MAT 670, Fall 2023, Stony Brook University

TOPICS IN COMPLEX ANALYSIS
DESSINS AND DYNAMICS
Part III: The folding theorem and applications

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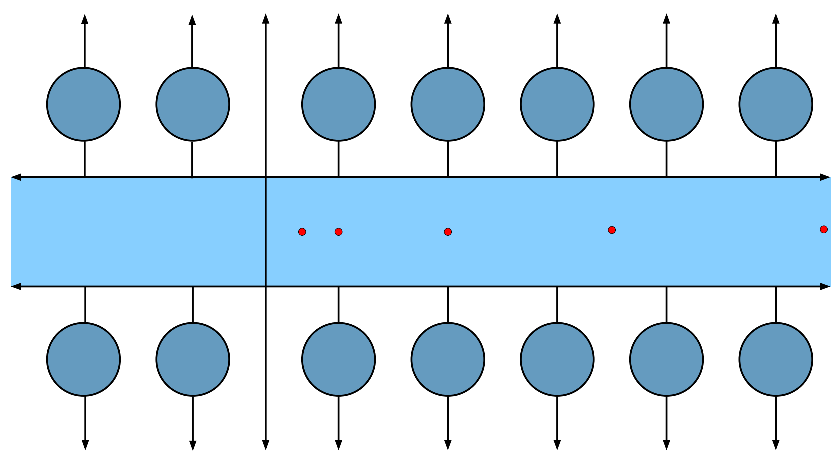

- Relevant definitions
- Statement of the quasiconformal folding theorem
- The proof (with most details)
- Applications:

Some definitions

We need three definitions before stating our result.

These adapt "obvious" properties of finite trees to infinite case.

(1) Tree neighborhoods: replaces Hausdorff metric ϵ-neighborhoods.

(2) Bounded geometry: nearby edges have comparable sizes.

(3) τ-lower bound: lower bound for measure of edges.

If e is an edge of T and $r > 0$ let

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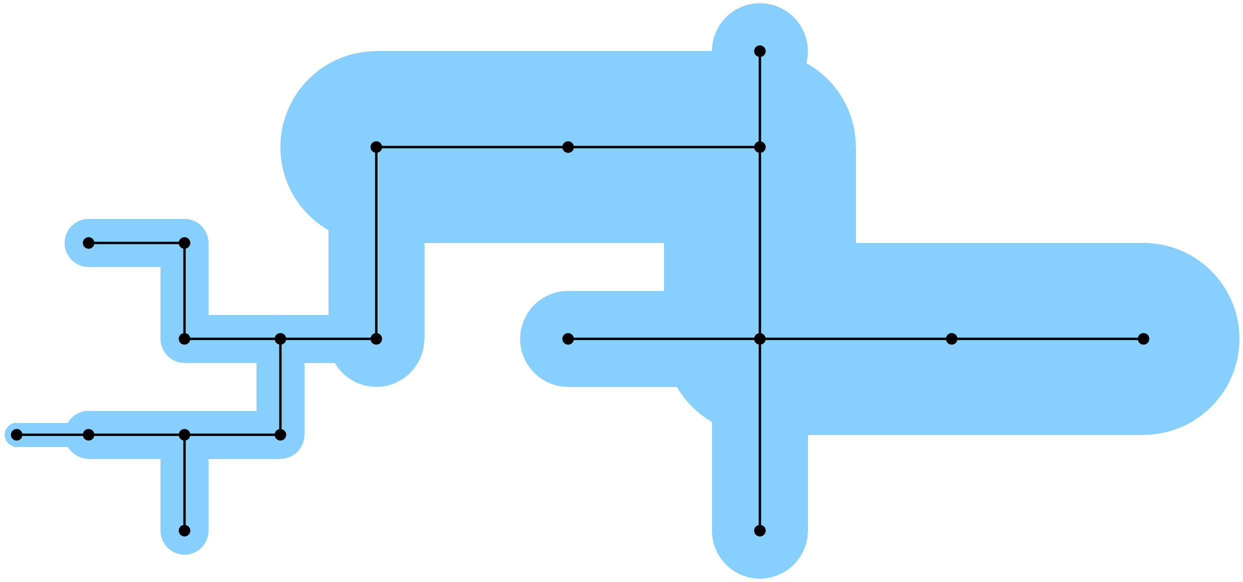

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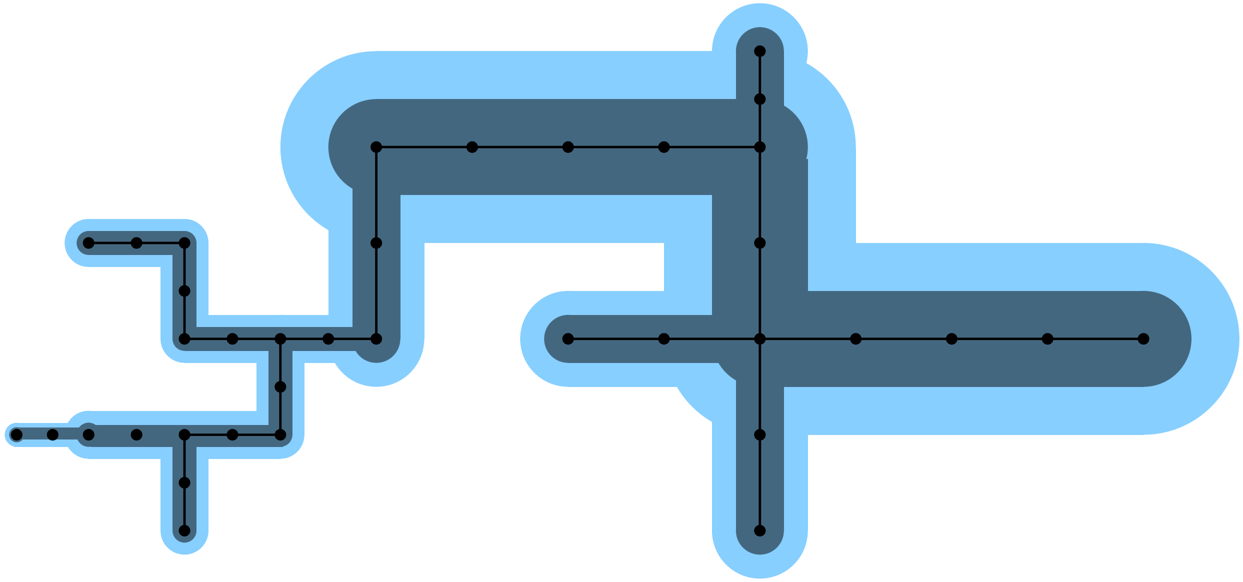
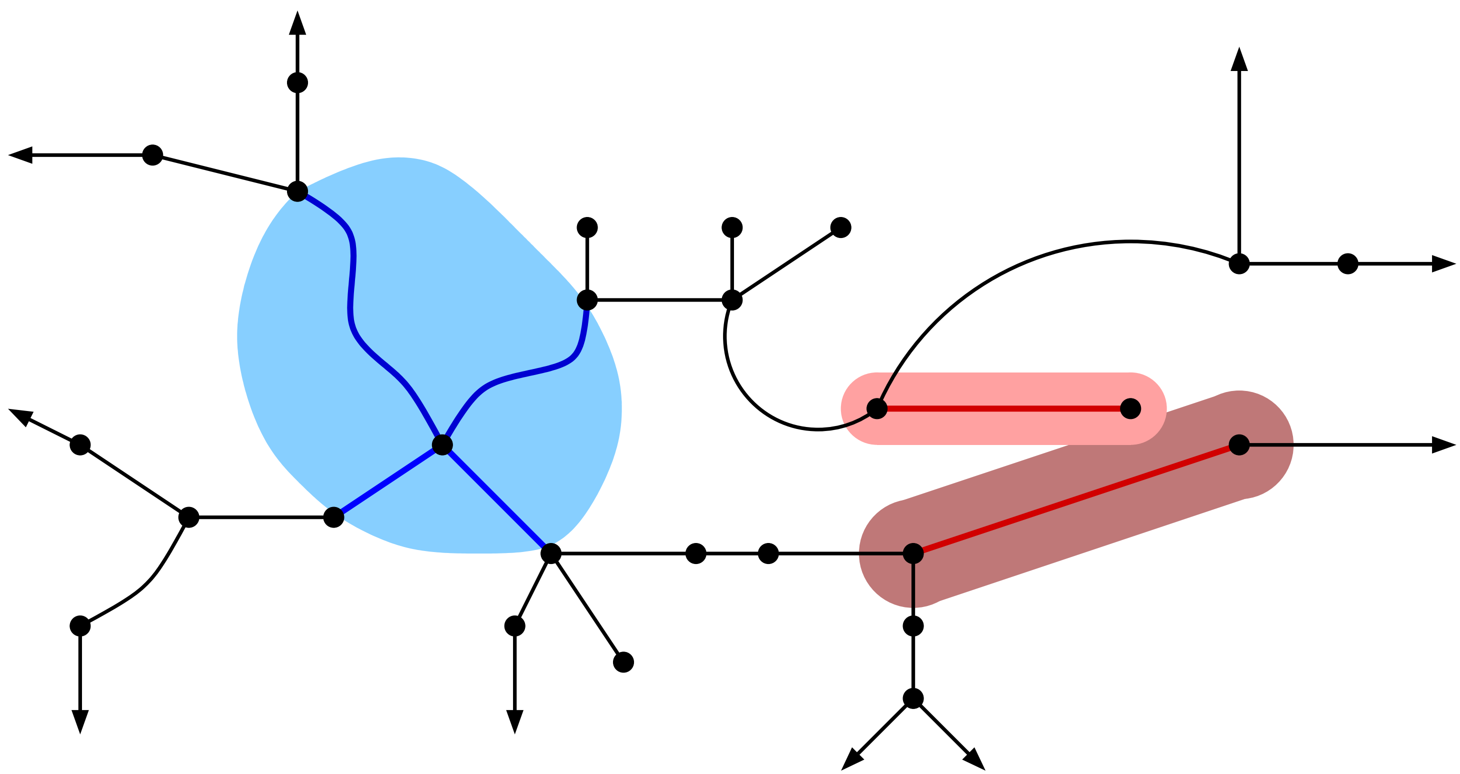


Adding vertices reduces $T(r)$. Useful scaling property.

Bounded Geometry (local condition; easy to verify):

- edges are uniformly smooth.
- adjacent edges form bi-Lipschitz image of a star $= \{z^n \in [0, r]\}$
- non-adjacent edges are well separated,

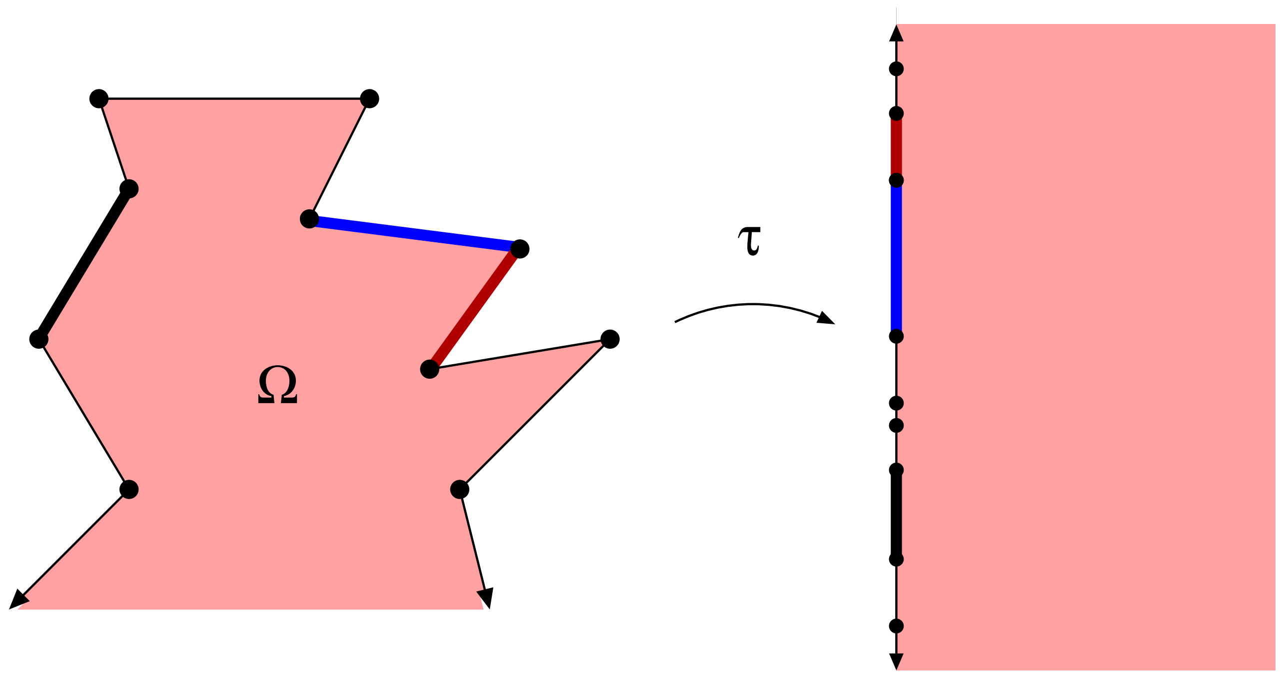
$$\mathrm{dist}(e, f) \geq \epsilon \cdot \min(\mathrm{diam}(e), \mathrm{diam}(f)).$$



τ-Lower Bound (global condition; harder to check):

Complementary components of tree are simply connected.

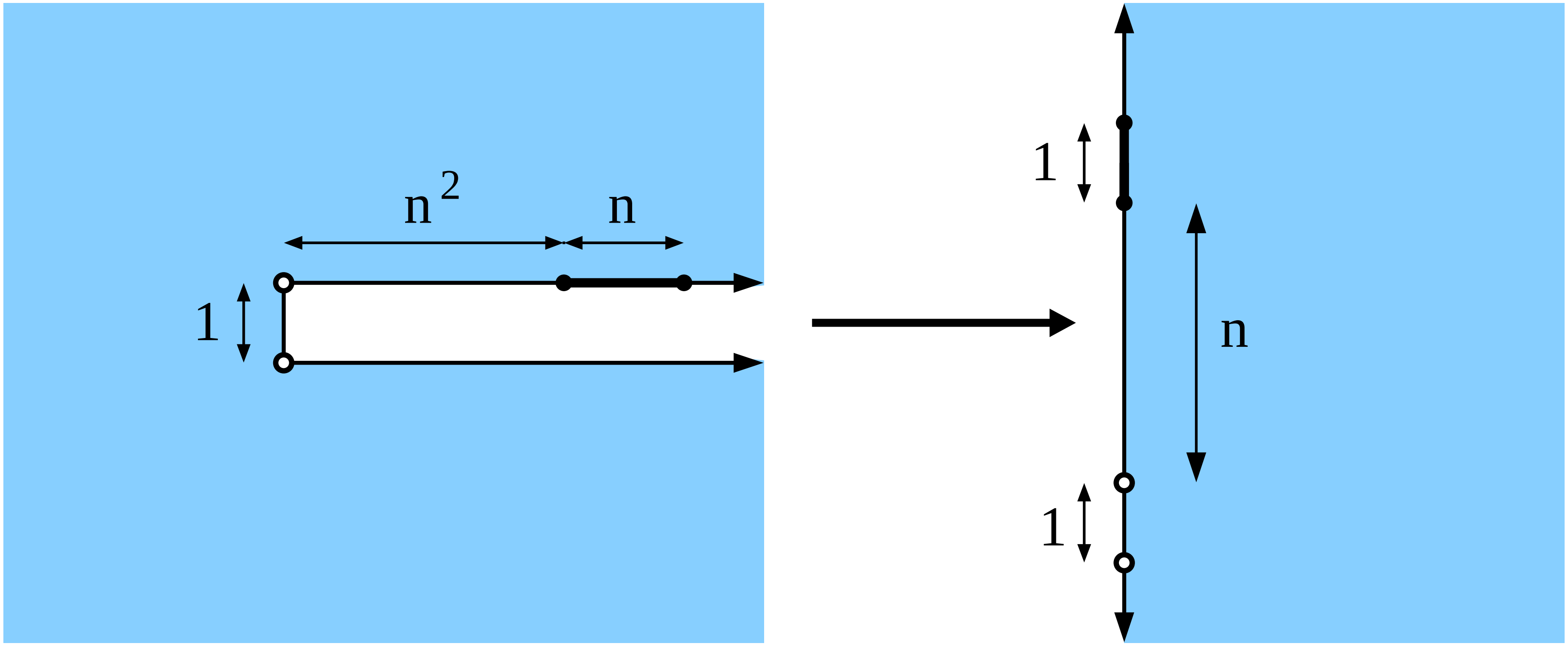
Each can be conformally mapped to right half-plane. Call map τ.



We assume all images have length $\geq \pi$.

Need positive lower bound; actual value usually not important.

Non-example: half-strip. "Inside" is OK, but ...



Conformal map of outside to half-plane is $\tau(z) \approx \sqrt{z}$.

Unit intervals on half-plane have pre-images $\simeq n$.

$\Rightarrow$ Bounded geometry and τ-condition can't both hold.

Assume that T is an unbounded, locally finite tree such that every component Ω_j of $\Omega = \mathbb{C} \setminus T$ is simply connected.

We also assume that $\Omega_j = \sigma_j(\mathbb{H}_r)$ where σ_j is a conformal map that extends continuously to the boundary and sends ∞ to ∞.

The inverses of these maps define a map $\tau : \Omega \to \mathbb{H}_r$ that is conformal on each component (we let $\tau_j = \sigma_j^{-1}$ denote the restriction of τ to Ω_j). Whenever we refer to a conformal map $\tau : \Omega \to \mathbb{H}_r$, we always mean a map that arises in this way.

Since T is a tree, it is bipartite and we assume the vertices have been labeled with ± 1 so that adjacent vertices always have different labels.

If V is the vertex set of T, let $V_j = \{z \in \partial \mathbb{H}_r : \sigma_j(z) \in V\}$; this is a closed set with no finite limit points. (It is tempting to write $V_j = \sigma_j^{-1}(V)$, but σ_j^{-1} is not defined on all of V and may be multi-valued where it is defined.)

The collection $\mathcal{I}_j$ of connected components of $\partial\mathbb{H}_r \setminus V_j$ is called the partition of $\partial\mathbb{H}_r$ induced by Ω_j (different choices of the map τ_j only change the partition by a linear map).

If $T = f^{-1}([-1,1])$ is the tree associated to an entire function with critical values ± 1, then the associated partition is $\partial\mathbb{H}_r \setminus \pi i\mathbb{Z}$, and partition elements have equal size.

Theorem 1. *Suppose T has bounded geometry and every edge has τ-size $\geq \pi$. Then there is an entire f and a K-quasiconformal ϕ so that $f \circ \phi = \cosh \circ \tau$ off $T(r_0)$. K only depends on the bounded geometry constants of T. The only critical values of f are ± 1 and f has no finite asymptotic values.*

The idea of the proof of Theorem 1 is to replace the tree T by a tree T' so that $T \subset T' \subset T(r_0)$ and to replace τ by a map η that is quasiconformal from each component of $\Omega' = \mathbb{C} \setminus T'$ onto $\mathbb{H}_r$.

We will prove we can do this with a map η such that $\eta(V) \subset \pi i \mathbb{Z}$, $\eta = \tau$ off $T(r_0)$ and so that $g = \cosh \circ \eta$ is continuous across T'.

The latter condition will imply g is quasiregular on the whole plane and hence, by the measurable Riemann mapping theorem, there is a quasiconformal $\phi : \mathbb{C} \to \mathbb{C}$ such that $f = g \circ \phi^{-1}$ is entire.

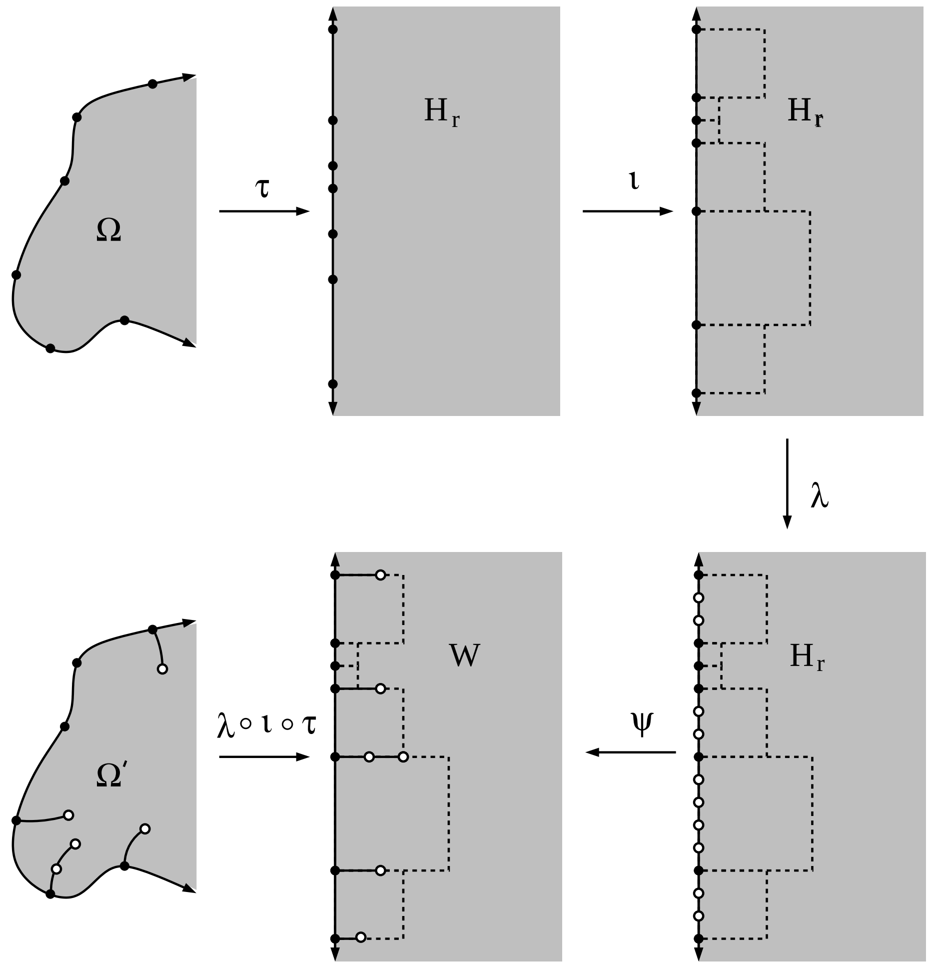
Since g is locally 1-to-1 except at the vertices of T, the only critical values are ± 1.

It is also easy to see there are no finite asymptotic values and this proves the theorem.

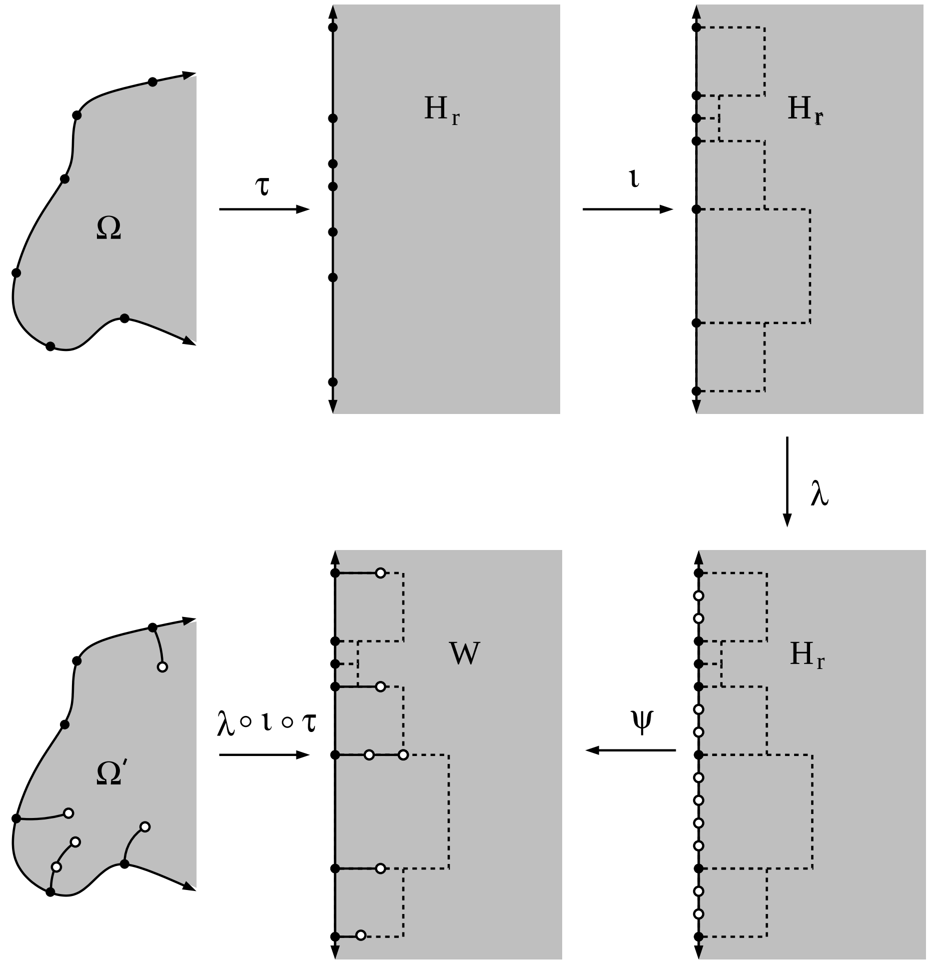
In fact, any preimage of any compact set K of diameter $r < 2$ will only have compact connected components. This condition rules out finite asymptotic values.

Let η_j denote the restriction of η to the component Ω_j. We build η_j by post-composing τ_j with quasiconformal maps

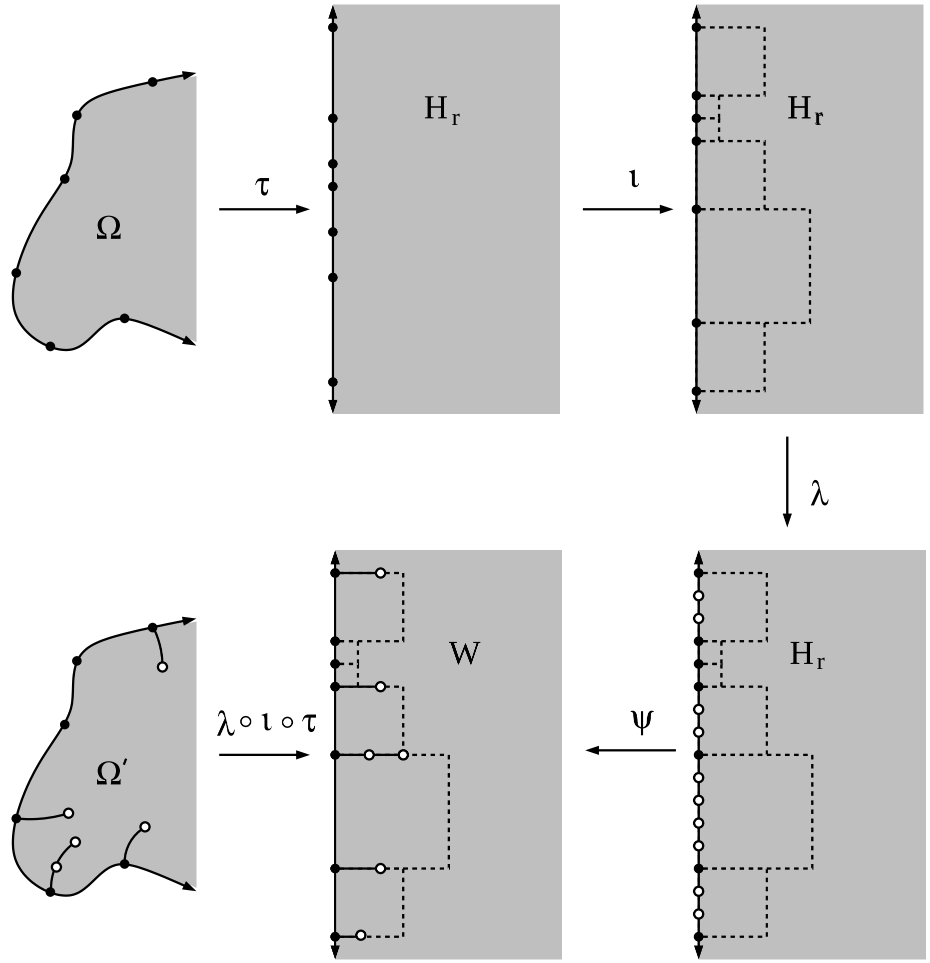
$$\eta_j : \Omega_j \xrightarrow{\tau_j} \mathbb{H}_r \xrightarrow{\iota_j} \mathbb{H}_r \xrightarrow{\lambda_j} \mathbb{H}_r \xrightarrow{\psi_j} W_j \subset \mathbb{H}_r.$$



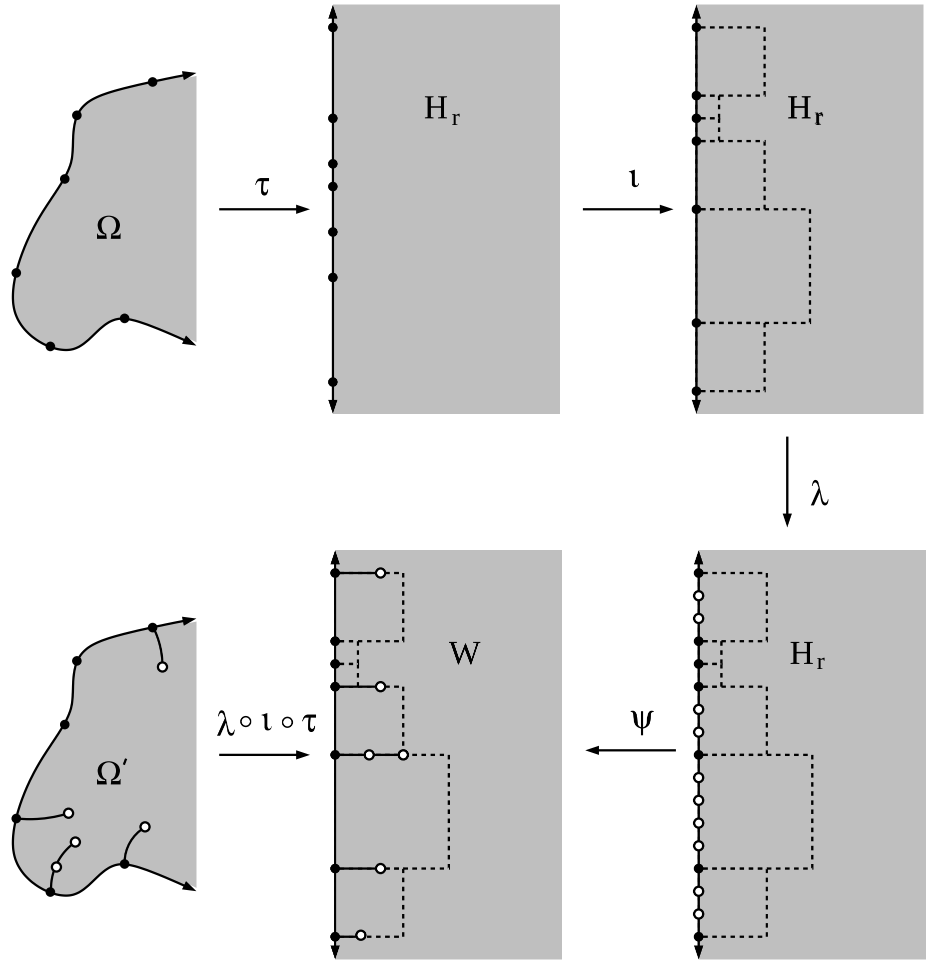
η is built as a composition: τ maps Ω to $\mathbb{H}_r$, ι sends vertices to integer points, λ makes the map preserve arclength and ψ "folds" the boundary. $V_{\mathcal{I}}$ is the union of dashed squares.



