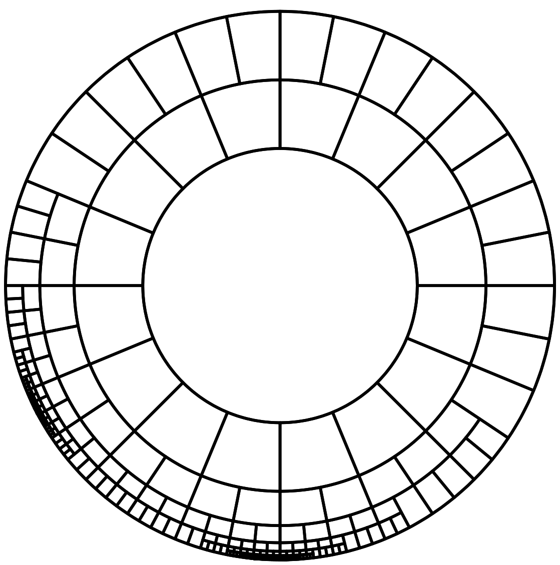
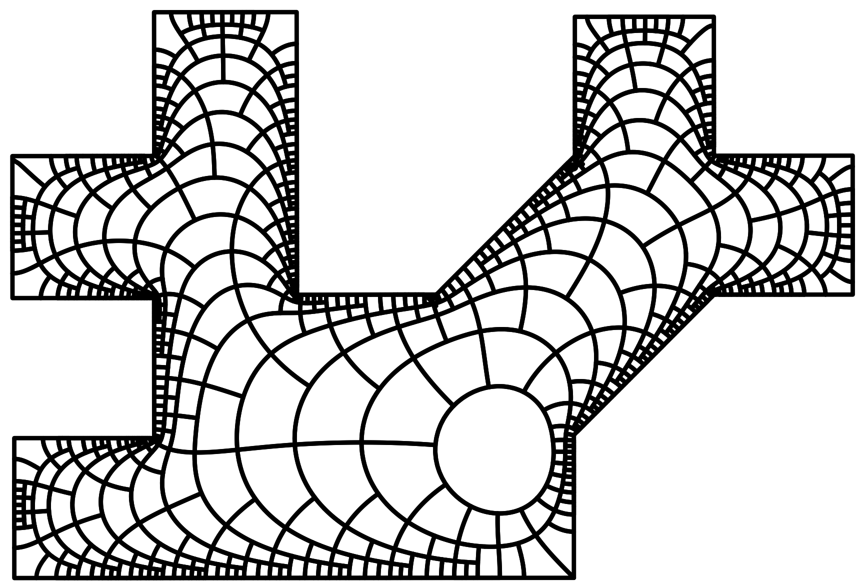
