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Entire functions arising from trees

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Abstract Given any infinite tree in the plane satisfying certain topological conditions, we construct an entire function f with only two critical values ± 1 and no asymptotic values such that $f^{-1}([-1,1])$ is ambiently homeomorphic to the given tree. This can be viewed as a generalization of the result of Grothendieck (see Schneps (1994)) to the case of infinite trees. Moreover, a similar idea leads to a new proof of the result of Nevanlinna (1932) and Elfving (1934).

Keywords entire function, tree, Shabat, Riemann surface, the type problem

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1 Introduction and main results

An entire function is called a *Shabat* entire function if it has only two critical values and no finite asymptotic values (Note that ∞ is always the asymptotic value of any entire function). In the case where it is a polynomial, it is called a Shabat polynomial. Without loss of generality, we may assume that the two critical values are ± 1. For a Shabat entire function f, we define $T_f := f^{-1}([-1,1])$, which can be considered as a graph embedded in $\mathbb{C}$ whose vertices are the preimages of ± 1 and whose edges are the preimages of $(-1,1)$. For example, the function $z \mapsto \sin(z)$ is a Shabat entire function. The following observation for Shabat entire functions is clear.

Observation. Let f be a Shabat entire function. Then T_f is a tree. Moreover,
(1) if f is a polynomial, then T_f is a finite tree;
(2) if f is transcendental, then T_f is an infinite tree.

We say that two trees T_1 and T_2 are *equivalent*, if there is an orientation-preserving homeomorphism $\varphi : \mathbb{C} \to \mathbb{C}$ such that $\varphi(T_1) = T_2$ with vertices and edges being mapped to vertices and edges, respectively. In this sense, we also say that T_1 is ambiently homeomorphic to T_2. If $T = T_f$ for some Shabat entire function f, then T is called a *true tree*.

An inverse problem to the above observation asks for the realization of Shabat entire functions from any given trees embedded in the plane, or equivalently, finding a true tree which is equivalent to the given one. If such a true tree exists, we say that it is a *true form* of the given tree. If a tree is finite, the inverse problem has a positive solution which is essentially due to Grothendieck (see, for example, [18,30]). That is to say, each finite tree T has a true form. This is an important aspect of Grothendieck's theory of dessins d'enfants. For other aspects of finite true trees, see [7]. However, in the case where a tree is infinite, it is possible that no corresponding true tree exists (see Corollary 2.2 and the examples in Section 5).

In this paper we shall explore some topological conditions on the tree that ensure the existence of true forms. As discussed above, the existence of true forms is equivalent to the existence of certain Shabat entire functions. Therefore, essentially we need to construct Shabat entire functions based on the information given by the tree.

Our work is motivated by Bishop's recent groundbreaking technique of quasiconformal folding [8]. More precisely, he proved that for any infinite tree T in the plane satisfying a certain *bounded geometry* condition, there is always a Shabat entire function f such that T_f approximates T in the sense that T_f is the quasiconformal image of a tree T' obtained by adding some branches to T. In general, T_f is not equivalent to T, but T_f can be chosen to lie in a small neighborhood of T.

For some basic knowledge on graph theory, we refer to [10]. To state our conditions, we first introduce some definitions.

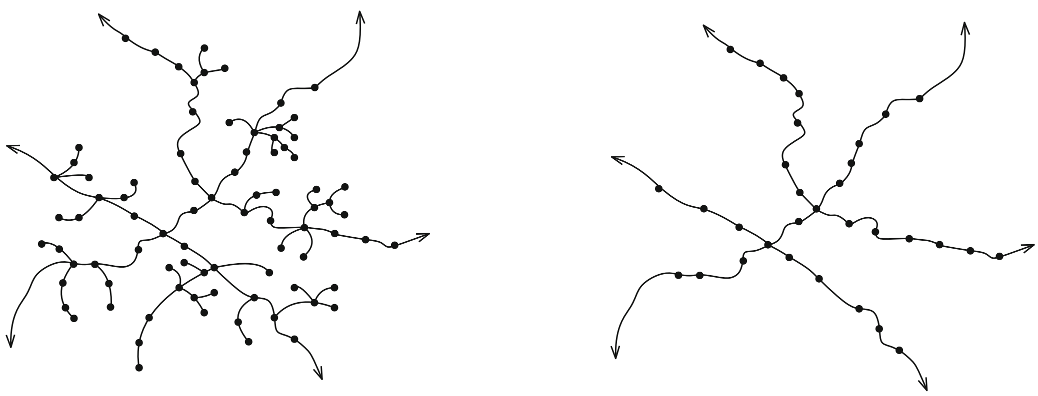
Definition 1.1 (Semi- and bi-infinite paths)**.** Let T be an infinite tree in the plane. A bi-infinite path γ in T is a homogeneous tree in T of valence two. A semi-infinite path is a tree of local valence two at every vertex except for one at which the local valence is one.

Definition 1.2 (Kernel)**.** Let T be an infinite tree in the plane. A *kernel* of T, denoted by $\mathcal{K}(T)$, is defined as a semi-infinite path if T has only one complementary component in the plane, or as the union of all bi-infinite paths in T otherwise.

Roughly speaking, the kernel of a tree is obtained by cutting off all finite branches attached to the tree (see Figure 1). It is easy to see that a homogeneous tree coincides with its kernel. (A tree is homogeneous if all its vertices have the same local valence.) The notion of kernel is well defined in the sense that if an infinite tree T has finitely many ($\geqslant 2$) complementary components in the plane, then the kernel $\mathcal{K}(T)$ is unique. In the case where the tree has exactly one unbounded complementary component (which indeed happens for some Shabat entire functions, such as $z \mapsto \cos\sqrt{z}$), there is no bi-infinite path. In this situation, we consider a path which connects any *fixed* finite vertex to ∞ without backtracking in the tree instead (see Figure 2).

An infinite tree (or more generally a connected graph) can be viewed as a metric space. To be more precise, a graph G is a pair (V, E), where V is a set of vertices and E is a set of pairs of elements in V which are called edges. Then the length of a path is defined to be the number of edges, and the distance of two vertices is the length of the shortest path connecting them. We call this metric the *word metric* of the tree. Thus we can speak of the *word distance* on the tree. Moreover, endowed with this metric any connected tree will be a geodesic metric space. In the following, we will use $d(A, B)$ to denote the word distance between two vertices or two subsets or a vertex and a set.

Now we can formulate our conditions as follows.



(a) (b)

Figure 1 (a) An infinite tree with more than one complementary component. (b) The kernel of the left-hand side tree

Figure 2 (a) An infinite tree with only one complementary component and (b) one of its kernels

Topological uniformness condition (TUC). Let T be an infinite tree in the plane, satisfying the following conditions:

(T1) the local valence of the tree is uniformly bounded above;

(T2) T has finitely many complementary components in the plane;

(T3) $d(v, \mathcal{K}(T))$ is uniformly bounded above, i.e., there exists $M \in \mathbb{N}$ such that $d(v, \mathcal{K}(T)) \leqslant M$ for all vertices v of T.

Our main result can be stated as follows.

Theorem 1.3 (Entire functions from trees). *Let T satisfy the topological uniformness condition. Then T has a true form.*

In other words, the above theorem says that there is a Shabat entire function f such that T_f is equivalent to T. The next question one may ask is the *uniqueness* of the true form. This follows from the fact that two entire functions with only two singular values which are topologically equivalent in the sense of Eremenko and Lyubich [17] are in fact conformally equivalent [15, Proposition 2.3]. Thus we see that the true form is unique up to affine maps in the plane.

We note that no condition in the TUC can be dropped. We will construct various examples to show this in Section 5. These examples also indicate that one can slightly extend the condition we imposed above.

