MAT 670, Fall 2023, Stony Brook University

TOPICS IN COMPLEX ANALYSIS
DESSINS AND DYNAMICS
Part II: Extremal length and quasiconformal maps

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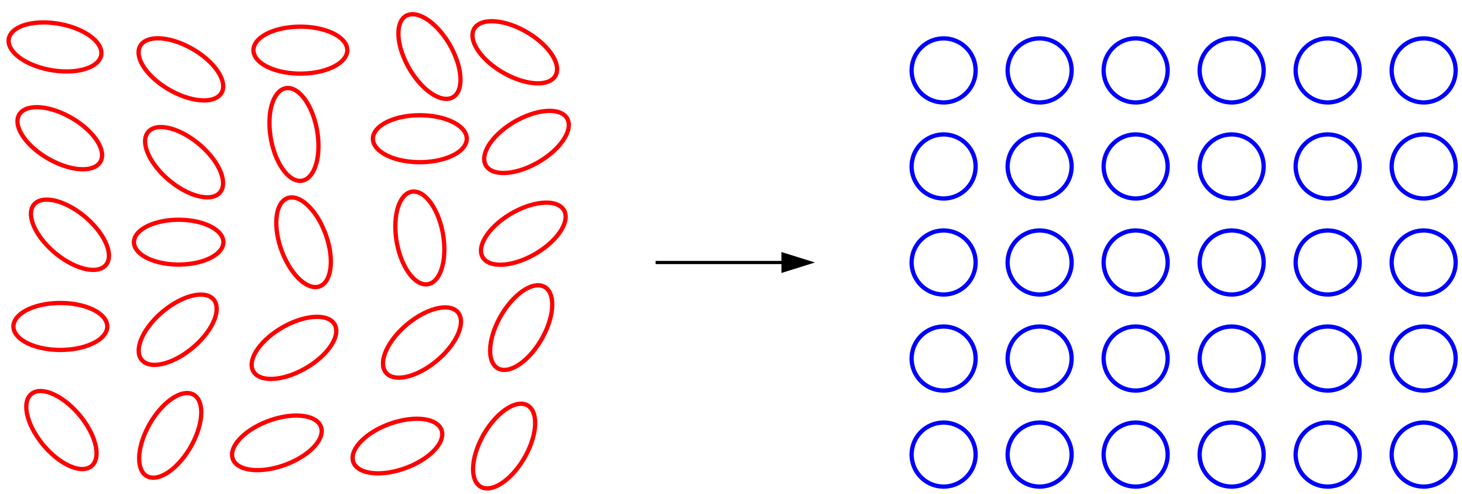


I will try to compress a semester-long course on quasiconformal maps in to a few weeks, giving many statements and few proofs. Among the topics I plan to touch on are:

- Extremal length and modulus
- Definitions of quasiconformal and quasisymmetric
- Compactness
- Measurable Riemann Mapping Theorem
- Estimates for QC maps
- Removability

Modulus and Extremal length

Suppose Γ is a family of locally rectifiable paths in a planar domain Ω and ρ is a non-negative Borel function on Ω.

We say ρ is admissible for Γ (and write $\rho \in \mathcal{A}(\Gamma)$) if

$$\ell(\Gamma) = \ell_\rho(\Gamma) = \inf_{\gamma \in \Gamma} \int_\gamma \rho ds \geq 1,$$

and define the modulus of Γ as

$$\mathrm{Mod}(\Gamma) = \inf_\rho \int_M \rho^2 dxdy,$$

where the infimum is over all admissible ρ for Γ.

The reciprocal of modulus is called the extremal length:

$$\lambda(\Gamma) = 1/M(\Gamma).$$

Lemma 1 (Conformal invariance)**.** *If* $\mathcal{F}$ *is a family of curves in a domain* Ω *and* f *is a one-to-one analytic mapping from* Ω *to* Ω' *then* $M(\mathcal{F}) = \mathcal{M}((\mathcal{F}))$.

Proof. This is just the change of variables formulas

$$\int_\gamma \rho \circ f |f'| ds = \int_{f(\gamma)} \rho ds,$$

$$\int_\Omega (\rho \circ f)^2 |f'|^2 dxdy = \int_{f(\Omega)} \rho dxdy.$$

These imply that if $\rho \in \mathcal{A}(f(\mathcal{F}))$ then $|f'| \cdot \rho \circ f^{-1} \in \mathcal{A}(f(\mathcal{F}))$, and thus $M(f(\mathcal{F})) \leq \mathcal{M}(\mathcal{F})$. We get the other direction by considering f^{-1}. $\square$

Lemma 2 (Monotonicity). *If* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *are collections such that every* $\gamma \in \mathcal{F}_1$ *contains some curve in* $\mathcal{F}_2$ *then*

$$M(\mathcal{F}_1) \leq M(\mathcal{F}_2)$$

and

$$\lambda(\mathcal{F}_1) \geq \lambda(\mathcal{F}_2).$$

The proof is immediate since $\mathcal{A}(\mathcal{F}_1) \supset \mathcal{A}(\mathcal{F}_2)$.

Lemma 3 (Grötzsch Principle)**.** *If* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *are families of curves in disjoint domains then*

$$M(\mathcal{F}_1 \cup \mathcal{F}_2) = M(\mathcal{F}_1) + M(\mathcal{F}_2).$$

Proof. Suppose ρ_1 and ρ_2 are admissible for $\mathcal{F}_1$ and $\mathcal{F}_2$. Take $\rho = \rho_1$ and $\rho = \rho_2$ in their respective domains. Then it is easy to check that ρ is admissible for $\mathcal{F}_1 \cup \mathcal{F}_2$ and $\int \rho^2 = \int \rho_1^2 + \int \rho_2^2$ so domains then $M(\mathcal{F}_1 \cup \mathcal{F}_2) \leq M(\mathcal{F}_1) + M(\mathcal{F}_2)$. By restricting an admissible metric ρ to each domain, a similar argument proves the other direction. $\square$

Lemma 4 (Series Rule)**.** *If $\mathcal{F}_1$ and $\mathcal{F}_2$ are families of curves in disjoint domains and every curve of $\mathcal{F}$ contains both a curve from $\mathcal{F}_1$ and $\mathcal{F}_2$, then $\lambda(\mathcal{F}) \geq \lambda(\mathcal{F}_1) + \lambda(\mathcal{F}_2)$.*

Proof. If $\rho_i \in \mathcal{A}(\mathcal{F}_i)$ for $i = 1, 2$, then $\rho = t\rho_1 + (1-t)\rho_2$ is admissible for $\mathcal{F}$. Since the domains are disjoint we may assume $\rho_1\rho_2 = 0$ everywhere so for $0 \leq t \leq 1$,

$$\rho^2 = t^2\rho_1^2 + (1-t)^2\rho_2^2.$$

Integrating ρ^2 then shows

$$M(\mathcal{F}) \leq t^2 M(\mathcal{F}_1) + (1-t)^2 M(\mathcal{F}_2),$$

for each t. To find the optimal t set $a = M(\mathcal{F}_1)$, $b = M(\mathcal{F}_2)$, differentiate the right hand side above, and set it equal to zero

$$2at - 2b(1-t) = 0.$$

Solving gives $t = b/(a+b)$ and plugging this in above gives

$$M(\mathcal{F}) \le t^2 a + (1-t)^2 b = \frac{b^2 a + a^2 b}{(a+b)^2}$$
$$= \frac{ab(a+b)}{(a+b)^2} = \frac{ab}{a+b} = \frac{1}{\frac{1}{a} + \frac{1}{b}}$$

or

$$\frac{1}{M(\mathcal{F})} \ge \frac{1}{M(\mathcal{F}_1)} + \frac{1}{M(\mathcal{F}_2)},$$

which, by definition, is the same as

$$\lambda(\mathcal{F}) \ge \lambda(\mathcal{F}_1) + \lambda(\mathcal{F}_2),$$

$\square$

Modulus of a rectangle.

So suppose $R = [0, b] \times [0, a]$ is a b wide and a high rectangle and Γ consists of all rectifiable curves in R with one endpoint on each of the sides of length a.

Then each such curve has length at least b, so if we let ρ be the constant $1/b$ function on R we have

$$\int_\gamma \rho ds \geq 1,$$

for all $\gamma \in \Gamma$. Thus this metric is admissible and so

$$\mathrm{Mod}(\Gamma) \leq \iint_T \rho^2 dxdy = \frac{1}{b^2} ab = \frac{a}{b}.$$

To prove a lower bound, we use the well known Cauchy-Schwarz inequality:

$$(\int fg dx)^2 \le (\int f^2 dx)(\int g^2 dx).$$

To apply this, suppose ρ is an admissible metric on R for γ. Every horizontal segment in R connecting the two sides of length a is in Γ, so since γ is admissible, the Cauchy-Schwarz inequality gives

$$1 \le \int_0^b (1 \cdot \rho(x, y)) dx \le \int_0^b 1^2 dx \cdot \int_0^b \rho^2(x, y) dx.$$

Now integrate with respect to y to get

$$a = \int_0^a 1 dy \leq b \int_0^a \int_0^b \rho^2(x, y) dx dy,$$

which implies $\mathrm{Mod}(\Gamma) \geq \frac{b}{a}$. Thus we must have equality.

Lemma 5. *If* $A = \{z : r < |z| < R\}$ *then the modulus of the path family connecting the two boundary components is*

$$\frac{1}{2\pi} \log \frac{R}{r}.$$

More generally, if $\mathcal{F}$ *is the family of paths connecting* $r\mathbb{T}$ *to a set* $E \subset R\mathbb{T}$*, then* $M(\mathcal{F}) \geq |E| \log \frac{R}{r}$.

Proof. By conformal invariance, we can rescale and assume $r = 1$. Suppose ρ is admissible for $\mathcal{F}$. Then for each $z \in E \subset \mathbb{T}$,

$$1 \leq (\int_1^R \rho dr)^2 \leq (\int_1^R \frac{dr}{r})(\int_1^R \rho^2 r dr) = \log R \int_1^R \rho^2 r dr$$

so

$$\int_0^{2\pi} \int_1^R \rho^2 r dr d\theta \geq \int_E \int_r^R \rho^2 r dr d\theta \geq |E| \int_1^R \rho^2 r dr \geq |E| \log R \quad \square$$

A quadrilateral Q is a Jordan curve in the plane with two distinguished, disjoint, closed subarcs. The modulus of Q is the modulus of the path family in Q connecting these two boundary arcs.

We will use without proof that there is a conformal map of the interior of Q to a rectangle that extends homeomorphically to the boundary with the four marked points mapping to the four corners of the rectangle.

If the rectangle has side lengths $a, b > 0$, and the distinguished arcs of Q map to the then the modulus of the quadrilateral is a/b.

Lemma 6. *[short connection, large modulus]*
Suppose Q is a quadrilateral with opposite pairs of sides E, F and C, D. Assume

1. *E and F can be connected in Q by a curve of diameter $\leq \epsilon$,*

2. *any curve connecting C and D in Q has diameter at least 1.*

Then the modulus of the path family connecting E and F in Q is larger than $M(\epsilon)$ where $M(\epsilon) \to \infty$ as $\epsilon \to 0$.

Proof. There is a segment $(a, b) \subset Q$ with $|a - b| \leq \epsilon$ and $a \in E$ and $b \in F$. Define a metric on Q by

$$\rho(z) = \frac{1}{2}|z - a|^{-1} / \log(1/2\epsilon)$$

for $\epsilon < |z - a| < 1/2$.

Any curve γ connecting C and D must cross S and since γ has diameter ≥ 1 it must leave the annulus where ρ is non-zero. As before, this shows that the modulus of the path family in Q separating E and F is small, hence the modulus of the family connecting them is large. $\square$

Lemma 7. *Suppose* $\Omega \subset \mathbb{C}$ *is a topological annulus of modulus* M *whose boundary consists of two Jordan curves* γ_1, γ_2 *with* γ_2 *separating* γ_1 *from* ∞. *Then* $\operatorname{diam}(\gamma_1) \le (1-\epsilon)\operatorname{diam}(\gamma_2)$ *where* $\epsilon > 0$ *depends only on* M.

Proof. Rescale so $\mathrm{diam}(\gamma_2) = \mathrm{diam}(\Omega) = 1$ and suppose $\mathrm{diam}(\gamma_1) > 1-\epsilon$. Then there are points $a \in \gamma_1$ and $b \in \gamma_2$ with $|a-b| \le \epsilon$. Let ρ be the metric on Ω defined by $\rho(z) = \frac{1}{|z-a| \log(1/2\epsilon)}$ for $\epsilon < |z-a| < 1/2$.

Then any curve $\gamma \subset \Omega$ that separates γ_1 and γ_2 satisfies $\int_\gamma \rho ds \ge 1$ and

$$M \le \int \rho^2 dx dy \le \frac{\pi}{4} \log^{-2} \frac{1}{2\epsilon}.$$

Thus the modulus of the path family separating the boundary components is bounded above by the right hand side, and the modulus of the reciprocal family connecting the boundary components is bounded below by $\frac{\pi}{4}\log^2\frac{1}{2\epsilon}$. Thus $\epsilon \geq \frac{1}{2}\exp(-\sqrt{\pi M/4})$. $\square$

Definitions of quasiconformal maps

Quasiconformality: Geometric definition

A homeomorphism $f : \Omega \to \Omega'$ is K-quasiconformal if there is a $K < \infty$ so that for every quadrilateral $Q \subset \Omega$

$$\frac{1}{K} M(Q) \leq M(f(Q)) \leq K M(Q).$$

Quasiconformality: Analytic definition

A homeomorphism $f : \Omega \to \Omega'$ is K-quasiconformal if f is absolutely continuous on almost all lines and there is a $0 \leq k = (K-1)/(K+1) < 1$ so that

$$|f_z| \leq k|f_{\bar{z}}|.$$

Quasiconformality: Metric definition

A homeomorphism $f : \Omega \to \Omega'$ is K-quasiconformal if there is a $K < \infty$ so that for every $x \in \Omega$

$$\limsup_{r \to 0} \frac{\max_{|y-x|=r} |f(y) - f(x)|}{\min |y - x| = r|f(y) - f(x)|} \leq K.$$

A homeomorphism $f : \Omega \to \Omega'$ is K-quasisymmetric if there is a $K < \infty$ so that for every $x \in \Omega$

$$\sup_{r > 0} \frac{\max_{|y-x|=r} |f(y) - f(x)|}{\min |y - x| = r|f(y) - f(x)|} \leq K.$$

Quasisymmetric $\Rightarrow$ quasiconformal

Sometimes converse holds, e.g., on $\mathbb{R}^n$, $n \geq 2$

Quasisymmetric $\neq$ Quasiconformal in $\mathbb{R}$.

All three definitions are equivalent for planar domains.

Geometric definition with modulus is convenient for proving compactness of K-QC maps.

Analytic definition is convenient for mapping theorem and estimating map in terms of dilatation.

Analytic $\Rightarrow$ Geometric is fairly easy.

Geometric $\Rightarrow$ Analytic is harder. Must prove differentiability.

Equivalence with metric definition is due to Heinonen and Koskela (in certain metric spaces, including Euclidean space).

Lemma 8. *If we have a piecewise differentiable K-quasiconformal map between a $1 \times a$ and $1 \times b$ rectangle with dilatation $\leq K$, then $\frac{a}{K} \leq b \leq Ka$.*

Proof. By integrating over horizontal lines in the first rectangle, we see

$$b \le \int_0^a (|f_z| + |f_{\overline{z}}|)dx,$$

and integrating in the other variable,

$$b \le \int_0^1 \int_0^a (|f_z| + |f_{\overline{z}}|)dxdy.$$

Thus by Cauchy-Schwarz

$$\begin{aligned}
b^2 &\leq \left(\int_0^1 \int_0^a (|f_z| + |f_{\overline{z}}|)(|f_z| - |f_{\overline{z}}|) dxdy\right)\left(\int_0^1 \int_0^a \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|} dxdy\right) \\
&\leq \left(\int_0^1 \int_0^a (|f_z|^2 - |f_{\overline{z}}|^2) dxdy\right)\left(\int_0^1 \int_0^a \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|} dxdy\right) \\
&\leq \left(\int_0^1 \int_0^a J_f dxdy\right)\left(\int_0^1 \int_0^a D_f dxdy\right) \\
&\leq baK,
\end{aligned}$$

so $b \leq Ka$. The other direction follows by considering the inverse map. $\square$

Corollary 9. *If f is a piecewise differentiable K-quasiconformal on the whole rectangle and $(1+\epsilon)$-quasiconformal except on a set of area δ, then $b/a \leq 1+\epsilon+K\delta$. In particular, a sequence of such maps whose dilatations satisfy $\sup_n \|\mu_n\|_\infty \leq k < 1$ and so that $\{\mu_n\}$ tends to 0 in measure, will tend to a 1-quasiconformal map.*

Equicontinuity of K-quasiconformal maps

Arzela-Ascoli theorem:

A family of continuous functions on a compact Hausdorff space has compact closure for the sup-norm topology (uniform convergence) if and only if

(1) the family is equicontinuous,
(2) the family is pointwise bounded.

