plane with bounded constant.

Many technical details. For example, attach finite trees (instead of spikes) to reduce τ-measure.



This allows arbitrarily large foldings with uniform QC bounds.

More general folding theorem: replace tree by graph, faces labeled D,L,R.



D = bounded Jordan domains (z^n, high degree critical points)

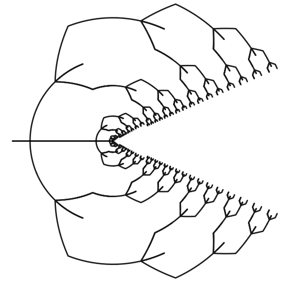
L = unbounded Jordan domains ($e^{-\tau(z)}$, finite asymptotic values)

RR-edges map to $[-1, 1]$, other edges map to $\mathbb{T}$.

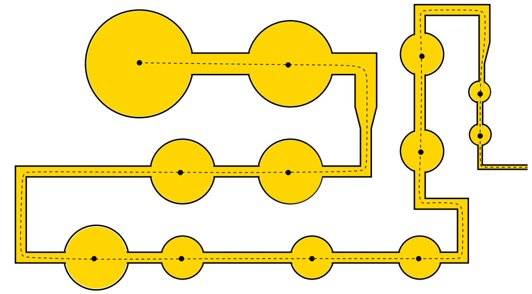
Another variation inverts the D-components, introduce poles.

APPLICATIONS OF QC-FOLDING

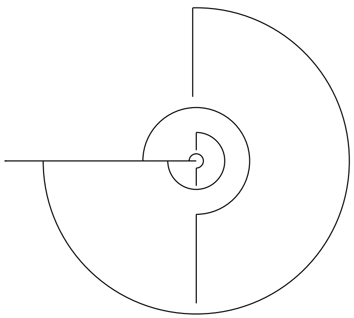
EL Models



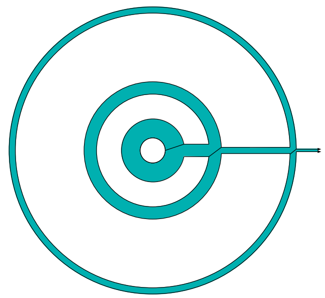
Order Conjecture



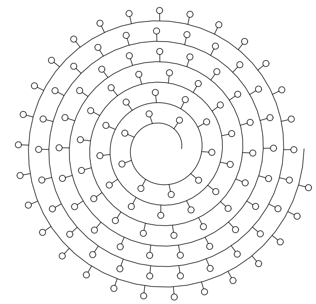
Post-singular dynamics



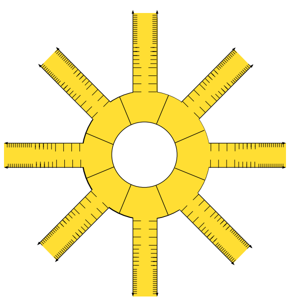
Eremenko Conjecture



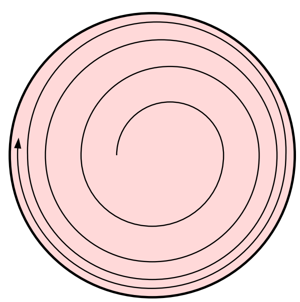
Near zero



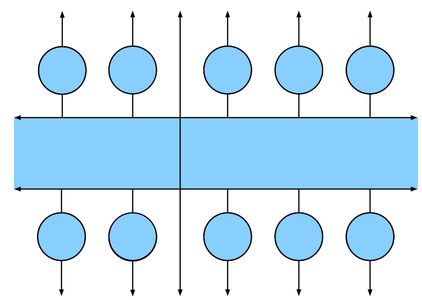
Wiman's Conjecture



Dimension near 1



Folding in disk



Wandering domain

$$\begin{aligned}\textbf{Singular set} &= \text{closure of critical values and finite asymptotic values} \\ &= \text{smallest set so that } f \text{ is a covering map onto } \mathbb{C} \setminus S\end{aligned}$$

Eremenko-Lyubich class = bounded singular set = $\mathcal{B}$

Speiser class = finite singular set = $\mathcal{S} \subset \mathcal{B}$

$$\begin{aligned}\textbf{Singular set} &= \text{closure of critical values and finite asymptotic values} \\ &= \text{smallest set so that } f \text{ is a covering map onto } \mathbb{C} \setminus S\end{aligned}$$

Eremenko-Lyubich class = bounded singular set = $\mathcal{B}$

Speiser class = finite singular set = $\mathcal{S} \subset \mathcal{B}$

"Plain" folding produces functions in the smaller Speiser class.

Using D and L components, we can create singular values other than ± 1.

There is an "easier" version of folding for the Eremenko-Lyubich class.

Application: wandering domains

Given an entire function f,

Fatou set $= \mathcal{F}(f) =$ open set where iterates are normal family.

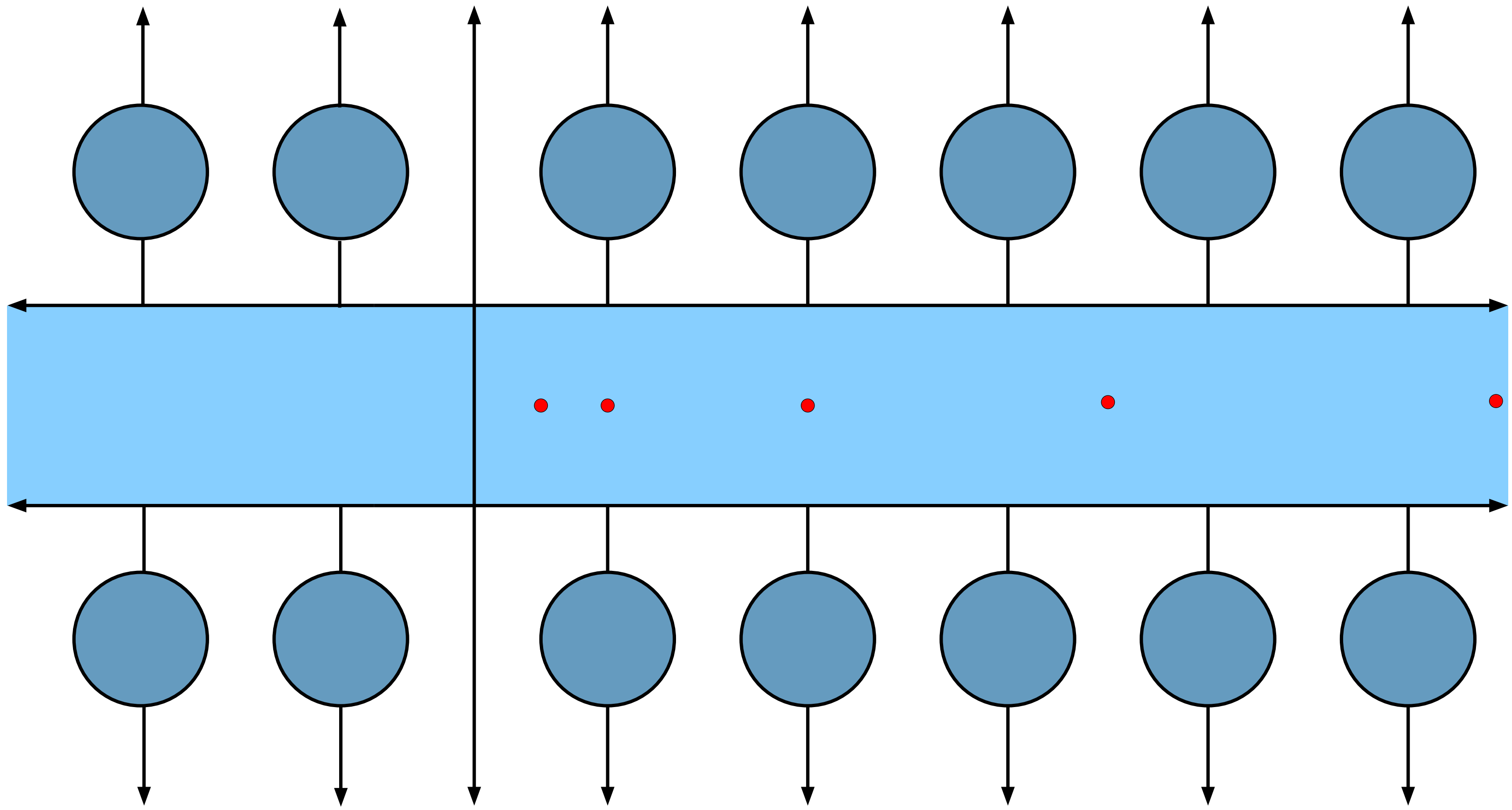
Julia set $= \mathcal{J}(f) =$ complement of Fatou set.

f permutes components of its Fatou set.

Wandering domain = Fatou component with infinite orbit.

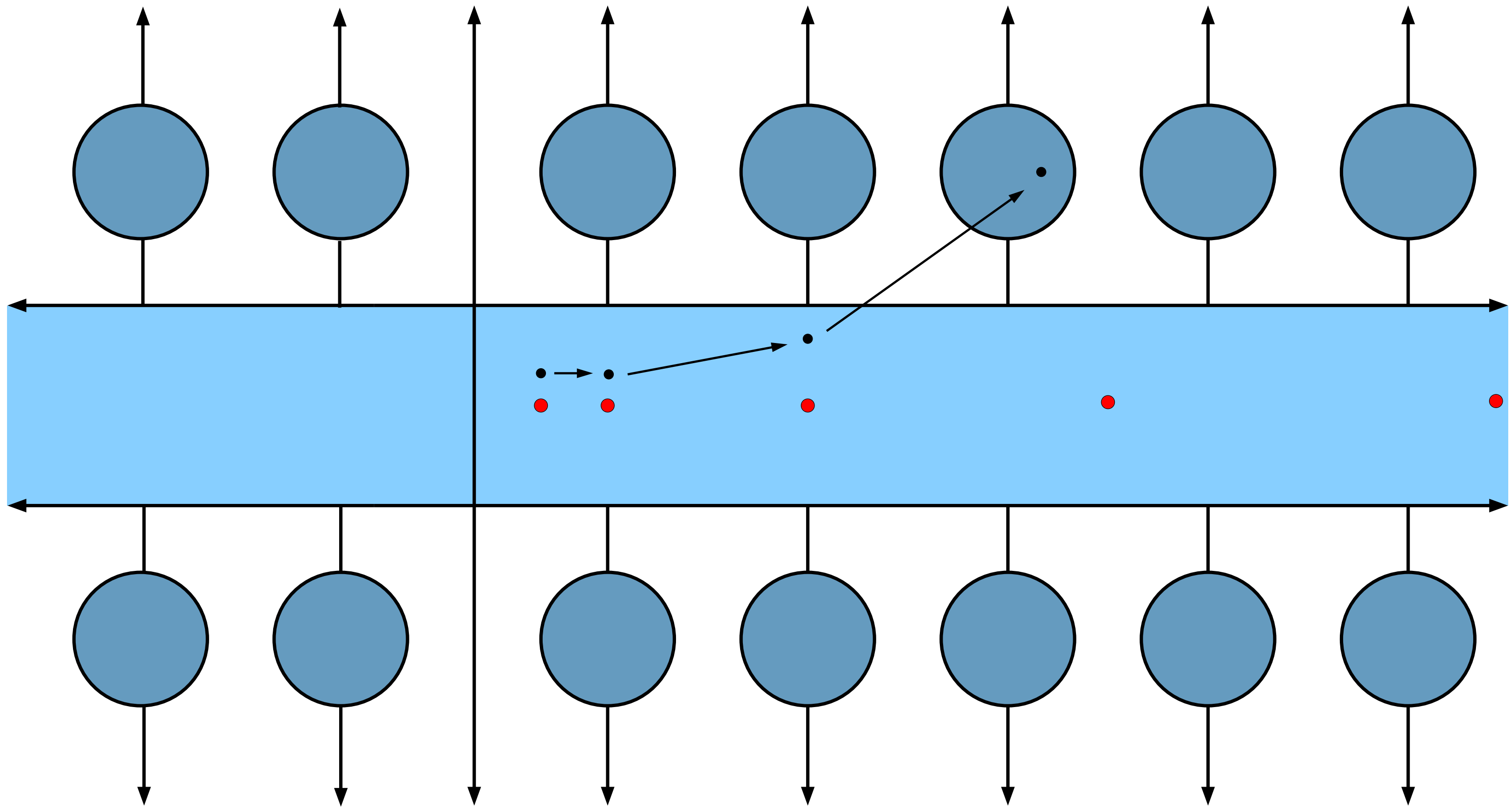
- Entire functions can have wandering domains (Baker 1975).
- No wandering domains for rational functions (Sullivan 1985).
- Also none in Speiser class (Eremenko-Lyubich, Goldberg-Keen).
- More generally, none for finite type maps (Epstein).