Idea of proof. The proof of Theorem 1.3 is based on the criterion of the classical type problem from the geometric theory of meromorphic functions (see Subsection 2.1). To each such tree T in the theorem, one can construct a Speiser graph Γ. Then it is well known that there is a surface (X, p) spread over the sphere of class $\mathcal{S}$ which corresponds to the Speiser graph Γ one to one. If the surface is parabolic, then there is a Shabat entire function f whose Speiser graph is Γ. Moreover, it will be clear from the proof that T_f will be equivalent to T. We will show that as long as the topological uniformness condition is satisfied, the surface will be of parabolic type, which hence ensures the existence of a Shabat entire function.

The rest of this article is organized as follows. Section 2 collects some notions and results required. Then the proof of Theorem 1.3 is given in Section 3. In the following section, we give a new proof of a result of Nevanlinna [26] and Elfving [14]. The last section is devoted to the construction of various examples showing the sharpness of our theorem.

2 Preliminaries

2.1 The type problem

We give here a brief introduction to the *type problem* in the geometric function theory. For a detailed description, we refer to [12, 16].

According to the uniformization theorem, every open simply connected Riemann surface X is conformally equivalent to either the complex plane $\mathbb{C}$ or the unit disk $\mathbb{D}$. The Riemann surface is said to be of *parabolic* or *hyperbolic* type, respectively. In other words, there exists a conformal map $\phi : X_0 \to X$, where X_0 is $\mathbb{C}$ or $\mathbb{D}$. The map ϕ is called the *uniformizing map*.

A *surface spread over the sphere* is an equivalence class $[(X, p)]$, where X is an open, simply connected, topological surface, and $p : X \to \widehat{\mathbb{C}}$ a topologically holomorphic map, i.e., an open, continuous and discrete map. Here, (X_1, p_1) is equivalent to (X_2, p_2) if there exists a homeomorphism $\psi : X_1 \to X_2$ such that $p_1 = p_2 \circ \psi$.

A theorem of Stoïlow [32] says that any topologically holomorphic map is locally modeled by the power map $z \mapsto z^k$ for some $k \in \mathbb{N}$. Thus there is a unique Riemann surface structure on X which makes it a Riemann surface. So there is a uniformizing map $\phi : X_0 \to X$, and hence $f := p \circ \phi$ is a meromorphic function in $\mathbb{C}$ or $\mathbb{D}$. The surface (X, p) is called the surface associated with f. For a given surface spread over the sphere, the type problem is the determination of the conformal type of the surface.

Some classical results can be viewed as criteria of the type problem, for example, Picard's theorem, Ahlfors' five islands theorem and others. In particular, we recall the following result due to Nevanlinna [27], which can also serve as a criterion for the type problem. Recall that a point $a \in \widehat{\mathbb{C}}$ is said to be a *totally ramified value* (of multiplicity $m \geqslant 2$) of a surface (X, p) spread over the sphere, if all preimages of a under p are multiple (of multiplicity at least m). Take $m = \infty$ if a is omitted. The following result is due to Nevanlinna.

Theorem 2.1. *If a surface spread over the sphere has q totally ramified values of multiplicities m_k for $1 \leqslant k \leqslant q$, and*

$$\sum_{k=1}^{q} \left(1 - \frac{1}{m_k}\right) > 2, \tag{2.1}$$

then the surface is hyperbolic. In particular, a parabolic surface has at most four totally ramified values.

We mentioned in the introduction that some trees do not have true forms. Indeed, the above theorem of Nevanlinna can be used to construct such trees. We only mention the following example.

Corollary 2.2. *Any homogeneous tree of valence greater than or equal to* 3 *does not have a true form.*

Proof. Suppose that such a tree indeed gives a Shabat entire function f. Then ± 1 will be two totally ramified values of f (and another one is ∞). Every vertex of the tree is a preimage of one of ± 1. Hence for $q = 3$, $m_1 = m_2 = 3$ and $m_3 = \infty$, the inequality in the above theorem is satisfied, which leads to a contradiction since the surface corresponding to a transcendental entire function is always parabolic. □

2.2 Speiser class

For a surface (X, p) spread over the sphere, the map $p : X \to \widehat{\mathbb{C}}$ is modeled on the power map $z \mapsto z^k$ in a neighborhood of each point $x \in X$ for some $k := k(z) \in \mathbb{N}$. If $k \geqslant 2$, then x is a critical point of p, and $p(x)$ is called a *critical value* of p. A point $b \in \widehat{\mathbb{C}}$ is called an *asymptotic value* of p if there exists a curve $\gamma : [0, \infty) \to X$, such that $\gamma(t)$ leaves every compact subset of X as $t \to \infty$, and $p(\gamma(t)) \to b$ as $t \to \infty$. When we say a is a *singular value*, we always mean that it is a critical value or an asymptotic value.

A surface (X, p) spread over the sphere belongs to the *Speiser class*[1)], denoted by $\mathcal{S}$, if p has only finitely many singular values. In other words, the set of singular values of (X, p) is the smallest closed subset Sing of $\widehat{\mathbb{C}}$ such that

$$p : X \setminus \{p^{-1}(\mathrm{Sing})\} \to \widehat{\mathbb{C}} \setminus \mathrm{Sing}$$

is a covering map. A surface in class $\mathcal{S}$ has a combinatorial representation in terms of a *Speiser graph* (also called the *line complex*). To give a definition of a Speiser graph, assume that $(X, p) \in \mathcal{S}$ has q singular values $a_1, \dots, a_q$. Then one can choose an oriented Jordan curve L on $\widehat{\mathbb{C}}$ passing through $a_1, a_2, \dots, a_q$ in the cyclic order. The curve L decomposes $\widehat{\mathbb{C}}$ into two components A and B such that A is to the left-hand side of positive orientation of L. Then we choose two base points $\circ \in A$ and $\times \in B$ and connect them by open Jordan arcs γ_j $(j = 1, \dots, q)$ such that $\gamma_j \cap \gamma_k = \emptyset$ for $j \neq k$ and γ_j intersects the segment (a_j, a_{j+1}) in L at exactly one point with indices modulo q. Then the Speiser graph Γ of the surface (X, p) is defined to be the embedding of $p^{-1}(\bigcup_j \gamma_j)$ into the plane $\mathbb{C}$. For some figures of Speiser graphs, see [19, Chapter 7, Section 4] and also the figures in Section 5.

1) We note here that the terminology *Speiser class* is also used in the other setting, where a function meromorphic in the plane belongs to the Speiser class if the function has finitely many singular values. But here the setting is larger in the sense that every Speiser class function gives a parabolic surface spread over the sphere, but a Speiser class surface need not be parabolic.

We consider Speiser graphs defined above as graphs in the sense that preimages of $\circ$ (marked as circles) and preimages of $\times$ (marked as crosses) serve as vertices of the graph while the set of edges contains all preimages of γ_j for all j. Therefore, Speiser graphs have the following properties: Each edge connects a preimage of $\circ$ and a preimage of $\times$; around each vertex there are exactly q edges emanating from this vertex. Moreover, the faces around a vertex have a certain cyclic order inherited from the orientation of L (which can be seen by marking faces using a_j). For a detailed description of graphic properties of Speiser graphs we refer to [19, Chapter 7], in which one can find applications of Speiser graphs in the theory of meromorphic functions. An important property is that, up to the choice of a base curve, there is a one-to-one correspondence between surfaces in class $\mathcal{S}$ and equivalence classes of Speiser graphs. Here, two Speiser graphs are equivalent if they are ambiently homeomorphic. For a detailed proof of this fact, we refer to [19].

Remark 2.3. Transcendental entire functions in class $\mathcal{S}$ attract a lot of interest in transcendental dynamics since the work of Eremenko and Lyubich [17], mainly due to the fact that these functions have certain similarities to polynomials from a dynamical point of view.

