MAT 639, Spring 2026, Stony Brook University

Topics in Real Analysis: Harmonic Measure

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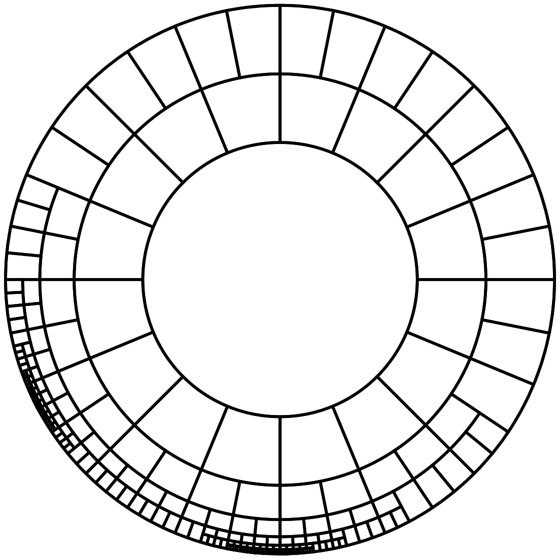
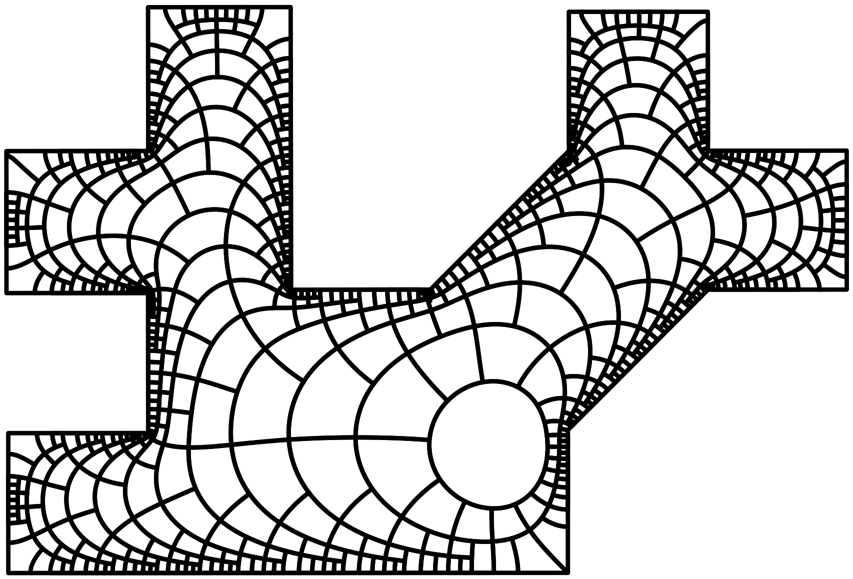