Are there wandering domains in Eremenko-Lyubich class?



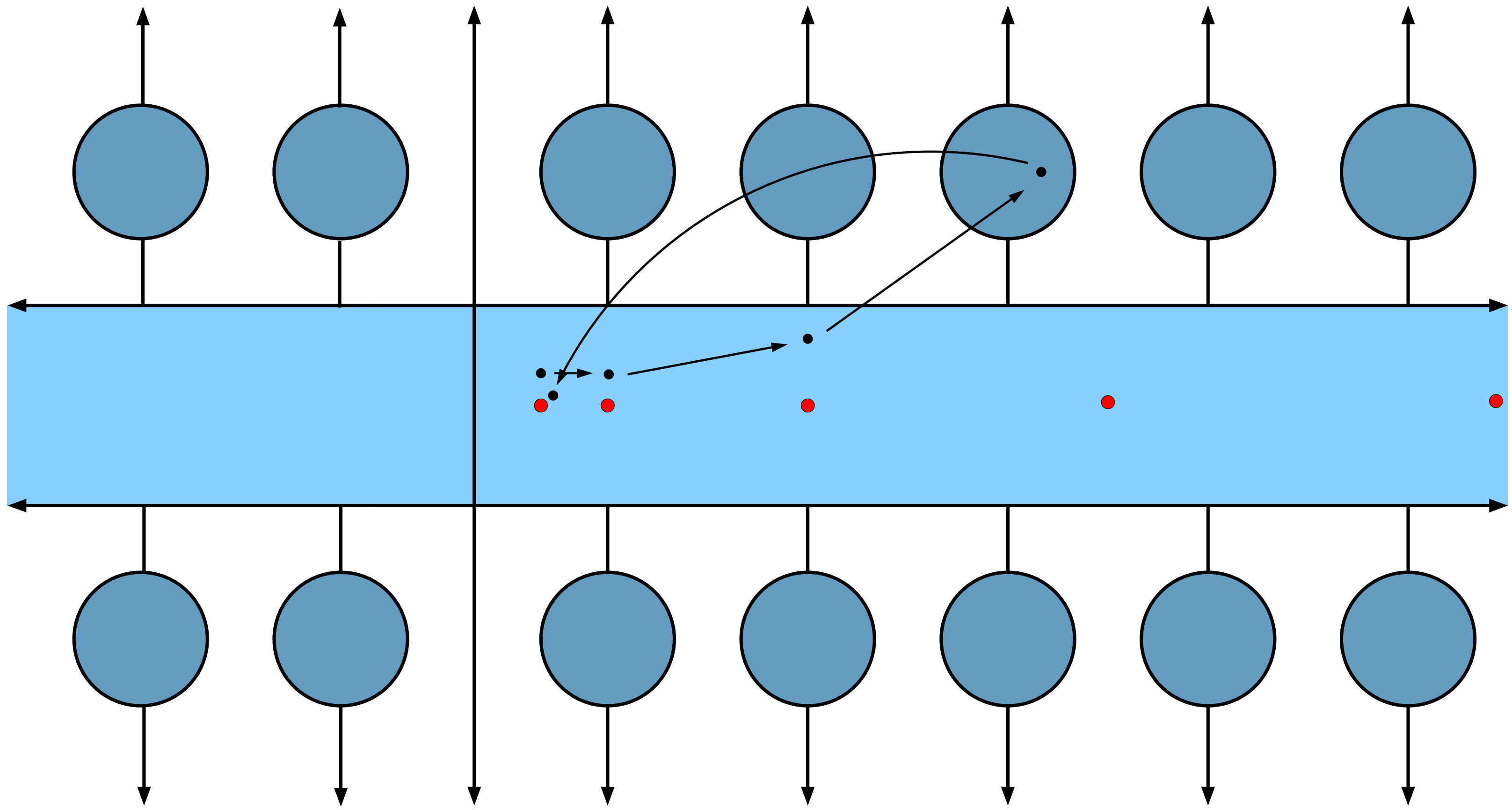
Graph giving wandering domain in Eremenko-Lyubich class.

Original proof corrected by Marti-Pete and Shishikura, who also give alternate construction.



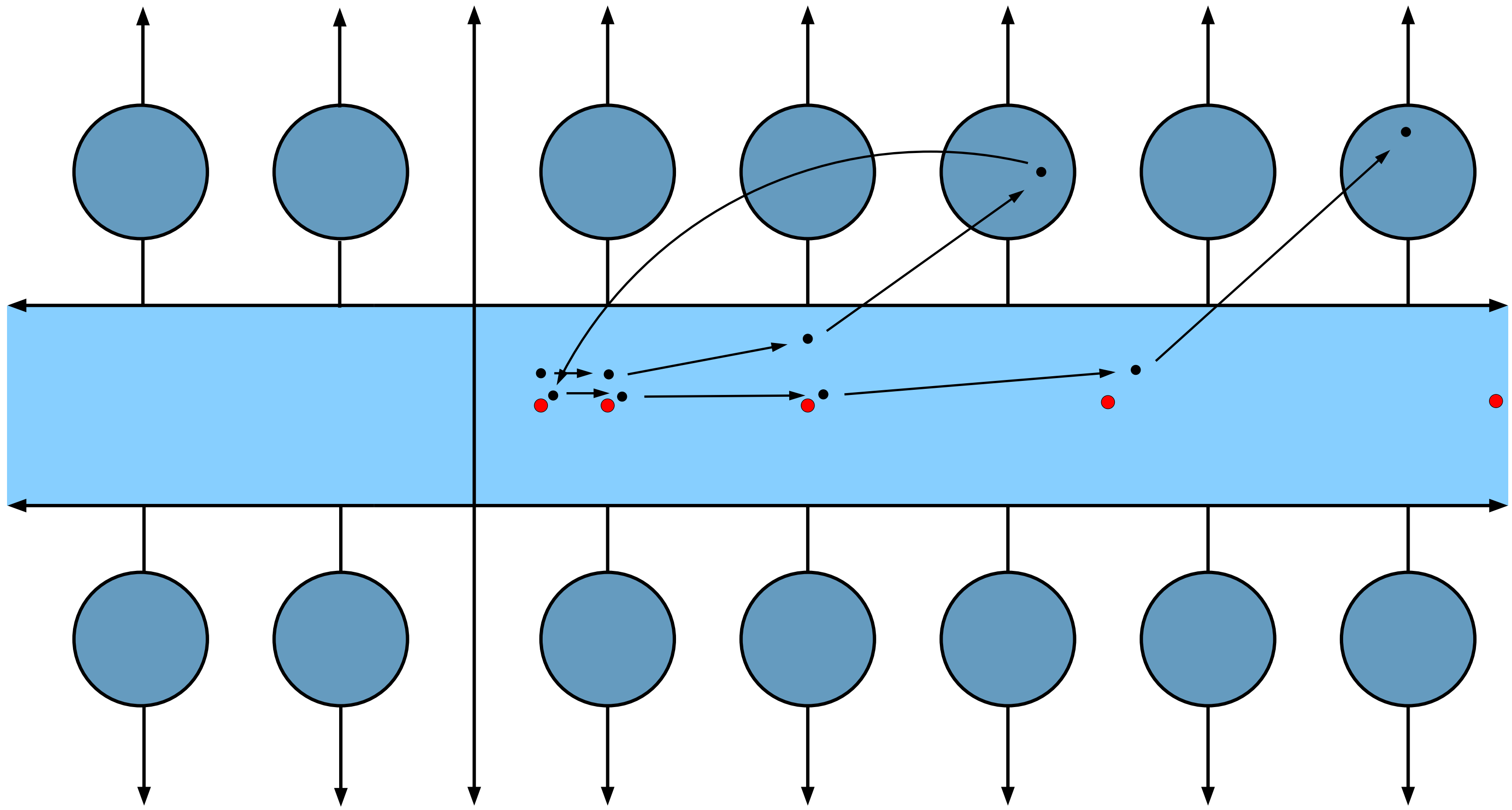
Graph giving wandering domain in Eremenko-Lyubich class.

Variations by Lazebnik, Fagella-Godillon-Jarque, Osborne-Sixsmith.



Graph giving wandering domain in Eremenko-Lyubich class.

Variations by Lazebnik, Fagella-Godillon-Jarque, Osborne-Sixsmith.

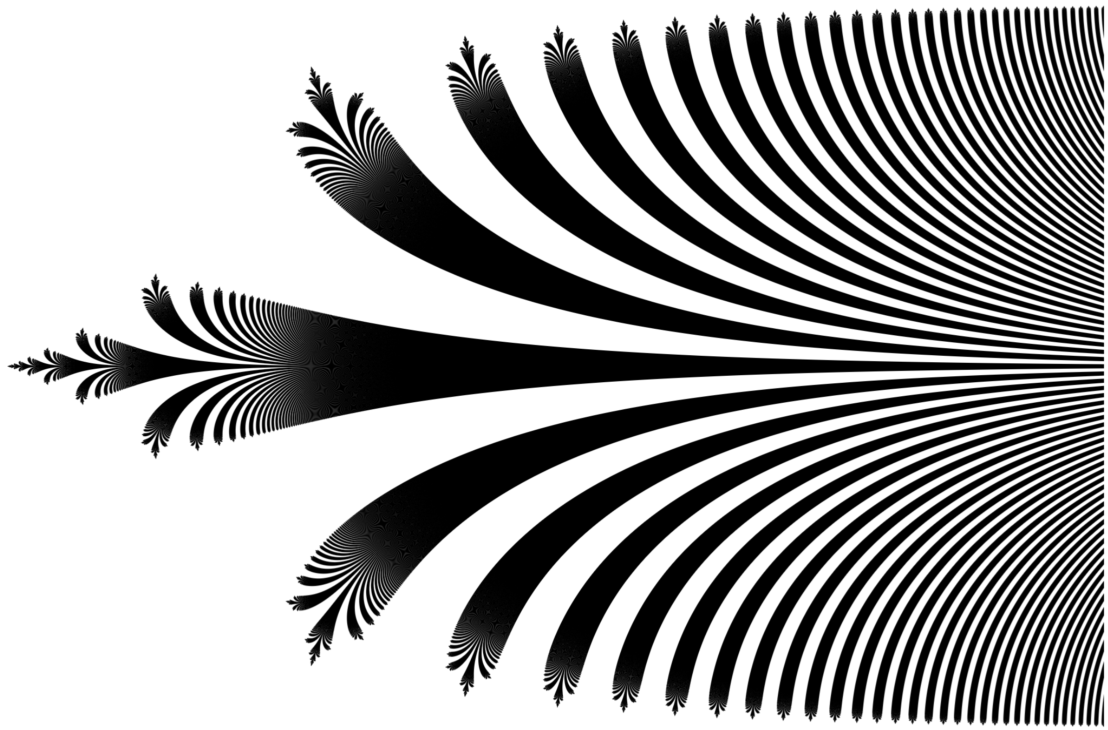


Graph giving wandering domain in Eremenko-Lyubich class.

Variations by Lazebnik, Fagella-Godillon-Jarque, Osborne-Sixsmith.

Application: dimensions of Julia sets Julia set is usually fractal. What is its (Hausdorff) dimension?

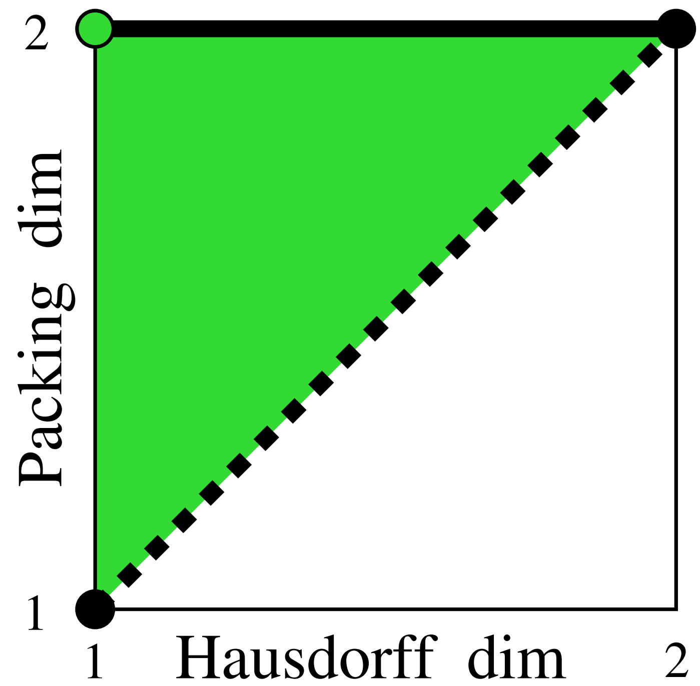
Packing dimension?