2.3 Doyle-Merenkov criterion

There are many criteria to determine the conformal type of a surface in class $\mathcal{S}$ by using Speiser graphs by Ahlfors, Nevanlinna, Ullrich, Wittich and others. For a detailed account, see [35]. Doyle [11] used a modification of the Speiser graph to give a necessary and sufficient condition. This was generalized to a larger setting by Merenkov [24] by using a geometric method.

We say that an infinite, locally finite, connected graph is parabolic/hyperbolic, if the simple random walk on the graph is recurrent/transient. The Doyle-Merenkov criterion connects the type of a surface and that of some extended Speiser graph which is defined as follows.

Let Λ denote a half-plane lattice with vertices at $\mathbb{Z}\times\mathbb{N}_0$, and $\Lambda_n := \Lambda/n\mathbb{Z}$ a half-cylinder lattice. Let Γ be a Speiser graph. Then the *extended Speiser graph* Γ_n is defined as follows: In each unbounded face we embed a half-plane lattice Λ by identifying the corresponding edges and vertices on the boundaries, and in each face of Γ with $2k$ edges on the boundary for $k \geqslant n$, we embed the half-cylinder lattice Λ_{2k}. The graph Γ_n obtained in this way is called an *extended Speiser graph* of Γ. The graph obtained by adding a half-plane lattice to each unbounded face and leaving all bounded faces unchanged is denoted by Γ_∞. As a matter of fact, $\Gamma_\infty = \Gamma_n$ for some $n \in \mathbb{N}$ if all bounded faces of Γ have an upper bound for the numbers of edges on the boundaries.

Theorem 2.4 (Doyle-Merenkov criterion). *Let $n \in \mathbb{N}$ be fixed. A surface spread over the sphere $(X,p) \in \mathcal{S}$ is parabolic if and only if Γ_n is parabolic.*

This criterion is easy to apply and has many applications, one of which is the construction of a parabolic surface with negative mean excess (see [3]). This answers a question of Nevanlinna [27] in the negative.

2.4 Quasi-isometry

We shall use in the proof of Theorem 1.3 the notion of the quasi-isometry between two infinite, locally finite graphs both endowed with the word metric.

Definition 2.5 (Quasi-isometry). Let (X_1, d_1) and (X_2, d_2) be two metric spaces. A map $\Phi : X_1 \to X_2$ is called a quasi-isometry, if it satisfies the following two conditions:

(1) for some $\varepsilon > 0$, the ε-neighborhood of the image of Φ in X_2 covers X_2;

(2) there are constants $k \geqslant 1$, $C \geqslant 0$ such that for all $x_1, x_2 \in X_1$,

$$\frac{1}{k}\cdot d_1(x_1,x_2) - C \leqslant d_2(\Phi(x_1),\Phi(x_2)) \leqslant k\cdot d_1(x_1,x_2) + C.$$

It is easy to check that being quasi-isometric is an equivalence relation. The notion of quasi-isometry was introduced by Gromov [20] and Kanai [21]. Many properties of metric spaces are preserved under quasi-isometries. For example, the Gromov hyperbolicity of geodesic metric spaces is stable under quasi-isometric maps. For us, a particularly important property of the quasi-isometry is that the type of an infinite, locally finite graph is stable under quasi-isometries. This is essentially due to Kanai [22, Corollary 7].

Theorem 2.6 (Stability of the type). *Let Γ_1 and Γ_2 be two connected, finite valence graphs endowed with the word metric, which are quasi-isometric. Then Γ_1 and Γ_2 are simultaneously hyperbolic or parabolic.*

2.5 Combinatorial modulus

To determine the type of a graph, we shall use the notion of combinatorial modulus, which can be viewed as a discrete version of the classical conformal modulus. The notion was used by Duffin [13] and Cannon [9]. Here, we will follow the presentation in [24, Section 5].

A *mass distribution* for a graph Γ is a non-negative function on the set E of edges of Γ. A chain in Γ is a sequence $\{e_j\}_{j=M}^{N}$ with $-\infty \leqslant M \leqslant N \leqslant \infty$, where e_j are edges and adjacent edges share a vertex. Let $\mathcal{C}$ be a family of chains in Γ. A mass distribution m is admissible for the family $\mathcal{C}$, if $\sum m(e_j) \geqslant 1$ for each chain $\{e_j\}$ in $\mathcal{C}$. Then the *combinatorial modulus* of the chain family $\mathcal{C}$ is defined as

$$\operatorname{mod} \mathcal{C} := \inf \left\{ \sum_{e \in E} m(e)^2 \right\},$$

where the infimum is taken over all admissible mass distributions. The reciprocal of the combinatorial modulus is called the *extremal length* of $\mathcal{C}$, denoted by $\lambda(\mathcal{C})$. The combinatorial modulus has properties similar to those of the conformal modulus (see [2, 31]).

A domain in a graph is a connected subset of the graph in the sense that every two vertices in the domain can be connected by a chain whose edges belong to this domain. An annulus in a graph is a subset of the set of edges whose complement in the graph has two disjoint domains. A sequence of annuli $\{A_n\}$ is nested if the annuli are pairwise disjoint, and A_{n+1} separates A_n from ∞. We will use $\operatorname{mod} A$ for an annulus A to denote the combinatorial modulus of the family of chains connecting the two boundary components of A. Here, the boundary of an annulus A in a graph Γ is the set of vertices that are not in A and each of which can be connected to a vertex in A by one edge.

A locally finite graph Γ is hyperbolic or parabolic if and only if $\lambda(\Gamma_v)$ is finite or infinite respectively for some vertex v (and hence for every vertex) in Γ, where Γ_v denotes the family of chains connecting v to infinity (see, for example, [31, Corollary 3.84]). Compare this with a similar result for the conformal modulus (see, for example, [12, Theorem 1]).

For our applications, to determine parabolicity it is sufficient to use the following proposition.

Proposition 2.7. *Let Γ be a connected finite valence graph. If there exists a sequence $\{A_n\}$ of disjoint nested annuli with*

$$\sum_{n=1}^{\infty} \frac{1}{\operatorname{mod} A_n} = \infty,$$

then Γ is parabolic.

This result will be used later to prove the parabolicity of surfaces associated with trees satisfying the TUC. As we cannot locate a proof of this result, we will include here a short proof.

Proof of Proposition 2.7. As noted above, to show that Γ is parabolic we can choose any vertex v and show that $\lambda(\Gamma_v) = \infty$, where Γ_v denotes the family of chains connecting v to infinity. Here, we choose v to be a vertex belonging to the inner complementary component of A_1. Then

$$\lambda(\Gamma_v) \geqslant \sum_{n=1}^{\infty} \lambda(A_n) = \sum_{n=1}^{\infty} \frac{1}{\operatorname{mod} A_n} = \infty.$$

The first inequality above follows from the definition of the extremal length. $\square$

3 TUC implies parabolicity

The proof of Theorem 1.3 is divided into two steps. First, we construct a Speiser graph Γ considered as the dual graph of a triangulation of the plane induced from the given tree T. This is a standard procedure. In the next step, we use the Doyle-Merenkov criterion to show that some extended Speiser graph Γ_∞ is of parabolic type, and thus the surface corresponding to Γ is parabolic. With this in hand we will obtain a Shabat entire function.

3.1 Construction of a Speiser graph from the tree

Suppose that T is any given tree in the plane which satisfies the topological uniformness condition. Then T can be colored in a bipartite pattern (by simply choosing one vertex as black and every vertex which has word distance one as white, and then proceeding). Connect each side of an edge to infinity by adding two curves at the two endpoints. This gives a triangulation of the sphere based on the given tree (see Figure 3). Then we consider the dual graph of this triangulation, denoted by Γ (i.e., choose a point in each triangle and then connect two points as long as the two triangles share a common edge). It is easy to see that Γ is a Speiser graph. Thus there is a surface (X, p) spread over the sphere in class $\mathcal{S}$ which corresponds to Γ.

