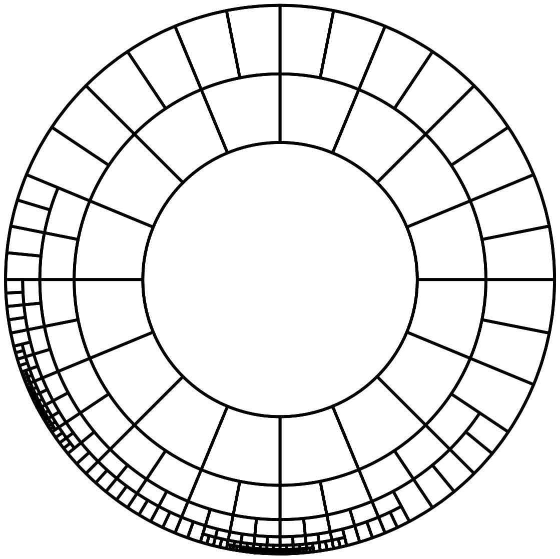
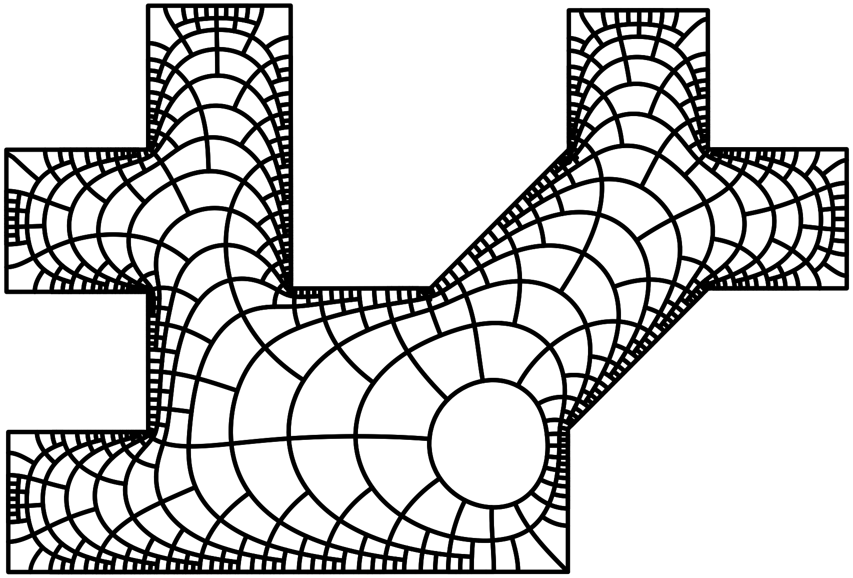
MAT 639, Spring 2026, Stony Brook University

Topics in Real Analysis: Harmonic Measure

Christopher Bishop





McMillan's Twist Point Theorem

Suppose γ is an analytic Jordan curve defined on $[0,1]$ such that $\gamma(0) = 0$ and $\gamma(1) = 1$. If x is a point not on γ we can define the winding $w(x,\gamma)$ of γ around x by taking

$$\arg(0 - x) - \arg(1 - x),$$

where we take a continuous branch of $\arg(z - x)$ defined on γ.

Since the curve is analytic it has a well defined tangent at each endpoint, so we can also define the windings at the endpoints by truncating the curve and taking limits.

We can also define the change of argument of γ' as $\arg(\gamma'(0)) - \arg(\gamma'(1))$ where again we choose a continuous branch of arg.

Lemma: $|2\pi[w(0,\gamma)+w(1,\gamma)]-[\arg(\gamma'(0)-\arg(\gamma'(1))]|\leq 4\pi.$

Proof. If γ is a line segment then there is nothing to do.

Otherwise, because of analyticity we may assume γ hits $[0, 1]$ only finitely often.

Replace γ by a homotopic smooth curve which intersects $[0, 1]$ the least number of times among all curves homotopic to γ by a homotopy which is the identity in some neighborhood of 0 and 1 (thus 0 and 1 are fixed and so are the tangent direction at these points).