$\mathcal{J}((e^z-1)/2)$, courtesy of Arnaud Chéritat

A short (and incomplete) history:

- Baker (1975): f transcendental $\Rightarrow \mathcal{J}$ contains a continuum $\Rightarrow$ dim ≥ 1.
- Misiurewicz (1981), McMullen (1987) dim =2 occurs (is common)
- Stallard (1997, 2000): $\{\dim(\mathcal{J}(f)) : f \in \mathcal{B}\} = (1, 2]$.
- Rippon-Stallard (2005) Packing dim = 2 for $f \in \mathcal{B}$.
- Albrecht-B. (2020) $1 <$ H-dim < 2 can occur in Speiser class
- B (2018) H-dim = P-dim = 1 can occur (outside $\mathcal{B}$).
- Burkart (2019) $1 <$ P-dim < 2 can occur (outside $\mathcal{B}$).
- Many other results known, e.g., relating growth rate to dimension.

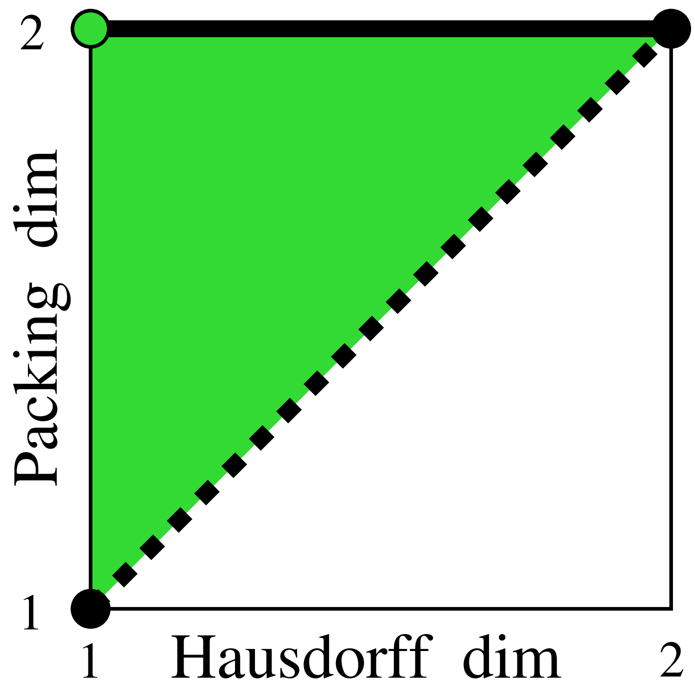


Hausdorff, upper Minkowski and packing dimension are defined as

$$\mathrm{Hdim}(K) = \inf\{s : \inf\{\textstyle\sum_j r_j^s : K \subset \cup_j D(x_j, r_j)\} = 0\},$$

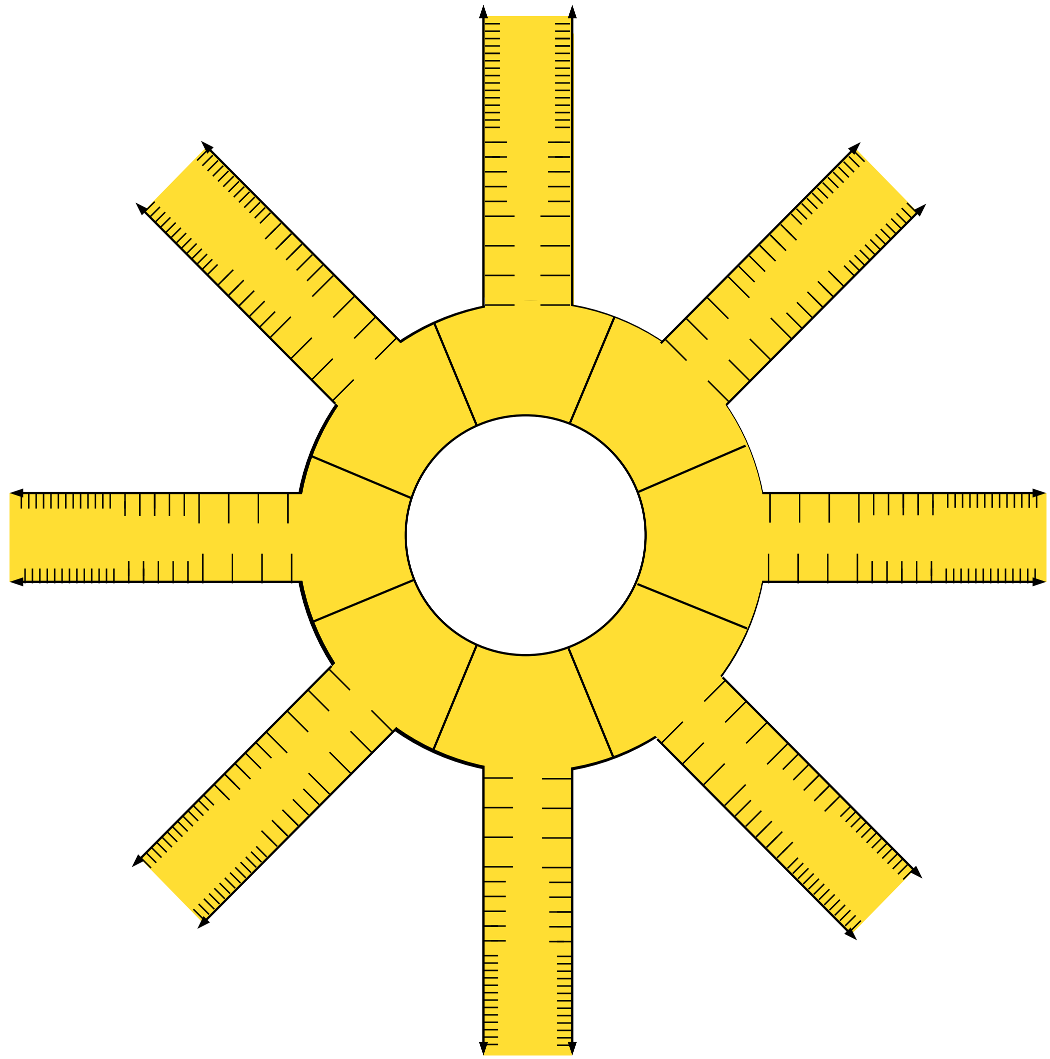
$$\overline{\mathrm{Mdim}}(K) = \inf\{s : \limsup_{r\to 0} \inf N(r) r^s = 0 : K \subset \cup_{j=1}^N D(x_j, r)\},$$

$$\mathrm{Pdim}(K) = \inf\{s : K \subset \cup_{j=1}^\infty K_j : \overline{\mathrm{Mdim}}(K_j) \le s \text{ for all } j\},$$



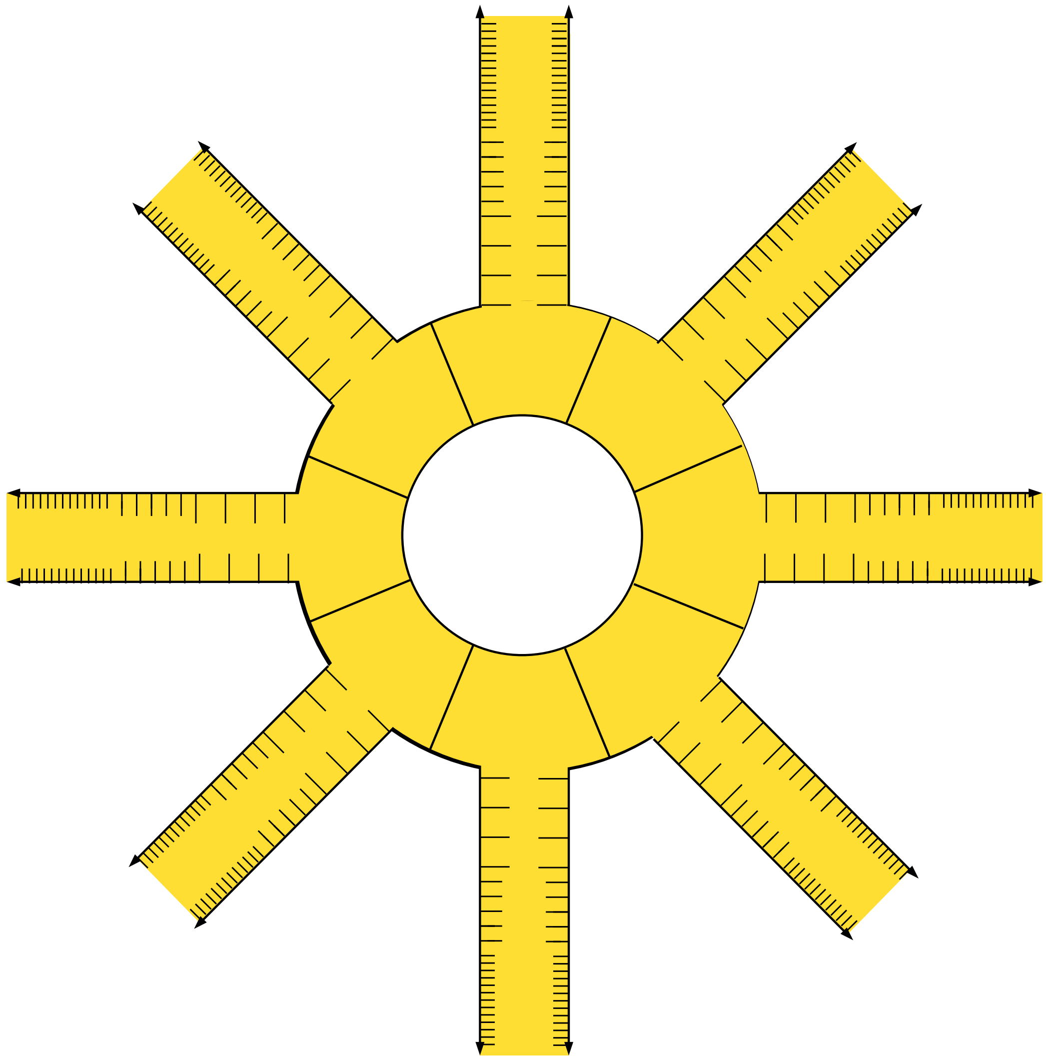
In polynomial dynamics it is difficult to construct examples with large dimension or positive area (Shishikura, Buff, Cheritat).

For entire functions, it is harder to find small Julia sets.



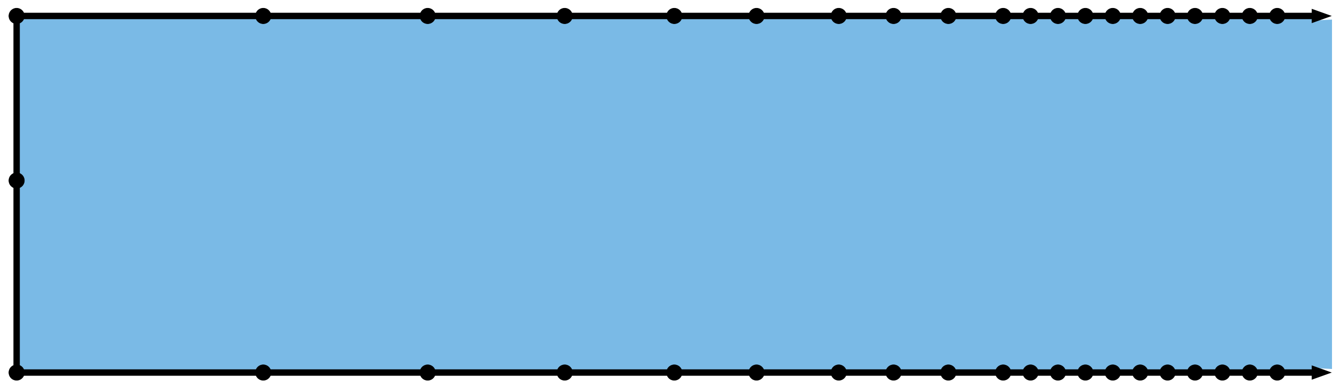
$$\inf\{\dim(\mathcal{J}(f)) : f \in \mathcal{S}\} = 1 \qquad \text{(B.-Albrecht, 2018).}$$

Uses upper and lower τ-bounds to control dimension.



