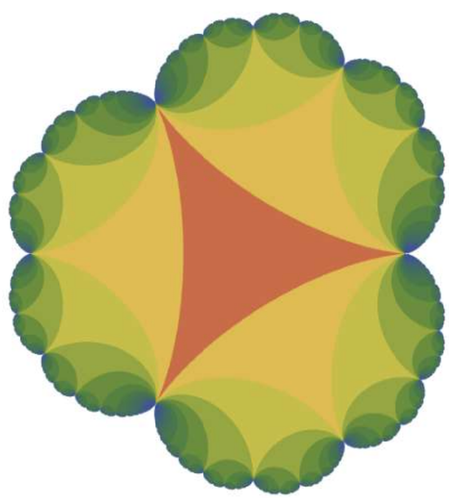
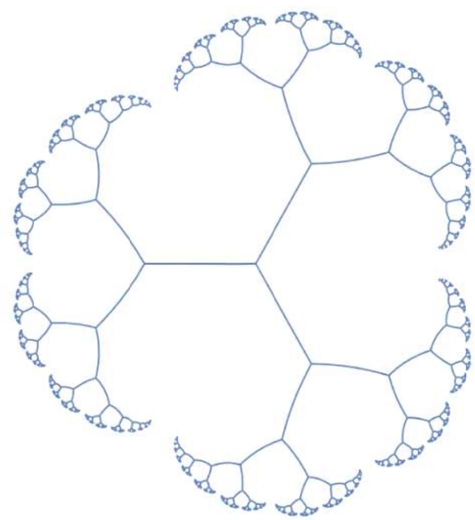
The infinite trivalent tree and the developed deltoid

Steffen Rohde, UW

Helsinki, August 14, 2023





joint with Oleg Ivrii, Peter Lin and Emanuel Sygal

Outline :

1) Deltoid

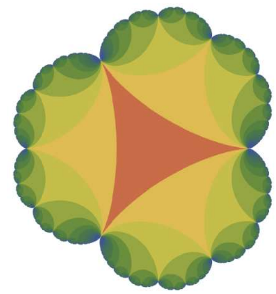
2) Trees

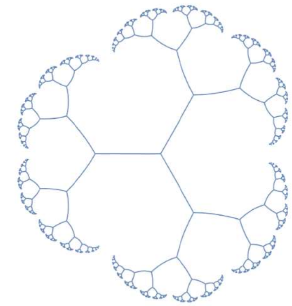
3) Theorem and remarks

4) Teleportation and obstacles

5) Proof strategy

6) Outlook

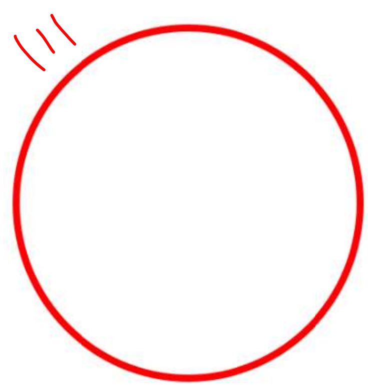




DYNAMICS OF SCHWARZ REFLECTIONS: THE MATING PHENOMENA

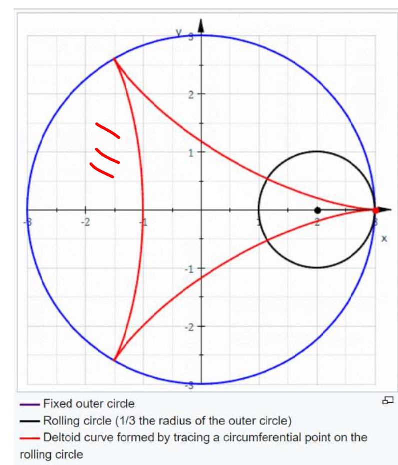
SEUNG-YEOP LEE, MIKHAIL LYUBICH, NIKOLAI G. MAKAROV, AND SABYASACHI MUKHERJEE

Deltoid curve :



$$\xrightarrow{\varphi}$$

$$2z + \frac{1}{z^2}$$



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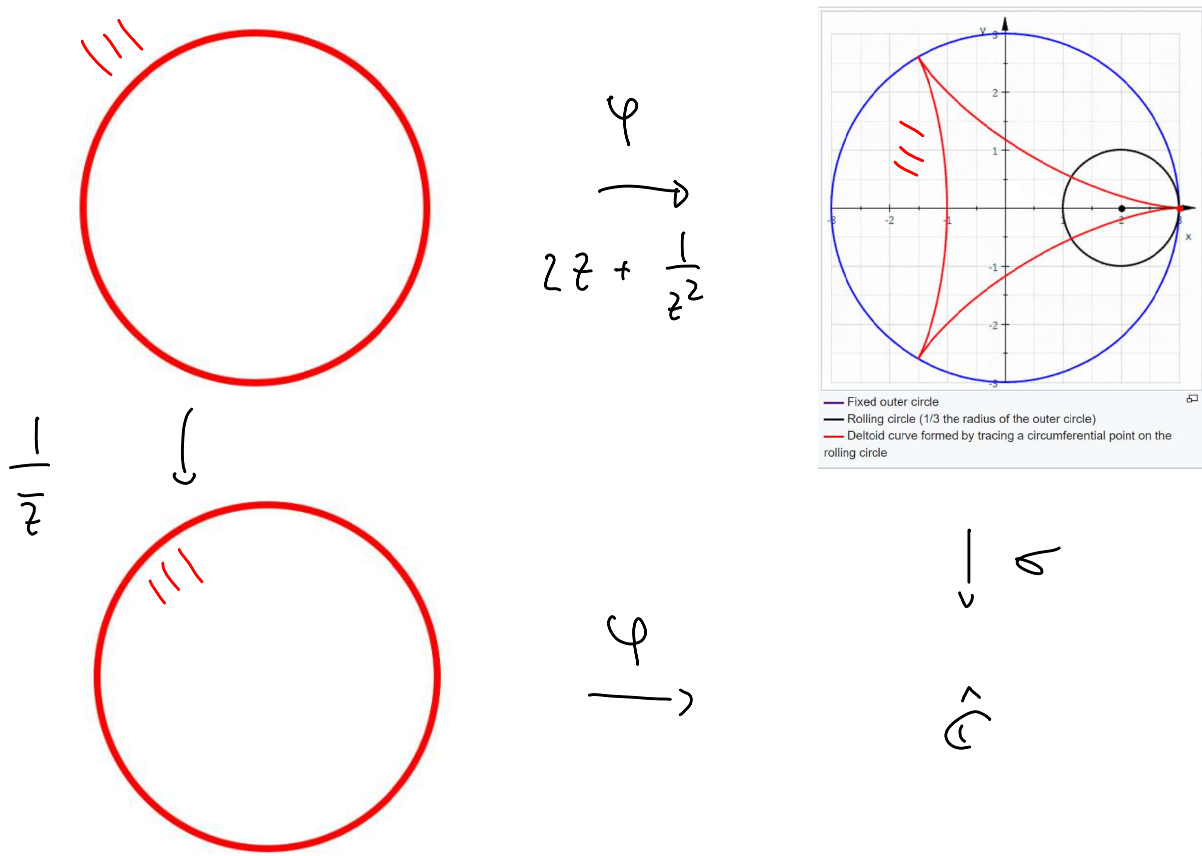
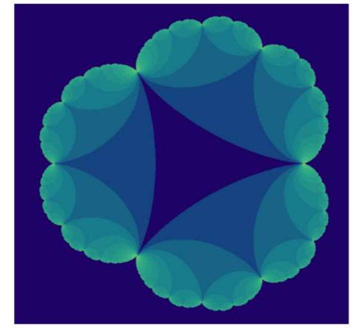


Developed deltoid $\Omega := \bigcup_{n \geq 0} \sigma^{-n}(\quad)$



Theorem 1.2. (i) *The boundary of the developed deltoid $\partial\Omega$ is the unique Jordan curve that realizes the mating of the ideal triangle group and $z \to \overline{z}^2$.*

(ii) *The developed deltoid Ω is a John domain. In particular, $\partial\Omega$ is conformally removable.*

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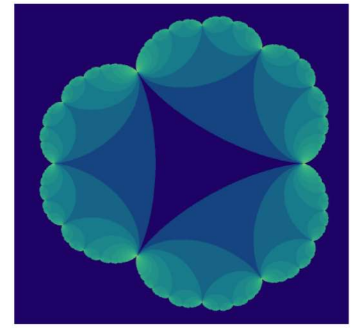

