Planar maps with at most six neighbors on average

Christopher Bishop, Stony Brook
Dennis Sullivan, Stony Brook
Michael Wigler, Cold Spring Harbor

AMS Sectional Meeting – March 19, 2016 Stony Brook NY
www.math.sunysb.edu/~bishop/lectures
Hexagon $\rightarrow$ triangles with probability $p$, $\text{ANS} = \frac{12p+6}{5p+1}$ ($p = 1/4$ shown).
Limit need not exist
average number sides $\rightarrow \infty$, areas $\rightarrow 0$, diameters bounded
average number sides $\uparrow \infty$, diameters $\rightarrow \infty$, areas bounded below
average number sides $\uparrow \infty$, forbid vertices of degree 1 and 2
Theorem 1: Suppose faces have areas that are bounded below and diameters that are bounded above, and every vertex had degree $\geq 3$. Then

$$\limsup_{t \to \infty} \text{ANS}(t) \leq 6.$$  

Average is over all faces contained in circle of radius $t$.  

Average over faces inside/hitting an expanding circle
Theorem false in three dimensions
Euler’s formula for finite planar graphs: \( V - E + F = C + 1 \)

\( V \) = number of vertices
\( E \) = number of edges
\( F \) = number of faces
\( C \) = number of components
Euler’s formula for finite planar graphs: $V - E + F = C + 1$

$V =$ number of vertices

$E =$ number of edges

$F =$ number of faces

$C =$ number of components
Euler’s formula for finite planar graphs: \( V - E + F = C + 1 \)

\( V = \) number vertices
\( E = \) number of edges
\( F = \) number of faces
\( C = \) number of components
**Sphere Theorem:** If $H$ is a finite graph on the 2-sphere and every vertex has degree $\geq 3$, then $E < 3F$. 
Sphere Theorem: If $H$ is a finite graph on the 2-sphere and every vertex has degree $\geq 3$, then $E < 3F$.

Proof: Clearly

$$\sum_{v \in H} \deg(v) = 2E.$$

By assumption, $\deg(v) \geq 3$ for all vertices so $3V \leq 2E$. Hence $V \leq \frac{2}{3}E$. 
**Sphere Theorem:** If $H$ is a finite graph on the 2-sphere and every vertex has degree $\geq 3$, then $E < 3F$.

**Proof:** Clearly

$$\sum_{v \in H} \deg(v) = 2E.$$  

By assumption, $\deg(v) \geq 3$ for all vertices so $3V \leq 2E$. Hence $V \leq \frac{2}{3}E$. Plugging this into Euler’s formula $V - E + F = C + 1$ gives

$$-\frac{1}{3}E + F \geq C + 1$$

or

$$E \leq 3F - 3(C + 1) < 3F.$$
**Sphere Theorem:** If $H$ is a finite graph on the 2-sphere and every vertex has degree $\geq 3$, then $E < 3F$.

**Corollary:** Number of sides $= 2E < 6F$. So for a finite planar graph (all degrees $\geq 3$), the average number of sides per face is $\leq 6$.

$E = 16, \quad F = 8, \quad \text{ANS} = 4$
This is for finite $H$. What about infinite maps?
Choose a piecewise smooth region $R$ in the plane and let $tR + x$ denote the region dilated by a factor of $t > 0$ and translated by $x$.

Let $H = H(R, t, x)$ be the sub-map of $G$ consisting of the 2-cells of $G$ that lie inside $tR + x$.

Let $\text{ANS}(t)$ be average number of sides over faces in $H(R, t, x)$.

Usually take circle around origin.
**Theorem 1:** Suppose all faces of $G$ have diameter $\leq D < \infty$, and have area $\geq A > 0$, and that every vertex has degree $\geq 3$. Then

$$\limsup_{t \to \infty} \text{ANS}(H(R, t, x)) \leq 6.$$
**Theorem 1:** Suppose all faces of $G$ have diameter $\leq D < \infty$, and have area $\geq A > 0$, and that every vertex has degree $\geq 3$. Then

$$\limsup_{t \to \infty} \text{ANS}(H(R, t, x)) \leq 6.$$ 

Average is taken over faces contained in $tR + x$.

Same conclusion holds for faces hitting $tR + x$.

Really only need $R$ to have non-empty interior and zero area boundary.
Definition: edge boundary: If $H \subset G$, then $\partial_E H$ is set of edges in $G \setminus H$ with at least one endpoint in $H$.

$$e(H) = \frac{|\partial_E H|}{F}.$$ 

Here $e(H) = 9/5$. Related to Cheeger constant of $G$. 

Lemma 1: If $H$ is a sub-map of $G$ then $\text{ANS}(H) \leq 6 + 4e(H)$.

Proof: Note that

$$2E + |\partial_E H| \leq \sum_{v \in H} \deg_G(v) \leq 2E + 2|\partial_E H|,$$

The left hand inequality would be an equality, except that some edges in $\partial_E H$ might have both endpoints on $\partial H$. 

Lemma 1: If $H$ is a sub-map of $G$ then $\text{ANS}(H) \leq 6 + 4e(H)$.

Proof continued: Every vertex has degree $\geq 3$, so

$$3V \leq \sum_{v \in H} \deg_G(v) \leq 2E + 2|\partial_E H|.$$ 

Divide by 3 and put estimate for $V$ into Euler’s formula:

$$\left(\frac{2}{3}E + \frac{2}{3}|\partial_E H|\right) - E + F \geq C$$

Simplifying gives

$$\frac{E}{F} \leq 3 + 2\frac{|\partial_E H|}{F} - 3\frac{C}{F} \leq 3 + 2e(H).$$

Hence

$$\text{ANS}(H) \leq 2E/F \leq 6 + 4e(H).$$
Definition: the face boundary: $\partial_F H$ is the set of faces in $G \setminus H$ that touch $H$.