Lemma 10. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is a K-quasiconformal map that fixes both 0 and 1. Then $|f(x)|$ is bounded with an estimate depending on $|x|$ and K, but not on f.*

Proof. First suppose $\mathrm{Re}(x) \leq 1/2$ and consider the topological annulus with boundary component $[0, x]$ and $[1, \infty)$.

The modulus of the path family separating the two boundary components is bounded below depending only on $|x|$. But if $R = |f(x)|$ then by using the metric $\rho(z) = 1/(|z| \log R)$, we see that the modulus of $f(\mathcal{F})$ is at most $1/\log R$.

This is a contradiction if R is too large.

For $\mathrm{Re}(x) > -1/2$ we consider $[1, x]$ and $[0, -\infty)$. $\square$

Theorem 11. *A K-quasiconformal map of the plane that fixes both 0 and 1 is locally Hölder continuous.*

Proof. Suppose f is as in the lemma and $x, y \in D(0, r)$. By Lemma 10, $D(0, 2r)$ is mapped into $D(0, R)$ for some $R = R(r, K)$. Surround $\{x, y\}$ by $N = \lfloor \log_2 \frac{r}{|x-y|} \rfloor$ annuli $\{A_j\}$ of modulus $\log 2$.

The image annuli $\{f(A_j)\}$ have moduli bounded away from zero, and hence $\mathrm{diam}(f(A_{j+1})) \leq (1-\epsilon)\mathrm{diam}(f(A_j))$ by Lemma 7.

Therefore

$$\begin{aligned} |f(x) - f(y)| &\leq R(1-\epsilon)^N \\ &\leq R2^{\log_2(1-\epsilon)(1+\log_2 R - \log_2 |x-y|)} \\ &\leq C(R)|x-y|^{\log_2(1-\epsilon)}. \quad \square \end{aligned}$$

Thus normalized K-QC maps form a equicontinuous family.

Arzela-Ascoli theorem may be applied.

One can prove the actual Hölder exponent is $\alpha = 1/K$ (Mori's theorem).

Lemma 12. *If $\varphi : \mathbb{D} \to \mathbb{D}$ is quasiconformal and onto, then φ extends continuously to a homeomorphism of $\mathbb{T} = \partial\mathbb{D}$ to itself.*

Proof. We may assume $f(0) = 0$; the general case follows after composing with a Möbius transformation.

Suppose $w, z \in \mathbb{D}$. We will show that

$$|f(z) - f(w)| \leq C|z - w|^{\alpha},$$

for constants $C < \infty$, $\alpha > 0$ that depend only on the quasiconstant K of f. This implies f is uniformly continuous and hence has a continuous extension to the boundary of $\mathbb{D}$.

Let $d = |z - w|$ and $r = \min(1 - |z|, 1 - |w|)$. There are several cases depending on the positions of the points z, w and the relative sizes of d and r.

To start, note that if $|z-w| \geq \frac{1}{10}$ we can just take $C = 20$ and $\alpha = 1$. So from here on, we assume $|z-w| < 1/10$.

Suppose $r > 1/4$, so $z, w \in \frac{3}{4}\mathbb{D}$. Surround the segment $[z, w]$ by $N \simeq \log d$ annuli with moduli $\simeq 1$. Then just as in the proof of Theorem 11, the image annuli have moduli $\simeq 1$ (with a constant depending on K) and hence

$$|f(z) - f(w)| \leq (1 - \epsilon(K))^N = O(|z-w|^\alpha),$$

for some $\alpha > 0$ depending only on K.

Next suppose $|z| \geq 3/4$ and $d > r$. Then separate $[z, w]$ from 0 by $N \simeq \log d$ disjoint quadrilaterals with a pair of opposite sides being arcs of $\mathbb{T}$, and all with moduli $\simeq 1$. Since $f(0) = 0$ and the image quadrilaterals have moduli $\simeq 1$, the diameters shrink geometrically, so $|z - w| = (1 - \epsilon(K))^N = O(d^\alpha)$, as desired.

Finally, if $r \leq d$ we combine the two previous ideas: we start by separating $[z, w]$ from 0 by $\simeq \log d$ quadrilaterals with as above. The smallest quadrilateral then bounds a region of diameter approximately r containing $[z, w]$ and we then construct $\simeq \log r/d$ disjoint annuli with moduli $\simeq 1$ that each separate $[z, w]$ from this smallest quadrilateral.

The same arguments as before now show

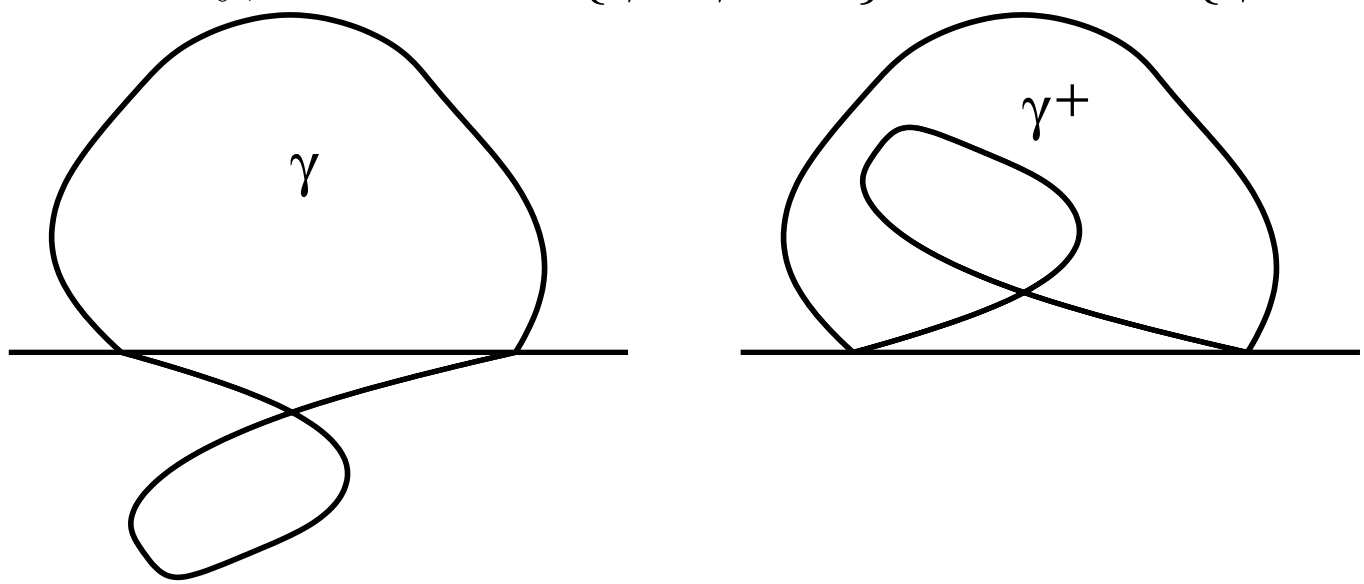
$$|z - w| = (1 - \epsilon(K))^{-\log r}(1 - \epsilon(K))^{\log r/d} = O(d^\alpha) = O(|z - w|^\alpha).$$

$\square$

Symmetry and Koebe's theorem

If γ is a path in the plane let $\bar{\gamma}$ be its reflection across the real line and let $\gamma^+ = (\gamma \cap \mathbb{H}) \cup \overline{\gamma \cap \mathbb{H}_l}$, where $\mathbb{H}, \mathbb{H}_l = \overline{\mathbb{H}}$ denote the upper and lower half-planes.

If Γ is a path family, define $\overline{\Gamma} = \{\bar{\gamma} : \gamma \in \Gamma\}$ and $\Gamma^+ = \{\gamma^+ : \gamma \in \Gamma\}$.



Lemma 13. *If* $\Gamma = \overline{\Gamma}$ *then* $M(\Gamma) = 2M(\Gamma^+)$.

Proof. We start by proving $M(\Gamma) \leq 2M(\Gamma^+)$. Given a metric ρ, define $\sigma(z) = \max(\rho(z), \rho(\bar{z}))$. Then for any $\gamma \in \Gamma$,

$$\int +\gamma^+ \sigma ds \geq \int_{\gamma^+} \rho ds \geq \inf_{\gamma \in \Gamma} \int_\gamma \rho ds.$$

Thus if ρ admissible for Γ^+, then σ is admissible for Γ

Thus, since $\max(a, b)^2 \leq a^2 + b^2$,

$$M(\Gamma) \leq \int \sigma^2 dxdy \leq \int \rho^2(z)dxdy + \int \rho^2(\bar{z})dxdy \leq 2\int \rho^2(z)dxdy.$$

Taking the infimum over admissible ρ's for Γ^+ makes the right hand side equal to $2M(\Gamma^+)$, proving the claim.

For the other direction, given ρ define $\sigma(z) = \rho(z) + \rho(\bar{z})$ for $z \in \mathbb{H}$ and $\sigma = 0$ if $z \in \mathbb{H}_l$. Then

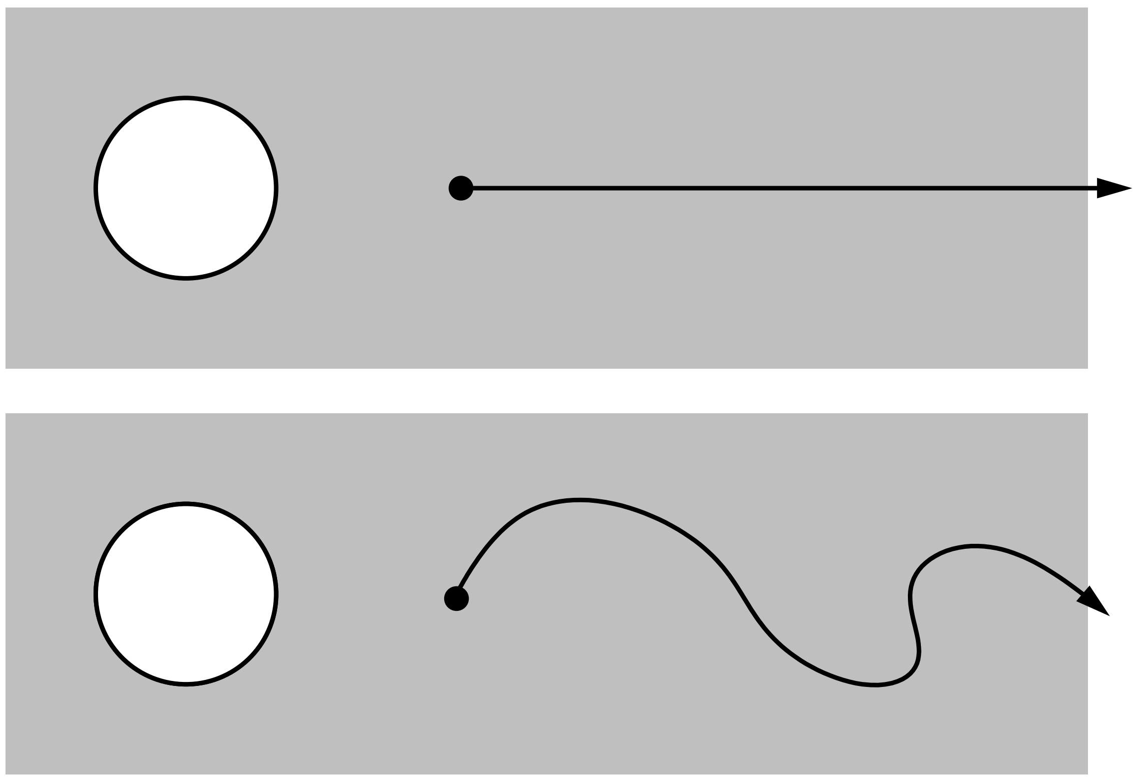
$$\begin{aligned}
\int_{\gamma^+} \sigma ds &= \int_{\gamma^+} \rho(z) + \rho(\bar{z}) ds \\
&= \int_{\gamma\cap\mathbb{H}} \rho(z) ds + \int_{\gamma\cap\mathbb{H}} \rho(\bar{z}) ds + \int_{\gamma\cap\mathbb{H}_l} \rho(z) + \int_{\gamma\cap\mathbb{H}_l} \rho(\bar{z}) ds \\
&= \int_{\gamma} \rho(z) ds + \int_{\bar{\gamma}} \rho(z) ds \\
&\geq 2 \inf_{\rho} \int_{\gamma} \rho ds.
\end{aligned}$$

Thus if ρ is admissible for Γ, $\frac{1}{2}\sigma$ is admissible for Γ^+. Hence, since $(a+b)^2 \leq 2(a^2+b^2)$,

$$\begin{aligned}
M(\Gamma^+) &\leq \int (\frac{1}{2}\sigma)^2 dxdy \\
&= \frac{1}{4}\int_{\mathbb{H}} (\rho(z)+\rho(\bar{z}))^2 dxdy \\
&\leq \tfrac{1}{2} \int_{\mathbb{H}} \rho^2(z)dxdy + \int_{\mathbb{H}} \rho^2(\bar{z})dxdy \\
&= \frac{1}{2}\int \rho^2 dxdy.
\end{aligned}$$

Taking the infimum over all admissible ρ's for Γ gives $\frac{1}{2}M(\Gamma)$ on the right hand side, proving the lemma. $\square$

Lemma 14 (Rays are extreme). *Let* $\mathbb{D}^* = \{z : |z| > 1\}$ *and* $\Omega_0 = \mathbb{D}^* \setminus [R, \infty)$ *for some* $R > 1$. *Let* $\Omega = \mathbb{D}^* \setminus K$*, where* K *is a closed, unbounded, connected set in* $\mathbb{D}^*$ *which contains the point* $\{R\}$. *Let* Γ_0, Γ *denote the path families in these domains with separate the two boundary components. Then* $M(\Gamma_0) \leq M(\Gamma)$.



Proof. We use the symmetry principle we just proved.

The family Γ_0 is clearly symmetric (i.e., $\Gamma = \overline{\Gamma}$, so $M(\Gamma^+) = \frac{1}{2}M(\Gamma_0)$.

The family Γ may not be symmetric, but we can replace it by a larger family that is.

Let Γ_R be the collection of rectifiable curves in $\mathbb{D}^* \setminus \{R\}$ which have zero winding number around $\{R\}$, but non-zero winding number around 0.

Clearly $\Gamma \subset \Gamma_R$ and Γ_R is symmetric so$M(\Gamma) \geq M(\Gamma_R) = 2M(\Gamma_R^+)$. Thus all we have to do is show $M(\Gamma_R^+) = M(\Gamma_0^+)$.

We claim $\Gamma_R^+ = \Gamma_0^+$. Clearly $\Gamma_0 \subset \Gamma_R$, so we only need $\Gamma_R^+ \subset \Gamma_0^+$.

Suppose $\gamma \in \Gamma_R$. Since γ has non-zero winding around 0 it must cross both the negative and positive real axes. If it never crossed $(0, R)$ then the winding around 0 and R would be the same, which false, so γ must cross$(0, R)$ as well.

Choose points $z_- \in \gamma \cap (-\infty, 0)$ and $z_+ \in \gamma \cap (0, R)$. These points divide γ into two subarcs γ_1 and γ_2.Then $\gamma^+ = \gamma_1^+ \cup \gamma_2^+$. But if we reflect γ_2^+ into the lower half-plane and join it to γ_1^+ it forms a closed curve γ_0 that is in Γ_0 and $\gamma_0^+ = \gamma^+$. Thus $\gamma^+ \in \Gamma_0^+$, as desired. $\square$

Let $\Omega_{\epsilon,R} = \{z : |z| > \epsilon\} \setminus [R, \infty)$. Thus $\Omega_{1,R}$ is the domain considered in the previous lemma. We can estimate the moduli of these domains using the Koebe map

$$k(z) = \frac{z}{(1+z)^2} = z - 2z^2 + 3z^3 - 4z^4 + 5z^5 - \ldots,$$

which is conformal $\mathbb{D} \to \mathbb{R}^2 \setminus [\frac{1}{4}, \infty)$ and $k(0) = 0$, $k'(0) = 1$.

Then $k^{-1}(\frac{1}{4R}z)$ maps $\Omega_{\epsilon,R}$ conformally to an annular domain in the disk whose outer boundary is the unit circle and whose inner boundary is trapped between the circle of radius

$$\frac{\epsilon}{4R}(1 \pm O(\frac{\epsilon}{R})).$$

Thus the modulus of $\Omega_{\epsilon,R}$ is $2\pi \log \frac{4R}{\epsilon} + O(\epsilon/R)$.

Lemma 15. *Suppose* $z, w \in \mathbb{D}$ *and* K *is a compact connected set in* $\mathbb{D}$ *which contains both these points. Let* Γ *be the path family that separates* K *and* $\mathbb{T}$. *Then the modulus of this family is maximized when* K *is the hyperbolic geodesic between* z *and* w *in which case the modulus is* $2\pi \log \frac{4}{\rho}(z, w) + O(\rho(z, w))$, *where* ρ *denotes the hyperbolic distance.*

Proof. By conformal invariance we may use a Möbius transformation to move z to 0 and w onto the positive axis. Applying an inversion, the path family is mapped to one as in Lemma 14, showing that the radial line from z to w maximizes the modulus. The estimate of the modulus follows from our previous remarks. $\square$

We now give an elegant proof of the Koebe $\frac{1}{4}$-theorem due to Mateljevic.