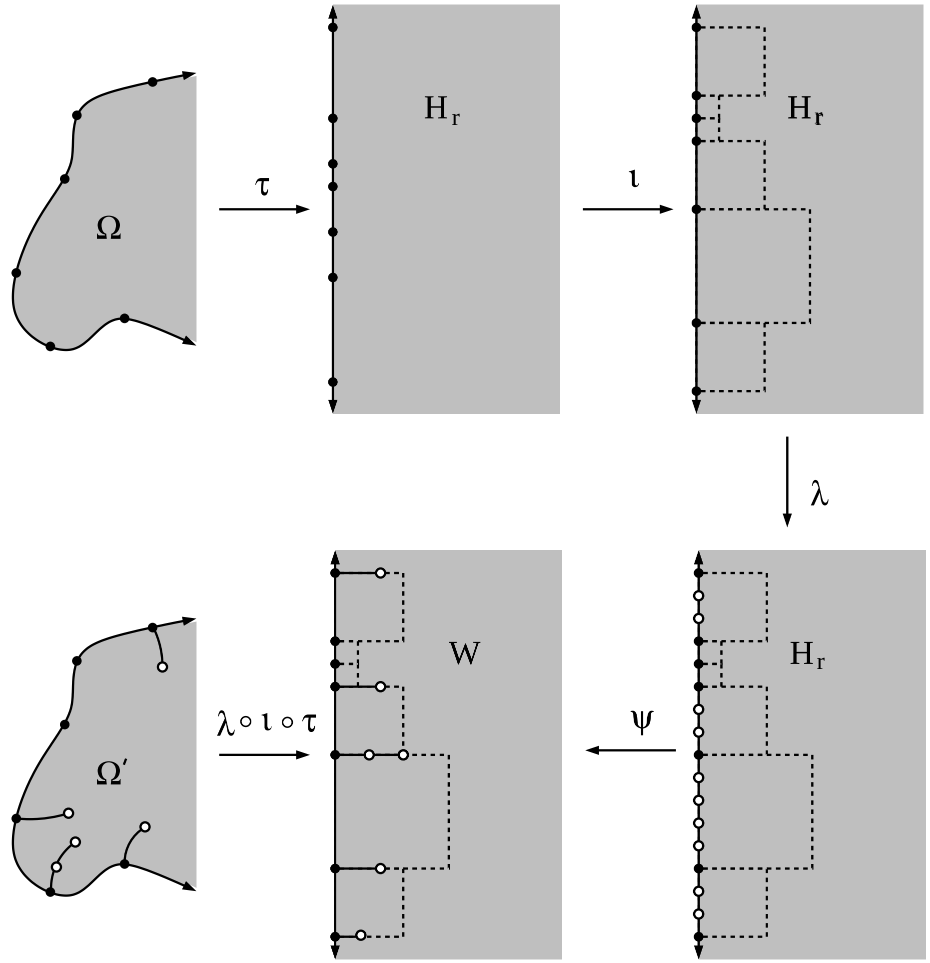
As noted earlier, the map τ_j sends vertices of T to a discrete set $V_j \subset \partial\mathbb{H}_r$ and $\mathcal{I}_j$ denotes the complementary components of V_j. By assumption all these intervals have length $\geq \pi$.



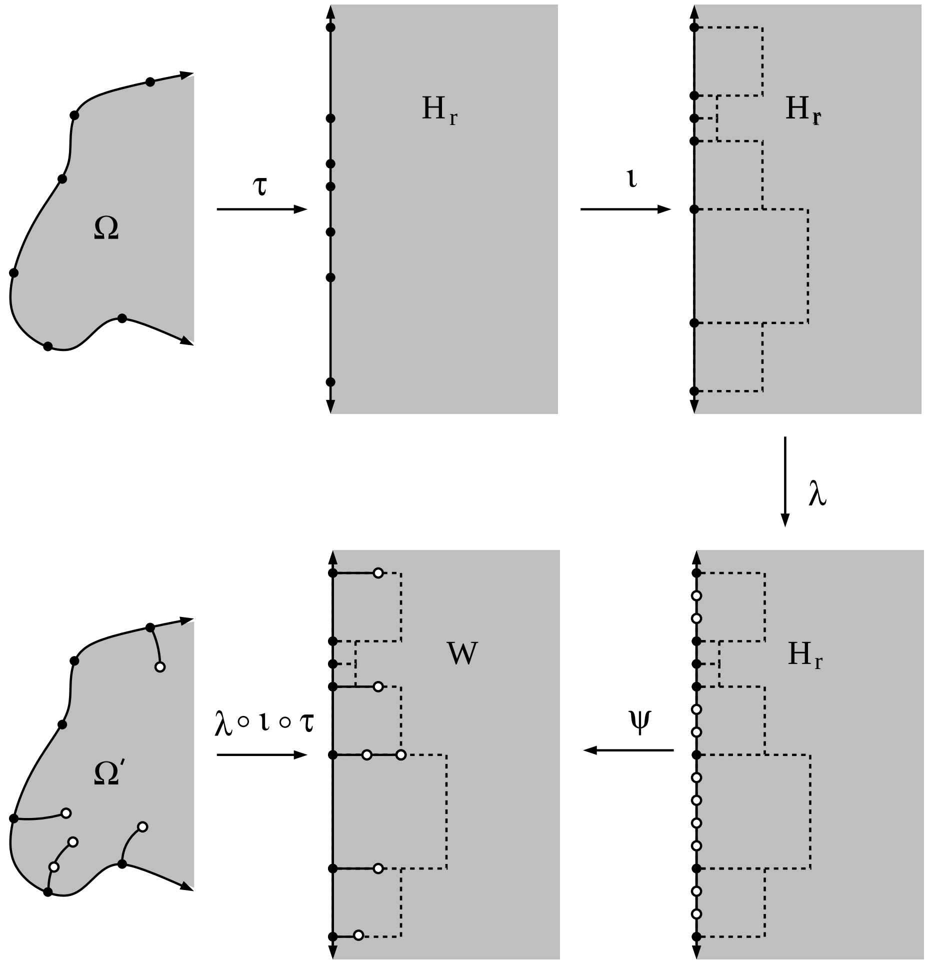
Let $\mathcal{Z}$ be the collection of connected components of $\partial\mathbb{H}_r \setminus \pi i\mathbb{Z}$.



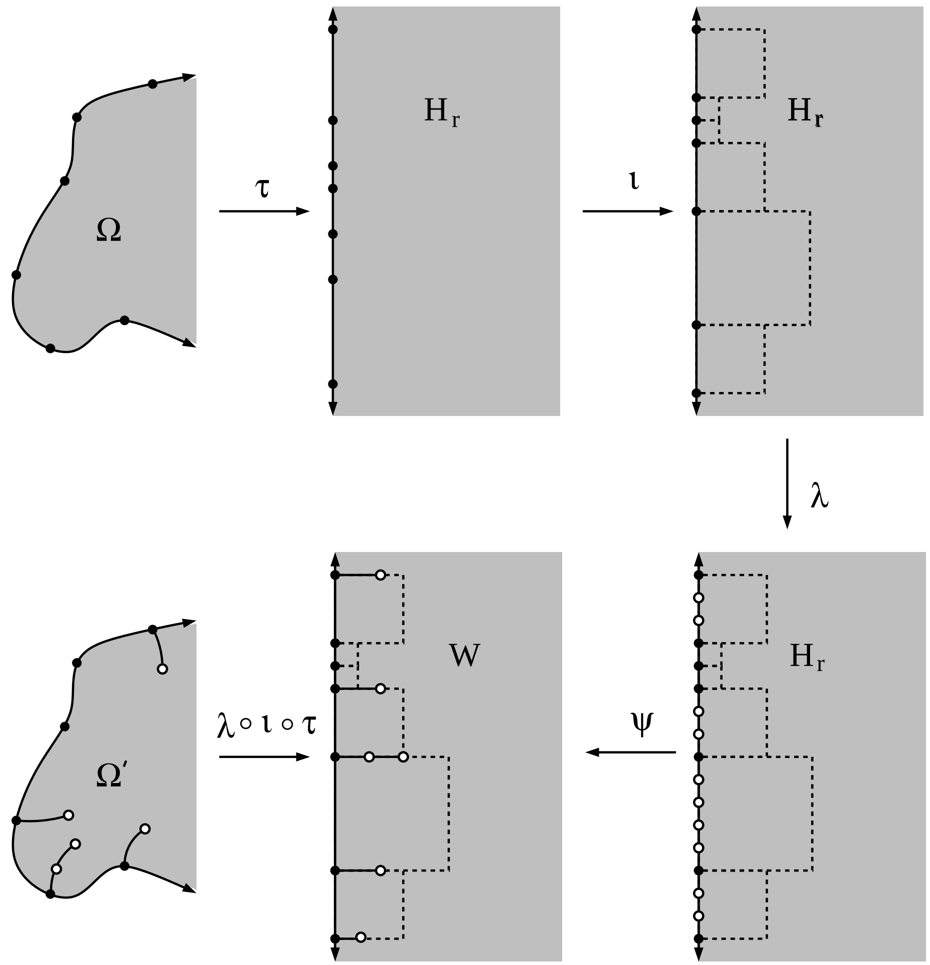
We will construct $\iota_j : \mathbb{H}_r \to \mathbb{H}_r$ to be a QC map that sends each point of V_j into $\pi i\mathbb{Z}$ and sends each interval of $\mathcal{I}_j$ to an interval of length $(2n+1)\pi$. Moreover, $\iota_j \circ \tau_j$ preserves vertex parity.



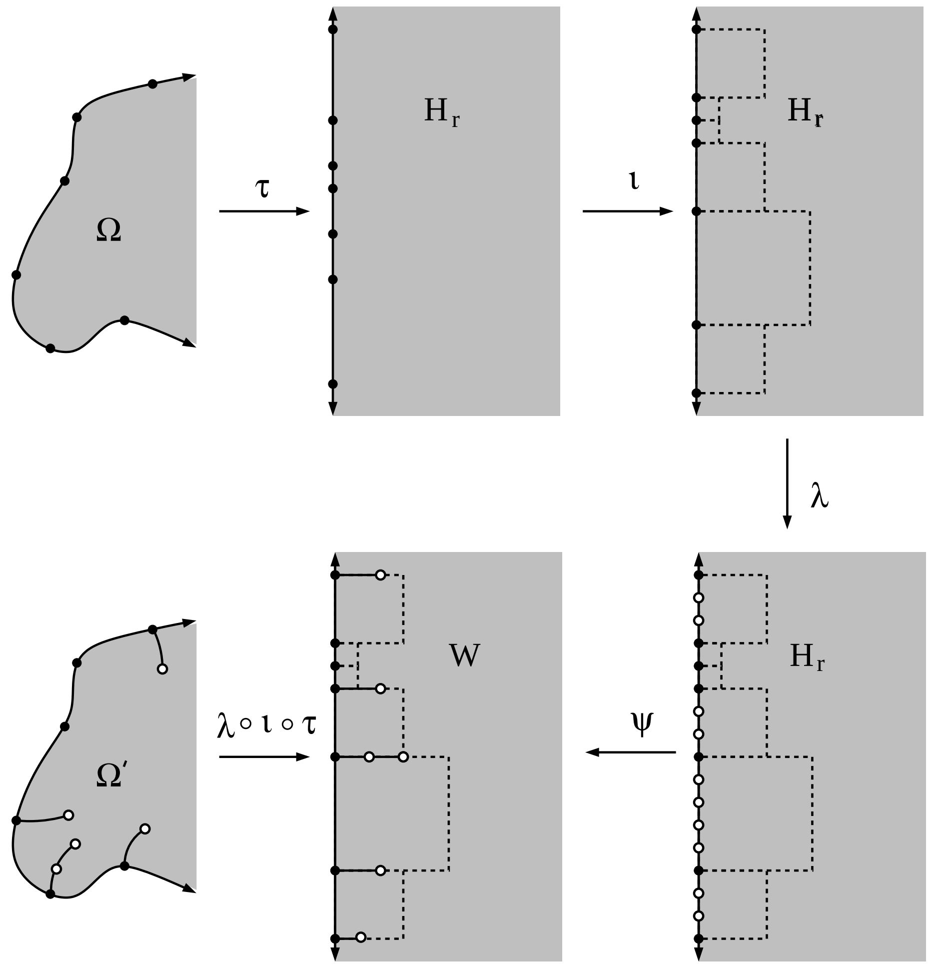
These "odd-length" intervals give a new partition of $\partial\mathbb{H}_r$ that we call $\mathcal{K}_j$.



Next, we construct a quasiconformal map $\lambda_j : \mathbb{H}_r \to \mathbb{H}_r$ that fixes the endpoints of $\mathcal{K}_j$ and such that $|(\lambda_j \circ \iota_j)'|/|\sigma_j'|$ is a.e. constant on each element of $\mathcal{K}_j$.



Informally, $\lambda_j \circ \iota_j \circ \tau_j$ multiplies length on each side of $\partial\Omega_j$ by a constant factor: if a side is mapped to an interval $K \in \mathcal{K}_j$ of length $(2n+1)\pi$, then normalized length on that side is multiplied by $(2n+1)$.

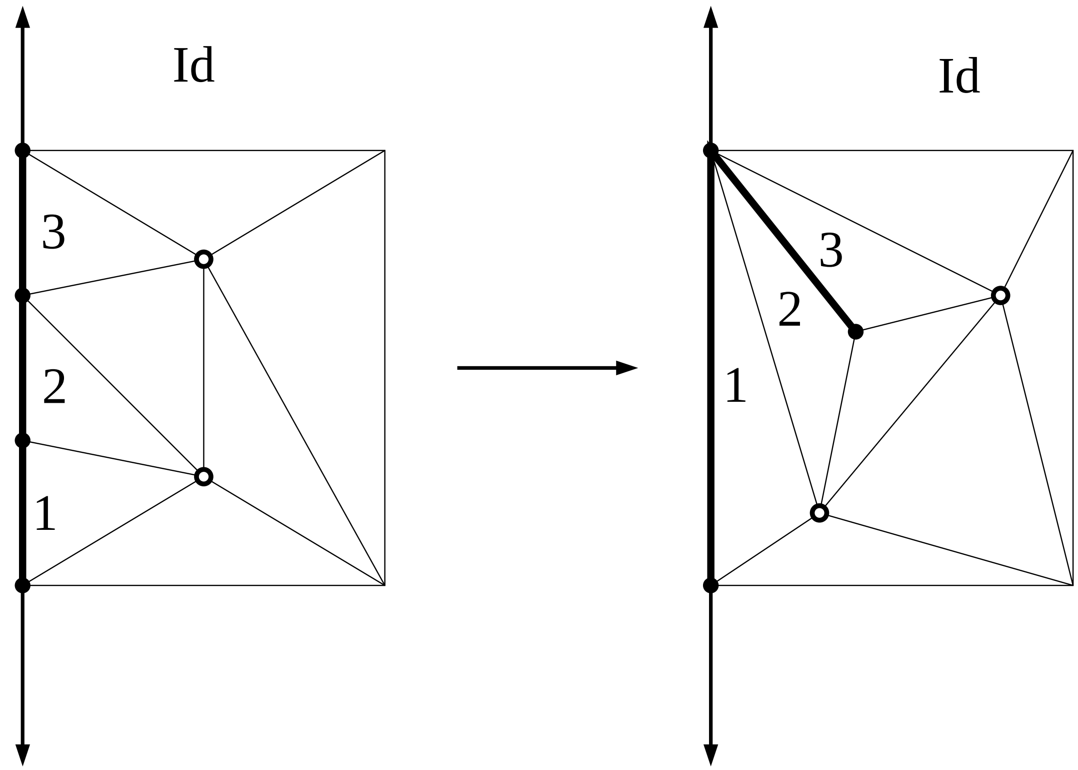


Each side of T maps to a union of $2n+1$ elements of $\mathcal{Z}$. We want it to map to a single element. We fix this by adding $2n$ extra sides to T.

This is accomplished with the following lemma that describes the "quasi-conformal folding".

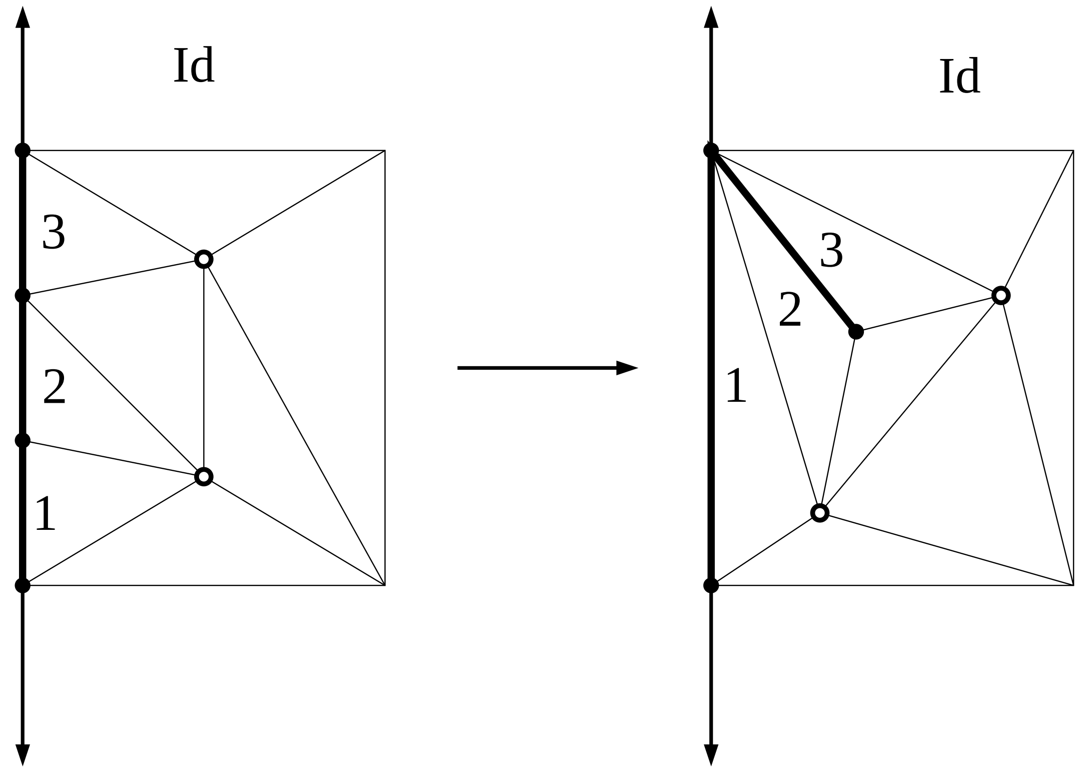
Lemma 2. *Suppose $\mathcal{K}$ is a partition of $\partial\mathbb{H}_r$ into intervals with end-points in $\pi i\mathbb{Z}$ and lengths in $(2\mathbb{N}+1)\pi$ and suppose adjacent intervals have comparable lengths, with a uniform constant M. Then there is a quasiconformal map $\psi : \mathbb{H}_r \to W \subset \mathbb{H}_r$ so that:*

1. *ψ is the identity off $V_{\mathcal{K}}$.*
2. *ψ is affine on each component of $\mathcal{Z} = \partial\mathbb{H}_r \setminus \pi i\mathbb{Z}$.*
3. *Each element $K \in \mathcal{K}$ contains an element of $\mathcal{Z}$ that is mapped to K by ψ.*
4. *For any $x, y \in \mathbb{R}$, $\psi(ix) = \psi(iy)$ implies $\cosh(ix) = \cosh(iy)$.*



This is a simple folding.

Here an interval K of length 3π is folded into $\mathbb{H}_r$ so that one interval is expanded and the other two are sent to two sides of a slit.



The map ψ is piecewise linear on the triangulations. Since there are only finitely many triangles, it is clearly quasiconformal.

Can fold up K sides onto slit but QC constant increases. We want bounds independent of K.

Suppose ψ_j is the map given by Lemma 2 when applied to the partition $\mathcal{K}_j$ corresponding to the component Ω_j. Let

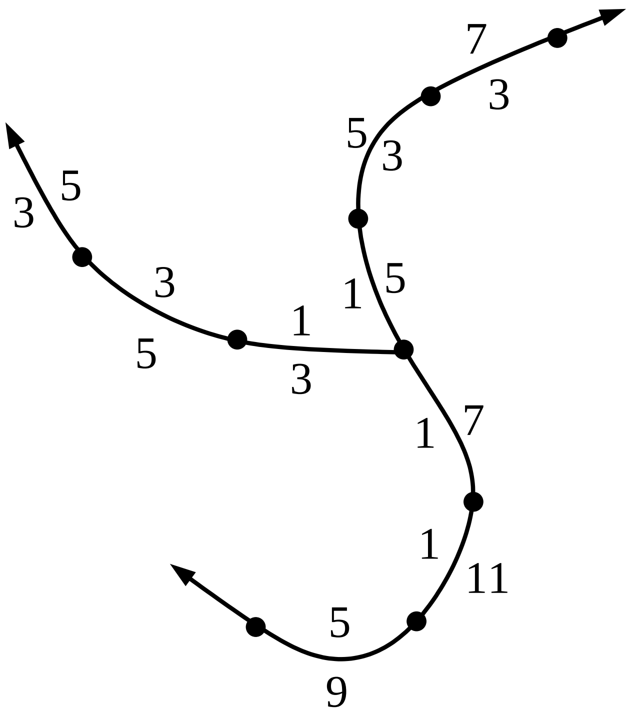
$$\Omega_j' = (\lambda_j \circ \iota_j \circ \tau_j)^{-1}(W) = (\psi_j \circ \lambda_j \circ \iota_j \circ \tau_j)^{-1}(\mathbb{H}_r) \subset \Omega_j.$$

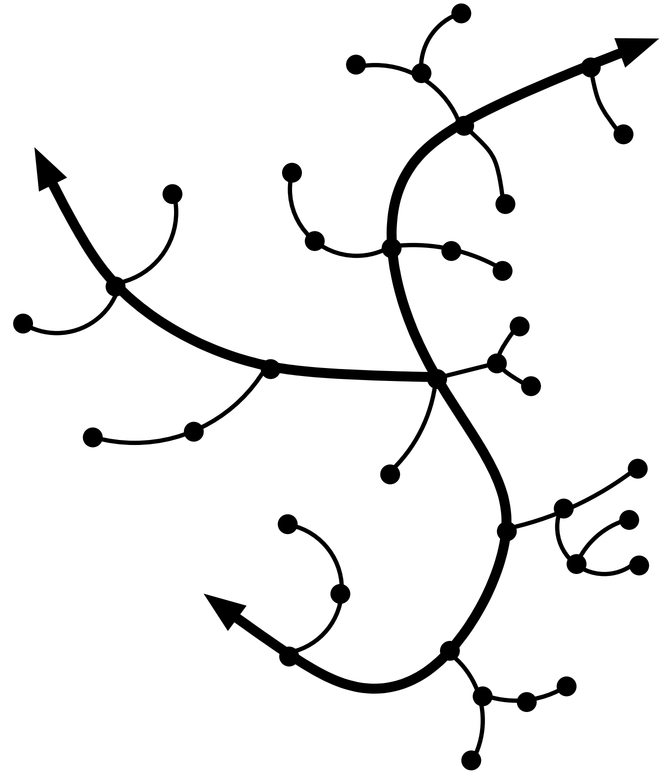
This is just Ω_j with countably many finite trees removed, each rooted a vertex of T.

The composition $\eta_j = \psi_j^{-1} \circ \lambda_j \circ \iota_j \circ \tau_j$ maps Ω_j to $\mathbb{H}_r$ and satisfies

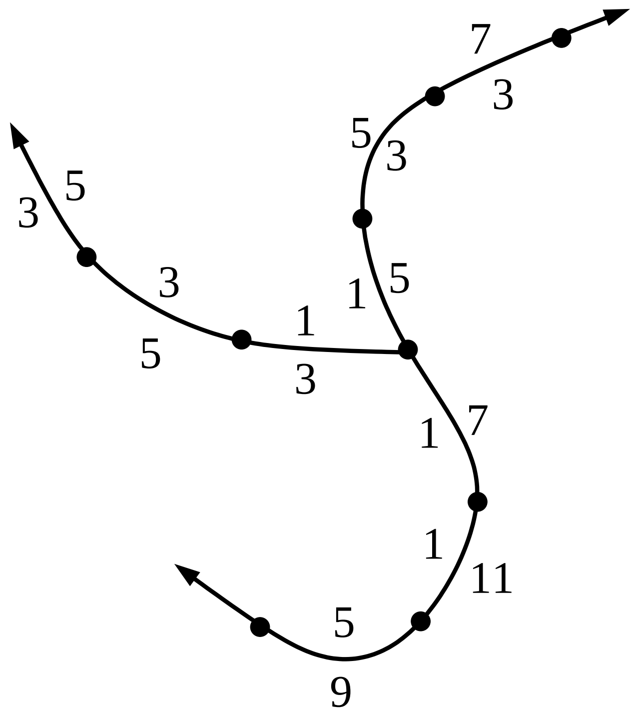
1. η_j is uniformly quasiconformal from each component of Ω' to $\mathbb{H}_r$.
2. η_j maps vertices of T to points in $\pi i \mathbb{Z}$ of the correct parity.
3. η_j preserves normalized length on all sides of T'.

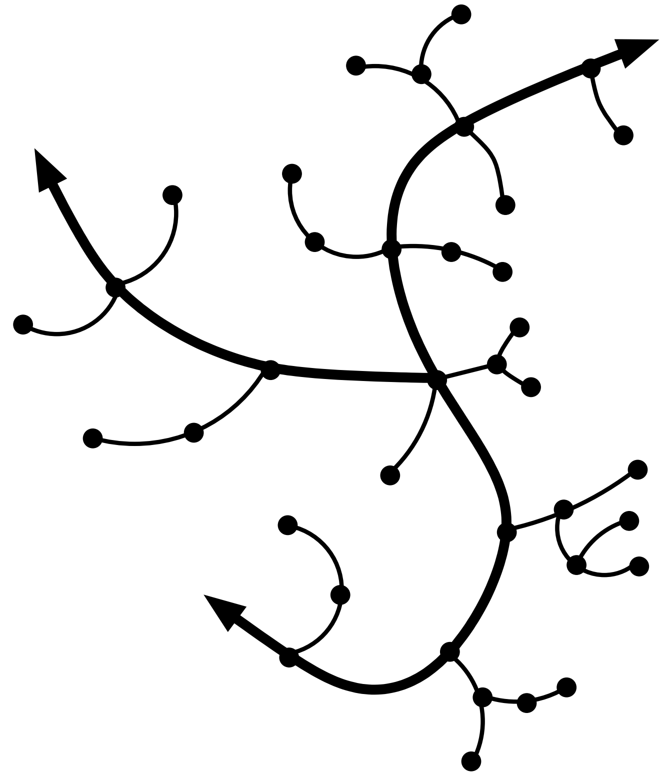
These conditions imply $g = \cosh \circ \eta$ is continuous across T'.





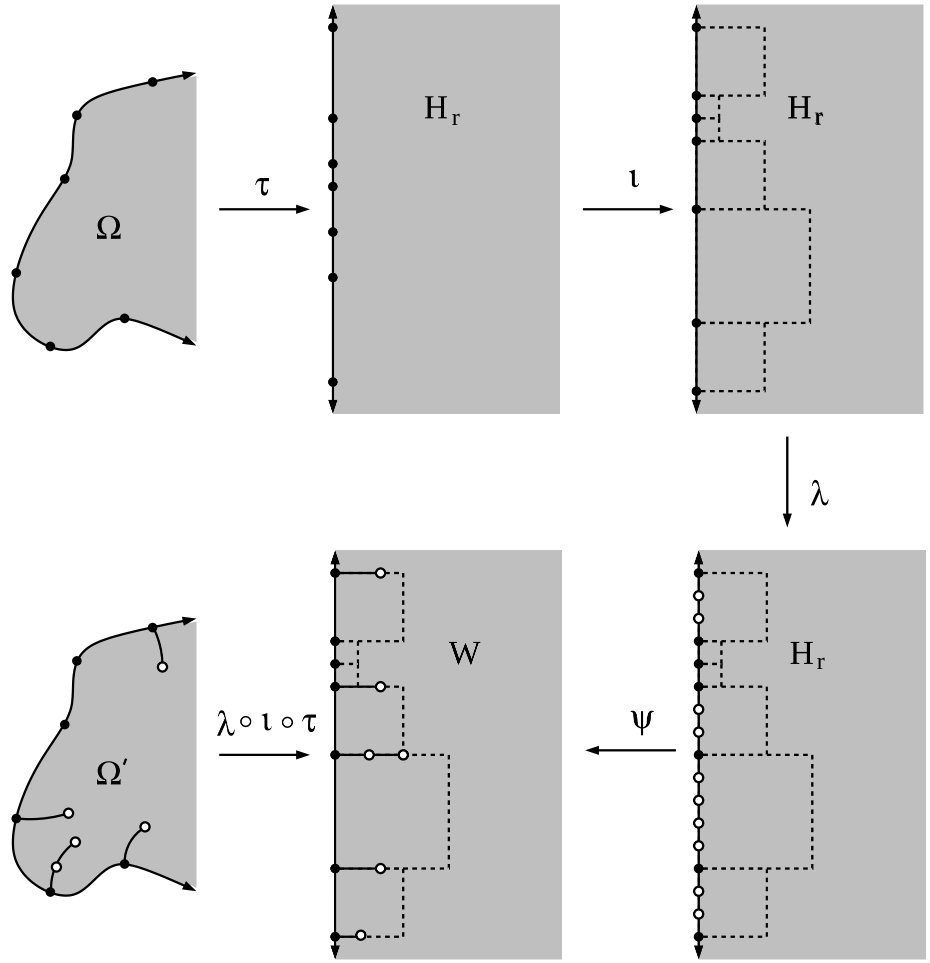
On the left is a tree T with possible τ-lengths of sides marked. On the right is the tree T' which is formed by adding a tree with n edges at one endpoint of a T-edge with label $(2n+1)$.





Every edge of T' is either an edge of T, in which case it is rectifiable, or it is a quasiconformal image of a line segment.

Thus T' is removable for quasiregular maps and hence g is quasiregular on the whole plane, as desired.



Finally, ι_j, λ_j and ψ_j are all the identity off $V_{\mathcal{I}}$. This will imply $\eta_j = \tau_j$ off $T(r_0)$ for some fixed r_0, and this completes the proof of Theorem 1 (except for proving the various results described above).

A neighborhood of the tree

Lemma 3. *$\tau^{-1}(V_{\mathcal{I}}) \subset T(r_0)$ for some $r_0 \leq 25.3$.*

Proof. For an interval $I \subset \partial\mathbb{H}_r$, let

$$W(I,\alpha) = \{z \in \mathbb{H}_r : \omega(z, I, \mathbb{H}_r) > \alpha\},$$

where ω denotes harmonic measure.

The set in $\mathbb{H}_r$ where I has harmonic measure bigger than α is the same as the set where I subtends angle $\geq \pi\alpha$; this is a crescent bounded by I and the arc of the circle in $\mathbb{H}_r$ that makes angle $\pi(1-\alpha)$ with I.

Some simple geometry shows that $W(I, \frac{1}{2}) \subset Q_I \subset W(I, \frac{1}{4})$ and hence $V_{\mathcal{I}} \subset \cup_{I \in \mathcal{I}} W(I, \frac{1}{4})$ (recall that Q_I is the square in $\mathbb{H}_r$ with I as one side).

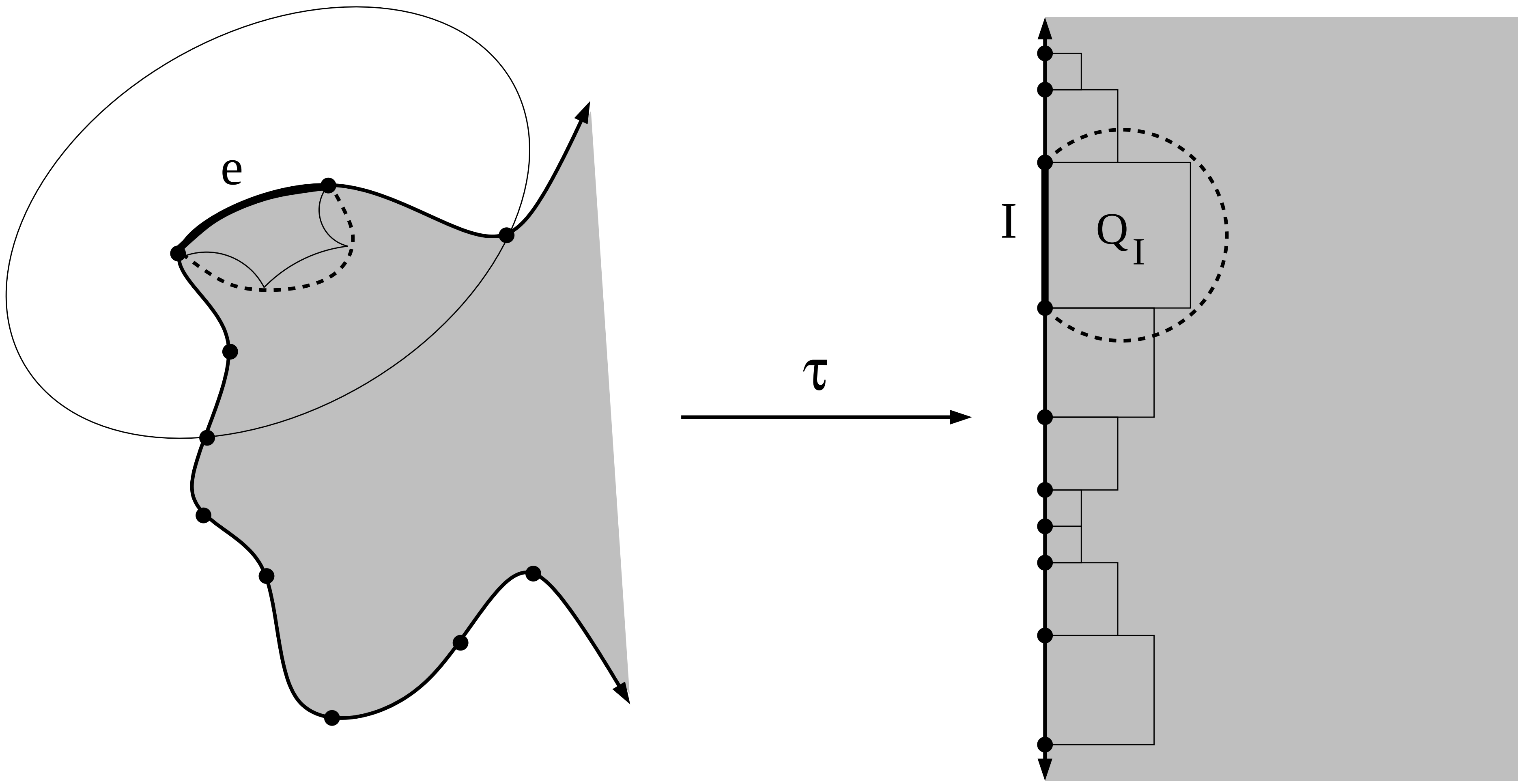
Thus $\tau^{-1}(V_{\mathcal{I}})$ is contained in the set of points z in Ω such that some single edge e of T has harmonic measure $\omega(z, e, \Omega) \geq 1/4$. Beurling's projection theorem (see Corollary III.9.3 of Garnett-Marshall "Harmonic Measure") then implies

$$\frac{1}{4} \leq \omega(z, e, \Omega) \leq \frac{4}{\pi} \tan^{-1} \sqrt{\frac{\operatorname{diam}(e)}{\operatorname{dist}(z, e)}}.$$

Hence

$$\operatorname{dist}(z, e) \leq (\tan(\frac{\pi}{16}))^{-2} \cdot \operatorname{diam}(e)$$

and so $\tau^{-1}(V_{\mathcal{I}}) \subset T(r_0)$, where $r_0 = \tan^{-2}(\frac{\pi}{16}) \approx 25.27$. $\square$


e
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Integerizing a partition

Lemma 4. *Suppose $\mathcal{I} = \{I_j\}$ is a bounded geometry partition of the real numbers (i.e., adjacent intervals have comparable lengths) so that every interval has length ≥ 1. Then there is second partition $\mathcal{J} = \{J_j\}$ so that*

1. *Every endpoint of $\mathcal{J}$ is an integer.*
2. *The length of J_j is an odd integer.*
3. *I_j and J_j have lengths differing by ≤ 2.*
4. *The left endpoints of I_j and J_j are within distance $5/2$ of each other. Similarly for the right endpoints.*

Proof. We use induction to create adjacent intervals $J_1, J_2, \ldots$ with integer endpoints, so that that the right endpoint of J_k is less than or equal to the right endpoint of I_k.