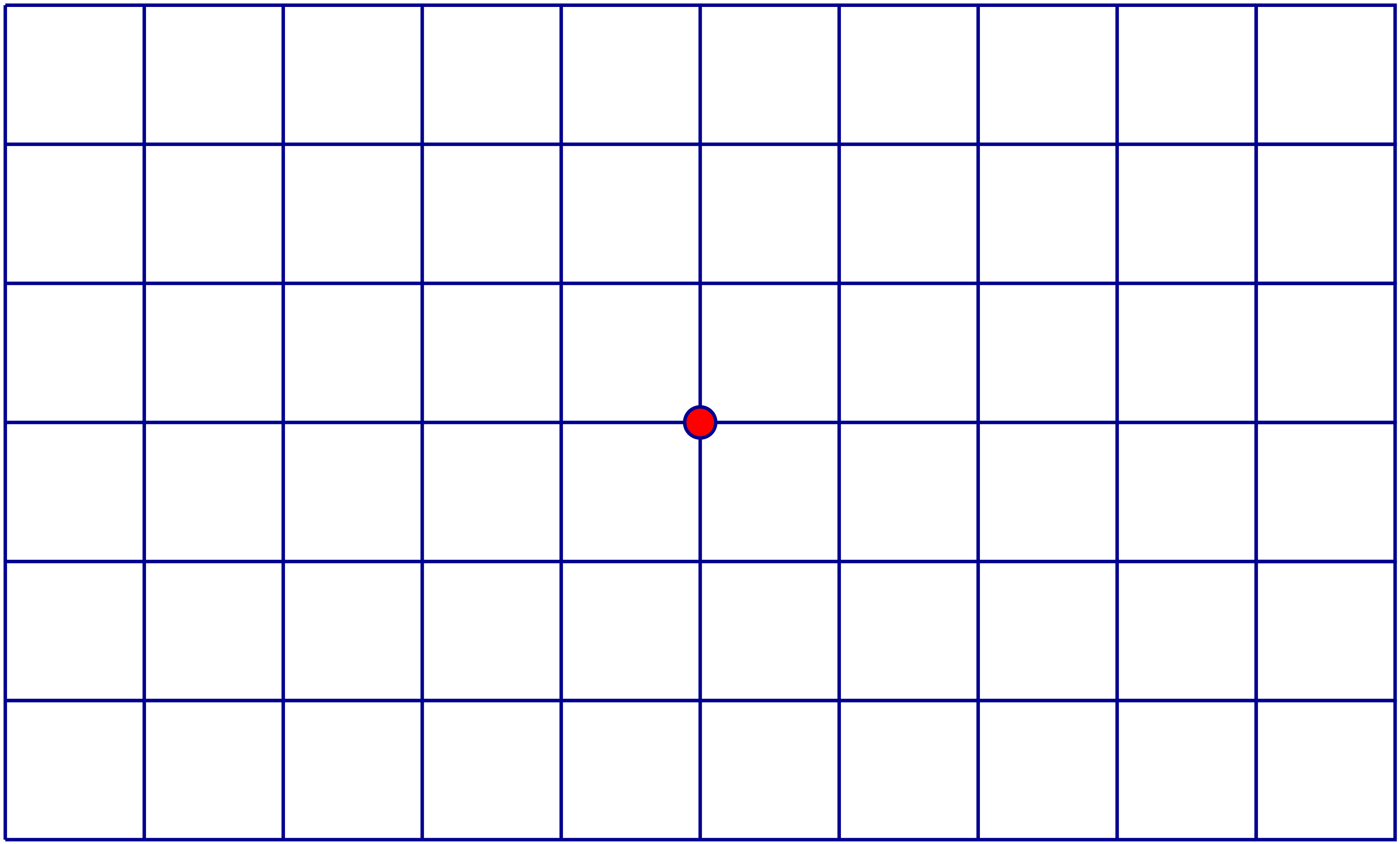
MAT 639, Spring 2026, Stony Brook University