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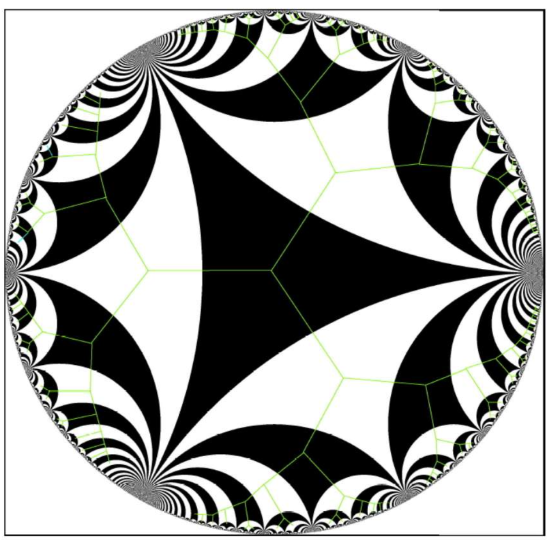

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Find "conformally natural" (eg QS) representations of 2d objects:

Surfaces: Bonk-Kleiner, Quasisymmetric parametrizations of two-dimensional metric spheres, Invent. Math. 2002

Carpets: Bonk, Uniformization of Sierpiński carpets in the plane, Invent. Math. 2011

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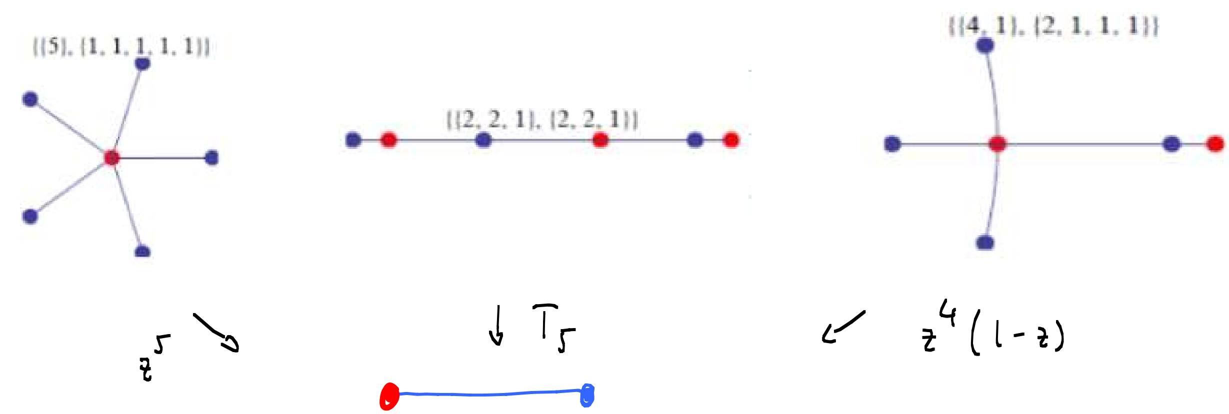
Bonk-Tran, The continuum self-similar tree, Progr. Probab. 2021

Bonk-Meyer, Uniformly branching trees, Trans. AMS 2022

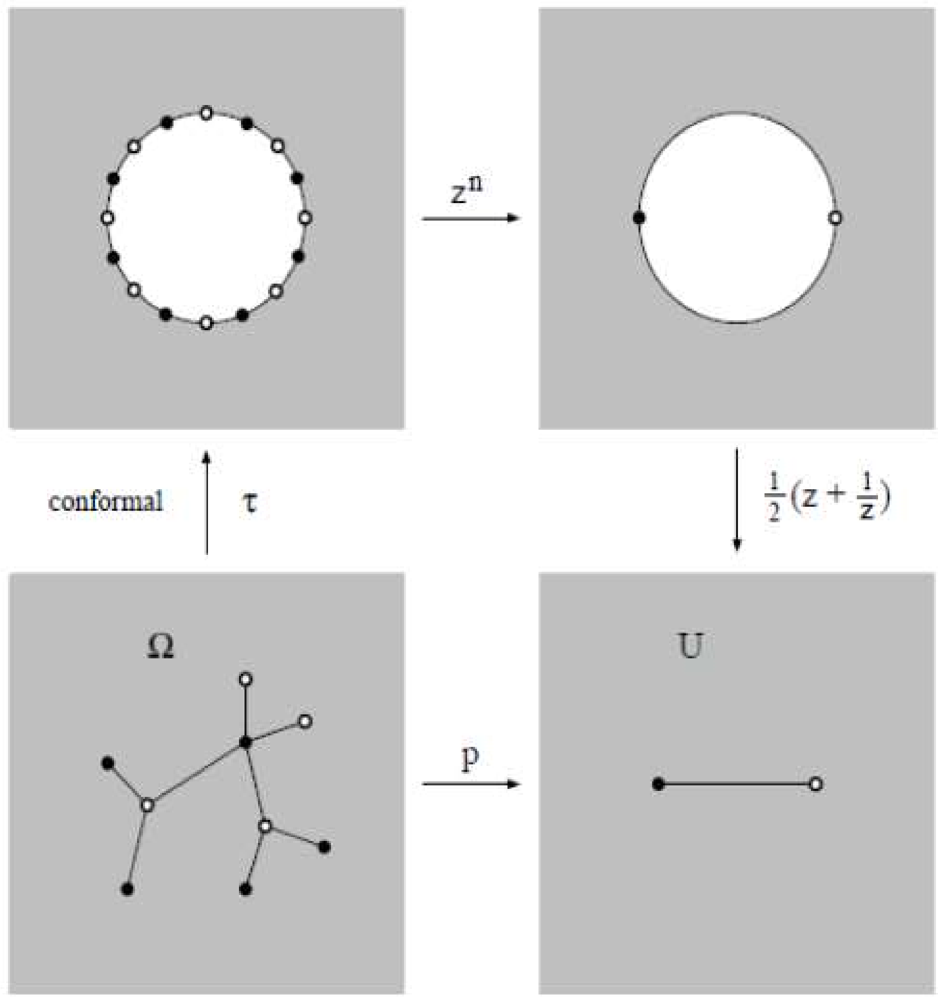
Thm (Shabat)

Every combinatorial tree can be embedded as $T \subset \mathbb{C}$ s.th. there is a polynomial p with $p^{-1}[0,1] = T$ and $p(\mathrm{Crit}) = \{0, 1, \infty\}$



Laminations and balanced trees :

Bishop, True trees are dense, Invent. Math. 2014



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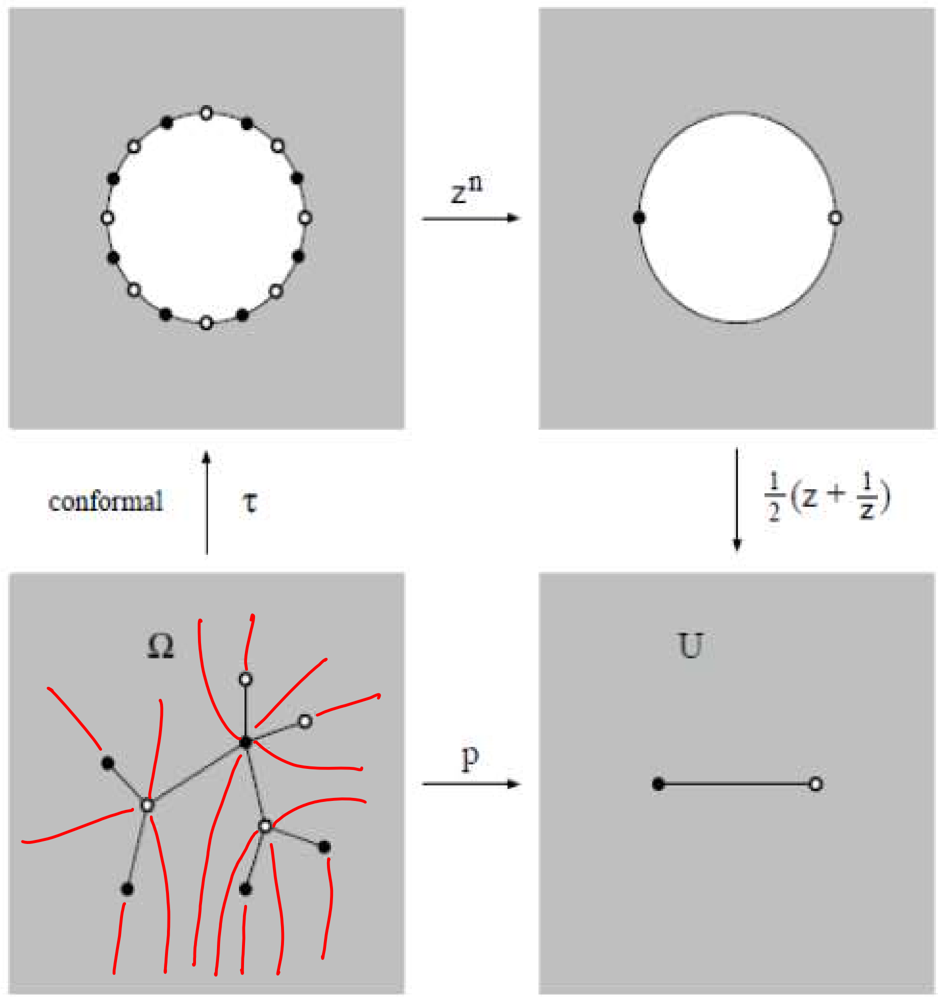

