DIMENSIONS OF TRANSCENDENTAL JULIA SETS

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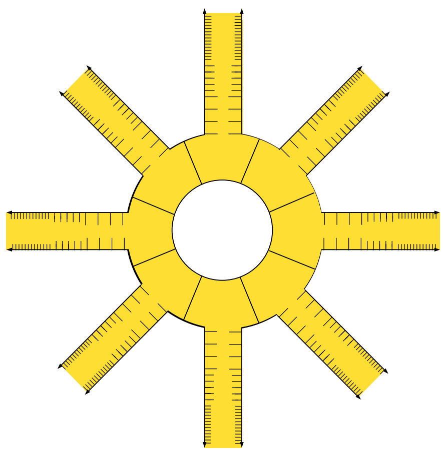
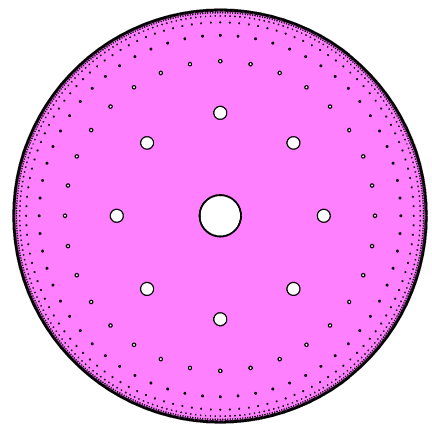
www.math.sunysb.edu/~bishop/lectures





THE PLAN

- Hausdorff and packing dimensions
- Polynomial versus transcendental
- Dimension $d = 2$
- Dimension $1 < d < 2$
- Dimension $d = 1$
- Open problems.

Defn: Minkowski dimension. If K is a bounded set, let $N(K, \epsilon)$ be the minimal number of cubes of diameter ϵ needed to cover K.

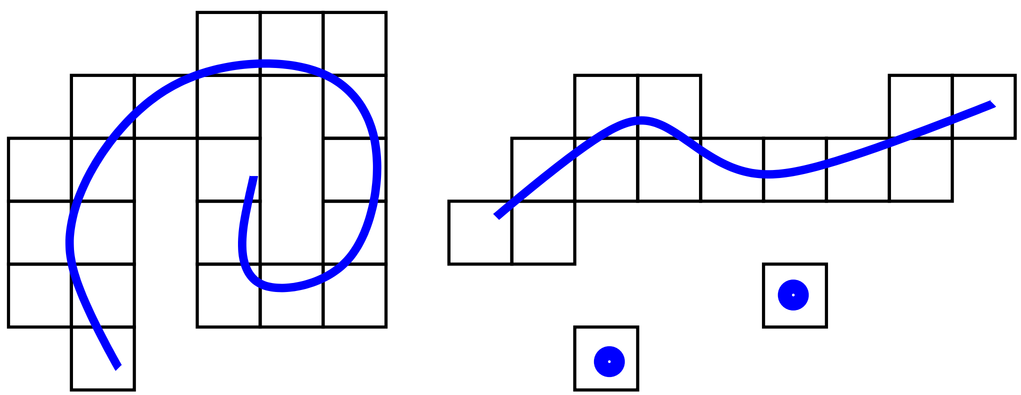
Upper Minkowski dimension:

$$\overline{\mathrm{Mdim}}(K) = \limsup_{\epsilon \to 0} \frac{\log N(K, \epsilon)}{\log 1/\epsilon},$$

Lower Minkowski dimension

$$\underline{\mathrm{Mdim}}(K) = \liminf_{\epsilon \to 0} \frac{\log N(K, \epsilon)}{\log 1/\epsilon}.$$

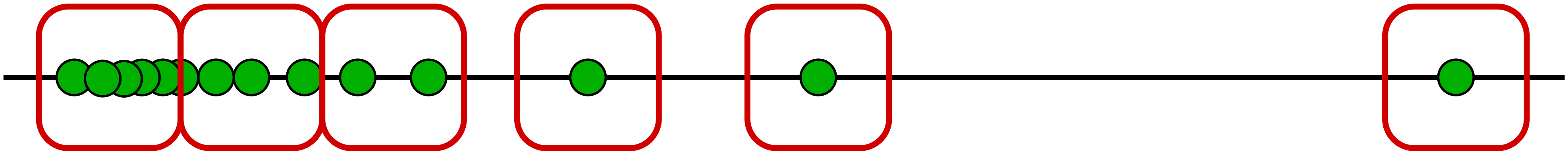
If these agree, common value is **Minkowski dimension**, $\overline{\mathrm{Mdim}}(K)$.



Two disadvantages:

- Not defined for unbounded sets

- Countable sets can have dimension > 0.



$\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ needs $\sqrt{n}$ balls of size $1/n$ balls to cover.

The packing dimension fixes these problems:

Defn: packing dimension

$$\mathrm{Pdim}(A) = \inf \left\{ \sup_{j \geq 1} \overline{\mathrm{Mdim}}(A_j) : A \subset \bigcup_{j=1}^{\infty} A_j \right\},$$

where the infimum is over all countable covers of A.

By definition $\mathrm{Pdim}(\cup_n A_n) = \sup_n \mathrm{Pdim}(A_n)$.

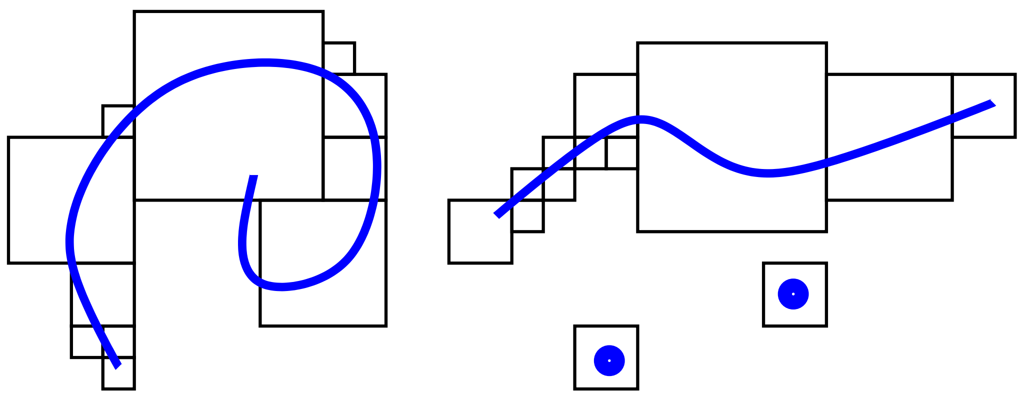
Defn: α-dimensional Hausdorff content

$$\mathcal{H}^\alpha_\infty(K) = \inf\{\sum_i |U_i|^\alpha\},$$

$\{U_i\}$ = cover of K, $|E|$ = diameter of a set E.

Like Minkowski dimension, but allows covering sets of different sizes.

Defn: Hausdorff dimension $\dim(K) = \inf\{\alpha : \mathcal{H}^\alpha_\infty(K) = 0\}$.



Always true that Hdim $\leq$ Pdim, but "$<$" can sometimes hold.

For polynomials, $\mathrm{Hdim}(\mathcal{J})$ can take any value in $(0, 2]$.

Same for meromorphic functions (Bergweiler-Cui).

For transcendental entire functions, $\mathrm{Hdim}(\mathcal{J}) \in [1, 2]$.

Today, I will only discuss entire functions:

- Sketch proof that $\mathrm{Hdim} \geq 1$
- Discuss examples for $d = 2$, $1 < d < 2$, $d = 1$.

Thm (Baker): Hdim ≥ 1 for transcendental entire functions.

Lemma: Non-trivial loops escape

- Suppose curve γ in Fatou set surrounds a point of $\mathcal{J}$.
- If $\{f^n\}$ bounded on γ, also bounded on interior by max principle.
- Hence interior of γ in Fatou set, a contradiction.
- So a point in γ escapes. By normality all γ escapes.

Lemma: Iterates of γ have non-zero index around 0

- Suppose not.
- Then minimum principle applies and interior of γ escapes.
- But γ surrounds $\mathcal{J}$ and hence surrounds a pre-periodic point.
- Contradiction.
- $\Rightarrow$ iterates of γ surround every compact set.

Lemma: Multiply connected Fatou components are bounded:

- Suppose Ω is multiply connected and unbounded.
- Suppose $\gamma \subset \Omega$ surrounds a Julia point.
- γ escapes, index non-zero $\Rightarrow \gamma_n = f^n(\gamma)$ intersects Ω for all large n.
- $\Rightarrow \gamma_n \subset \Omega$ for all n.

...

Lemma: Multiply connected Fatou components are bounded:

- Suppose Ω is multiply connected and unbounded.
- Suppose $\gamma \subset \Omega$ surrounds a Julia point.
- γ escapes, index non-zero $\Rightarrow \gamma_n = f^n(\gamma)$ intersects Ω for all large n.
- $\Rightarrow \gamma_n \subset \Omega$ for all n.
- Schwarz lemma $\Rightarrow$ hyperbolic distance from γ_{n+1} to γ_n is bounded.
- Implies $\mathrm{diam}(\gamma_{n+1}) \leq C \cdot \mathrm{diam}(\gamma_n)$.
- Implies that f grows polynomially. Contradiction.
- Hence multiply connected Fatou components are bounded.

Thm: $\mathcal{J}$ contains non-trivial continuum, so $\mathrm{Hdim}(\mathcal{J}) \geq 1$

- Suppose not. Then $\mathcal{J}$ is totally disconnected.
- $\Rightarrow$ one multiply connected Fatou component.
- Such a component is bounded. Contradiction.
- Hence $\mathcal{J}$ contains a continuum.

So $\mathrm{Hdim}(\mathcal{J}) \in [1, 2]$. Next we will discuss examples of

- $\mathrm{Hdim}(\mathcal{J}) = 2$
- $1 < \mathrm{Hdim}(\mathcal{J}) < 2$
- $\mathrm{Hdim}(\mathcal{J}) = 1$

Many transcendental functions have $\dim(\mathcal{J}) = 2$:

Thm (Misiurewicz): $\dim(\mathcal{J}) = \mathbb{C}$ for $f(z) = e^z$.

Thm (McMullen): $\dim(\mathcal{J}) = 2$ and $\text{area}(\mathcal{J}) = 0$ for $f(z) = \lambda e^z$.

Thm (McMullen): $\text{area}(\mathcal{J}) > 0$ for $f(z) = \lambda \cdot \cosh(z)$.

$$\begin{aligned}\textbf{Singular set} \;&=\; \text{closure of critical values and finite asymptotic values}\\ &=\; \text{smallest set so that } f \text{ is a covering map onto } \mathbb{C}\setminus S\end{aligned}$$