Easy to see $|\partial_F H| \leq |\partial_E H|$. We need converse direction.
One adjacent face, many adjacent edges
Lemma 2: Suppose the faces of $G$ have diameters $\leq D < \infty$ and $H$ is a sub-map. Let $N$ be the number of faces that lie inside a $3D$-neighborhood of $\partial H$. Then $|\partial_E H| \leq 3N$. 
**Lemma 2:** Suppose the faces of $G$ have diameters $\leq D < \infty$ and $H$ is a sub-map. Let $N$ be the number of faces that lie inside a $3D$-neighborhood of $\partial H$. Then $|\partial_E H| \leq 3N$.

**Proof:**

Let $G'$ be the finite graph on the sphere consisting of the faces of $G$ that lie within the $2D$-neighborhood of $\partial H$, together with their edges and vertices. Check that edges of $G'$ include all edges in $\partial_E H$.

Each face of $G'$ is either a face of $G$ or contains a face of $G$ that is within $3D$ of $\partial H$. Thus number of faces of $G'$ is at most $N$. 
**Lemma 2:** Suppose the faces of $G$ have diameters $\leq D < \infty$ and $H$ is a sub-map. Let $N$ be the number of faces that lie inside a $3D$-neighborhood of $\partial H$. Then $|\partial E H| \leq 3N$.

**Proof continued:**

Define $G''$ by removing any vertices of degree 2 from $G'$; combine edges. $G'$ and $G''$ have same number of faces, $\leq N$.

If $e \in \partial E H$, endpoints have degree $\geq 3$ in $G'$ (any edge touching $e$ in $G$ is in $G'$). So $e$ is an edge of $G''$.

By Sphere Theorem, edges in $G''$ bounded by three times faces in $G''$. Hence $|\partial E H| \leq 3N$. 
Theorem 1: Suppose all faces of $G$ have diameter $\leq D < \infty$, and have area $\geq A > 0$, and that every vertex has degree $\geq 3$. Then

$$\limsup_{t \to \infty} \text{ANS}(H(R, t, x)) \leq 6.$$
**Theorem 1:** Suppose all faces of $G$ have diameter $\leq D < \infty$, and have area $\geq A > 0$, and that every vertex has degree $\geq 3$. Then

$$\limsup_{t \to \infty} \operatorname{ANS}(H(R, t, x)) \leq 6.$$ 

**Proof:** Only need show $e(H(R, t, x)) \to 0$. 
**Theorem 1:** Suppose all faces of $G$ have diameter $\leq D < \infty$, and have area $\geq A > 0$, and that every vertex has degree $\geq 3$. Then

$$\limsup_{t \to \infty} \text{ANS}(H(R, t, x)) \leq 6.$$  

**Proof continued:** The number of faces in $H(R, t, x)$ is $\geq ct^2$. 

![Diagram](https://via.placeholder.com/150)
**Theorem 1:** Suppose all faces of $G$ have diameter $\leq D < \infty$, and have area $\geq A > 0$, and that every vertex has degree $\geq 3$. Then

$$\limsup_{t \to \infty} \text{ANS}(H(R, t, x)) \leq 6.$$ 

**Proof continued:** Area within $3D$ of $\partial H$ is $O(t \cdot D)$, so $|\partial_E H| = O(tD/A)$. 

![Diagram of a circle with a radius of 3D]
**Theorem 1:** Suppose all faces of $G$ have diameter $\leq D < \infty$, and have area $\geq A > 0$, and that every vertex has degree $\geq 3$. Then

$$\limsup_{t \to \infty} \text{ANS}(H(R, t, x)) \leq 6.$$ 

**Proof continued:** Hence $e(H) = O(t/t^2) = O(1/t)$. 
**Theorem 2:** If $G$ satisfies the area lower bound and diameter upper bounded and every vertex has degree 3, then

$$\lim_{t \to 0} \text{ANS}(t) = 6.$$
Voronoi diagram of Poisson point process
Ω is $\delta$-thick if for all $0 < r < \text{diam}(\Omega)$ and all $x \in \partial \Omega$ we have
\[
\text{area}(\Omega \cap D(x, r)) \geq \delta r^2.
\]

True if $\Omega$ is convex, with a $\delta$ that depends on the aspect ratio.

Holds for quasidisks.
Theorem 3: Suppose $G$ is a planar map and every face is $\delta$-thick, for some $\delta > 0$. Then there is a nested, increasing sequence of sub-maps $\{H_n\}$ so that $\lim_{n \to \infty} \text{ANS}(H_n) \leq 6$. 
**Theorem 3:** Suppose $G$ is a planar map and every face is $\delta$-thick, for some $\delta > 0$. Then there is a nested, increasing sequence of sub-maps $\{H_n\}$ so that $\lim_{n \to \infty} \text{ANS}(H_n) \leq 6$.

**Idea of proof:**

If $\lim \inf > 6$, then $e(H(t)) > \epsilon > 0$ for all $t > t_0$.

We can show this forces $|\partial_F H(t)| \nearrow \infty$ in a finite time.

This contradicts local finiteness of $G$. 
\[
\limsup = \infty, \quad \liminf \leq 6.
\]
Aspect Ratio\((K) = \inf R/r\) where \(D(x, r) \subset K \subset D(y, R)\).

Is there a “6” theorem if faces have uniformly bounded aspect ratios?
Aspect Ratio($K$) = $\inf \frac{R}{r}$ where $D(x, r) \subset K \subset D(y, R)$.

Is there a “6” theorem if faces have uniformly bounded aspect ratios?

No.
Out-radius/in-radius is bounded, but number of neighbors $\uparrow \infty$
Out-radius/in-radius is bounded, but number of neighbors ↗ ∞.