Glue equilateral triangles, uniformize

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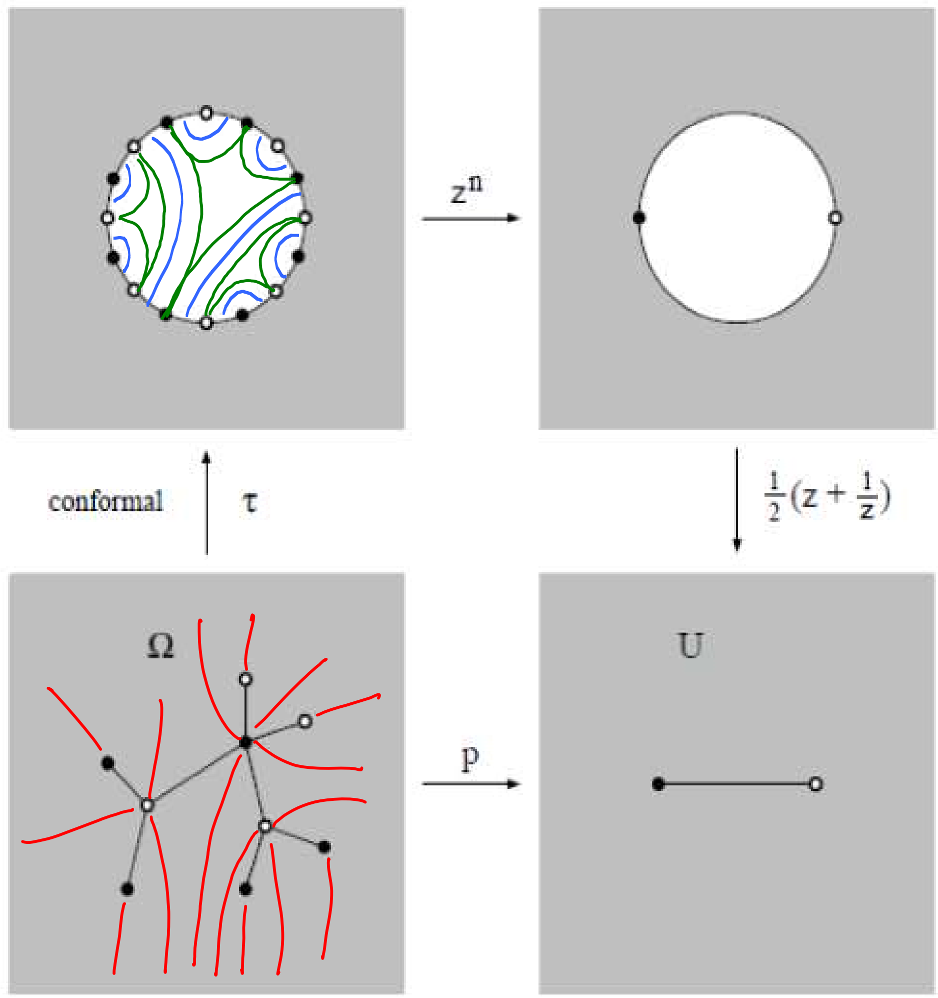

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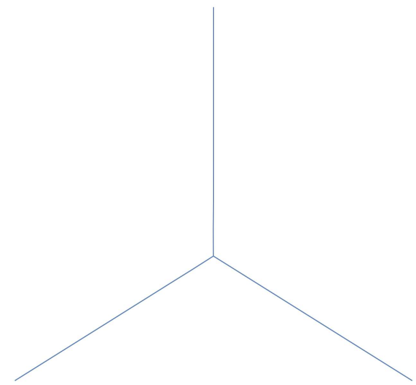