Since we start with an infinite tree in the plane with finitely many complementary components, the Speiser graph Γ will have finitely many unbounded faces, and every other face has a uniformly bounded number of edges on its boundary.

For later purposes, we also consider the same process for the kernel $\mathcal{K}(T)$ of T, and thus obtain another Speiser graph Σ. The surface corresponding to Σ will be parabolic. Note that the kernel of a tree satisfying the TUC also satisfies the TUC. If we knew Theorem 1.3 already, we could conclude that the surface of the kernel is parabolic. Instead, we will prove this fact by using Goldberg's theory of almost periodic ends. More precisely, except for a finite portion in the corresponding Speiser graph every component of the rest is a sine-end (which is a special case of periodic ends). Since there are only finitely many such ends, the corresponding surface is parabolic (see [19, Chapter 7, Section 7]). Thus the existence of Shabat entire functions is assured.

3.2 Determination of the type

Now we consider an extended Speiser graph by embedding a half-plane lattice in each unbounded face of Γ to get an extended Speiser graph Γ_∞, which is infinite and locally finite. We also denote by Γ^*_∞ the dual graph of Γ_∞.

Similarly, the Speiser graph Σ corresponding to the kernel $\mathcal{K}(T)$ is used to obtain an extended Speiser graph Σ_∞ and its dual is denoted by Σ^*_∞.

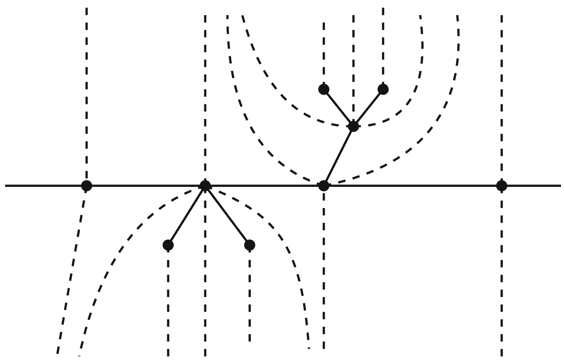


Figure 3 A triangulation of the plane based on a given tree. Every triangle has one side of an edge on the tree as one edge. The dual graph of this triangulation will be the Speiser graph associated with this tree

As can be observed from the construction, Γ_∞^* is nothing but topologically an extension of the tree T by embedding a half-plane lattice in each complementary component; so is Σ_∞^*.

Proposition 3.1. *Γ_∞^* is quasi-isometric to Γ_∞ and Σ_∞^* is quasi-isometric to Σ_∞.*

Proof. All graphs here have finite valences. Then we define a map which sends a vertex v of Γ_∞^* (resp. Σ_∞^*) to any vertex on the boundary of the face of Γ_∞ (resp. Σ_∞) corresponding to v. Then it is easy to check that this map is a quasi-isometry. For details, we refer to [25, Theorem C]. □

Theorem 3.2. *Γ_∞^* is quasi-isometric to Σ_∞^*.*

Proof. To prove the theorem, we need to construct a quasi-isometry φ between Γ_∞^* and Σ_∞^*, both of which are endowed with the word metric. As we noted before, the graph Γ_∞^* can be obtained by embedding a half-plane lattice in each face of the tree T, while Σ_∞^* is obtained by embedding a half-plane lattice in each face of the kernel $\mathcal{K}(T)$ of the tree T. Recall that $\mathcal{K}(T)$ is obtained from T by cutting all finite trees attached to some vertices.

The quasi-isometry will be constructed by gluing two maps: The first one is a quasi-isometry which maps the tree T to its kernel $\mathcal{K}(T)$ and the second one is essentially a quotient map on half-plane lattices.

To define the required maps, we call every vertex v on the kernel $\mathcal{K}(T)$ a *parent*, and a vertex v' on $T\setminus\mathcal{K}(T)$ is said to be a *child* of v if v' can be connected to v through a path without intersecting with the kernel. See Figure 4 as an illustration. Therefore, by definition every child has exactly one parent since the tree endowed with the word metric is a geodesic metric space, where two vertices can be connected by a unique geodesic. Moreover, it follows from the topological uniformness condition that every parent has at most finitely many children.

We will define a map $\varphi : \Gamma_\infty^* \to \Sigma_\infty^*$ by

$$\varphi(v) = \begin{cases} \varphi_1(v), & v \in T, \\ \varphi_2(v), & v \in \Gamma_\infty^* \setminus T \end{cases} \tag{3.1}$$

such that φ is a quasi-isometry. We start with the construction of the quasi-isometry $\varphi_1 : T \to \mathcal{K}(T)$. The map is defined locally. To be more precise, with v as a parent the map φ_1 sends the union of v and all its children to v. Then the topological uniformness condition implies that φ_1 is actually a quasi-isometry. The first condition in Definition 2.5 is easy to check. For the second one we choose two arbitrary vertices v_1 and v_2 on T. By (T3) in the TUC there is a constant M not depending on v_1 and v_2 such that $\mathrm{dist}(v_j, \mathcal{K}(T)) \leqslant M$ for $j = 1, 2$. Moreover, suppose that the parents of v_j are u_j for $j = 1, 2$. Then by our definition of φ_1 we see that $\varphi_1(v_j) = u_j$. Thus

$$d(\varphi_1(v_1), \varphi_1(v_2)) = d(u_1, u_2) \leqslant d(v_1, v_2). \tag{3.2}$$

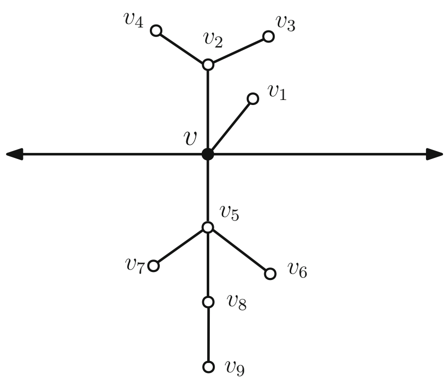


Figure 4 An illustration of the notion of parents and children. Shown is a portion of the tree T around a vertex on the kernel in which the arrowed line denotes part of the kernel $\mathcal{K}(T)$. By definition, the vertex v is a parent, while all the other vertices marked, namely $v_1, v_2, \ldots, v_9$ in the figure, are the children of v

On the other hand we know that

$$d(v_1, v_2) = d(u_1, u_2) + d(v_1, \mathcal{K}(T)) + d(v_2, \mathcal{K}(T)) \leqslant d(u_1, u_2) + 2M.$$

Therefore, we see that

$$d(\varphi_1(v_1), \varphi_1(v_2)) = d(u_1, u_2) \geqslant d(v_1, v_2) - 2M. \tag{3.3}$$

Combining the above two estimates (3.2) and (3.3) we immediately see that $\varphi_1 : T \to \mathcal{K}(T)$ is a quasi-isometry.