The two quantities

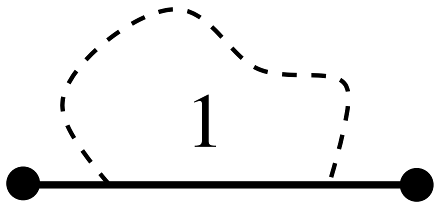
$$w(0,\gamma)+w(1,\gamma), \qquad \arg(\gamma'(0)-\arg(\gamma'(1)),$$

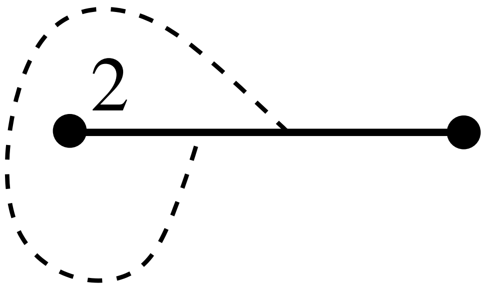
are invariant under such homotopies (since they can only take a discrete set of values, they can not be changed under continuous deformations), so it suffices to prove the result for the new curve.

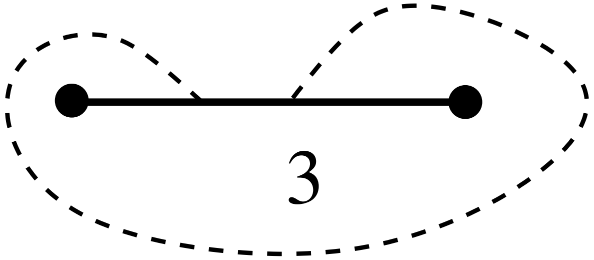
So we assume γ has the minimum number of intersections with $[0,1]$, say $\{0 = y_0, y_1, y_2, \dots y_n = 1\}$, which map via γ^{-1} to points say $\{0 = x_0, x_1, x_2, \dots x_n = 1\} \subset [0,1]$.

Divide γ into oriented subarcs $\gamma_i = \gamma|_{[x_i,x_{i+1}]}$. Then γ_i is a Jordan arc with endpoints on $[0,1]$, but otherwise disjoint from $[0,1]$.

The three possible types of arcs (up to a homeomorphism of the plane mapping $[0,1]$ to itself) are shown below and denoted types 1,2 and 3.







Except at the points 0 and 1 its makes sense to say that γ_i approaches it endpoints from either "above" or "below" $[0, 1]$.

For each x_i with $0 < i < n$ it is easy to see that γ_{i-1} and γ_i approach from different sides; otherwise there would be a smooth homotopy which removes the intersection at x_i, thus lowering the total number of intersections.

Similarly, none of these subarcs can be of type 1.

Otherwise, using the fact that γ_{i-1} and γ_{i+1} approach x_i and x_{i+1} respectively from the opposite side we can homotopy γ_i across $[0, 1]$ thus removing the intersections at both x_i and $x_i + 1$.

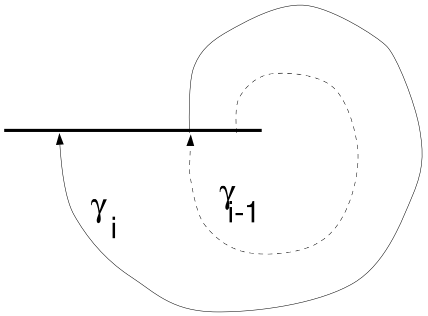
Thus the subarcs of γ must be either type 2 or 3.

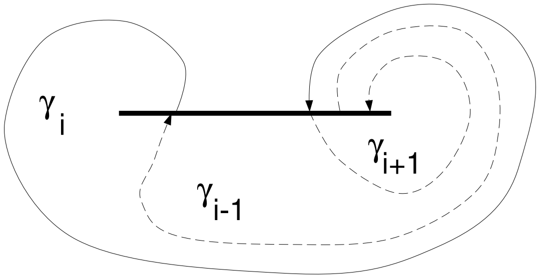
We say that γ_i is "good" if $y_{i+1} > y_i$ and is "bad" if $y_{i+1} < y_i$. We first claim that the minimality of γ implies there are no bad subarcs.

Suppose that there are bad arcs. We will consider two cases.