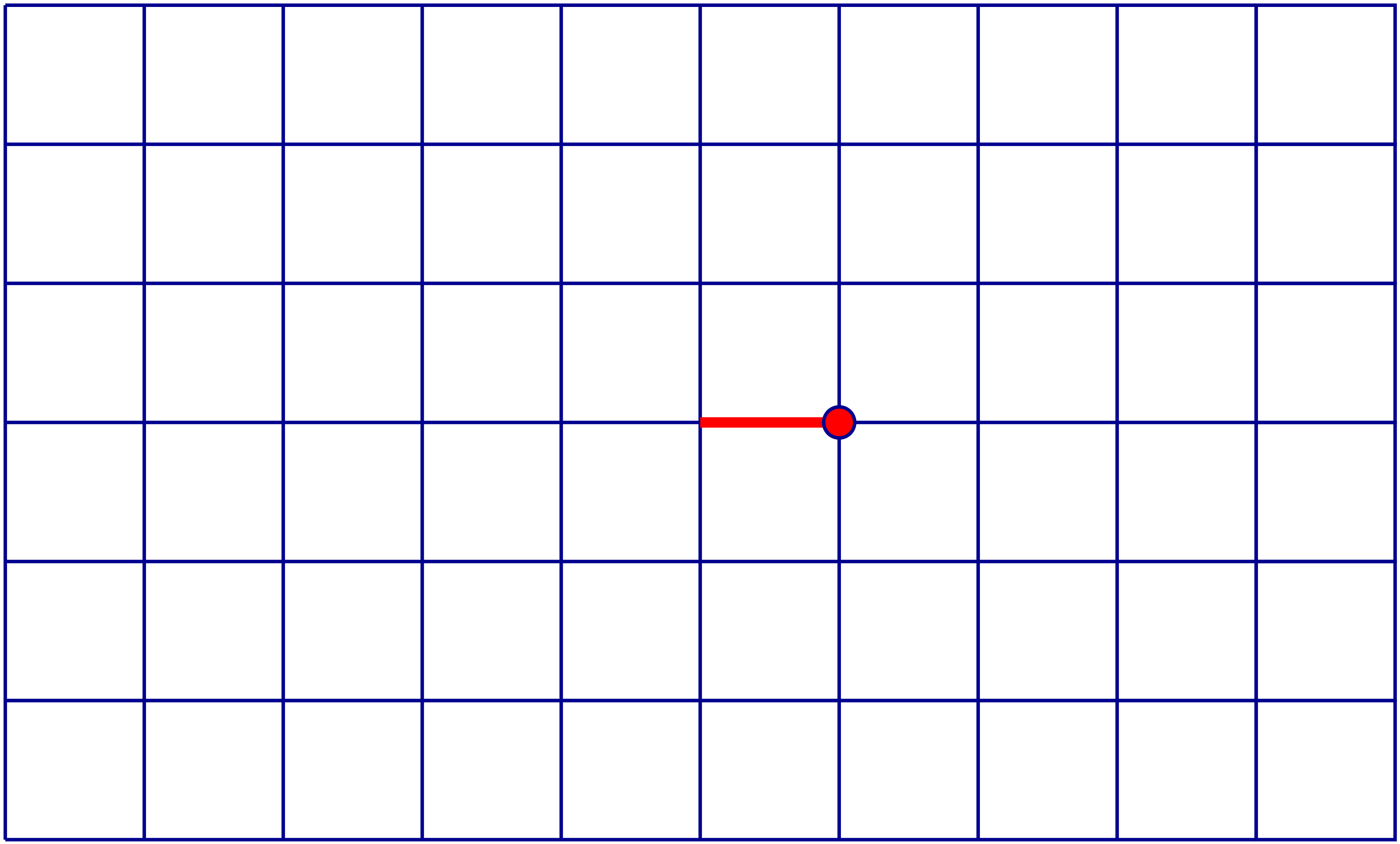
Topics in Real Analysis: Harmonic Measure