After translating by at most $\frac{1}{2}$ we can assume I_0 contains a non-trivial interval with integer endpoints and odd length. Let J_0 be the maximal such interval in I_0.

For $j > 0$, let the left endpoint of J_j be the right endpoint of J_{j-1}. Choose its right endpoint to be the largest integer that is less than or equal to the right endpoint of I_j and so that J_j has odd length.

Since I_j has length ≥ 1, there is such a choice.

Then (1)-(3) all hold and (4) holds with constant 2. A similar argument holds for $j < 0$. When we undo the initial translation, (1)-(3) all hold with the same constants and (4) holds with 5/2. $\square$

Lemma 5. *There is a quasiconformal map* ι *of the upper half-plane* $\mathbb{H} = \{x+iy : y > 0\}$ *to itself that sends the partition* $\mathcal{I}$ *in Lemma 4 to the partition* $\mathcal{J}$. *The map* ι *is the identity on* $\mathbb{H}+i = \{x+iy : y > 1\}$ *and the dilatation is bounded independent of* $\mathcal{I}$.

Proof. We now define a map $\psi_1 : \mathbb{R} \to \mathbb{R}$ as the piecewise linear map that sends I_j to J_j. This is clearly bi-Lipschitz.

This boundary mapping ψ_1 can be extended to a quasiconformal mapping of $\mathbb{H}$ that is the identity off the strip $S\{x + iy : 0 < y < 1\}$ by linearly interpolating the identity on $\{y = 1\}$ with ψ_1 on $\mathbb{R}$.

It is easy to see this defines a bi-Lipschitz (hence quasiconformal) map of S to itself, that extends to the identity on the rest of $\mathbb{H}$. $\square$

Length respecting maps

Lemma 6. *If T has bounded geometry tree then adjacent partition elements of $\mathcal{I}_j$ have comparable length.*

Proof. Adjacent intervals $I, J \subset \partial\mathbb{H}_r$ correspond to sides of adjacent edges e, f of T will have comparable lengths iff there is a point $z \in \mathbb{H}_r$ from which the harmonic measures of I, J and both components of $\partial\mathbb{H}_r \setminus (I \cup J)$ are all comparable.

But if we take a point $w \in \Omega$ that with

$$\operatorname{dist}(w, e) \simeq \operatorname{dist}(w, f) \simeq \operatorname{dist}(w, \partial\Omega)$$

the bounded geometry assumption and the conformal invariance of harmonic measure imply this is true for $z = \tau(w)$. $\square$

We say that a homeomorphism h of one rectifiable curve γ_1 to another rectifiable curve γ_2 respects length if it is absolutely continuous with respect to arclength and $|h'|$ is a.e. constant, i.e., $\ell(\tau(E)) = \ell(E)\ell(\gamma_2)/\ell(\gamma_1)$, for every measurable $E \subset \gamma_1$.

Lemma 7. *Suppose* $\eta : \Omega \to \mathbb{H}_r$ *is quasiconformal on each of its connected components, maps the vertices of* T *into* $\pi i \mathbb{Z}$ *and is length respecting on each side of* T*. Also suppose that for each edge* e *in* T*, the two sides of* e *have equal* τ*-length. If* $\cosh \circ \eta$ *is continuous at all vertices of* T*, then it is continuous across all edges of* T*.*

Proof. Suppose v, w are the endpoints of e and $z \in e$.

By assumption the two possible images of e under η have the same length and have their endpoints in $\pi i\mathbb{Z}$.

Since $\cosh \circ \eta$ is continuous at w, both of its images have the same parity. Similarly for v. Therefore the length respecting property implies both images of z have the same distance from $2\pi i\mathbb{Z}$, which implies the result. $\square$

Theorem 8. *Suppose T is a bounded geometry tree, Ω_j is a component of $\Omega = \mathbb{C} \setminus T$ and $\sigma_j : \mathbb{H}_r \to \Omega_j$ is the inverse to τ for this component. Suppose the partition of $\partial\mathbb{H}_r$ induced by Ω_j has bounded geometry. Then there is a quasiconformal map $\beta : \mathbb{H}_r \to \mathbb{H}_r$ so that $\sigma_j \circ \beta$ is a length respecting on every element of $\mathcal{I}_j$. The map β is the identity on $V_j \subset \partial\mathbb{H}_r$ and on $\mathbb{H}_r \setminus V_{\mathcal{I}}$.*

Proof. Consider adjacent intervals $I, J \in \mathcal{I}_j$ corresponding to edges e, f of T with a common vertex v. The bounded geometry condition states that e and f have comparable length and Lemma 6 says I and J have comparable length.

Suppose θ is the interior angle of Ω formed by the edges e and f and let $\alpha = \theta/\pi$. Then

$$|\frac{d}{dx}\sigma_j(x)| \simeq \frac{\ell(e)}{\ell(I)}(x-a)^{\alpha-1},$$

on both I and J near the endpoint a.

Let K be the interval centered at a with length $\ell(K) = \frac{1}{4}\min(\ell(I), \ell(J))$. Normalize so $a = 0$ and $\ell(K) = 1$ and consider the map $\varphi(z) = z|z|^{\alpha-1}$ for $|z| \leq 1$ and the identity for $|z| > 1$.

Then $\varphi \circ \tau$ has a derivative that is bounded and bounded away from zero on $\sigma_j(K)$ The map φ is the identity outside the disk with diameter $\ell(K)$, so is certainly the identity outside $V_{\mathcal{I}}$.

Now build a version of φ for every pair of adjacent edges to get a quasiconformal map $\varphi : \mathbb{H}_r \to \mathbb{H}_r$ that fixes every endpoint of our partition $\mathcal{I}$ and is the identity outside $V_{\mathcal{I}}$.

For any interval $I \in \mathcal{I}$, we can use integration to define a bi-Lipschitz map $\kappa : I \to I$ fixing each endpoint of I and so that the derivative of $\kappa \circ \varphi \circ \tau$ has constant absolute value.

By simple linear interpolation this κ can be extended to a bi-Lipschitz map of Q_I (the square in $\mathbb{H}_r$ with I as one side) that is the identity on the other three sides of Q_I.

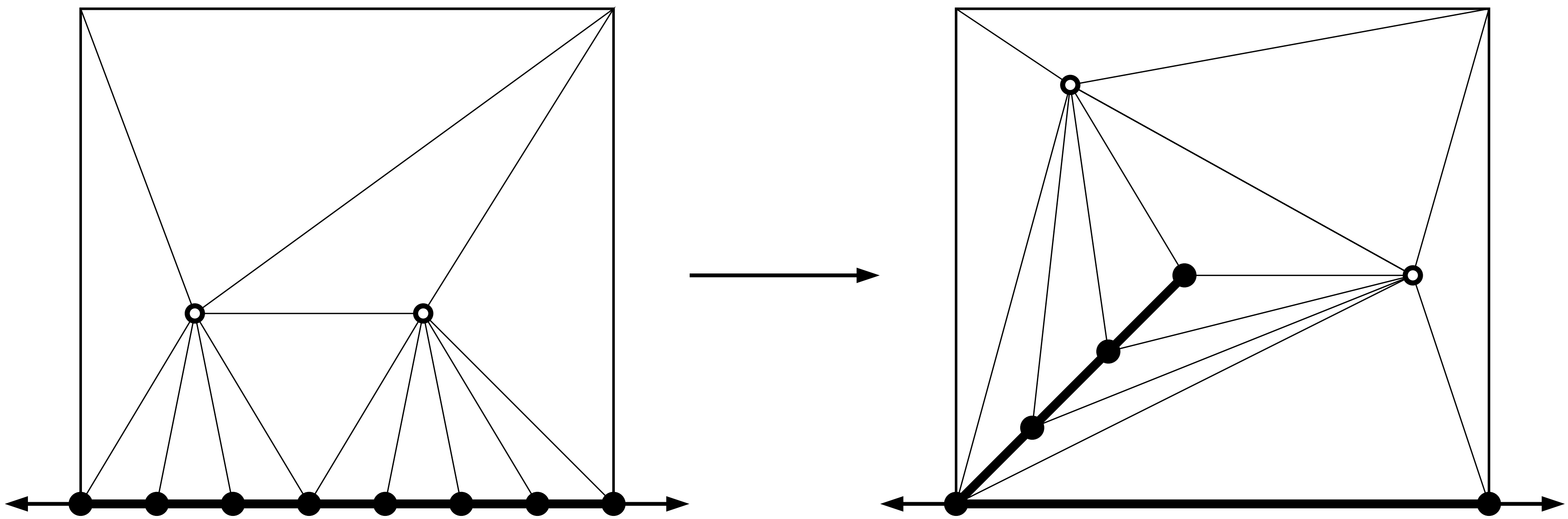
Doing this for every interval in the partition defines a quasiconformal κ on $\mathbb{H}_r$ that is the identity off $V_{\mathcal{I}}$. Clearly $\beta = \kappa \circ \varphi$ satisfies the conclusions of Theorem 8, completing the proof. $\square$

Building the tree

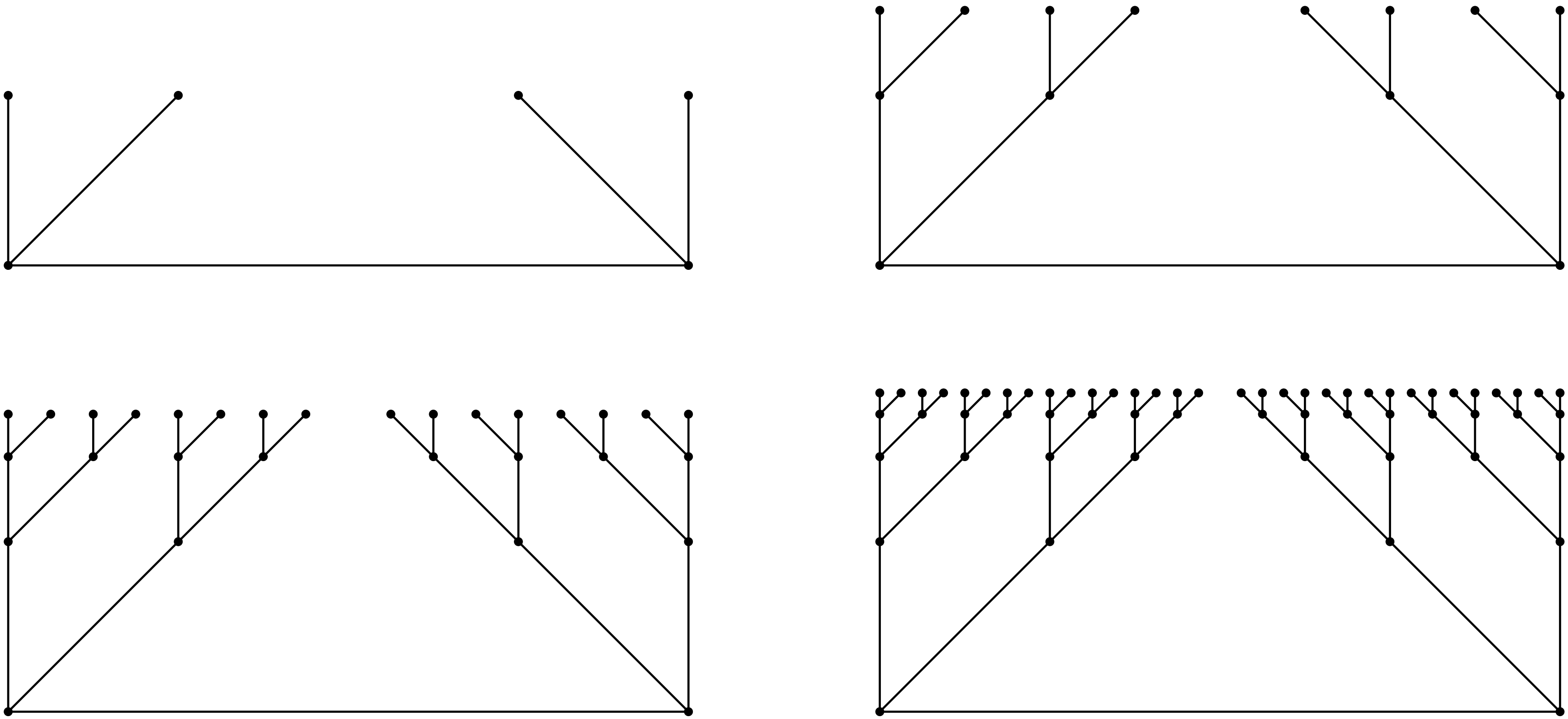
This is a slight reformulation of something stated earlier.

Lemma 9. *Suppose $\mathcal{J} = \{J_j\}$ is a partition of $\mathbb{R}$ into intervals with endpoints in $\mathbb{Z}$ and all odd lengths. Assume that any two adjacent elements have lengths within a factor of $M < \infty$ of each other. Then there is a map ψ of $\mathbb{H} = \{x + iy : y > 0\}$ into itself and intervals $J'_j \subset J_j$, so that the following all hold:*

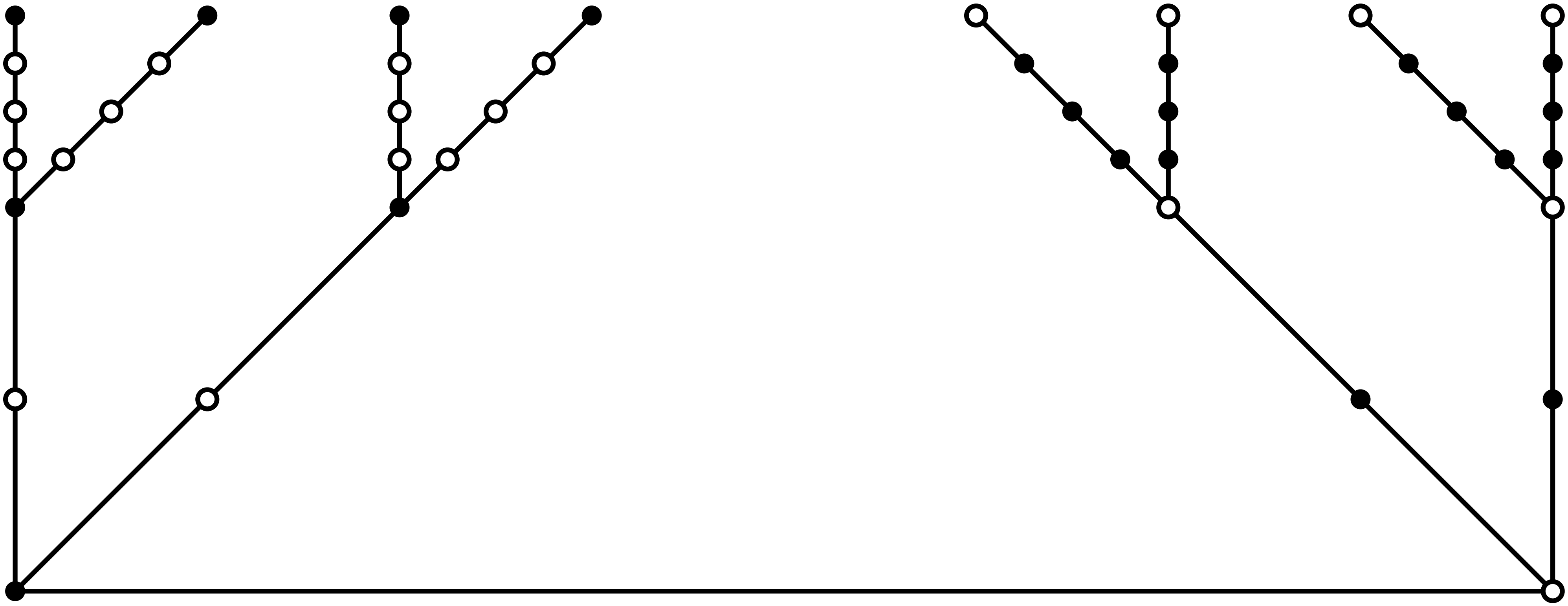
1. *each J'_j has integer endpoints and length 1.*
2. *ψ is the identity off $V_{\mathcal{J}}$.*
3. *ψ is quasiconformal with a constant depending only on M.*
4. *ψ is affine on each component of $\mathbb{R} \setminus \mathbb{Z}$.*
5. *$\psi(J'_j) = J_j$ for all j.*
6. *$\psi(x) = \psi(y)$ implies $x, y \in \mathbb{R}$ have the same distance to $2\mathbb{Z}$.*



This simple tree is just a slit in the upper half-plane partitioned into n edges. The triangulations show how $\mathbb{H}$ can be mapped to the complement of the slit by a piecewise linear map that is the identity outside the indicated square.



Shown are the trees $\hat{T}_1, \hat{T}_2, \hat{T}_3$ and $\hat{T}_4$.



We add vertices to $\hat{T}_j$ to get T_j. The jth level is divided into 2^j equal sub-edges by adding extra vertices. We illustrate only the $j = 2$ case, since its hard to see individual vertices at higher levels.

How many edges are in T_j? How many sides?

The number of edges is

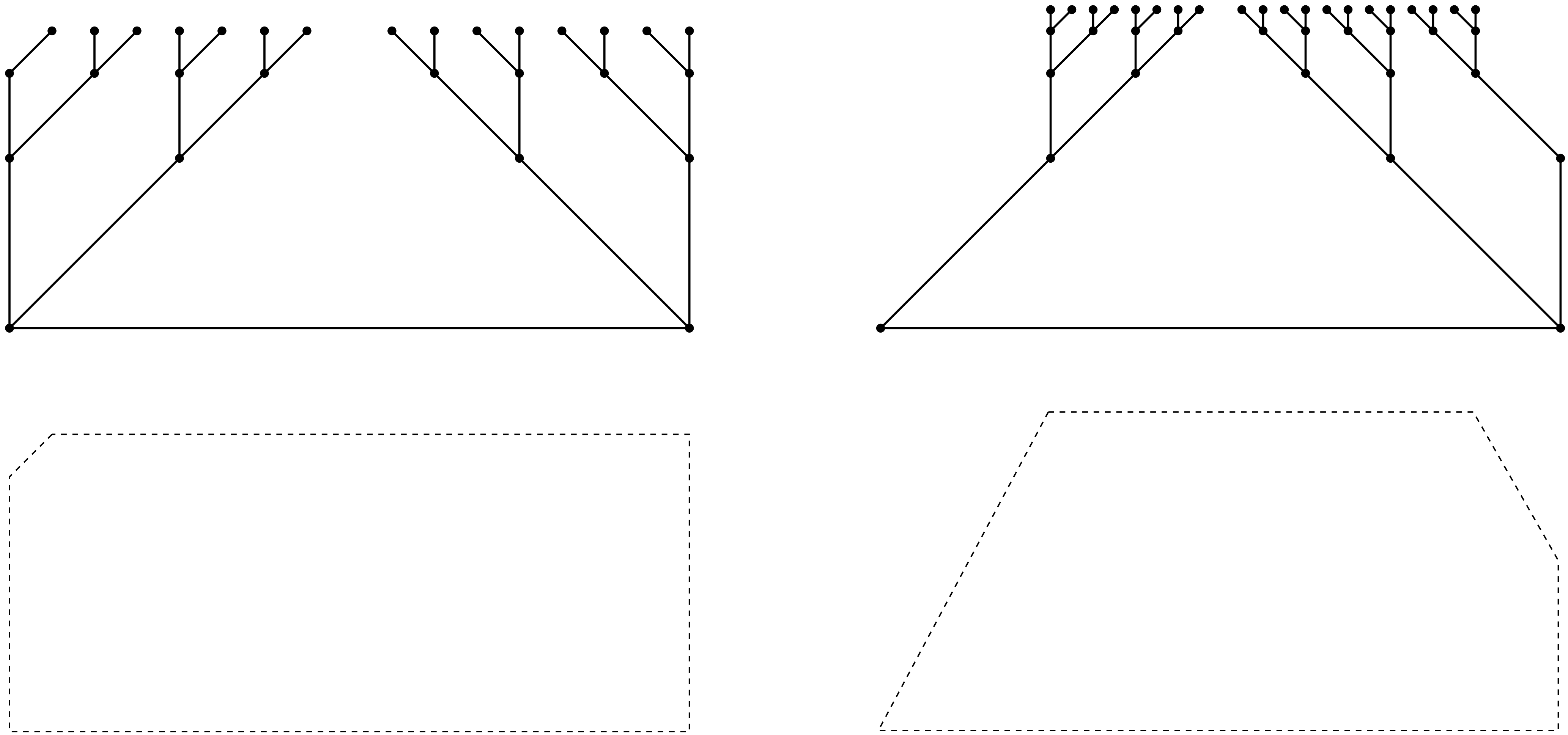
$$1 + \sum_{k=1}^{j} 2^{2k+1} = -1 + 2\sum_{k=0}^{j} 4^k = \frac{2}{3}(4^{j+1} - 1) - 1.$$

Normally, the number of sides would be twice the number of edges, but for our purposes, we only want to count a side of T_j if it is accessible from the interior of R_j, the convex hull of T_j.

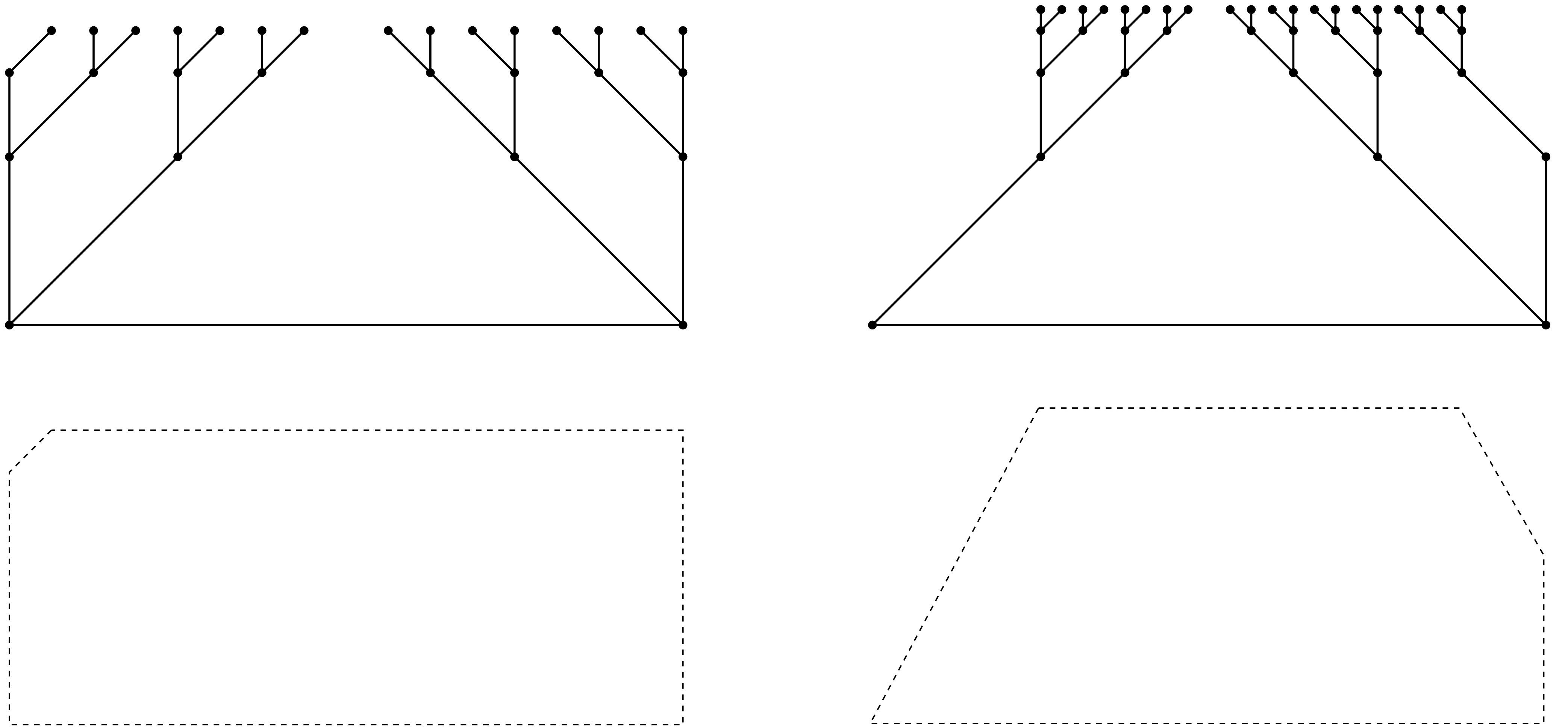
Thus we have to subtract the "inaccessible" sides belonging to the bottom and sides of R_j. After a little arithmetic, this gives

$$N_j = [\frac{4}{3}(4^{j+1} - 1) - 2] - [1 + 2\sum_{k=1}^{j} 2^k] = \frac{4}{3}(4^{j+1} - 1) + 1 - 2^{j+2}$$

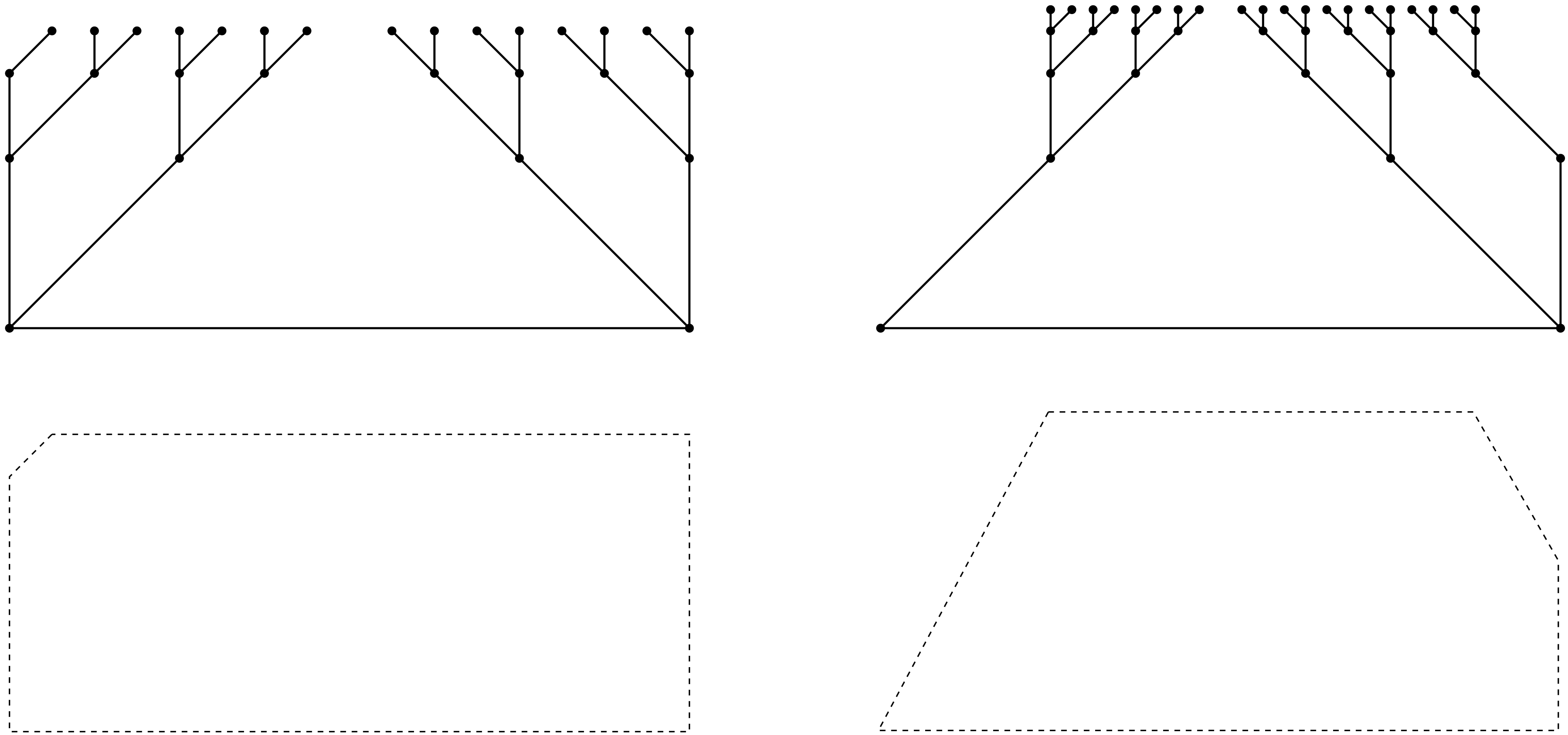
The first few values are $13, 69, 309, \ldots$. Because of symmetry, we know the answer is odd and less than 4^j.



We let $T_j^{i,k}$ be the tree T_j with the top i levels of the the left-hand side removed, together with all the other edges that are disconnected from the base. We also remove the top k levels of the right-hand side.



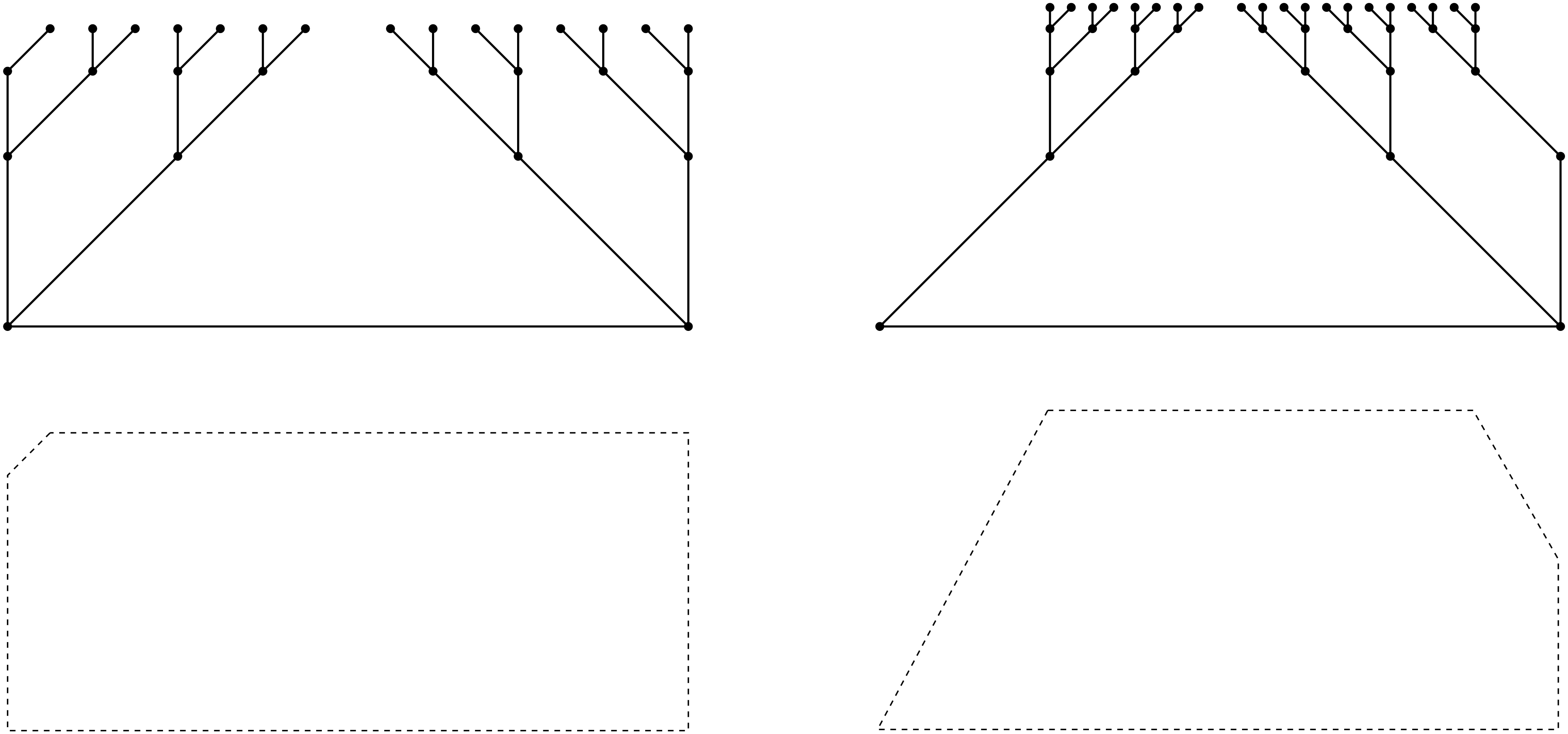
Let $R_j^{i,k}$ be the convex hull of the remaining tree.