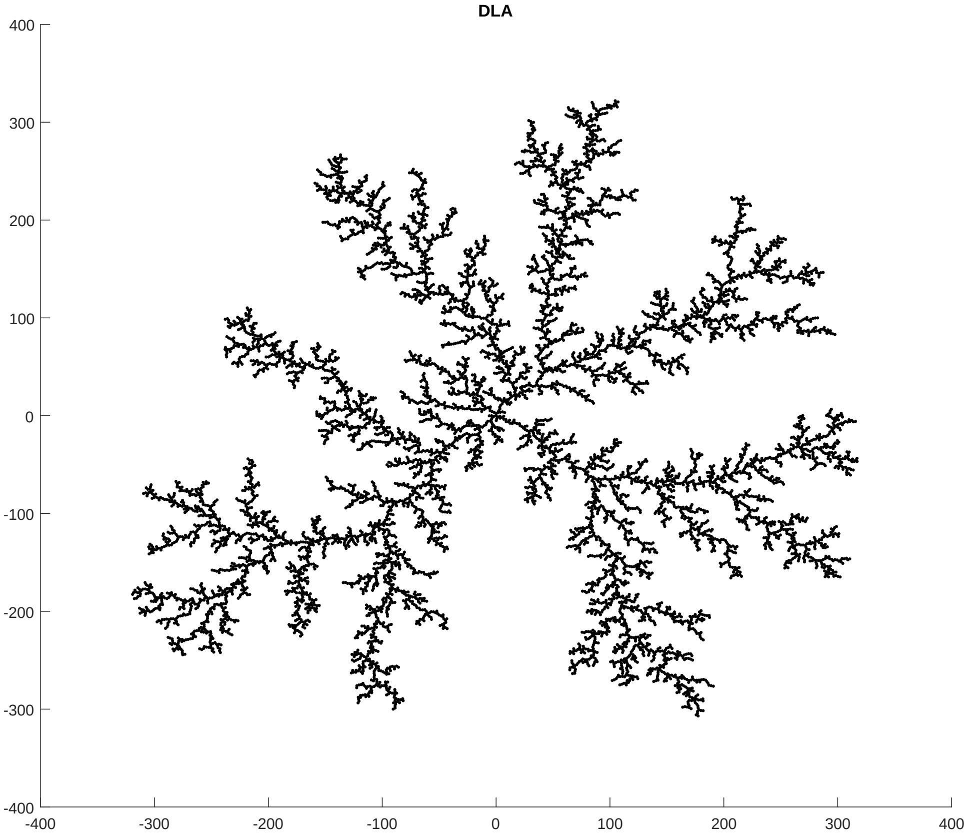


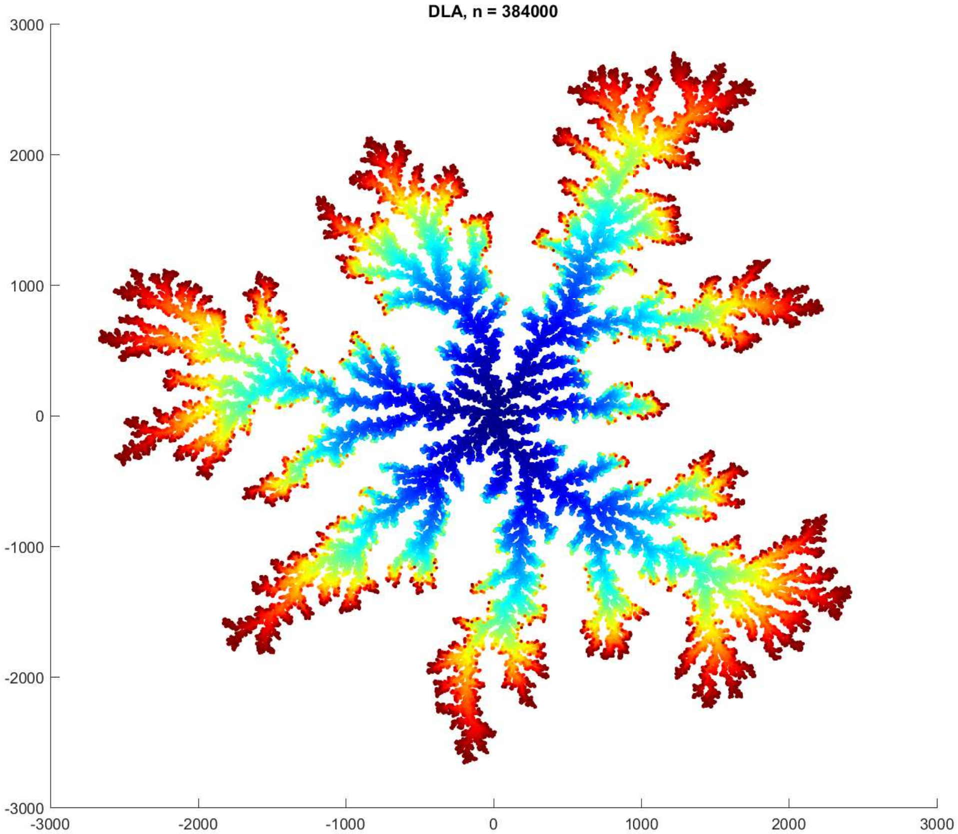
Kesten's theorem on growth of DLA