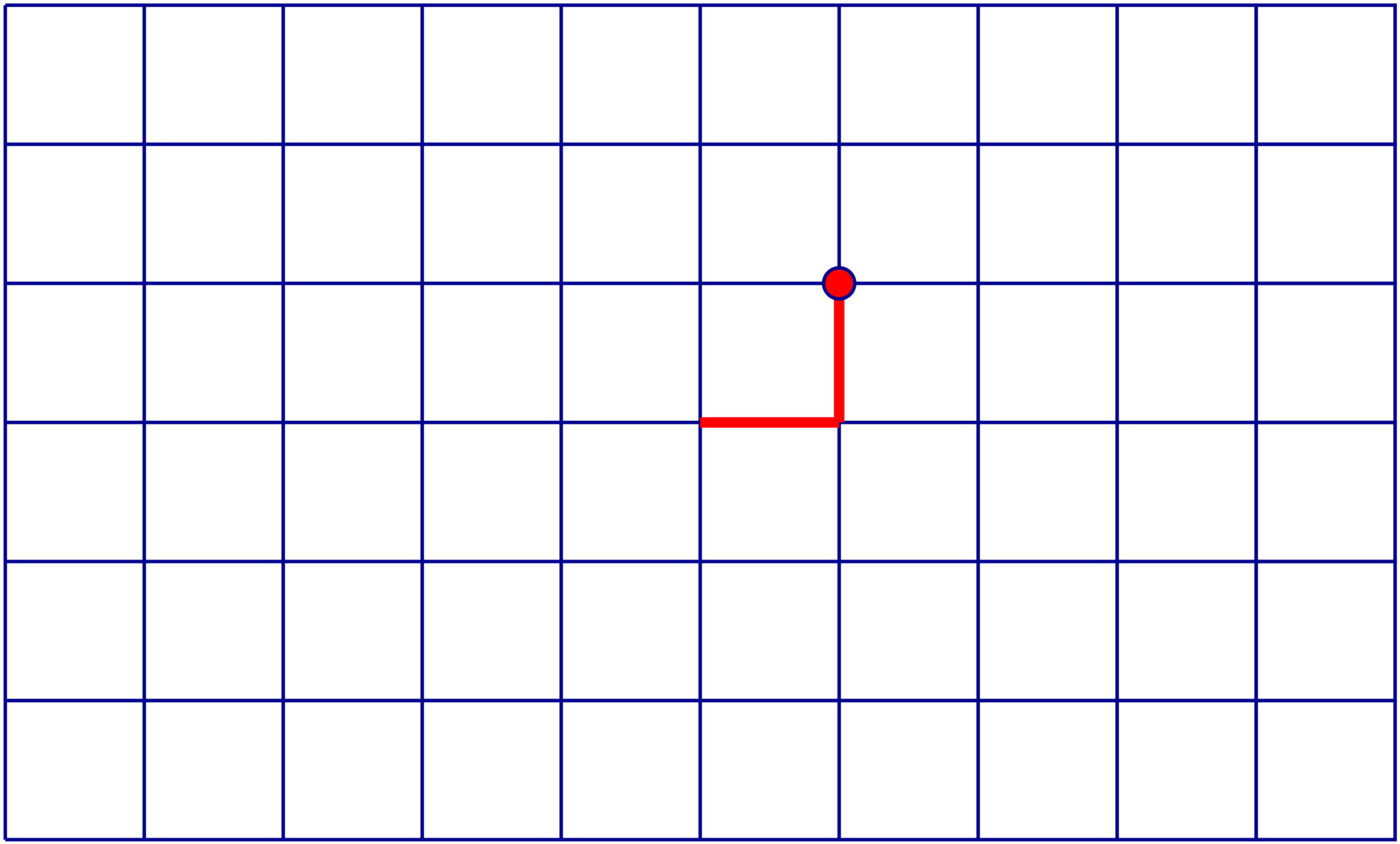
Christopher Bishop

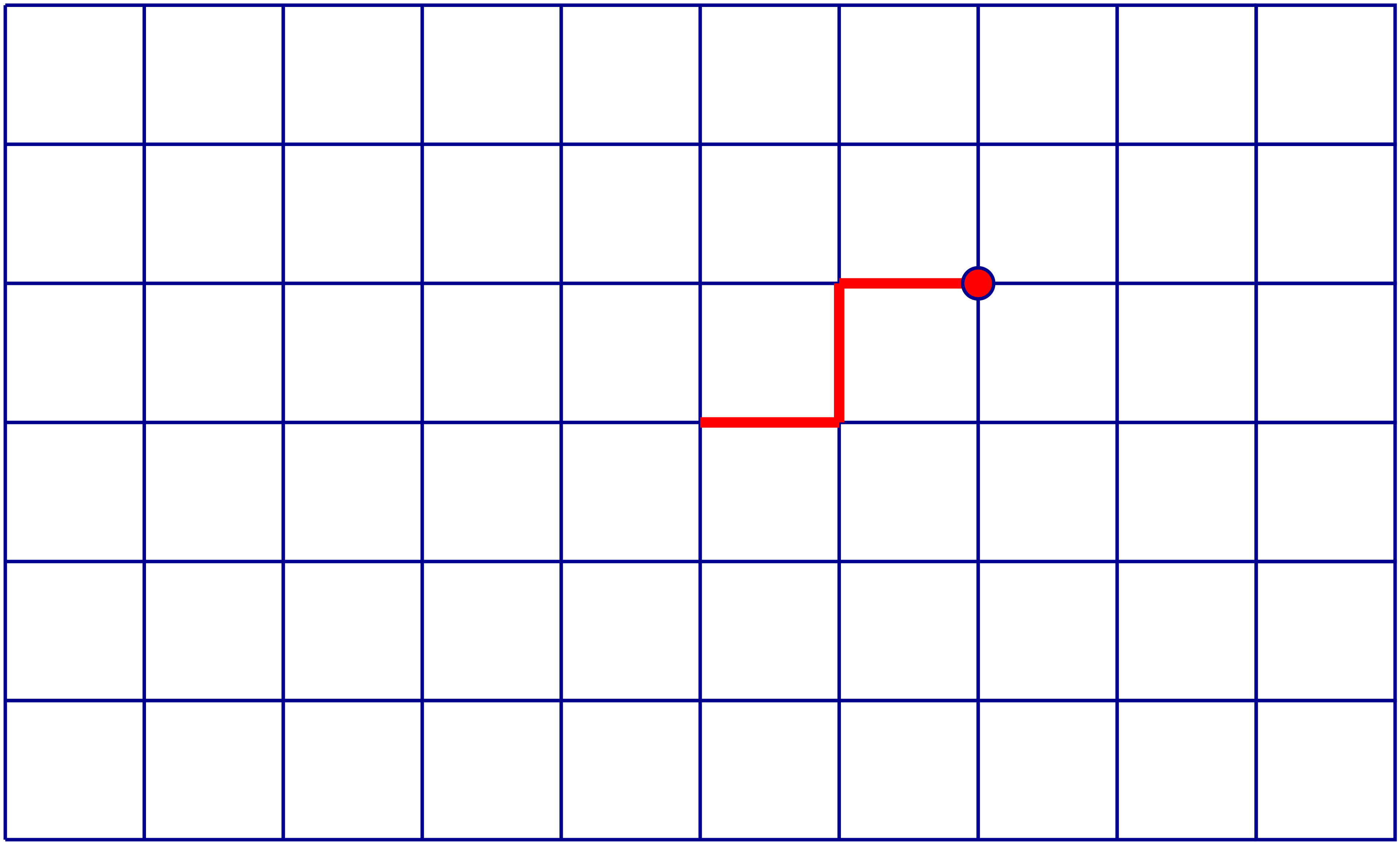