The number of sides in $T_j^{i,k}$ is

$$N_{j,i,k} = N_j - [2^j + \cdots + 2^{j-i+1}] - [2^j + \cdots + 2^{j-k+1}] \geq N_j - 2^{j+2} + 2.$$

The exact number is not important, but we will need that it is odd and comparable to 4^j.

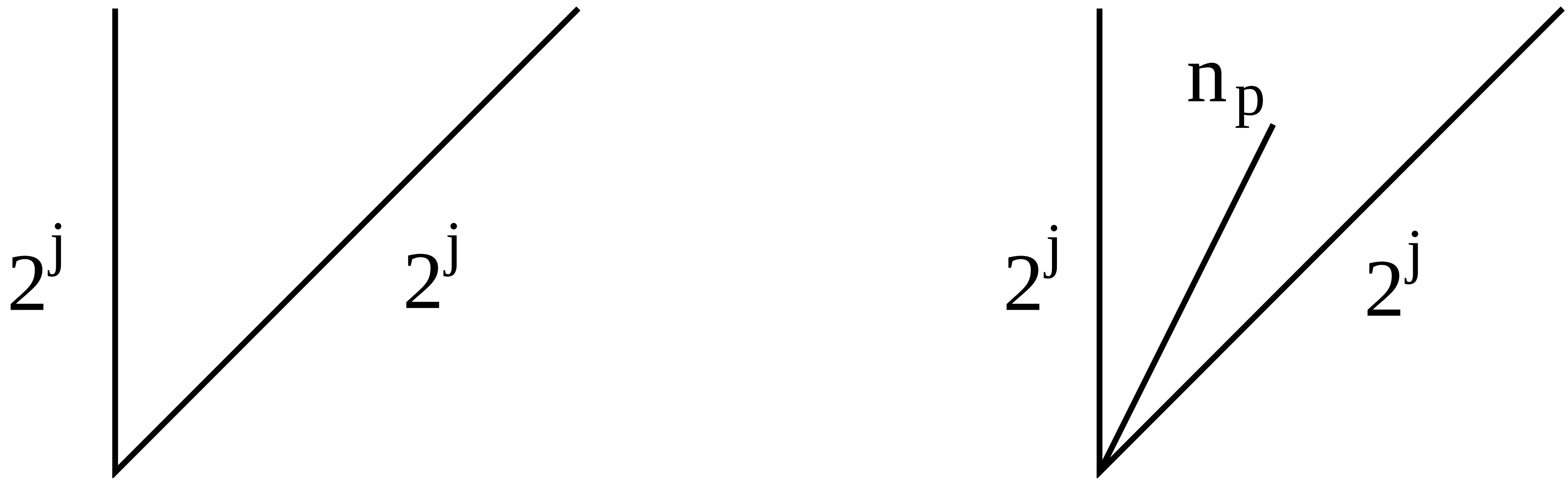


To get oddness, it is important to remember that this is the tree T_j, not $\hat{T}_j$, so there are an even number of edges on T_j in each level along the left and right sides of R_j.

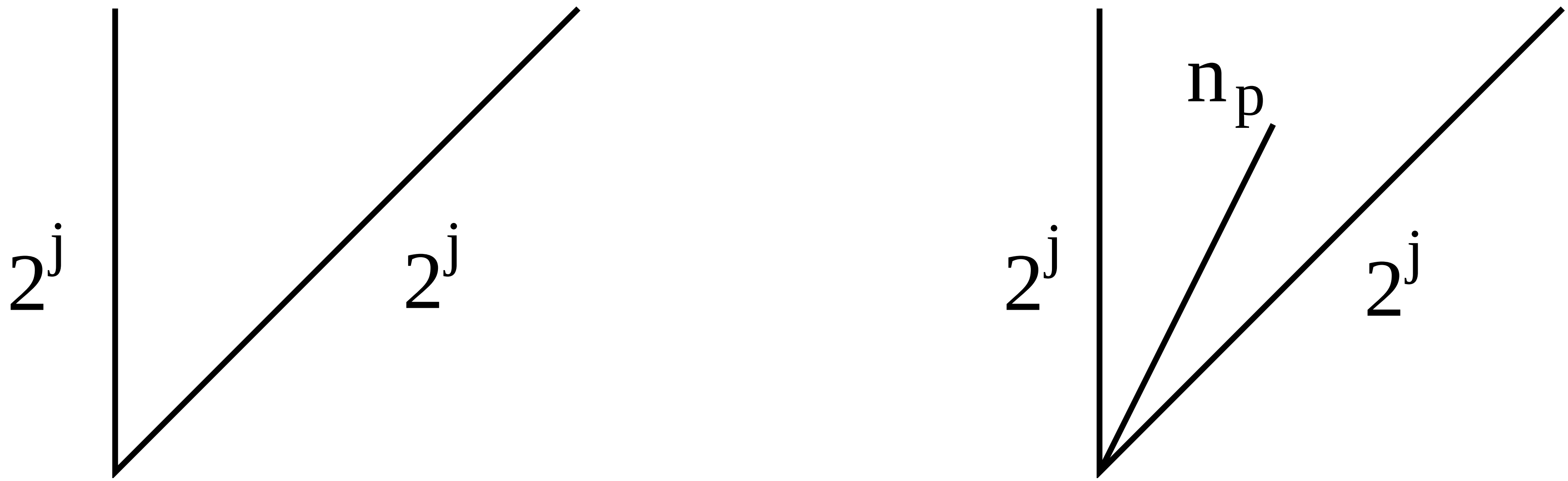
So far, we have built trees that have an exponentially growing odd number of sides. We want to be able to achieve any odd number, and to do this, we will add edges to our clipped trees.

Suppose we are given an odd, positive integer m and define the level of m as the value of j such that $N_j \leq m < N_{j+1}$, where we set $N_0 = 1$ and $N_j \simeq 4^j$ was defined earlier.

Suppose we are also given non-negative integers i, k that are both less than the level j of m. We will add edges to the clipped tree $T_j^{i,k}$ so that the total number of edges is m.



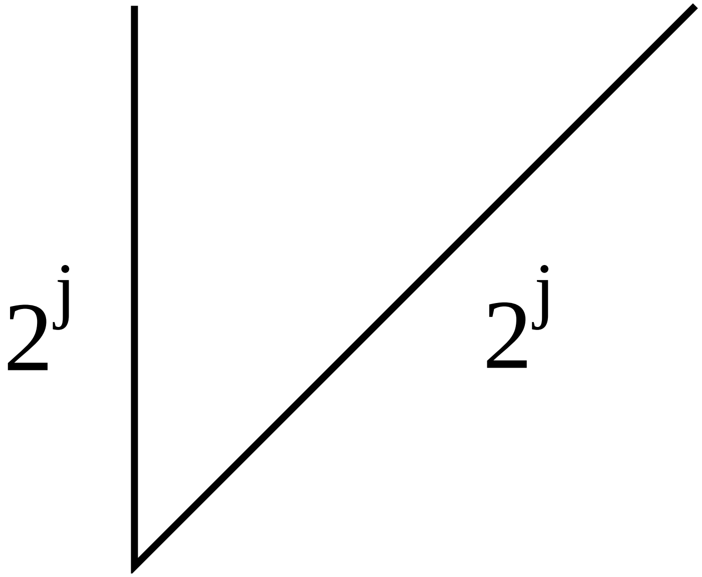
First suppose $j \geq 2$. Then there are $\sim 2^j$ triangular components of $R_j^{i,k} \setminus T_j^{i,k}$, and we add a segment connecting the center of the pth triangle to its bottom vertex and divide it into n_p equal sub-segments. We call thus segment a “spike”.

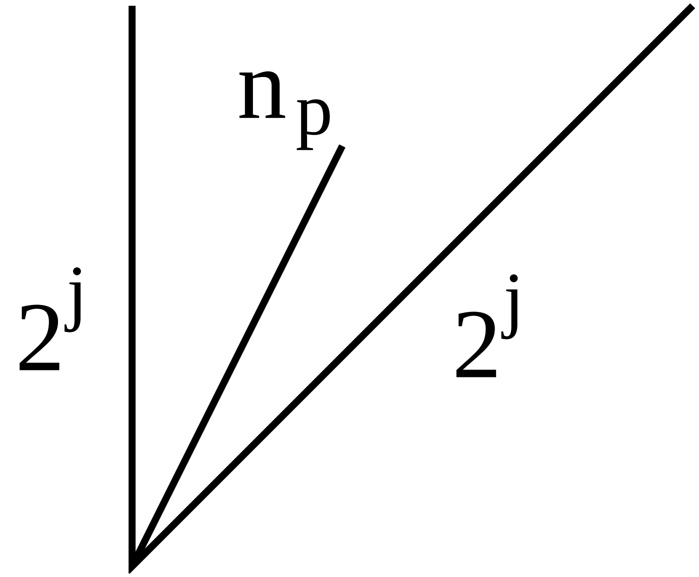


We choose the integers $\{n_p\}$ so that

$$2\sum_p n_p = m - N_{j,i,k} \quad \text{and} \quad n_p = O(2^j),$$

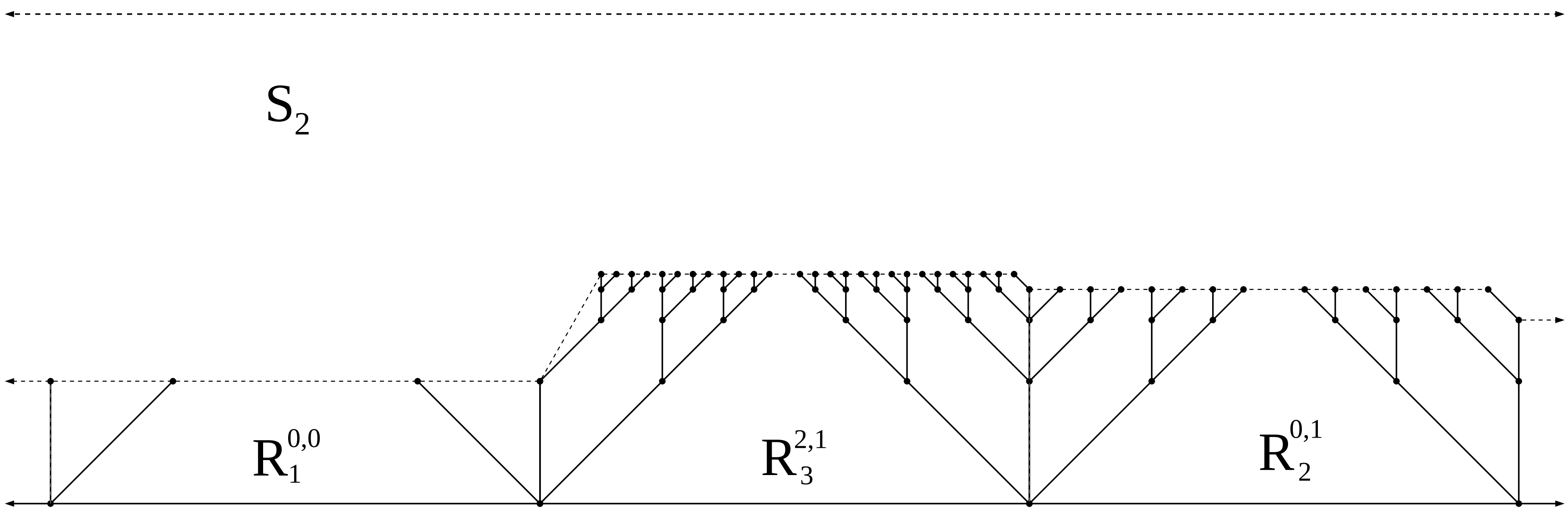
where the constant is allowed to depend on i, k (eventually both of these will be chosen to be $O(1)$, so the constant above will also be $O(1)$).





If $j = 1$ and $i = 0$ or $k = 0$ then there is at least one triangular component where we can add a spike.

If $j = 1$ and $i = k = 1$ then instead of adding a spike, use a simple folding in place of $T_1^{1,1,}$.

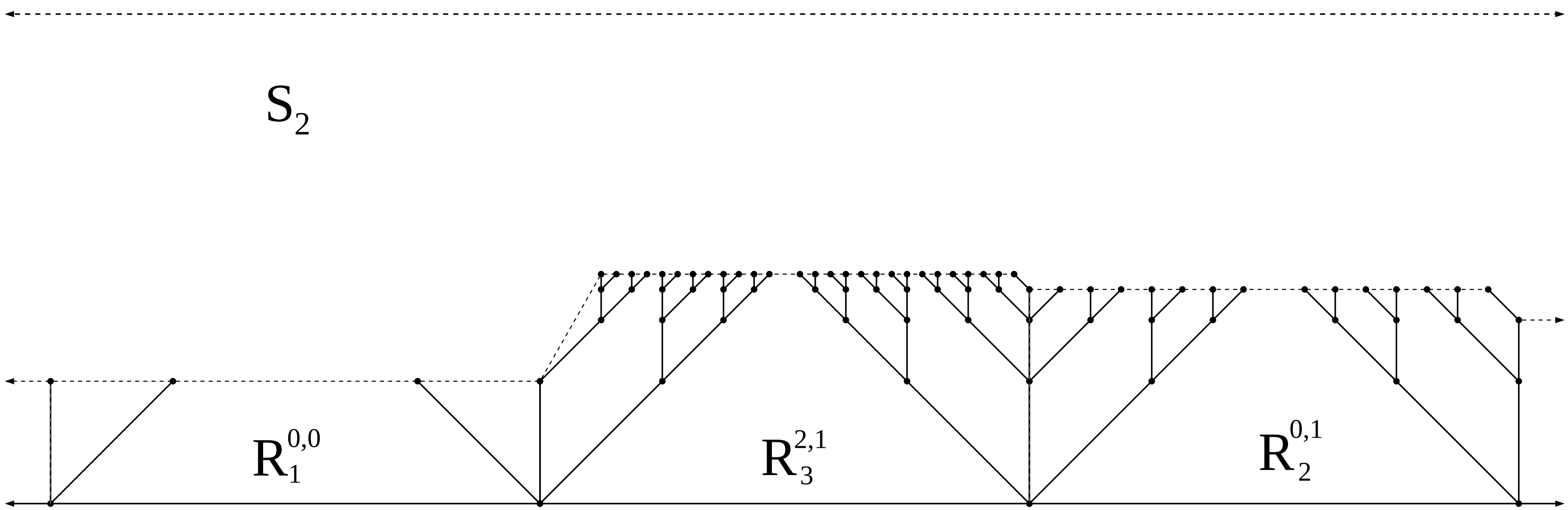


Suppose $J \in \mathcal{J}$ and let m be its length (an odd, positive integer). Let j be the level of m and let j_1 and j_2 be the levels of the elements of $\mathcal{J}$ that are adjacent to J and to its left and right respectively.

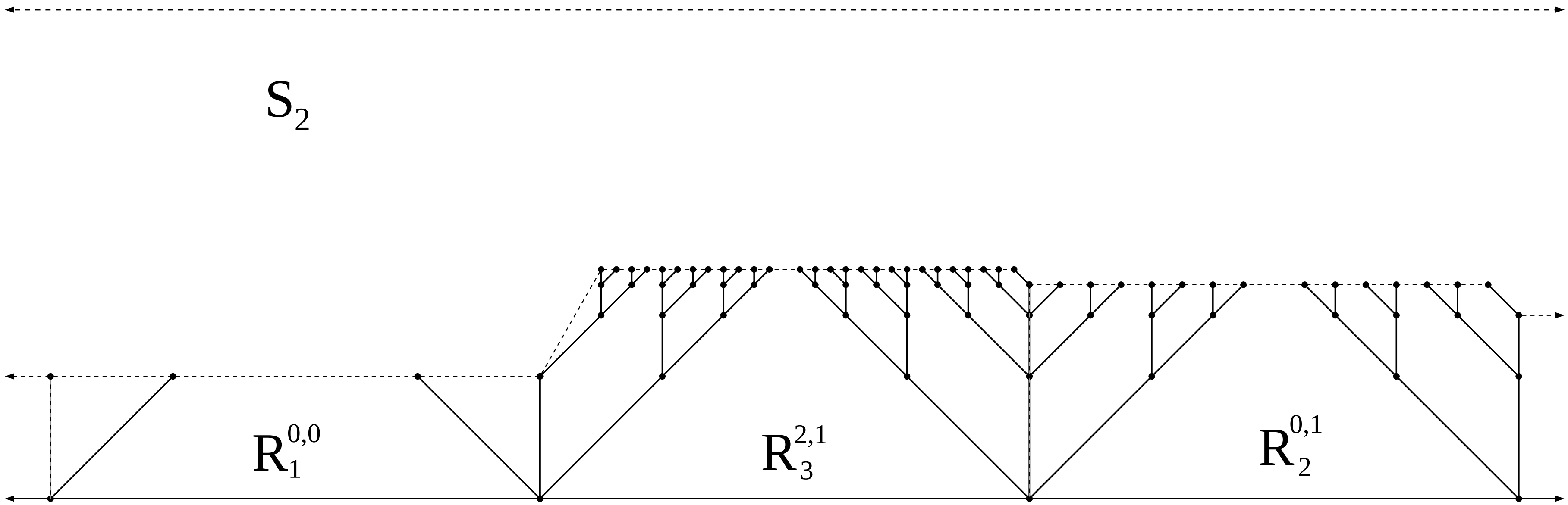
Let

$$i = \max(0, j - j_1), \quad k = \max(0, j - j_2),$$

and associate to J the tree $T_{j,m}^{i,k}$.

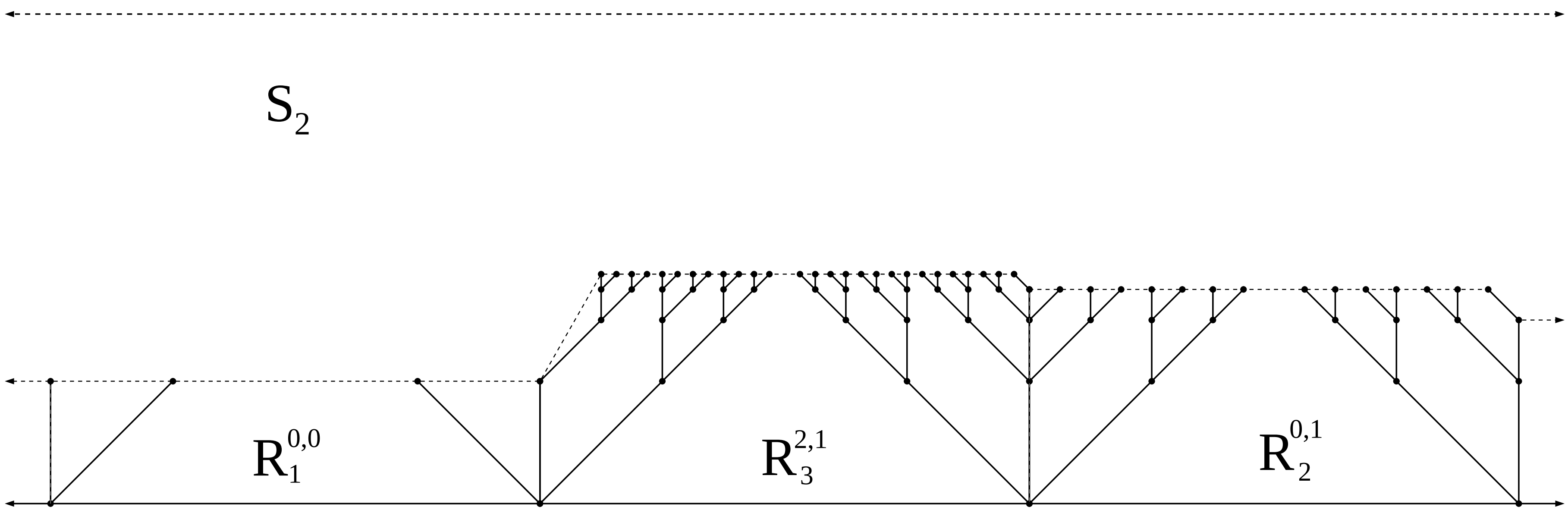


The indices i, k have been chosen so that when two intervals are adjacent, and the corresponding trees have different levels, then the higher tree has been clipped to match the level of its lower neighbor.

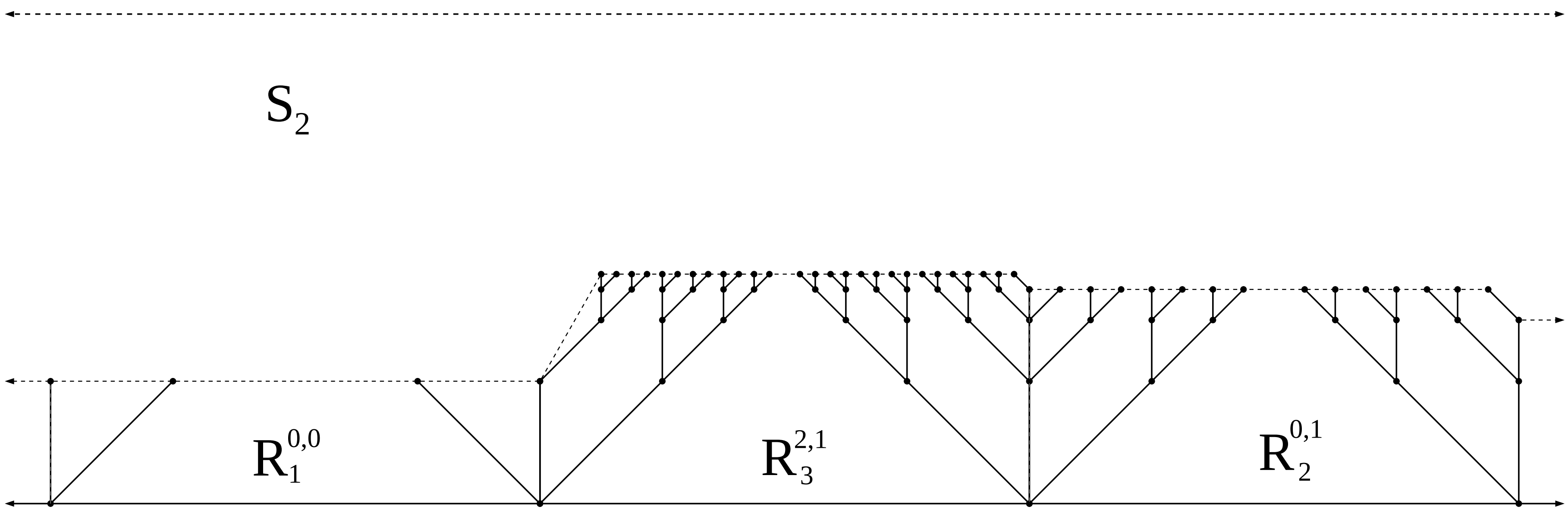


Thus the union of the clipped convex hulls $\cup R_j^{i,k}$ has an upper edge that is a Lipschitz graph γ (the graph coincides with the real line on intervals where we use a simple folding).

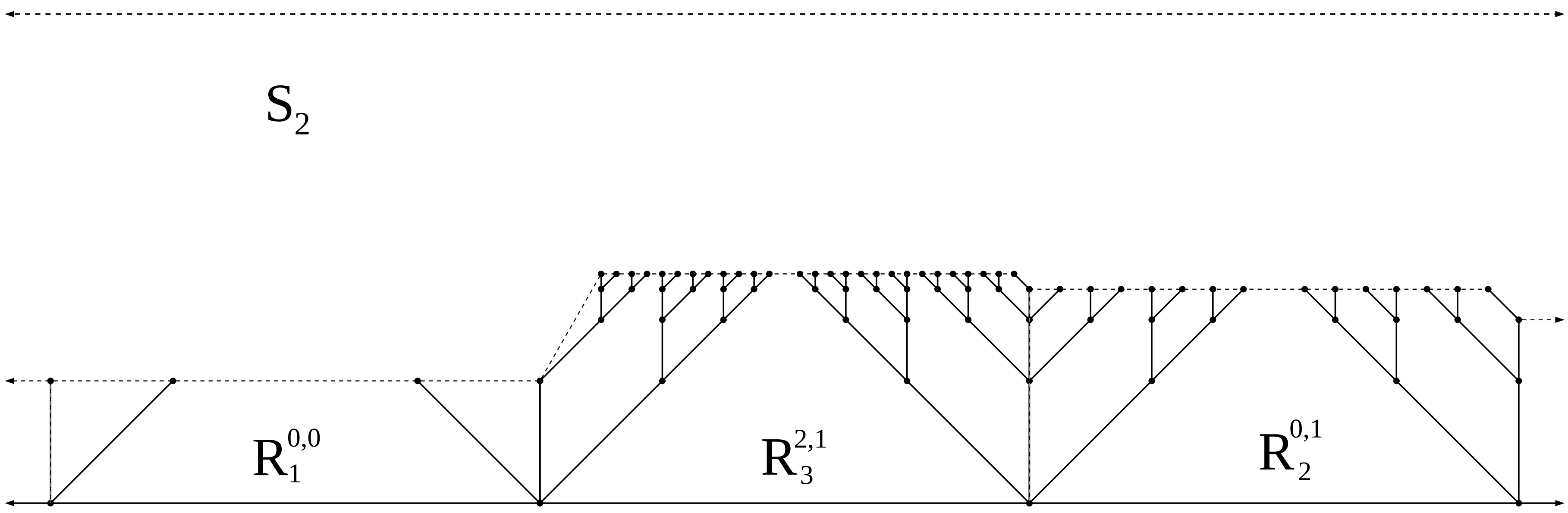
The region above Γ and below height 2 is a variable width strip that we denote S_2.



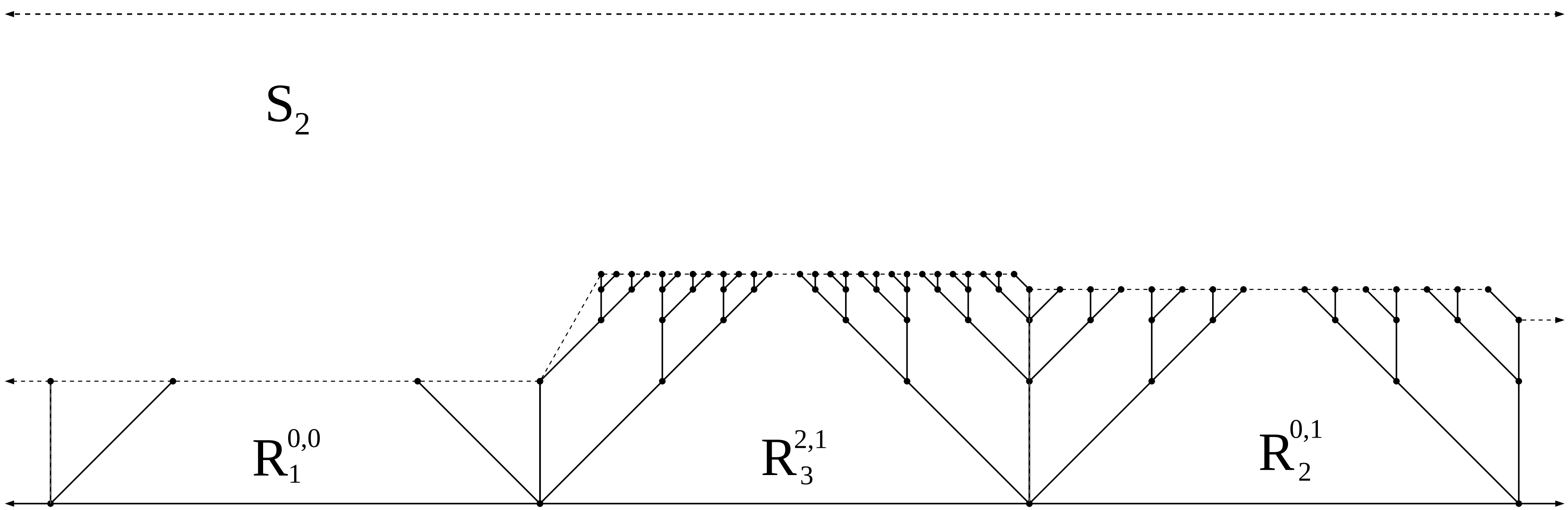
If m has level ≥ 1, then inside the copy of $R_j^{i,k}$ with base J we place a copy of the tree $T_{j,m}^{i,k}$ and remove this tree from the upper half-plane.



If m has level 0, or if we are in the case when $j = 1 = k$ discussed earlier, $R_j^{i,k}$ is a line segment on $\mathbb{R}$ and we remove a diagonal line segment divided into $\frac{1}{2}(m-1)$ edges; above these intervals the map will be a simple folding of size m.

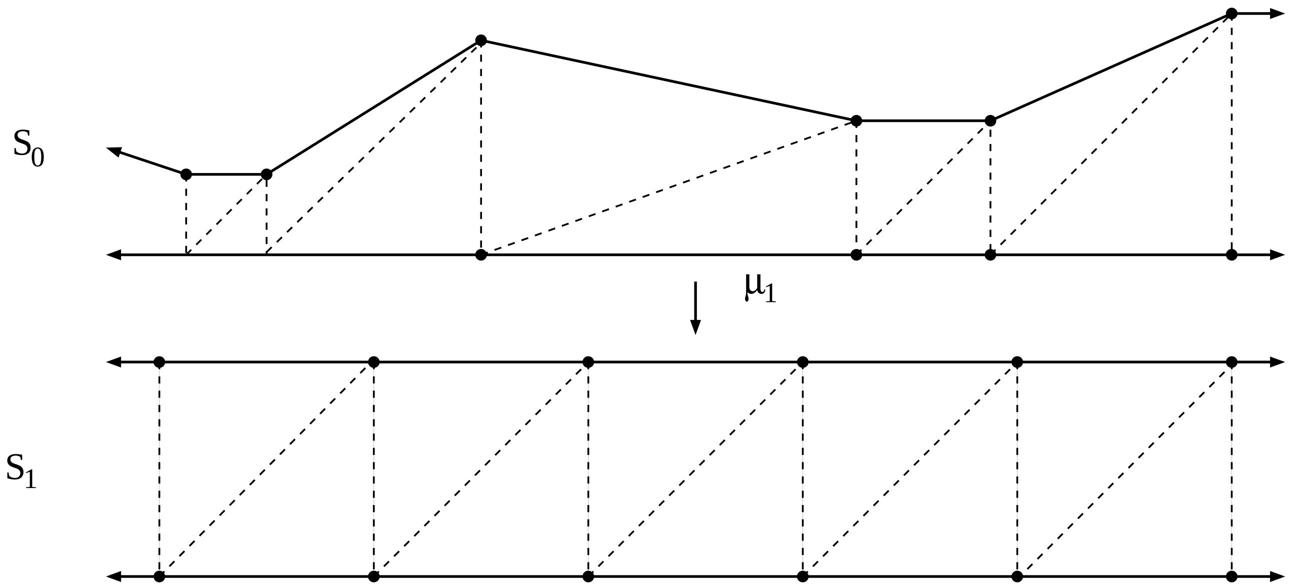


Doing one of these steps for every element of the partition defines the simply connected region $W = \mathbb{H} \setminus \Gamma$.



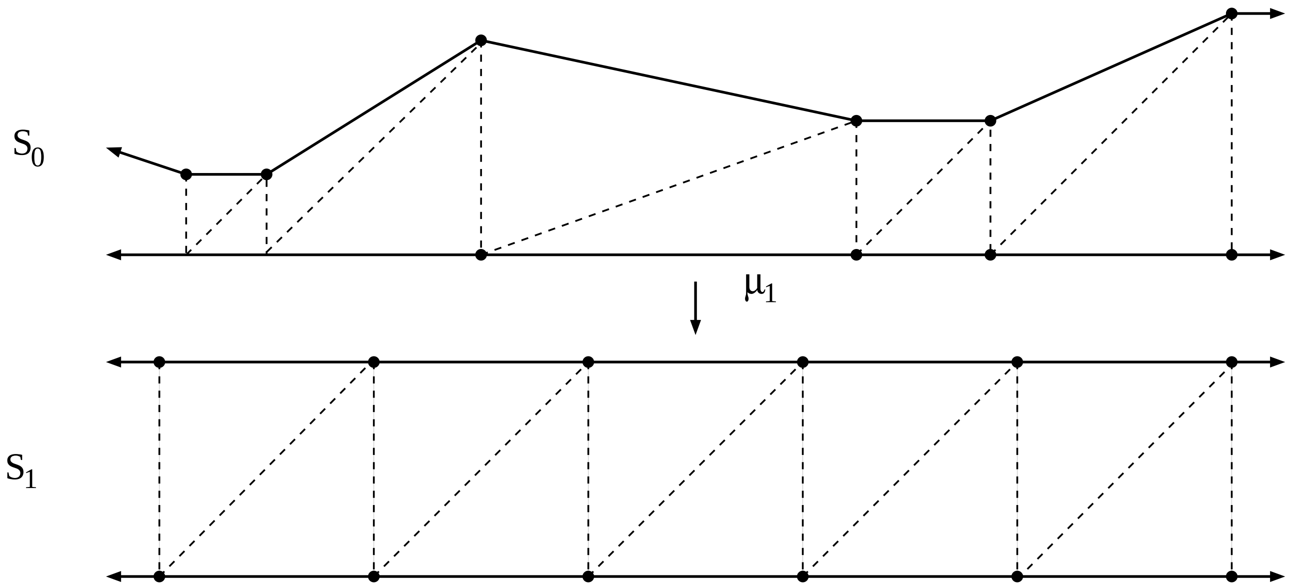
Next we want to triangulate the region W and use piecewise linear maps on the triangles to define the map ψ.