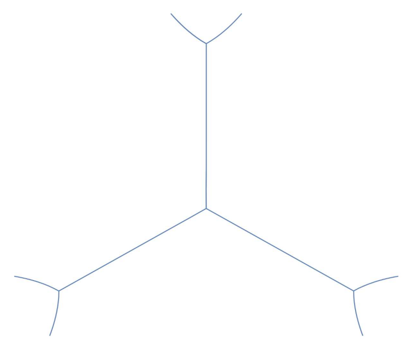

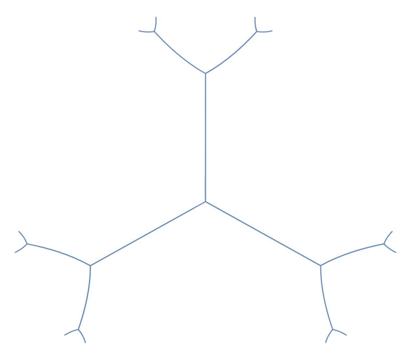

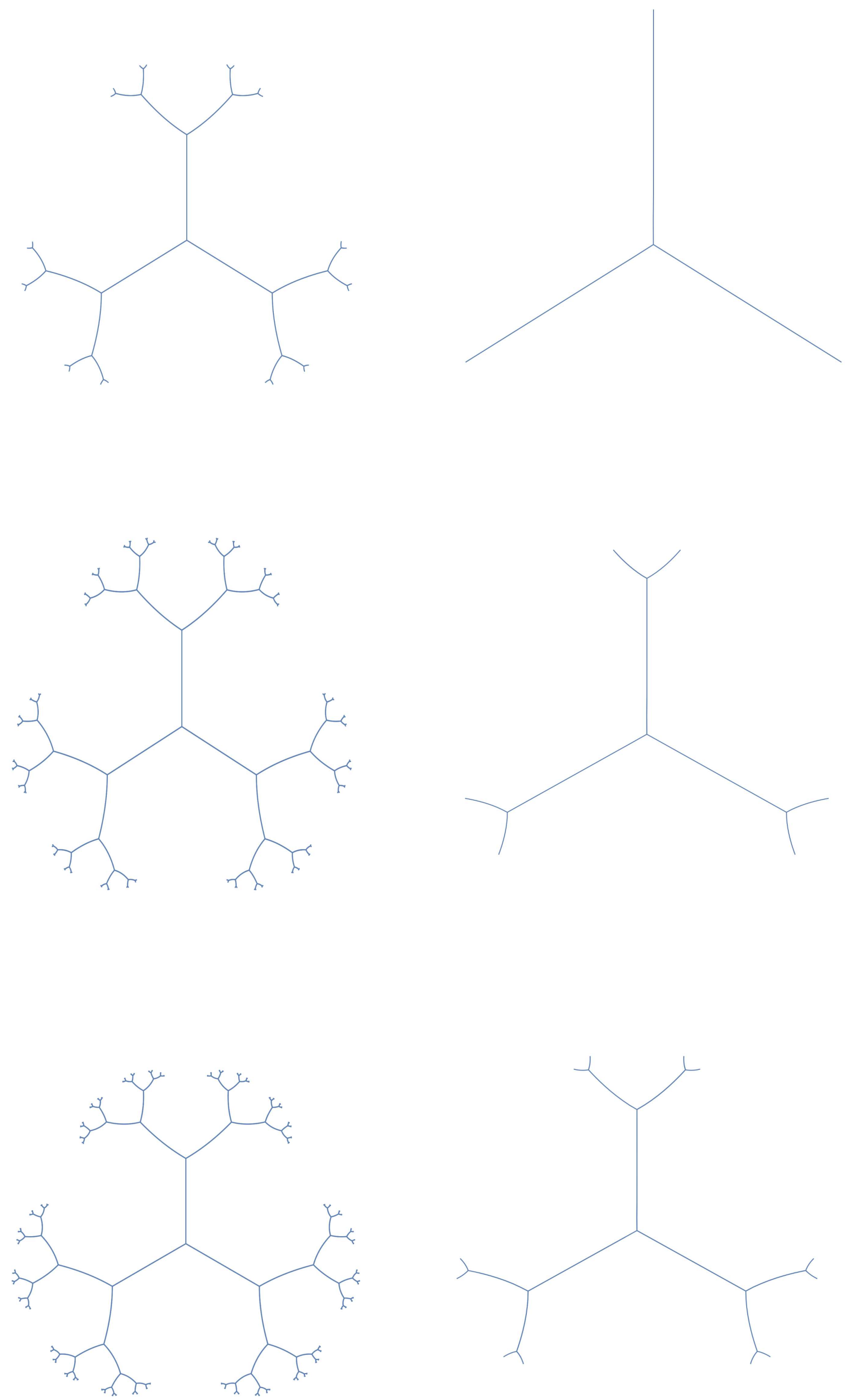

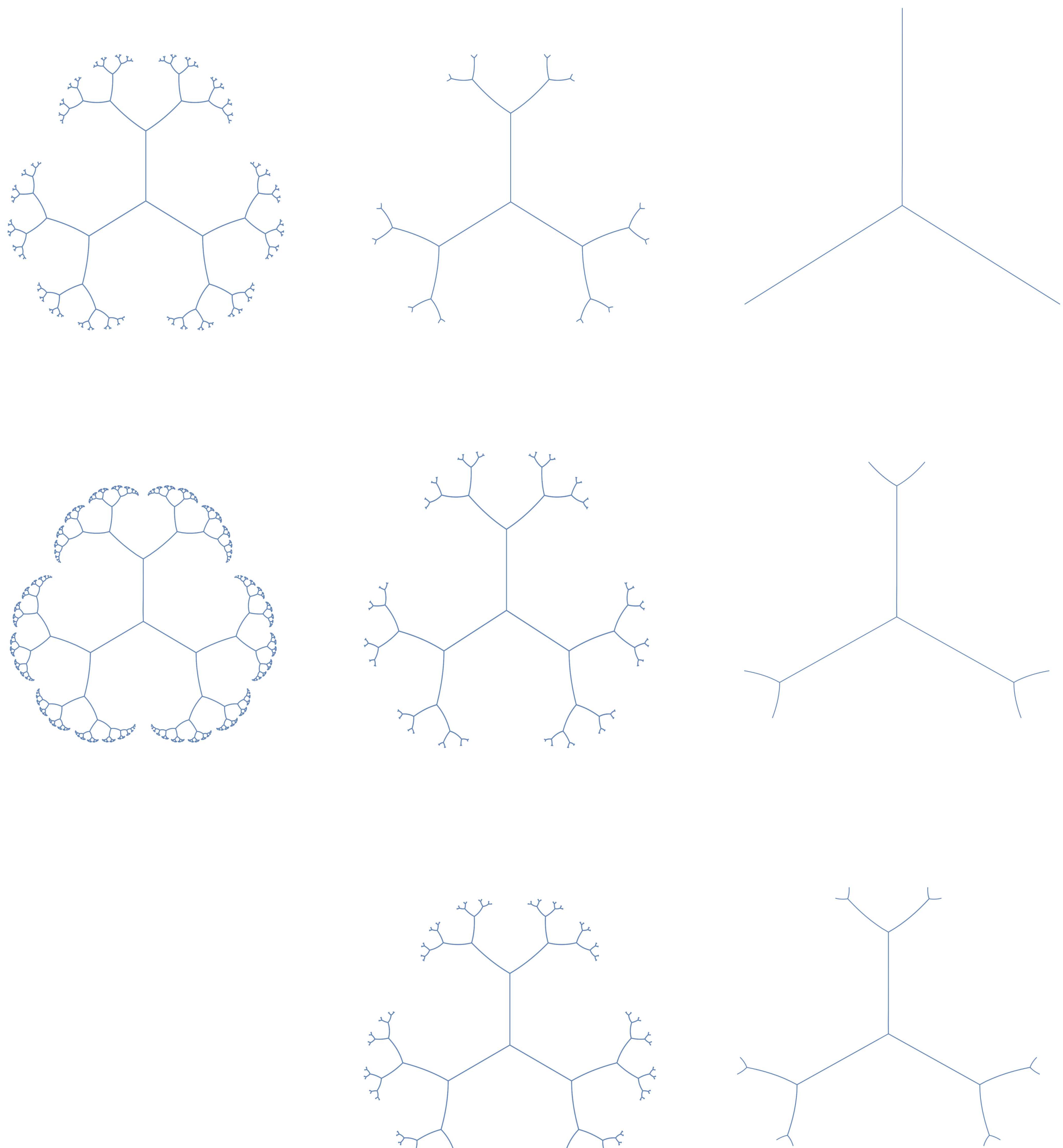

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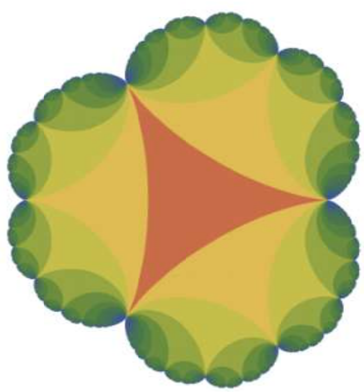
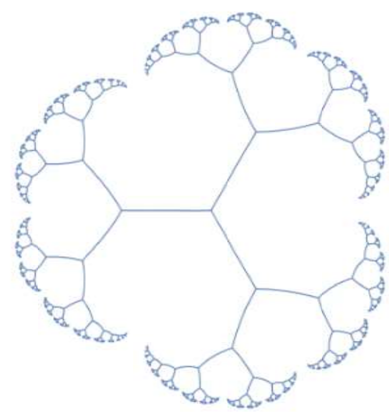
joint with Oleg Ivrii, Peter Lin and Emanuel Sygal :

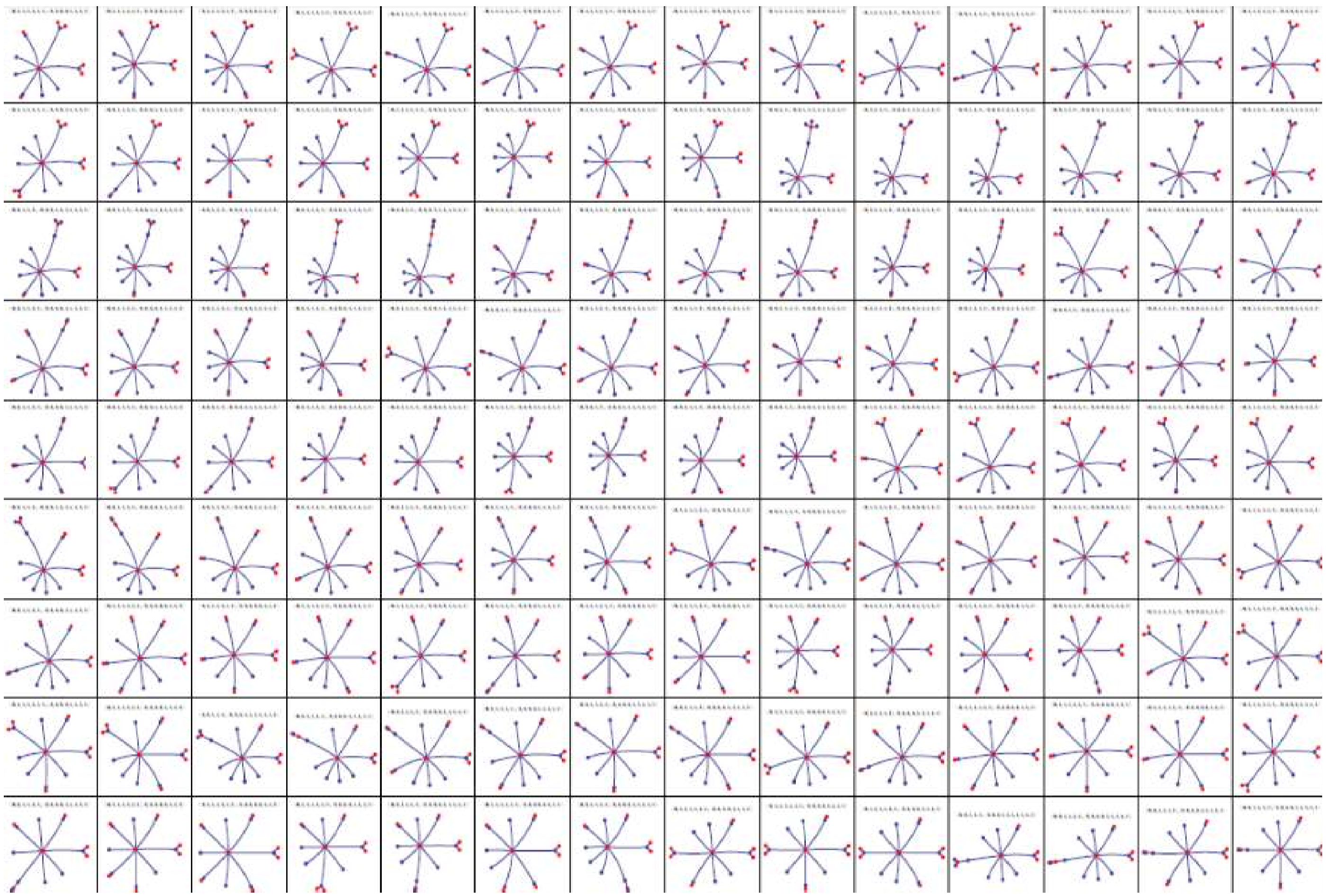
Theorem 1.1. *The trees $\mathcal{T}_n$ converge in the Hausdorff topology to an infinite trivalent tree union a Jordan curve $\mathcal{T}_\infty \cup \partial\Omega$. The domain Ω enclosed by $\partial\Omega$ is the developed deltoid. The Shabat polynomials p_n converge to $F \circ R^{-1}$ where F is a modular function invariant under an index 2 subgroup of* $\mathrm{PSL}(2,\mathbb{Z})$ *and $R : \mathbb{D} \to \Omega$ is the Riemann map.*