Theorem 16 (The Koebe $\frac{1}{4}$ Theorem)**.** *Suppose f is holomorphic, 1-1 on $\mathbb{D}$ and $f(0) = 0$, $f'(0) = 1$. Then $D(0, \frac{1}{4}) \subset f(\mathbb{D})$.*

Proof. Let $R = \operatorname{dist}(0, \partial f(\mathbb{D}))$. Let $A_{\epsilon,r} = \{z : \epsilon < |z| < r\}$ and note that by conformal invariance

$$2\pi \log \frac{1}{\epsilon} = M(A_{\epsilon,1}) = M(f(A_{\epsilon,1})).$$

Let $\delta = \min_{|z|=\epsilon} |f(z)|$. Since $f'(0) = 1$, $\delta = \epsilon + O(\epsilon^2)$. Note that $f(\mathbb{D}) \setminus D(0, \delta) \supset f(A_{\epsilon,1})$, so

$$M(f(\mathbb{D}) \setminus D(0, \delta)) \geq M(f(A_{\epsilon,1})).$$

By Lemma 14 ("rays are extreme")

$$M(f(\mathbb{D}) \setminus D(0,\delta)) \leq M(\Omega_{\delta,R}) = 2\pi \log \frac{4R}{\delta} + O(\frac{\delta}{R}).$$

Putting these together gives

$$2\pi \log \frac{4R}{\delta} + O(\frac{\delta}{R}) \geq 2\pi \log \frac{1}{\epsilon},$$

or

$$\log 4R - \log(\epsilon + O(\epsilon^2)) + O(\frac{\epsilon}{R}) \geq -\log \epsilon.$$

Taking $\epsilon \to 0$ shows $\log 4R \geq 0$, or $R \geq \frac{1}{4}$. $\square$

Capacity and boundary values of conformal maps

Theorem 17 (Gehring-Hayman inequality)**.** *There is an absolute constant $C < \infty$ to that the following holds. Suppose $\Omega \subset \mathbb{C}$ is hyperbolic and simply connected. Given two points in Ω, let γ be the hyperbolic geodesic connecting these two points and let γ' be any other curve in Ω connecting them. Then $\ell(\gamma) \leq C\ell(\gamma')$.*

"Hyperbolic geodesics are approximately Euclidean geodesics".

Proof. Let

$$Q_n = \{z \in \mathbb{D} : 2^{-n-1} < |z-1| < 2^{-n}\},$$

and let

$$\gamma_n = \{z \in \mathbb{D} : |z-1| = 2^{-n}\},$$

$$z_n = \gamma_n \cap [0,1).$$

Let $f : \mathbb{D} \to \Omega$ be conformal, normalized so that γ is the image of $I = [0,r] \subset \mathbb{D}$ for some $0 < r < 1$.

WLOG, we may assume $r = z_{N+1}$ for some N (if not we truncate a segment of the form $J = [z_{N+1}, r]$ and use Koebe's theorem to compare the lengths of $f(J)$ and $\gamma' \cap f(Q_{N+1})$).

Let $Q_n' \subset Q_n$ be the sub-quadrilateral of points with $|\arg(z-1)| < \pi/6$. Each of these has bounded hyperbolic diameter and hence by Koebe's theorem its image is bounded by four arcs of diameter $\simeq d_n$ and opposite sides are $\simeq d_n$ apart.

In particular, this means that any curve in $f(Q_n)$ separating γ_n and γ_{n+1} must cross $f(Q'_n)$ and hence has diameter $\gtrsim d_n$. Since Q_n has bounded modulus, so does $f(Q_n)$ and so Lemma 6 ("short connection, large modulus") says that the shortest curve in $f(Q_n)$ connecting γ_n and γ_{n+1} has length $\ell_n \simeq d_n$.

Any γ' in Q connecting γ_n and γ_{n+1} has length at least ℓ_n, and so

$$\ell(\gamma) = O(\sum d_n) = O(\sum \ell_n) \leq O(\ell(\gamma')).$$

$\square$

Given $E \subset \mathbb{T}$ we will denote the capacity of E to be the modulus of the path family in the annulus $\{\frac{1}{2} < |z| < 1\}$ that has one endpoint on $\{|z| = \frac{1}{2}\}$ and one endpoint on E.

This definition of capacity is non-standard, and is a substitute for the usual logarithmic capacity $\operatorname{cap}(E)$ of E.

Lemma 18. *If E has zero capacity, then it has zero length.*

Proof. We prove the contrapositive. If E has positive length, suppose ρ is an admissible metric for the corresponding path family. Considering the radial segments connecting E to $\{|z| = 1/2\}$, we see

$$\begin{aligned}
|E| &\leq 2\int_E\int_{1/2}^1 \rho(z)drd\theta \leq 4\int_E\int_{1/2}^1 \rho(z)rdrd\theta \\
&\leq 4\left(\int_E\int_{1/2}^1 \rho^2(z)dxdy\right)^{1/2} \cdot \left(\int_E\int_{1/2}^1 1dxdy\right)^{1/2} \\
&\leq 2\left(\int_E\int_{1/2}^1 \rho^2(z)dxdy\right)^{1/2} \cdot \sqrt{|E|}.
\end{aligned}$$

Hence $\int \rho^2 dxdy \geq \frac{1}{4}|E|$. $\square$

Actually, sets of zero capacity have zero Hausdorff dimension.

Lemma 19 (Bounded radial images)**.** *Suppose* $f : \mathbb{D} \to \Omega$ *is conformal, and for* $R \geq 1$,

$$E_R = \{x \in \mathbb{T} : |f(x) - f(0)| \geq R \operatorname{dist}(f(0), \partial\Omega)\}.$$

Then E_R *has capacity* $O(1/\log R)$ *if* R *is large enough.*

Proof. Assume $f(0) = 0$ and $\operatorname{dist}(0, \partial\Omega) = 1$ and let $\rho(z) = |z|^{-1}/\log R$ for $z \in \Omega \cap \{1 < |z| < R\}$.

Then ρ is admissible for the path family $\mathcal{F}$ connecting $D(0, 1/2)$ to $\partial\Omega \setminus D(0, R)$ and $\iint \rho^2 dxdy \leq 2\pi/\log R$.

By definition $M(\mathcal{F}) \leq 2\pi/\log R$ and $\lambda(\mathcal{F}) \geq (\log R)/2\pi$.

By Koebe distortion $K = f^{-1}(D(0, 1/2))$ is contained in a compact subset of $\mathbb{D}$, independent of Ω.

One can show that the extremal length connecting K to the E is comparable to the extremal length connecting $\{|z| = 1/2\}$ to E. $\square$

Corollary 20. *Suppose* $f : \mathbb{D} \to \Omega$ *is conformal and* $z \in \mathbb{D}$*. For any direction* θ *at* z *and any* $\epsilon > 0$ *there is a hyperbolic geodesic ray* γ *start at* z *within* ϵ *of direction* θ *so that the Euclidean length of* $f(\gamma)$ *is* $\leq C_\epsilon \mathrm{dist}(f(z), \partial\Omega)$*.*

Lemma 21. *There is a* $c > 0$ *so that the following holds. Suppose* $f : \mathbb{D} \to \Omega$ *and* $\frac{1}{2} \leq r < 1$. *Let*

$$E(\delta, r) = \{x \in \mathbb{T} : |f(sx) - f(rx)| \geq \delta \text{ for some } s \in (r, 1)\}.$$

Then the extremal length of the path family $\mathcal{P}$ *connecting* $D(0, r)$ *to* E *is bounded below by* $c\delta^2/a(r)$ *where* $a(r) = \text{area}(\Omega \setminus f(D(0, r))$.

Proof. Suppose $z, w \in \Omega$, suppose γ is the hyperbolic geodesic connecting z and w and suppose $\tilde{\gamma}$ is any path in Ω connecting these points. By the Gehring-Hayman inequality there is a universal $C < \infty$ such that $\ell(\gamma) \leq C\ell(\tilde{\gamma})$ (here $\ell(\gamma)$ denotes the length of γ).

Now suppose we apply this with $z = f(sx)$ and $w \in f(D(0, r))$. By the Gehring-Hayman estimate, the length of any curve from w to z is at least $1/C$ times the length of the hyperbolic geodesic γ between them.

But this geodesic has a segment γ_0 that lies within a uniformly bounded distance of the geodesic γ_1 from $f(rx)$ to z. By the Koebe distortion theorem γ_0 and γ_1 have comparable Euclidean lengths, and clearly the length of γ_1 is at least δ.

Thus the length of any path from $f(D(0,r))$ to $f(sx)$ is at least δ/C. Now let $\rho = C/\delta$ in $\Omega \setminus f(D(0,r))$ and 0 elsewhere. Then ρ is admissible for $f(\mathcal{P})$ and $\iint \rho^2 dxdy$ is bounded by $C^2 a(r)/\delta^2$.

Thus $\lambda(\mathcal{P}) \geq C^{-2}\delta^2/a(r)$. $\square$

Corollary 22. *If $f : \mathbb{D} \to \Omega$ is conformal, then f has radial limits except on a set of zero capacity (and hence almost everywhere).*

Proof. Let $E_{r,\delta} \subset \mathbb{T}$ be the set of $x \in \mathbb{T}$ so that $\mathrm{diam}(f(rx, x)) > \delta$, and let $E_\delta = \bigcap_{0<r<1} E_{r,\delta}$. If f does not have a radial limit at $x \in \mathbb{T}$, then $x \in E_\delta$ for some $\delta > 0$, and this has zero capacity by Lemma 21.

Taking the union over a sequence of δ's tending to zero proves the result. The set where f has a radial limit ∞ has zero capacity by Lemma 19, so we deduce f has finite radial limits except on zero capacity. $\square$

Continuity of modulus and limits of K-QC maps

Lemma 23. *Suppose* $\{f_n\}$ *are conformal maps of* $\mathbb{D} \to \Omega_n$ *that converge uniformly on compact subsets of* $\mathbb{D}$ *to a conformal map* $f : \mathbb{D} \to \Omega$*. Suppose that the boundary of each* Ω_n *is the homeomorphic image* $\partial\Omega_n = \sigma_n(\mathbb{T})$ *and that* $\{\sigma_n\}$ *converges uniformly on* $\mathbb{T}$ *to a homeomorphism* $\sigma : \mathbb{T} \to \partial\Omega$*. Then* $f_n \to f$ *uniformly on the* $\overline{\mathbb{D}}$*.*

This implies modulus is a continuous function of the quadrilateral.

Proof. Fix $\epsilon > 0$ and choose n so large that if we divide $\mathbb{T}$ into n equal sized intervals $\{J_j\}_1^n$, then σ maps each of them to a set I_j of diameter at most $\epsilon/2$.

Let $I_j^k = \sigma_k(J_j)$. Because $\sigma_k \to \sigma$ uniformly, the sets I_j all have diameter at most ϵ, if k is large enough.

Next choose $\eta > 0$ so small that if $k, m > 1/\eta$ and $\sigma_m(J_j)$ and $\sigma_k(J_i)$ contain points at most distance $C\eta$ apart, then J_i and J_k are the same or adjacent to each other.

We can do this because of the uniform convergence and the fact that σ is 1-to-1. By passing to the limit the same property holds for σ.

Next choose m so large that $f(\mathbb{D}) \setminus f(\{|z| < 1 - \frac{1}{m}\})$ is contained in an η-neighborhood of $\partial\Omega$.

Choose m points $\{z_j\}$ equally spaced on the circle $|z| = 1 - \frac{1}{m}$, and let $K_j \subset \mathbb{T}$ be the arc centered at $z_j/|z_j|$ of length $4\pi/m$.

Fix a small number $\delta > 0$ (determined below, depending only on η).

By Lemma 19 ("long curves, small capacity") choose a point $w_j \in K_j$ so that $|w_j - z_j| \leq 2/m$ and

$$|f(w_j) - f(w_j(1 - \frac{1}{m}))| \leq C\delta.$$

This is possible if m is large enough depending on δ.

Similarly, choose points $w_j^k \in K_j$ so that

$$|f_k(w_j^k) - f_k(z_j)| \leq 2C\delta.$$

This is possible since $f_k \to f$ uniformly on the compact set $\{|z| \leq 1 - \frac{1}{m}\}$ and thus $\partial f_k(\mathbb{D})$ is contained in an 2δ-neighborhood of $\partial\Omega$ for k large enough and $\partial\Omega_k$ is contained in a δ-neighborhood of $\partial\Omega$ because of the uniform convergence of the parameterizations.

By taking even m larger, if necessary, we can also arrange that each I_j contains at least one of the points $f(z_m/|z_m|)$. (Recall $\{I_j\}$ are images n equal size arcs $\{J_j\}$ on circle.)

Thus each $f(K_j)$ is mapped into the union of at most 2 of the I_j and hence its image has diameter at most 2ϵ.

Also, the points $f(w_p^k)$ and $f(w_{p+1}^k)$ are at most $C\delta$ apart, so belong to the same or adjacent sets I_j.

Thus $f_k(K_p)$ is a union of at most 4 such adjacent sets and hence has diameter $O(\epsilon)$.

For each w_p^k there is an arc J_j so that $f_k(w_p^k) \subset \sigma_k(J_j)$. Similarly, there is an arc J_i so that $f(w_p) \in I_i = \sigma(J_i)$.

Since $f_k \to f$ uniformly on the finite set $\{z_n\}$, we have, for k sufficiently large

$$\begin{aligned}
|f_k(w_n^k) - f(w_n)| &\leq |f_k(w_n^k) - f_k(z_n)| \\
&\quad +|f_k(z_n) - f(z_n)| \\
&\quad\quad +|f(z_n) - f(w_n)| \\
&\leq (2C + 1 + C)\delta.
\end{aligned}$$

This is less than η if δ is small enough (determines δ).

Since I_i and I_j each have diameter at most ϵ, their union has diameter $< 2\epsilon$ and the union of the intervals adjacent to these is at most 4ϵ.

Similarly for I_i^k and J_j^k. Thus $f_k(K_p)$ and $f(K_p)$ are contained in $O(\epsilon)$-neighborhoods of each other. Thus $f_k \to f$ uniformly on $\mathbb{T}$.

By the maximum principle, this implies uniform convergence on the closed disk, as desired. $\square$

Corollary 24. *Suppose* $\{f_n\}$ *are homeomorphisms* $\mathbb{C} \to \mathbb{C}$ *that converge uniformly to a homeomorphism* f *and suppose that* Q *is a quadrilateral. Then the moduli of* $Q_n = f_n(Q)$ *converge to the modulus of* $f(Q)$

Corollary 25. *Suppose* $\{f_n\}$ *are* K*-QC homeomorphisms on* $\Omega \subset \mathbb{C}$ *that converge uniformly to a homeomorphism* f*. Then* f *is* K*-QC.*

Proof. If $Q \subset \Omega$, then $M(f(Q)) = \lim_n M(f_n(Q)) \leq KM(Q)$. $\square$

Lemma 26. *If* $\{f_n\}$ *is a sequence of* K*-quasiconformal maps on* Ω *that converge uniformly on compact subsets to a homeomorphism* f*, then* f *is* K*-quasiconformal.*

Proof. Any quadrilateral $Q \subset \Omega$ has compact closure in Ω so $Q' = \lim_n f_n(Q)$ is a quadrilateral in $f(\Omega)$ and we need only check that if Q is a quadrilateral then $M(\lim_n f_n(Q)) = \lim_n M(f_n(Q))$. However, this follows from Lemma 23. $\square$

Corollary 27. *Suppose $\{f_n\}$ are K-QC homeomorphisms on $\mathbb{C}$ normalized to fix $0, 1$. The there is a subsequence that converges uniformly to a K-QC map f.*

Proof. By earlier estimates we get equicontinuity, so can apply Arzela-Ascoli to get uniformly convergent subsequence. Limit is K-QC by continuity of modulus. $\square$

If f_n is K_n-QC and $K_n \searrow 1$ then the limit f is 1-QC.