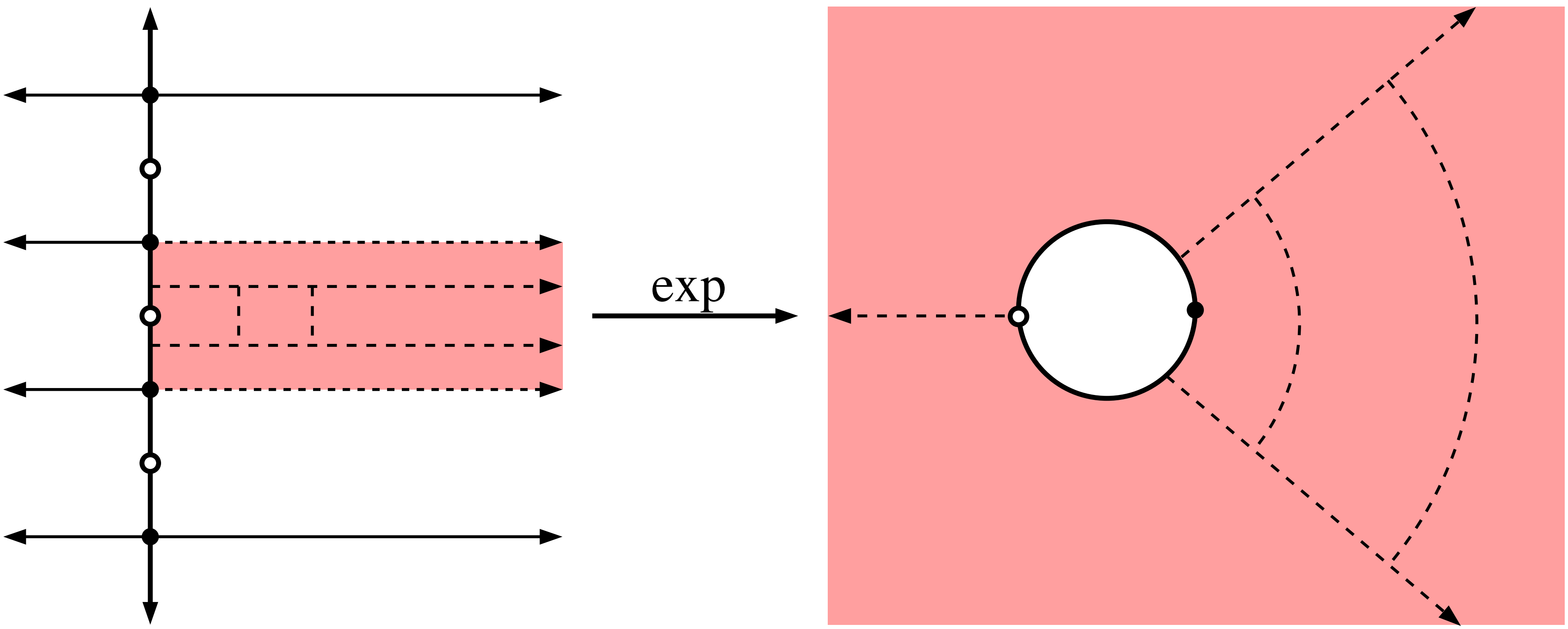
Eremenko-Lyubich class = bounded singular set $= \mathcal{B}$

Speiser class = finite singular set $= \mathcal{S} \subset \mathcal{B}$

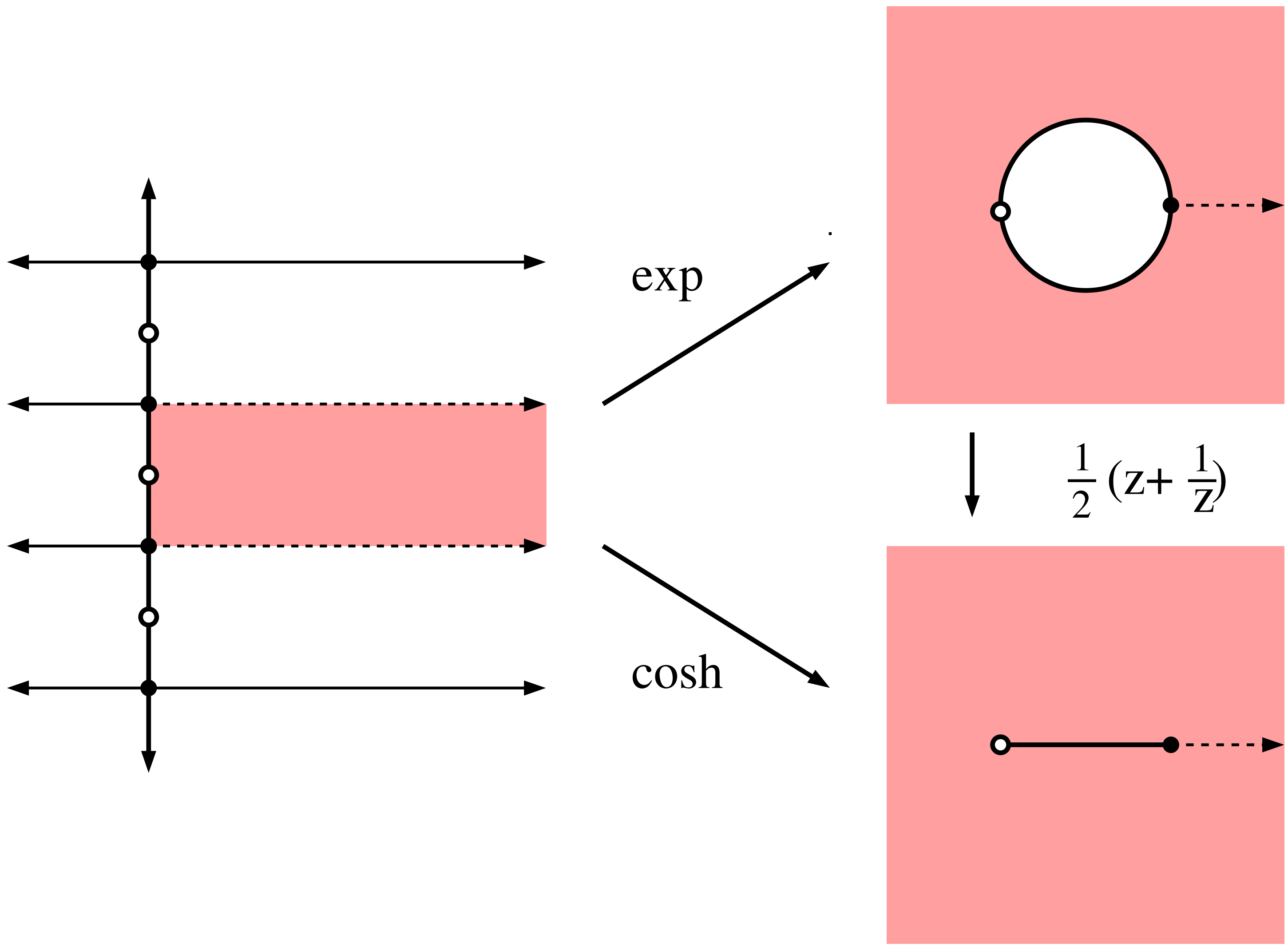
$\lambda \cdot \exp(z)$, and $\lambda \cdot \cosh(z)$ are in Speiser class.

Defn: Escaping set $I(f) = \{z : f^n(z) \to \infty\}$.

Fact: In general, $\mathcal{J}(f) = \partial I(f)$. For f in EL-class, $\mathcal{J}(f) = \overline{I(f)}$.



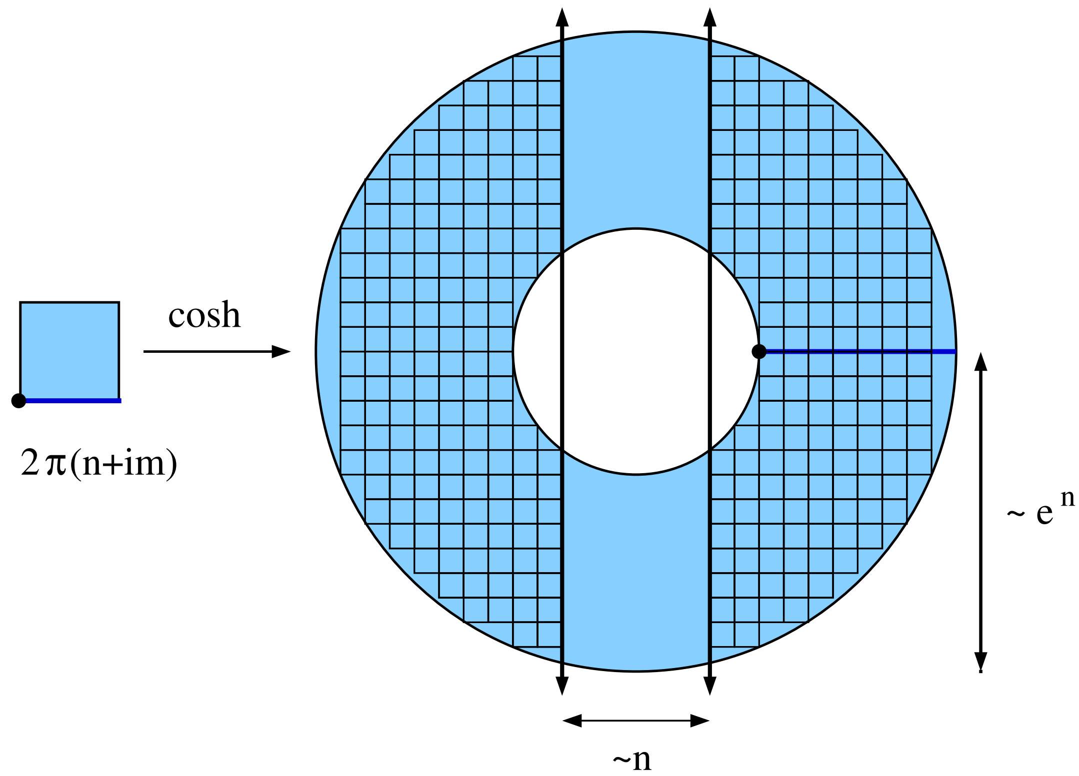
Definition of $\exp(z)$.



Definition of $\cosh(z)$.

$$\cosh(-x+iy) = \overline{\cosh(x+iy)}$$

Proof that $\operatorname{area}(\mathcal{J}) > 0$ **for cosh:**

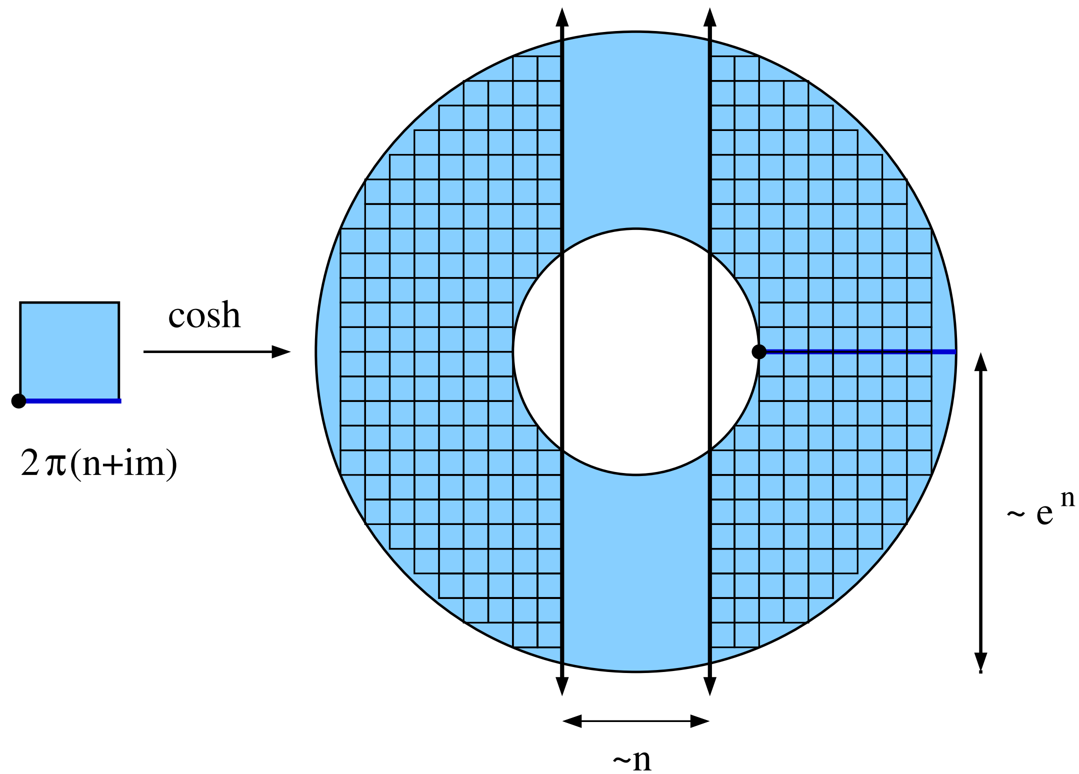


Let $S = 2\pi(n + im) + [0, 2\pi]^2$.

$\cosh(S)$ approximately covers annulus A_n of area $\simeq 2^{2|n|}$.

Annulus contains $\simeq e^{2|n|}$ disjoint translates of S.

Proof that $\text{area}(\mathcal{J}) > 0$ **for cosh:**



Omit $\simeq |n| \cdot e^{|n|}$ squares near y-axis, $\simeq e^{|n|}$ near ∂A_n.

Remaining squares cover $1 - O(|n| \cdot e^{-|n|})$ area of annulus.

$\sum_{n>0} ne^{-n} < \infty \Rightarrow$ positive area escapes (so is in $\mathcal{J}$.)