Exponential imbalance. “Spikes” are chosen to give bounded imbalance.
Do this to control number and sizes of preimages of a disk.

Theorem (Stallard): There are Eremenko-Lyubich functions whose Julia sets have Hausdorff dimension close to 1.



There is EL function with tract $\{z : |f(z)| > 1\} \approx$ half-strip.

Use "EL-folding" to define QR g so that:

- $g(0) = 0$ and $|g(z)| < 1$ outside S = half-strip.
- Inside S, $g(z) = \exp(\exp(z - K))$ (conformal $S \to \mathbb{H}_r$, then exp).

Fixed point $g(0) = 0$ attracts everything in complement of S

Thus the Julia set is inside S. More precisely,

$$\mathcal{J}(g) \subset \bigcap X_n,$$

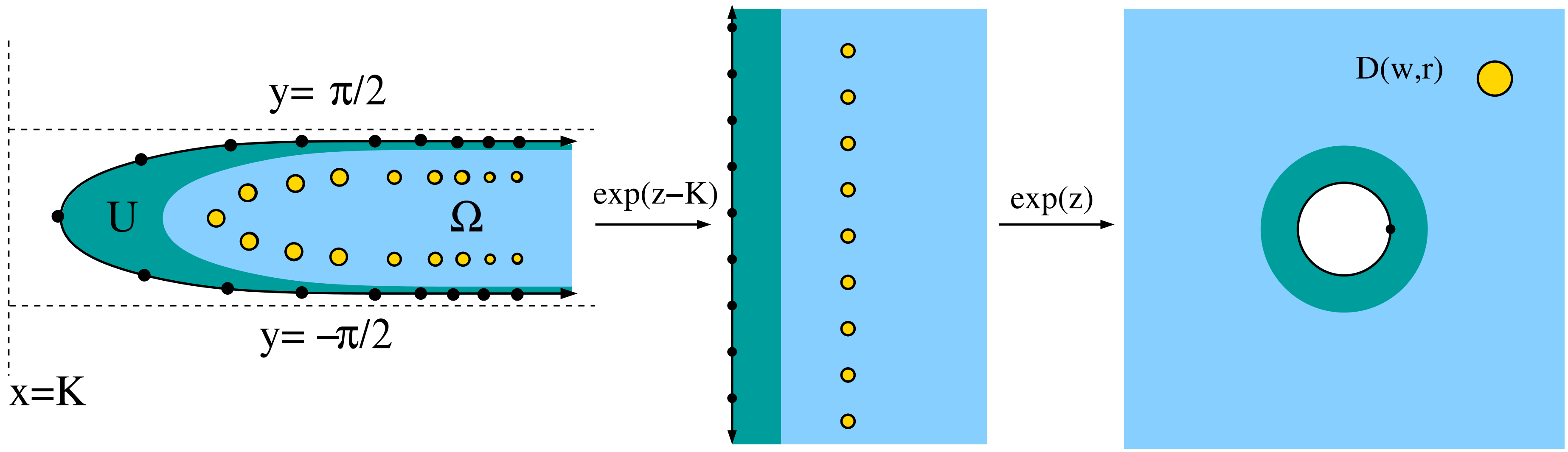
$$X_n = \{z : |g^k(z)| \geq K, k = 1, \ldots, n\}.$$

To prove $\dim(\mathcal{J}) \leq 1+\delta$, it suffices to show:

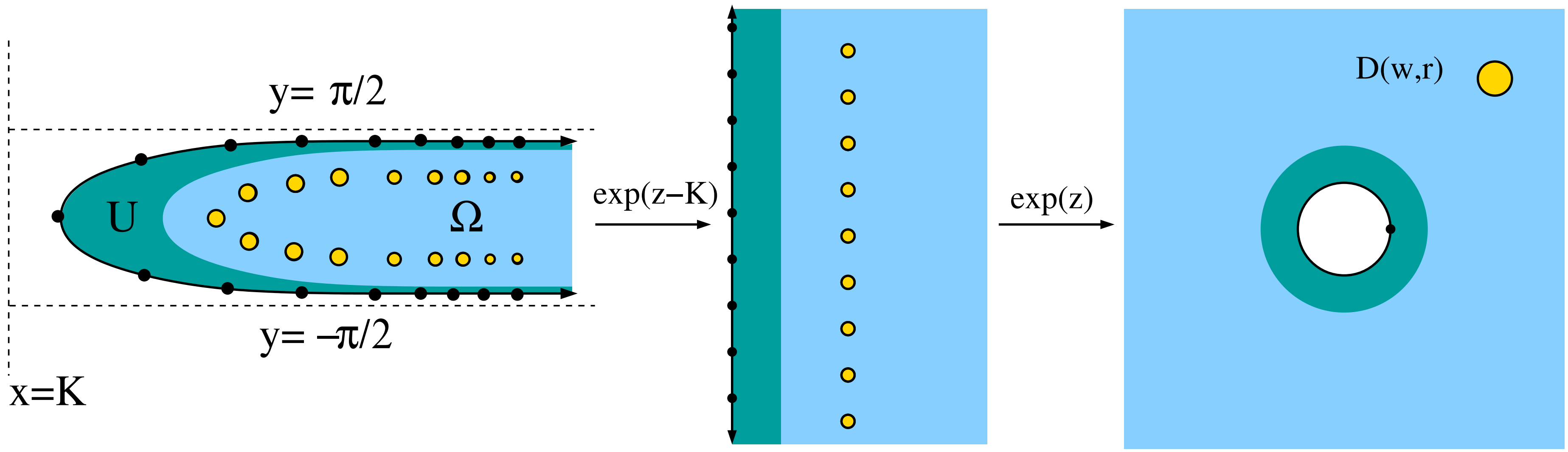
(1) X_1 can be covered by disks $\{D_j\}$ so that $\sum_j \operatorname{diam}(D_j)^{1+\delta} < \infty$,

(2) if D hits $\mathcal{J} \cap \{|z| > K\}$, then its preimages satisfy

$$\sum_{W_j \in f^{-1}(D)} \operatorname{diam}(W_j)^{1+\delta} \leq \epsilon \cdot \operatorname{diam}(D)^{1+\delta}.$$



This is what preimages of one disk look like.

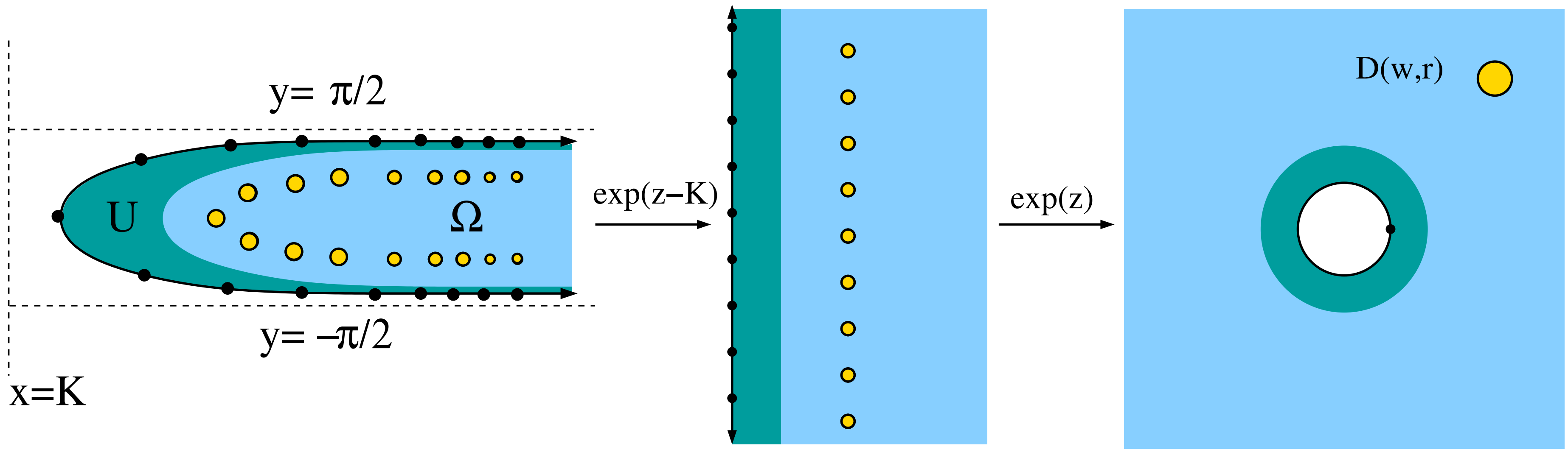


Preimage of gold disk $D = D(w, r)$ defined in two steps:

- stack of regions of diameter $O(r/|w|)$ on line $\{x = \log|w|\}$.
- region at height $2\pi k$ in stack has single preimage U_k of diameter

$$O\left(\frac{r}{|w|(\log|w| + 2\pi|k|)}\right).$$

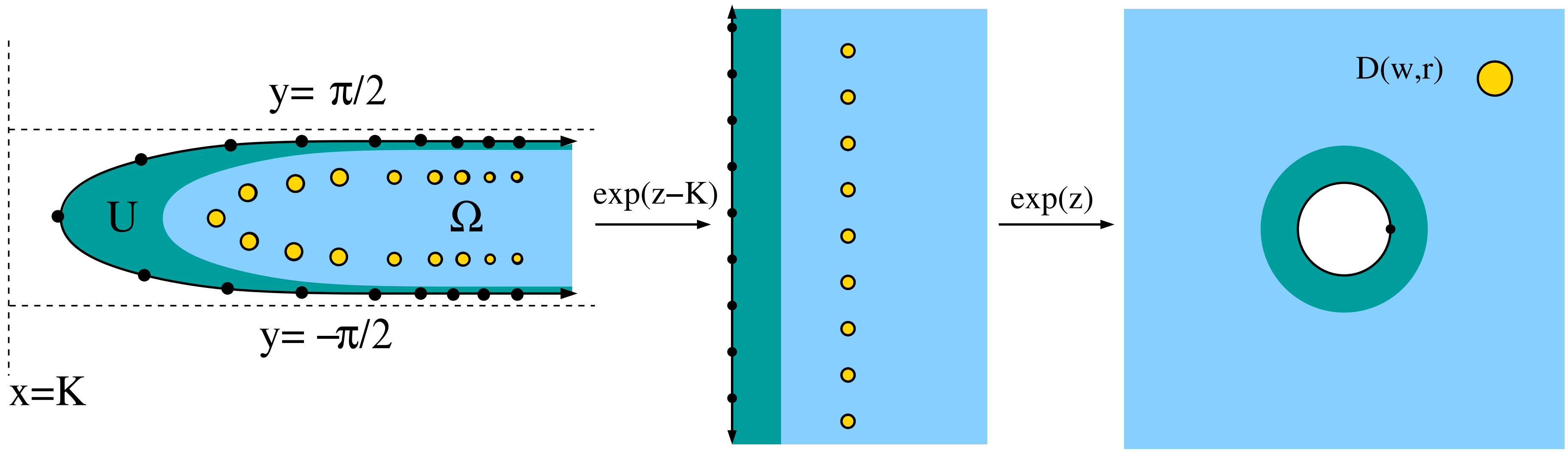
These estimates only use $(\log z)' = 1/z$.