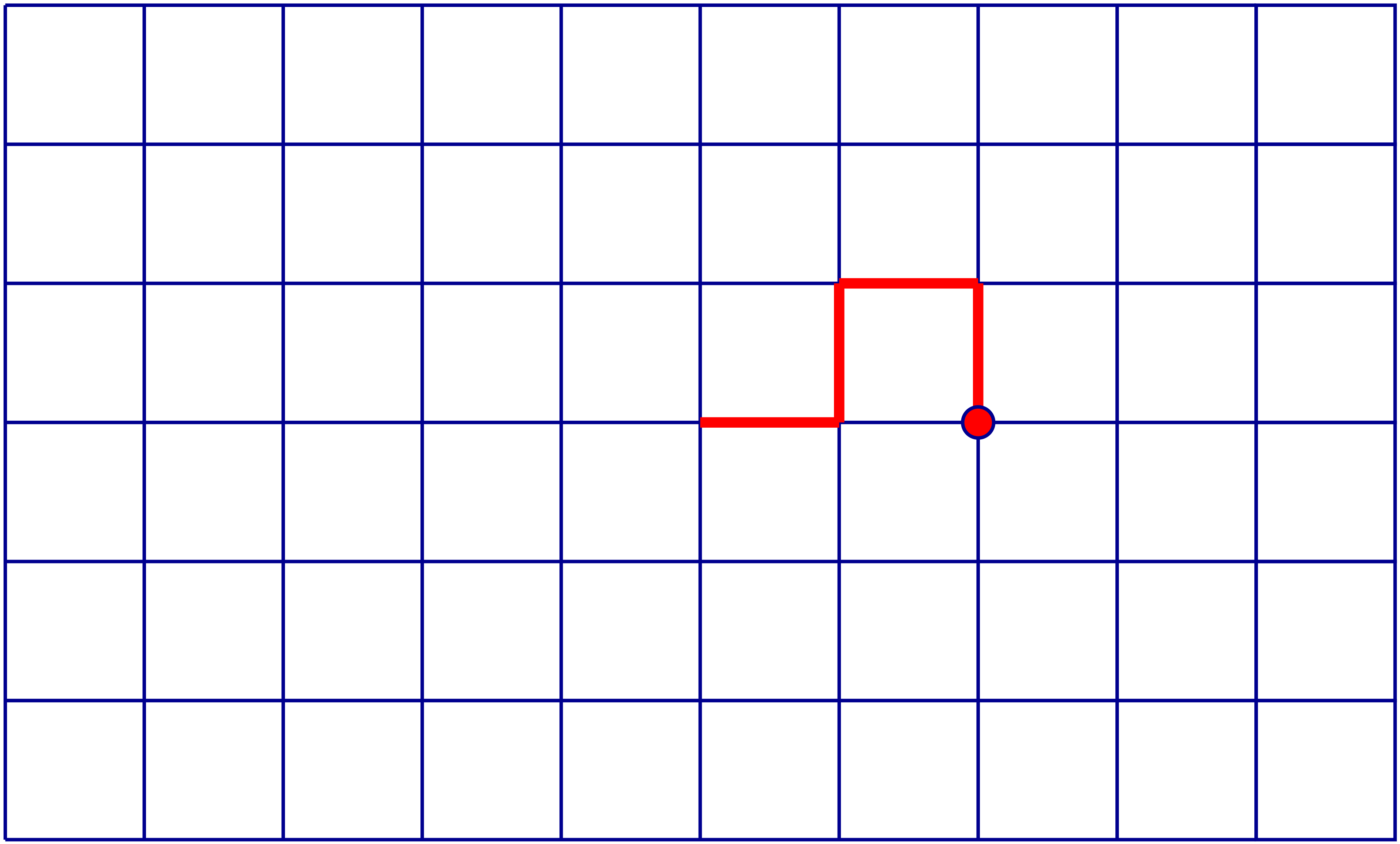


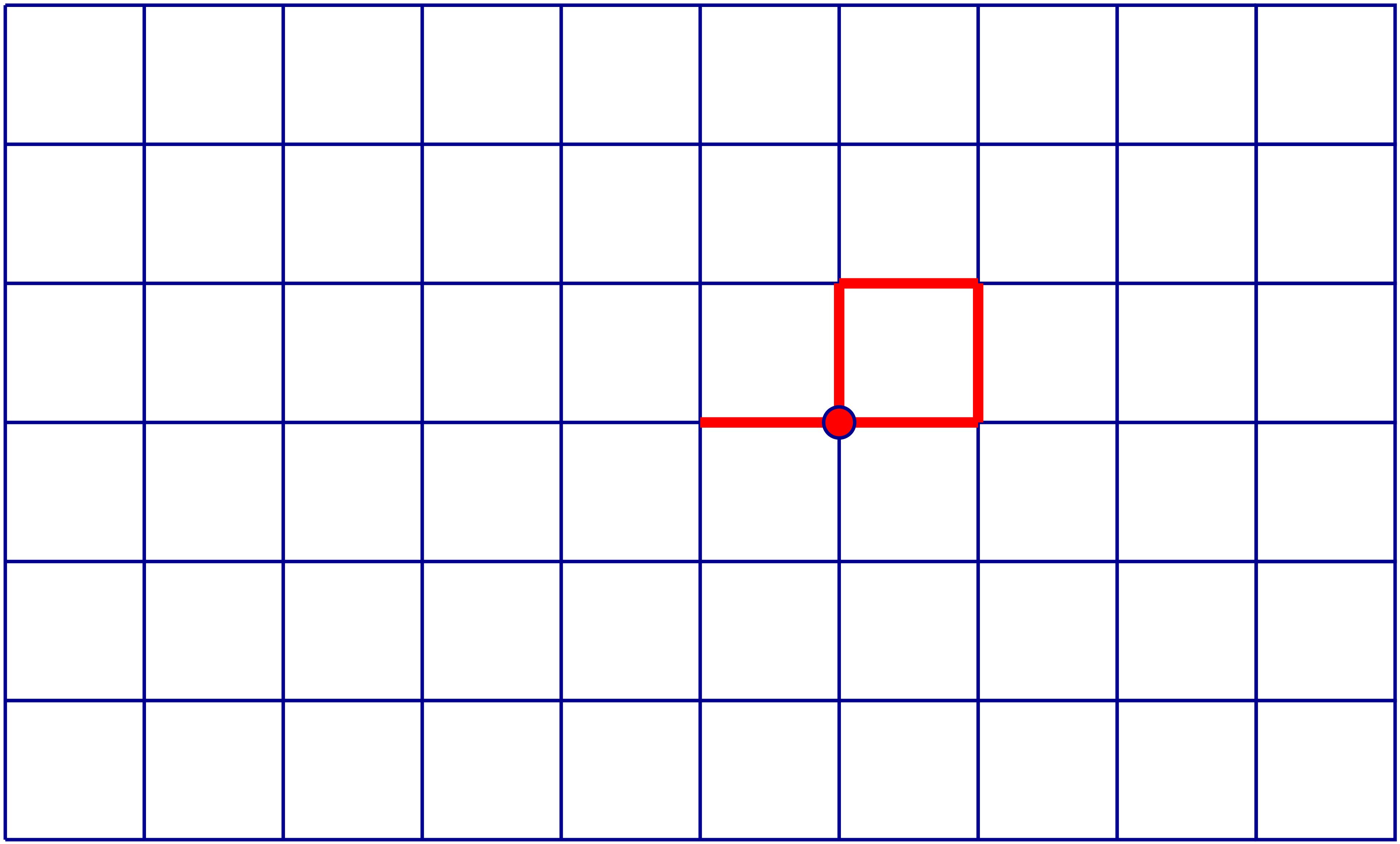