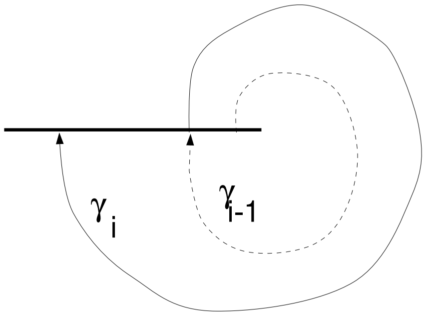
First suppose there is a bad arc of type 2. Then there is a bad arc γ_i of type 2 with endpoint y_i as close to 1 as possible (i.e., farthest to the right among all bad type 2 arcs).

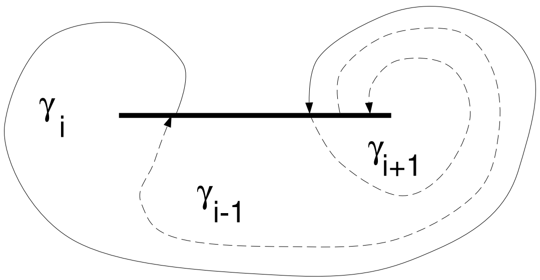
Then the preceding arc must be type 2 as well (see the figure), but this is only possible it is bad as well with a larger endpoint. This is a contradiction and implies that there are no bad arcs of type 2.





Now suppose all the bad arcs are type 3. Choose γ_i to be the last bad arc in the ordering of γ. Then γ_{i+1} is good and must be type 2. Topologically, the only possibilities for γ_{i-1} are that it is type 1 or is a bad arc of type 2. Both are ruled out by hypothesis, so we deduce there are no bad arcs.





We can now finish the proof. Replace γ by a homotopic arc where the homotopy is the identity except in small neighborhoods of each intersection point y_i, $0 < 1 < n$ and in those neighborhoods the curve is changed so that γ' is horizontal and points to the right as γ crosses $[0, 1]$.

Then for each subarc γ_i the tangents points the same direction at either endpoint. Thus the change in argument of γ' along each γ_i is a multiple of 2π.

There are only a few cases and it is each of them it is trivial to check that the change in argument of γ' is 2π times $w(0, \gamma_i) + w(1, \gamma_i)$ where $w(z, \gamma)$ denotes the change in $\arg(y - z)$ for some branch of the argument function on γ.

By summing over i we now get that the change of argument of γ' on $[x_1, x_{n-1}$ is equal to 2π times $w(0, \gamma|_{[x_1,x_{n-1}]}) + w(1, \gamma|_{[x_1,x_{n-1}]})$.

Adding in the two end intervals γ_0 and γ_{n-1} can only alter the equality by a factor of at most 2π each so we obtain the lemma. $\square$

If Ω is simply connected we say $x \in \partial\Omega$ is an *inner tangent* point of Ω if for any $\epsilon > 0$ x is the vertex of a cone in Ω with angle $\pi - \epsilon$, but is the vertex of no cone with angle $> \pi$.

We say that x is a *cone point* if it is the vertex of some cone in Ω.

Lemma: *If* Ω *is simply connected then the set of cone points has* σ*-finite* 1*-dimensional measure and almost every cone point is an inner tangent point.*

Proof. By considering cones with rational angles and radius, we can write the set of cone points as a countable union of sets, each of each are the vertices of cones in Ω with fixed side directions and diameters. It clearly suffices to prove the claims for any such set.

Let $F \subset \partial\Omega$ be the set of points $x \in \partial\Omega$ so that

$$W_x = \{x + z : |z| < r, |\arg(-iz)| \leq \epsilon\}\} \subset \Omega.$$

By again dividing into a countable number of subsets we may assume that F is contained in the rectangle $R = \{z : |\mathrm{Im}(z)| < r/10, |\mathrm{Re}(z)| < r\epsilon/10\}$.

Let $W = \cup_{x \in F} W_x$. Then $R \cap \partial W$ is graph of a Lipschitz function (norm depending only on ϵ), and hence is rectifiable. Since it contains F, F has finite 1-dimensional measure and a.e. point of F is a tangent point of the arc.