If $\delta > 0$ is fixed and R is large enough, then

$$\sum_k \operatorname{diam}(U_k)^{1+\delta} \lesssim \left(\frac{r}{|w|}\right)^{1+\delta} \sum_k \frac{1}{(\log|w| + 2\pi|k|)^{1+\delta}}$$

$$\lesssim \left|\frac{r}{w}\right|^{1+\delta} \frac{1}{\delta \log^{1+\delta}|w|} \ll \left|\frac{r}{w}\right|^{1+\delta} \ll r^{1+\delta}$$



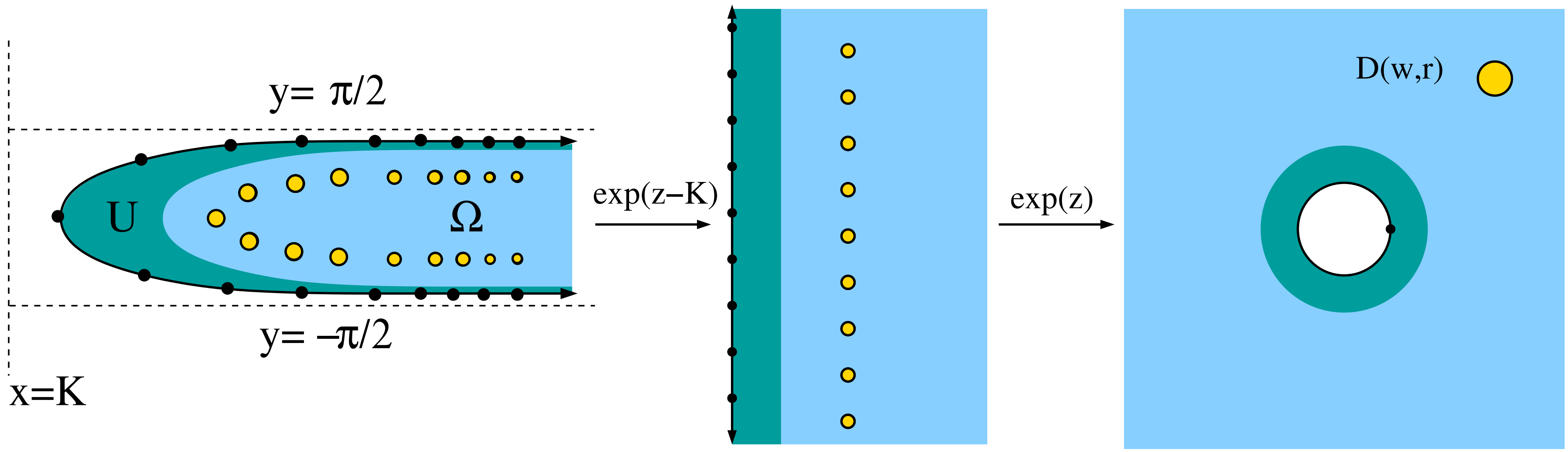
If $\delta > 0$ is fixed and R is large enough, then

$$\sum_k \operatorname{diam}(U_k)^{1+\delta} \lesssim \left(\frac{r}{|w|}\right)^{1+\delta} \sum_k \frac{1}{(\log|w| + 2\pi|k|)^{1+\delta}}$$

$$\lesssim \left|\frac{r}{w}\right|^{1+\delta} \frac{1}{\delta \log^{1+\delta}|w|} \ll \left|\frac{r}{w}\right|^{1+\delta} \ll r^{1+\delta}$$

This proves $\dim(\mathcal{J}(g)) \leq 1 + \delta$.

QC correction φ is bi-Lipschitz on $\mathcal{J}$ so $f = g \circ \varphi$ works too.

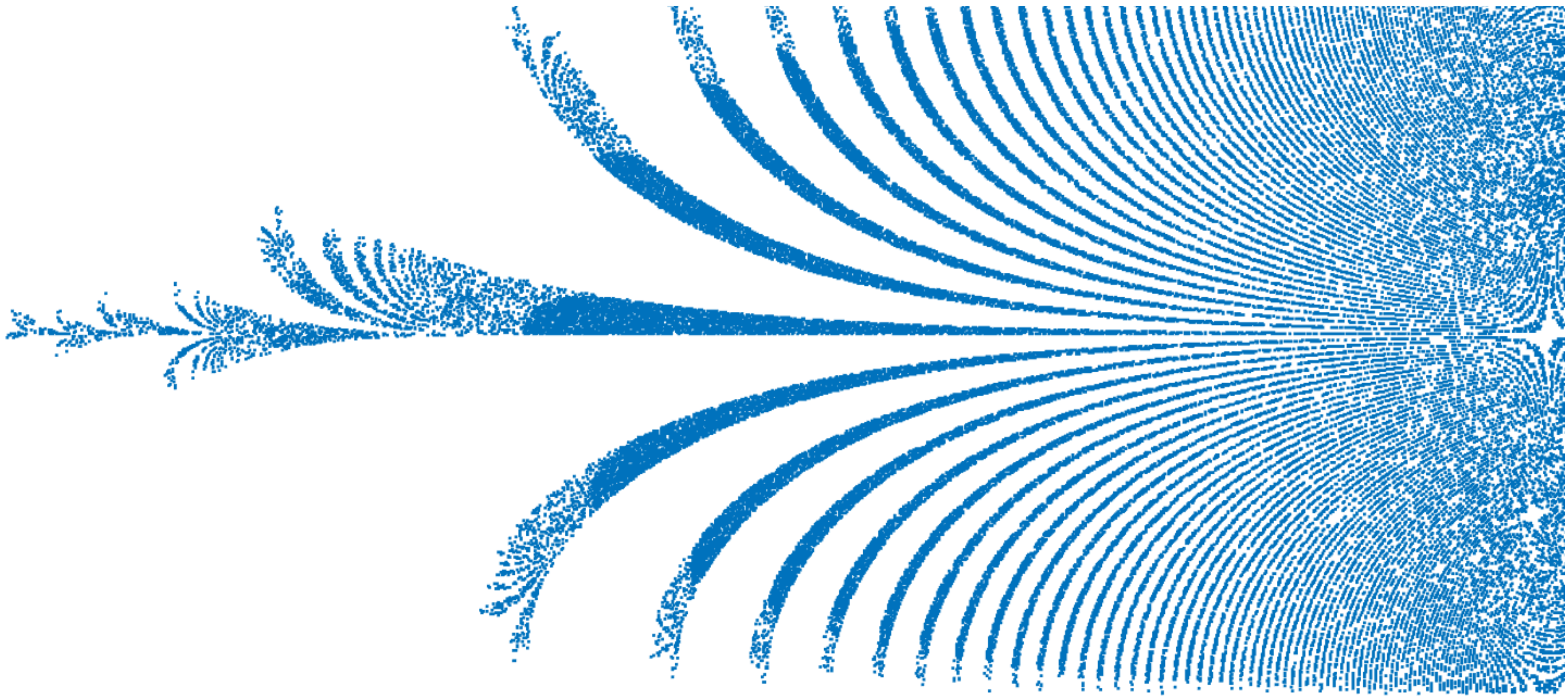


Lemma: If A, B are disjoint, planar sets and $|\mu_\varphi| \le k\chi_A$ where

$$\int_A \frac{dxdy}{|z-w|^2} \le C < \infty, \quad \forall w \in B,$$

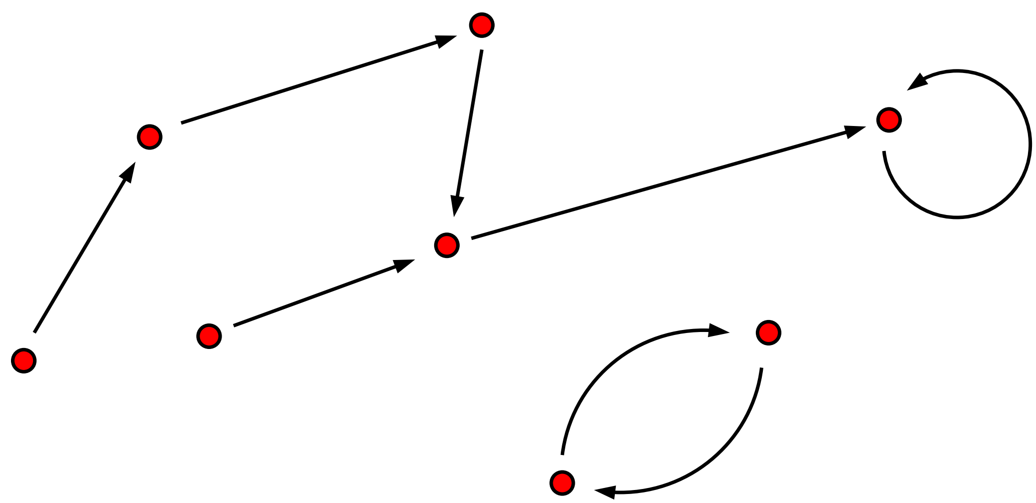
then φ is biLipschitz on B with constant $M(C, K)$.

Follows from work of Bojarski, Lehto, Tecihmüller and Wittich.



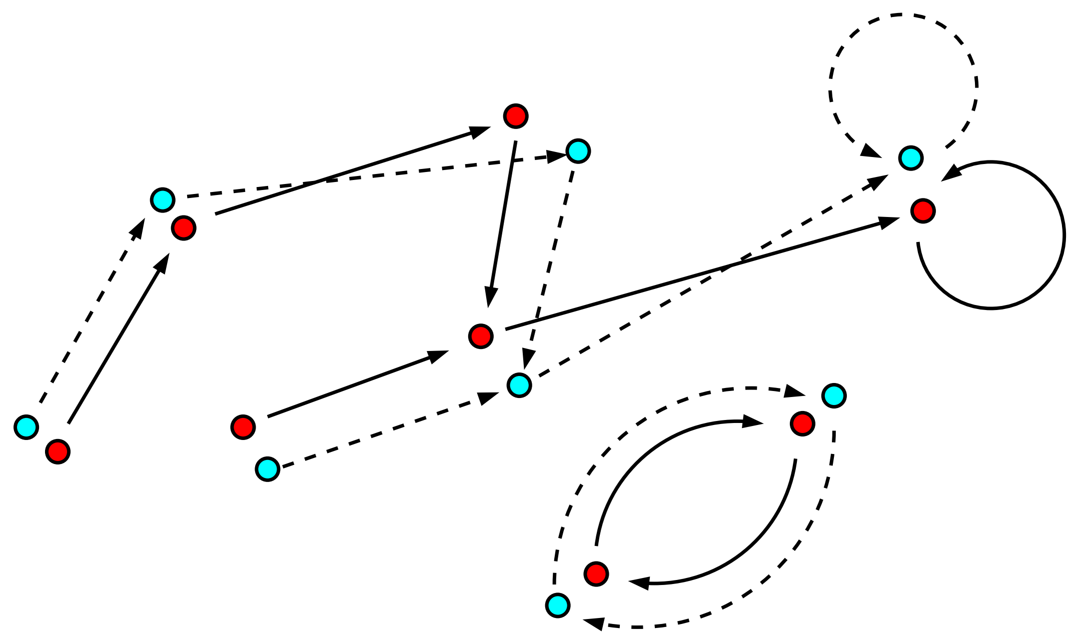
Application: prescribing postcritical dynamics

DeMarco, Koch and McMullen proved that a post-critically finite rational map can have **any** given dynamics.

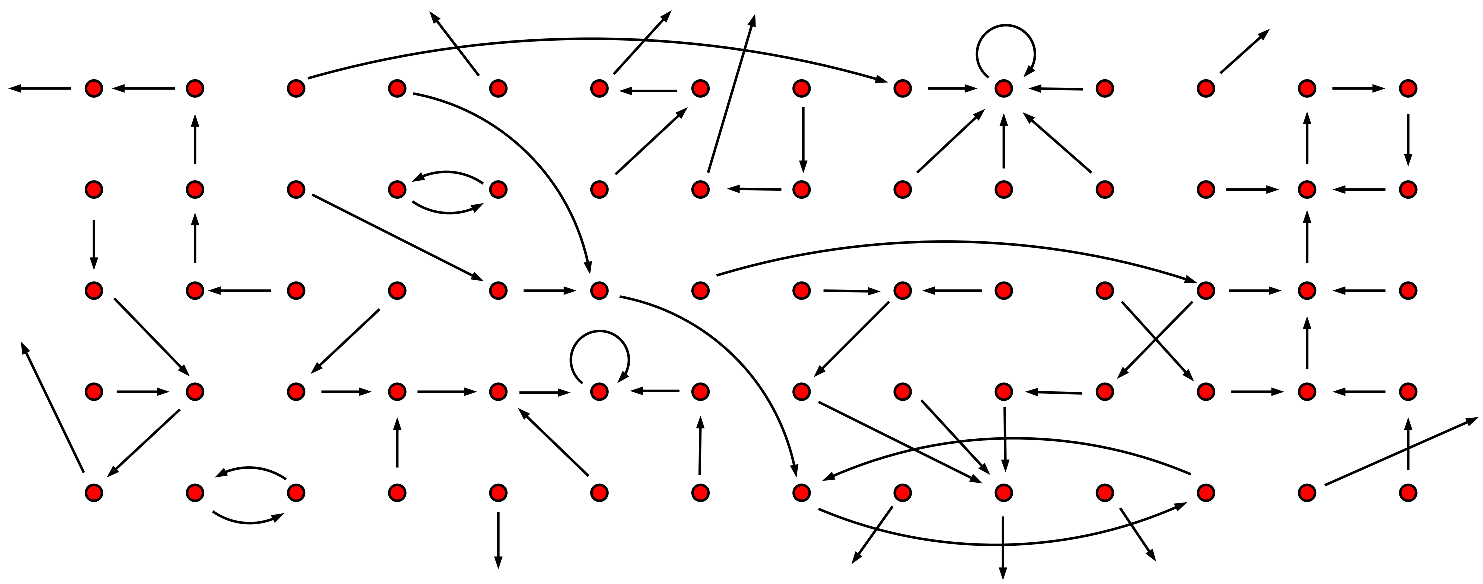


Application: prescribing postcritical dynamics

DeMarco, Koch and McMullen proved that a post-critically finite rational map can have **any** given dynamics.