Order of growth:

$$\rho(f) = \limsup_{|z|\to\infty} \frac{\log\log|f(z))|}{\log|z|}.$$

Barański (2008) and Schubert (2007) proved that the Julia set of any finite-order Eremenko-Lyubich function has Hausdorff dimension 2.

Transcendental examples with $1 < \mathrm{Hdim}(\mathcal{J}) < 2$:

Gwyneth Stallard gave examples in EL-class: all $1 < d < 2$.

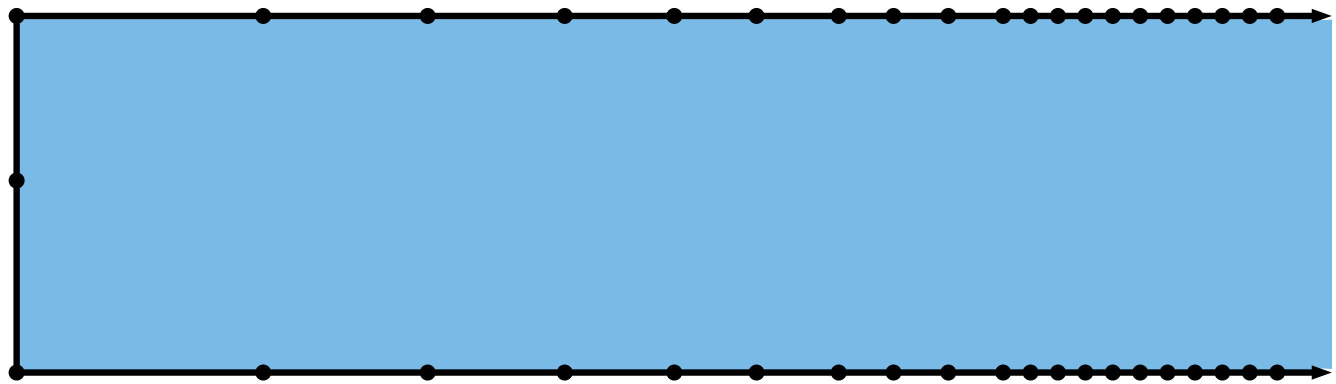
Rippon-Stallard proved $\mathrm{Pdim}(\mathcal{J}) = 2$ in EL-class.

$\Rightarrow$ Hdim $\neq$ Pdim can occur.

Simon Albrecht and I gave sequence in Speiser class with Hdim $\to 1$.

Open problem: do all values $(1, 2]$ occur for the Speiser class?

Theorem (Stallard): There are Eremenko-Lyubich functions whose Julia sets have Hausdorff dimension close to 1.



There is EL function with tract $\{z : |f(z)| > 1\} \approx$ half-strip.

- Cauchy integrals
- Solve $\overline{\partial}$-equation
- Use models theorem for EL-class.

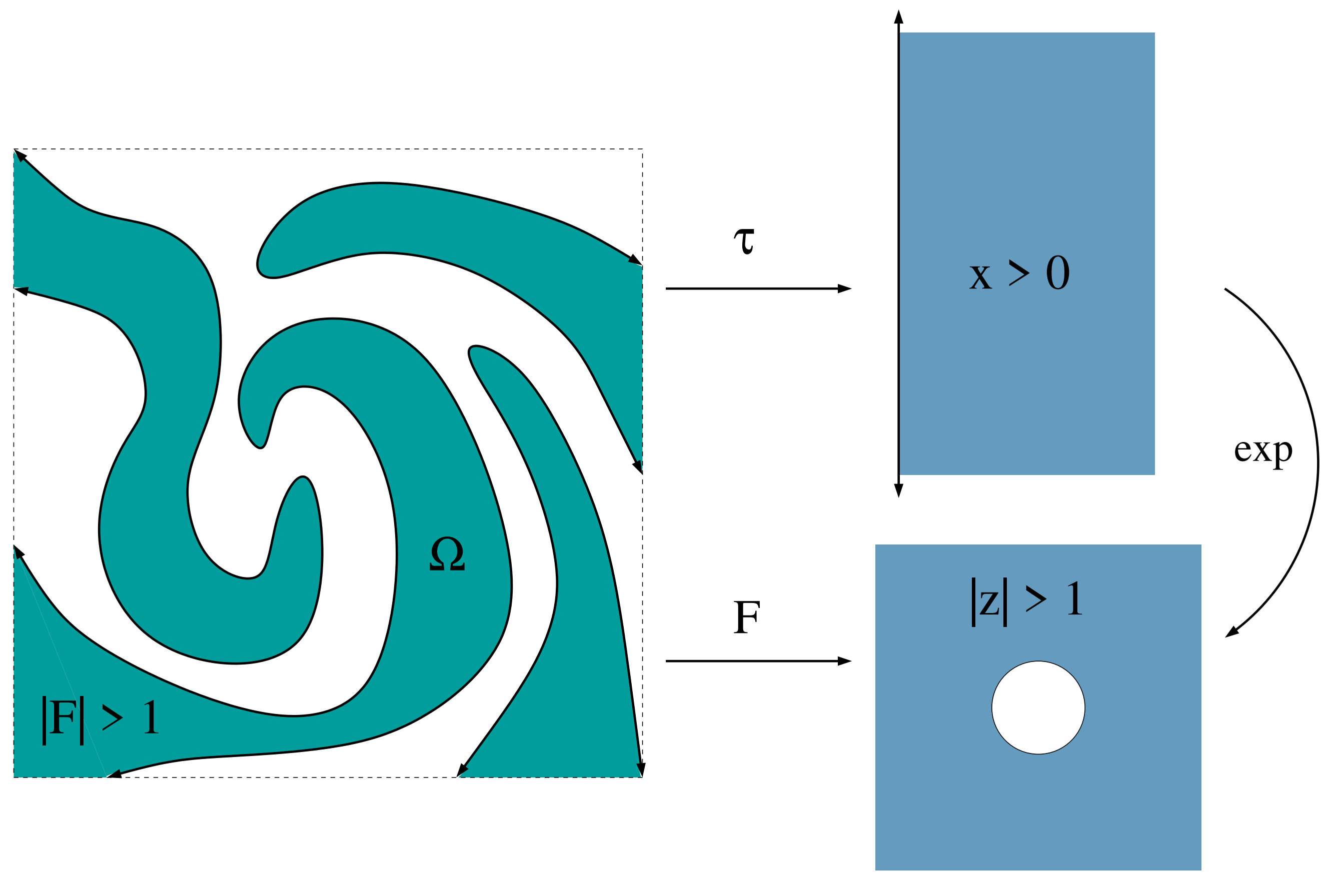
There is no Speiser function with this tract (even approximately).

Models theorem:

Suppose $F \in$ EL-class and $S(F) \subset \mathbb{D} = \{|z| < 1\}$.

$\Omega = \{|F| > 1\}$ has simply connected components, called **tracts**

$W = \mathbb{C} \setminus \overline{\Omega} = \{|F| < 1\}$ is connected, simply connected



F is a covering map $\Omega \to \mathbb{D}^* = \{|z| > 1\}$, $F = \exp \circ \tau$.

τ is conformal from each tract to $\mathbb{H}_r$ = right half-plane

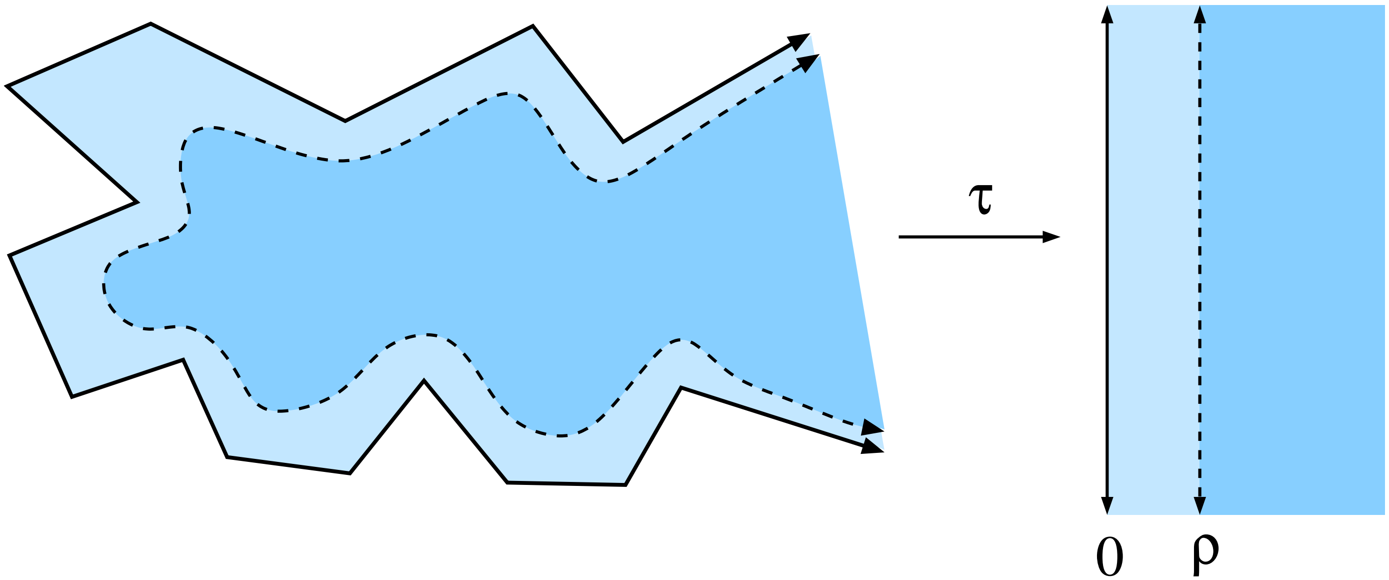
$W = \mathbb{C} \setminus \Omega$ is like the tree in the folding theorem.

A **model** is a pair (Ω, τ) where

- $\Omega = \cup \Omega_j$ is a disjoint union of unbounded Jordan domains
- τ is conformal from each Ω_j to $\mathbb{H}_r$ ($\infty \to \infty$).

Every Eremenko-Lyubich function F gives a model with $\Omega = \{|F| > R\}$.

Does every model give an Eremenko-Lyubich function?



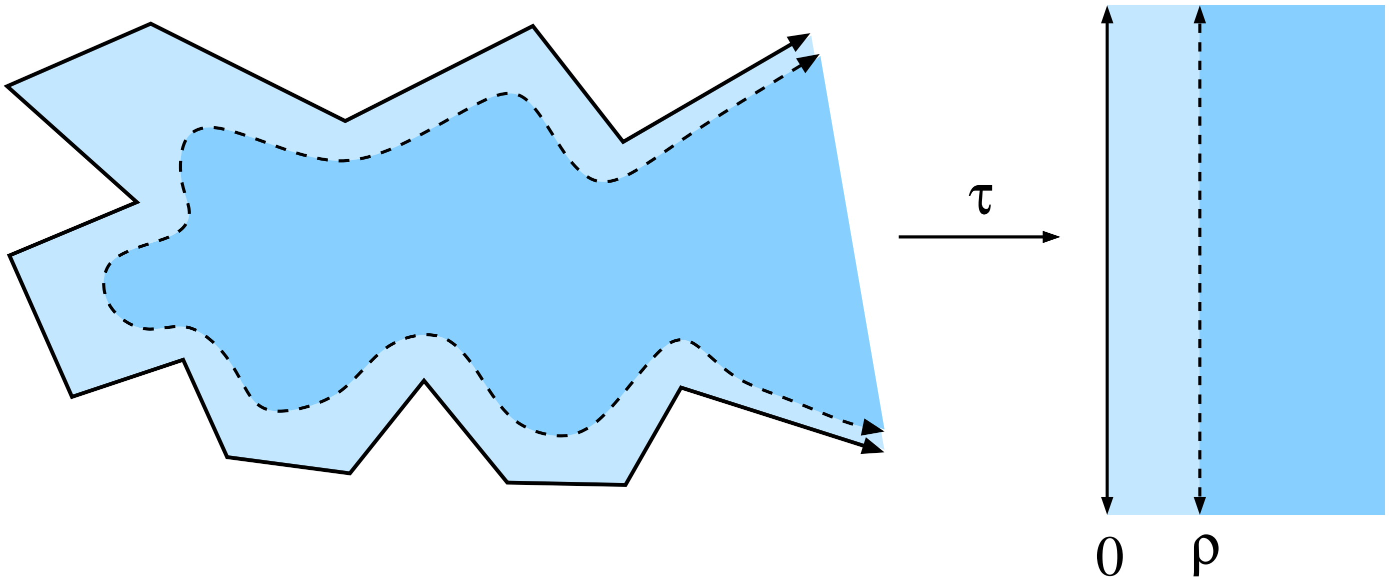
Theorem: Suppose (Ω, τ) is a model and $\rho > 0$. Define

$$\Omega(\rho) = \tau^{-1}(\{x + iy : x > \rho\}) \subset \Omega.$$

Then there is a quasiregular g so that

(1) $g = e^\tau$ on $\Omega(\rho)$,

(2) $|g| \leq e^\rho$ off Ω.