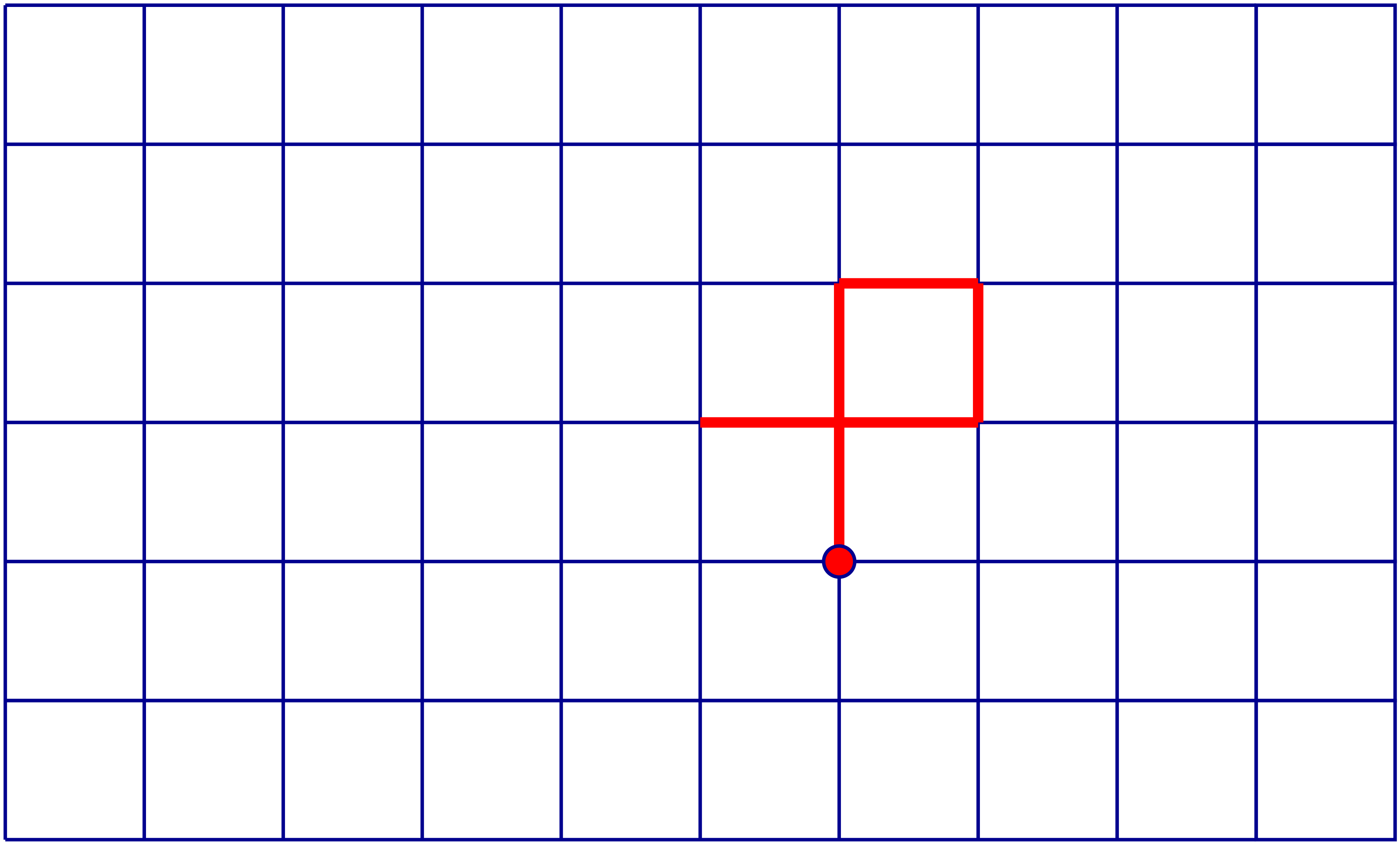


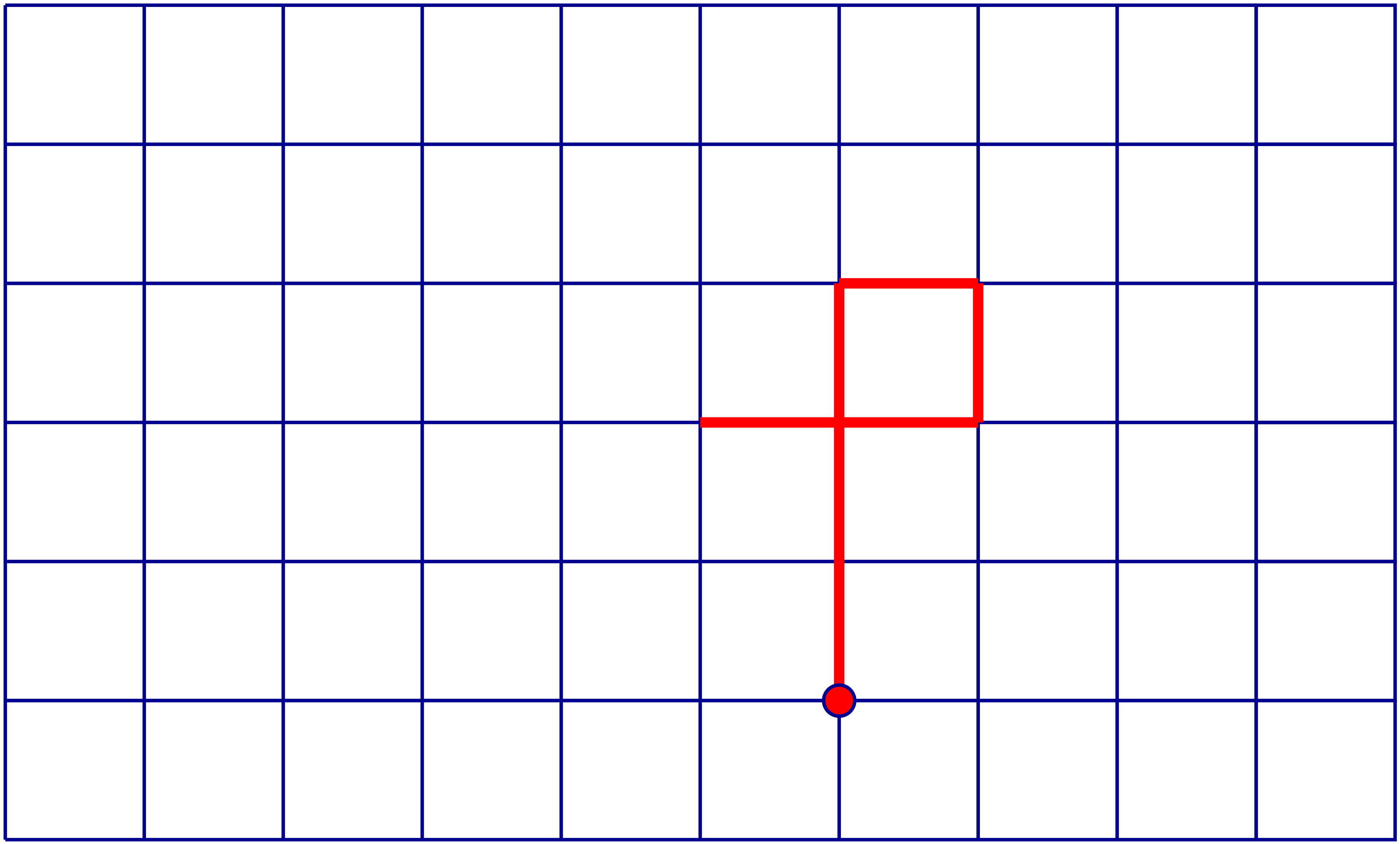