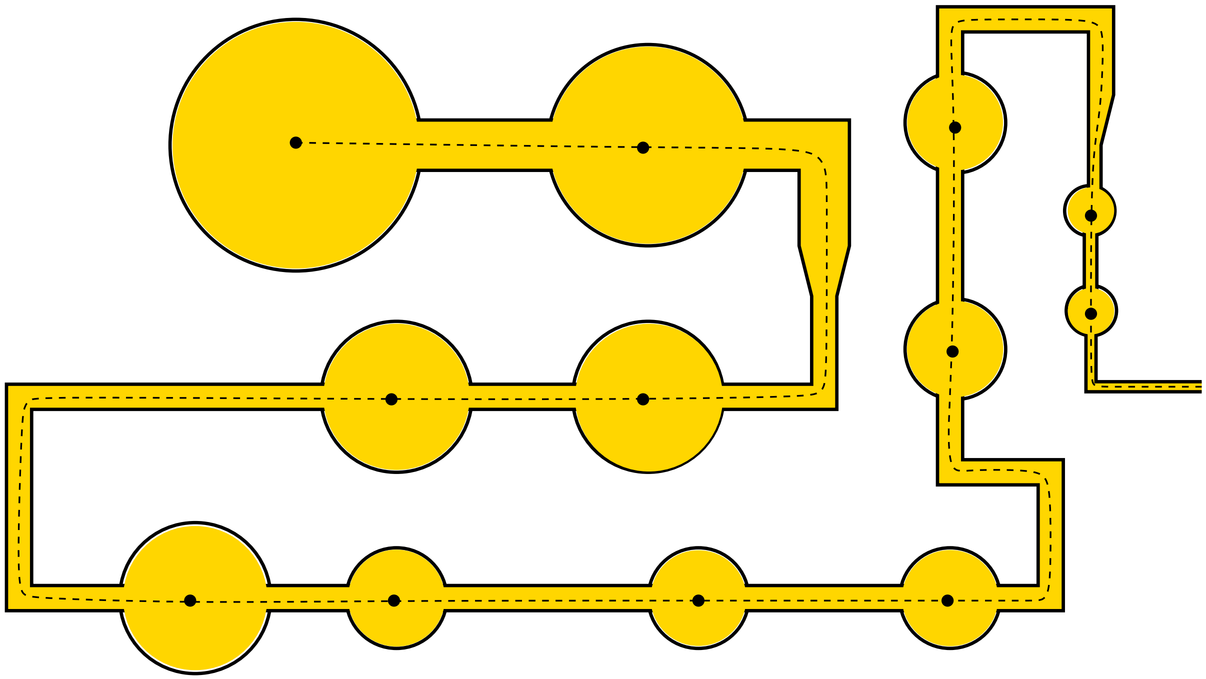
More precisely, given any $\epsilon > 0$, any finite set X and any map $h : X \to X$ there is a rational r whose post-singular set $P(r)$ is the same size as X, $P(r)$ ϵ-approximates X in Hausdorff metric, and $|r - h| < \epsilon$ on P.



Theorem (Lazebnik, B.): Let X be discrete ($\geq$ 4 points), let $h : X \to X$ be any map, and let $\epsilon > 0$. Then there is a transcendental meromorphic function f and a bijection $\psi : X \to P(f)$ so that

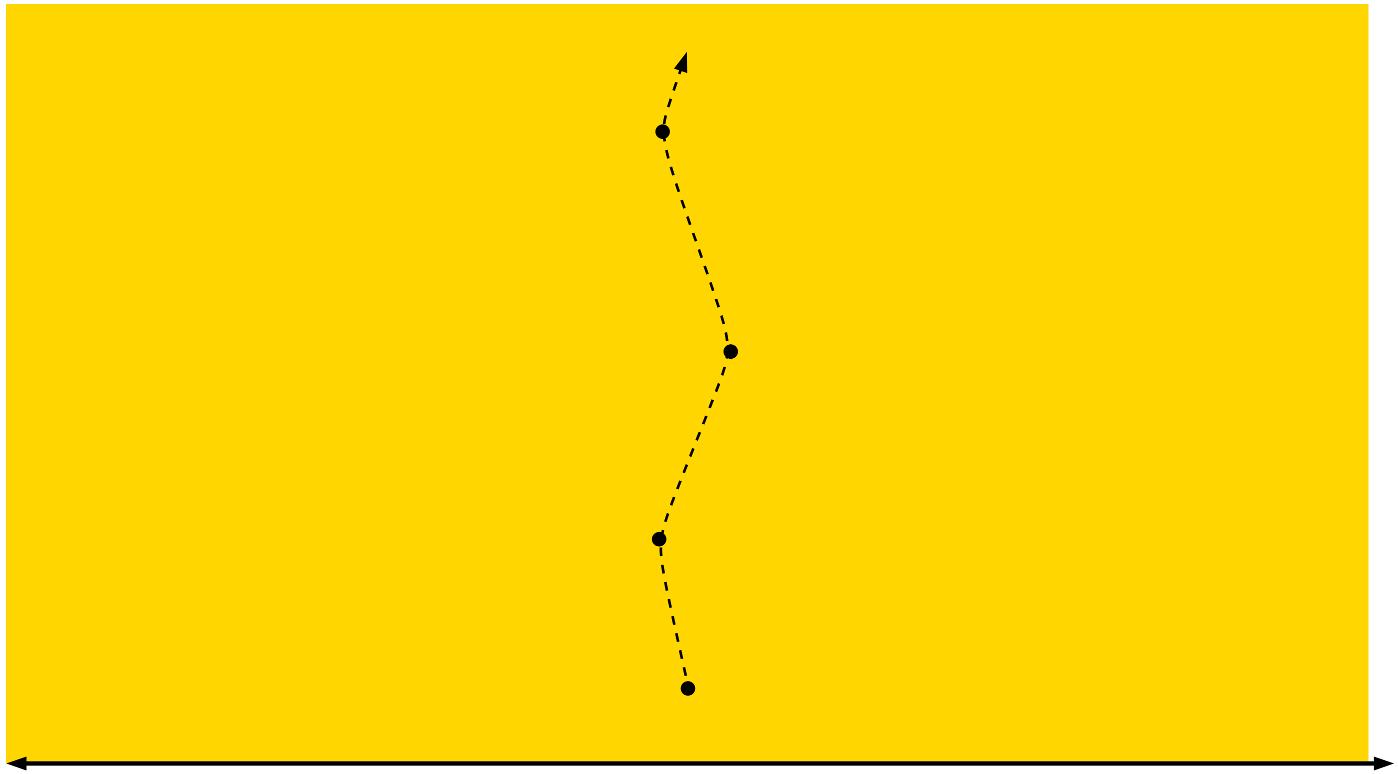
$$|\psi(z) - z| \leq \max(\epsilon, o(1)) \quad \text{and} \quad f = \psi \circ h \circ \psi^{-1} \text{ on } P(f).$$

Main argument (folding + fixed point thm) due to Kirill.

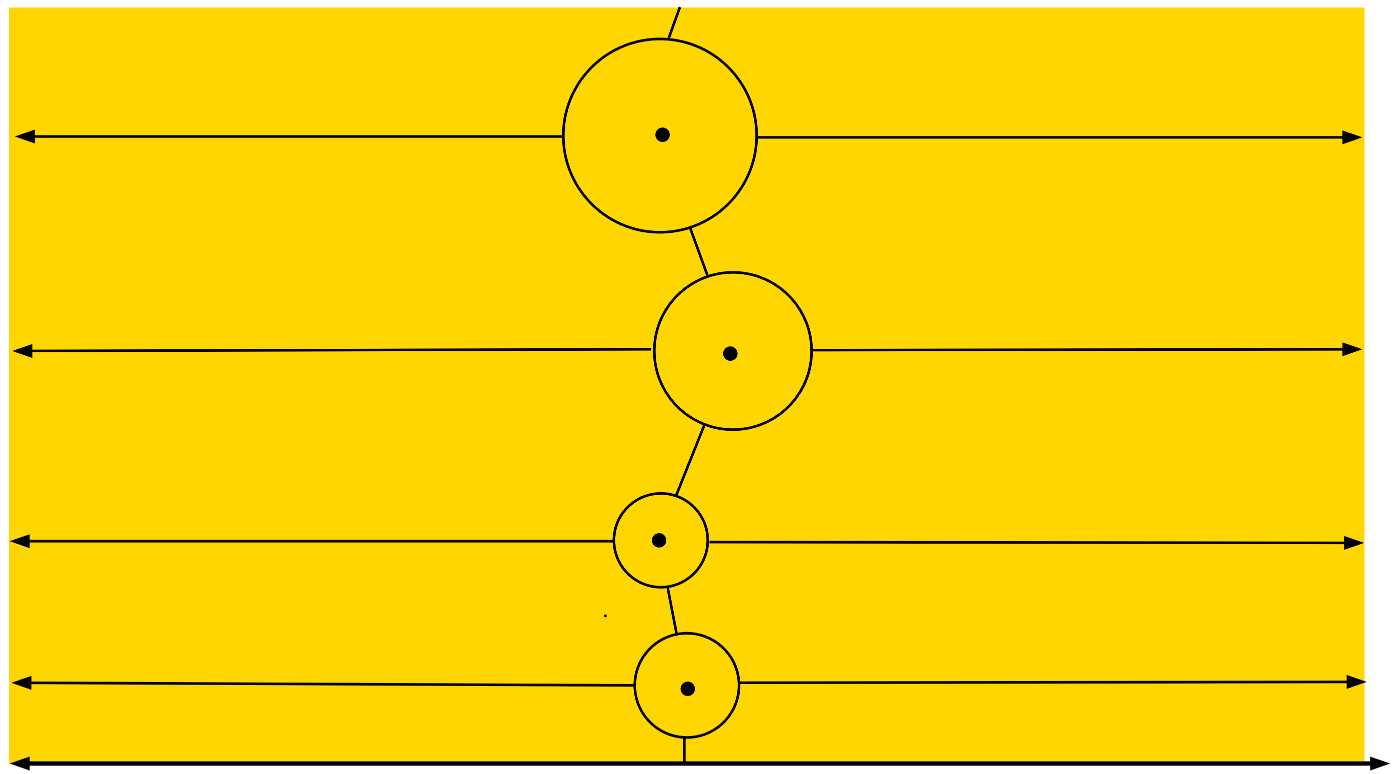


Reduce to case where ± 1 are in post-singular set and all other post-singular points are off the unit circle.

Construct a domain W that contains the given set, and so that these points lie very close to hyperbolic geodesic to ∞.