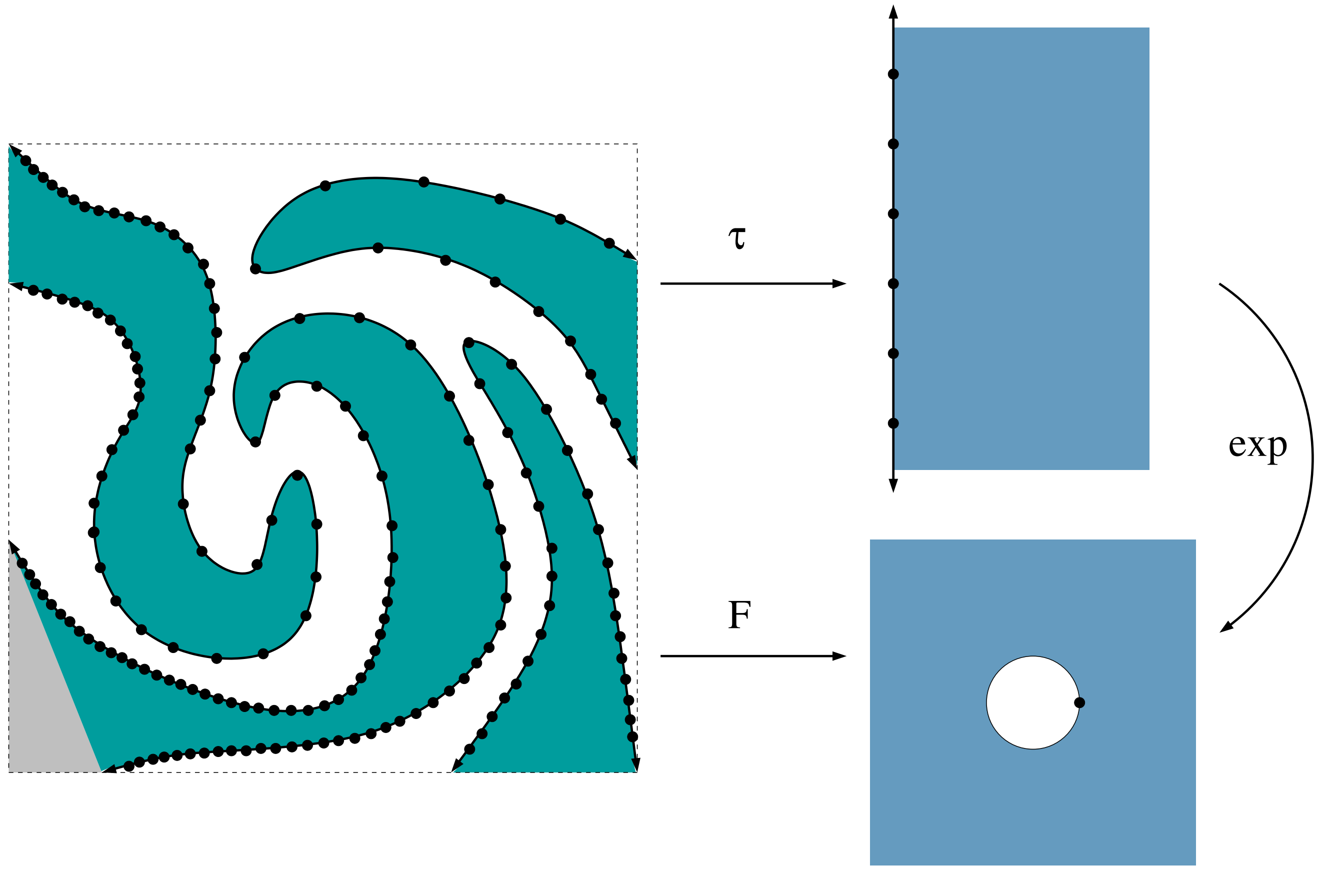


The QR constant depends on ρ, but not on Ω.

There is a quasiconformal φ so that $f = g \circ \varphi \in \mathcal{B}$.

g is holomorphic except on $\Omega(\rho) \setminus \Omega(\rho/2)$ (often has finite area).

Tracts of f correspond to components of Ω. Very similar shapes.



Idea of proof:

F maps each tract (component of Ω) to outside of disk.

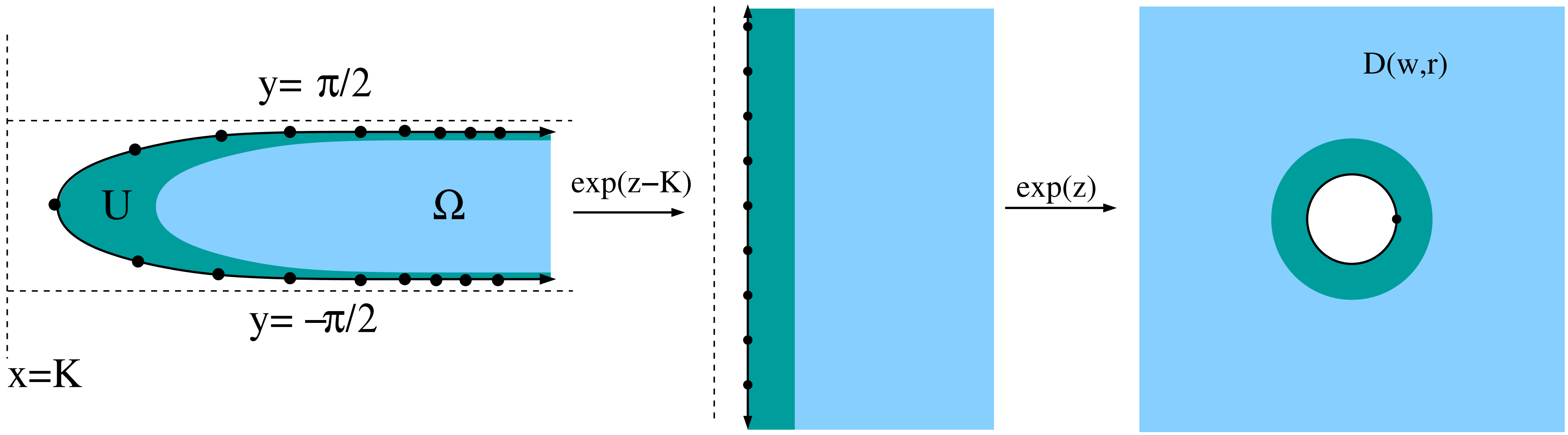
Riemann map W to disk, follow by Blaschke product.

Choose Blaschke product so two maps almost match along $\partial W = \partial\Omega$.

Match exactly with QC deformation, then apply MRMT.

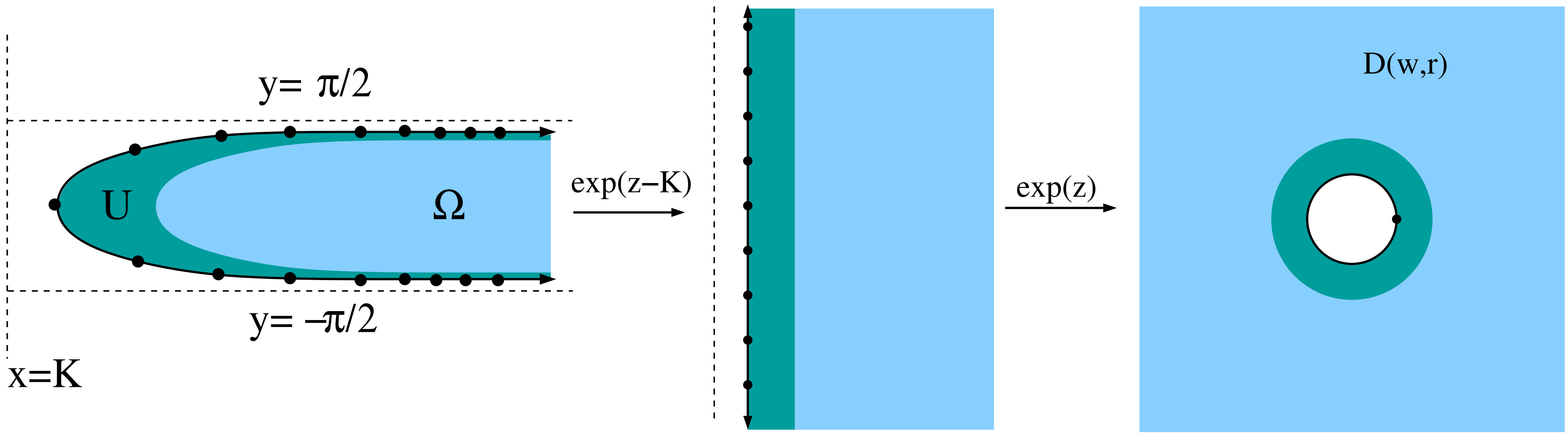
Assume we have g in EL-class so that:

- $g(0) = 0$ and $|g(z)| < 1$ outside $S =$ half-strip.
- $\{|z| < 1\}$ attracted to 0 (in Fatou set).
- Inside S, $g(z) \approx \exp(\exp(z - K))$



Inside S, $g(z) \approx \exp(\exp(z - K))$

This is conformal map of S to half-plane, followed by exp.