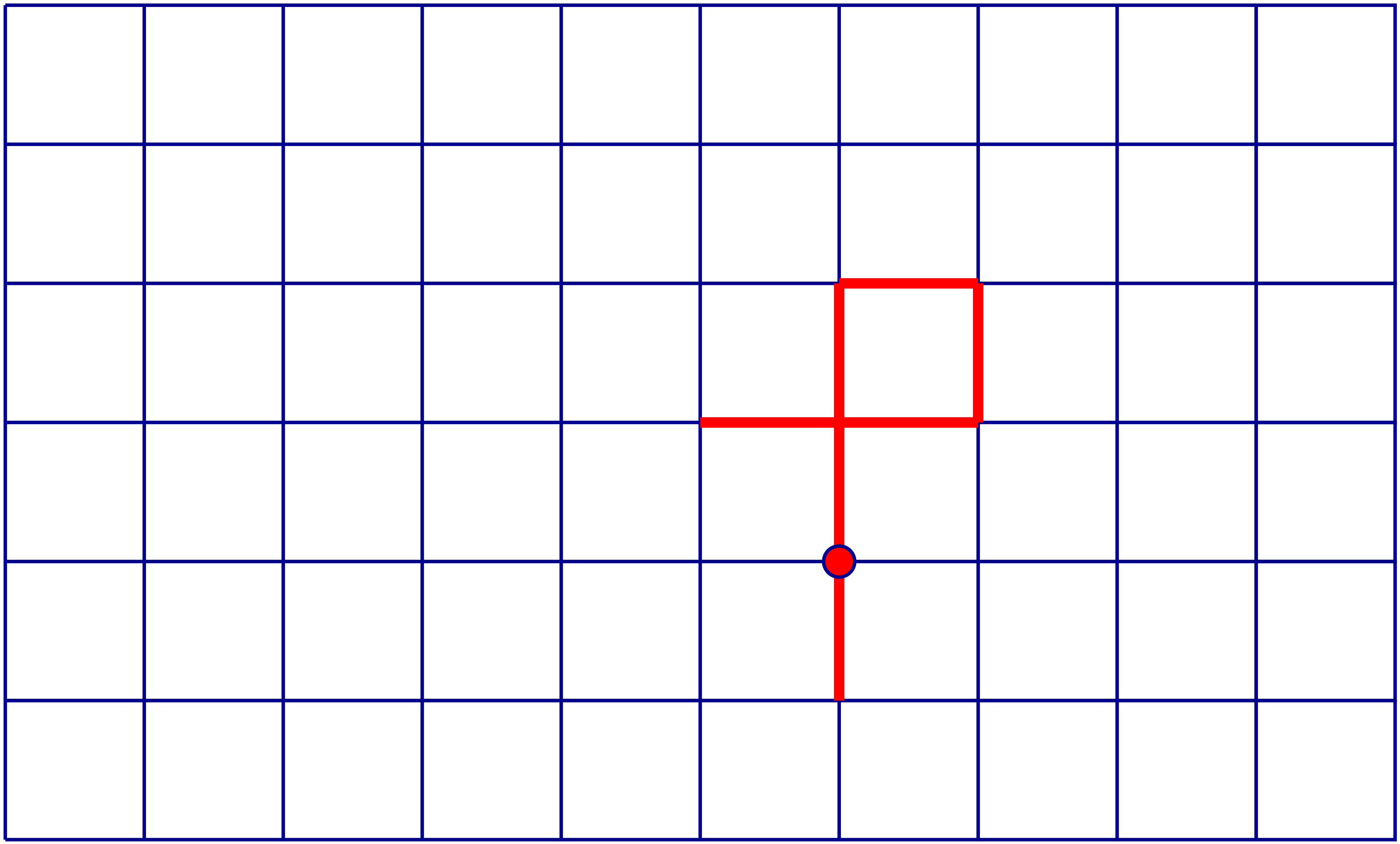


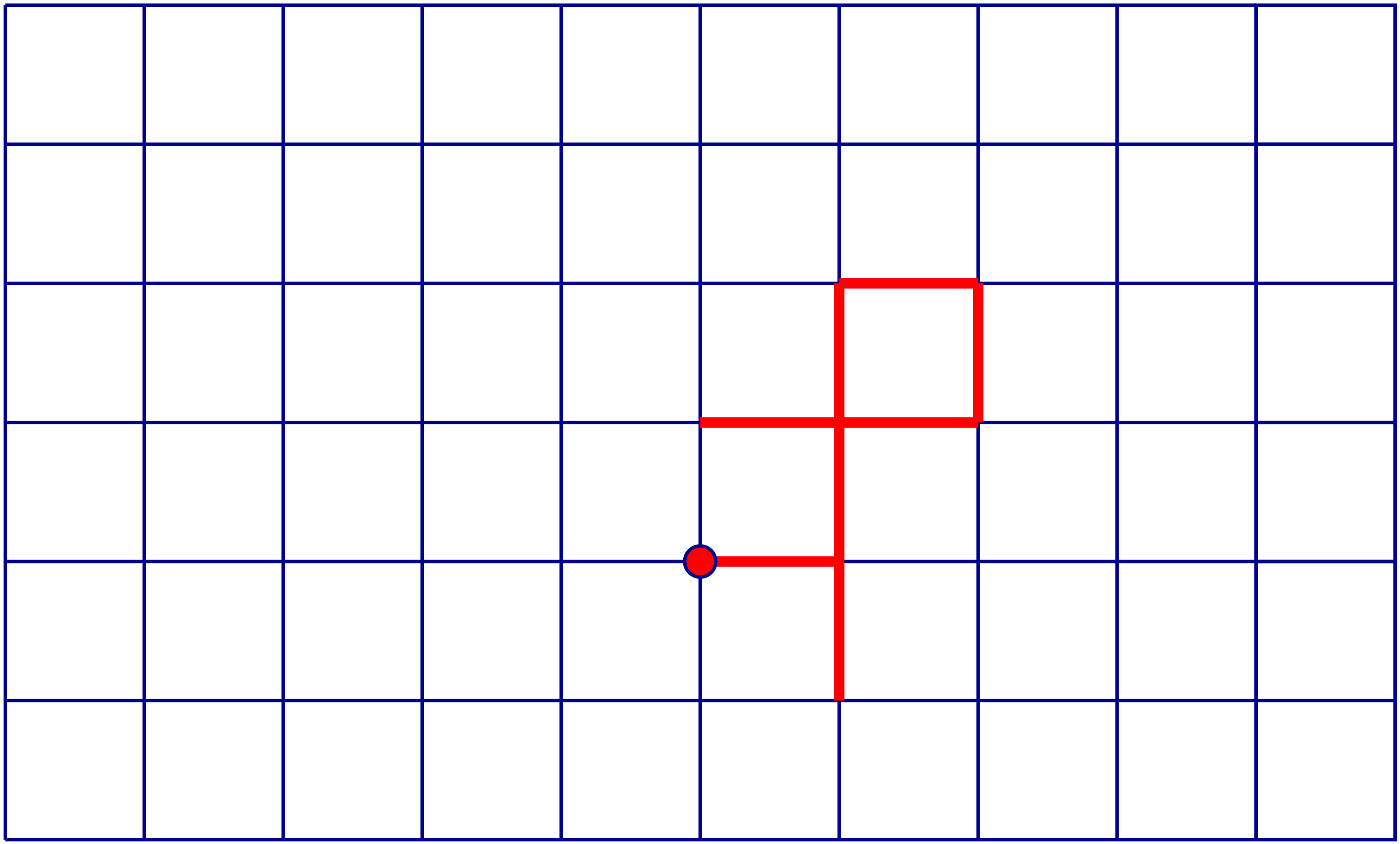