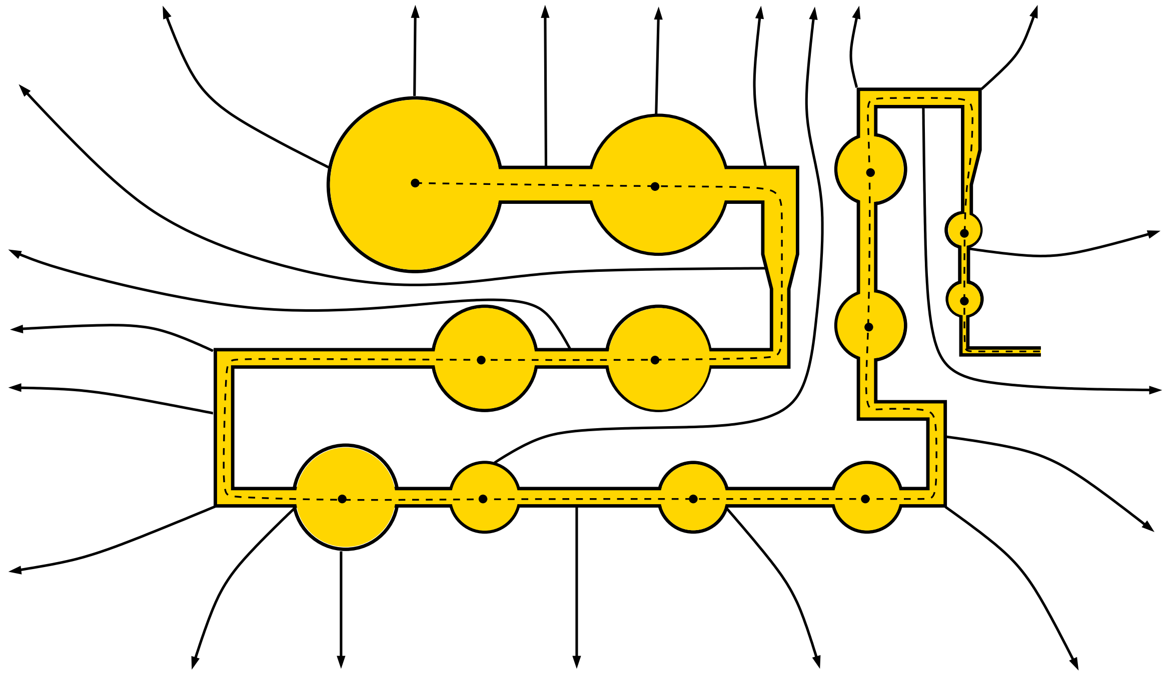
Conformally map W to upper half-plane; points almost on vertical ray.



Divide upper half-plane into R and D components.

Folding map sends centers anywhere we want in $\mathbb{D}$.

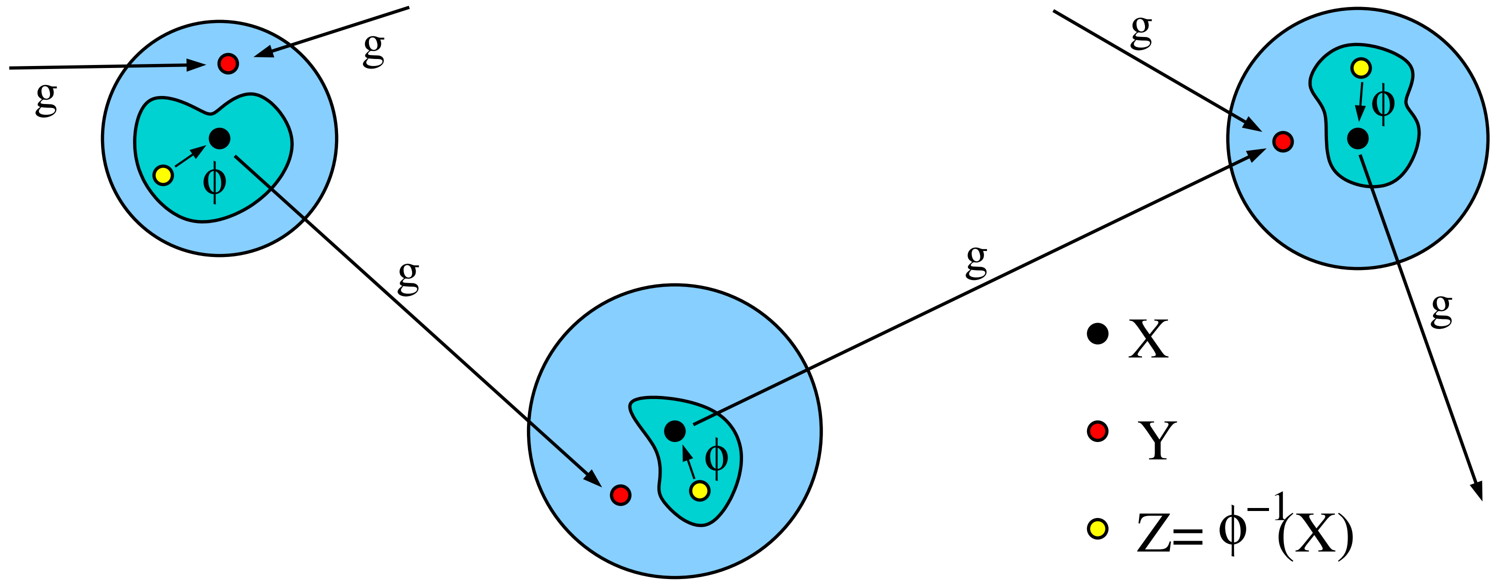
Analogous construction maps centers outside $\mathbb{D}$; introduces poles.



Transfer graph back to W. Decompose complement of W into R-components (not hard) and define map on whole plane by folding.

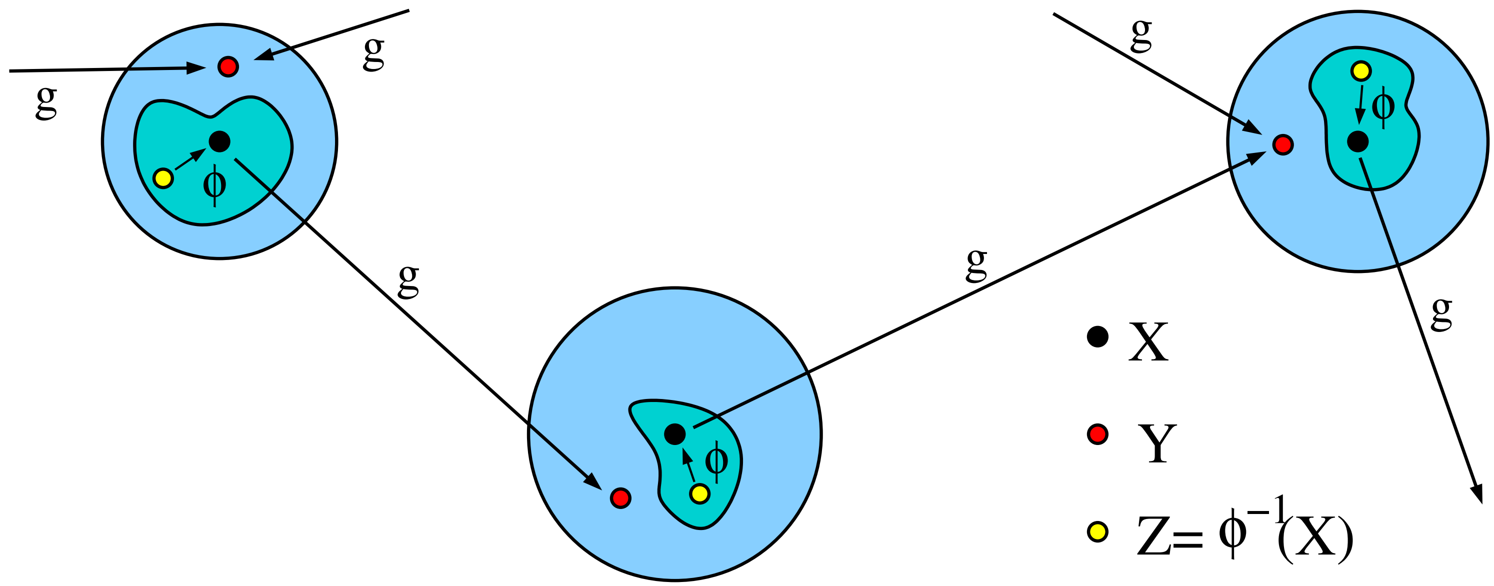
Gives quasi-meromorphic map g with desired post-singular behavior.

Mapping theorem gives a meromorphic $f = g \circ \phi$. However, because of ϕ, the post-singular set might not be invariant under f. How to fix this?



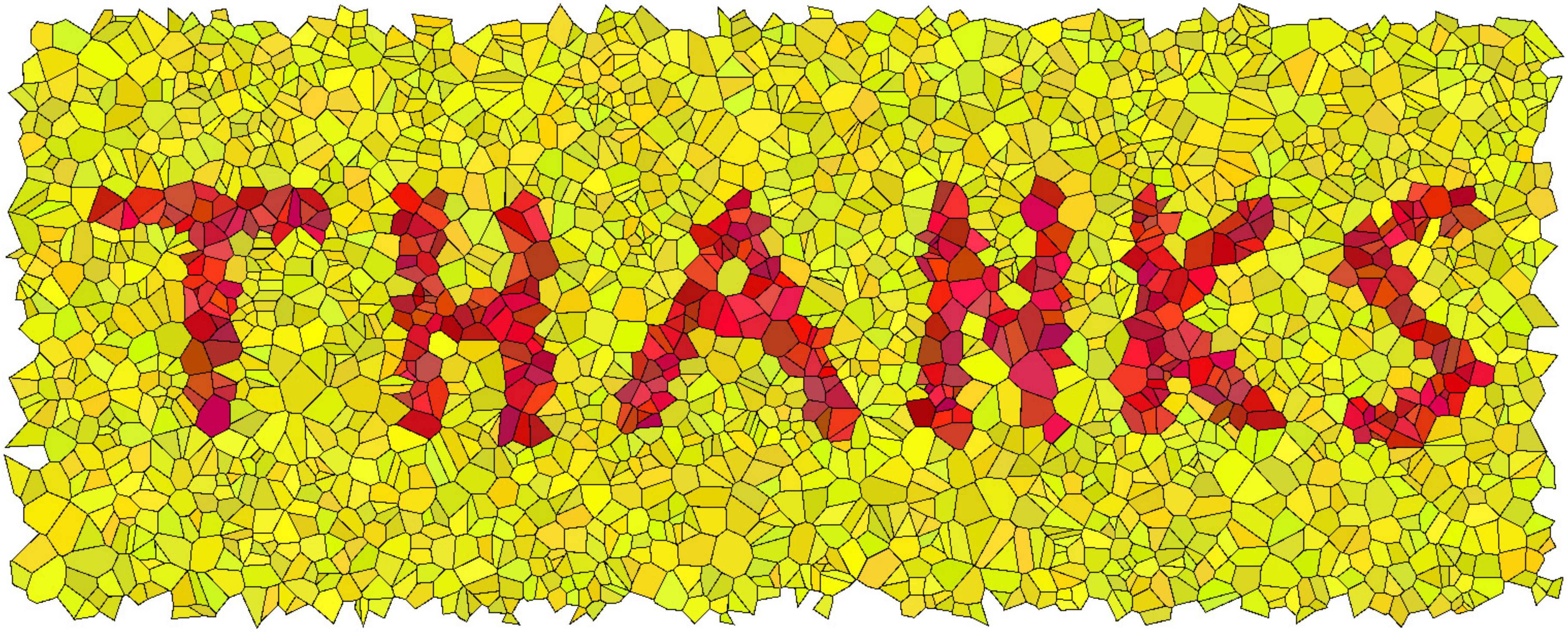
Replace X (black) by a union of disjoint disks (blue), and let $Y \approx X$ be one point from each disk (red). As above, construct g so $X \to Y$. Get meromorphic $f = g \circ \phi$ that has singular points at $Z = \phi^{-1}(X)$ (yellow).