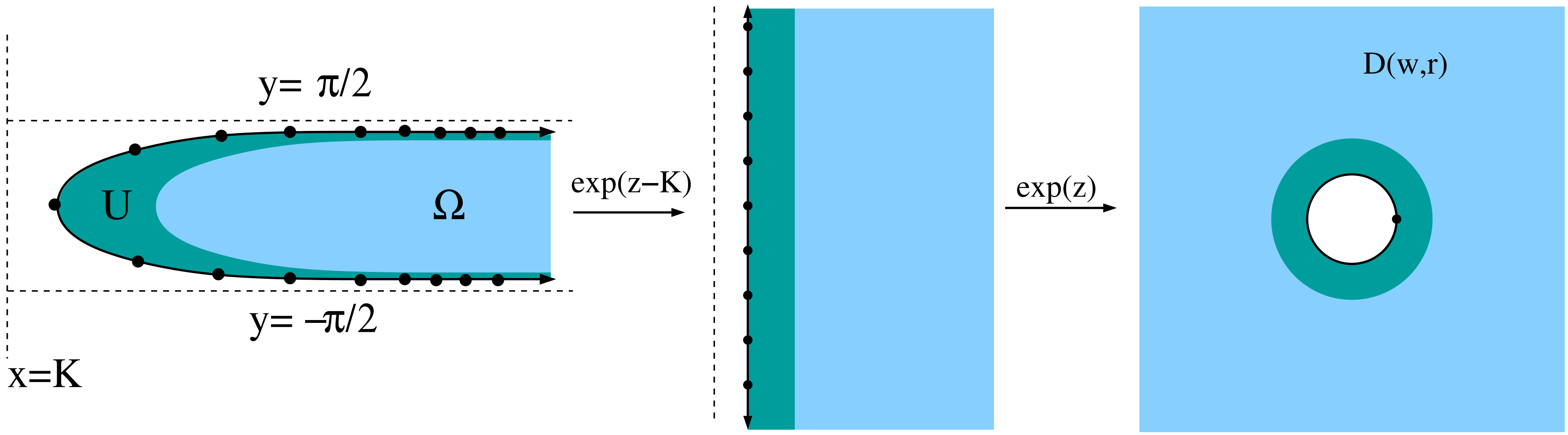


Fixed point $g(0) = 0$ attracts everything in complement of S

Thus the Julia set is inside S. More precisely,

$$\mathcal{J}(g) \subset \bigcap X_n,$$

$$X_n = \{z : |g^k(z)| \geq K, k = 1, \ldots, n\}.$$



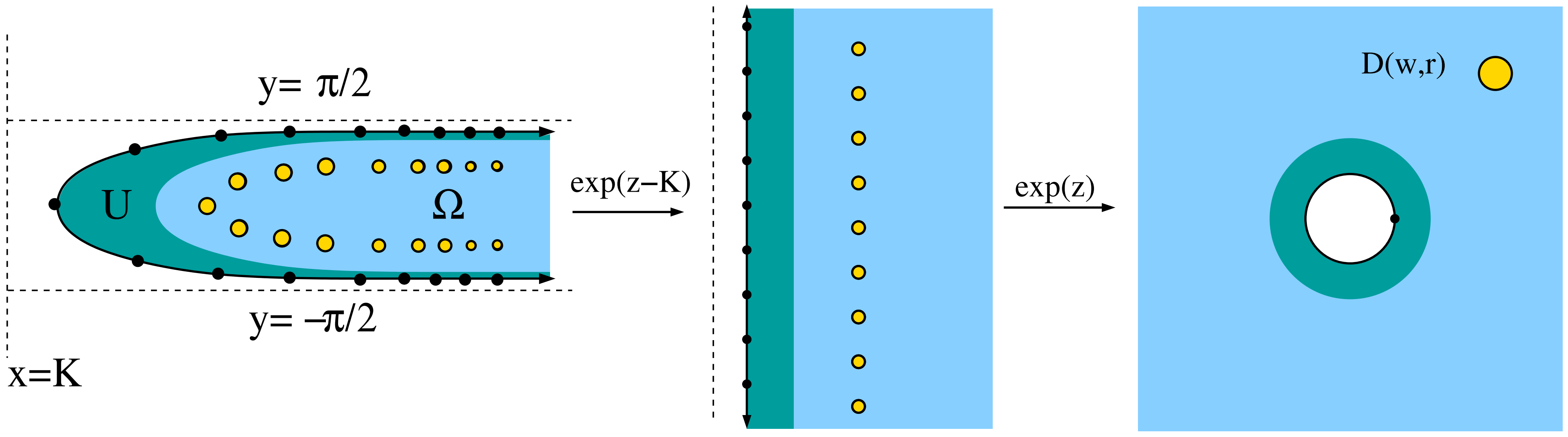
To prove $\dim(\mathcal{J}) \leq 1+\delta$, it suffices to show: if K large enough, then

(1) X_1 can be covered by disks $\{D_j\}$ so that $\sum_j \mathrm{diam}(D_j)^{1+\delta} < \infty$,

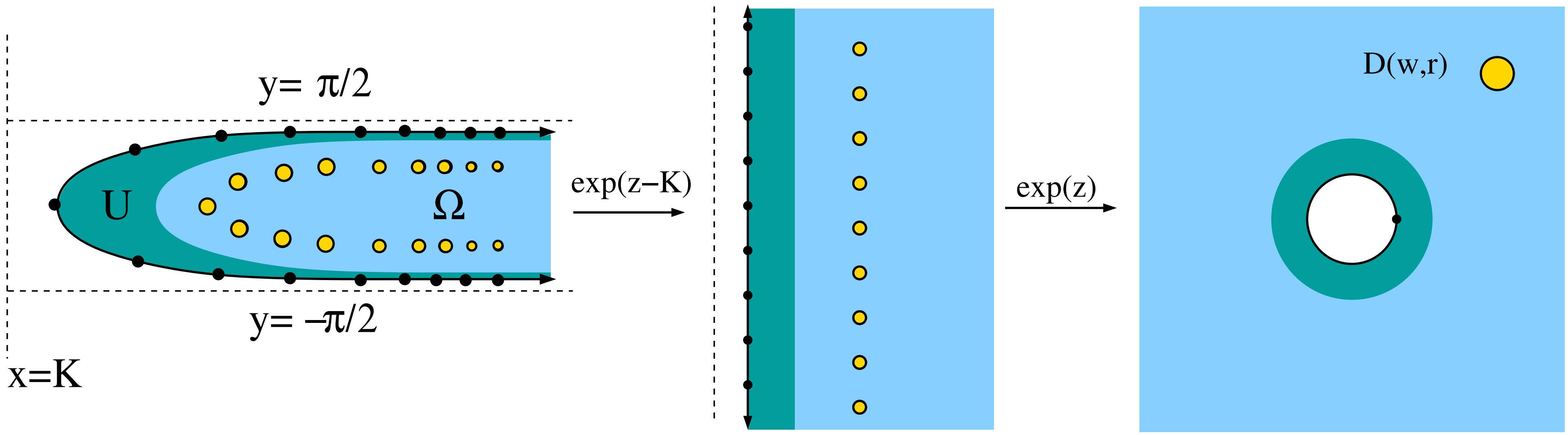
(2) if D hits $\mathcal{J} \cap \{|z| > K\}$, then its preimages satisfy

$$\sum_{W_j \in f^{-1}(D)} \mathrm{diam}(W_j)^{1+\delta} \leq \epsilon \cdot \mathrm{diam}(D)^{1+\delta}.$$

If K is large, we may take $\epsilon > 0$ as small as we wish.



This is what preimages of one disk look like.

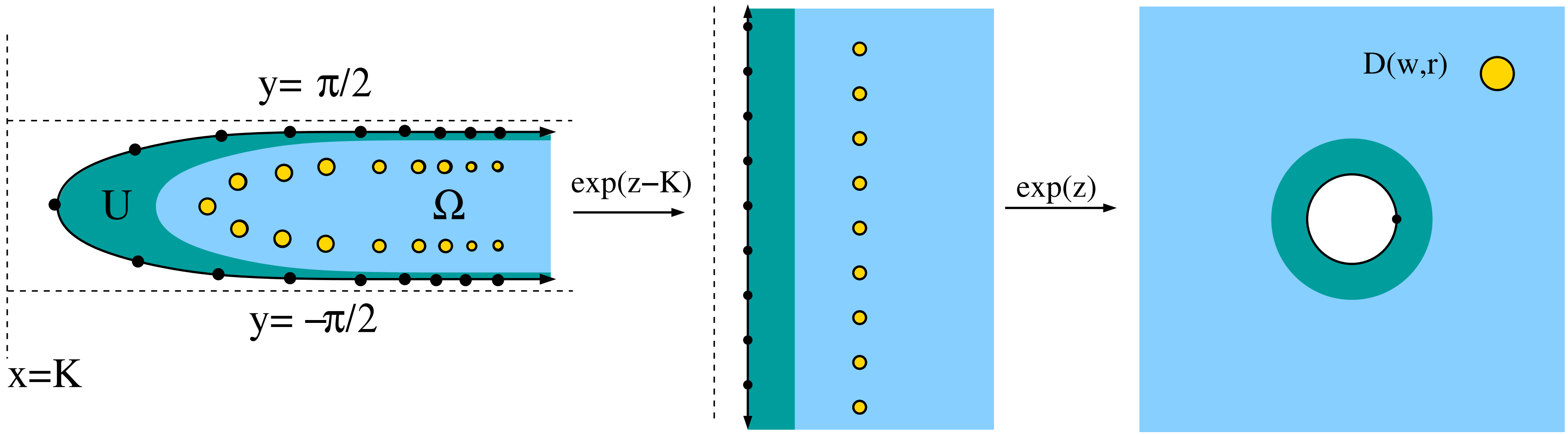


Preimage of gold disk $D = D(w, r)$ defined in two steps:

- stack of regions of diameter $O(r/|w|)$ on line $\{x = \log |w|\}$.
- region at height $2\pi k$ in stack has single preimage U_k of diameter

$$O\left(\frac{r}{|w|(\log |w| + 2\pi |k|)}\right).$$

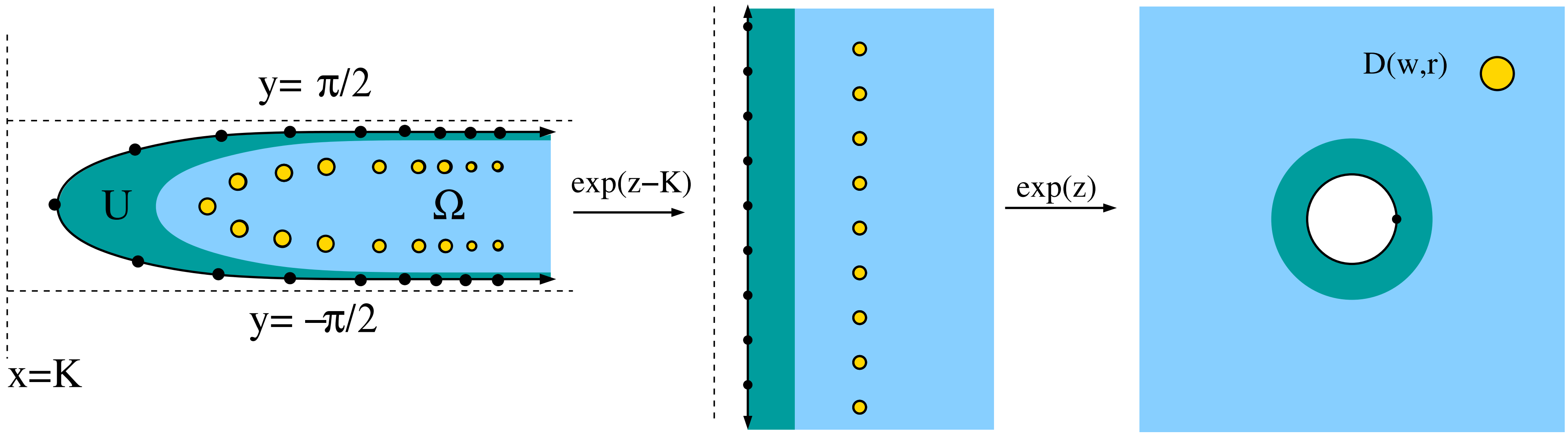
These estimates only use $(\log z)' = 1/z$.



If $\delta > 0$ is fixed and K is large enough, then

$$\sum_k \operatorname{diam}(U_k)^{1+\delta} \lesssim \left(\frac{r}{|w|}\right)^{1+\delta} \sum_k \frac{1}{(\log|w| + 2\pi|k|)^{1+\delta}}$$

$$\lesssim \left|\frac{r}{w}\right|^{1+\delta} \frac{1}{\delta \log^{1+\delta}|w|} \ll \left|\frac{r}{w}\right|^{1+\delta} \ll r^{1+\delta}$$