To get the map φ we still need to find a map φ_2 which is defined on the remaining part of Γ_∞^*. For this purpose, let u be a vertex on $\Gamma_\infty^* \setminus T$ (which is equivalent to saying that u belongs to the embedded half-plane lattice). Then there is exactly one vertex on T which minimizes the distance between u and T. Thus as we defined before, this unique vertex has a unique parent v on $\mathcal{K}(T)$. In this sense, we also say that v is a parent of u. Denote by T_v the union of v and all its children on T. Moreover, suppose that the distance between u and T_v is $k := k(u) \in \mathbb{N}$. Note that for the parent v of u, in each component of $\mathbb{C} \setminus \mathcal{K}(T)$ there is exactly one vertex u' in $\Sigma_\infty^* \setminus \mathcal{K}(T)$ such that $\mathrm{dist}(u', v) = \mathrm{dist}(u', \mathcal{K}(T)) = k(u)$. Then we define the map

$$\varphi_2 : \Gamma_\infty^* \setminus T \to \Sigma_\infty^* \setminus \mathcal{K}(T), \quad \varphi_2(u) = u',$$

where u' is the unique vertex corresponding to u as discussed above. Now we show that φ_2 is a quasi-isometry in each component of $\Gamma_\infty^* \setminus T$. Notice that $\Gamma_\infty^* \setminus T$ is not connected but consists of finitely many components due to (T2) in the TUC. We only need to check the second condition in Definition 2.5. Let u_1 and u_2 be two vertices in one component of $\Gamma_\infty^* \setminus T$ whose parents are v_1 and v_2, respectively. First, note that

$$d(\varphi_2(u_1), \varphi_2(u_2)) \leqslant d(u_1, u_2), \tag{3.4}$$

since basically the map φ_2 is a shrinking of the half-plane lattice. From the other side, without loss of generality we assume that $k(u_1) \geqslant k(u_2)$. Since by the TUC, at each v_j there are only finitely many vertices on each component of $T \setminus \mathcal{K}(T)$ and the number is uniformly bounded above by some constant A depending only on the TUC, now we have

$$\begin{aligned} d(u_1, u_2) &\leqslant d(v_1, v_2) + 2A \cdot B + k(u_1) - k(u_2) \\ &= d(v_1, v_2) + 2A \cdot B + d(\varphi_2(u_1), v_1) - d(\varphi_2(u_2), v_2) \\ &\leqslant d(\varphi_2(u_1), \varphi_2(u_2)) + 2A \cdot B, \end{aligned} \tag{3.5}$$

where B is a constant which bounds the local valence of T. By (T1) this constant depends only on the tree. Therefore, φ_2 is a quasi-isometry on each component by (3.4) and (3.5).

Now it is left to show that the map φ defined in (3.1) is a quasi-isometry. Clearly it is only necessary to check the case where two vertices are chosen in different components of $\Gamma_\infty^* \setminus T$. Let u_1 and u_2 be two such vertices with parents v_1 and v_2, respectively. Then

$$\begin{aligned} d(u_1, u_2) &\leqslant d(u_1, T_{v_1}) + M + d(v_1, v_2) + d(u_2, T_{v_2}) + M \\ &= d(\varphi(u_1), v_1) + d(v_1, v_2) + d(\varphi(u_2), v_2) + 2M \\ &= d(\varphi(u_1), \varphi(u_2)) + 2M. \end{aligned} \tag{3.6}$$

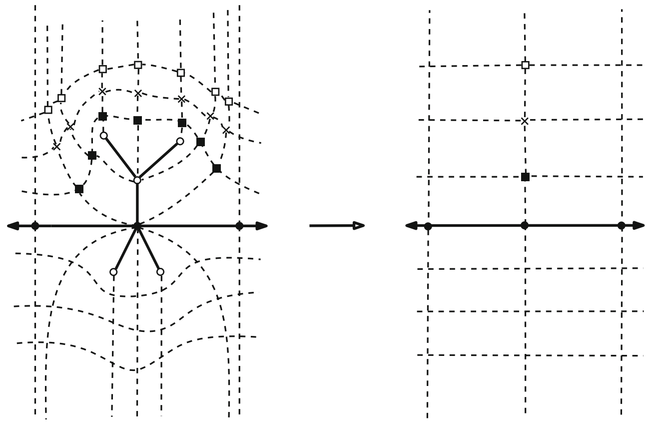
Moreover, it is easy to see that

$$d(\varphi(u_1), \varphi(u_2)) \leqslant d(u_1, u_2). \tag{3.7}$$

Thus, by the above discussion the map φ is a quasi-isometry. For an illustration of the map φ, see Figure 5. □

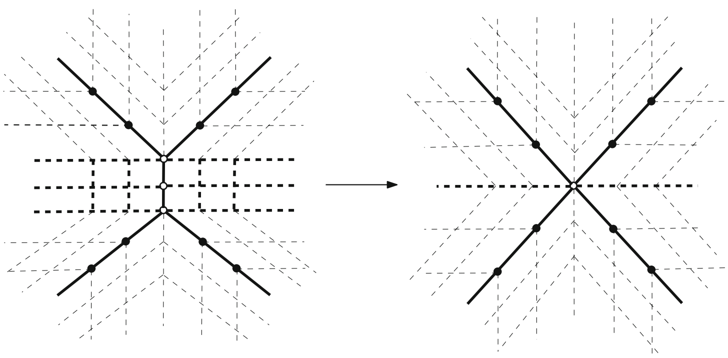
To prove Theorem 1.3 we will show that Σ_∞^* is of parabolic type. To this aim, we introduce a standard model Σ_s for each Σ_∞^*. Let N be the number of complementary components of $\mathcal{K}(T)$ in the plane. Denote by T_s an infinite tree in the plane which satisfies:

(1) T_s has N complementary components in the plane;



(a) (b)

Figure 5 The quasi-isometric map is defined by sending every vertex in T to its parent, and every vertex which is not in T and has combinatorial distance k to the unique vertex which is not in $\mathcal{K}(T)$ and has combinatorial distance k to the kernel as shown in the figure. In the figure, all boxes, crosses and square vertices in (a) are sent to the box, cross and square vertex in (b), respectively. All vertices on the tree but not on the kernel (marked as circles) are sent to their parents



(a) (b)

Figure 6 (a) The kernel $\mathcal{K}(T)$ of a tree T and Σ_∞^*. (b) The standard model Σ_s. The map is then defined as identifying lines marked as deep dark dashed lines

(2) T_s has exactly one vertex of valence N and all the other vertices are of local valence two.
Then the standard model Σ_s is defined from T_s by embedding in each complementary component of T_s a half-plane lattice, identifying the boundary vertices of T_s with the boundary vertices of the half-plane lattice. See the right-hand side of Figure 6 in the case where $N = 4$, in which T_s is the graph marked as solid lines.

Proposition 3.3. *Σ_∞^* is quasi-isometric to Σ_s.*

Proof. We only sketch the idea of proof. First define a quasi-isometry between $\mathcal{K}(T)$ and T_s, which is, roughly speaking, sending all vertices with local valence greater than two and all vertices in between to the only vertex in T_s with local valence greater than two. Then one needs to define a map from $\Sigma_\infty^* \setminus \mathcal{K}(T)$ to $\Sigma_s \setminus T_s$, which is similar to the map φ_2 defined in the proof of Theorem 3.2 (see Figure 6 for an explanation). The details are left to the reader. □

Theorem 3.4. *Σ_s is parabolic.*

Proof. We sketch a proof. Suppose that $\mathcal{K}(T)$ has N complementary components in the plane. According to Proposition 2.7, it is enough to find a sequence of disjoint annuli $\{A_n\}$ in Σ_s such that

$$\sum \frac{1}{\operatorname{mod} A_n} = \infty. \tag{3.8}$$

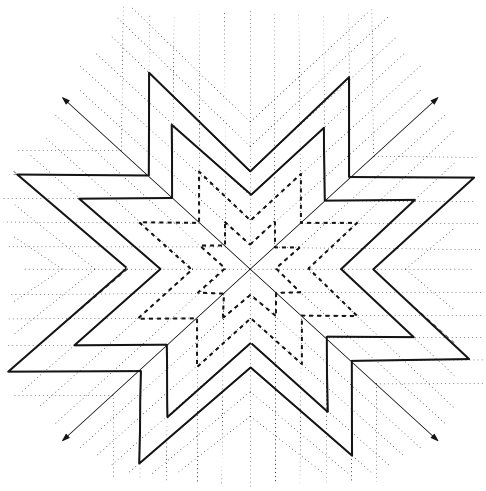


Figure 7 The first annulus is A_1 with inner and outer boundaries marked as dashed lines, while the second annulus is A_2 with boundaries marked as solid lines. Both annuli have combinatorial width 1