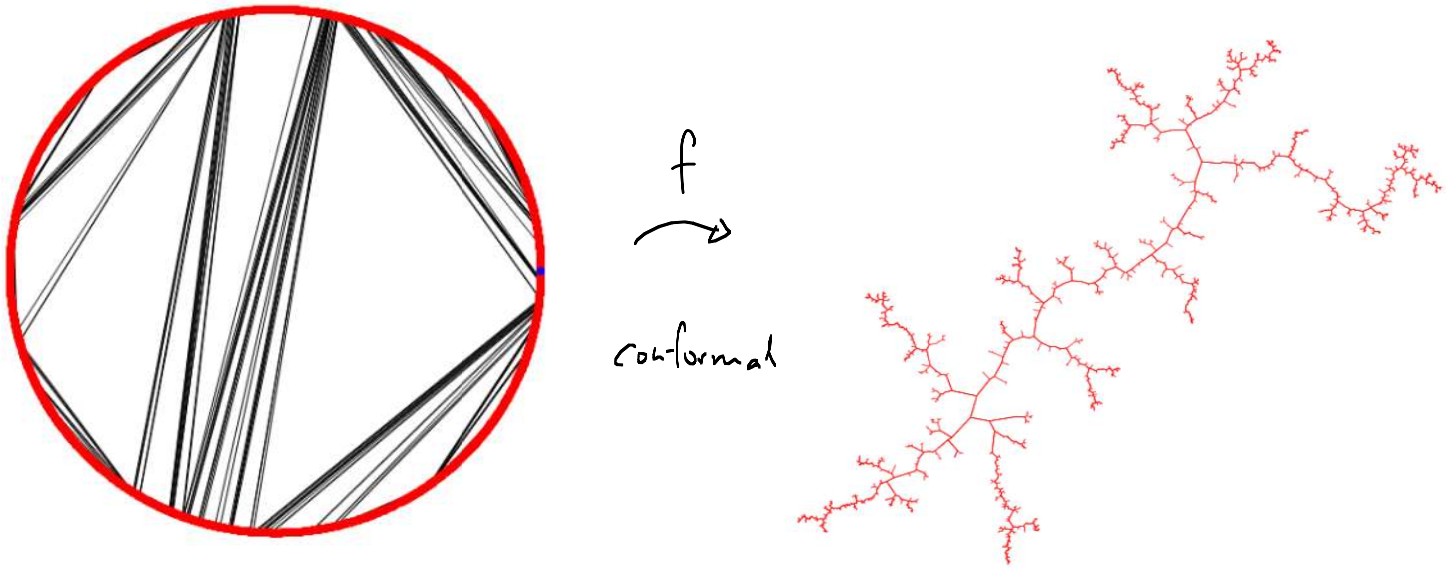


How does a large random tree look like?

Bishop's theorem: True trees are dense (Invent. Math. 2014)



Conformal welding of Aldous' Continuum Random Tree



Thm (Lin - R.) 1) The CRT can be welded

2) $f_n \to f$, Hölder a.s. , $\mathcal{L}_f = CRT$

In particular, dia(edges) $\to 0$ $(n \to \infty)$.

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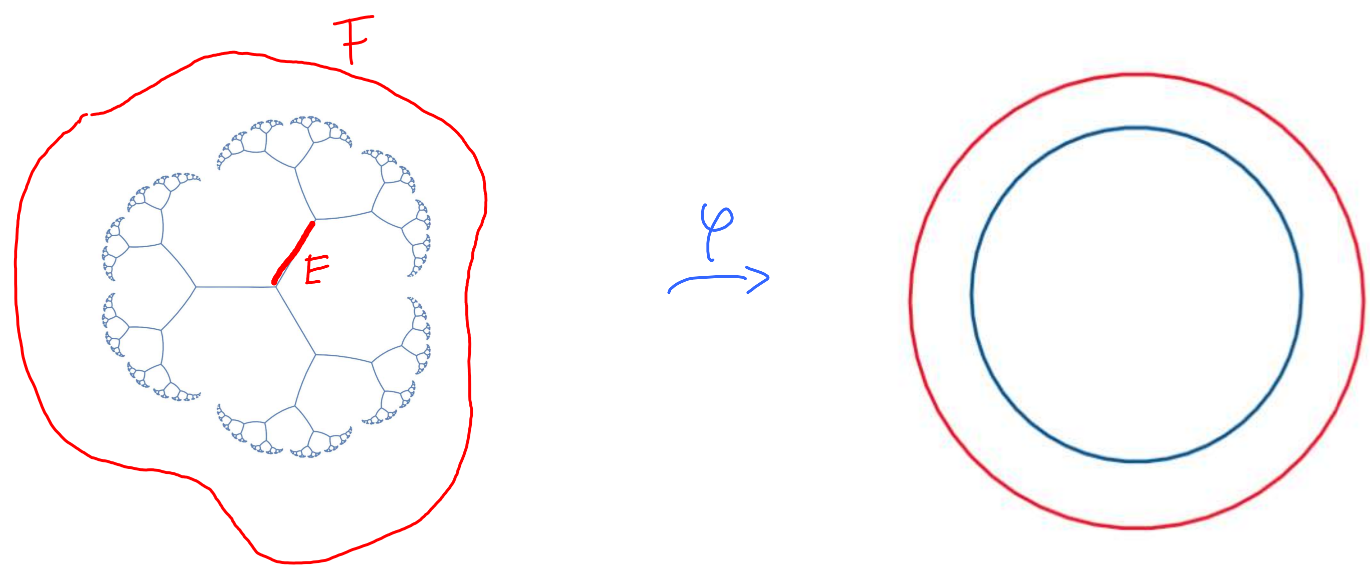
Key technique: A variant of conformal modulus; reminiscent of Oded Schramm's "transboundary extremal length"

see Bonk, Uniformization of Sierpiński carpets in the plane, Invent. Math. 2011

$\mathrm{dia}(E) \not\to 0$:

$\mathrm{diam}(E) \to 0$:



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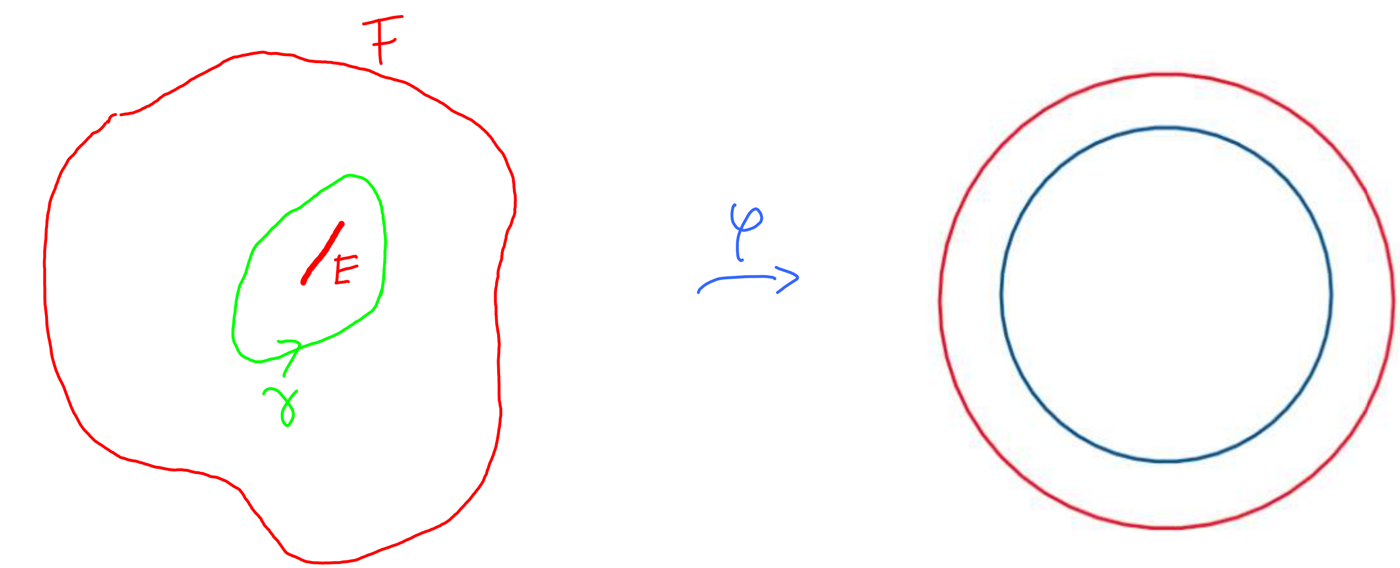

$$\frac{\mathrm{dist}(E,F)}{\mathrm{dia}\,E} \asymp M(\Gamma) := \inf_{\rho} \int \rho^2\,dx\,dy \;, \quad \int_{\gamma} \rho\,|dz| \geq 1$$

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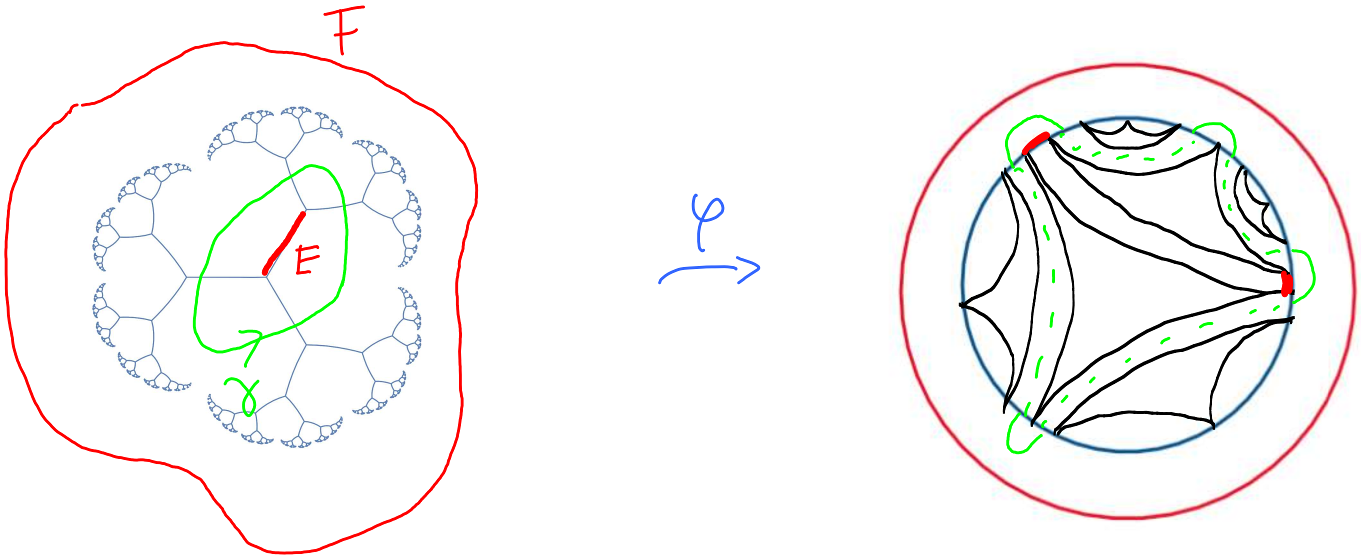

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teleporting through e cuts $\alpha_e \asymp 2^{-d_{T_n}(\mathrm{root},\, e)}$

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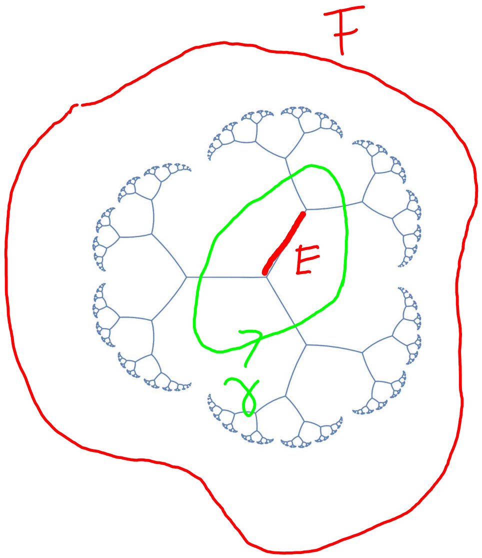
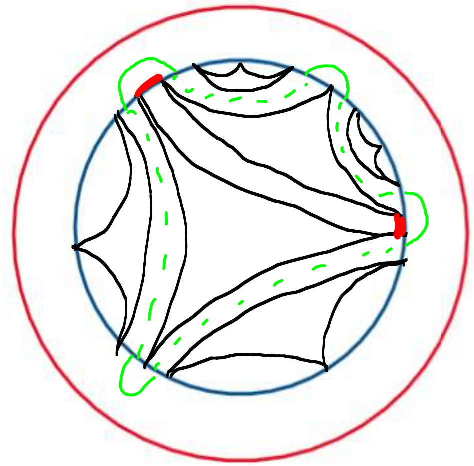


$\xrightarrow{\varphi}$



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$$\rho := \rho_0 + \sum_{e \in T} \alpha_e \rho_e \ , \quad \rho_0 = 1_{\{1<|z|<2\}} \ , \quad \rho_e = 1_{N_{1/2}(e)}$$

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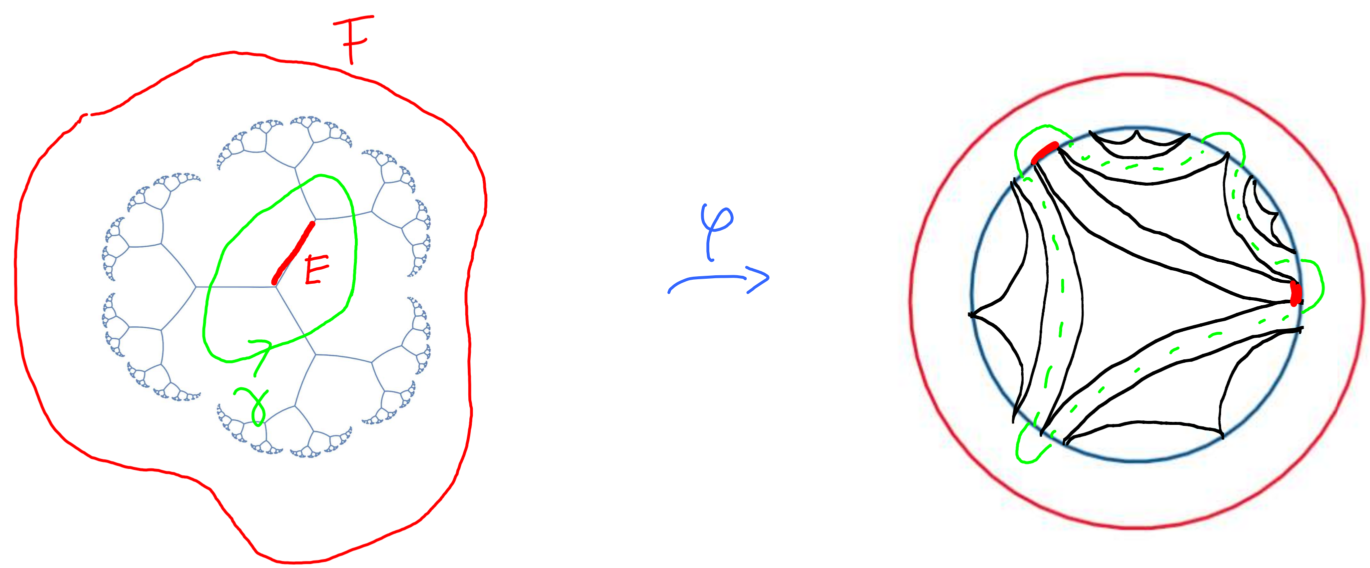