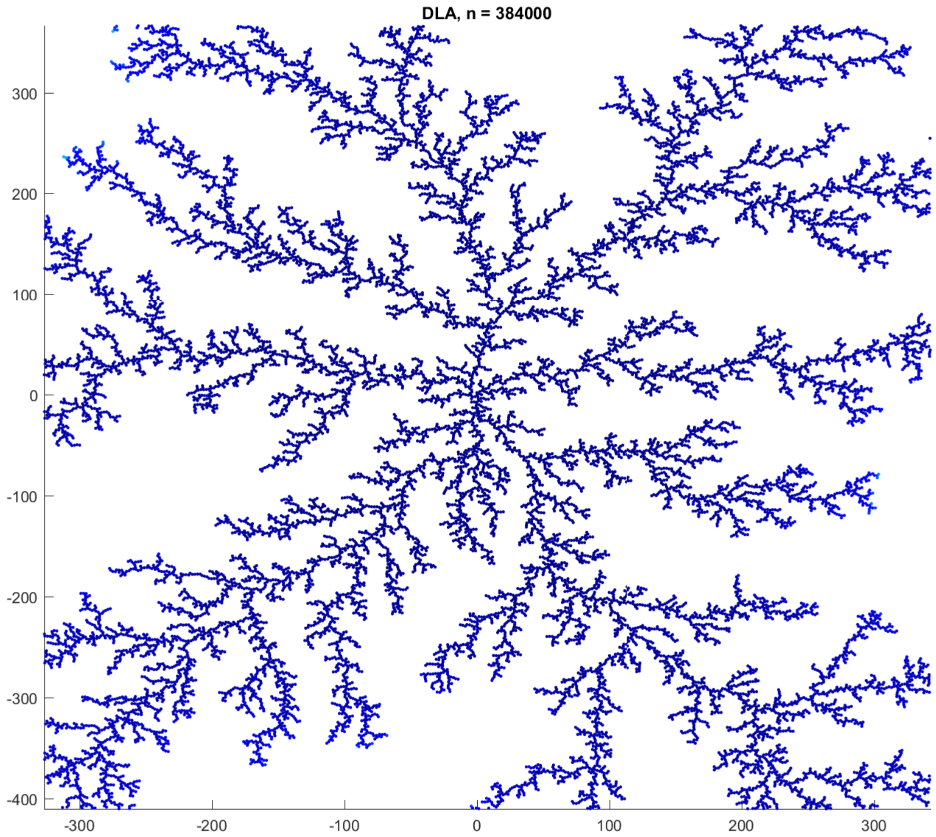
Start with a unit disk centered at the origin.