To this aim, let A_n be a finite graph which has combinatorial width 1 in each component of $\Sigma_s \setminus T_s$ (see Figure 7). We consider a mass distribution on Σ_s which assigns mass one to every edge. It is easy to check that this mass distribution is admissible for every chain. Then we can see that

$$\operatorname{mod} A_n = \mathcal{O}(n),$$

which implies (3.8). This completes the proof of the theorem. □

Proof of Theorem 1.3. In some sense, our theorem says that the type of the surface is determined by the kernel of the tree. It follows from the Doyle-Merenkov criterion that the surface (X, p) corresponding to Γ is parabolic if and only if Γ_∞ is parabolic. By Theorem 2.6 and Proposition 3.1 we see that Γ_∞ is parabolic if and only if Γ_∞^* is parabolic. Proposition 3.3 and again Theorem 2.6 say that this holds if and only if Σ_s is parabolic. Theorem 3.4 says that this is indeed the case. Thus Γ_∞ is parabolic and hence the surface (X, p) is parabolic. Therefore, we obtain an entire function f with two critical values ± 1 and no asymptotic values such that T_f is homeomorphic to the given tree T. □

4 Nevanlinna's theorem revisited

Using similar ideas to those in the proof of Theorem 1.3, we give here a topological and combinatorial proof of a result due to Nevanlinna [26] and Elfving [14]. For this purpose we need the notion of a logarithmic singularity. We follow the definition in [4].

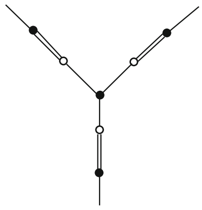
Let (X, p) be a surface spread over the sphere in class $\mathcal{S}$. Denote by A the set of singular values of p. Then

$$p : X \setminus p^{-1}(A) \to \widehat{\mathbb{C}} \setminus A$$

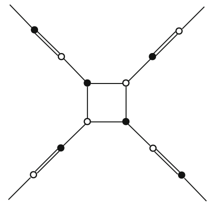
is a covering map. Let $b \in A$ be fixed and $D_\chi(b, r)$ be a spherical disk centered at b with radius r. Consider a function $U : r \mapsto U(r)$ which to each $r > 0$ assigns a component $U(r)$ of $p^{-1}(D_\chi(b, r))$ in such a way that $r_1 < r_2$ implies that $U(r_1) \subset U(r_2)$. If $\bigcap_{r>0} U(r) = \emptyset$, then we say that $r \mapsto U(r)$ defines a singular point over b. A singular point over b is said to be *logarithmic* if for some $r > 0$ the map $p|_{U(r)} : U(r) \to D_\chi(b, r) \setminus \{b\}$ is a universal covering.

Nevanlinna's theorem is then stated as follows.

Theorem 4.1 (See [14, 26]). *Let (X, p) be a surface spread over the sphere in the Speiser class with finitely many logarithmic singularities ($\geqslant 2$) and finitely many critical points. Then the surface is parabolic.*



(a)



(b)

Figure 8 (a) Speiser graph with three logarithmic singularities and without critical points; the corresponding meromorphic function has polynomial Schwarzian derivative. (b) Speiser graph with one critical point (corresponding to the face with four edges on the boundary) and with four logarithmic singularities; the function is actually $z \mapsto e^{z^2}$. Both surfaces are ramified over three values

The above form is due to Elfving [14]. Nevanlinna [26] proved his theorem without allowing critical points. His proof uses Speiser graphs and some theory from complex differential equations (see also [27, Chapter XI]). Right after the paper [26], Ahlfors [1] gave another proof, in which the connection with quasiconformal mappings is implicitly mentioned. Recently, a purely analytic approach to the above theorem is given by Langley [23] by using the Wiman-Valiron property of meromorphic functions with logarithmic singularities [6]. Here, we supply a purely combinatorial proof, which also uses Speiser graphs, but the theory of complex differential equations is not used.

Since surfaces satisfying the condition in the theorem are parabolic, there exist meromorphic functions in the plane corresponding to these surfaces. The importance of these surfaces lies in the fact that the corresponding meromorphic functions are extremal in the sense of Nevanlinna's inverse problem (see [19, Chapter 7] for explanations and more details, and also a proof of the above theorem).

Proof of Theorem 4.1. The proof will be similar to that of Theorem 1.3. The key observation is that up to a large disk in the graph which contains all critical points, the Speiser graph is essentially a "tree" (when "observed from far away").

We assume that the surface (X, p) has q logarithmic singularities and m critical points (see Figure 8 for some examples with and without critical points).

We denote by Γ the corresponding Speiser graph corresponding to (X, p). By Theorem 2.4, to show that (X, p) is parabolic we only need to embed a half-plane lattice in each unbounded face of Γ to obtain an extended Speiser graph Γ_∞ and show that Γ_∞ is parabolic (noting that we can do so since there is a uniform bound on the number of edges of faces). To achieve this, we need to do some surgery on the Speiser graph Γ. We first replace each face with finitely many edges on the boundary by a single vertex. Then if two vertices are connected by more than one edge, we replace all these edges by a single edge. After these operations, what is left is actually a tree which coincides with its kernel, denoted by T. Now in each unbounded face of T we, as before, embed a half-plane lattice. Then we get a graph, denoted by Σ_∞. It follows from the proof of Theorem 1.3 that Σ_∞ is parabolic.

The rest of the proof proceeds as that of Theorem 1.3. Since there are only finitely many critical points, it is easy to establish a quasi-isometry between Γ and T. Then we establish a quasi-isometry between each component of $\Gamma_\infty \setminus \Gamma$ and the corresponding component of $\Sigma_\infty \setminus T$. This is similar as before. Then combining all these quasi-isometries we can define a quasi-isometry between Γ_∞ and Σ_∞. The parabolicity of Σ_∞ implies that Γ_∞ is parabolic. Thus (X, p) is parabolic. The theorem follows. $\square$

5 Examples

This section is devoted to the construction of some examples concerning the items in the TUC. On the one hand, we construct Shabat entire functions which show that every item in the TUC can be slightly extended. On the other hand, we show that if any one of the items is dropped, there may not be Shabat entire functions.

5.1 Growing local valence

We consider trees with growing local valences. First, we show that there exist Shabat entire functions such that the local valence of preimages of one critical value tends to infinity.

5.1.1 *Parabolicity*

There exist Shabat entire functions with unbounded local valences. Such functions can actually be constructed by using the so-called MacLane-Vinberg method. Moreover, entire functions constructed in this way belong to the *Laguerre-Pólya class*, denoted by $\mathcal{LP}$, which consists of entire functions that are locally uniform limits of real polynomials with only real zeros. Thus such entire functions have only real zeros.

Let us consider the example in [5, Example 1.7], which was constructed by using the above MacLane-Vinberg method and hence belongs to the Laguerre-Pólya class. More precisely, the function, denoted by g, has two critical values 0 and 1 and no finite asymptotic values. It is easy to see that the map $f(z) := 2g(z) - 1$ is an entire function with two critical values ± 1 and no finite asymptotic values, and hence is a Shabat entire function. By the properties of $\mathcal{LP}$ functions, the preimages of -1 are all real (which correspond to all zeros of g) whose multiplicities tend to ∞ quite fast (see Figure 9).