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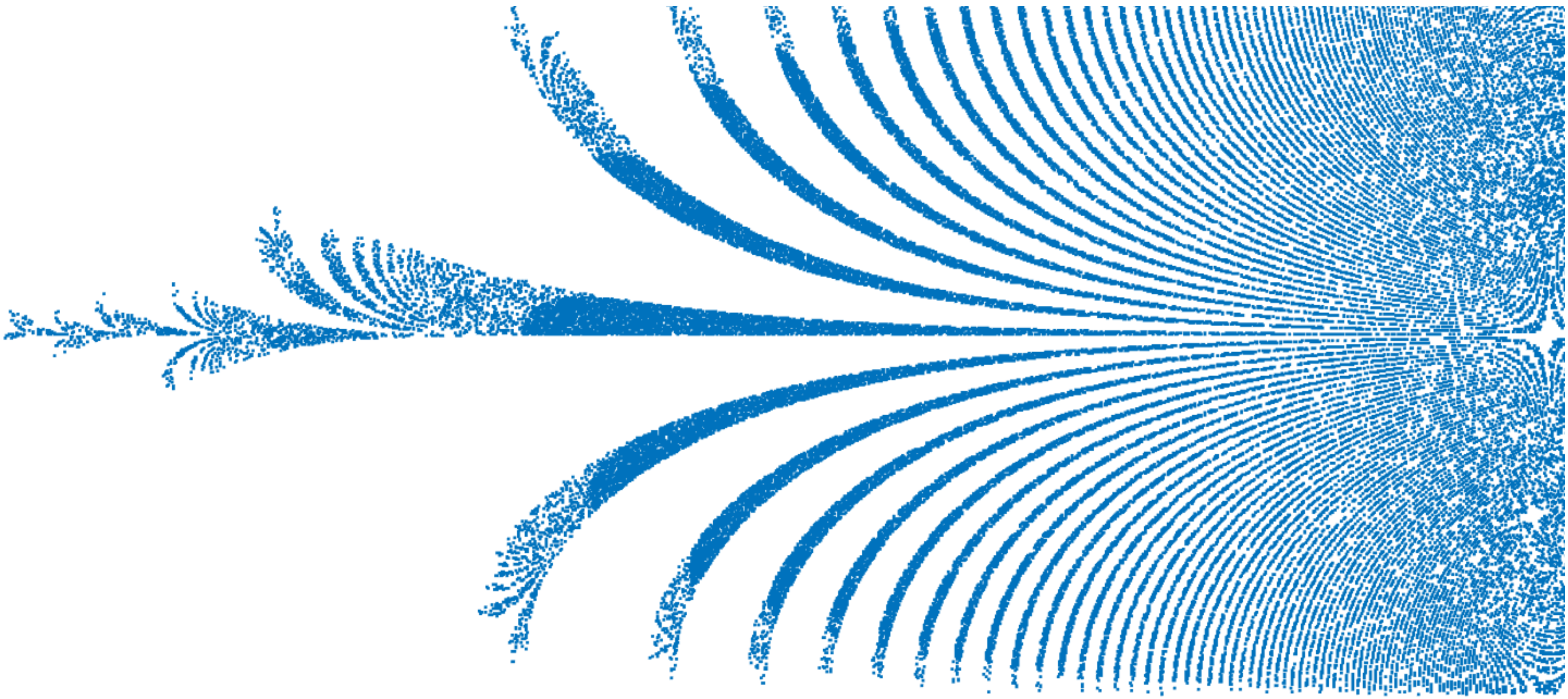
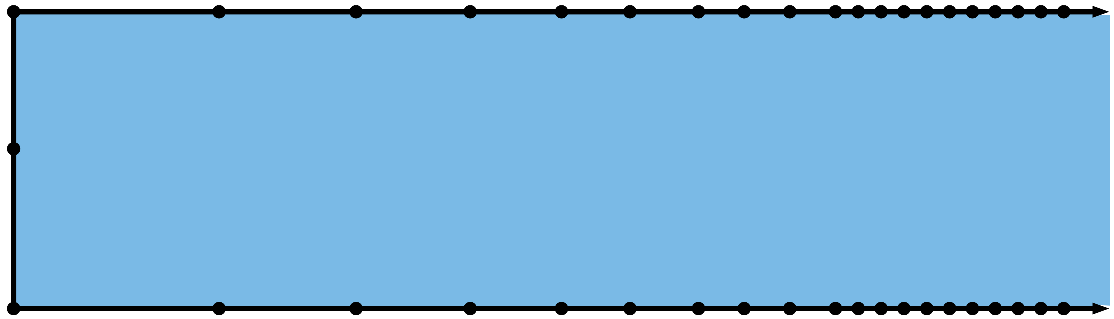
$$\lesssim \left|\frac{r}{w}\right|^{1+\delta} \frac{1}{\delta \log^{1+\delta}|w|} \ll \left|\frac{r}{w}\right|^{1+\delta} \ll r^{1+\delta}$$

This proves $\dim(\mathcal{J}(g)) \leq 1 + \delta$.

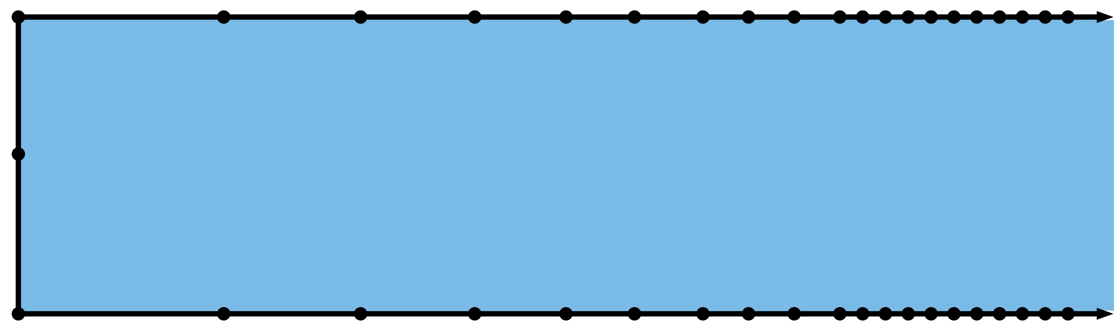




To get $1 < d < 2$ in Speiser class, we want to repeat same argument.

Find g is Speiser class so that

- $g(z) \approx \exp(\exp(z - K))$ in half-strip
- $|g| < 1$ outside n half-strip



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Find g is Speiser class so that

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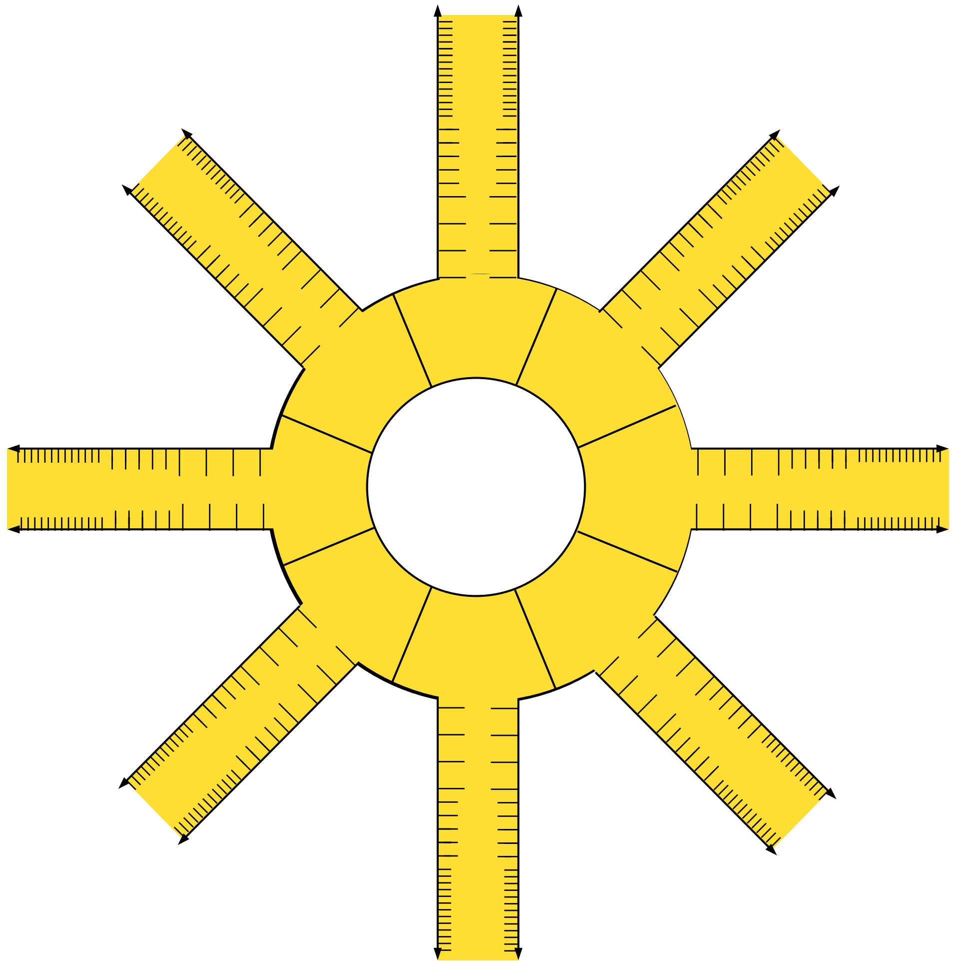
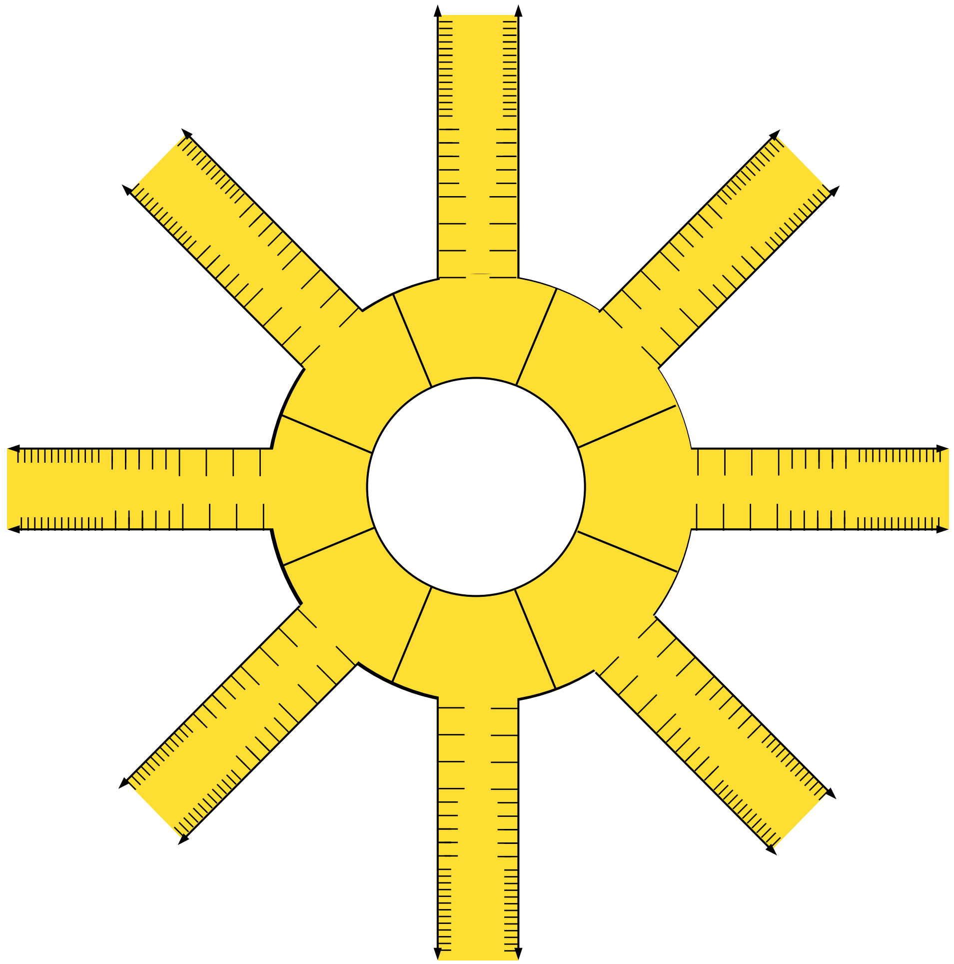
- $|g| < 1$ outside n half-strip

Unfortunately, no such g exists (B. 2017).

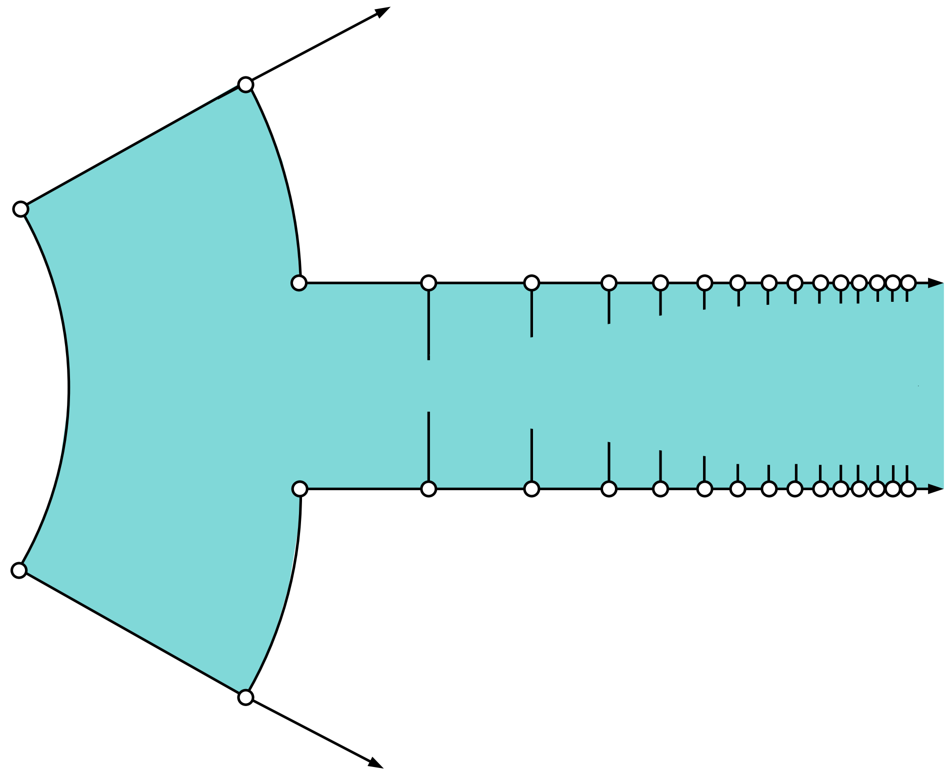


Instead, we use several strip-like tracts.