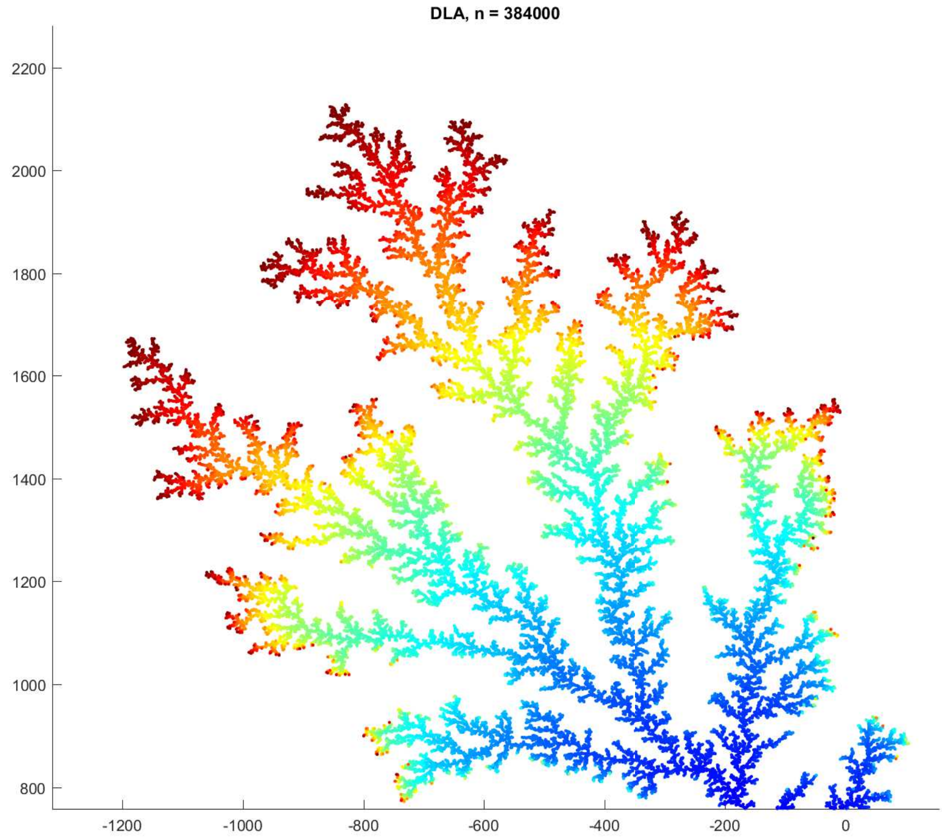
Imagine another unit disk, whose center moves as a Brownian motion starting near infinity unit the it hits the first disk and the stops.

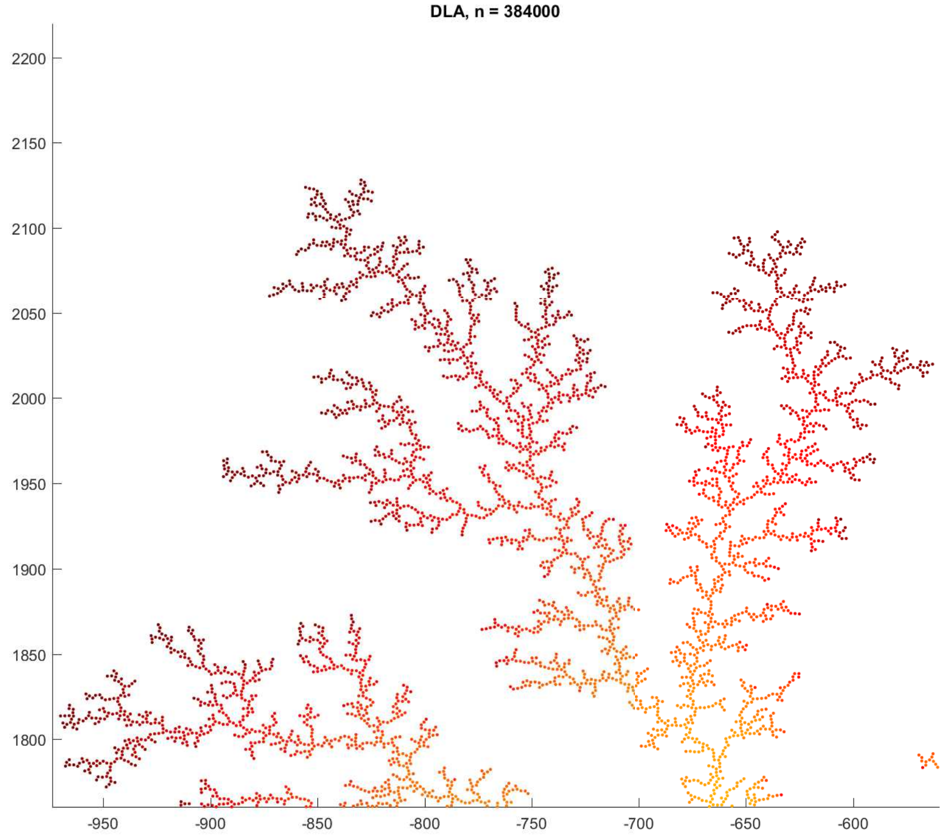
Now send in another random disk until it hits one of the first two. Continue in this way until n disks have accumulated to form a connected set.