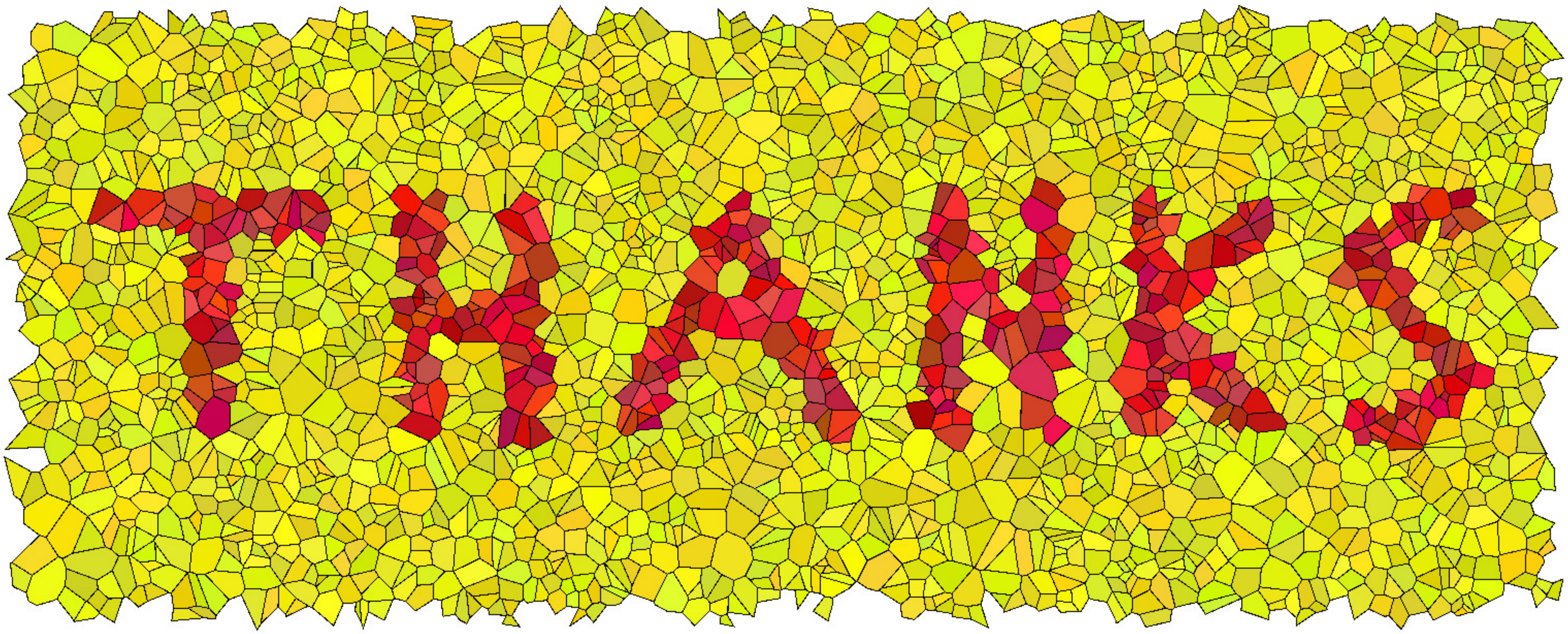
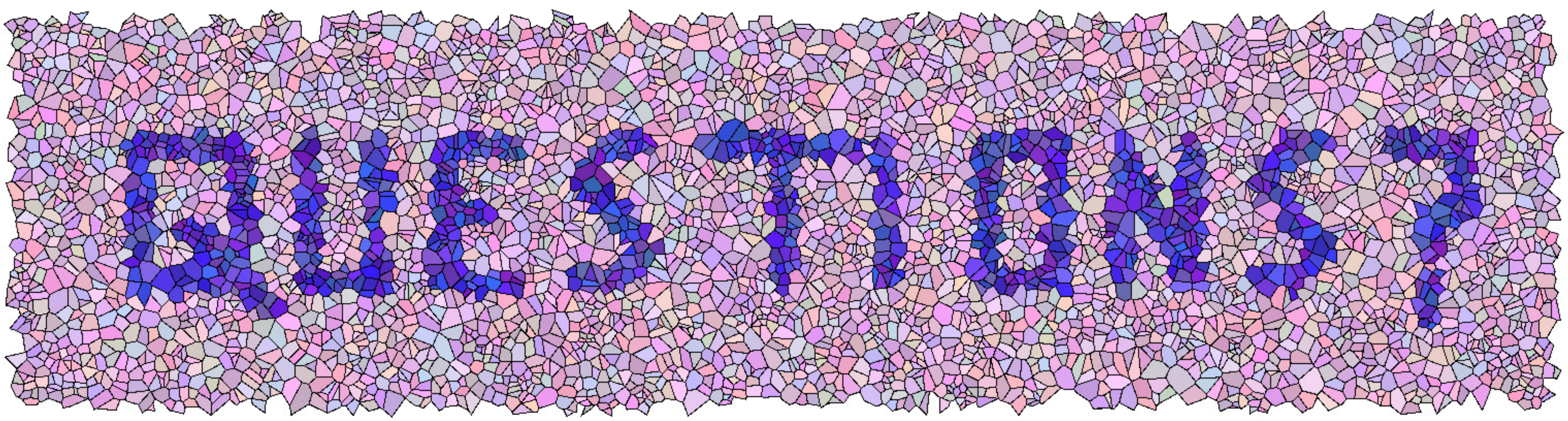
We want $Z = Y$ (yellow points = red points).

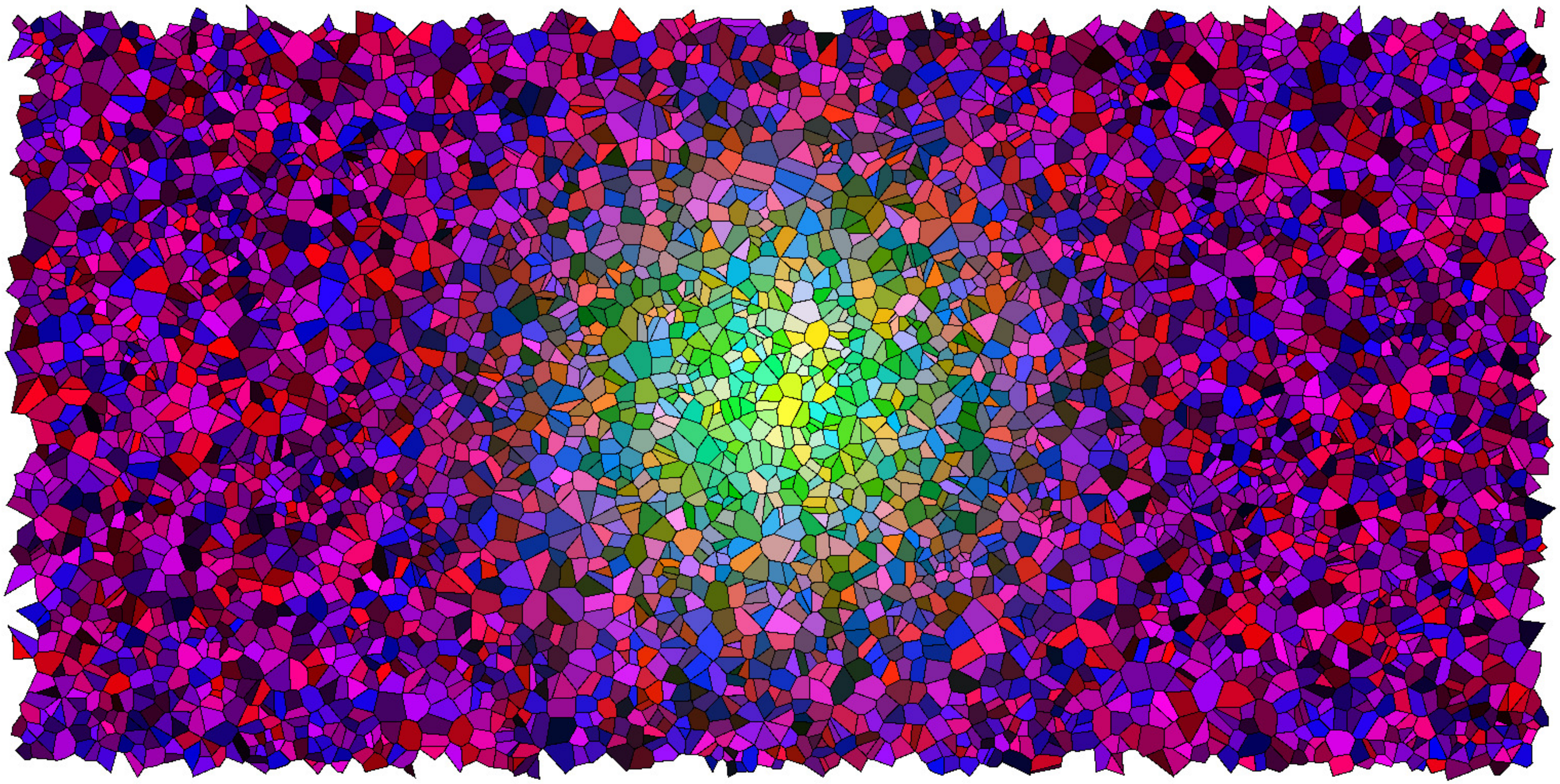


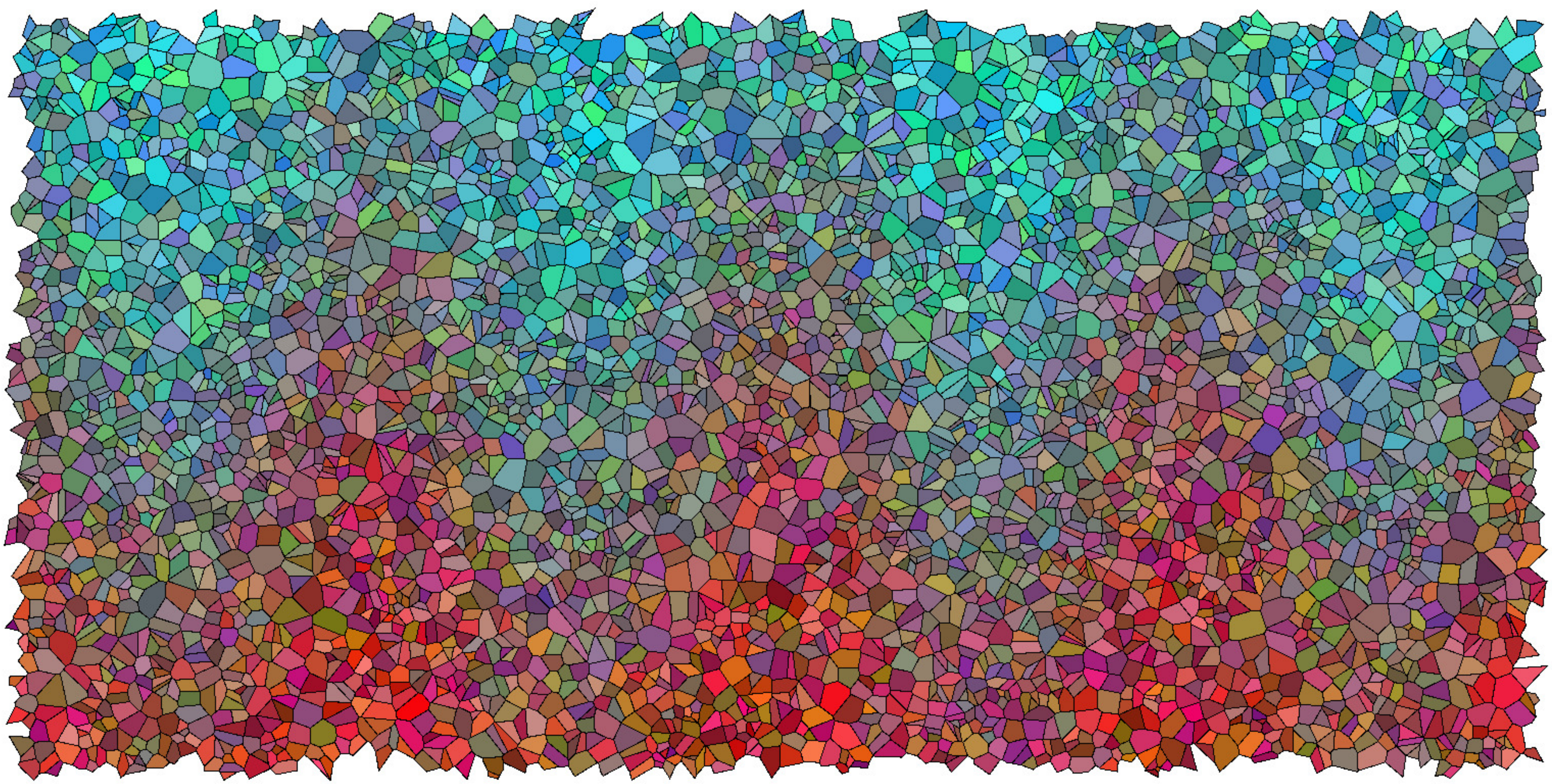
Replace X (black) by a union of disjoint disks (blue), and let $Y \approx X$ be one point from each disk (red). As above, construct g so $X \to Y$. Get meromorphic $f = g \circ \phi$ that has singular points at $Z = \phi^{-1}(X)$ (yellow).

Z (yellow) is continuous function of Y (red). Because ϕ is close to the identity, Z remain inside the disks. An infinite-dimensional fixed point theorem gives choice of Y so that $Z = Y$, as desired.


THANKS


THANKS

QUESTIONS?





Coming Attractions

"Mappings and Meshes: connections between continuous and discrete geometry"

2 lectures: 11am Fri May 7, and 11am Sat May 8 (NY time)

FRG Workshop on Geometric Methods for Analyzing Discrete Shapes

https://cmsa.fas.harvard.edu/frg-2021/

Abstract: I will give two lectures about some interactions between conformal, hyperbolic and computational geometry. The first lecture shows how ideas from discrete and computational geometry can help compute conformal mappings, and the second lecture reverses the direction and shows how conformal maps can give meshes of polygonal domains with optimal geometry.