Building the map ψ

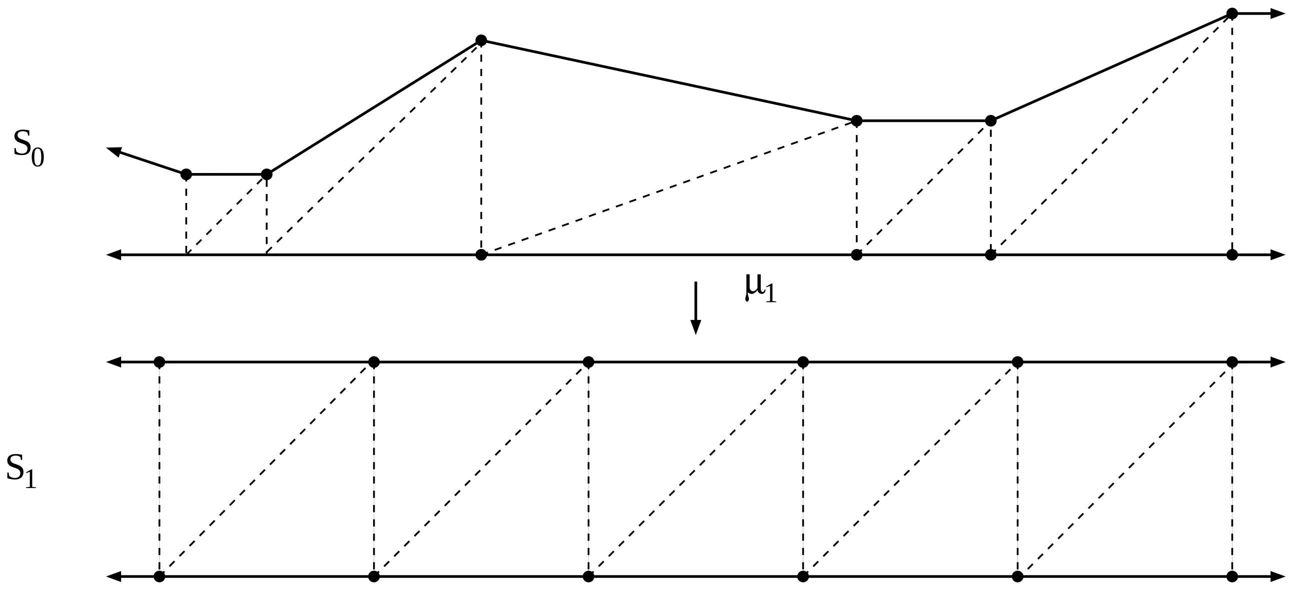


Given two adjacent intervals J_k, J_{k+1} of our partition $\mathcal{J}$ with common endpoint x_k, let $h_j = \min(\ell(J_k), \ell(J_{k+1}))$ be the length of the shorter one and let $z_k = x_k + ih_k$.

Form an infinite polygonal curve by joining these points in order, and let S_0 be the region bounded between this curve and the real axis.

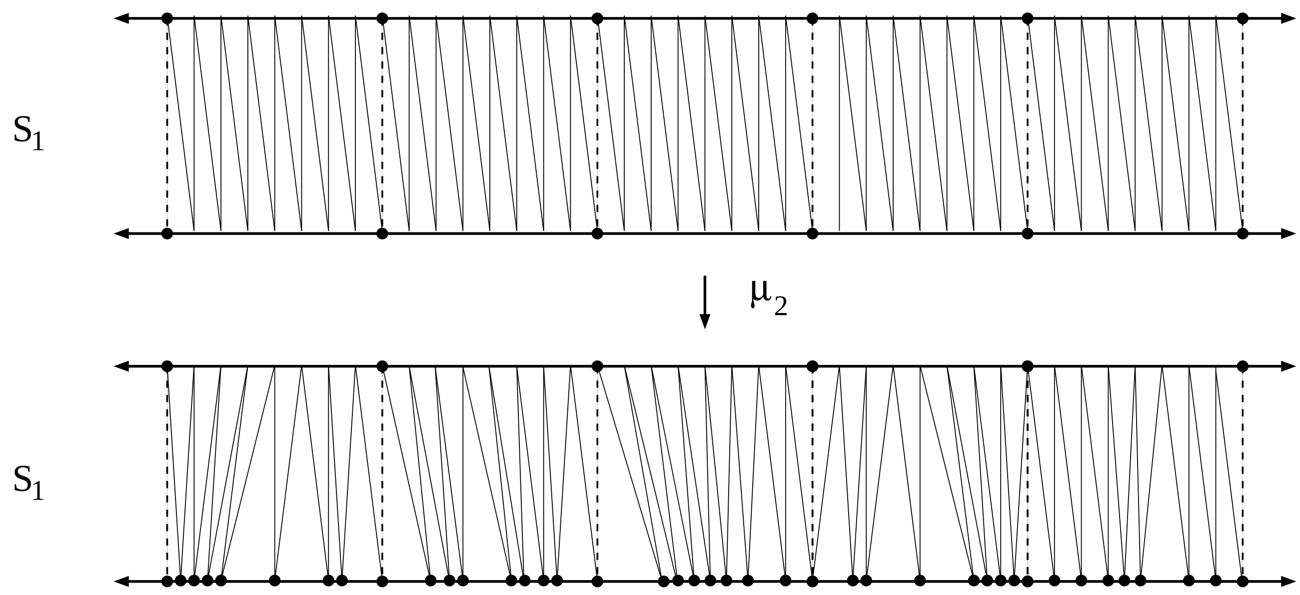


The vertical crosscuts at the points x_k cut the region into trapezoids and because of our assumption about the lengths of adjacent elements of $\mathcal{J}$ being comparable, only a compact family of trapezoids occur.



It is easy to QC map S_0 to the strip $S_1 = \{x + iy : 0 < y < 2\}$ by sending each trapezoid to a square of side length 2

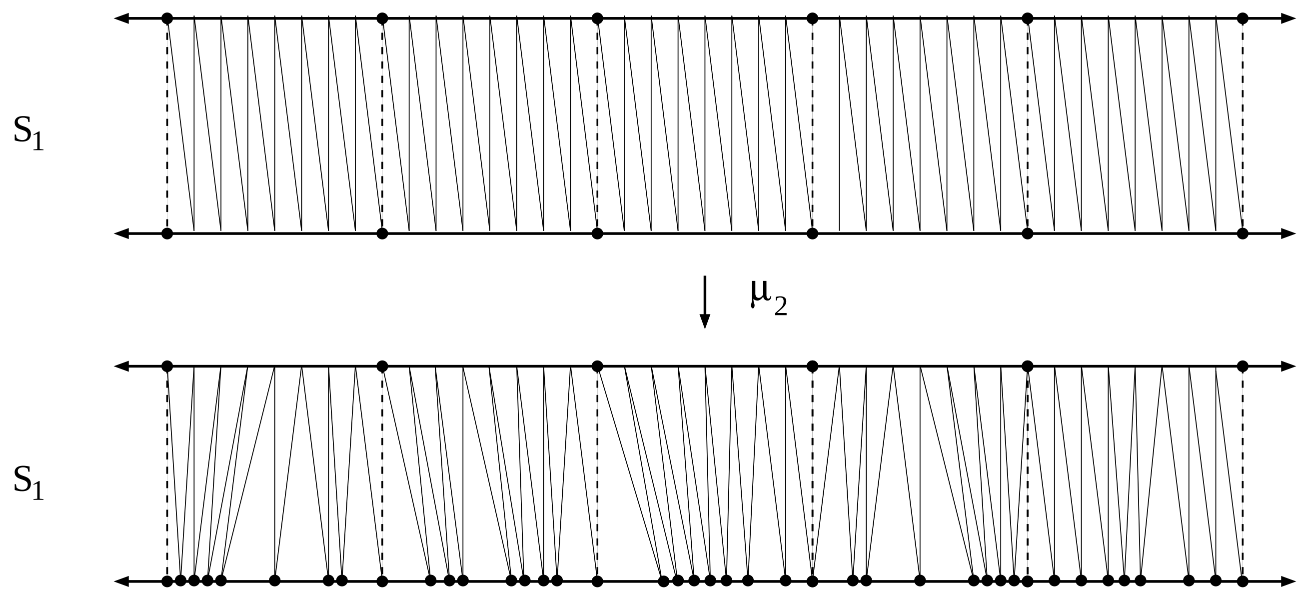
Cut each trapezoid into triangles by a diagonal and map these linearly to the right triangles obtained by cutting the square by a diagonal). Denote this map by μ_1



Next we define a map $\mu_2 : S_1 \to S_1$ that is the identity on the top edge of S_1 and will be biLipschitz on the bottom edge.

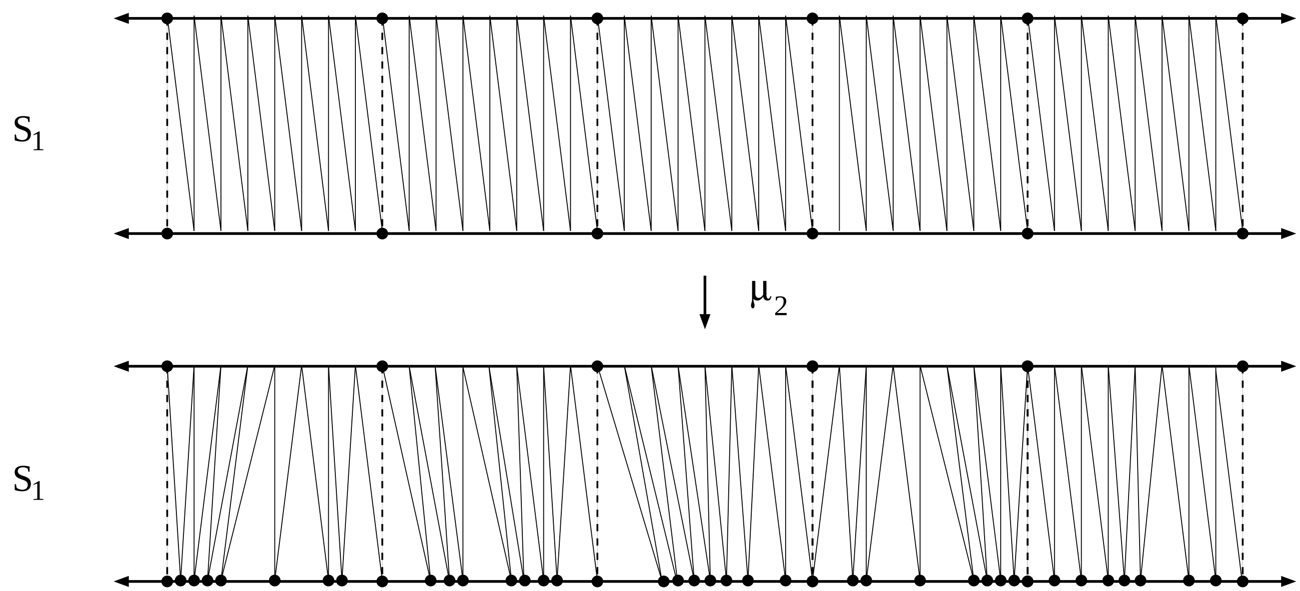
Such a map clearly has a biLipschitz extension to the interior of the strip.

On the bottom edge μ_2 fixes the even integer points.



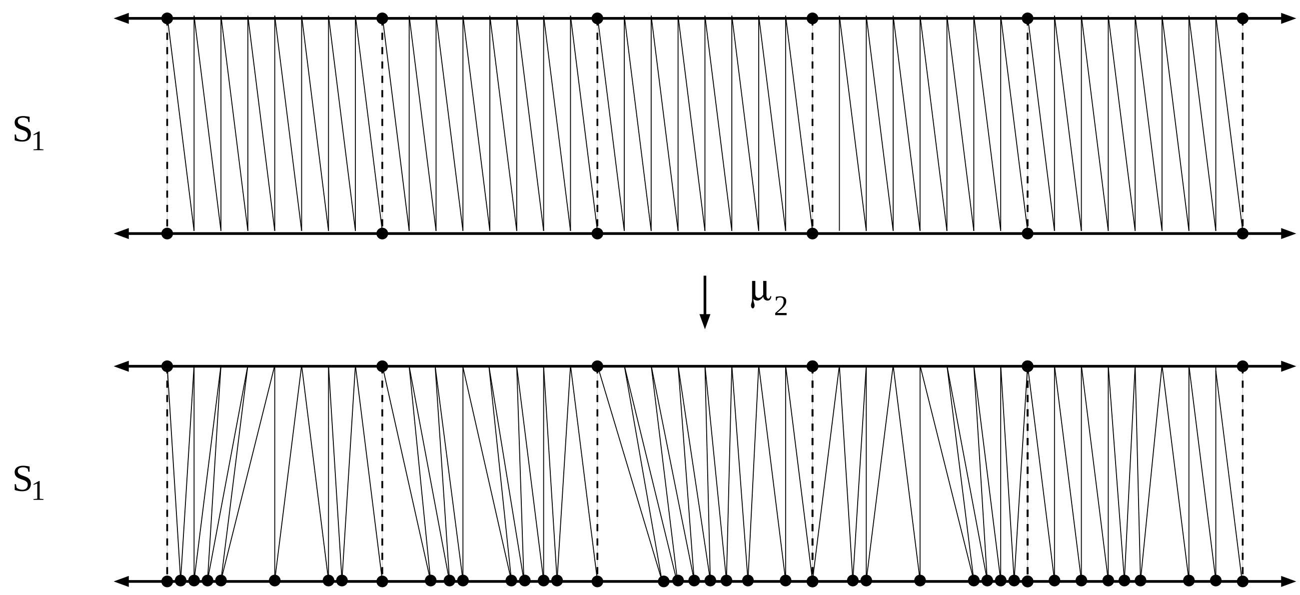
Suppose I is an interval of length 2 on the bottom edge of S_1 that corresponds to a an interval $J \in \mathcal{J}$ of length m and that $T^{i,k}_{j,m}$ is the corresponding clipped tree.

If this tree is a simple folding, we just take μ_2 to be the identity on I.



Otherwise, project the degree 1 vertices of $T^{i,k}_{j,m}$ vertically onto I. These points partition I into subintervals $\{I_p\}$ that correspond 1-to-1 to the components $\{V_p\}$ of $V = R^{i,k}_{j,m} \setminus T^{i,k}_{j,m}$.

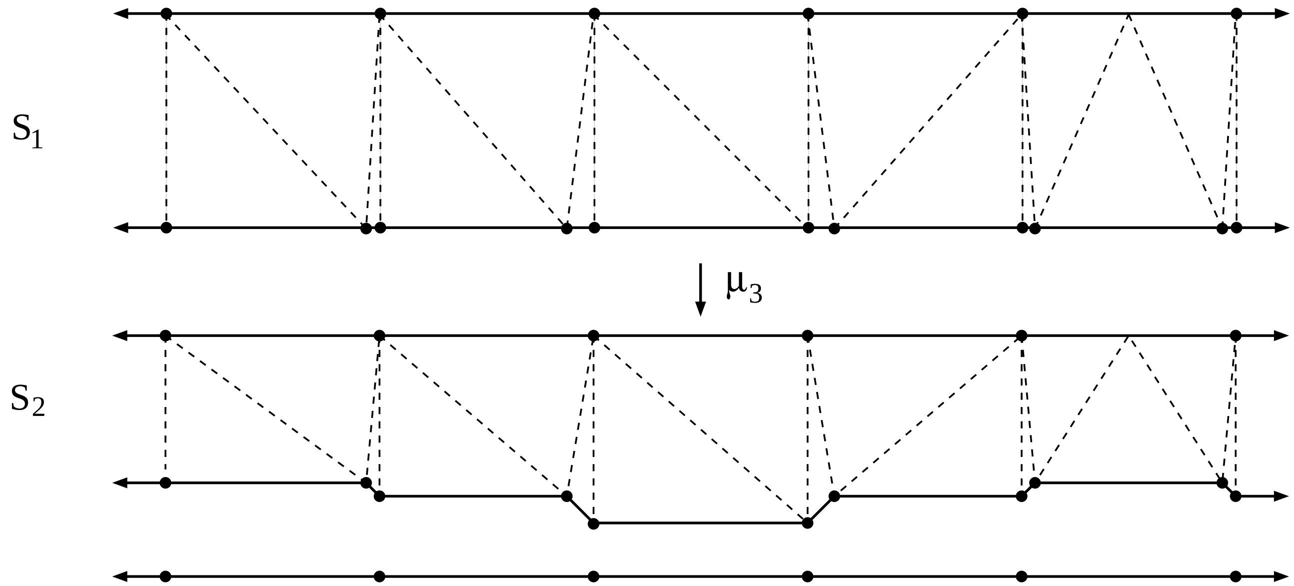
If V_p has m_p sides, divide I_p into m_p equal subintervals. This gives a partition of I into $m = \sum_p m_p$ intervals (of possibly different sizes).



Define μ_2 to map the "even" partition of I into m equal length intervals to "unequal" partition.

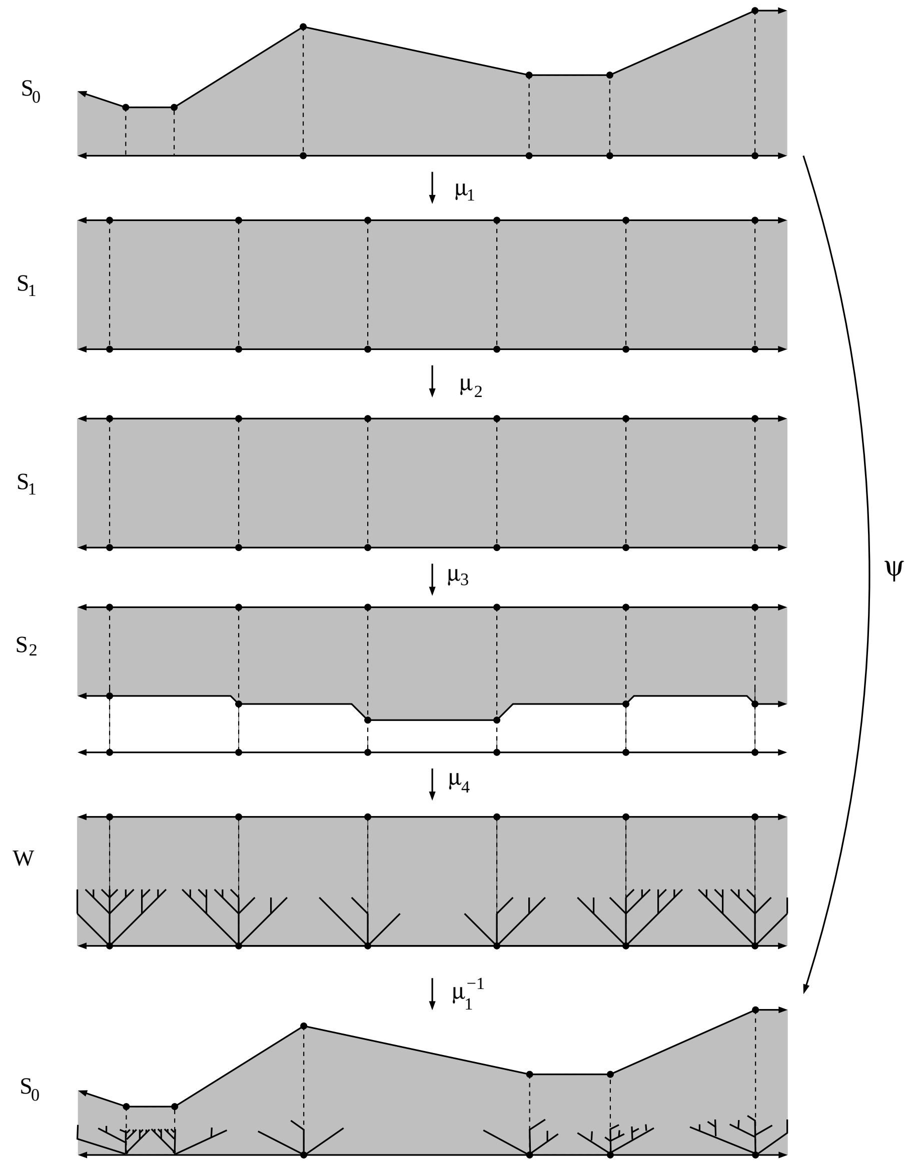
The map is biLipschitz because each interval in the "unequal" partition has length comparable to $|I|/m$.

To see this, recall that if m has level $j \geq 1$ then $m \simeq 4^j$, there are $\simeq 2^j$ components of V and each contains $\simeq 2^j$ sides.

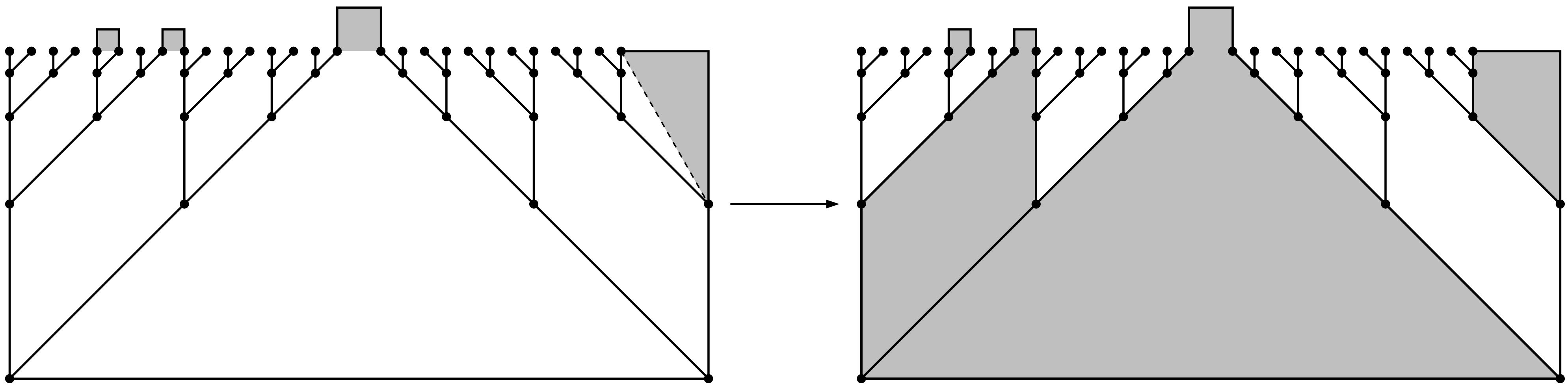


Define a variable width strip S_2 whose upper boundary is $\{y = 2\}$ and whose lower boundary is the upper envelope γ of the union of the regions $R^{i,k}_{j,m}$.

We let $\mu_3 : S_1 \to S_2$ be a biLipschitz map that is the identity on the top boundary of S_2 and agrees with vertical projection onto γ on the bottom edge (again, easy to define using triangulations).

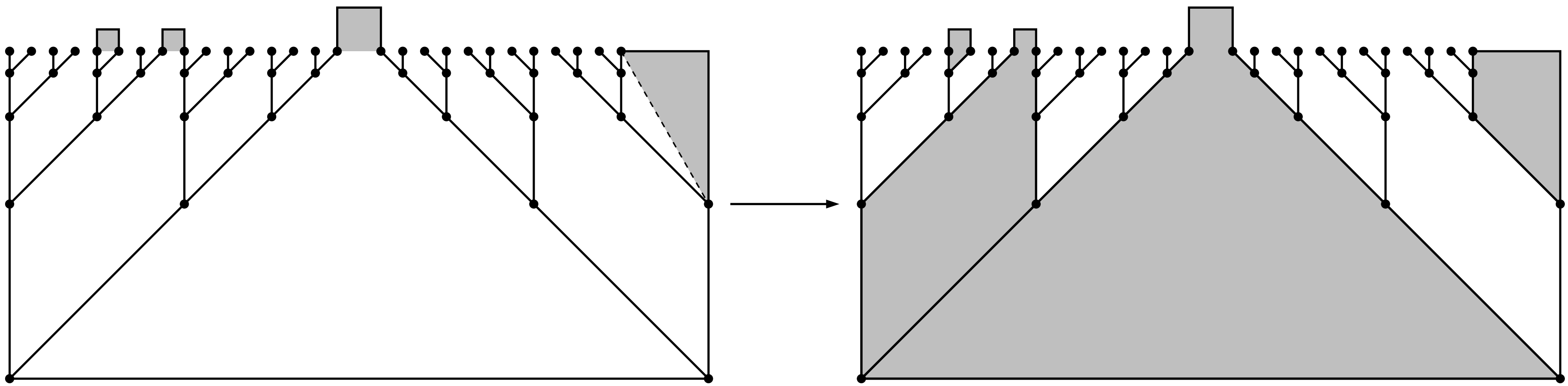


The maps $S_0 \xrightarrow{\mu_1} S_1 \xrightarrow{\mu_2} S_1 \xrightarrow{\mu_3} S_2 \xrightarrow{\mu_4} W \xrightarrow{\mu_1^{-1}} S_0$ define ψ.

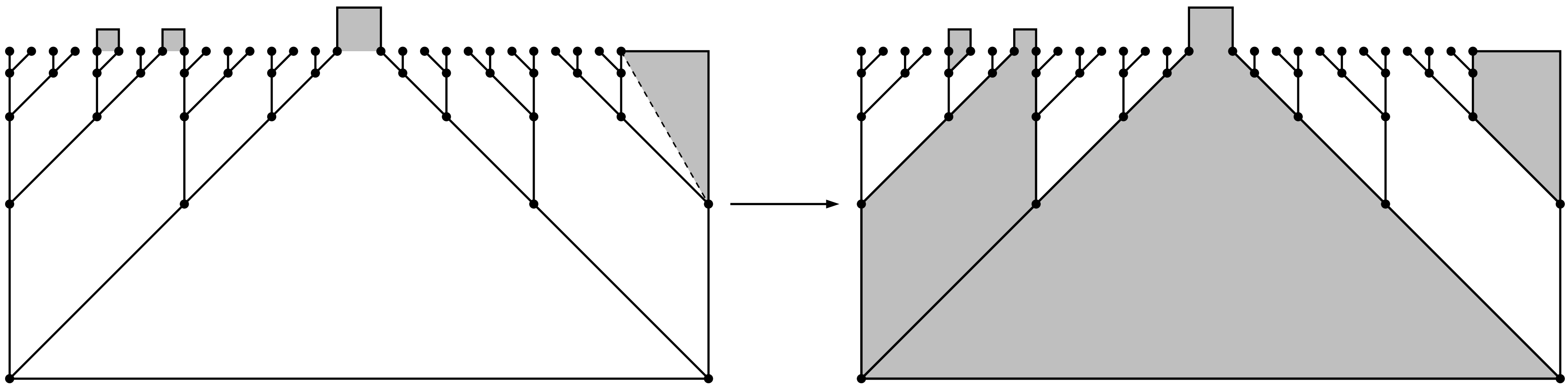


The final step is to construct the map μ_4.

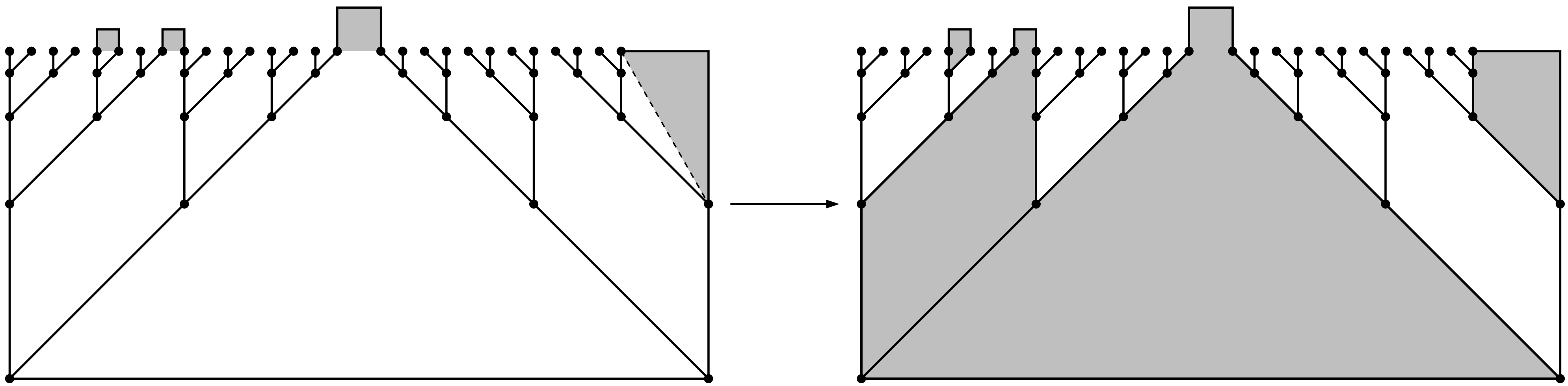
Suppose I is an interval of length 2 corresponding to some $J \in \mathcal{J}$ and let $Q \subset S_1$ be the 2×2 square with base I.



The map μ_4 is the identity above S_2, so we only need to define it inside each such Q so that it is the identity on $\partial Q \cap S_2$ (then the definitions on different squares will join to form a quasiconformal map on S_2.

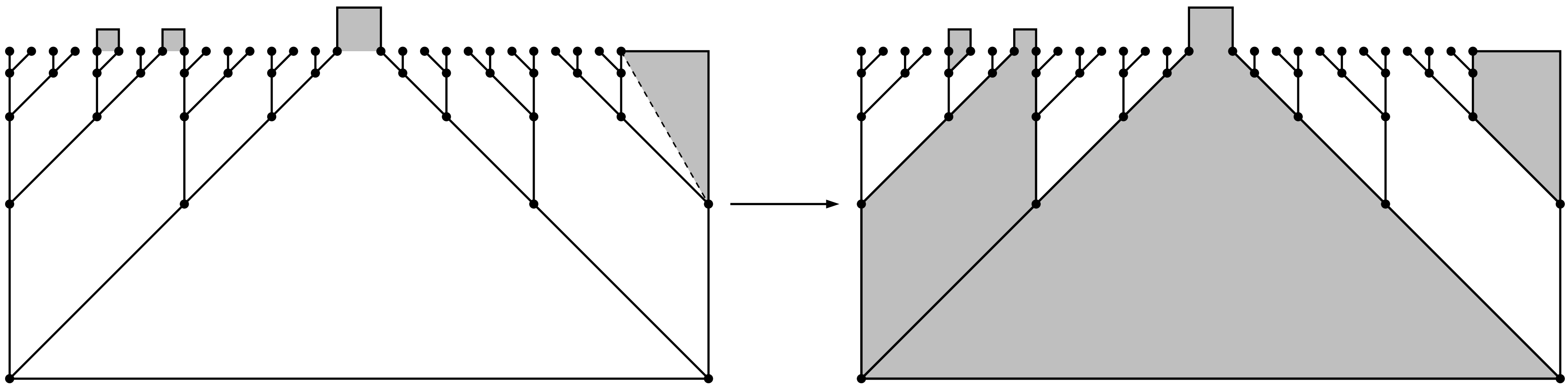


If $W \cap Q$ is a simple folding, we have already seen how to define μ_4 in a previous figure. Otherwise, suppose Q contains the convex hull $R = R_j^{i,k}$ of a the tree $T = T_{j,m}^{i,k}$. Let $R' = R_j$ be the "unclipped" version of R.



As noted earlier, $\partial R \setminus T$ consists of intervals, and each interval I_p has been partitioned into m_p equal length intervals where m_p is the number of sides of the corresponding component of $R \setminus T$.