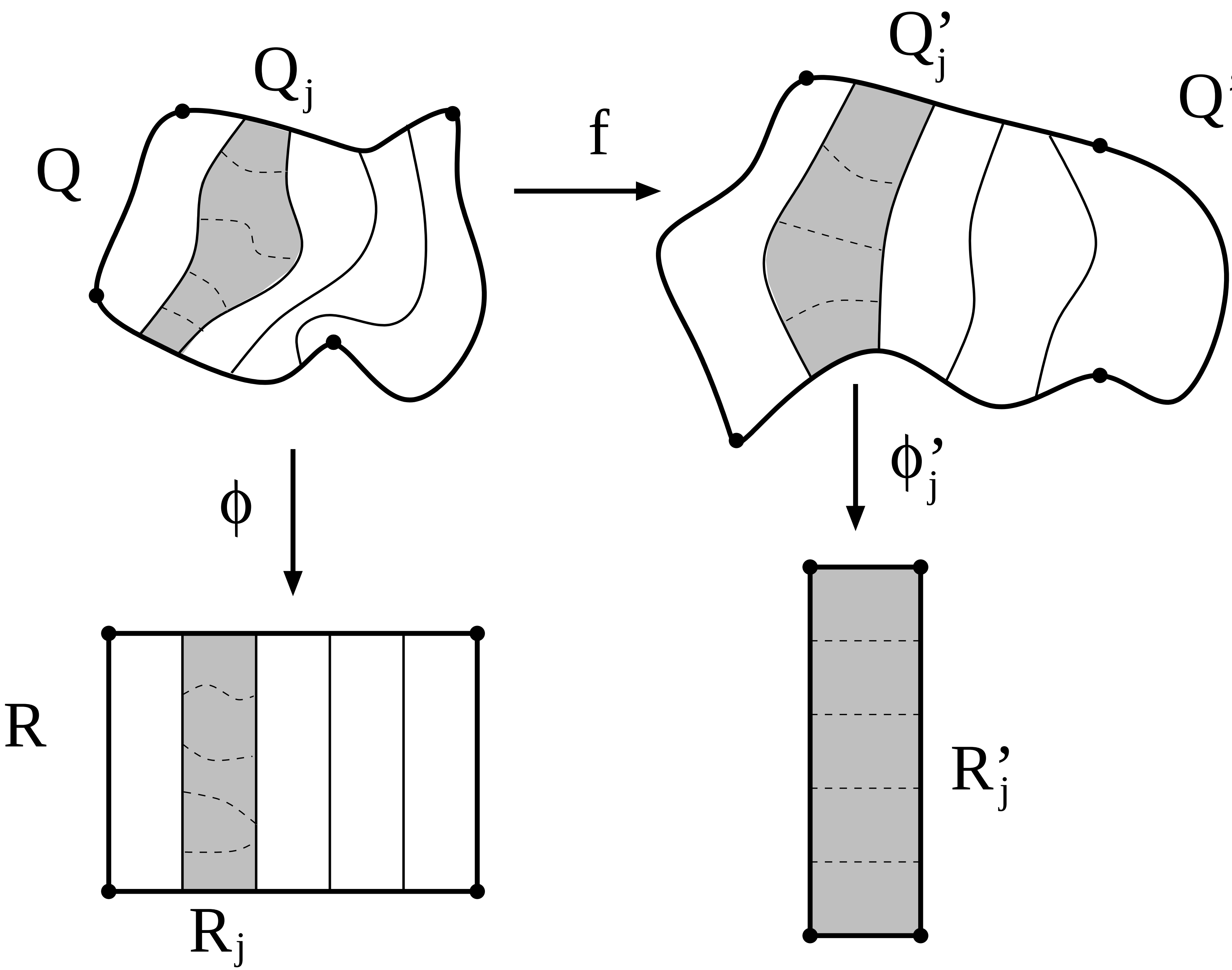
If we knew f was conformal then it is also linear, and since it fixes $0, 1$, f would be the identity.

1-quasiconformal = conformal

Lemma 28. *If f is a homeomorphism of $\Omega \subset \mathbb{C}$ that is K-QC in a neighborhood of each point of Ω, then f is K-QC on all of Ω.*

Proof. Suppose $Q \subset \Omega$ is a quadrilateral that is conformally equivalent via a map ϕ to a $1 \times m$ rectangle R and $Q' = f(Q)$ is conformally equivalent a $1 \times m'$ rectangle R'.

Divide R into M equal vertical strips $\{S_j\}$ of dimension $1 \times m/M$. We have to choose M sufficiently large that several things happen.



Q
Q_j
f
Q'_j
Q'
ϕ
ϕ'_j
R
R_j
R'_j

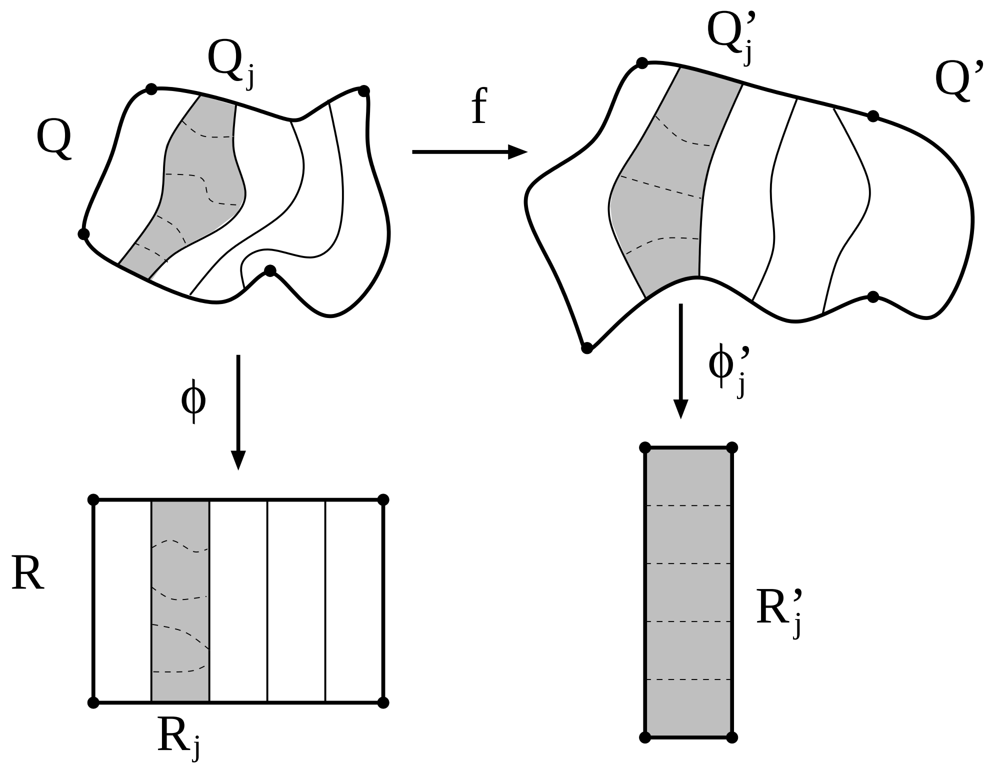
First choose $\delta > 0$ so that f^{-1} is K-quasiconformal on any disk of radius δ centered at any point of Q' (possible since Q' has compact closure in Ω).

The closure of Q' is a union of Jordan arcs γ corresponding via $f \circ \phi^{-1}$ to vertical line segments in R.

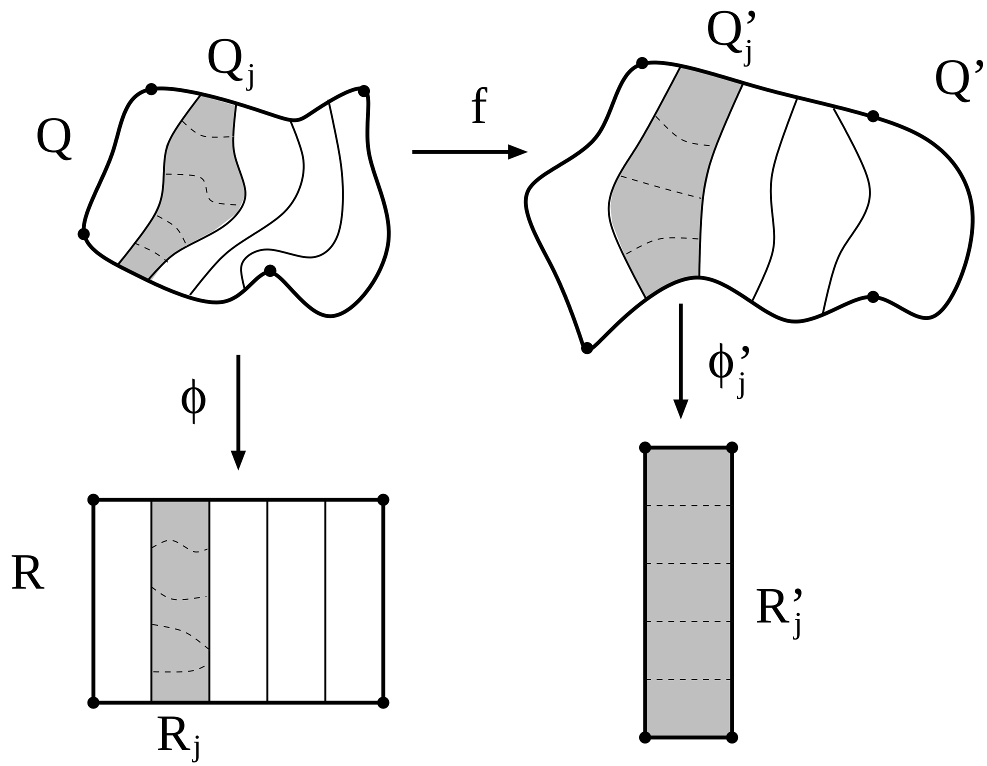
By the continuity of $f \circ \phi^{-1}$ there is an $\eta > 0$ so that if $z \in R$ then $f(\phi^{-1}(D(z, \eta)))$ has diameter $\leq \delta$.

By the continuity of the inverse map, there is an $\epsilon > 0$ so that $x, y \in Q'$ and $|x - y| < \epsilon$ implies $|\phi(f^{-1}(x)) - \phi(f^{-1}(y))| \leq \eta$.

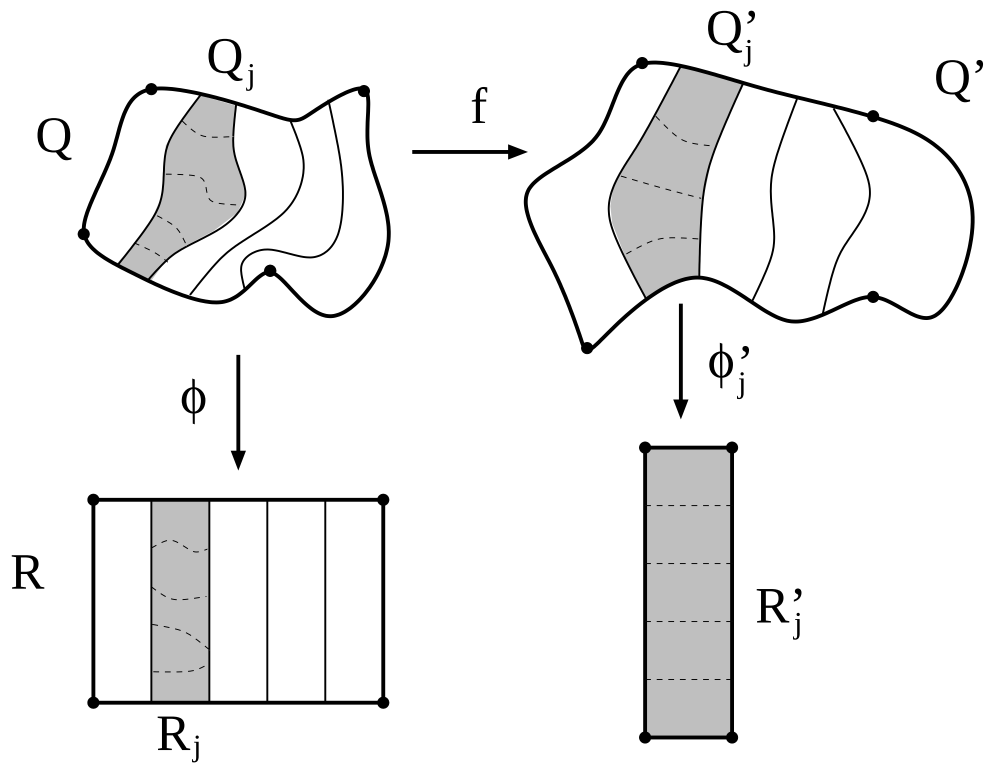
Thus for any $\delta > 0$ there is an $\epsilon > 0$ so that if $x, y \in \gamma \subset Q'$ are at most distance ϵ apart, then the arc of γ between then has diameter at most δ (and ϵ is independent of which γ we use).



Choose M so large that each region $Q'_j = f(\phi^{-1}(S_j))$ contains a disk of radius at most ρ, where ρ will be chosen small depending on ϵ.



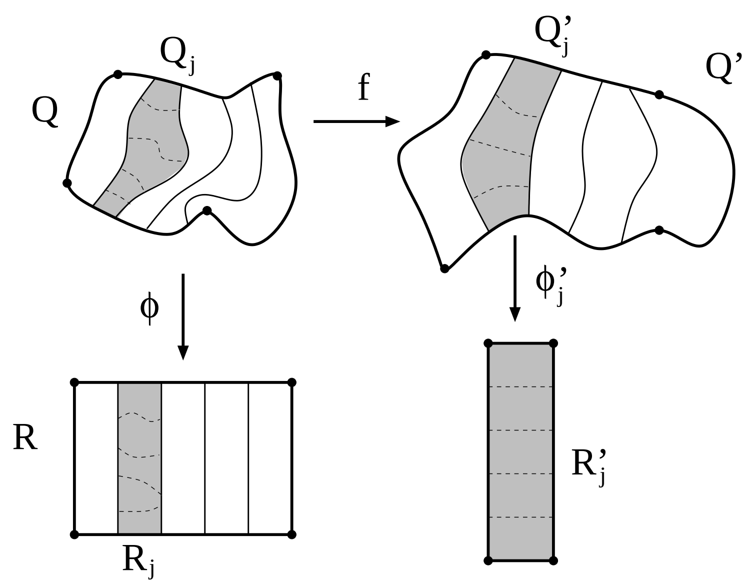
Map Ω_j conformally to a $1 \times m'_j$ rectangle R'_j. There is an absolute constant C so that every for every $y \in [0,1]$, there is a $t \in (0,1)$ with $|t-y| \leq Cm_j$ and so that the horizontal cross-cut of R'_j at height t maps via ϕ_j^{-1} to a Jordan arc of length $\leq C\rho$.



Thus we can divide R_j' by horizontal cross-cuts into rectangles $\{R_{ij}'\}$ of modulus $m_{ij}' \simeq 1$ so that the preimages of these rectangles under ϕ_j are quadrilaterals with two opposite sides of length $\leq C\rho$ and which can be connected inside the quadrilateral by a curve of length $\leq C\rho$.

Taking δ as above, choose ϵ as above corresponding to $\delta/4$ and choose ρ so that $3C\rho < \min(\epsilon, \delta/4)$.

Then all four sides of the quadrilateral Q'_{ij} have diameter $\leq \delta/4$ and hence Q'_{ij} has diameter less than δ and hence lies in a disk where f^{-1} is K-quasiconformal. Let m_{ij} be the modulus of corresponding preimage quadrilateral $Q_{ij} = f^{-1}(Q'_{ij})$.

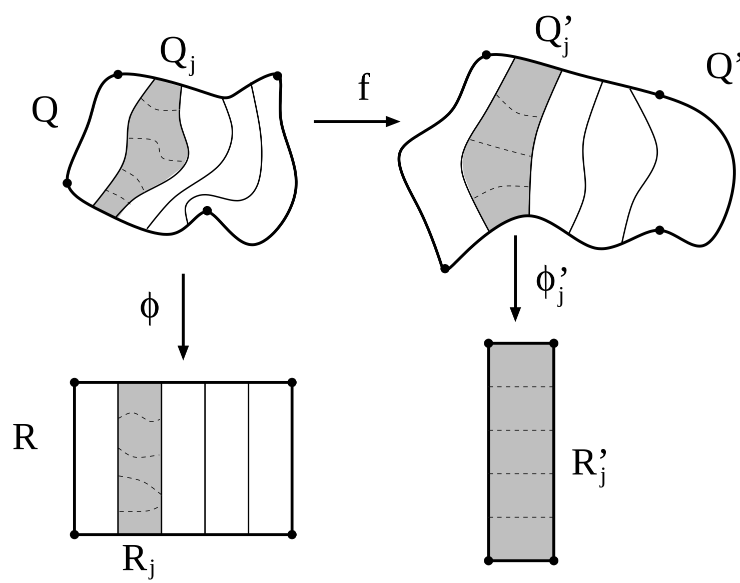


Then using the rules of extremal length

$$\frac{M}{m} \geq \sum_i \frac{1}{m_{ij}}, \qquad \frac{1}{m'_j} = \sum_i \frac{1}{m'_{ij}}, \qquad m' \geq \sum_j m'_j,$$

and by the definition of K-quasiconformal,

$$\frac{1}{K} \leq \frac{m_{ij}}{m'_{ij}} \leq K.$$



Hence

$$\frac{M}{m} \geq \sum_i \frac{1}{m_{ij}} \geq \frac{1}{K} \sum_i \frac{1}{m'_{ij}} = \frac{1}{Km'_j}$$

or

$$\frac{m}{M} \leq Km'_j$$

for every j. Thus

$$m = \sum_{j=1}^{M} \frac{m}{M} \leq \sum_j Km'_j \leq Km'.$$

Applying the same result to the inverse map shows f is K-quasiconformal.