DLA
400
300
200
100
0
-100
-200
-300
-400
-400
-300
-200
-100
0
100
200
300
400


DLA, n = 384000
3000
2000
1000
0
-1000
-2000
-3000
-3000
-2000
-1000
0
1000
2000
3000


DLA, n = 384000
300
200
100
0
-100
-200
-300
-400
-300
-200
-100
0
100
200
300


DLA, n = 384000
2200
2000
1800
1600
1400
1200
1000
800
-1200
-1000
-800
-600
-400
-200
0


DLA, n = 384000
2200
2150
2100
2050
2000
1950
1900
1850
1800
-950
-900
-850
-800
-750
-700
-650
-600

It is easy to show that The diameter of DLA grows less than n and greater than $\sqrt{n}$.

Harry Kesten proved:

Theorem 0.1. *Almost surely, the diameter of DLA(n) is* $O(n^{2/3})$.

No non-trivial lower bound is yet known.

Proof. A moving disk will hit a set E when the center is precisely distance 1 from that set. In our case, the set is a union of n unit disks centered at a finite set of points $P_n = \{p_1, \ldots, p_n\}$.

Thus the process of adding the next disk by letting it wander by Brownian motion, is precisely the same as choosing a point p_{n+1} on the set

$$E_n = \{z : \mathrm{dist}(z, P_n) = 2\},$$

with respect to harmonic measure at ∞ for the domain Ω_n that is the unbounded complementary component of E_n.

Since E_n is, by definition, a connected set, Ω_n is simply connected and will be bounded by a finite number of circular arcs.

Almost surely (with probability one) Ω_n will be the entire complement of E_n.

Otherwise, we must have chosen a disk that made contact with two or more earlier disks. But there are only a finite number of points on E_k where this happens, and finite sets have harmonic measure zero (e.g., Beurling's theorem), so the probability of making such a choice is zero.

Thus, almost surely, each disk in the cluster (except the one at the origin) hits exactly one previously chosen disk, although it may be hit by several (at most four, almost surely) later ones.

Consider

$$\mathrm{rad}(n) = \max\{|p| : p \in P_n\},$$

which measures the size of the DLA cluster in terms of a disk around the origin, and its inverse

$$\mathrm{exit}(m) = \max\{n : \mathrm{rad}(n) \leq m\},$$

which measures how soon the cluster grows beyond a given radius.

The theorem is stated in terms of an upper bound for $\mathrm{rad}(n)$, but is equivalent to a lower bound for $\mathrm{exit}(m)$:

$$\liminf_{m\to\infty} \frac{\mathrm{exit}(m)}{m^{3/2}} \geq \beta, \tag{0.1}$$

holds almost surely for some constant $\beta > 0$.

More precisely, we define

$$V_m = \{\mathrm{exit}(m) \leq \beta m^{3/2}\},$$

and we will prove that $\sum_m \mathbb{P}(V_m) < \infty$ if $\beta > 0$ is small enough.

The Borel-Cantelli lemma then implies that the probability that V_m occurs infinitely often is zero. Thus almost surely V_m only occurs finitely often, which gives (0.1).