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$$M(\Gamma) \leq \int \rho^2 \lesssim 1 + \sum_{e \in T} \alpha_e^2 \asymp \sum_{m=1}^{n} \sum_{d_{T_n}(\mathrm{root},e)=m} \left(2^{-m}\right)^2 \asymp \sum_{m=1}^{n} 2^m \cdot \left(2^{-m}\right)^2 \lesssim 1$$

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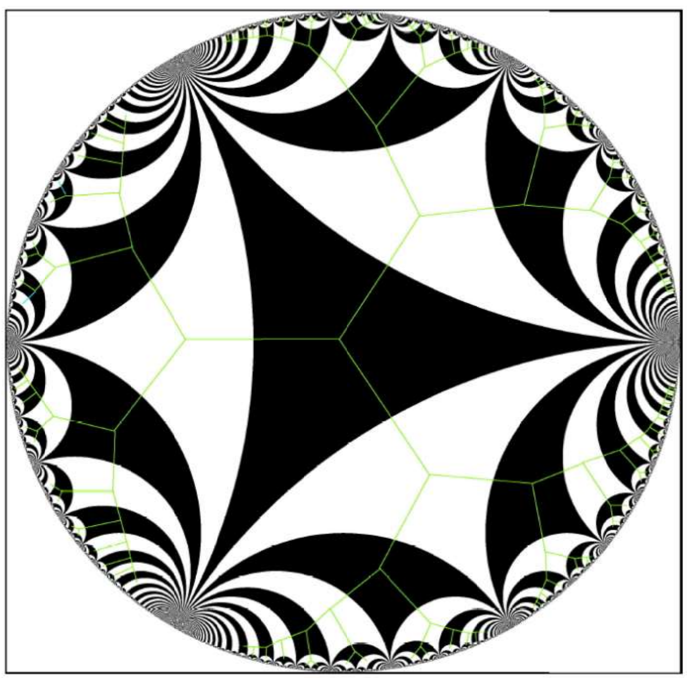
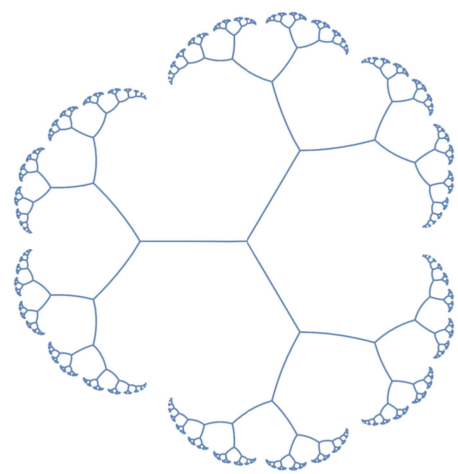
Proof Strategy:

1. Subsequential Hausdorff limits have the right topology (smooth tree+Jordan curve $\mathcal{T}_\infty \cup \partial\Omega$)

2. Identify the conformal welding of $\partial\Omega$

3. Show that the conformal welding determines the curve (and the tree)





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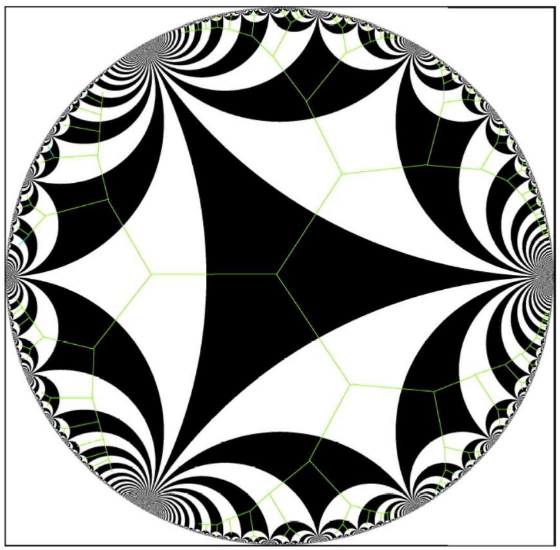

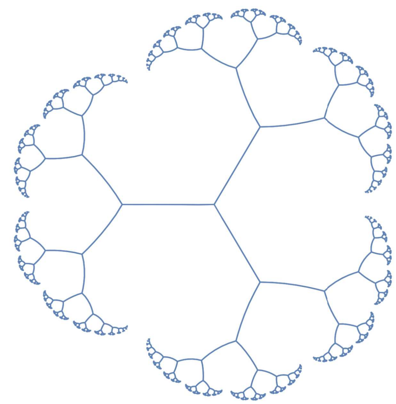

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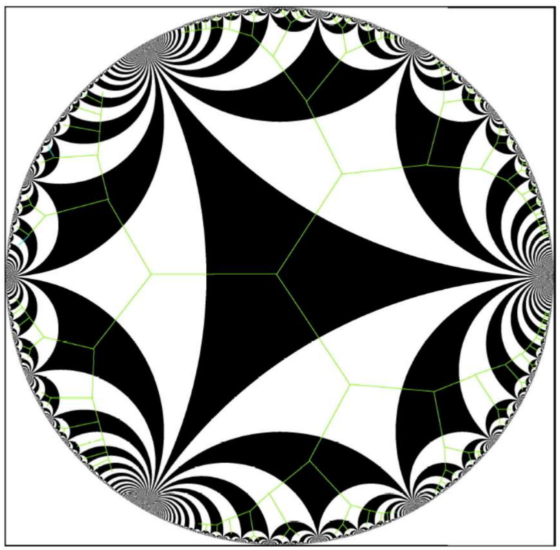

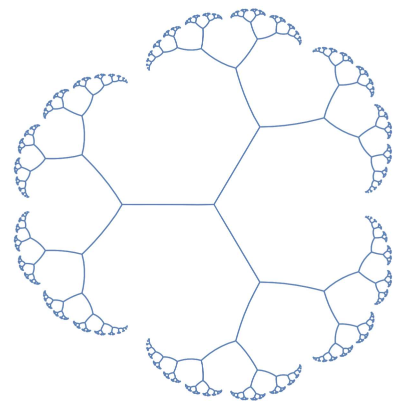

Interior: Limit T_∞ of trivalent tree $\xleftrightarrow{\text{conf.}}$ "modular tree"

Why? Conformal maps between "tiles" glue up along edges

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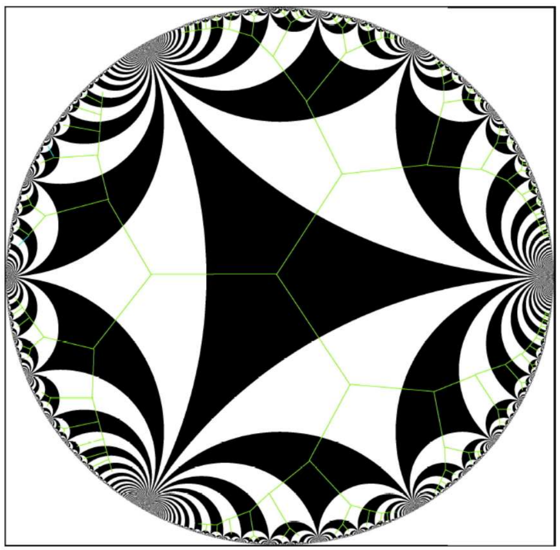

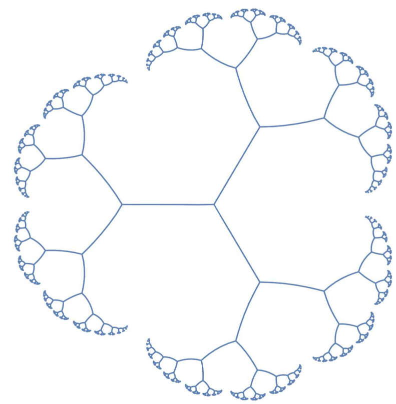

Interior: Limit T_∞ of trivalent tree $\xmapsto{\text{conf.}}$ "modular tree"