□

If $K = 1$, then $m = m'$ the last line of the above proof becomes

$$m' = m \leq \sum_j \frac{m}{M} \leq \sum_j m'_j \leq m'.$$

so we deduce

$$\sum_j m'_j = m',$$

whereas in general, we only have $\sum_j m'_j \leq m'$.

We'll use this to deduce that a 1-QC map must be conformal.

Lemma 29. *Consider a* $1 \times m$ *rectangle* R *that is divided into two quadrilaterals* Q_1, Q_2 *of modulus* m_1 *and* m_2 *by a Jordan arc* γ *the connects the top and bottom edges of* R. *Then if* $m = m_1 + m_2$, *the curve* γ *is a vertical line segment.*

Proof. Let φ_1, φ_2 be the conformal maps of Q_1, Q_2 onto $1 \times m_1$ and $1 \times m_2$ rectangles R_1, R_2 respectively.

Set $\rho = |f_1'|$ on Q_1 and $\rho = |f_2'|$ in Q_2 and zero elsewhere. Then each horizontal line is cut by γ into pieces one of which connects the left vertical edge of R to γ, and another that connects γ to the right edge of R.

The images of these connect the vertical edges of R_1 and R_2 respectively. Thus the images have lengths at least m_1 and m_2 respectively. Thus the length of the image of the entire horizontal segment in Q is $\geq m_1 + m_2$.

If we integrate over all horizontal segments in Q, we see

$$\int_Q (\rho - 1) dx dy \geq m_1 + m_1 - m = 0.$$

Similarly,

$$\begin{aligned}\int_Q (\rho^2 - 1)dxdy &= \text{area}(f_1(Q_1) + \text{area}(f_2(Q_2)) - \text{area}(Q) \\ &= (m_1 + m_2) - m = 0.\end{aligned}$$

Thus

$$\begin{aligned}\int_Q (\rho - 1)^2 dxdy &= \int_Q (\rho^2 - 1) - 2(\rho - 1)dxdy \\ &= \int_Q (\rho^2 - 1) - 2\int_Q (\rho - 1)dxdy \\ &= 0 - 2\int_Q (\rho - 1)dxdy \\ &\leq 0.\end{aligned}$$

Since $(\rho - 1)^2 \geq 0$, this implies

$$\int_Q (\rho - 1)^2 dxdy = 0$$

and hence $\rho = 1$ a.e., f_1, f_2 are linear, and γ is a vertical segment. $\square$

Lemma 30. *If f is 1-QC on Ω, then it is conformal on Ω.*

Proof. If f is 1-quasiconformal in the proof of Theorem 28, then as noted before Lemma 29, we must have

$$\frac{M}{m} = \sum_i \frac{1}{m_{ij}}, \qquad \frac{1}{m'_j} = \sum_i \frac{1}{m'_{ij}}, \qquad m' = \sum_j m'_j,$$

Thus the map $\psi = \varphi' \circ f \circ \varphi^{-1}$ between identical rectangles must be the identity map. Thus $f = (\varphi')^{-1} \circ \varphi$ is a composition of conformal maps, hence conformal. $\square$

Small dilatation implies almost linear:

Corollary 31. *For any* $\delta > 0$ *and and any* $r > 0$ *there is an* $\epsilon > 0$ *so that the following holds. If* $f : \mathbb{C} \to \mathbb{C}$ *is* $(1+\epsilon)$*-quasiconformal and* f *fixes* 0 *and* 1*, then* $|z - f(z)| \leq \delta$ *for all* $|z| < r$.

Proof. If not, there is a sequence of $(1 + \frac{1}{n})$-quasiconformal maps that all fix 0 and 1 and points $z_n \in D(0, r)$ so that $|z_n - f_n(z_n)| > \delta$.

However, there is a subsequence that converges uniformly on compact subsets of the plane to a 1-quasiconformal map that fixes 0 and 1 and that moves some point by at least δ.

However a 1-quasiconformal map is conformal on $\mathbb{C}$, hence of form $az + b$ and since it fixes both 0 and 1, it is the identity and hence doesn't move any points, a contradiction. $\square$

Geometric definition implies analytic definition

Theorem 32. *If f is quasiconformal, then f is absolutely continuous on almost every line in any given direction.*

Proof. After a Euclidean similarity, we may consider horizontal lines in $Q = [0, 1]^2$. Define

$$A(y) = \text{area}(f([0, 1] \times [0, y])).$$

Then $A(0) = 0$, $A(1) = \text{area}(f(Q)) < \infty$ and A is increasing.

Thus A is continuous except on a countable set and has a finite derivative almost everywhere. Fix a value of y where both this things happen, and we will show that f is absolutely continuous on the horizontal line $L_y = [0, 1] \times \{y\}$.

The main idea is that if this failed, then modulus estimates relating length to area will force $A'(y) = \infty$.

Divide $R = [0,1] \times [y, y+\frac{1}{n}]$ into $m << n$ disjoint $\frac{1}{m} \times \frac{1}{n}$ sub-rectangles denoted $\{R_j\}$.

Let $R'_j = f(R_j)$ and let the "left", "right", and "bottom" edges of R'_j be the images under f of corresponding edges of R_j.

Let b_j be length of $f(L_y \cap \partial R_j)$, i.e., the length of the bottom edge of R'_j. This number might be finite or infinite.

Fix $\epsilon > 0$.

If m is fixed, as $n \to \infty$, any curve in $f(R_j)$ joining the opposite "vertical" sides limits on the bottom edge.

Hence the liminf of the lengths of such curves is at least the length of the bottom edge of R'_j.

If b_j is finite, by taking n large enough, we can insure that any curve in $f(R_j)$ than joins the images of the vertical sides of R_j has length $\geq b_j - \epsilon$.

If b_j is infinite, we can insure these curves all have length $\geq 1/\epsilon$.

By quasiconformality we know

$$M(R'_j) \geq M(R_j)/K = \frac{m}{Kn},$$

and using the metric $\rho = 1$ on R'_j, shows

$$M(R'_j) \leq \frac{\text{area}(R'_j)}{b_j^2}.$$

Thus by Cauchy-Schwarz,

$$\begin{aligned}
(\sum_{j=1}^{m} b_j)^2 &\leq (\sum_{j=1}^{m} b_j^2 m)(\sum_{j=1}^{m} \frac{1}{m}) \\
&\leq m \sum_{j=1}^{m} \frac{\text{area}(R'_j)}{M(R'_j)} \\
&\leq m \sum_{j=1}^{m} \frac{\text{area}(R'_j)}{m/Kn} \\
&\leq \sum_{j=1}^{m} \text{area}(R'_j) Kn \\
&\leq K \frac{A(y+\frac{1}{n}) - A(y)}{1/n} \\
&\to KA'(y).
\end{aligned}$$

Since we assumed $A'(y) < \infty$, $f(L_y)$ has finite length for our choice of y.

Given a compact set E of the horizontal segment L_y, suppose E is hit by N of the rectangles R_j and that m has been chosen so large that $N/m \leq 2m_1(E)$.

Then repeating the argument above, but only summing over the j's so that the bottom edges of R_j hit E,

$$\begin{aligned}
(\sum_j b_j)^2 &\leq (\sum_j b_j^2 m)(\sum_j \frac{1}{m}) \\
&\leq (m)(N/m) \sum_j \frac{\text{area}(R'_j)}{M(R'_j)} \\
&\leq N \sum_j \frac{\text{area}(R'_j)}{m/Kn} \\
&\leq \frac{N}{m} \sum_{j=1}^{m} \text{area}(R'_j) Kn \\
&\leq Km_1(E) \frac{A(y+\frac{1}{n}) - A(y)}{1/n} \\
&\to Km_1(E) A'(y).
\end{aligned}$$

Thus $m_1(E)$ small, implies $\sum b_j$ is small, and hence $f(E)$ has small 1-dimensional measure. Hence f is absolutely continuous on L_y, as desired. $\square$

We have shown that quasiconformal maps are absolutely continuous on almost every horizontal and almost every vertical line, so f_x, f_y exist almost everywhere and hence $f_z, f_{\overline{z}}, \mu_f = f_{\overline{z}}/f_z$ are all well defined almost everywhere.

We want f to be differentiable a.e., i.e.,

$$f(z) = f(w) + f_z(w)(z - w) + f_{\overline{z}}(w)(\overline{z} - \overline{w}) + o(|z - w|),$$

This requires more work.

A remarkable theorem of Gehring and Lehto says this is true.

Theorem 33. *If f is a homeomorphism of $\Omega \subset \mathbb{C}$ and has partials almost everywhere, then it is differentiable almost everywhere.*

Proof. By Egorov's theorem the limits

$$f_x(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h},$$

$$f_y(z) = \lim_{h\to 0} \frac{f(z+ih) - f(z)}{h},$$

are uniform and converge to a continuous functions on a compact set $E \subset \Omega$ so that $\operatorname{area}(\Omega \setminus E)$ is as small as we wish.

Almost every point of E is a point of density for the intersection of E with both the vertical and horizontal lines through z_0, so if suffices to prove differentiability at such points.

For simplicity we assume 0 is such a point.

The proof follows the usual case where we assume the partials are continuous. Here we replace continuous on a neighborhood of 0 with continuous on a set E such that E has density 1 at 0.

Because of the continuity and uniform convergence on E, for any $\epsilon > 0$ there is a $\delta > 0$ so that

$$|f_x(0) - f_x(z)|, \quad |f_y(0) - f_y(z)| < \epsilon,$$

if $z \in E \cap D(0, \delta)$-neighborhood of 0 and

$$|f_x(z) - \frac{f(z+h) - f(z)}{h}|, \quad |f_y(z) - \frac{f(z+ih) - f(z)}{h}| < \epsilon,$$

if $z \in E \cap D(0, \delta)$ and $h \in [-\delta, \delta]$.

Note that

$$\begin{aligned}
f(z) - f(0) - xf_x(0) - yf_y(0) &= [f(z) - f(x) - yf_y(0)] \\
&\quad +[f(x) - f(0) - xf_x(0)] \\
&\quad +[yf_y(x) - yf_y(0)] \\
&= I + II + III.
\end{aligned}$$

If $|z| < \delta$ and $x \in E$, then by the inequalities above, $I < \epsilon|y|$, $II < \epsilon|x|$ and $III < \epsilon y$, so the term on the far left is bounded by $3\epsilon|z|$, which proves differentiability if $x \in E$. A similar proof works if $iy \in E$.

Fix $\epsilon > 0$ and choose δ so small that if $0 < x < \delta$, then

$$E \cap (\frac{x}{1+\epsilon}, x) \neq \emptyset, \qquad E \cap (\frac{iy}{1+\epsilon}, iy) \neq \emptyset.$$

This is possible since 0 is point of density.

Thus if $0 < |x|, |y| \leq \delta/(1+\epsilon)$ can find points

$$x_1, x_2 \in E \cap (\frac{x}{1+\epsilon}, (1+\epsilon)x)$$

and

$$iy_1, iy_2 \in E \cap i(\frac{y}{1+\epsilon}, (1+\epsilon)y)$$

and so that

$$x + iy \in R = (x_1, x_2) \times (y_1, y_2).$$

Thus for each point $z = (x, y)$ on the boundary of R we have

$$|f(z) - f(0) - xf_x(0) - yf_y(0)| \leq \epsilon|z|$$

Since f is a homeomorphism (all we need is that it is continuous and open), $|f|$ takes its maximum on the boundary, so

$$\begin{aligned}
\sup_{z=x+iy\in R} & |f(z) - f(0) - xf_x(0) - yf_y(0)| \\
& \leq \sup_{w=u+iv\in\partial R} |f(w) - f(0) - xf_x(0) - yf_y(0)| \\
& \leq 3\epsilon|w| + \sup_{w=u+iv\in\partial R} |x-u||f_x(0)| + |y-v||f_y(0)| \\
& \leq 3\epsilon(1+\epsilon)|z| + \epsilon|f_x(0)||z| + \epsilon|f_y(0)||z|.
\end{aligned}$$

$\square$

Partials are in L^2

Lemma 34. *If f is K-quasiconformal then*
$$\int_Q J_f dxdy \leq \text{area}(f(Q)) \leq \pi \text{diam}(f(Q))^2,$$
for every square Q.

Proof. Second inequality is trivial.

We claim first holds for any map that is differentiable almost everywhere.

At any point x where f is differentiable we can choose a small square Q_x containing x such that

$$\mathrm{area}(f(Q')) \geq (1-\epsilon) J_f(x) \mathrm{area}(Q'),$$

and by the Lebesgue differentiation theorem, for almost every x we have

$$\int_{Q'} J_f dx dy \leq (1+\epsilon) J_f(x) \mathrm{area}(Q'),$$

for all small enough squares centered at x.

Combining these two estimates and using the Vitali covering theorem to extract a collection of disjoint squares $\{Q_j\}$ with centers x_j and with these properties that cover almost every point of Q, we get

$$\begin{aligned}
\int_Q J_f dxdy &\leq \sum_j \int_{Q_j} J_f dxdy \\
&\leq (1+\epsilon) J_f(x_j) \text{area}(Q_j) \\
&\leq \frac{1+\epsilon}{1-\epsilon} \text{area}(f(Q_j)) \\
&\leq \frac{1+\epsilon}{1-\epsilon} \text{area}(f(Q)).
\end{aligned}$$

Taking $\epsilon \searrow 0$, gives $\text{area}(f(E)) \geq \int_E J_f dxdy$. $\square$

Lemma 35. *If f is K-quasiconformal then*
$$\int_Q |f_z|^2 dxdy \leq \frac{\pi}{1-k^2} \mathrm{diam}(f(Q))^2,$$
for every square Q.

Proof. Follows from previous result since for K-QC maps
$$J_f = |f_z|^2 - |f_{\bar{z}}|^2 \geq |f_z|^2 - k^2|f_z|^2 = (1-k^2)|f_z|^2$$
or
$$|f_z|^2 \leq J_f/(1-k^2).$$
$\square$

Later we will show that partials are in L^p for some $p > 2$.

A weak version of the mapping theorem

Theorem 36. *Suppose* Γ *is a triangulation of the plane,* $0 \leq k < 1$ *and* $\mu(z)$ *is constant on the interior of each triangle with* $|\mu| < k$*. Then there is a homeomorphism* f *of the plane with* $\mu_f = \mu$*.*

Proof. For each triangle T let A be the affine map with dilatation $\mu(T)$ and T_μ be the image of T under A.

Form an Riemann surface by identifying the triangles T_μ along the same edges as in Γ. This defines a Riemann surface that is quasiconformally equivalent to the plane via the map $\Phi : R \to \mathbb{C}$ that is affine on each triangle.

By the uniformization theorem, there is also a conformal map $\Psi : R \to \mathbb{C}$.

Since R is simply connected and not-compact, it is conformally equivalent to either the disk or the plane and since it quasiconformally equivalent to the plane we know the extremal length of the path family connected an disk to ∞ on R is infinite, and hence it must be conformally equivalent to the plane.

Then $\Psi \circ \Phi^{-1} : \mathbb{C} \to \mathbb{C}$ is quasiconformal with dilatation μ. $\square$

Weak Measurable Riemann Mapping Theorem:

Theorem 37. *For any measurable* μ *on the plane with* $|\mu| \leq k < 1$*, there is a quasiconformal map* f *with* $f = \lim_n f_n$ *and* $\mu_n = \mu_{f_n}$ *where* $\{\mu_n\}$ *satisfy the conditions of Theorem 36,* $\mu_n \to \mu$ *almost everywhere, and* $\{f_n\}$ *are the corresponding maps.*

Proof. Take the standard equilateral triangulation of the plane and a series of refinements by recursively subdividing each triangle into four equilateral sub-triangles.

Define a piecewise constant dilatation on the nth triangulation by taking the average of μ on each triangle and let $\{f_n\}$ be the corresponding sequence of quasiconformal maps, normalized to fix $0, 1, \infty$.

Since these are all quasiconformal with the same bound, they form an equicontinuous family and we can extract a subsequence that converges uniformly on compact subsets of the plane.

The limit function f is also K-quasiconformal by Lemma 26.

If μ is continuous on a disk D, then the dilatations μ_n converge uniformly to μ on compact subsets of the plane. $\square$

Is μ the dilatation of f? Yes, but we have not proved this yet.

We need a few other facts.

Note that $f_n \to f$ uniformly does not imply dilatations converge pointwise.

We will show they do converge weakly and, assuming they converge pointwise to something, that something must be the dilatation of f.

We will need the following fact, to be proven later:

Lemma 38. *If f is quasiconformal, then f_z and $f_{\overline{z}}$ are locally in L^p.*

Due to Bojarskii 1955 in plane, Gehring 1973 in $\mathbb{R}^n$ (partials in L^p for some $p > n$). We will follow Gehring's proof, in case $n = 2$.

The Pompeiu formula for QC maps and weak convergence

Assuming the partials of a QC map are in L^p for some $p > 2$, we can prove a.e. convergence of the dilatations.

We start with some useful formulas and estimates.