Thus almost every point of F is an inner tangent of W and hence of Ω. $\square$

A point x is called a twist point for Ω if for any branch of $\arg(z-x)$ defined on Ω we have

$$\limsup_{z\to x, z\in\Omega} \arg(z-x) = \infty,$$

and

$$\liminf_{z\to x, z\in\Omega} \arg(z-x) = -\infty.$$

Thus to approach a twist point x through Ω we must "twist around" x arbitrarily far in both directions. It is difficult to draw a twist point, but they do exist, e.g., "most" points on the von Koch snowflake.

McMillan's Twist Point Theorem *If Ω is a simply connected domain then almost every point on $\partial\Omega$ (with respect to harmonic measure) is either an inner tangent point or a twist point.*

Proof. The proof is essentially Plessner's theorem. Let $\Phi : \mathbb{D} \to \Omega$ be a Riemann mapping and apply Plessner's theorem to the derivative Φ'.

Plessner's theorem says that we can write $\mathbb{T} = E_0 \cup E_1 \cup E_2$ where E_0 has measure zero, Φ' has non-zero non-tangential limits at every point of E_1 and Φ' is non-tangentially dense at every point of E_2.

Clearly the set $\Phi(E_1) \cup \Phi(E_2)$ has full harmonic measure on $\partial\Omega$.

Moreover, we know that $\Phi(E_1) \subset \partial\Omega$ consists of inner tangents. Thus if we can show that $\Phi(E_2)$ consists of twist points almost everywhere (with respect to harmonic measure) we will be done.

In fact, all we have to do is produce a sequence of points $z_n \in \Omega$ with $\arg(z_n - x) \to +\infty$ and another with arguments tending tending to $-\infty$.

To prove this, suppose it fails. Then there is a set F of positive measure on $\mathbb{T}$ on which Φ has nontangential limits, Φ' is non-tangentially dense but

$$\arg[\Phi(re^{i\theta}) - \Phi(e^{i\theta})],$$

remains bounded above as $r \to 1$. We will show this is impossible.

Since Φ' is non-tangentially dense on F, so is $\log \Phi' = \log |\Phi'| + i \arg \Phi'$. Hence $\arg \Phi'$ must be non-tangentially unbounded above and below by Plessner's theorem.

By the Bonk-Balogh theorem, there is a M so that the images of the rays $[0, e^{i\theta})$ have length less than M except on a set of measure $|F|/2$. So be replacing F by a set of half the measure we may assume of the associated rays have bounded length.

Let $\theta_0 \in F$ be a point of density and consider a sequence $\{z_n\} \to 1$ so that $\arg \Phi(z_n) \to +\infty$.

Fix a large N and choose an n so large that $\arg \Phi'(z_n) > 4\pi(N+1)$. Let $\gamma = \Phi([0, z_n)$ be the image of the radial segment from 0 to z_n.

The change of argument of γ' from one endpoint to the other is $|\arg \Phi'(z_n)|$. By our earlier Lemma, the curve γ "winds around" one of its endpoints at least N times.

The curve does not wind around the origin arbitrarily often since Φ has a non-zero radial limit at x.

More precisely, we proved earlier that if the argument of γ' changes by more than $4\pi(N+1)$ between its two endpoints then

$$w(a,\gamma)+w(b,\gamma)\geq 2N,$$

By rescaling we may assume $\Phi'(0) = 1$. So by Koebe's theorem there is a disk $D_0 \subset \Omega$ of diameter similar to 1 that γ never re-enters once it leaves. Moreover, γ does not wind around 0 inside this disk.

Since γ has length at most M, it can wind around 0 at most $M/2\pi$ times. If N was chosen large enough, we see that most of the winding of γ must be around the other endpoint $b = \Phi(rx)$.

We would like to deduce that the curve γ also winds around the point $\Phi(e^{i\theta_0})$, but this may not be true. Instead we will show that there is another point in F near b around which the curve does wind.

Recall that θ_0 was chosen to be a point of density of F. So if $|z_n|$ is close enough to 1, more that half the interval of length $1 - |z_n|$ centered at $e^{i\theta_0}$ consists of points in F.

We can find a point x in F so that x can be connected to b in Ω by a curve of length at most $C\mathrm{dist}(b, \partial\Omega)$, and this curve does not intesect γ.

This implies that the winding of γ around b and around x can differ by at most a bounded factor.

Thus the winding of γ around x must be very large. This contradicts the assumption that $x \in F$, proving the theorem. $\square$