Then method of quasiconformal folding can be applied.



- $g(z) \approx \exp(\exp(\omega \cdot z - K))$ in each half-strip,
- $|g| < 1$ on sectors.
- Zero attracts central disk and all sectors.



A single R-component.

For QC-folding experts: vertical slits chosen to give $\tau \simeq 1$ on segments between strips and sectors.

Preimages of a disk follow the boundary.

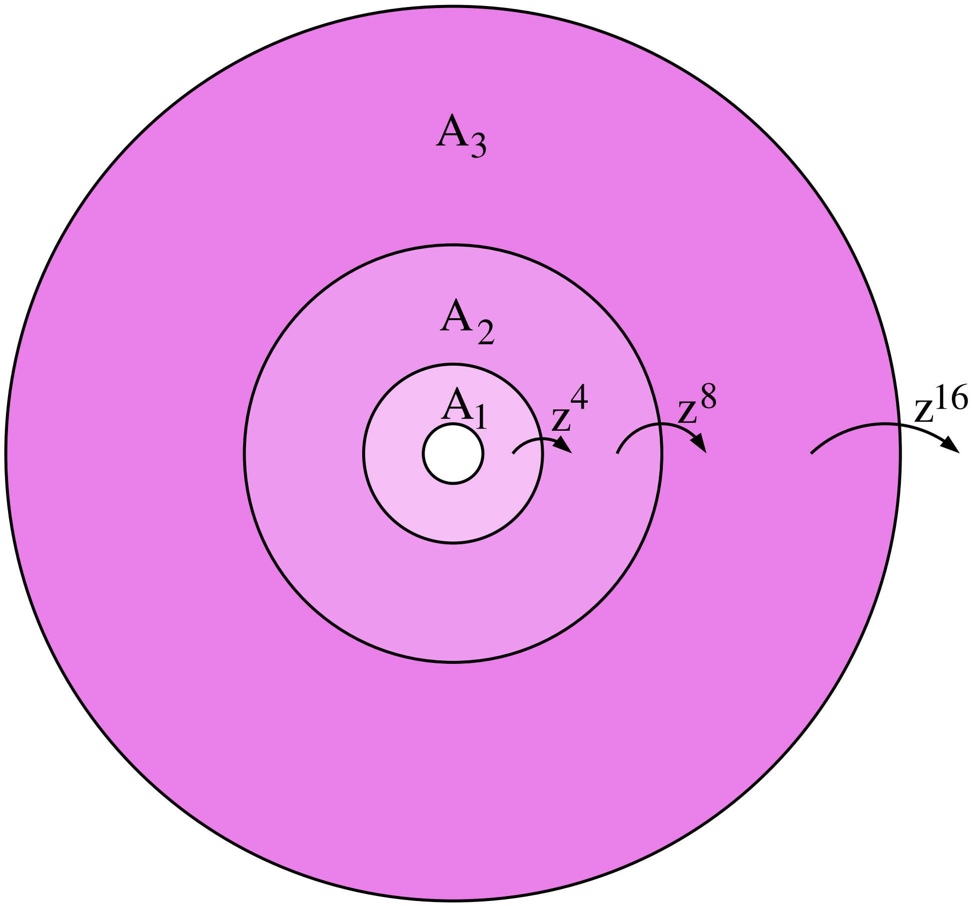
Estimates like before, but more intricate.

Transcendental Julia sets with dimension 1

Theorem: $\text{Hdim}(\mathcal{J}) = \text{Pdim}(\mathcal{J}) = 1$ is possible. I gave example using infinite products. Somewhat technical.

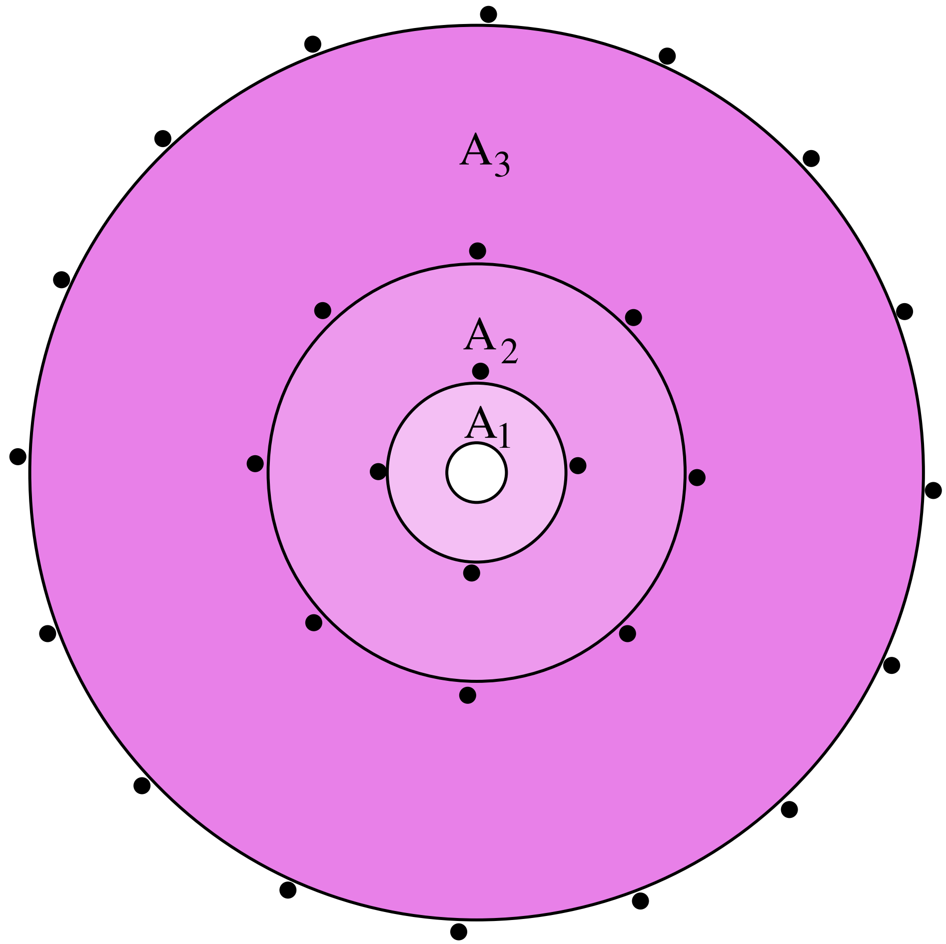
New, more geometric, proof by Burkart and Lazebnik ("folding-like").

Both proofs based on similar geometry.



Suppose we have annuli $\{r_k < |z| < r_{k+1}\}$.

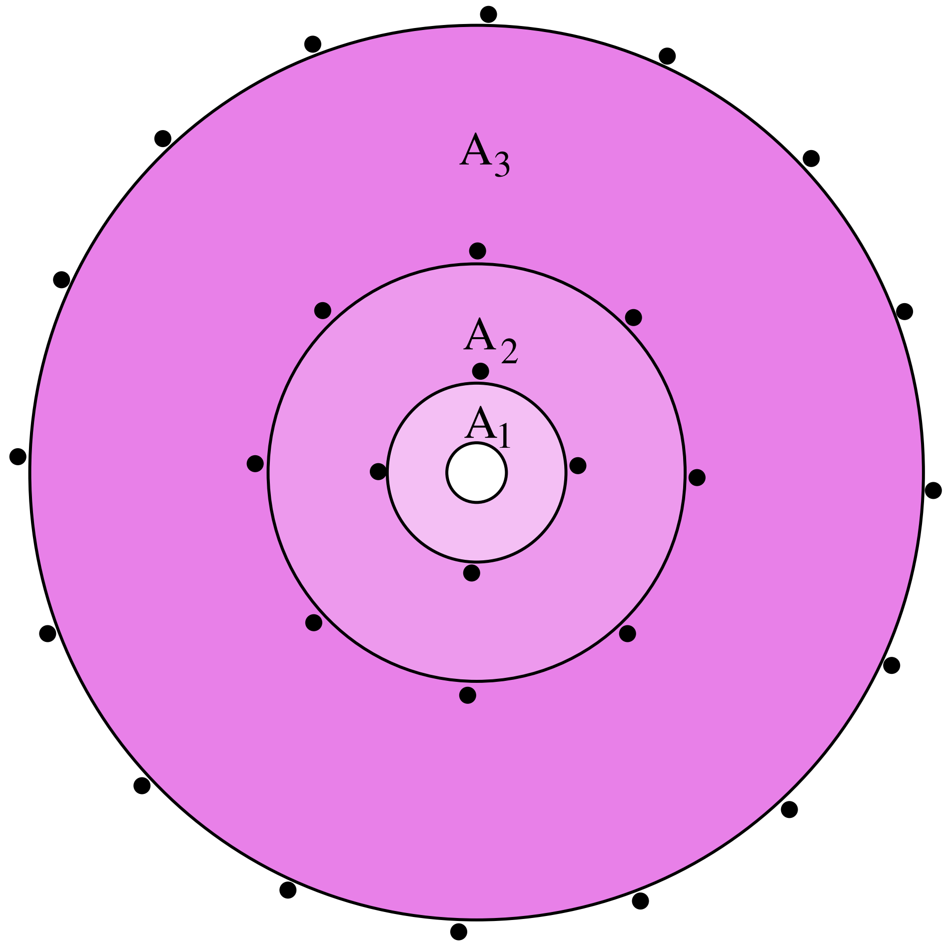
maps $z \to C_k \cdot z^{2^k}$ from A_k to A_{k+1}.



Approximate this by placing 2^k zeros evenly around kth circle.

Approximate polynomial p near origin so there is some Julia set near origin.

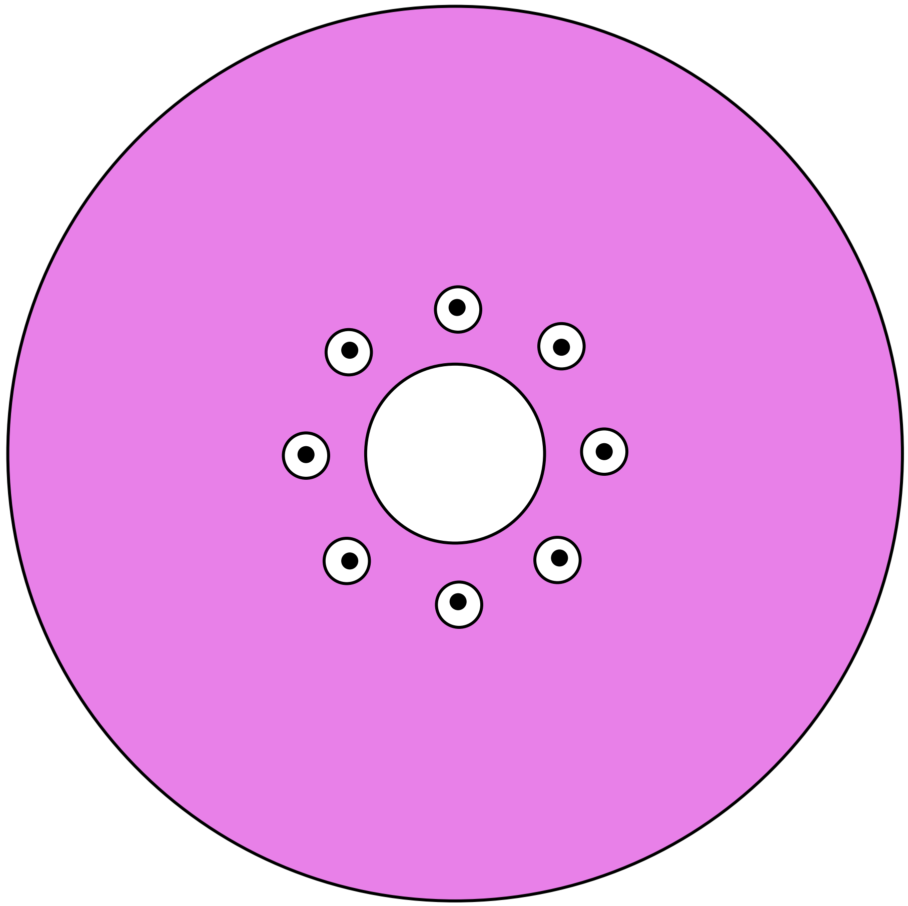
Annuli escape, each corresponds to a different Fatou component.