Pompeiu formula:

Corollary 39. *If Ω has a piecewise C^1 boundary and f is quasiconformal on Ω, then*

$$f(w) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{z-w} dz - \frac{1}{\pi} \iint_{\Omega} \frac{f_{\overline{z}}}{z-w} dxdy. \tag{1}$$

Proof. For smooth functions this follows from Green's formula.

In general, smooth f using radial bump function to get sequence $\{f_n\}$ converging uniformly to f.

Note that $(f_n)_{\overline{z}} \to f_{\overline{z}}$ a.e. and $\sup(f_n)_{\overline{z}} \in L^p$ by Hardy-Littlewood maximal theorem.

(We are assuming for moment that $f_z, f_{\overline{z}} \in L^P$, some $p > 2$.)

Note that

$$f_n(w) \to f(w)$$

and

$$\int_{\partial\Omega} \frac{f_n(z)}{z-w} dz \to \int_{\partial\Omega} \frac{f(z)}{z-w} dz$$

by uniform convergence.

Finally, we want last term to converge:

$$\iint_\Omega \frac{(f_n)_{\overline{z}}}{z-w} dxdy \to \iint_\Omega \frac{f_{\overline{z}}}{z-w} dxdy$$

or

$$\iint_\Omega \frac{f_{\overline{z}} - (f_n)_{\overline{z}}}{z-w} dxdy \to 0.$$

Choose $p > 2$ so that $f_{\overline{z}} \in L^p$ and $q < 2$ the conjugate exponent. Since $1/z \in L^2$, Hölder's inequality gives

$$\iint_\Omega \frac{f_{\overline{z}} - (f_n)_{\overline{z}}}{z - w} dxdy$$
$$\leq \left(\iint_\Omega |f_{\overline{z}} - (f_n)_{\overline{z}}|^p dxdy \right)^{1/p} \left(\iint_\Omega |z - w|^{-q} dxdy \right)^{1/q}.$$

Second term is fixed, finite constant.

As noted earlier, the first term tends to zero a.e. and is in L^p by Hardy-Littlewood maximal theorem.

Thus by Lebesgue dominated convergence theorem integral tends to zero.

$\square$

Lemma 40. *Suppose* $\{g_n\} \in L^p(R, dxdy)$ *for some* $p > 2$ *and*

$$\lim_n \iint_R \frac{g_n(z)}{z-w} dxdy = 0$$

for all $w \in R$. *Then* $\lim_n \iint_R g_n dxdy = 0$.

It is "well known" that if μ is a finite measure whose Cauchy transform

$$\hat{\mu}(w) = \int \frac{d\mu(z)}{z-w} = 0$$

almost everywhere, then μ is the zero measure.

Proof. Fix rectangles $R'' \subset R' \subset R$, each compactly contained in the interior of the next.

The Cauchy integral formula on $\partial R'$ implies we can uniformly approximate the constant function 1 on R'' by a finite (Riemann) sum

$$s(z) = \sum \frac{a_k}{z - w_k} \approx \frac{1}{2\pi i} \int_{\partial R'} \frac{1 \cdot dz}{z - w}$$

with $w_k \in \partial R'$ and $\sum |a_k|$ is uniformly bounded.

Then

$$\iint_R g_n(z)dxdy = \iint_R g_n(z)s(z)dxdy + \iint_R g_n(z)(1-s(z))dxdy.$$

By assumption the first integral is small if n is large enough. Thus is suffices to fix n and show the following are small:

$$\iint_{R''} g_n(z)(1-s(z))dxdy + \iint_{R\backslash R''} g_n(z)(1-s(z))dxdy.$$

$$\iint_{R''} g_n(z)(1 - s(z))dxdy + \iint_{R\setminus R''} g_n(z)(1 - s(z))dxdy.$$

For a fixed n, the first integral can be made as close to zero as we wish by taking s close to 1 on R''.

$$\iint_{R''} g_n(z)(1-s(z))dxdy + \iint_{R\setminus R''} g_n(z)(1-s(z))dxdy.$$

The second integral can be made small by taking area$(R \setminus R'') \to 0$; this implies the L^p norm of g_n on $R \setminus R''$ tends to zero (hence so does its L^1 norm) whereas the L^q norm of s remains uniformly bounded (it is a convex combination of L^q functions with bounded norm).

Thus, by Hölder's inequality, we can make $\iint_R g_n dxdy$ as small as we wish if n is large, proving the lemma. $\square$

Weak convergence of derivatives:

Lemma 41. *If* $\{g_n\}$ *are* K*-quasiconformal maps that converge uniformly on compact sets to a quasiconformal map* g*, then for any rectangle* R*.*

$$\iint_R [(g_n)_z - g_z] dxdy \to 0,$$

$$\iint_R [(g_n)_{\overline{z}} - g_{\overline{z}}] dxdy \to 0.$$

and $(g_n)_z \to g_z$ *and* $(g_n)_{\overline{z}} \to g_{\overline{z}}$ *weakly (as measures).*

Proof. First consider the $\overline{z}$-derivative. Let $h_n = (g_n)_{\overline{z}} - g_{\overline{z}}$.

By the Pompeiu formula and the fact that $g_n \to g$ uniformly on R, we deduce that

$$\lim_{n\to\infty} \iint_R \frac{h_n(z)}{z-w} dxdy = 0$$

for any $w \in R$. Then by Lemma 40

$$\iint_R h_n dxdy \to 0.$$

To prove weak conference, take any continuous f of compact support and uniformly approximate it by a function $\tilde{f}$ that is constant on finite collection of rectangles. Then

$$\iint fh_n dxdy = \iint (f - \tilde{f})h_n dxdy + \iint \tilde{f}h_n dxdy.$$

The first integral is bounded by $\epsilon \iint |h_n| dxdy$.

This is small since $\|h_n\|_1 \leq C\|h_n\|_p$ is uniformly bounded on a large ball containing the support of both f and $\tilde{f}$.

$$\iint fh_n dxdy = \iint (f - \tilde{f})h_n dxdy + \iint \tilde{f}h_n dxdy.$$

The second integral tends to zero since is a finite linear combination of integrals of h_n over rectangles.

The result for z-derivatives follows from the same proof applied to the complex conjugates of g and $\{g_n\}$, using the fact that $(\bar{f})_{\overline{z}} = \overline{f_z}$. $\square$

Almost everywhere convergence of dilatations

Theorem 42. *Suppose $\{f_n\}$, f are all K-quasiconformal maps on the plane with dilatations $\{\mu_n\}$, μ_f respectively, that $f_n \to f$ uniformly on compact sets and that $\mu_n \to \mu$ pointwise almost everywhere. Then $\mu_f = \mu$ almost everywhere.*

Recall that $f_n \to f$ uniformly by itself does not imply μ_n has any a.e. pointwise limit.

By earlier arguments, this theorem implies MRMT. We prove it assuming partials are in L^p for some $p > 2$.

Proof. We restrict attention to some domain Ω with compact closure.

We know that $f_{\bar{z}} = \mu_f f_z$ almost everywhere and we know that f_z is non-zero almost everywhere, so it suffices to show that

$$f_{\bar{z}}(w) - \mu(w) f_z(w) = 0,$$

almost everywhere.

We claim the integral of $f_{\bar{z}}(w) - \mu(w) f_z(w)$ over any rectangle R is zero.

The theorem then follows by an application of the Lebesgue differentiation theorem: at almost every point an integrable function is the limit of its averages over rectangles shrinking down to that point.

We re-write this function as

$$
\begin{aligned}
f_{\bar{z}}(w) - \mu(w) f_z(w) &= [f_{\bar{z}}(w) - (f_n)_{\bar{z}}(w)] \\
&\qquad +[(f_n)_{\bar{z}}(w) - \mu_n \cdot (f_n)_z(w)] \\
&\qquad\qquad +[\mu_n(w)(f_n)_z(w) - \mu(w)(f_n)_z(w)] \\
&\qquad\qquad\quad +[\mu(w)(f_n)_z(w) - \mu(w) f_z] \\
&= I + II + III + IV.
\end{aligned}
$$

Term I has integral tending to 0 by Lemma 41 ($(f_n)_z \to f_z$ weakly).

Term II equals zero almost everywhere by the definition of f_n.

Term III: By Cauchy-Schwarz, the integral over R is bounded by

$$(\iint_R (\mu - \mu_n)^2 dxdy)^{1/2} (\iint_R |(f_n)_z|^2 dxdy)^{1/2}.$$

The first integrand tends to zero pointwise and is bounded above by 2 almost everywhere, so the integrals tend to zero by the Lebesgue dominated convergence theorem.

In the second integral, Lemma 35 implies

$$\left(\iint_R |(f_n)_x|^2 dxdy\right)^{1/2} \simeq \mathrm{diam}(f_n(R)),$$

Since $\{f_n\}$ converges uniformly on compact sets, this remains bounded.

Thus the product of the two terms tends to zero.

Hence Term III tends to zero.

Term IV: First, we argue as for Term I, but applied to $f_z = (\bar{f})_{\bar{z}}$. Using the fact that $(\bar{f})_{\overline{z}} = \overline{(f_z)}$, we can show that

$$\iint_R (f_z - (f_n)_z) dxdy \to 0$$

for every rectangle R.

However, what we want is

$$\iint_R \mu(f_z - (f_n)_z) dxdy \to 0$$

for every rectangle R.

Now approximate μ in the $L^q(R, dxdy)$ norm by a function ν that is constant on a finite collection of disjoint squares (such functions are dense in L^q). Then

$$\iint_R \nu \cdot (f_z - (f_n)_z) dxdy \to 0$$

for every rectangle where ν is constant. For such rectangles,

$$\begin{aligned}\int_n \iint_R \mu((f_z - (f_n)_z) dxdy &= \lim_n \iint_R (\mu - \nu)((f_z - (f_n)_z) dxdy \\ &\leq \lim_n \|\mu - \nu\|_q \|(f_z - (f_n)_z)\|_p.\end{aligned}$$

The first term is as small and the second is uniformly bounded, so the product is small. Thus the limit must be zero. □

MRMT is now proven, except for showing partials are in L^p, some $p > 2$.

Reverse Hölder estimates imply $f_z \in L^p$

We want to show $f_{\overline{z}}$ is in L^p for some $p > 2$.

Since $|f_{\overline{z}}| \leq k|f_z|$, enough to show $f_z \in L^p$ for some $p > 2$.

We already know $f_z \in L^2$, recall

Lemma 43. *If f is K-quasiconformal then*

$$\int_Q |f_z|^2 dxdy \le \frac{\pi}{1-k^2} \mathrm{diam}(f(Q))^2,$$

for every square Q.

This implies

$$\frac{1}{\mathrm{area}(Q)} \int_Q |f_z|^2 dxdy \le K \left(\frac{1}{\mathrm{area}(Q)} \int_Q |f_z| dxdy \right)^2$$

for every square Q. This is known as a reverse Hölder inequality.

The usual Hölder inequality is

$$\begin{aligned}\left(\int_Q |f_z| dxdy\right)^2 &\leq \int_Q |f_z|^2 dxdy \cdot \int_Q 1^2 dxdy \\ &= \int_Q |f_z|^2 dxdy \cdot \mathrm{area}(Q)\end{aligned}$$

or

$$\left(\frac{1}{\mathrm{area}(Q)} \int_Q |f_z| dxdy\right)^2 \leq \frac{1}{\mathrm{area}(Q)} \int_Q |f_z|^2 dxdy$$

Result of Gehring, 1973:

Theorem 44. *Let $p > 1$. If $v(x) \geq 0$ and $v \in L^p(Q, dxdy)$, and if the "reverse Hölder inequality"*

$$\frac{1}{\text{area}(Q)} \int_Q v^p dxdy \leq K \left(\frac{1}{\text{area}(Q)} \int_Q vdxdy \right)^p,$$

holds for all subsquares of a square Q_0, then there is an $r > p$ so that

$$\left(\frac{1}{\text{area}(Q_0)} \int_{Q_0} v^r dxdy \right)^{1/r} \leq C(K, p, r) \frac{1}{\text{area}(Q_0)} \int_{Q_0} vdxdy,$$

Proof requires several preliminary lemmas.

Lemma 45 (The Calderon-Zygmund lemma)**.** *Suppose Q is a square, $u \in L^1(Q, dxdy)$ and suppose*

$$\alpha > \frac{1}{\mathrm{area}(Q)} \int_Q |u| dxdy.$$

Then there is a countable collection of pairwise disjoint open dyadic subsquares of Q so that

$$\alpha \leq \frac{1}{\mathrm{area}(Q_j)} \int_{Q_j} |u| dxdy < 4\alpha, \tag{2}$$

$$|u| \leq \alpha \text{ almost everywhere on } Q \setminus \cup_j Q_j, \tag{3}$$

$$\sum \mathrm{area}(Q_j) \leq \frac{1}{\alpha} \int_Q |u| dxdy \tag{4}$$

Proof. We say a subsquare of Q is type 1 if the average of f over Q is $< \alpha$.

Define a collection of subsquares by iteratively dividing type 1 squares into four, equal sized disjoint subsquares, and stopping if the average is $\geq \alpha$.

If the average of u over a square is less than α then average over each of the four subsquares is $< 4\alpha$, so every stopped square has property (2).

Any point not in a stopped square is a limit of squares where the average of u is $< \alpha$, so by the Lebesgue differentiation theorem $u \leq \alpha$ at almost every such point. This is (3).

Finally, (4) follows because

$$\int_Q |u| dx dy \geq \sum_j \alpha \cdot \text{area}(Q_j).$$

$\square$

Lemma 46. *Suppose that $p > 1$, $v \geq 0$, $E_\lambda = \{z : v(z) > \lambda\}$, and*

$$\int_{E_\lambda} v^p dxdy \leq A\lambda^{p-1} \int_{E_\lambda} vdxdy,$$

for all $\lambda \geq 1$. Then there is $r > p$ and $C < \infty$ so that

$$(\int_Q v^r dxdy)^{1/r} \leq C(\int_Q v^p dxdy)^{1/p}.$$

Proof. This is basically just arithmetic with distribution functions. Note that it suffices to assume area$(Q) = 1$ and $\int_Q v^p dxdy = 1$. Then

$$
\begin{aligned}
\int_{E_1} v^r dxdy &= \int_{E_1} v^p v^{r-p} dxdy \\
&= (r-p)\int_{E_1} v^p (1 + \int_1^v \lambda^{r-p-1} d\lambda) dxdy \\
&= (r-p)\int_{E_1} v^p + (r-p)\int_1^\infty \lambda^{r-p-1} \int_{E_\lambda} v^p dxdyd\lambda \\
&\leq (r-p)\int_{E_1} v^p + A(r-p)\int_1^\infty \lambda^{r-2} \int_{E_\lambda} v dxdyd\lambda \\
&\leq (r-p)\int_{E_1} v^p + A(r-p)\int_{E_1} v(\int_0^v \lambda^{r-2} d\lambda) dxdy \\
&\leq (r-p)\int_{E_1} v^p + A\frac{r-p}{r-1}\int_{E_1} v^r dxdy \\
&\leq (r-p)\int_{E_1} v^p + \frac{1}{2}\int_{E_1} v^r dxdy
\end{aligned}
$$

where the last inequality holds if r is close enough to p (depending on A and p). Subtracting the last term of the last step from the first step gives

$$\int_{E_1} v^r dxdy \leq 2(r-p) \int_{E_1} v^p dxdy.$$

Off E_1 we have $v \leq 1$ so $v^r \leq v^p$ and hence

$$\int_Q v^r dxdy \leq (1 + 2(r-p)) \int_Q v^p dxdy.$$

Because of our normalizations, this proves the lemma. $\square$

Proof of "reverse Hölder" theorem.