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Exterior: leaves of T_∞ $\xmapsto{\text{conf.}}$ leaves of "dyadic tree"

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$\varphi : \hat{\mathbb{C}} \setminus X \to \hat{\mathbb{C}} \setminus X'$ conformal, extends to homeomorphism

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Lemma 2.3. *Suppose that there is a countable exceptional set $E \subset X$ and a countable collection of closed subsets $s_1, s_2, \dots$ of X, called shadows, such that every point in $X \setminus E$ belongs to infinitely many sets s_i. If*

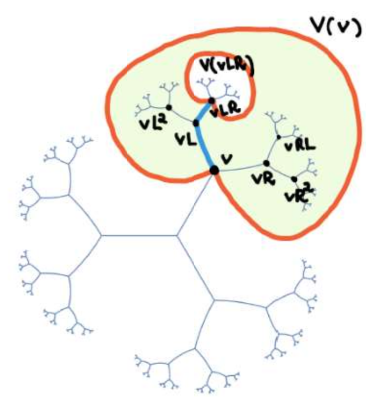
$$\sum_{i=1}^{\infty} \operatorname{diam}^2 s_i < \infty, \qquad \sum_{i=1}^{\infty} \operatorname{diam}^2 \varphi(s_i) < \infty, \tag{2.3}$$

then φ is a Möbius transformation.

Lemma 4.9. *The sums*

$$\sum_{v \in \mathcal{T}_n,\, v \neq v_{\text{root}}} \left\{ \operatorname{diam}^2 V(vRL) + \operatorname{diam}^2 V(vLR) \right\}$$

are uniformly bounded above, independent of n.



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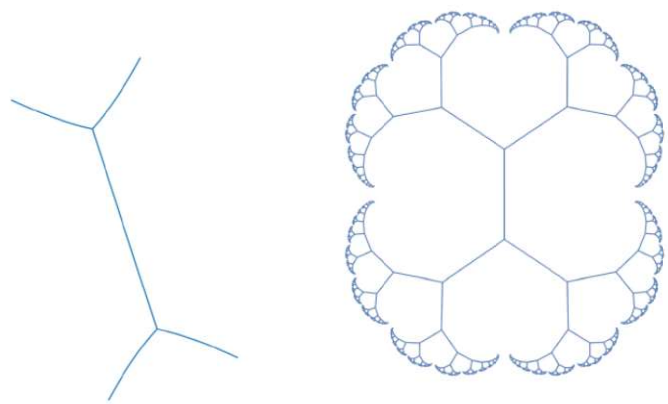
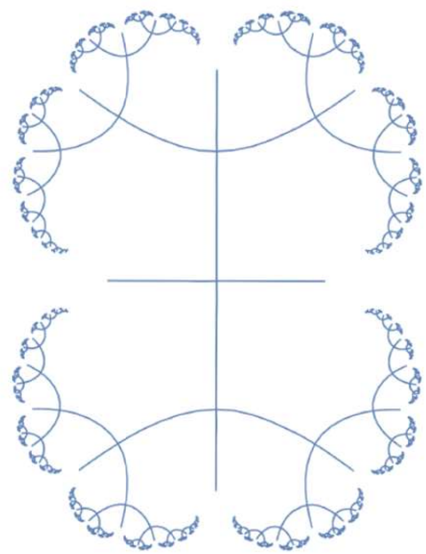
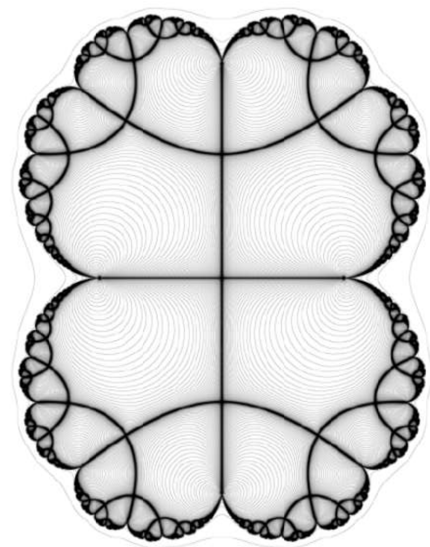


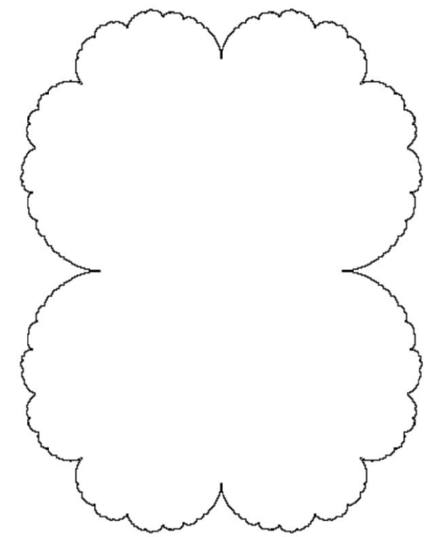
Figure 5: Unbalanced truncations of the infinite trivalent tree.

Corollary: Bishop's theorem (true trees are dense).

2. Complex dynamics:

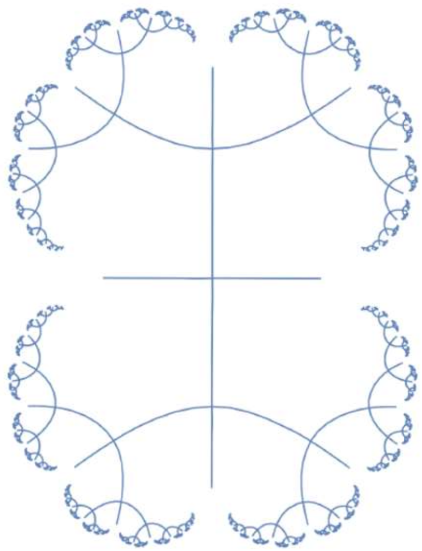


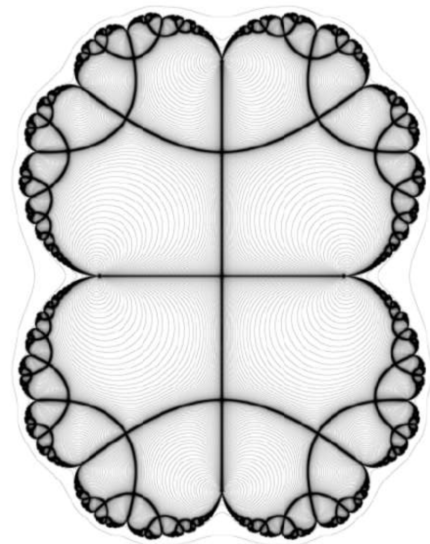


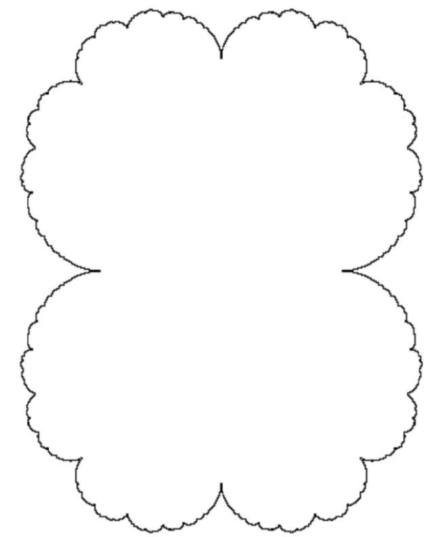


$\mathcal{J}(z^2 + 1/4)$

2. Complex dynamics:







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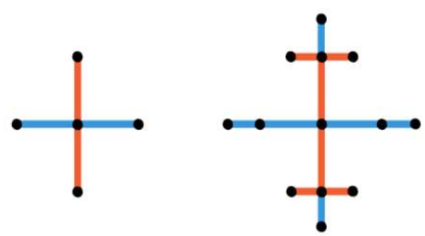

- If a leaf edge is blue, we attach another blue edge at the leaf vertex.
- If a leaf edge is red, we attach three edges, coloured red-blue-red in counter-clockwise order.

3. Random trees: Is there a (random) Jordan curve associated with the "Markovian hyperbolic triangulation" of Curien-Werner, J. Eur. Math. Soc. 2013 ?

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4. Develop the complex analysis to prove mating of trees (both deterministic and random)

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4. Develop the complex analysis to prove mating of trees (both deterministic and random)

5. Continue Mario's line of research on the probabilistic side:

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Thank you for your attention,

and

Happy birthday, Mario!