The kth annulus looks rotationally invariant.

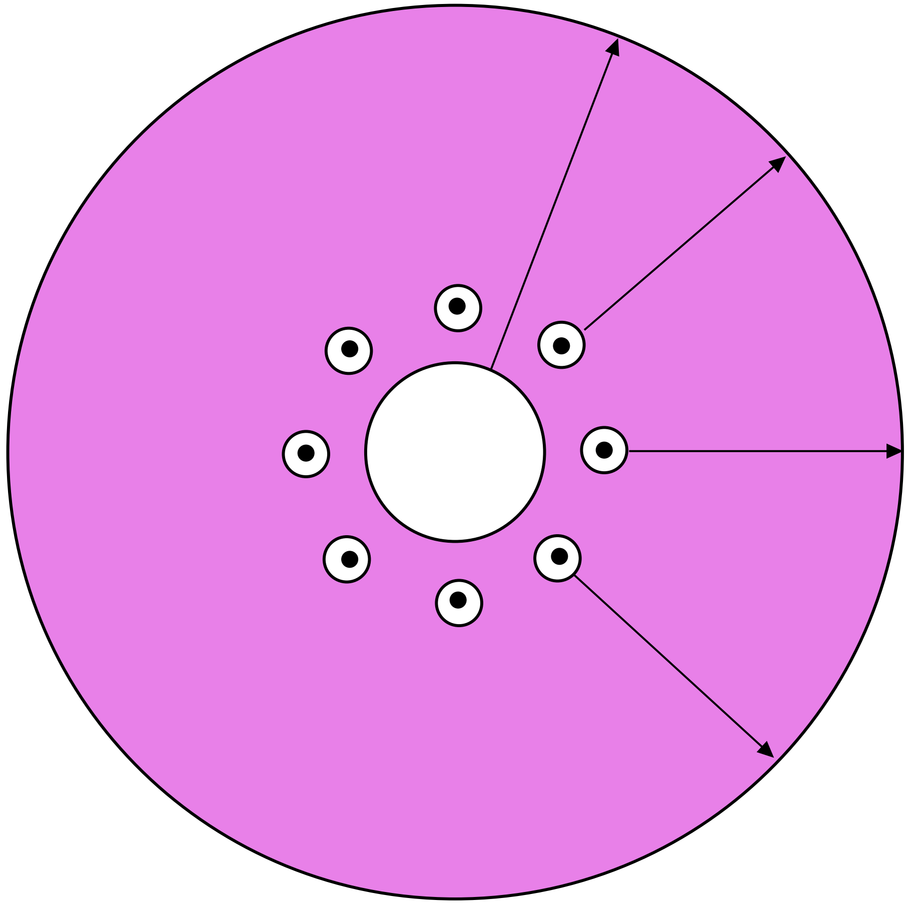
Other zeros are very, very close to 0 or ∞.

Its inner and outer boundaries should be nearly circular.

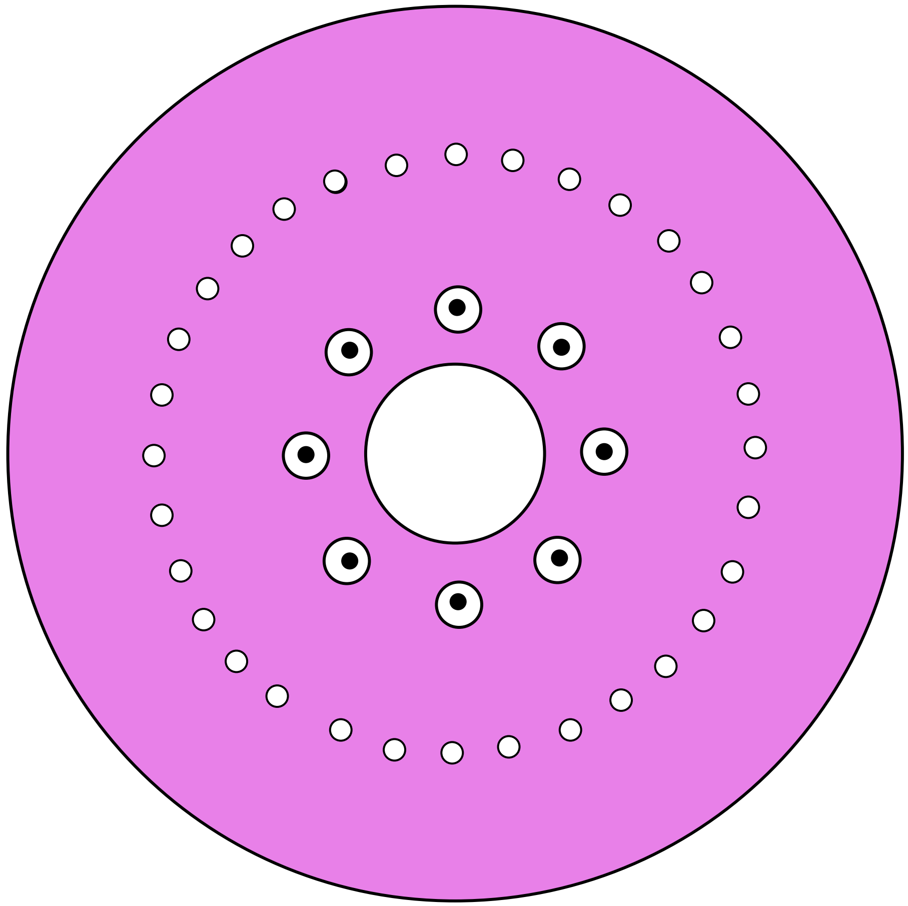


Fatou component has a boundary around each zero.

Must surround pre-image of component at zero.



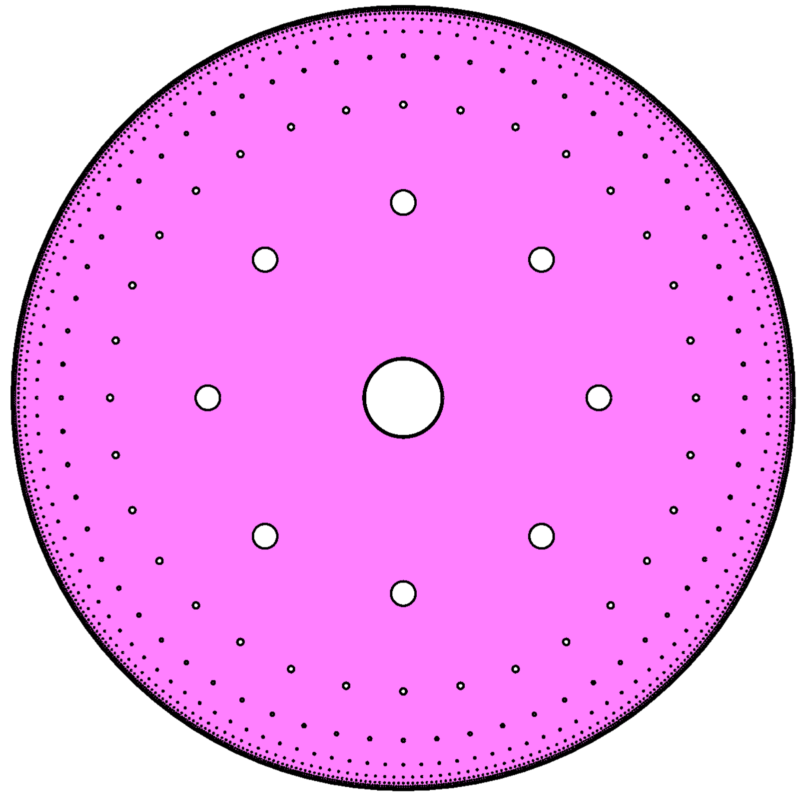
These boundaries and inner boundary map to outer boundary.



The kth annulus maps to $(k+1)st$ annulus.

The $(k+1)$st annulus also has ring of boundary components.

These have a preimage in the kth annulus; a second ring.



There is an infinite sequence of rings converging to outer boundary.

Estimates show component boundary is countable union of C^1 curves.

"Buried" points have small dimension $\Rightarrow \dim(\mathcal{J}) = 1$.

Open questions:

My example has finite spherical 1-measure, but infinite packing 1-measure.

$\Rightarrow$ not subset of rectifiable curve on sphere.

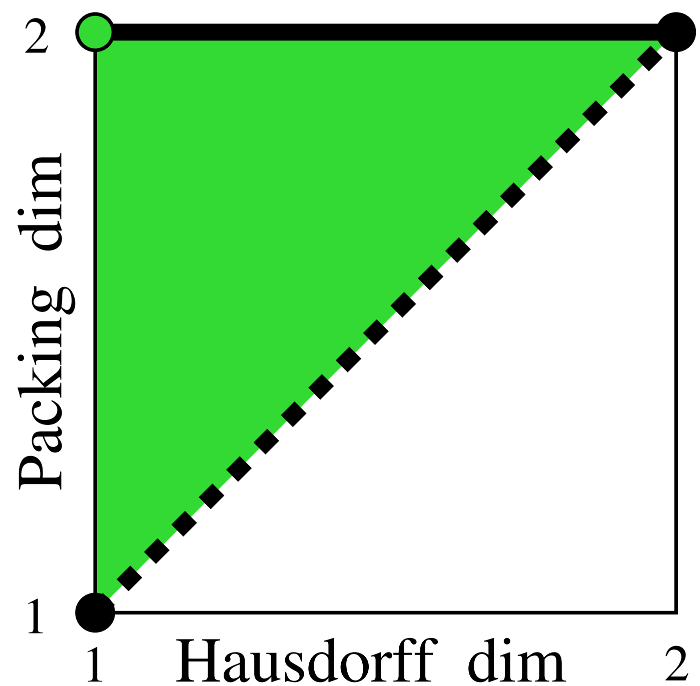
Can a transcendental entire Julia set lie on such a curve?

True for meromorphic. Julia set of $\tan(z)$ is a line.

Are the boundary components better than C^1?

Any Jordan curve is boundary of s.c. wandering domain – Boc Thaler.

Open questions: Black = known, Green = unknown



Which pairs $(\mathrm{Hdim}, \mathrm{Pdim})$ can occur for a transcendental entire function?

Burkart: any pair $(s, s), 1 < s < 2$ can be approximated.

Can we have Hdim = 1, Pdim = 2?

Does Hdim = Pdim hold for all polynomials?

Open questions:

Speiser class Julia sets take dimensions as close to 1 as desired.

Do they take all dimensions in $(1, 2]$?

Can the escaping set have dimension 1?

Open questions:

Eremenko and Lyubich showed Speiser class functions QC equivalent to f (i.e., $g = \psi \circ f \circ \varphi$) are a finite dimensional manifold M_f.

Think of $\dim(\mathcal{J}(g))$ as a function on M_f.

Often this is the constant 2 (e.g., finite order of growth).

Otherwise is it always non-constant?

Is the supremum over M_f always 2?

Do Shishikura's methods for Mandelbrot set apply?


THANKS
ANSWERS?