Proof. We need only verify the hypothesis of Lemma 46. Fix λ and set $\beta = 2K\lambda$. We will split the integral

$$\int_{E_\lambda} v^p dxdy = \int_{E_\lambda \setminus E_\beta} v^p dxdy + \int_{E_\beta} v^p dxdy$$

into two pieces. The second piece is trivial to bound by the correct estimate because

$$\int_{E_\lambda \setminus E_\beta} v^p dxdy \leq \beta^{p-1} \int_{E_\lambda \setminus E_\beta} v dxdy \leq (2K\lambda)^{p-1} \int_{E_\lambda} v dxdy.$$

To bound the other piece of the integral, we use the Calderon-Zygmund lemma (Lemma 45) to find a sequence of disjoint squares $\{Q_j\}$ so that

$$\beta^p \leq \frac{1}{\text{area}(Q_j)} \int_{Q_j} v^p dxdy < 2\beta^p,$$

and $v \leq \beta$ a.e. off $\cup Q_j$. Thus $E_\beta \setminus \cup Q_j$ has measure zero and

$$\int_{E_\beta} v^p dxdy \leq \sum_j \int_{Q_j} v^p dxdy \leq 2\beta^p \sum \text{area}(Q_j).$$

We now make use of the reverse Hölder hypothesis to write

$$\beta^p \leq \frac{1}{\text{area}(Q_j)} \int_{Q_j} v^p dxdy \leq (\frac{K}{\text{area}(Q_j)} \int_{Q_j} vdx)^p$$

or

$$\beta \leq \frac{K}{\text{area}(Q_j)} \int_{Q_j} vdx$$

Hence (recall $\beta = 2K\lambda$):

$$\begin{aligned} \mathrm{area}(Q_j) &\le \frac{K}{\beta}\int_{Q_j} v\,dxdy \\ &\le \frac{K}{\beta}\left(\int_{Q_j\cap E_\lambda} v\,dxdy + \lambda\mathrm{area}(Q_j)\right) \\ &\le \frac{K}{\beta}\int_{Q_j\cap E_\lambda} v\,dxdy + \frac{1}{2}\mathrm{area}(Q_j). \end{aligned}$$

Solving for area(Q_j) gives

$$\mathrm{area}(Q_j) \le \frac{2K}{\beta}\int_{Q_j\cap E_\lambda} v\,dxdy \le \frac{1}{\lambda}\int_{Q_j\cap E_\lambda} v\,dxdy.$$

Thus by the defining property of the Q_j's,

$$\int_{E_\beta} v^p dxdy \le \sum_j \int_{Q_j} v^p dxdy \le 2\beta^p \sum_j \mathrm{area}(Q_j)$$

Using the estimate from the previous slide, this is less than

$$\le 2\beta^p \lambda^{-1} \sum_j \int_{Q_j \cap E_\lambda} vdx \le 2^{p+1} K^p \lambda^{p-1} \int_{E_\lambda} vdx.$$

Thus the hypothesis of Lemma 46 holds with

$$A = (2K)^{p-1} + 2^{p+1} K^p,$$

and we deduce that $v \in L^r(Q, dxdy)$ for some $r > p$. $\square$

This completes the proof of the MRMT.

Distortion estimates for QC maps

Theorem 47. *Suppose* $f : \mathbb{R}^n \to \mathbb{R}^n$ *is* K*=quasiconformal and* $f(0) = 0$*. Suppose also that*

$$I(r) = \frac{1}{\omega_n} \int_{|x|<r} \frac{L_f(x) - 1}{|x|^n} dx \to 0$$

as $r \to 0$*. Then* $|f(x)| \sim C|x|$ *as* $|x| \to 0$ *and*

$$\min_{|x|=1} |f(x)| e^{-I(1)} \leq C \leq \max_{|x|=1} |f(x)| e^{-I(1)}$$

Lemma 48. *Suppose* A, B *are disjoint, planar sets and*

$$\int_A \frac{dxdy}{|z-w|^2} \leq C < \infty,$$

for all $w \in B$*. If* φ *is a* K*-quasiconformal map that is conformal off* A*, then* φ *is* M*-bi-Lipschitz on* B *with* M *depending only on* C *and* K*, i.e., for all* $w, z \in B$*,*

$$0 < \frac{1}{M(C,K)} \leq \frac{|\varphi(z) - \varphi(w)|}{|z-w|} \leq M(C,K) < \infty.$$

"Speiser class Julia sets with dimension near one", with Simon Albrecht, Journal d'Analyse, vol 141, issue 1, 2020, pages 49–98.

We say a measurable set $E \subset \mathbb{C}$ is (ϵ, h)-thin if $\epsilon > 0$ and

$$\text{area}(E \cap D(z, 1)) \leq \epsilon h(|z|)$$

for all $z \in \mathbb{C}$, where $h : [0, \infty) \to [0, \pi]$ is a bounded, decreasing function, such that

$$\int_0^\infty h(r) r^n dr < \infty,$$

for every $n > 1$.

If $a > 0$, the function $h(r) = \exp(-ar)$ satisfies this condition, and this example suffices for many applications.

Recall that a quasiconformal map $F : \mathbb{C} \to \mathbb{C}$ is often normalized by post-composing by a conformal linear map in one of two ways. First, we can assume $F(0) = 0$ and $F(1) = 1$.

We call this the 2**-point normalization**.

Second, if the dilatation of F is supported on a bounded set, then F is conformal in a neighborhood of ∞ and then we can choose R large and post-compose with a linear conformal map so that

$$|F(z) - z| = O(\frac{1}{|z|}),$$

for $|z| > R/2$. We say that such an F is normalized at ∞.

This is also called the **hydrodynamical normalization** of F.

Theorem 49. *Suppose $F : \mathbb{C} \to \mathbb{C}$ is K-quasiconformal, and $E = \{z : \mu(z) \neq 0\}$ is bounded (so F is conformal near ∞) and F is normalized so*

$$|F(z) - z| \leq M/|z|,$$

near ∞. Assume E is (ϵ, h)-thin. Then for all $z \in \mathbb{C}$,

$$|F(z) - z| \leq \frac{\epsilon^{\beta}}{|z| + 1},$$

where $\beta > 0$ depends only on K and h. In particular, as $\epsilon \to 0$, F converges uniformly to the identity on the whole plane.

Corollary 50. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is K-quasiconformal, $f(0) = 0$, $f(1) = 1$, and $E = \{z : \mu(z) \neq 0\}$ is (ϵ, h)-thin. Then*

$$(1 - C\epsilon^{\beta})|z - w| - C\epsilon^{\beta} \leq |f(z) - f(w)| \leq (1 + C\epsilon^{\beta})|z - w| + C\epsilon^{\beta} \quad (5)$$

where C and β only depend on $\|\mu\|_\infty$ and h.

"Quasiconformal maps with thin dilatations", Publicacions Matematiques, vol 66, 2022, 715–727

Removable sets for QC maps

We say that a compact set $E \subset \mathbb{R}^2$ is conformally removable if any homeomorphism of the plane to itself that is conformal off E is conformal everywhere.

This is equivalent to being quasiconformally removable.

Set of finite (or even sigma-finite) length are removable.

Sets of positive length are non-removable (MRMT).

Quasicircles are removable.

A **Whitney decomposition** of an open set Ω consists of a collection of dyadic squares $\{Q_j\}$ contained in Ω so that

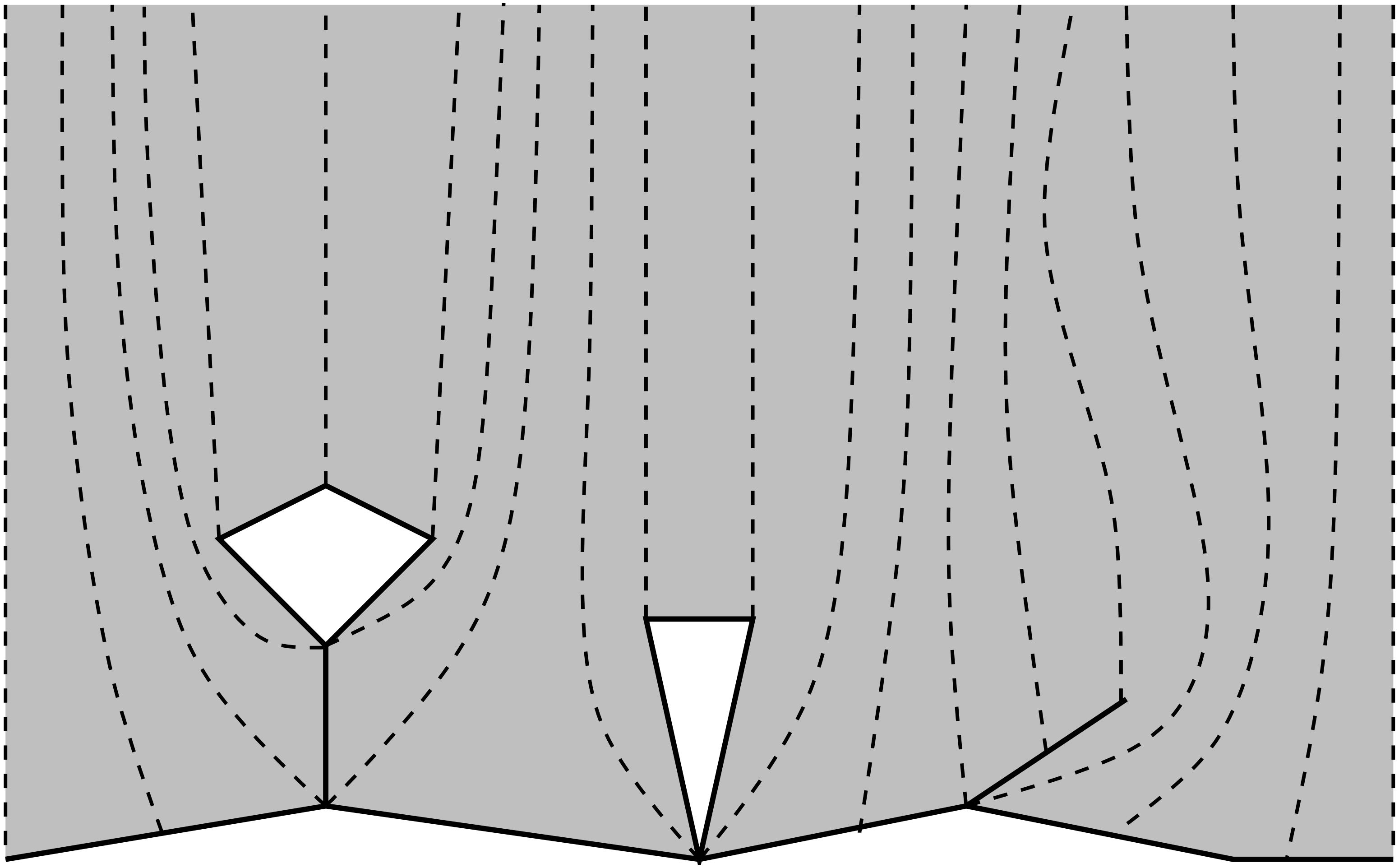
1. the interiors are disjoint,
2. the union of the closures is all of Ω,
3. for each Q_j, $\mathrm{diam}(Q_j) \simeq \mathrm{dist}(Q_j, \partial\Omega)$.

For existence, take the set of maximal dyadic squares Q so that

$$\mathrm{diam}(Q) \leq \frac{1}{4}\mathrm{dist}(Q, \partial\Omega),$$

(maximal = the parent square fails this condition).

Suppose K is compact, $\delta > 0$ and for each $x \in K$ let γ_x be a Jordan arc in $\Omega = \mathbb{C} \setminus K$ that connects x to $\Omega_\delta = \{z \in \Omega : \operatorname{dist}(z, K) \geq \delta\}$. For a single x, γ_x may consist of several arcs that connect x to Ω_δ.

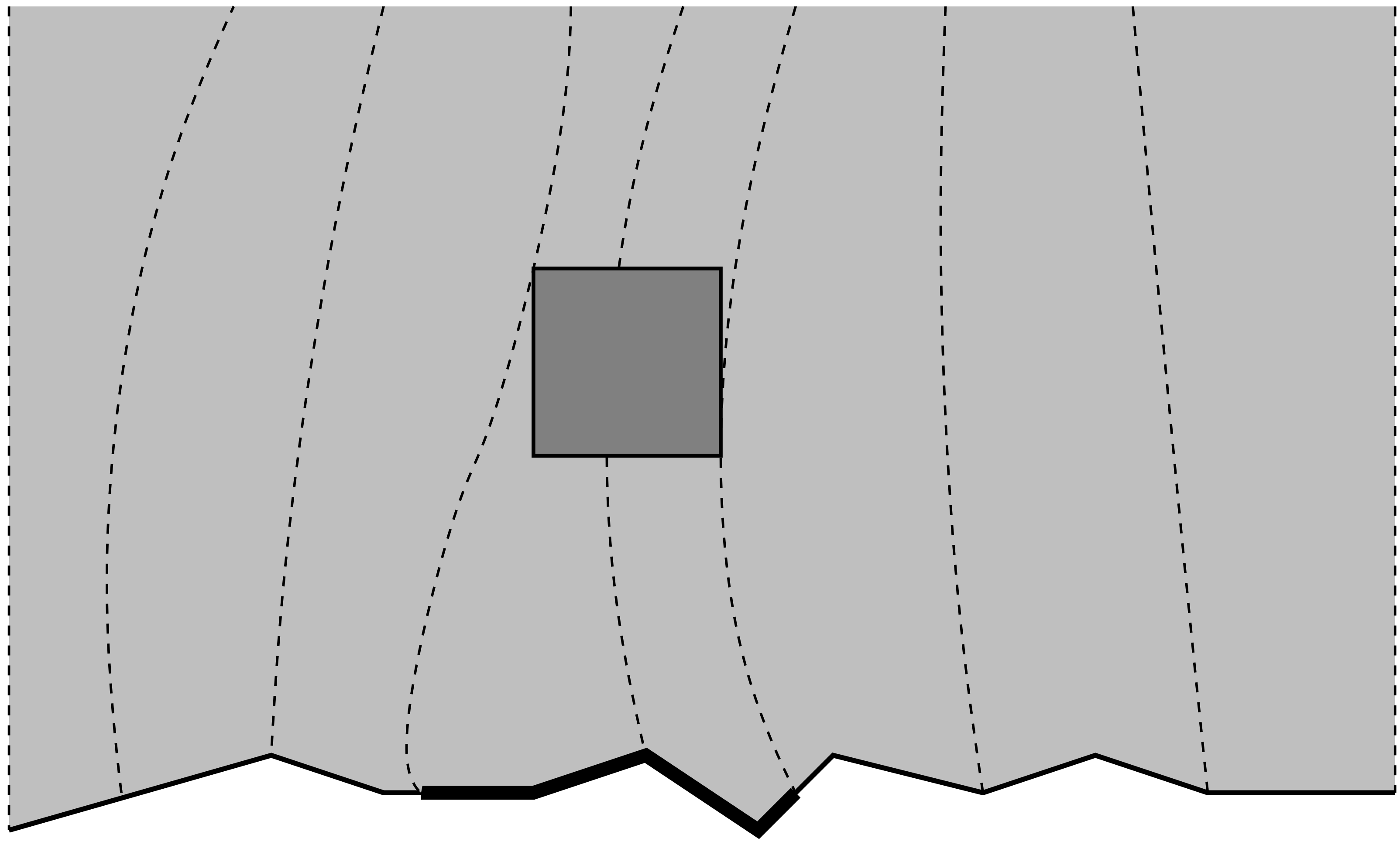


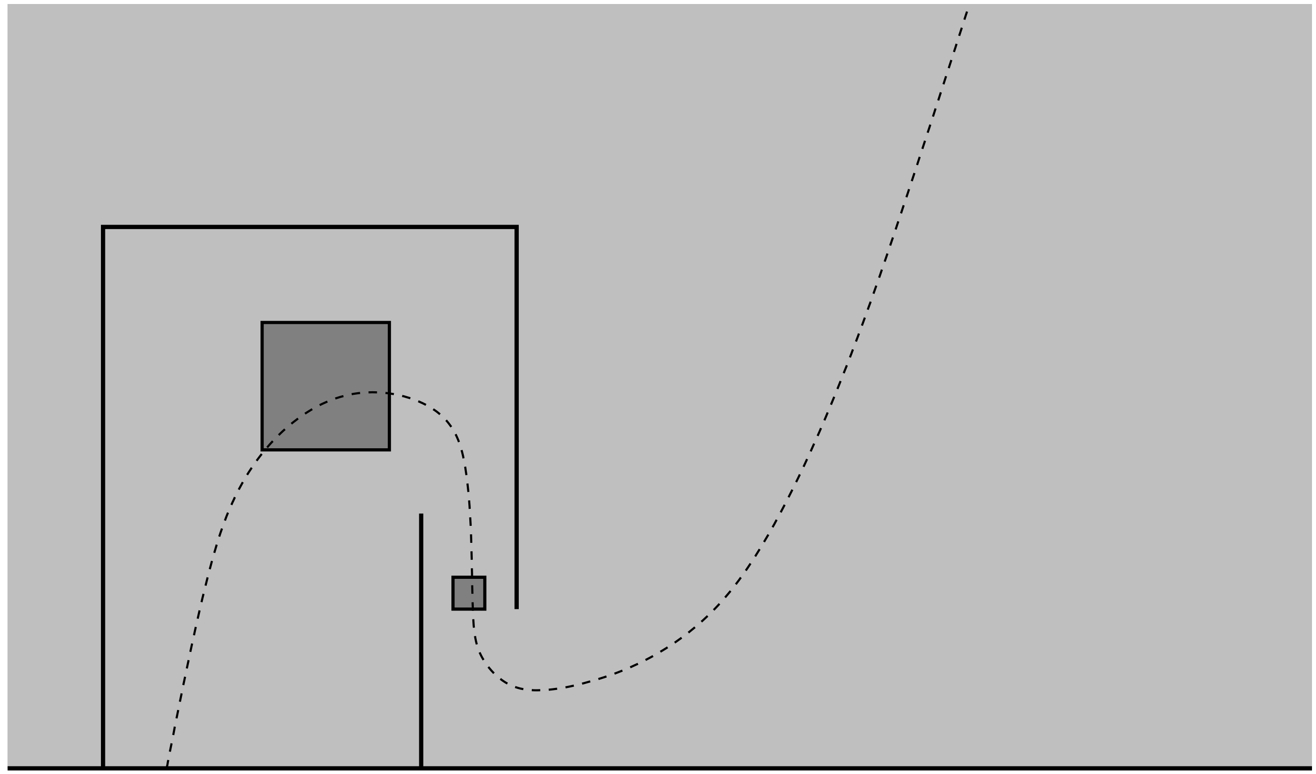
For each Whitney square $Q \subset \Omega$, let

$$S(Q) = \{x \in K : \gamma_x \cap Q \neq \emptyset\}.$$

This is called the "shadow" of Q on K.