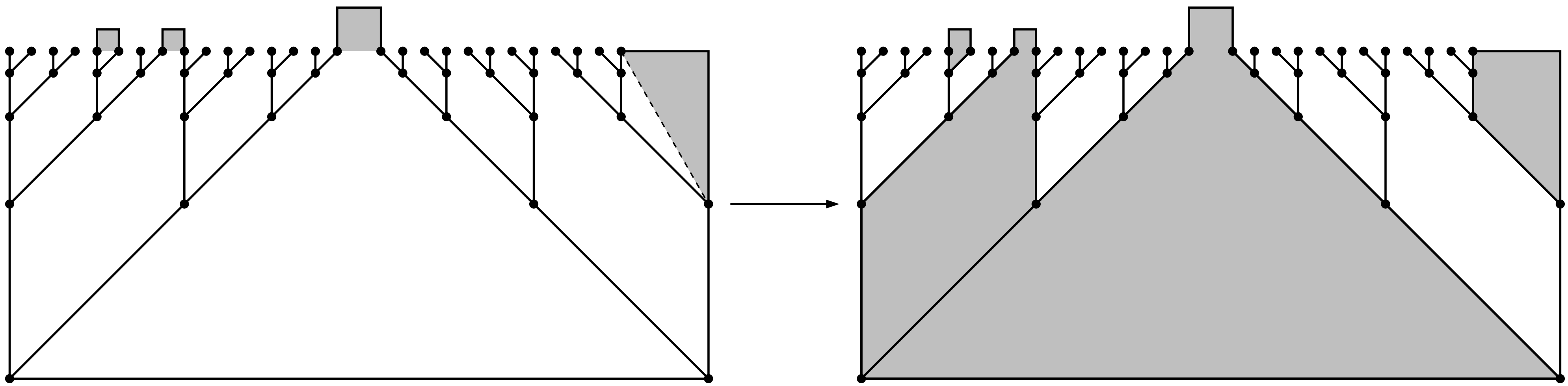
The interval I_p is horizontal unless it is the leftmost or rightmost interval, in which case it may be sloped (the "clipped" part of R).



For horizontal intervals I_p we let $Q_p \subset Q \setminus R$ be the square with base I_p. For sloped intervals we let Q_p denote the triangular component of $R' \setminus R$ containing the interval.

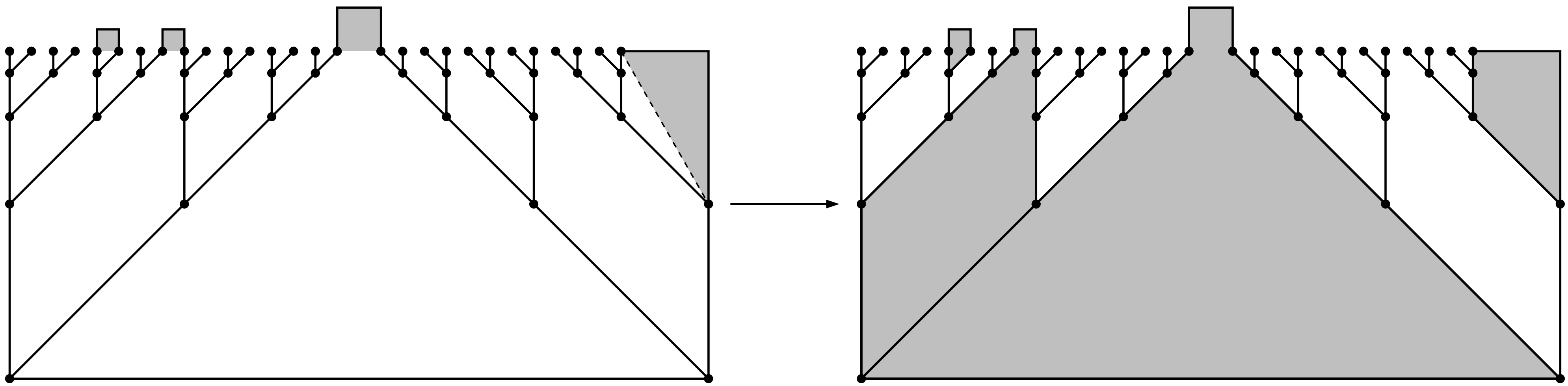
Let W_p be the component of $R \setminus T$ with I_p as its top edge.

We want to define $\mu_4 : Q_p \to U_p = Q_p \cup W_p$ to be quasiconformal, to be the identity on $\partial Q_p \setminus \overline{W_p}$, and to map each interval in our partition of I_p to an side of W_p.

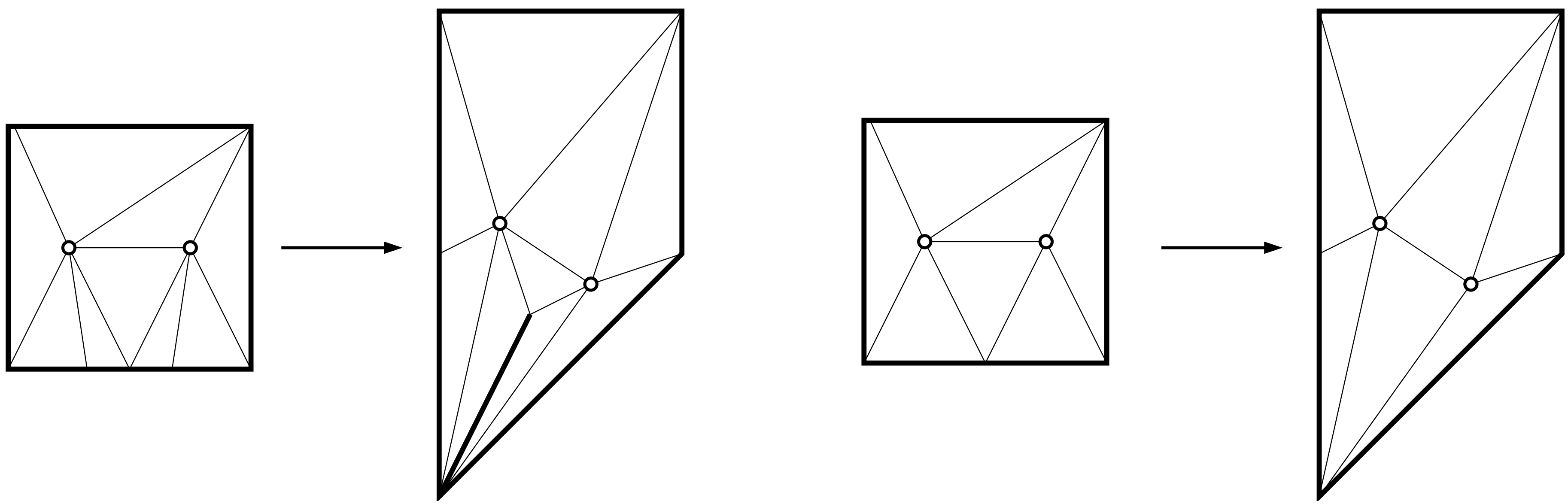


The four types of components W_p that have to be considered:

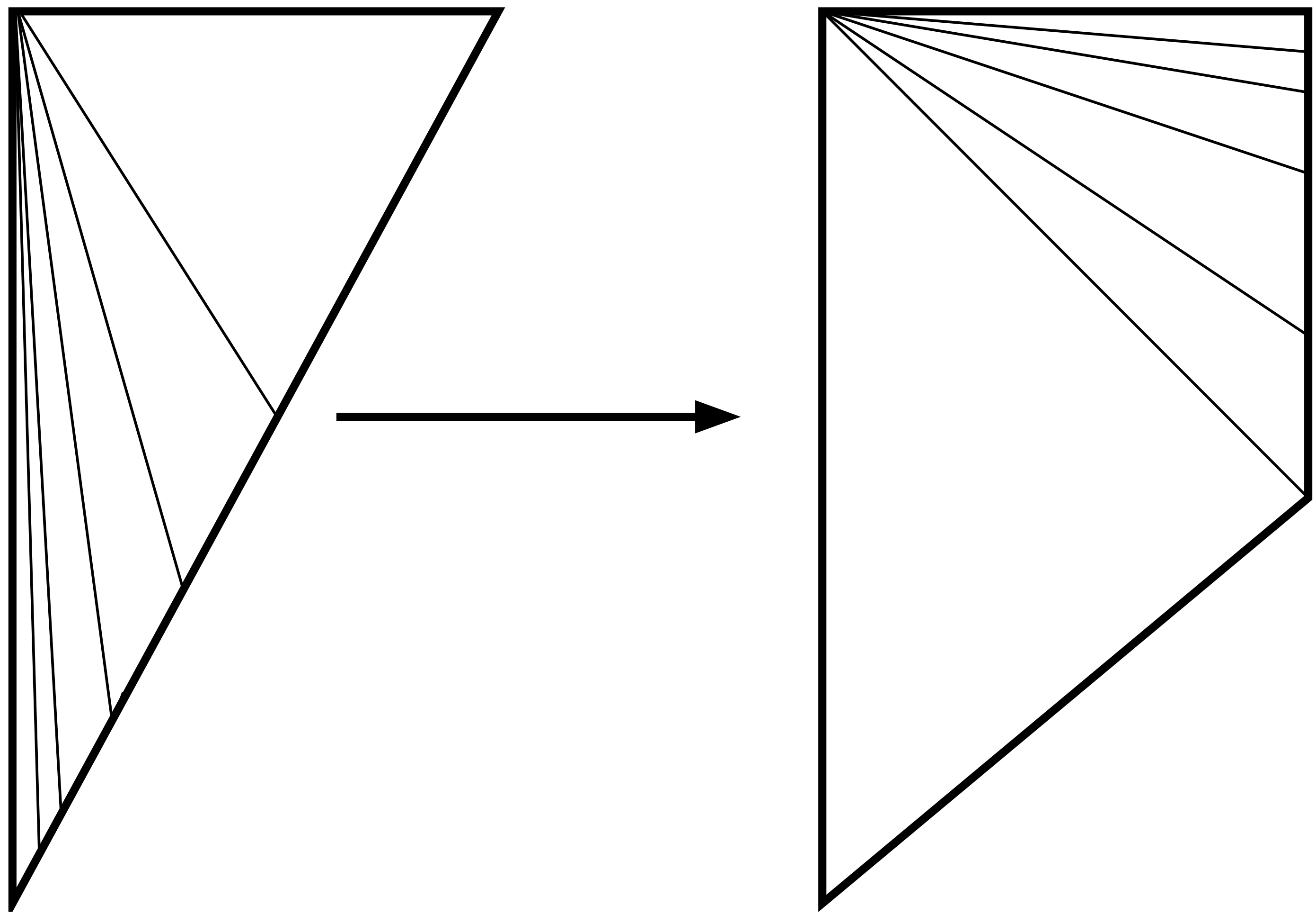
1. top triangles,
2. corner triangles,
3. parallelograms,
4. the center triangle,



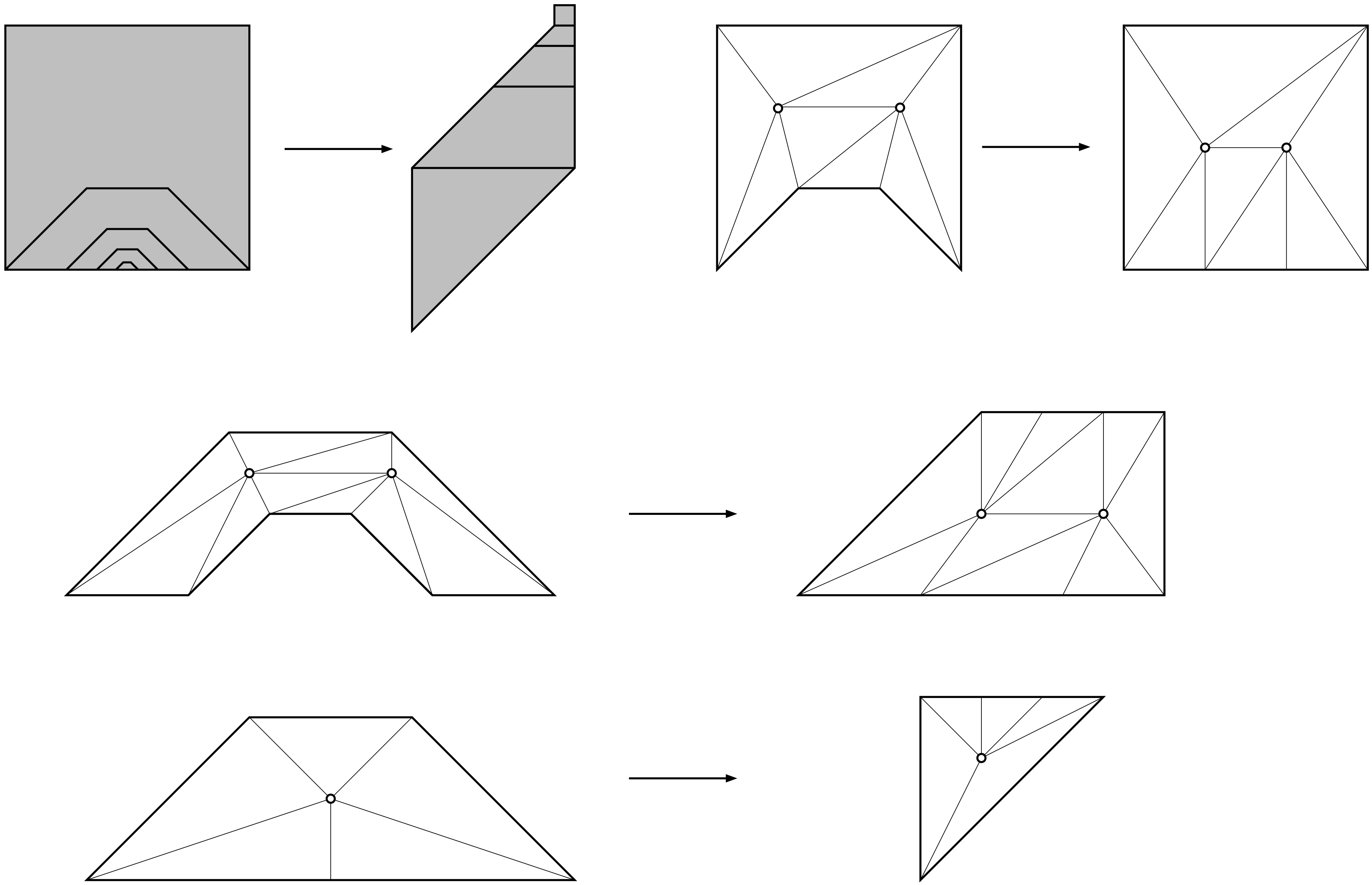
In each case, the map from Q_p to U_p is specified by drawing compatible triangulations of the two regions and then taking the piecewise affine map between these triangulations.



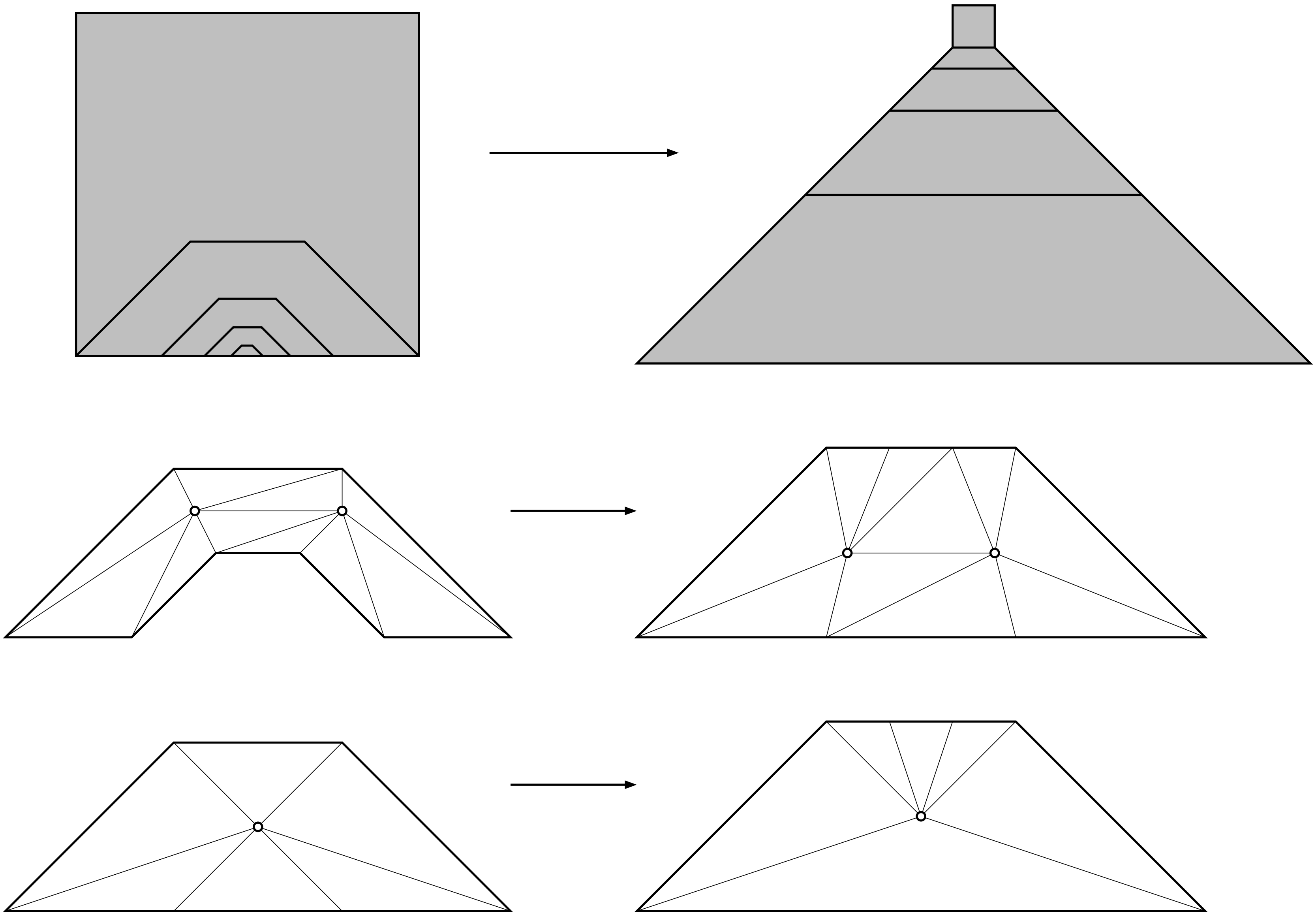
Case 1: top triangles with and without slits



Case 2: side triangles



Case 3: parallelograms



Case 4: central component

This completes the proof of the “plain” Folding Theorem.

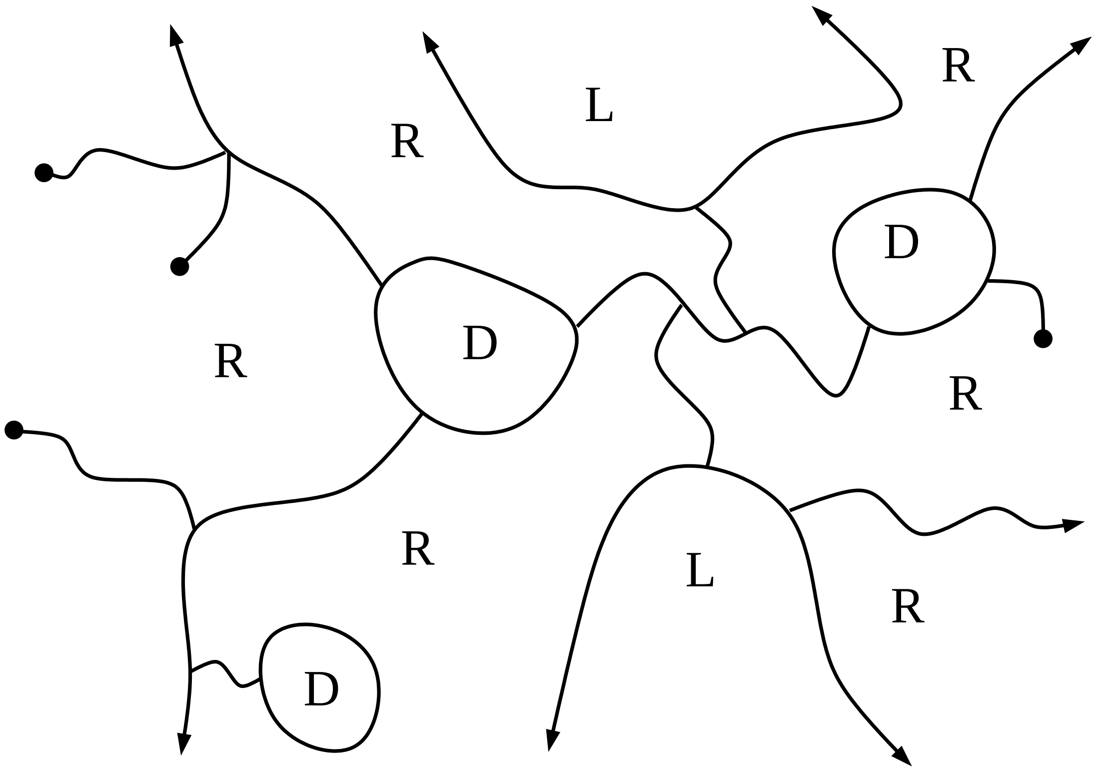
We will state, but not prove a more general version used in applications.

In our generalization, the tree T is replaced by a connected graph whose complementary components are each mapped to one of three possible standard domains:

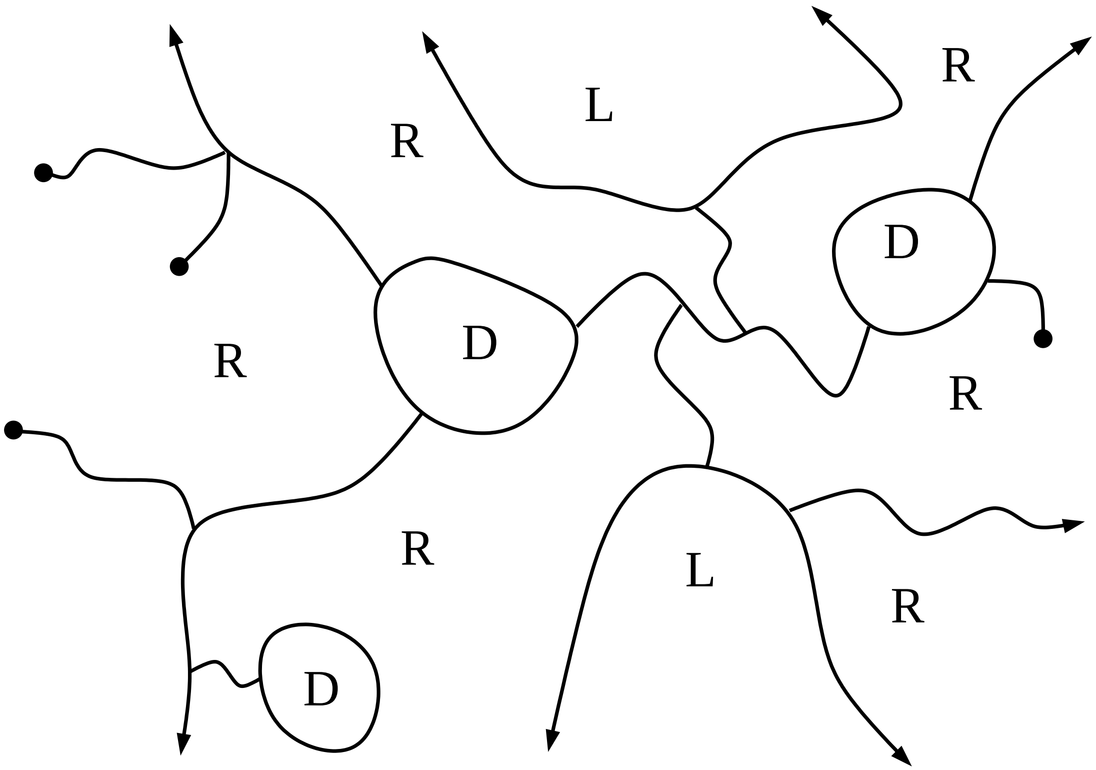
1. the unit disk, $\mathbb{D}$.
2. the left half-plane, $\mathbb{H}_l$.
3. the right half-plane, $\mathbb{H}_r$.

We shall refer to these as D-components, L-components and R-components respectively.

If only L- and R-components are used then the graph T is still a tree.



D-components that are bounded Jordan domains, L-components that are unbounded Jordan domains and R-components that are unbounded simply connected domains (they need not be Jordan).



D-components and L-components may only share an edge with a R-component and QC folding will only be applied on the R-components.

D-components: Ω is bounded and $\partial\Omega$ is a closed Jordan curve that is the union of a finite number of edges of T, say d. We are given a length respecting (on the boundary) quasiconformal map $\eta : \Omega \to \mathbb{D}$ and we assume the n vertices on $\partial\Omega$ map to the nth roots of unity on the circle.

The map $\sigma : \mathbb{D} \to \mathbb{D}$ is $z \mapsto z^d$ followed by a quasiconformal map $\rho : \mathbb{D} \to \mathbb{D}$ that is the identity on $\partial\mathbb{D}$.

We often take ρ to be the identity, and this gives a critical point of degree d with critical value 0.

If a critical value a is desired, then ρ is chosen so $\rho(0) = a$. If $|a| < 1/2$, then ρ can be chosen to be conformal on $\{|z| < 3/4\}$, so in this case, the dilatation of ρ is supported on $\{z : \frac{1}{4} < |z| < 1\}$. Thus, in all cases, the dilatation of σ is bounded by $O(|a|)$ and is supported on $\{z : 1 - \frac{1}{d} \log 4 < |z| < 1\}$.

L-components: Here Ω is an unbounded Jordan domain and we are given a length respecting, quasiconformal $\eta : \Omega \to \mathbb{H}_l$.

The map $\sigma : \mathbb{H}_l \to \mathbb{D} \setminus \{0\}$ is just $z \mapsto \exp(z)$.

This gives a component with finite asymptotic value 0. If a different asymptotic value a with $|a| < 1/2$ is desired, we post-compose this map with quasiconformal map $\rho : \mathbb{D} \to \mathbb{D}$ such that $\rho(0) = a$ and ρ is the identity on $\partial\mathbb{D}$ (just as for critical values for D-components).

R-components: This is what we used in Theorem 1. Here Ω is simply connected and unbounded and we are given a length respecting, quasiconformal map $\eta : \Omega \to \mathbb{H}_r$.

The boundary may be a tree instead of a Jordan curve.

In the Folding Theorem 1, we took $\sigma = \cosh$, but now we have to allow more general maps.

Under the map $\tau_j^{-1} : \mathbb{H}_r \to \Omega_j$, each interval I in the partition is mapped to one side of an edge e of T and either the other side of this edge also faces the same component Ω_j, or it faces a different component Ω_k, $k \neq j$.

In the latter case, the second component Ω_k could be a D, L or R-component.

Lemma 10 (exp-cosh interpolation)**.** *There is a quasiregular map* $\nu_j : \mathbb{H}_r \to \mathbb{C} \setminus [-1, 1]$ *so that*

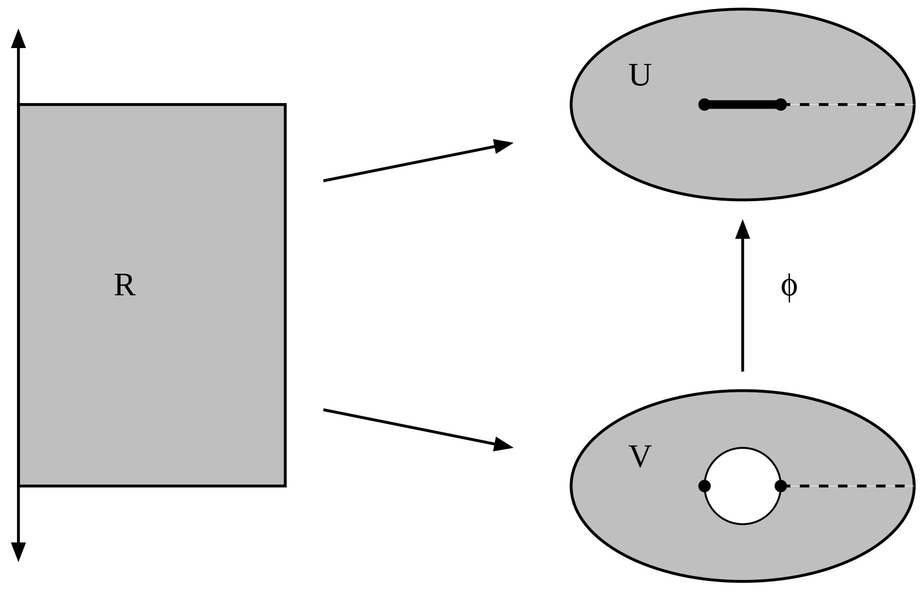
$$\nu_j(z) = \begin{cases} \cosh(z), & z \in J \in \mathcal{J}_1^j, \\ \exp(z), & z \in J \in \mathcal{J}_2^j, \\ \cosh(z), & z \in \mathbb{H}_r + 1 = \{x + iy : x > 1\}. \end{cases}$$

The quasiconstant of ν_j *is uniformly bounded, independent of all our choices.*

Proof. The proof is basically a picture.

Suppose J is one of our partition intervals and let $R = [0,1] \times J \subset \mathbb{H}_r$. The cosh map sends R into a topological annulus bounded by the unit circle and the ellipse $E = \{x + iy : (x/s)^2 + (y/t)^2 = 1\}$ where $x = \frac{1}{2}(e + \frac{1}{e}), y = \frac{1}{2}(e - \frac{1}{e})$.

The left side of R maps to the unit circle, the right side maps to E and and the top and bottom edges of R map to the real segment $[1, e]$. Let U be the region bounded by the ellipse and $V = U \setminus \mathbb{D}$ be the annular region.

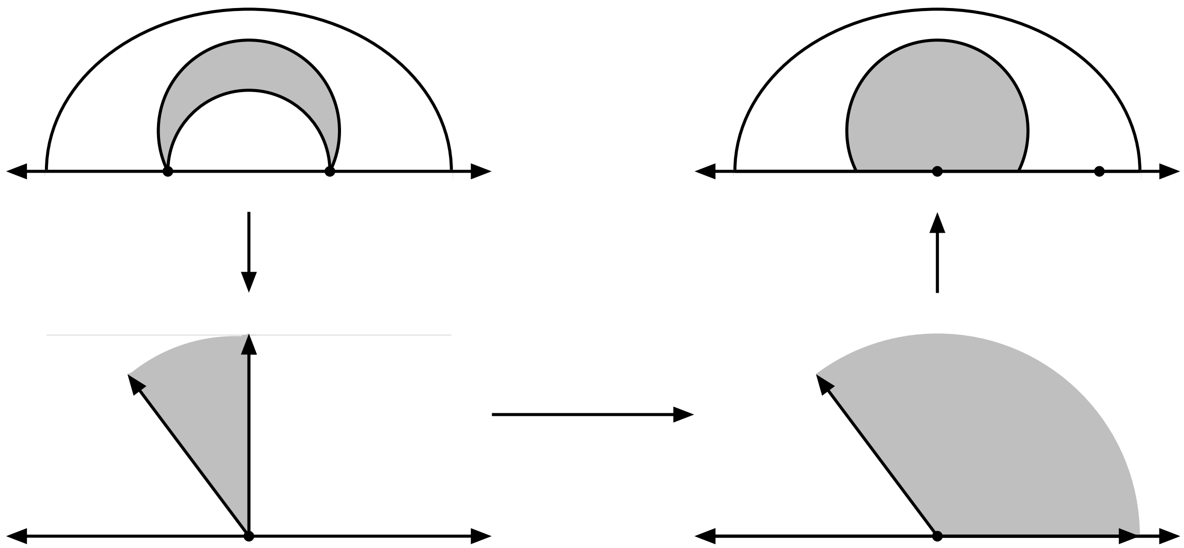


The cosh map sends the rectangle R to an ellipse minus the unit disk.

On some rectangles we modify it to map to the ellipse minus $[-1, 1]$.

Now define a quasiconformal map $\phi : V \to U$ that is the identity on E and on $[1, e]$, but that maps $\{|z| = 1\}$ onto $[-1, 1]$ by $z \to \frac{1}{2}(z + \frac{1}{z})$ (this is just the Joukowsky map that conformal maps the exterior of the unit circle to the exterior of $[-1, 1]$ and identifies complex conjugate points).

This map can clearly be extended from the boundary of V to the interior as a quasiconformal map.



We show only the construction in the upper half-plane; it is defined symmetrically in the lower half-plane.

The region V contains contains a crescent with vertices at ± 1 as shown. The crescent can be Möbius mapped to a sector which can be quasiconformally mapped to a larger sector by fixing radii and and expanding arguments.

In $\mathbb{H}_r + 1 = \{x + iy : x > 1\}$ and in rectangles corresponding to $J \in \mathcal{J}_1^j$, we set $\nu(z) = \cosh(z)$. In the rectangles corresponding to elements of $\mathcal{J}_2^j$ we let $\nu(z) = \phi(\cosh(z))$. This clearly has the properties stated in the lemma.