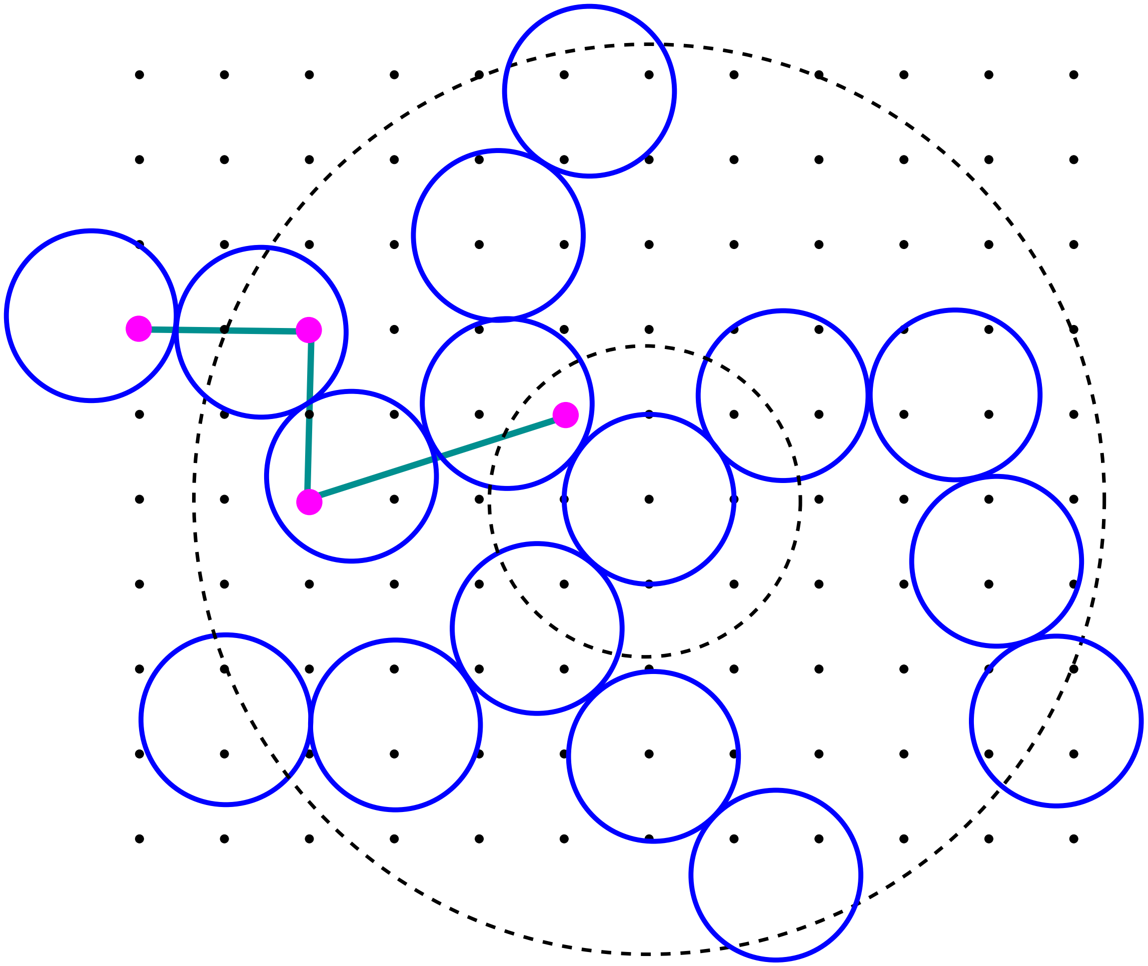
We estimate the probability of V_m by placing these events inside larger events and estimating those.

If V_m occurs, it means that the DLA cluster contains a path of at most $\beta m^{3/2}$ disks $\{D_1, \dots D_N\}$ that starts at the origin and ends with a disk that hits the circle $\{|z| = m\}$.

Moreover, every D_{j+1}, $j = 1, \dots N-1$ was selected after D_j in the growth process.

Otherwise suppose D_{j+1} is the first counterexample in the path. Then D_{j+1} is the unique earlier disk hit by D_j, so D_{j-1}, which also touches D_j, must have been chosen later than D_j, making D_j a counterexample too.

Every unit disk contains a point in the lattice $\mathbb{Z} \times \mathbb{Z}$, so for each path of unit disks as above, we can choose a sequence of lattice points $\mathbf{z} = \{z_1, \dots z_N\}$ such $z_j \in D_j$, $j = 1, \dots z_N$ and $|z_j - z_{j+1}| \leq 4$ since the union of two touching unit disks has diameter 4.