The name comes from the special case when K is connected and does not separate the plane and γ_x is a hyperbolic geodesic connecting x to ∞. If we think of ∞ as the "sun" and the geodesics as light rays, then $S(Q)$ is the part of K that blocked from ∞ by Q, i.e., it is Q's shadow.





The paths connecting a Whitney square to its shadow can sometimes hit larger Whitney squares. However this path will hit a largest square, and there after only hit smaller squares.

The immediate shadow $I(Q) \subset S(Q)$ is the closure of all $x \in S(Q)$ so that Q is the first Whitney square of that size hit by γ_x as we traverse it from x to Ω_δ.

Given $x \in I(Q)$, we let $\mathcal{I}(x, Q)$ to be all the dyadic squares for Ω that are hit by γ_x between x and Q, i.e., this is an infinite chain of Whitney squares that starts at Q and accumulates on x and has Q as its unique largest square.

We will assume three things about shadows:

1. $I(Q)$ is closed.
2. $\lim_{n\to\infty} \sum_{Q\in\mathcal{D}_n(\Omega)} \operatorname{diam}(I(Q))^2 = 0$ where the sum is over all Whitney squares for Ω of side length 2^{-n}.
3. $\operatorname{dist}(I(q), Q) \to 0$ as $\operatorname{diam}(Q) \to 0$,

These will hold in most situations we are interested in. For example, if Ω is simply connected and we take γ_x to be arcs of hyperbolic geodesics connecting some base point $z_0 \in \Omega$ to x, then (2) always holds, (3) holds if $\partial\Omega$ is locally connected, and (1) holds if Ω is a John domain.

An open, connected set Ω in $\mathbb{R}^2$ is called a **John domain** if any two points $a, b \in \Omega$ can be connected by a path γ in Ω with the property that $\operatorname{dist}(z, \partial\Omega) \gtrsim \min(|z-a|, |z-b|)$.

Lemma 51. *Suppose Q is a square, $\lambda > 1$ and f is K-quasiconformal on λQ. Then*

$$\operatorname{area}(f(Q)) \geq \epsilon \operatorname{diam}(f(Q))^2,$$

where $\epsilon > 0$ depends only on λ and K.

Proof. By rescaling by conformal linear maps we may assume the square Q is $[-2,2] \times [-2,2]$ and the map f fixes 0 and 1. Choose $x \in \partial Q$ and connect x to 0 and connect 1 to $\lambda \partial Q$ by disjoint curves γ_0, γ_1 so that the annular region $\lambda Q \setminus (\gamma_0 \cup \gamma_1)$ has modulus $\simeq 1$ with a constant that depends on λ (and decreases as λ increases.

The image of this annular region has modulus bounded away from 0 and ∞ and this implies $f(\gamma_0)$ is bounded in terms of K (otherwise, as in the proof of Lemma 10 we could define a metric $\rho(z) = 1/|z|$ on $1 < |z| < R$ and show that the path family separating $f(\gamma_0)$ from $f(\gamma_1)$ has very small modulus). Thus $\mathrm{diam}(f(Q))$ is bounded in terms of K alone.

Now consider the modulus of $A = Q \setminus [0,1]$. Again this is a fixed number $\simeq 1$, so the modulus of $f(A)$ is bounded away from zero. But every curve surrounding $f([0,1])$ has length at least 2, so the metric $\rho = 1/2$ is admissible, so

$$\text{mod }(f(A)) \leq \frac{1}{4}\text{area}(f(Q)).$$

Since the left hand side is bounded away from zero depending only on K, so is right hand side. $\square$

Theorem 52. *Suppose* Ω *has a Whitney decomposition so that the corresponding shadow sets satisfy conditions (1)-(3) above. Suppose that* f *is a homeomorphism of the plane that is* K*-quasiconformal on each component of* $\mathbb{R}^2 \setminus \partial\Omega$ *and that there is an* $M < \infty$ *so that*

$$\operatorname{dist}(f(Q_j), f(Q_{j+1})) \leq M \max(\operatorname{diam}(Q_j), \operatorname{diam}(Q_{j+1})), \tag{6}$$

whenever Q_j, Q_{j+1} *are consecutive squares in the chain associated to some* $x \in \partial\Omega$. *Then* f *is a* C*-quasiconformal map on the whole plane where* C *depends only on* K *and* M.

If the chain associated to each $x \in \partial\Omega$ consists of adjacent squares (i.e., Q_j touches Q_{j+1}, then the same is true for their images under f, so condition (6) is automatically satisfied. Thus we obtain:

Corollary 53. *Suppose Ω has a Whitney decomposition so that the corresponding shadow sets satisfy conditions (1)-(3) above and all the Whitney chains are connected. The $\partial\Omega$ is removable to quasiconformal homeomorphisms, i.e., any homeomorphism of the plane that is K-QC off $\partial\Omega$ is quasiconformal on the whole plane.*

Proof of Theorem 52. Suppose that W is any bounded quadrilateral in the plane, say of modulus m and that $W' = F(W)$ has modulus m'. We want to show that $m' \leq Cm$ where $C < \infty$ depends only on K and M as in the statement of the theorem. We will do this by mimicking the proof of Theorem 8, that showed that any piecewise differentiable map with bounded dilatation was quasiconformal (in the geometric sense).

Let $\varphi : W \to R = [0, m] \times [0, 1]$ and $\psi : W' \to [0, m'] \times [0, 1]$ be conformal maps of the quadrilaterals W, W' to rectangles R, R' of the same modulus. Let $X = \varphi(\partial\Omega \cap W) \subset R$.

The main difficulty with the proof is that we are going to consider three different Whitney decompositions: one for W, one for Ω and one for $U = R \setminus X$. To try to differentiate the different Whitney cubes we we let

$\{W_j\}$ denote a Whitney decomposition for W,

$\{Q_j\}$ a Whitney decomposition for Ω and

$\{U_j\}$ a Whitney decomposition for U.

Fix some $\epsilon > 0$.

Fix a Whitney cube W_j for W. We assume the decomposition is chosen so that $2W_j \subset W$.

Suppose $\delta > 0$ is so small (depending on our choice of W_j) that the following two conditions all hold:

1. If Q_k is a Whitney square for Ω with diameter less than δ and the shadow $I(Q_k)$ hits W_j, then $I(Q_k) \subset 2W_j$ and the entire Whitney chain connecting any point $x \in I(Q_k)$ to Q_k is contained in $2W_j$. This is possible by condition (3) on shadow sets.

2. Let $\mathcal{S}(W_j)$ denote the collections of all Whitney squares Q_k for Ω so that $\mathrm{diam}(Q_k) \leq \delta$ and $I(Q_k)) \cap W_j \neq \emptyset$. Then
$$\sum_{Q_k \in \mathcal{S}(W_j)} \mathrm{diam}(I(Q_k))^2 \leq \epsilon \mathrm{area}(W_j).$$
This holds for small enough δ, because by condition (2) on shadows, this sum over all Whitney squares for Ω is finite, so removing all the squares bigger than δ gives a sum that tends to 0 as δ tends to zero. Thus we can make is less than $\epsilon \cdot \mathrm{area}(W_j)$ by taking δ small enough (depending on W_j).

Let $\mathcal{S} = \cup_j \mathcal{S}(W_j)$ be the collection of all shadow sets of all Whitney squares for Ω that are in some $\mathcal{S}(W_j)$ for some Whitney square of W.

Claim: $\partial\Omega \cap W_j$ is covered by a finite number of the shadow sets $I(Q_k)$ with $Q_k \in \mathcal{S}(W_j)$.

Proof of Claim. Each point $x \in \partial\Omega \cap W_j$ is associated to a Whitney chain that contains a square with diameter comparable to δ. There are only finitely many such squares, so their shadows form a finite collection that covers $\partial\Omega \cap W_j$. $\square$

Suppose $L = [a+iy, b+iy]$ is a horizontal segment, compactly contained in the interior of R at height y. We wish to show that

$$\int_0^1 |g(b+iy) - g(a+iy)|dy \leq Cm,$$

where C depends only on K and M. If we can do this, then by letting $a \to 0$ and $b \to m$ we get

$$m' \leq \lim_{a \to 0, b \to m} |g(b+iy) - g(a+iy)|,$$

and hence

$$m' \leq \lim_{a \to 0, b \to m} \int_0^1 |g(b+iy) - g(a+iy)|dy \leq Cm,$$

which is the desired inequality.

Since L is compactly contained in the interior of R and X is relatively closed in the interior of R, $L \cap X$ is compact.

Thus $\varphi^{-1}(L \cap X)$ is a compact set of W, hence covered by finitely many whitney squares for W and hence is covered by finitely many shadows sets in $\mathcal{S}$.

Let $\mathcal{X}$ be the image of the elements of $\mathcal{S}$ under φ. Then $L \cap X$ is covered by finitely many elements of $\mathcal{X}$, say $X_1, \ldots X_n$.

For $k = 1, \ldots, n$, let $Y_k = [a_k, b_k]$ be the smallest closed interval in L that contains X_n (this is the convex hull of X_k, the interval with the same leftmost and rightmost point as X_k).

Then $Y_1, \ldots, Y_n$ also cover $L \cap X$ and we can extract a subcover with the property that $Y_j \cap Y_k \neq \emptyset$ implies $|j - k| \leq 1$.

Since the points a_k, b_k are both in the same set X_k, the preimage points $\varphi^{-1}(a_k), \varphi^{-1}(b_k)$ are both in the same element of $\mathcal{S}$.

Thus they are both in the shadow set of some Whitney square for Ω and are associated to a two sided chain of distinct Whitney squares $\{Q_m\}_{-\infty}^{\infty}$ of Whitney squares for Ω.

If two chains arising in this way, say from Y_k and Y_m with $m > k$, have a Whitney square in common, then we can combine the chains to form a chain connecting a_k to b_m consisting of distinct squares.

After doing this for all intersections, we end up with a finite collection of closed intervals Z_k in L which covers the same set as the union of the Y_k's and such that the two endpoints of each Z_k correspond to a two-sided Whitney chain in Ω and that different intervals use different Whitney squares (no overlapping chains).

Moreover, if Z_k has endpoints c_k, d_k and the corresponding chain is $\{Q_n\}$, then

$$|g(c_k) - g)d_k)| \leq (M+1) \sum_n \mathrm{diam}(\psi(f(Q_n))).$$

The set $V = L \backslash \cup_k Z_k$ consists of finitely many open intervals in $U = R \backslash X$ with their endpoints in X.

We break V into countable many sub-intervals by intersecting it with the Whitney squares for U (without loss of generality, we can assume the endpoints of L occur on the boundary of a Whitney square for U).

On each Whitney square U_k for U we define the constant function

$$Dg = \frac{\operatorname{diam}(g(U_k))}{\operatorname{diam}(U_k)}.$$

Then if $L_j = L \cap U_j$,

$$\int_{L_j} Dg dx = \operatorname{diam}(g(U_j))/\sqrt{2}.$$

Thus

$$\int_{L \setminus Z_L} Dg dx \simeq \sum_j \operatorname{diam}(g(U_j)),$$

where the sum is over Whitney squares for U that hit L.

Thus

$$|g(b+iy) - g(a+iy)| \lesssim \int_{L\cap U} Dg dx + \sum_n \operatorname{diam}(\psi(f(Q_n))).$$

Now integrate in y to get

$$\int_0^1 |g(b+iy) - g(a+iy)| dy \lesssim \iint_U Dg dx + \sum_n \operatorname{diam}(\psi(f(Q_n)))\mu_n,$$

where μ_n is the Lebesgue measure in $[0,1]$ of the set of lines L_y that use the Whitney square Q_n in at least one of the two-sided chains associated to a interval $Z \subset L_y$.

The measure of this set is no more than its diameter, which is no more than the diameter of $X_n = \varphi(I(Q_n))$. Thus

$$\begin{aligned}\int_0^1 &|g(b+iy) - g(a+iy)|dy \\ &\lesssim \iint_U Dg dx dy + \sum_n \mathrm{diam}(\psi(f(Q_n)))\mathrm{diam}(X_n),\end{aligned}$$

Estimate each term using the Cauchy-Schwarz inequality. First,

$$
\begin{aligned}
&\sum_n \mathrm{diam}(\psi(f(Q_n)))\mathrm{diam}(X_n) \\
&\quad \le (\sum_n \mathrm{diam}(\psi(f(Q_n)))^2)^{1/2}(\sum_n \mathrm{diam}(X_n)^2)^{1/2} \\
&\quad \le A(\sum_n \mathrm{area}(\psi(f(Q_n))))^{1/2}(\sum_{W_k}\sum_{Q_n\in\mathcal{S}(W_k)}[\frac{\mathrm{diam}(\varphi(W_k))}{\mathrm{diam}(W_k)}\mathrm{diam}(I(Q_n))]^2) \\
&\quad \le A(\sum_n \mathrm{area}(\psi(f(Q_n))))^{1/2}(\sum_{W_k}\sum_{Q_n\in\mathcal{S}(W_k)}[\frac{\mathrm{diam}(\varphi(W_k))}{\mathrm{diam}(W_k)}\epsilon\mathrm{area}(W_k))]^2)^{1} \\
&\quad \le A(\sum_n \mathrm{area}(R')^{1/2}\cdot\epsilon\cdot(\mathrm{area}(R)^{1/2}.
\end{aligned}
$$

where A just depends on the distortion estimate for conformal maps and ϵ is as small as we wish. Thus this term is small.

The other term is also bounded by Cauchy-Schwarz

$$\begin{aligned}
\iint_U Dg dx &= \sum_k \iint_{U_k} Dg dx dy \\
&\le (\sum_k \iint_{U_k} Dg^2 dx dy)^{1/2} (\sum_k \iint_{U_k} dx dy)^{1/2} \\
&\le (\sum_k (\mathrm{diam}(g(U_k))^2)^{1/2} (\mathrm{area}(R))^{1/2} \\
&\le A(\sum_k (\mathrm{area}(g(U_k)))^{1/2} (\mathrm{area}(R))^{1/2} \\
&\le A(\mathrm{area}(R')^{1/2} (\mathrm{area}(R))^{1/2} \\
&\le A\sqrt{m'm}.
\end{aligned}$$

Thus

$$\int_0^1 |g(b+iy) - g(a+iy)|dy \lesssim \sqrt{m'm} + O(\epsilon),$$

Taking $\epsilon \to$ and squaring gives

$$\left(\int_0^1 |g(b+iy) - g(a+iy)|dy\right)^2 \lesssim m'm.$$

Since we already know that

$$\int_0^1 |g(b+iy) - g(a+iy)|dy \gtrsim m'$$

this implies

$$\int_0^1 |g(b+iy) - g(a+iy)|dy \lesssim m.$$

as desired. $\square$

Lemma 54. *The Riemann map φ from the unit disk to a bounded John domain satisfies*

$$\operatorname{diam}(\varphi(I(Q))) \leq C \operatorname{diam}(\varphi(Q)),$$

$$\operatorname{dist}(\varphi(Q), \varphi(I(Q))) \leq C \operatorname{diam}(\varphi(Q)),$$

for some $C < \infty$, and any Whitney square Q and its shadow $I(Q)$.

Proof. The second inequality follows directly from Lemma 19 by considering the path family of radial lines connecting Q to I.

To prove the first, consider the Whitney-Carleson boxes Q_1 and Q_2 that are adjacent to Q and of the same size. By Lemma 19 each is connected to its shadow by a radial segment whose image under f has length comparable to $\operatorname{diam}(f(Q))$.

Thus there is a geodesic crosscut γ of the disk that passes through Q and whose image has length comparable to $\operatorname{diam}(f(Q))$.

Now suppose x is in the shadow of Q. Any curve connecting 0 to x crosses γ, so any curve Γ connecting $f(0)$ and $f(x)$ crosses $f(\gamma)$ and hence contains a point $z \in f(\gamma) \cap \Gamma$ that is at most distance $O(\text{diam}(f(Q))$ from $\partial\Omega$.

Thus by the definition of John domain, either

$$\text{dist}(f(0), z) = O(\text{diam}(f(Q))),$$

or

$$\text{dist}(f(x), z) = O(\text{diam}(f(Q))).$$

In a bounded domain, the first can only happen for finitely many Qs; for the remainder, the second must hold and hence $f(I(Q))$ is contained in a $O(\text{diam}(f(Q))$ neighborhood of $f(Q)$. $\square$

Corollary 55. *Boundaries of John domains are removable.*

Proof. The conclusions of the Lemma 54 imply (1)-(3) in Theorem 52. ☐