The map can be visualized as a map from $\mathbb{H}_r$ to a Riemann surface with sheets of the form either $\mathbb{C} \setminus [-1, 1]$ or $\mathbb{C} \setminus \overline{\mathbb{D}}$ attached along $[1, \infty)$ and chosen according to the type of the corresponding partition element. $\square$

Theorem 11. *Suppose T is a bounded geometry graph and suppose τ is conformal from each complementary component to its standard version.*

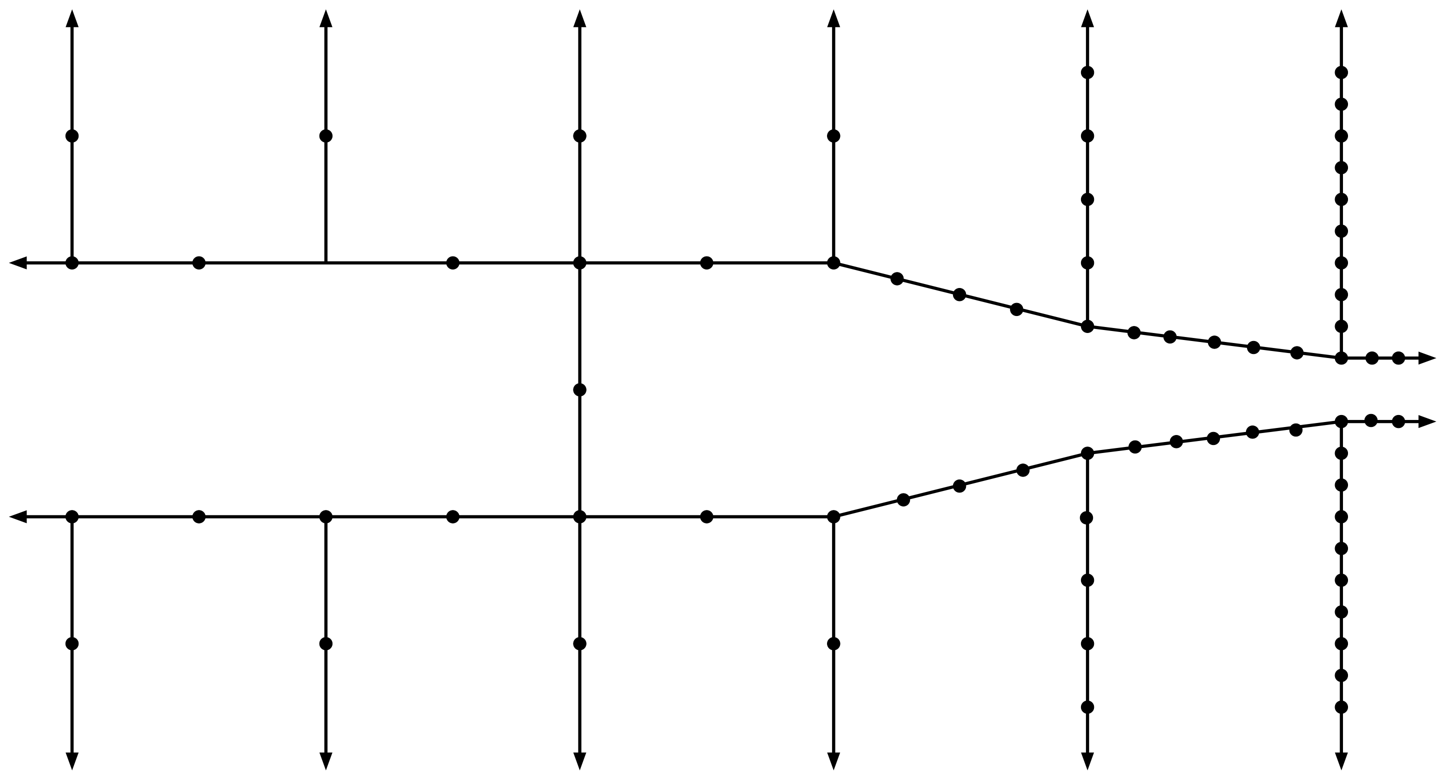
(1) Assume that D and L components only share edges with R components.

(2) Assume that τ on a D-component with n edges maps the vertices to nth roots of unity and on L-components it maps edges to intervals of length 2π on $\partial\mathbb{H}_l$ with endpoints in $2\pi i\mathbb{Z}$.

(3) On R-components assume that the τ-sizes of all edges are $\geq 2\pi$. Then there is an entire function f and a quasiconformal map ϕ of the plane so that $f \circ \phi = \nu \circ \tau$ off $T(r_0)$. The only singular values of f are ± 1 (critical values coming from the vertices of T) and the critical values and singular values assigned by the D and L-components.

Application: rapid growth

Example:

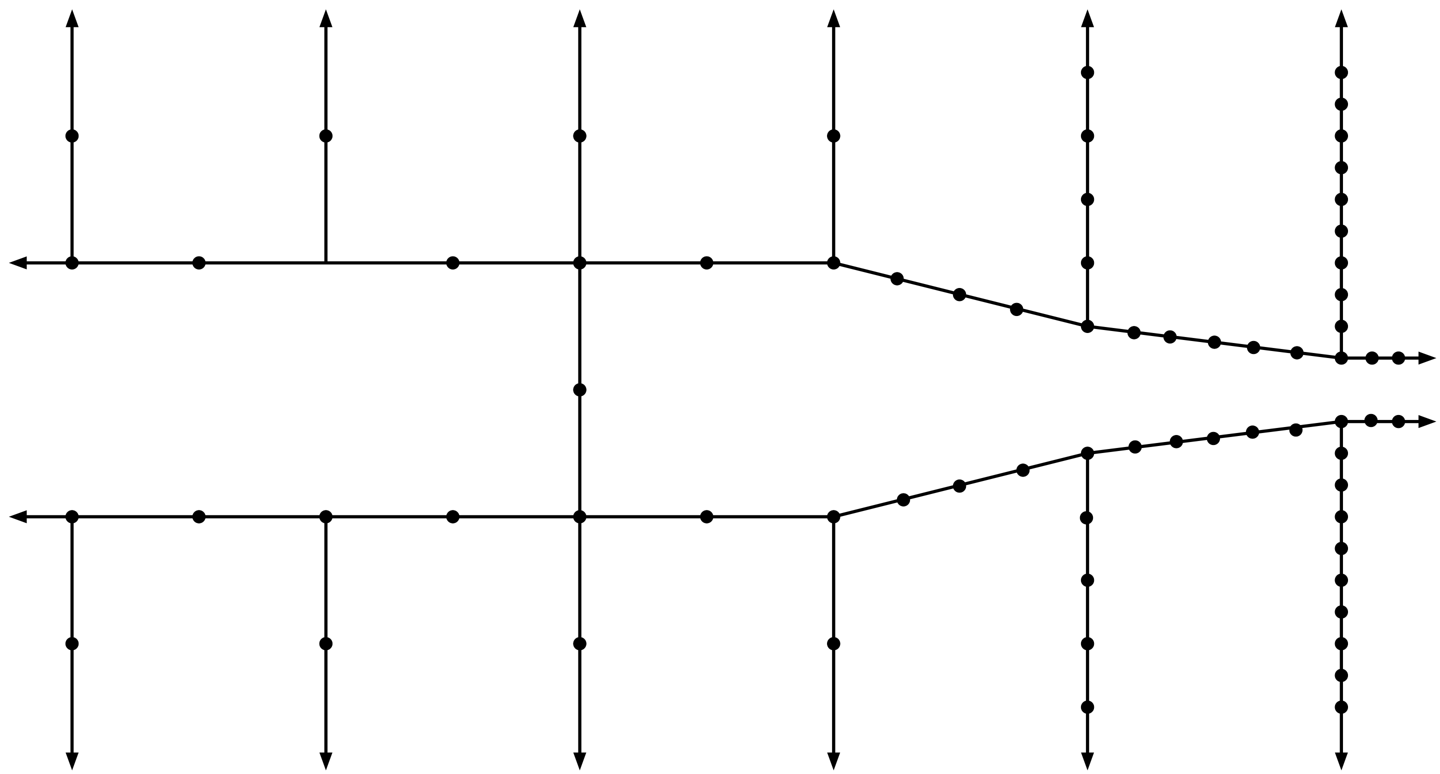


Check that this tree has:

(1) bounded geometry,

(2) the τ-lower bound.

Rapid increase in Speiser class



We get f with 2 singular values, $f(x) \nearrow \infty$ as fast as we wish.

Correction map φ is Hölder, only slows growth a little.

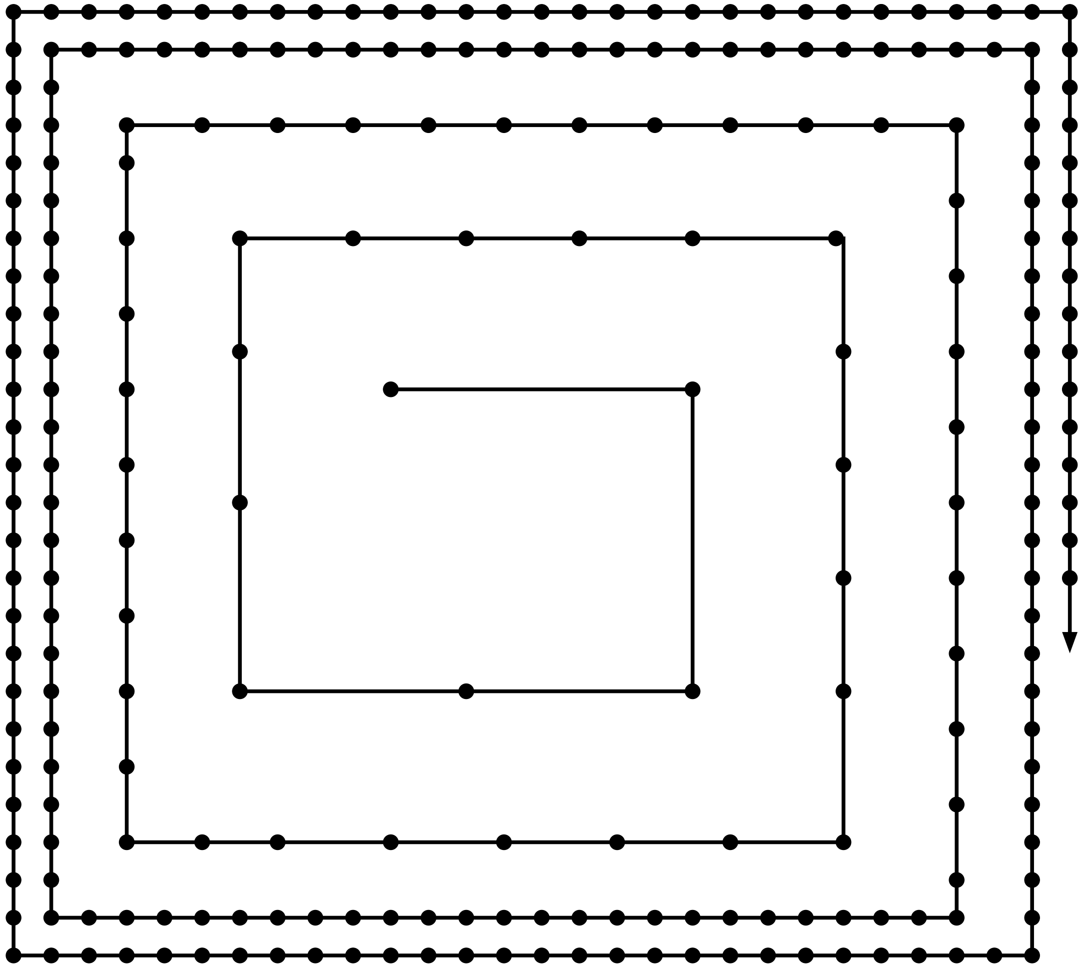
Similar examples due to Sergei Merenkov (2008) (3 singular values).

Application: fast spirals

Corollary 12. *For any function $\phi : [0,\infty) \to [0,\infty)$ that increases to ∞ there is a $f \in \mathcal{S}_{\in,\prime}$, a $t_0 < \infty$ and a curve $\gamma : [0,\infty) \to \mathbb{C}$ along which f tends to infinity, such that for all $t > t_0$,*

$$\arg(\gamma(t)) \geq \phi(|\gamma(t)|),$$

where $\arg$ is a continuous branch of the argument on the simply connected domain $\Omega = f^{-1}(\mathbb{C} \setminus [-1,1])$.

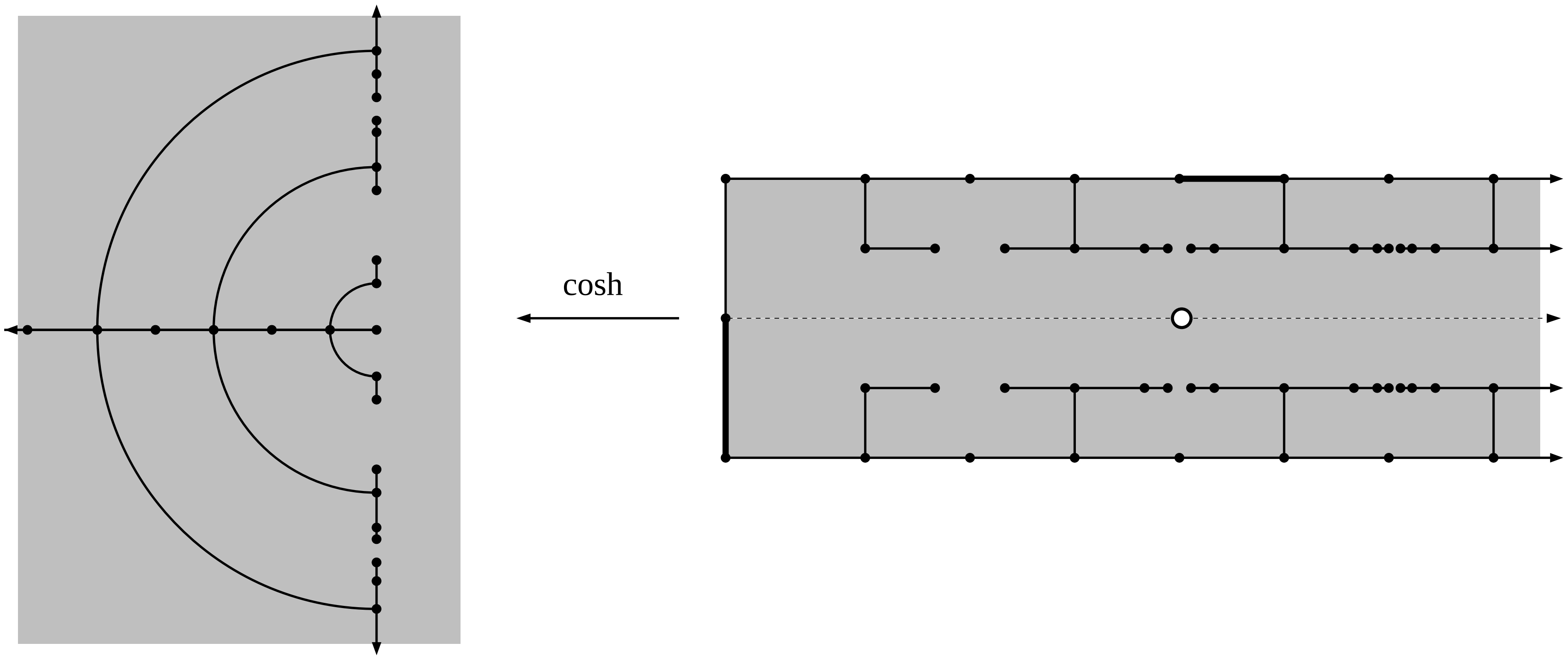


Application: the area conjecture

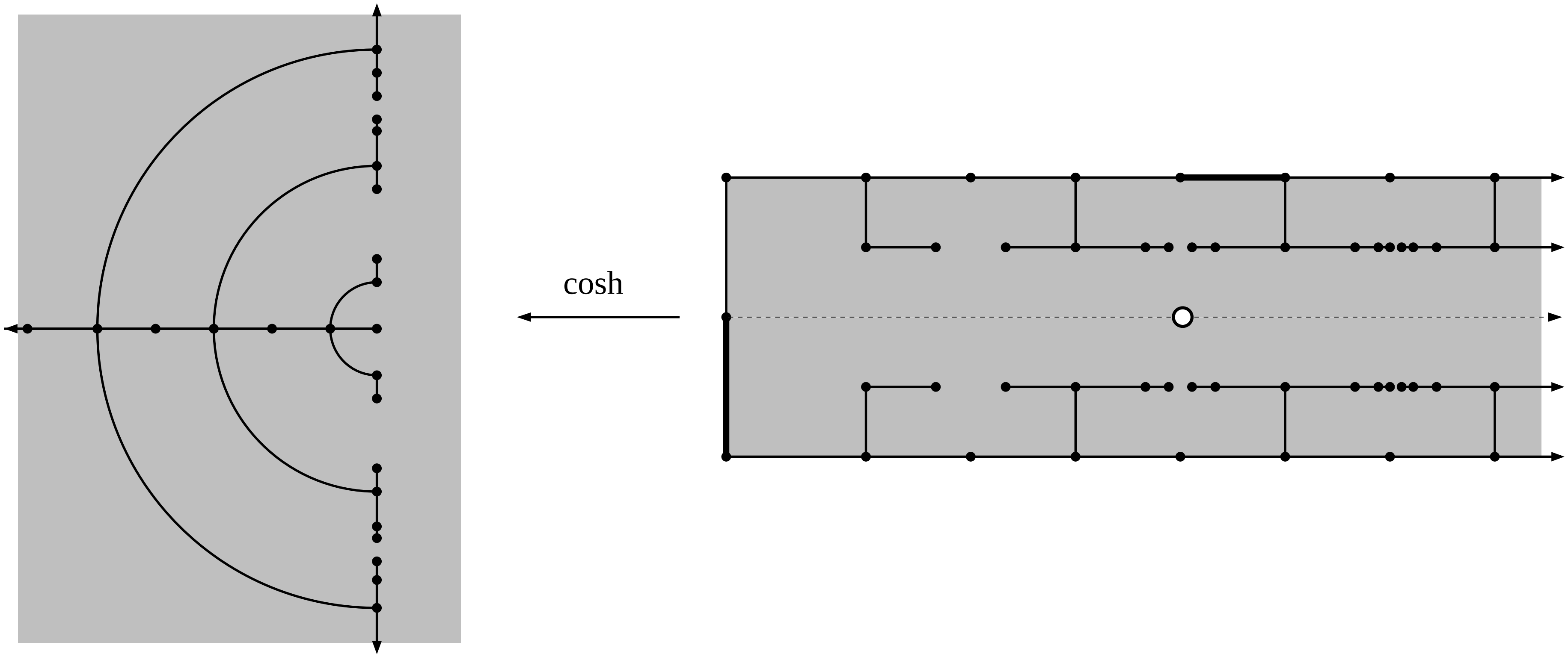
The logarithmic area of a set E in the plane is defined

$$\operatorname{logarea}(E) = \int_E \frac{dxdy}{x^2 + y^2}.$$

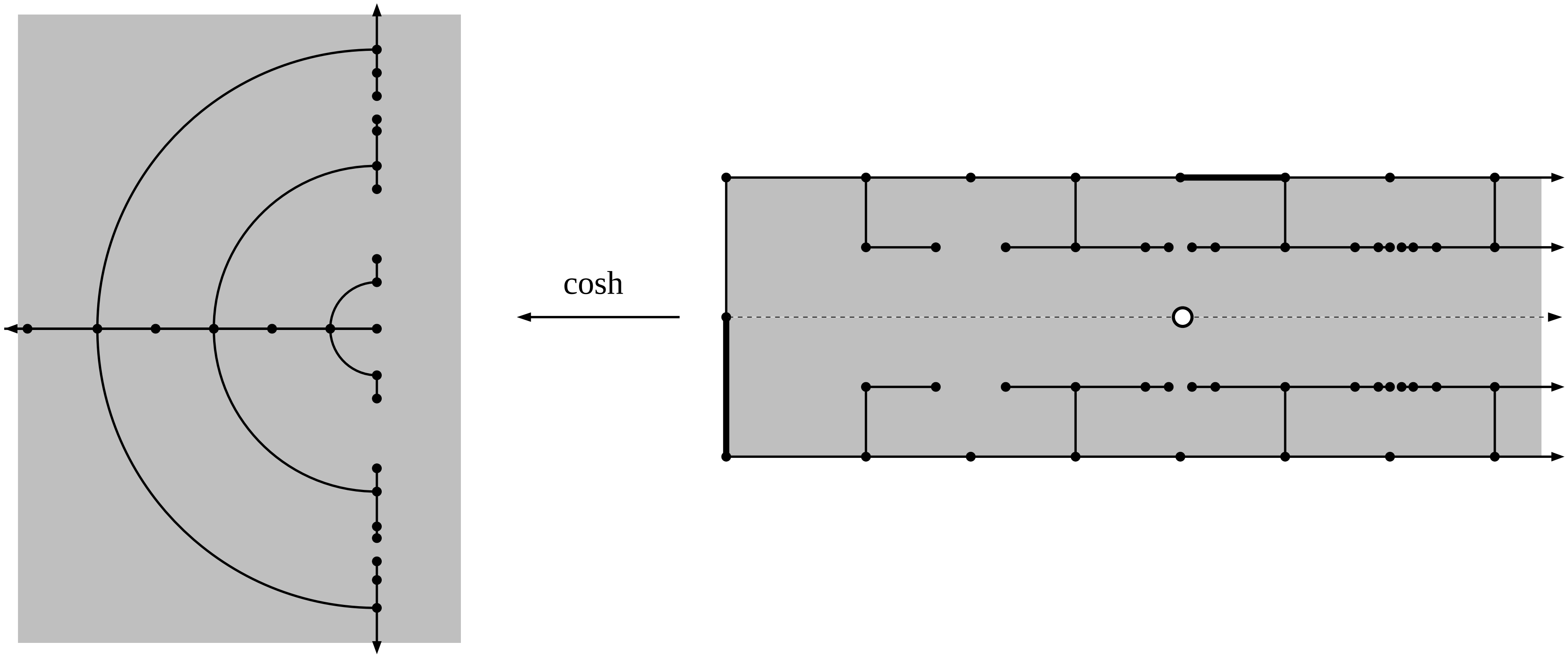
The area conjecture asks if $\operatorname{logarea}(f^{-1}(K)) < \infty$ whenever K is a compact set of $\mathbb{C} \setminus S(f)$ (recall $S(f)$ are the singular values of f).



On the left is Ω, the tract of the area conjecture counterexample and on the right is $\Omega' = \cosh^{-1}(\Omega)$; the same example in cosh-coordinates.

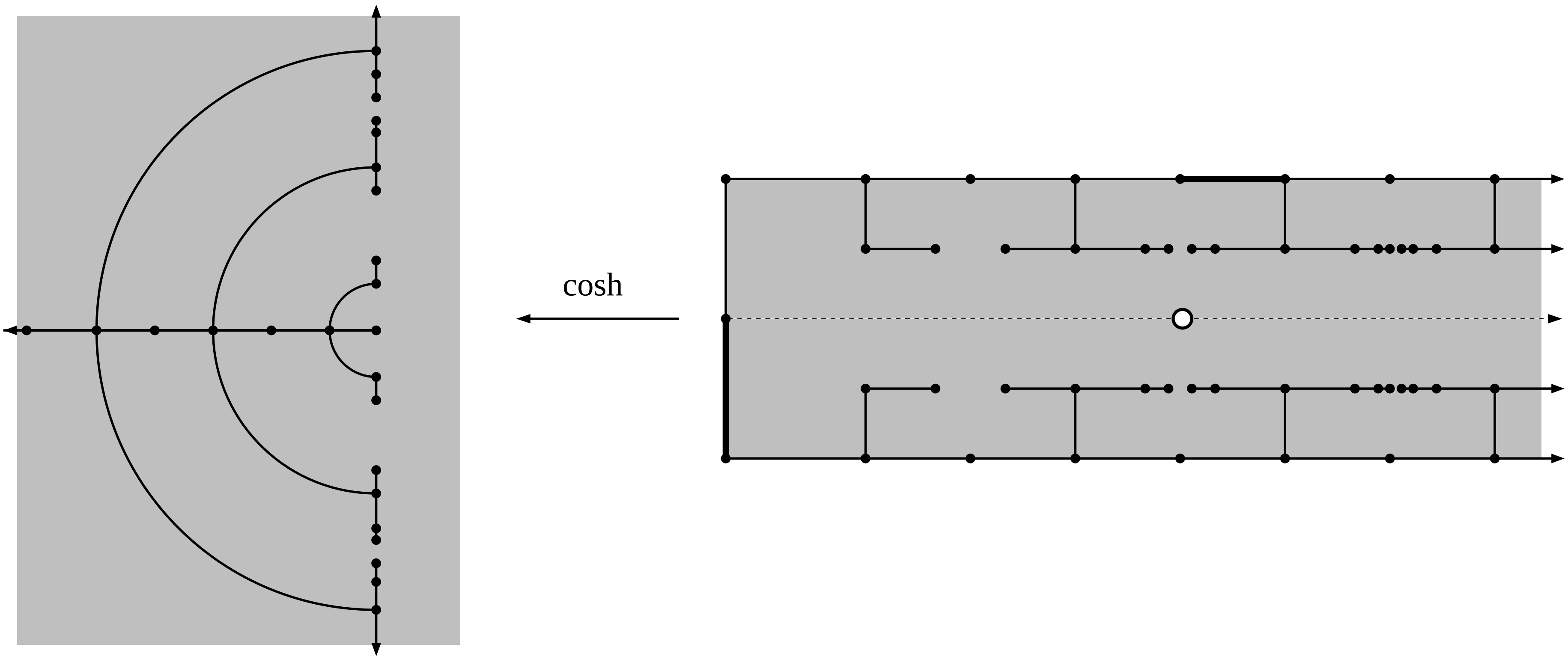


In the second picture, "rooms" are attached along a central strip by small gaps whose size is chosen so that edges on the top and bottom of the strip (thick edge) have approximately the same harmonic measure as the left side of the strip (thick edge) when viewed from a point (white dot) on the axis of the domain (dashed line).



With these choices, τ will have bounded derivative near the middle of each room and along the top edge.

This implies that $\{z : |g(z)| < R\}$ will contain a disk of radius comparable to 1 in each "room" and the union of these disks has infinite logarithmic area.

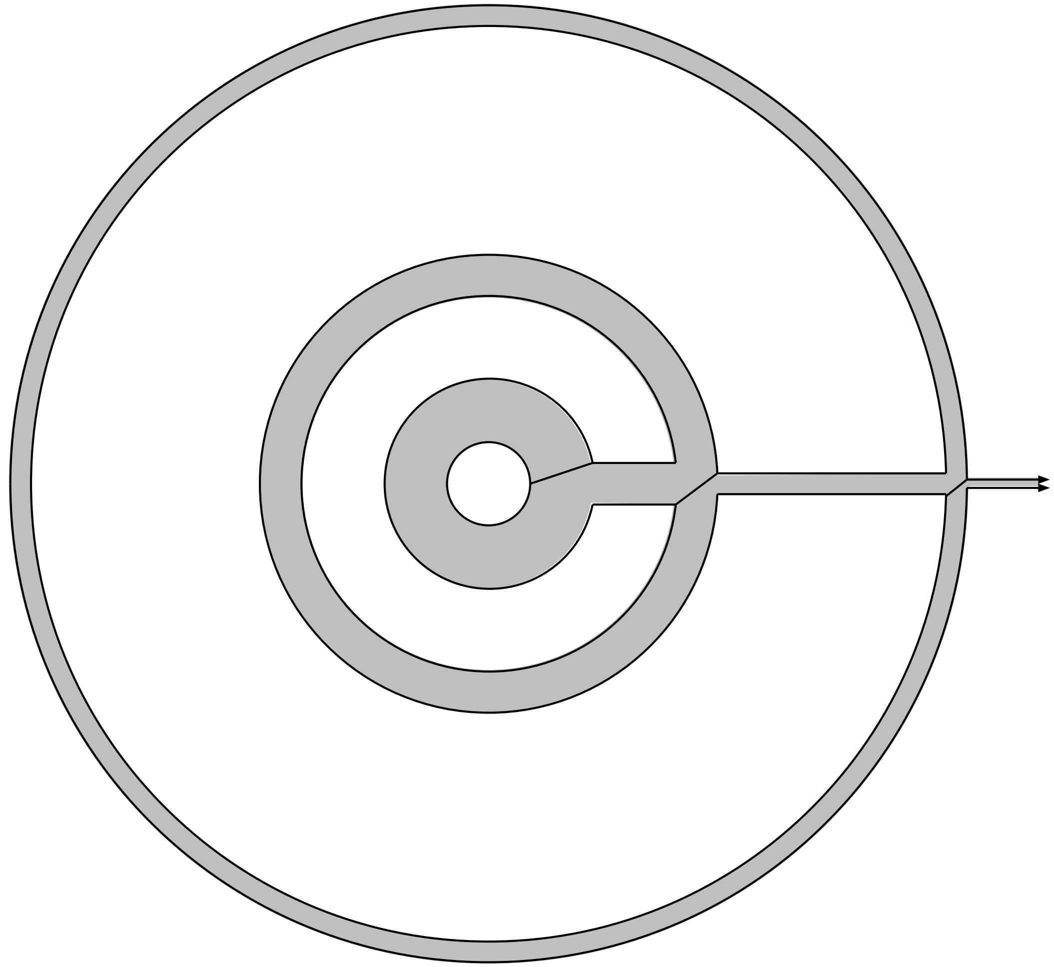


The quasiconformal change of variable ϕ preserves the

strip in cosh-coordinates and maps these disks to regions of Euclidean comparable area, so the entire function $f = g \circ \phi^{-1}$ disproves the area conjecture.

The following is a stronger counterexample to the area conjecture.

Corollary 13. *There is a function $f \in \mathcal{S}_{\ni}$ with critical values $\{-1, 0, 1\}$ and no finite asymptotic values so that* $\text{area}(\{z : |f(z)| > \epsilon\}) < \infty$ *for every $\epsilon > 0$.*



For an entire function f, we let

$$m(r) = \min_{|z|=r} |f(z)|, \qquad M(r) = \max_{|z|=r} |f(z)|.$$

By definition $m(r) \leq M(r)$, but it is interesting to ask how much smaller can m be compared to M?

Obviously we have to avoid zero's of f, but it is reasonable to ask if there is a finite α so that for any entire function, $m(r) \geq M(r)^{-\alpha}$ along some sequence of radii tending to infinity? The function e^z shows we can't take $\alpha < 1$.

Wiman proved that for any ϵ and any non-vanishing entire function f

$$m(r) > M(r)^{-1-\epsilon},$$

for some sequence of r's tending to ∞.

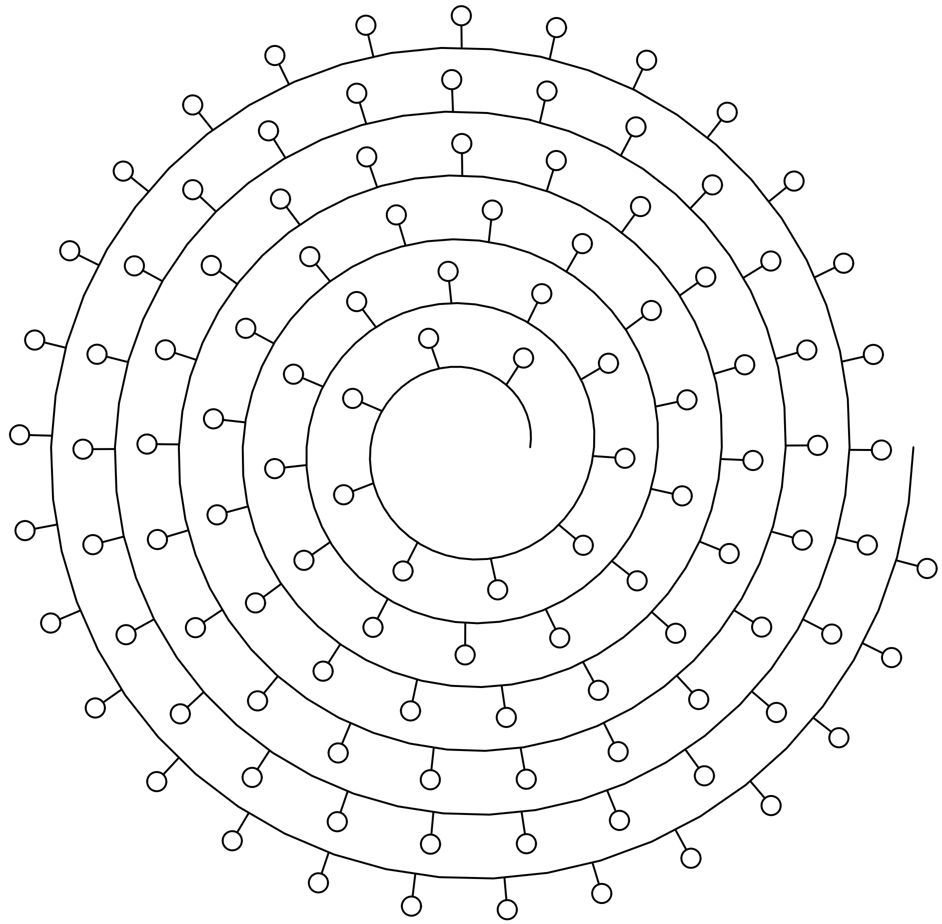
He conjectured this was true in general and this was verified by Beurling in the special case $|f(r)| = m(r)$ (i.e., the minimal values are attained along $\mathbb{R}^+$).

General case was disproved by Hayman.

Corollary 14. *There are* $A > 0, r_0 < \infty$ *and an entire function* $f \in \mathcal{S}_{\ni,\prime}$ *so that*

$$m(r) < M(r)^{-A \log\log\log M(r)} \tag{1}$$

for all $r > r_0$. *Hence* $m(r) < M(r)^{-C}$ *for every* C *and* r *large enough.*



Very rapid introduction to transcendental dynamics

The **Fatou set**, $\mathcal{F}(f)$, of an entire function f is the union of open disks on which $\{f^n\}$ forms an open family.