We say that sequence of distinct lattice points $\{z_1, \ldots, z_k\}$ is m-admissible if

$$|z_1| \leq m/2, \quad |z_k| \geq m, \quad |z_j - z_{j+1}| \leq 4.$$

Note that there are at most $m^2 80^{k-1}$ m-admissible sequences of length k :

- there are m^2 possible choices for z_1,
- each following choice is made from a 9×9 square, omitting the center.

Moreover, the length of an m-admissible sequence is at least $m/8$ since the first and last points are at least distance $m/2$ apart.

Given an m-admissible sequence $\mathbf{z}$ of length k, we define $W_m(\mathbf{z})$ to be the set of DLA clusters so that:
(1) the cluster contains at most $\beta m^{3/2}$ disks,
(2) the cluster contains the sequence $\mathbf{z}$, and
(3) the disk containing z_{j+1} was chosen after the disk containing z_j.

By our comments above each cluster in V_m contained in the event $W_m(\mathbf{z})$ for some m-admissible sequence of length $k \le \beta m^{3/2}$.

Thus all of V_m is contained W_m, the union of $W_m(\mathbf{z})$ over all m-admissible sequences of length at most $\beta m^{3/2}$.

We claim that if $\mathbf{z}$ has length k, then

$$\mathbb{P}(W_m(\mathbf{z})) \leq (C\beta)^k. \tag{0.2}$$

We will finish the proof of the theorem assuming this is true, and then prove the estimate.

Given (0.2)

$$\begin{aligned}\mathbb{P}(W_m) &\le \sum_{\mathbf{z}} \mathbb{P}(W(\mathbf{z}))\\ &\le \#(m - \text{admissible } \mathbf{z}) \cdot (C\beta)^k\\ &\le m^2 80^{k-1}(C\beta)^k\\ &\le m^2(80C\beta)^k\\ &\le m^2(80C\beta)^{m/4},\end{aligned}$$

since $k \ge m/4$. Thus if $\beta < 1/80C$,

$$\sum_m \mathbb{P}(V_m) \le \sum_m \mathbb{P}(W_m) \le \sum_m m^2(80C\beta)^{m/4} < \infty.$$

This completes the proof of Theorem 0.1, except for the proof of (0.2).

First we explain the general idea for proving (0.2). We will make it precise later.

Suppose we have already grown a cluster that contains the points $z_1, \ldots, z_j$. How long do we have to wait before the cluster contains z_{j+1}? We must add a disk within distance 4 of the disk containing z_j.

Since the cluster has diameter at least $m/2$, by Beurling's estimate the probability of choosing such a disk is less that $C/\sqrt{m}$.

Therefore the expected number of disks we add before covering z_{j+1} is at least $\sqrt{m}/C$. This has to happen k times, so we expect that $k\sqrt{m}/C$ disks need to be added to the cluster before the whole sequence $\mathbf{z}$ is covered.

Since $k \geq m/8$, we therefore expect to need about $m^{3/2}/C$ disks to be added. However, clusters in the event $W_m(\mathbf{z})$ only use $\beta m^{3/2}$ disks to cover $\mathbf{z}$.

If β is small compared to $1/C$, this event should have small probability.

To make this idea precise, let $D_1, \ldots$ be an enumeration of the disks in the cluster, in the order they are added. Suppose z_j is contained in disk $D_{k(j)}$ and let $w(j) = k(j+1) - k(j)$; the "waiting time" between covering z_j and z_{j+1}.

Then

$$\mathbb{P}(w(j) > t) \geq (1-p)^t,$$

where $p \leq C/\sqrt{m}$ is probability of hitting disk containing z_j (this is where we use Beurling's estimate).