We provide here another example. For this and also later purposes, we start with the Speiser graph of the sine function, which is shown in Figure 10. Now on the upper and lower edges of face ② we add two vertices respectively such that the type of the vertices is arranged to be compatible and the newly produced graph is still a Speiser graph (Actually this can be done as along as we add an even number of vertices on the edge). We do a similar surgery to the face -2. Then we continue this to all faces marked as $2k$, where $2|k|$ vertices are added on the upper and lower edges of face $2k$ respectively and then certain additional edges are added to make the graph a Speiser graph (of local valence 3). Denote this Speiser graph by Γ. We obtain the following graph, shown in Figure 11. Now we need to show that Γ corresponds to a surface spread over the sphere of parabolic type. Note first that associated with Γ is a tree which is locally finite, and satisfies the conditions (T2) and (T3) in the TUC but with a growing local valence (i.e., (T1) is violated). We use a criterion due to Nevanlinna and Wittich [36, Chapter VII], which can also be used to prove Nevanlinna's theorem in the case where there are no critical points. Since every vertex in the graph can be connected to ∞ with a path in $\mathbb{C}$ without intersecting with Γ, to use the above mentioned criterion we only need to fix a vertex as generation 0. Then the generation 1 will be the vertices that can be connected to the generation 0. Let Γ^n be the portion of Γ with vertices of generation at most n. Then a vertex in Γ^n is in $\partial\Gamma^n$ if it can be connected to infinity with a path in $\mathbb{C}$ without an intersection with Γ. Denote by $s(n)$ the number of vertices on $\partial\Gamma^n$. Nevanlinna-Wittich's criterion is as follows: The surface spread over the sphere corresponding to Γ is parabolic if $\sum \frac{1}{s(n)} = \infty$. It is easy to check that the graph in Figure 11 satisfies that $s(n) = \mathcal{O}(n)$ and hence the existence of a Shabat entire function with a growing local valence follows.

Remark 5.1. Another method to show that the surface associated with the Speiser graph in Figure 11 is parabolic is by using Theorem 2.4: We consider the extended Speiser graph Γ_4 and then find a sequence of disjoint annuli of width 1 to show that Proposition 2.7 is satisfied. Details are omitted.

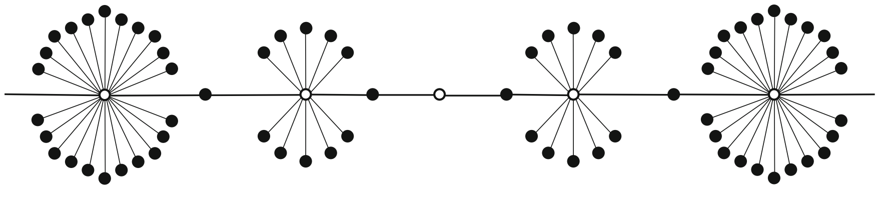


Figure 9 A rough sketch of the tree corresponding to the functions constructed in [5]. The local valence of critical points (marked as circles) corresponding to the critical value -1 tends to infinity fast

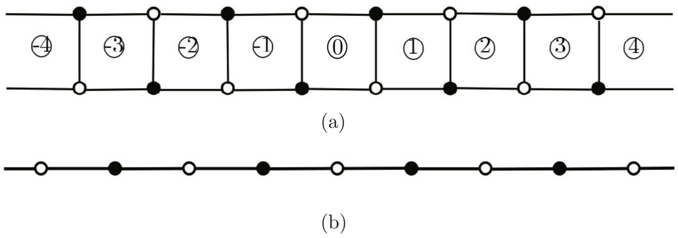


Figure 10 The Speiser graph of the sine function (a) and the associated tree (b). In the graph, we choose one face as a starting face marked as ⓪ for our constructions afterwards. The faces to the right and to the left are marked as ① and -①, respectively. We continue this marking as shown in the figure

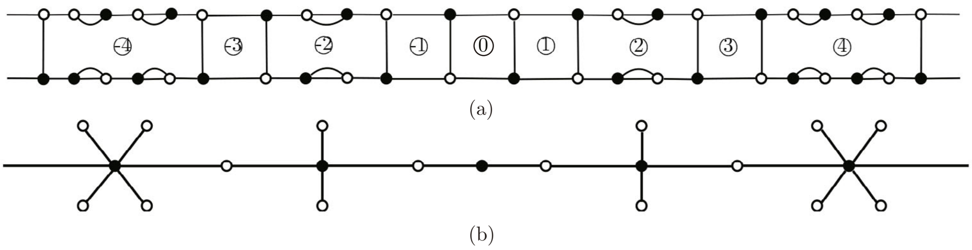


Figure 11 A Speiser graph with a growing local valence (a) and its corresponding tree (b)

5.1.2 *Hyperbolicity*

In this part, we will construct a tree which satisfies the TUC except (T1). Our example here is based on the surface given by Sutter [33] (see also [28]). The Speiser graph of this surface, denoted by Σ, is shown in Figure 12. However, this surface does not meet our requirement since it has a logarithmic singularity over some finite value. Nevertheless, we can modify this example for our aim. More precisely, we consider the Speiser graph Γ shown in Figure 13, where two unbounded faces represent two logarithmic singularities over infinity while faces with finitely many edges on the boundary correspond to critical points which are preimages of critical values ± 1. The corresponding tree is also shown in the figure.

We need to show that the surface corresponding to our Speiser graph is hyperbolic. For this aim, we consider the extended Speiser graphs Σ_3 and Γ_3 of Σ and Γ, respectively. By Theorem 2.4 and the hyperbolicity of the surface corresponding to Σ we know that Σ_3 is hyperbolic. Moreover, to show that the surface corresponding to Γ is hyperbolic, again by Theorem 2.4 it is enough to show that Γ_3 is hyperbolic. Now by Theorem 2.6 we only need to show that Γ_3 is quasi-isometric to Σ_3. But this is easy to establish and we omit the details. Moreover, we can arrange the markings of the Speiser graph Γ such that the two unbounded faces represent two logarithmic singularities over ∞ while all finite faces correspond to critical values ± 1.

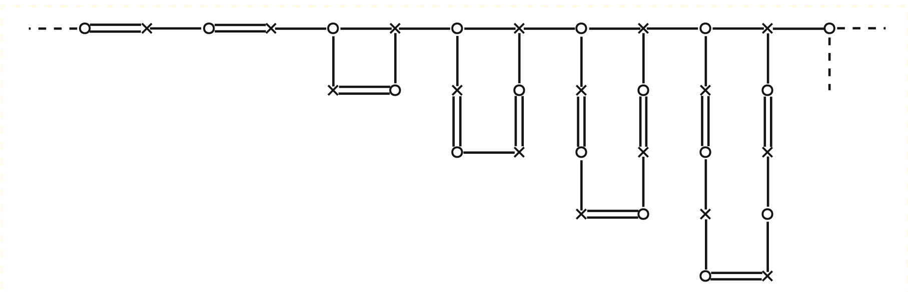


Figure 12 The Speiser graph used by Sutter [33] to construct a hyperbolic surface

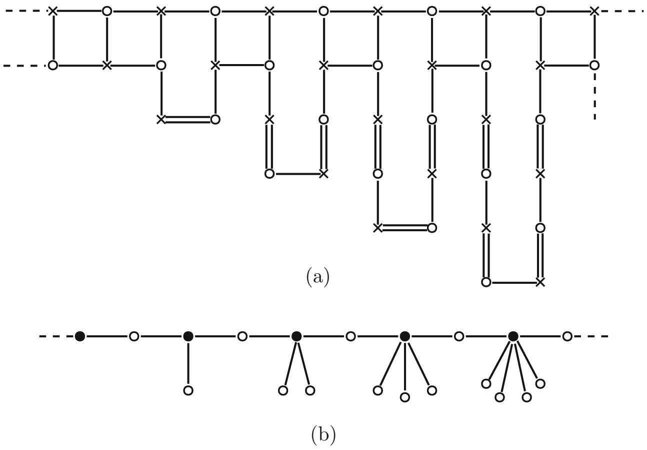


Figure 13 The Speiser graph (a) and the corresponding tree (b) that we use to construct a hyperbolic surface. There exists a sequence of critical points (marked as disks in the tree) with multiplicities tending to infinity

5.2 Non-uniform distance from the kernel

5.2.1 *Parabolicity*

To show that there exists a tree which has a true form, we consider the example in [24, Subsection 3.3]. The example constructed there has exactly two critical values 0 and 1 and no finite asymptotic values. Moreover, there is a sequence of vertices on the tree whose distance to the kernel tends to infinity. The surface spread over the sphere corresponding to this tree is parabolic, and thus there exists a Shabat entire function (by using affine maps which map 0 and 1 to -1 and 1, respectively). The vertices of this tree do not have a uniform distance to the kernel.