It is also clear that $f(\mathcal{F}(f)) \subset \mathcal{F}(f)$ (forward invariance), but equality need not hold if f has an omitted value.

For example, $\frac{1}{10}e^z$ has a Fatou component that contains 0, but $0 \notin f(\mathcal{F}(f))$.

It turns out that if U is a Fatou component that is mapped into a component V then $V \setminus U$ can have at most one point and $U = V$ if U is bounded.

A Fatou component is **wandering** if all images land in different components.

The **escaping set** $I(f)$ are all points whose orbits tend to ∞.

Eremenko proved this set is non-empty.

The complement $\mathcal{J}(f) = \mathbb{C} \setminus \mathcal{F}(f)$ is called the **Julia set** of f and is clearly a closed, totally invariant set and satisfies $\mathcal{J}(f) = \mathcal{J}(f^n)$ for every $n \in \mathbb{N}$.

The $\mathcal{J}(f)$ is non-empty.

Indeed, $\mathcal{J}(f) = \partial I(f)$ and $I(f) \neq \mathbb{C}$.

The Julia set is the closure of the repelling fixed points.

The Julia set is either nowhere dense or is the whole plane.

If f is entire and V is any neighborhood of any point $z \in \mathcal{J}(f)$ then $\cup_n f^n(V)$ covers the whole plane with at most one exception.

Theorem 15 (Baker)**.** *If f is a transcendental entire function, then every multiply connected component of the Fatou set is bounded.*

Corollary 16. *If f is a transcendental entire function then every multiply connected component of the Fatou set is a wandering domain.*

Corollary 17. *The Julia set of a transcendental entire function contains a non-trivial continuum.*

Corollary 18. *The Julia set of a transcendental entire function has Hausdorff dimension at least* 1.

Wandering domains

Theorem 19 (Baker). *There exists an entire function with a multiply connected Fatou component, hence with a wandering domain.*

The function will be

$$f(z) = z^2 \prod_{k=1}^{\infty} (1 + \frac{z}{R_k}),$$

where $R_k \nearrow \infty$ is a sequence of positive real numbers that are defined inductively.

Theorem 20 (Herman). *$f(z) = z - 1 + e^{-z} + 2\pi i$ has a wandering domain.*

Theorem 21 (Baker). *$f(z) = z + \sin z + 2\pi$ has a bounded, simply connected wandering domain.*

Suppose f is a transcendental entire function. A critical point of f is a zero of f' and a critical value is $f(z)$ where z is a critical point.

A asymptotic value is a $w \in S^2$ so that $\lim f(z) = w$ along a curve $\gamma : [0, \infty)$ that tends to ∞.

The singular values of f, denoted $S(f)$, is defined as the closure or the union of critical values and finite asymptotic values.

Lemma 22. *Suppose f is entire and U contain no critical values. Then f is a smooth covering map from $V = f^{-1}(\Omega)$ to Ω.*

Corollary 23. *Suppose f is entire and $S(f) \subset \mathbb{D}_R = \{z : |a| < R\}$. Then f is covering map from $\Omega = f^{(}\overline{\mathbb{D}}_R^c) = \{z : |f(z)| > R\}$ to $\overline{\mathbb{D}}_R^c = \{z : |z| > R\}$. Each connected component of Ω (called a tract of f) is an unbounded, simply connected domain whose boundary is an analytic Jordan curve that tends to ∞ in both directions.*

Suppose f is a transcendental entire function. If $S(f)$ is finite, we say f is finite type or in the Speiser class, denoted $\mathcal{S}$.

If $S(f)$ is bounded, we say f is bounded type or in the Eremenko-Lyubich class, denoted $\mathcal{B}$.

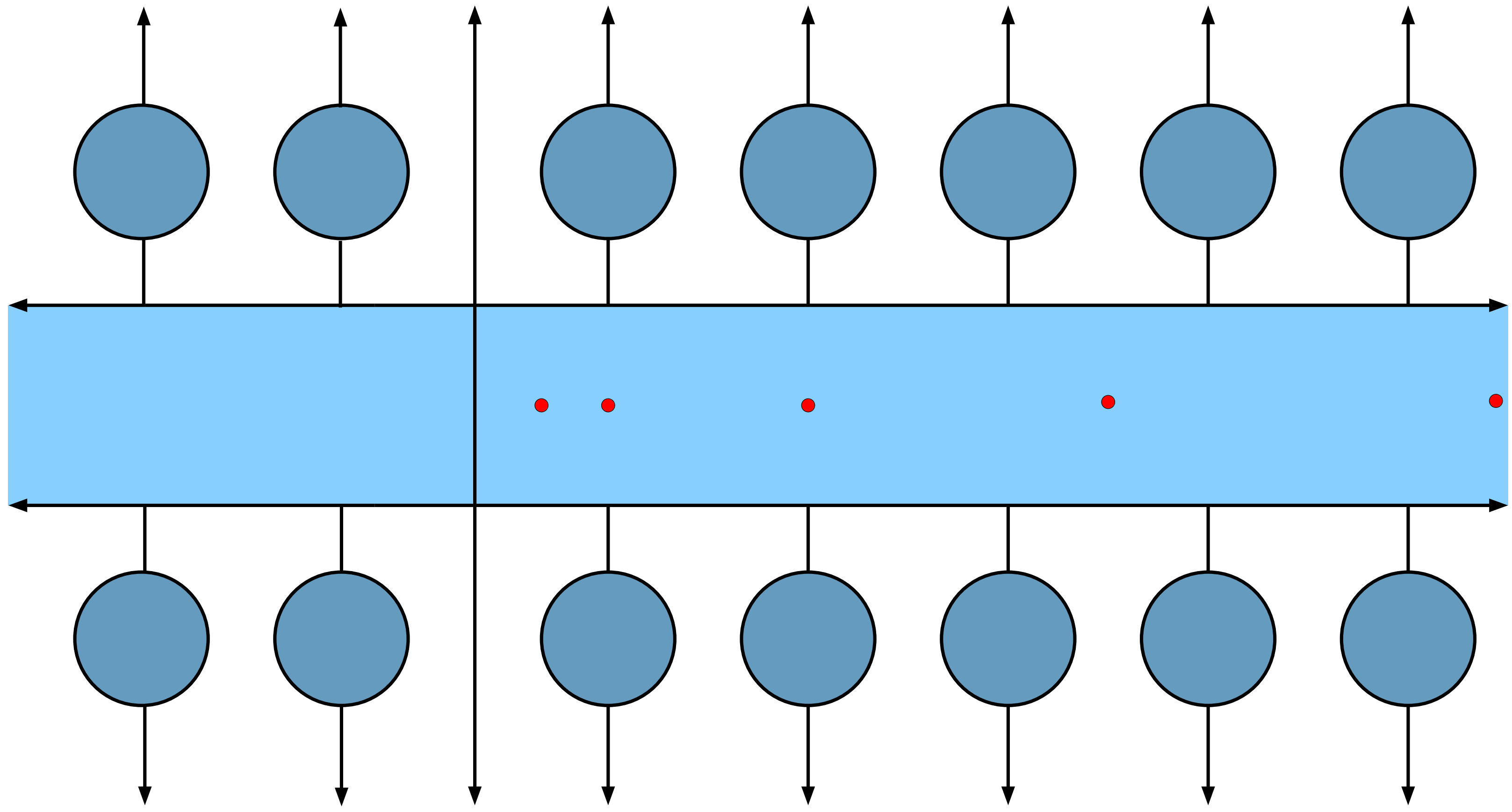
A little care needs to be taken with the terms “finite type” and “bounded type” since these are also used to mean something different in Nevanlinna theory. We will use “EL-type” be more precise.

Lemma 24. *If $f \in \mathcal{B}$, then every component of $\mathcal{F}(f)$ is simply connected.*

If f is Speiser class, the f has no wandering domains.

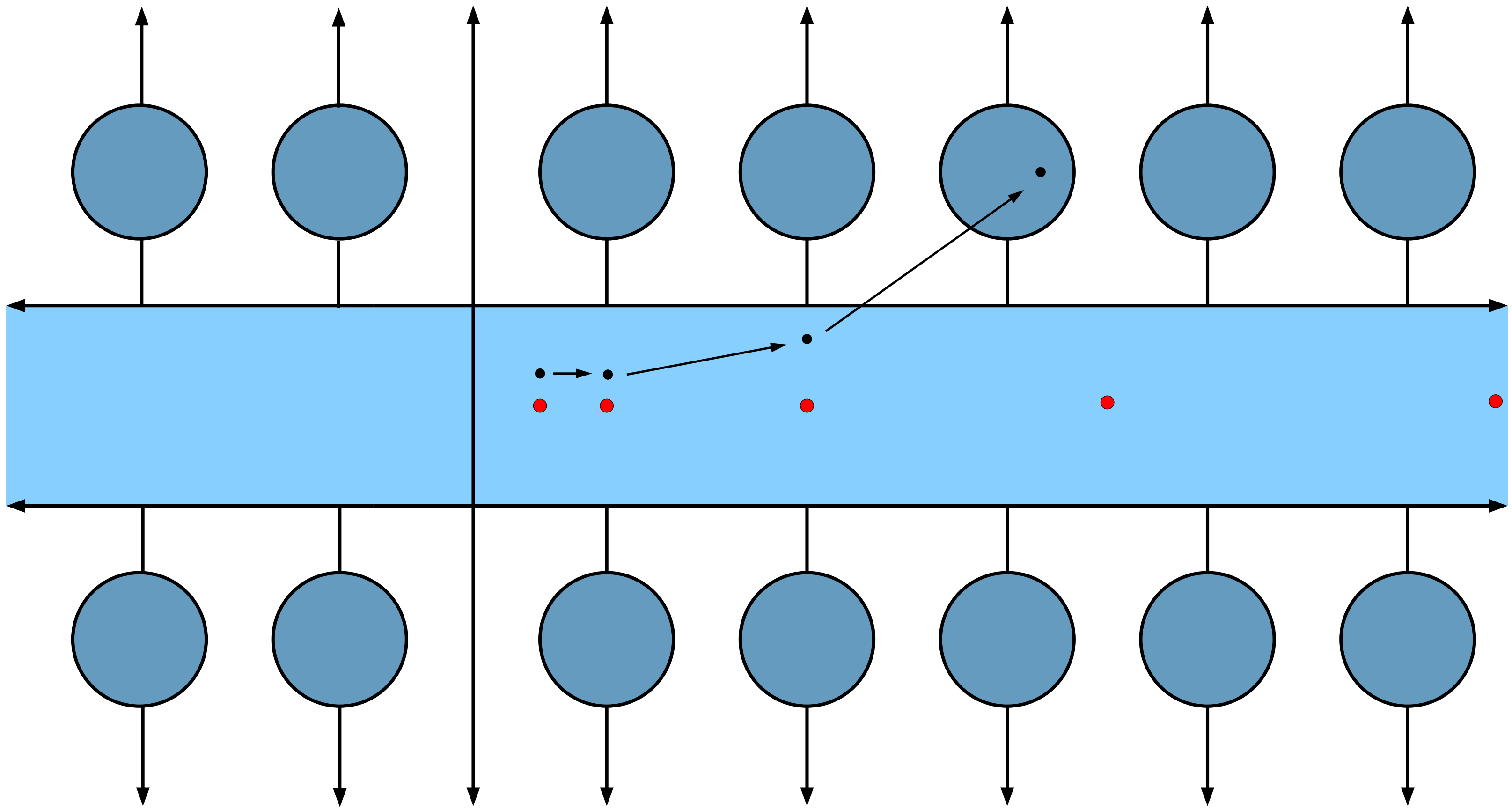
Sullivan's proof for polynomials extends with minor changes to this case, e.g. Eremenko-Lyubich or Goldberg-Keen.

Whether Eremenko-Lyubich functions have wandering domains remained open until 2015: yes, QC-folding gives examples.



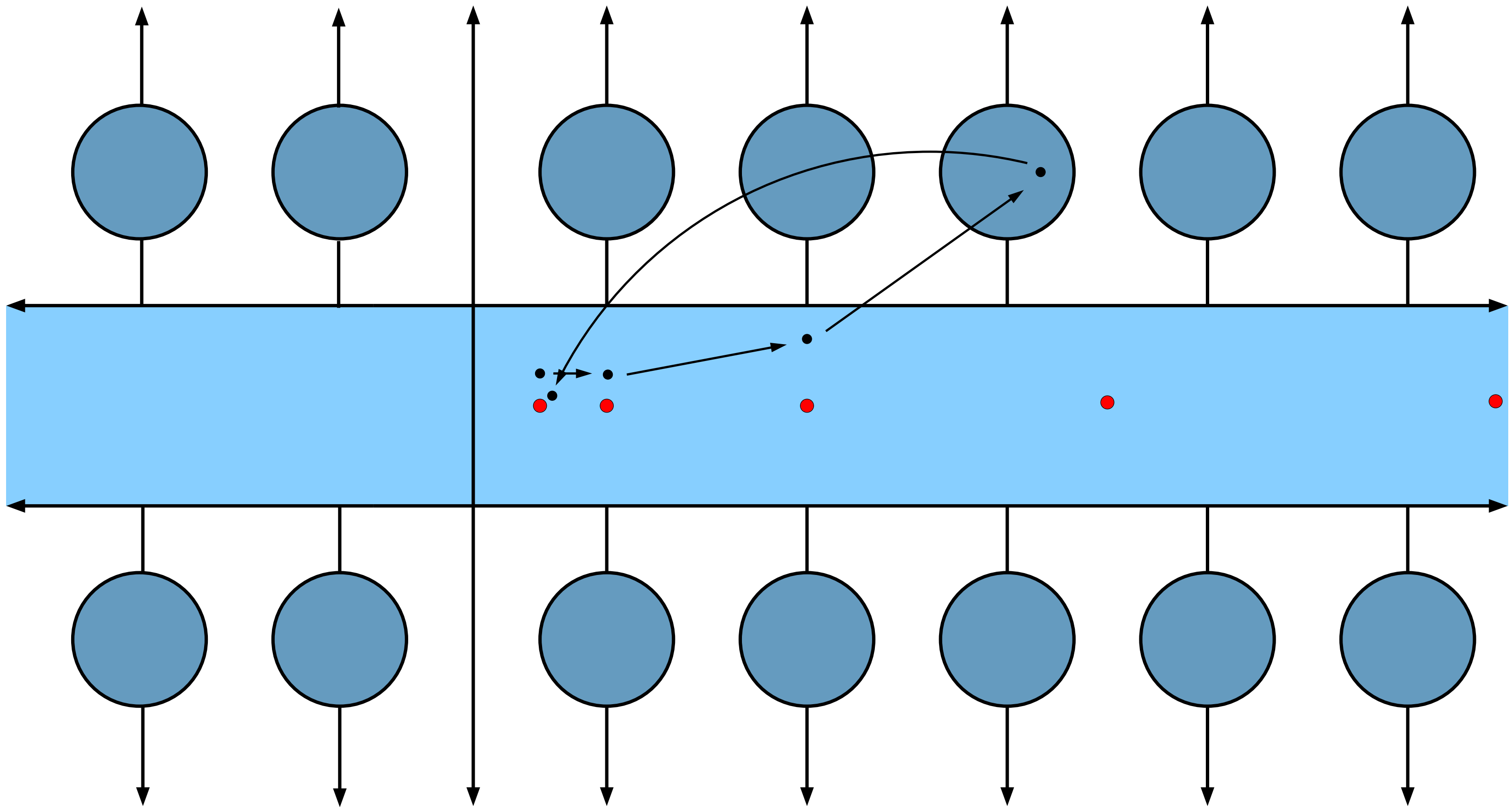
Graph giving wandering domain in Eremenko-Lyubich class.

Original proof corrected by Marti-Pete and Shishikura, who also give alternate construction.



Graph giving wandering domain in Eremenko-Lyubich class.

Variations by Lazebnik, Fagella-Godillon-Jarque, Osborne-Sixsmith.



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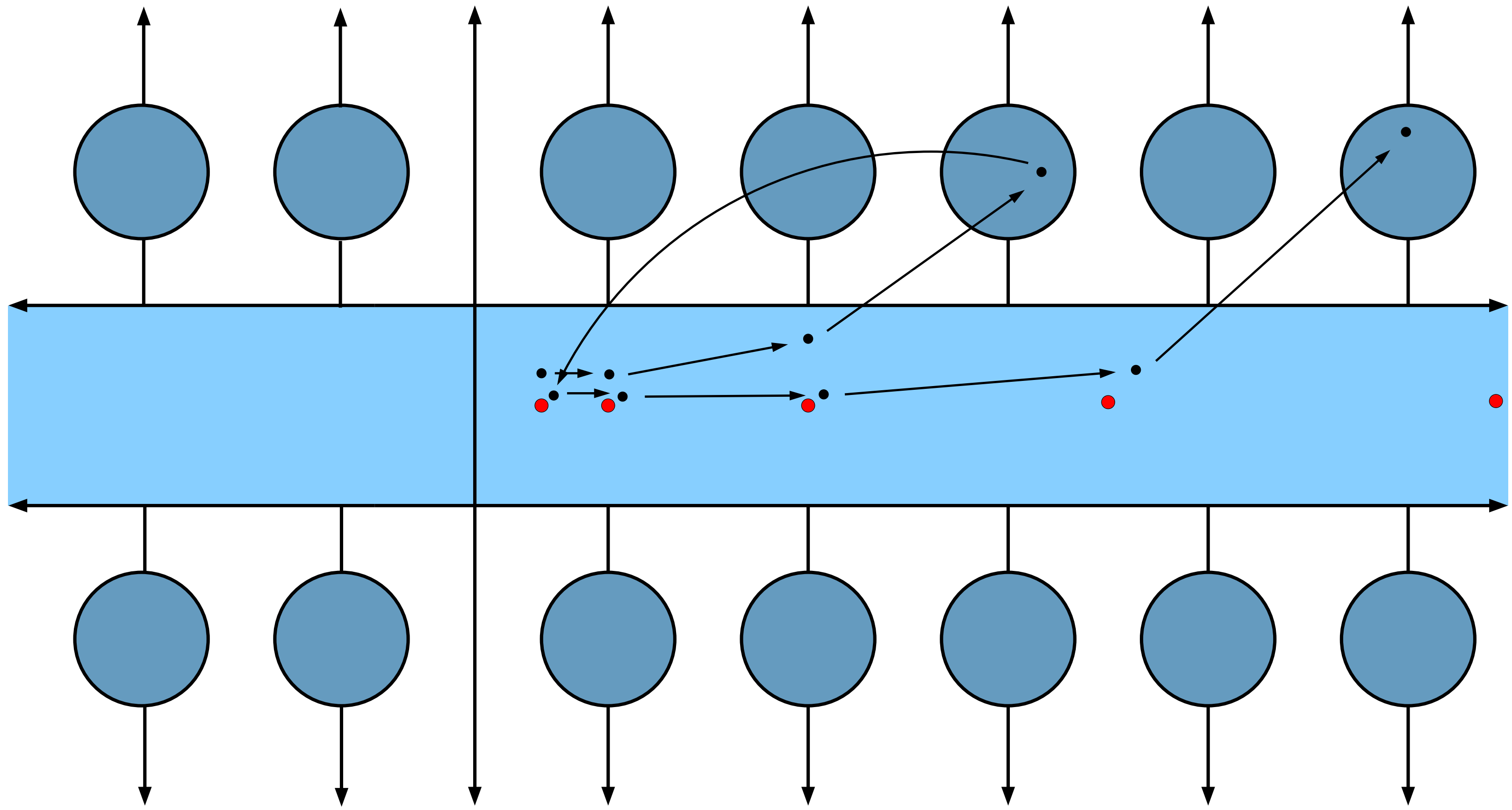


Graph giving wandering domain in Eremenko-Lyubich class.

Variations by Lazebnik, Fagella-Godillon-Jarque, Osborne-Sixsmith.

Models

Suppsoe f is entire.

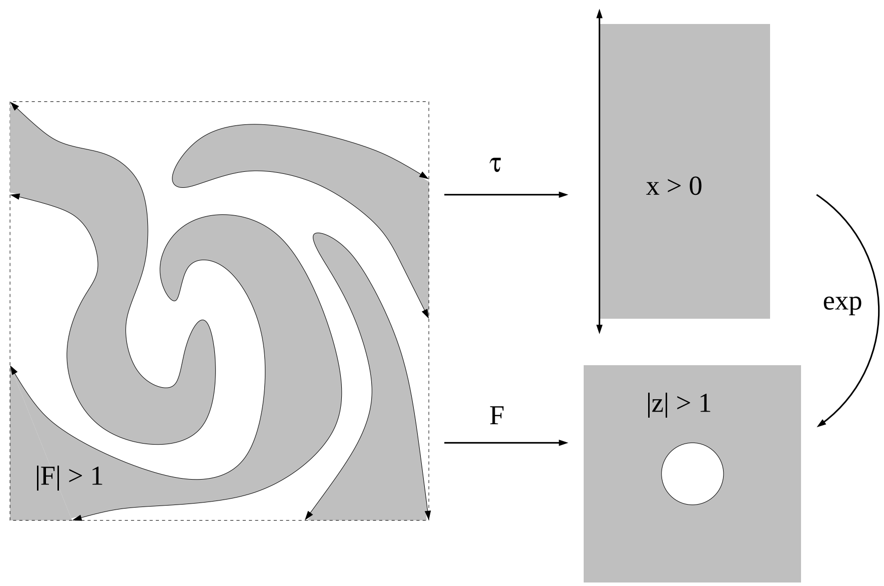
Eremenko and Lyubich showed that if $S(f) \subset \mathbb{D}_R = \{z : |z| < R\}$, then the inverse image Ω of $\mathbb{D}_R^* = \{z : |z| > R\}$ under f is a disjoint union of analytic, unbounded simply connected domains and that f acts a covering map $f : \Omega_j \to \mathbb{D}_R^*$ on each component Ω_j of Ω.

Which disjoint unions of analytic, unbounded simply connected domains can arise in this way?

Essentially, they all do.

If $f \in \mathcal{B}$ and $S(f) \subset \mathbb{D}_R$, we call $\Omega = \{z : |f(z)| > R\}$ a $\mathcal{B}$-level-set and each connected component is called a tract of f.

By normalizing f, we can assume that $R = 1$. On each tract there is a conformal map $\tau_j : \Omega_j \to \mathbb{H}_r = \{x + iy : x > 0\}$ so that $f(z) = \exp(\tau_j(z))$ on Ω_j. The collection of these conformal maps defines a holomorphic map $\tau : \Omega \to \mathbb{H}_r$.



Since $S(f)$ is compact, there is a $\rho > 0$, $S(f) \subset \{z : |z| \leq e^{-\rho}\}$ and hence $\Omega' = \{z : |f| > e^{-\rho}\}$ contains Ω and also consists of simply connected components.

It is locally finite (only a finite number of components meet any compact set) and on each component τ is continuous and 1-to-1 at infinity ($z_n \to \infty$ in Ω_j iff $\tau_j(z_n) \to \infty$).

Conversely, we claim these conditions essentially characterize $\mathcal{B}$-level-sets, at least in a quasiconformal sense:

Theorem 25. *Suppose $\rho > 0$ and Ω' is a union of disjoint, locally finite, unbounded simply connected regions and $\tau : \Omega' \to \mathbb{H}_r - \rho = \{x + iy : x > -\rho\}$ is conformal and continuous and 1-to-1 at ∞ on each component of Ω'. Then there is a quasi-regular function g that equals e^τ on $\Omega = \tau^{-1}(\mathbb{H}_r)$ and $|g| \leq 1$ off Ω. In particular, $\Omega = \{z : |g(z)| > 1\}$ is the level-set of a quasi-regular function of EL-type.*

Instead of defining a quasi-regular function of EL-type directly, we simply note that the measurable Riemann mapping theorem implies that any quasi-regular function g is of the form $g = f \circ \phi$ for some entire function f and some quasiconformal map $\phi : \mathbb{R}^2 \to \mathbb{R}^2$.

We say that g has type EL if f does. Thus every Ω in Theorem 25 is the QC image of some $\mathcal{B}$-level-set.

This is what we meant above when we said that this condition “essentially” characterizes bounded type level-sets.

We can be much more precise about the quasiconformal map ϕ that takes Ω to a $\mathcal{B}$-level-set.

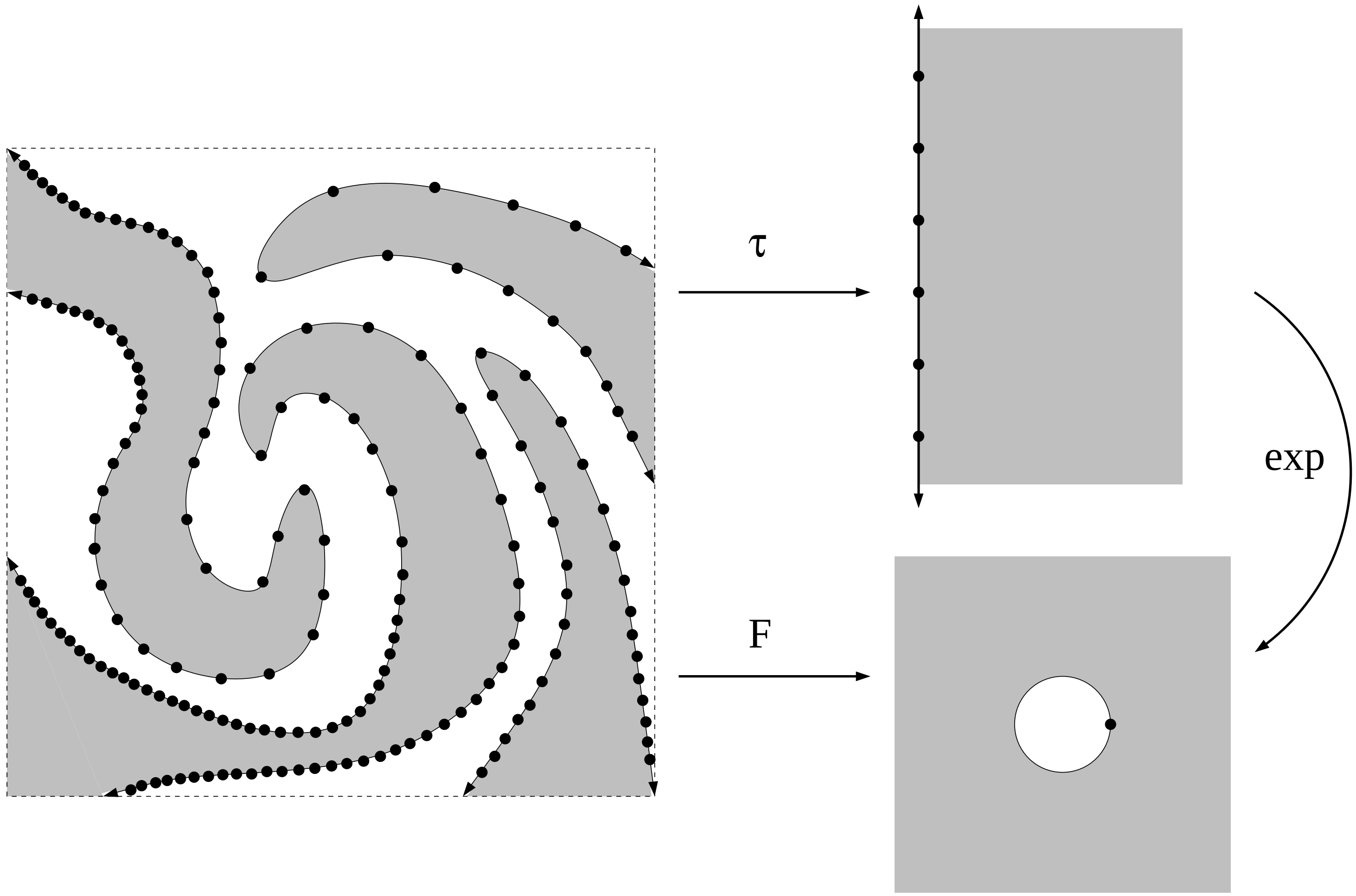
Note that the points $2\pi i\mathbb{Z} \subset \partial\mathbb{H}_r$ partition the boundary of $\mathbb{H}_r$ into equal sized segments. Thus the points $f^{-1}(1) = \tau^{-1}(2\pi i\mathbb{Z})$ partition $\partial\Omega$ into arcs. We call this a conformal partition of $\partial\Omega$, or the partition induced by τ.

Given an arc J in the partition, let

$$J(r) = \{z : \mathrm{dist}(z, J) < r \cdot \mathrm{diam}(J)\}.$$

We call this an r-neighborhood of J.

The union of r-neighborhoods over all partition arcs defines an open neighborhood of $\partial\Omega$ that we denote $T_\Omega(r)$. We just write $T(r)$ if the set Ω is clear from context.


τ
exp
F

Theorem 26. *Suppose* Ω *is as in Theorem 25. Then there is a* $f \in \mathcal{B}$ *and a* K*-quasiconformal map* ϕ *of the plane so that* $f \circ \phi = e^{\tau}$ *on* Ω*,* $f \circ \phi$ *is bounded off* Ω *and* ϕ *is conformal off* $T(r) \setminus \Omega$ *(in particular, it is conformal on* Ω*). The constants* $K, r < \infty$ *depend on* ρ *but are otherwise independent of* Ω *and* τ*.*

Corollary 27. *Suppose Ω is as in Theorem 25. Then there is a sequence $\{f_n\} \in \mathcal{B}$ and quasiconformal maps $\{\phi_n\}$ with uniformly bounded quasiconstant K so that $\Omega_n = \{z : |f_n(z)| > 1\} = \phi_n(\Omega)$ converges to Ω in the Hausdorff metric on any bounded subset of the plane.*

Under certain circumstances, one can actually prove Ω_n converges to Ω in the Hausdorff metric on the whole plane. For example, a result of Dyn'kin on pointwise differentiability of quasiconformal maps implies this is true if $\text{area}(T(r) \cap \mathbb{D}_t^*) = O(t^{2-4K-\epsilon})$ for some $\epsilon > 0$.

Estimates like this can often be proven with explicit calculations if Ω is "thin" near infinity. For example, when the tracts have finite in-radius, and we can use this to prove that $\text{area}(T(r) \cap \mathbb{D}_t^*)$ tends to zero exponentially fast in t. Hence these domains can be uniformly approximated (on the whole plane) by $\mathcal{B}$-level-sets.

Theorem 28. *Suppose Ω is as in Theorem 25. Then there is a $f \in \mathcal{S}$ and a K-quasiconformal map ϕ of the plane so that $f \circ \phi = e^{\mathcal{T}}$ on Ω and ϕ is conformal on Ω^c. The constants $K, r < \infty$ depend on ρ but are otherwise independent of Ω and τ. We may take f to have no finite asymptotic values, exactly two critical values, $\pm \exp(-\rho/2)$, and so that every critical point has degree ≤ 4.*

This is very similar to Theorem 26, but with two important differences.

First, the dilatation of ϕ is now supported on $\mathbb{C} \setminus \Omega$ instead of $T(r) \setminus \Omega$.

Second, Theorem 28 omits the phrase "and $f \circ \phi$ is bounded off Ω". Thus Ω need not be the entire level-set of f; it is merely a union of connected components of $\{z : |f| > 1\}$.

Thus any $\mathcal{B}$-tract is the QC image of a $\mathcal{S}$-tract, but (as we shall see below) not every $\mathcal{B}$-level-set is the QC image of a $\mathcal{S}$-level-set.

In other words, functions in $\mathcal{S}$ and $\mathcal{B}$ do not differ because of the geometry of individual tracts, but because of how the tracts "fit together" to form a level-set.

Theorem 29. *There is a $\mathcal{B}$-level-set that is not the QC image of any $\mathcal{S}$-level-set.*

Suppose $f \in \mathcal{S}$ and $S(f) \subset \mathbb{D}$.

Assume $\mathrm{dist}(S(f), \partial\mathbb{D}) = 1 - e^{-\rho}$ and let

$$\delta = \min\{|a - b| : a, b \in S(f), a \neq b\},$$

and $\eta = \min(1 - e^{-\rho}, \delta)$.

For $\epsilon < \eta/4$ the disks of radius ϵ centered at points of $S(f)$ are pairwise disjoint (even have disjoint doubles) and all lie inside $\mathbb{D}$. Thus the pre-image of such a disk is disjoint from $\Omega = \{z : |f(z)| > 1\}$ and consists of simply connected components.

If $a \in S(f)$ let $\Omega(a, \epsilon) = f^{-1}(D(a, \epsilon))$ be such a pre-image. A component of $\Omega(a, \epsilon)$ is either bounded and contains a critical point with critical value a, or is unbounded and has asymptotic value a along some unbounded path γ in the component.

Let $X = \overline{\mathbb{D}} \setminus \bigcup_a D(a,\epsilon)$, where the union is over $a \in S(f)$. Then X is a "Swiss cheese", i.e., disk with finitely many disjoint subdisks removed. For functions $f \in \mathcal{S}$, the preimage of this set must be "small" in the sense that is lies close to $\partial\Omega = \{z : |f(z)| = 1\}$:

Theorem 30. *For any $\epsilon < \eta/4$, there is a $r < \infty$ so that $f^{-1}(X) \subset T_\Omega(r)$. For each partition arc I of $\partial\Omega(a,\epsilon)$ there is a partition arc J of $\partial\Omega$ so that $I \subset J(r)$ and $J \subset I(r)$; thus $|I| \simeq |J| \simeq \mathrm{dist}(I,J)$.*