Therefore $w(j)$ is bounded below by a geometric random variable (the same one for each j), and $\sum_j w(j)$ will be bounded below by the corresponding sum of geometric variables. We estimate this distribution using:

Lemma 0.2. *Suppose* $X_1, \ldots X_n$ *are independent geometric random variables, i.e.,* $\mathbb{P}(X_j = s) = p(1-p)^{s-1}$ *for some* $0 < p < 1/2$, *and* $Y = \sum_{j=1}^n X_j$. *If* $a \geq 2p$, *then*

$$\mathbb{P}(Y \leq an/p) \leq (2e^2 a)^n.$$

Proof. Define the moment generating function of the random variable Y as the expected value of $\exp(tY)$. If X is a geometric random variable, then

$$\mathbb{E}(e^{tX}) \;=\; \sum_{j=1}^{\infty} e^{tj} p(1-p)^{j-1} = pe^t \sum_{j=0}^{\infty} (e^t(1-p))^j = \frac{pe^t}{1-e^t(1-p)}.$$

Since Y is a sum of independent copies of X,

$$\mathbb{E}(e^{tY}) \;=\; \prod_{j=1}^{\infty} \mathbb{E}(e^{tX}) = \left[\frac{pe^t}{1-e^t(1-p)}\right]^n.$$

By Chebyshev's inequality

$$\mathbb{P}\left(Y < \frac{\ln \lambda}{-t}\right) = \mathbb{P}(e^{-tY} > \lambda) \le \frac{1}{\lambda}\mathbb{E}(e^{-tY}).$$

Set $\lambda = \exp(-ant/p)$ to get

$$\begin{aligned}
\mathbb{P}(Y < an/p) &\le \exp(ant/p)\mathbb{E}(e^{-tY}) \\
&= \frac{\exp(ant/p)e^{-nt}p^n}{(1 - e^{-t}(1-p))^n} \\
&= \frac{\exp(ant/p)p^n}{(e^t - (1-p))^n}
\end{aligned}$$

Now set $t = \ln(a(1-p)/(a-p)$ and this becomes

$$\begin{aligned}
\mathbb{P}(Y < an/p) \ &\leq \ \frac{p^n \left(\frac{a(1-p)}{a-p}\right)^{an/p}}{(\frac{a(1-p)}{a-p} - (1-p))^n} \\
&\leq \ \frac{p^n \left(\frac{a(1-p)}{a-p}\right)^{an/p}}{(1-p)^n(\frac{a}{a-p} - 1)^n} \\
&\leq \ \frac{p^n \left(\frac{a(1-p)}{a-p}\right)^{an/p}}{(1-p)^n(\frac{p}{a-p})^n} \\
&\leq \ \left(\frac{a(1-p)}{a-p}\right)^{an/p} \left(\frac{a-p}{1-p}\right)^n .
\end{aligned}$$

Using $p < 1/2$ and $a \geq 2p$, we get $a \leq 2(a-p)$ and $1-p > 1/2$, so

$$\begin{aligned}
\mathbb{P}(Y < an/p) &\leq \left(\frac{a(1-p)}{a-p}\right)^{an/p}(2a)^n \\
&\leq \left(\frac{a}{a-p}\right)^{an/p}(2a)^n \\
&\leq \left(1+\frac{p}{a-p}\right)^{an/p}(2a)^n \\
&\leq \left(1+\frac{p}{a-p}\right)^{2(a-p)n/p}(2a)^n \\
&\leq (e^2 2a)^n,
\end{aligned}$$

since $(1+\frac{1}{x})^x \leq e$. $\square$

To finish the proof of (0.2), apply Lemma 0.2 with $a = \beta k/p \geq C_1 \beta m^{3/2}$

$$\begin{aligned}
\mathbb{P}(W_m) &\leq \mathbb{P}(\sum_{j=1}^{k} w(j) < \beta m^{3/2}) \\
&\leq \mathbb{P}(\sum_{j=1}^{k} X_j < C_1 \beta k/p) \\
&\leq (2e^2 C_1 \beta)^k = (C_2 \beta)^k,
\end{aligned}$$

as desired. This completes the proof of (0.2) and hence of Theorem 0.1. ☐

This completes the proof of Kesten's theorem.