Another example can be constructed as follows. We start again with the Speiser graph of the sine function, as shown in Figure 10. The marking stays the same. As before, on the upper and lower edges of the face $2k$ we add $2|k|$ vertices, respectively. But this time we obtain a different Speiser graph by connecting these newly added vertices in another way, which is shown in Figure 14. Denote this graph by Γ.

The parabolicity of the surface corresponding to this Speiser graph follows from the Nevanlinna-Wittich criterion as we did before. As a matter of fact, we can also use Theorem 2.4. This is due to the fact that every face in the graph has a uniformly bounded number of edges on the boundary ($\leqslant 8$) except for those faces with infinitely many edges on the boundary. Then one can consider the extended Speiser graph Γ_5, which is actually obtained by embedding in each unbounded face a half-plane lattice. To show the parabolicity of Γ_5 it is necessary to find a sequence of disjoint nested annuli and use Proposition 2.7. We leave the details to the reader.

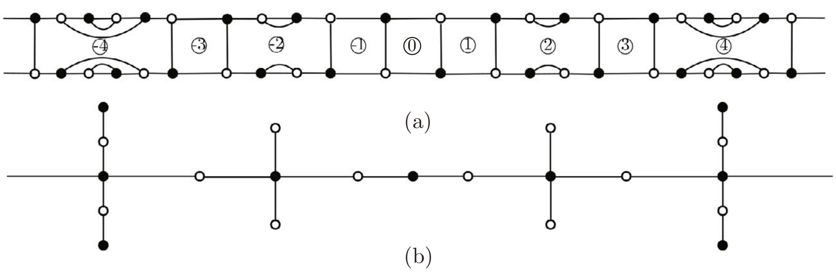


Figure 14 A Speiser graph (a) associated with a tree (b) with an unbounded distance to the kernel

5.2.2 *Hyperbolicity*

As before, we start with the Speiser graph of the sine function (see Figure 10). On the upper edges of faces ① and ⊖1 we add A_1 vertices, respectively. On the upper edges of faces ② and ⊖2 we add A_2 vertices, respectively. More generally, on the upper edge of face ⓚ we add A_k vertices. Here, $A_k \in \mathbb{N}$ and $A_k \to \infty$ as $k \to \infty$. Connecting these newly added vertices properly, we can get a Speiser graph of valence 3. Denote by Γ the Speiser graph obtained. In Figure 15, it is shown that a Speiser graph is constructed from the above surgery.

The Speiser graph Γ has two faces with infinitely many edges on the boundaries and all remaining faces have a uniformly bounded number of edges on the boundaries ($\leqslant 6$). Therefore, we can consider the extended Speiser graph Γ_4, which is actually obtained by embedding a half-plane lattice in the two unbounded faces. Now if Γ_4 is hyperbolic, then by Theorem 2.4 there does not exist any entire function.

To show that Γ_4 is hyperbolic, we shall use an isoperimetric inequality. Let V be a vertex set in a graph G and ∂V the boundary of V consisting of vertices with neighbours outside of V. Let f be a non-decreasing positive real function defined on $\mathbb{N}$. We say that G satisfies an f-isoperimetric inequality if there exists a constant $c > 0$ such that, for each finite vertex set V of G,

$$|\partial V| > cf(|V|), \tag{5.1}$$

where $|\cdot|$ denotes the cardinality. If the above inequality holds for all finite vertex sets V which contain a fixed vertex (root) v and induce connected subgraphs in G, then G is said to satisfy a rooted, connected f-isoperimetric inequality. A theorem by Thomassen [34, Theorem 3.4] says that each connected graph with a uniformly bounded local valence satisfying a rooted, connected f-isoperimetric inequality for $f(k) = k^{1/2+\varepsilon}$ and some $\varepsilon > 0$ is hyperbolic[2)]. It is easy to see that, for the graph Γ_4 if we let A_k tend to ∞ fast, then the inequality (5.1) is satisfied for some $\varepsilon > 0$. Hence Γ_4 is hyperbolic.

5.3 Infinitely many complementary components

As we discussed before, any homogeneous tree of valence no less than 3 does not give us any entire function. On the contrary, to show that there exists a Shabat entire function from a tree with infinitely many complementary components, one can use Bishop's technique of quasiconformal folding. As a simple example, we can consider the tree in [8, Figure 5], with which Bishop constructed a Shabat entire function with rapid growth. In fact, it follows from the Denjoy-Carleman-Ahlfors theorem (see [19, Chapter 5, Theorem 1.4]) that any Shabat entire function constructed from trees with infinitely many complementary components must be of infinite order.

We will consider the tree obtained from the one in [8, Figure 5] after the application of the quasiconformal folding. It follows from this technique that this tree will satisfy our conditions (T1) and (T2) but not (T3). The details are omitted.

Another more direct example of trees with infinitely many complementary components but satisfying (T1) and (T2) is obtained by considering the function $f(z) = \cos(\pi\cos(z))$. A sketch of the corresponding tree is shown in Figure 16.

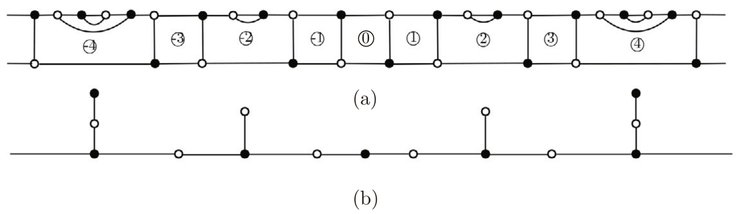


Figure 15 A Speiser graph (a) associated with a tree (b) with a non-uniform distance to the kernel. The tree below is just half of the one in Figure 14. The extended Speiser graph will be hyperbolic

[2)] Here, ε cannot be equal to 0, since the two-dimensional lattice is parabolic by the theorem of Pólya (see [31]) and in this case $\varepsilon = 0$.

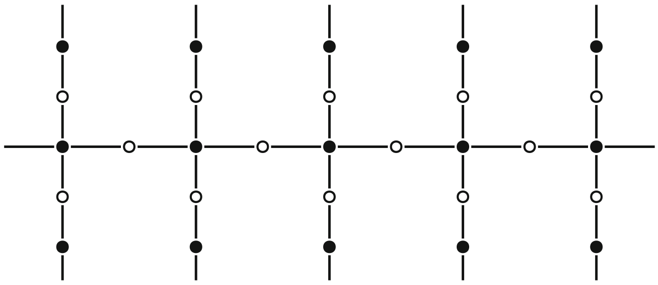


Figure 16 A sketch of the tree T_f for $f(z) = \cos(\pi\cos(z))$. Here, dots are the preimages of -1 and circles are the preimages of $+1$

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