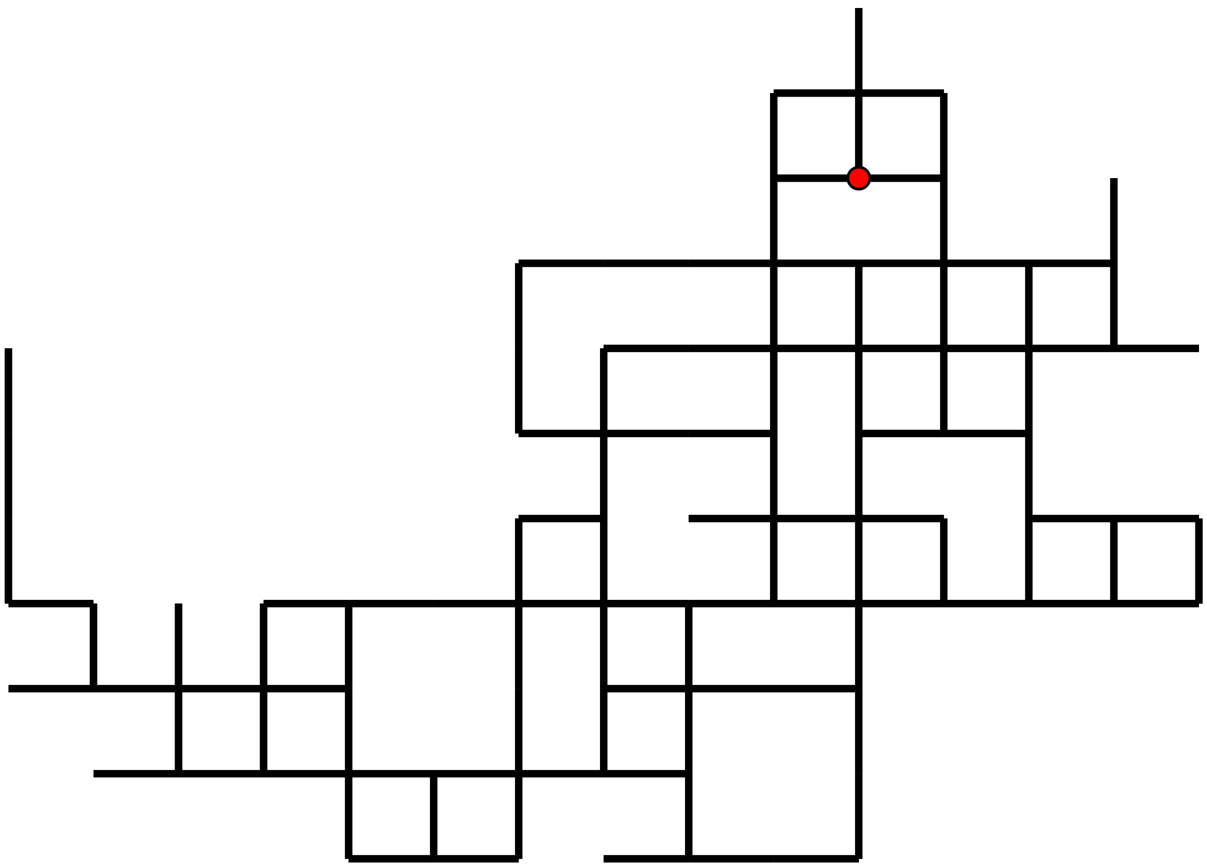




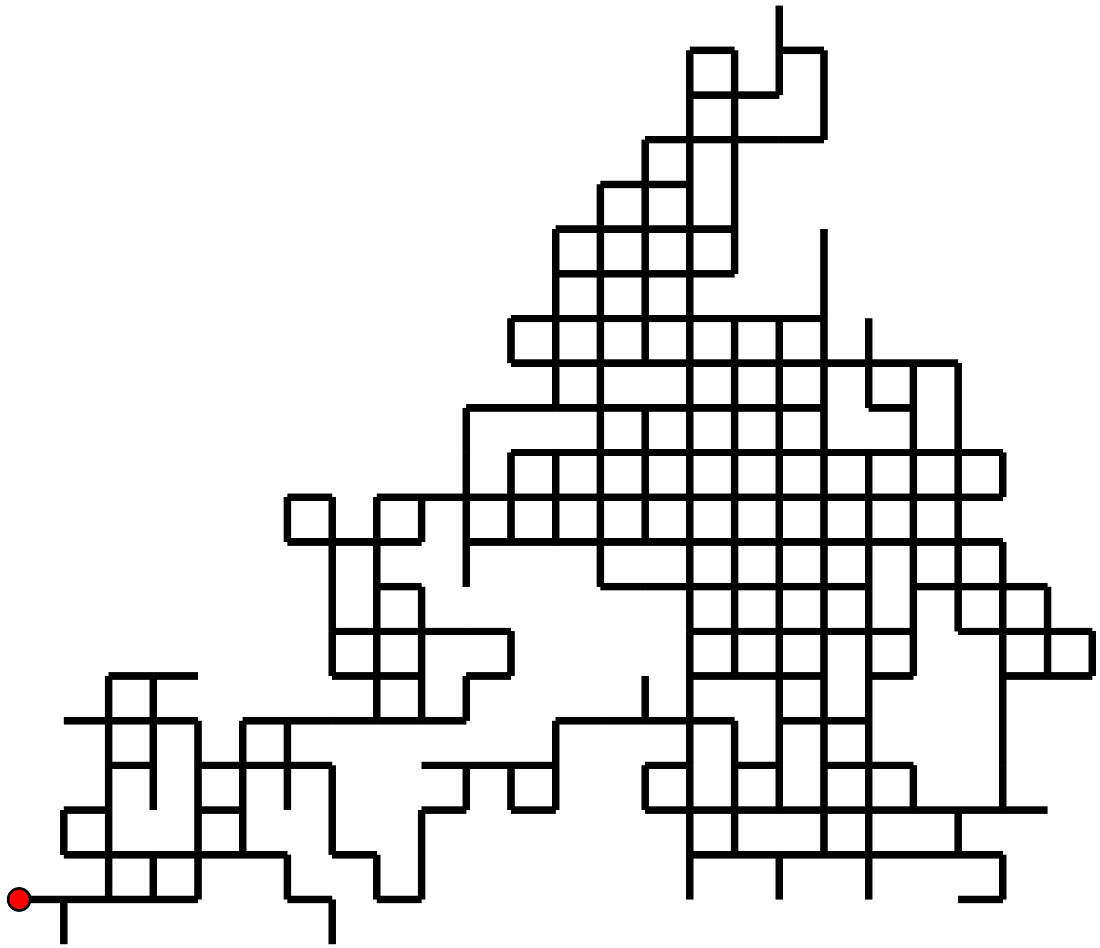




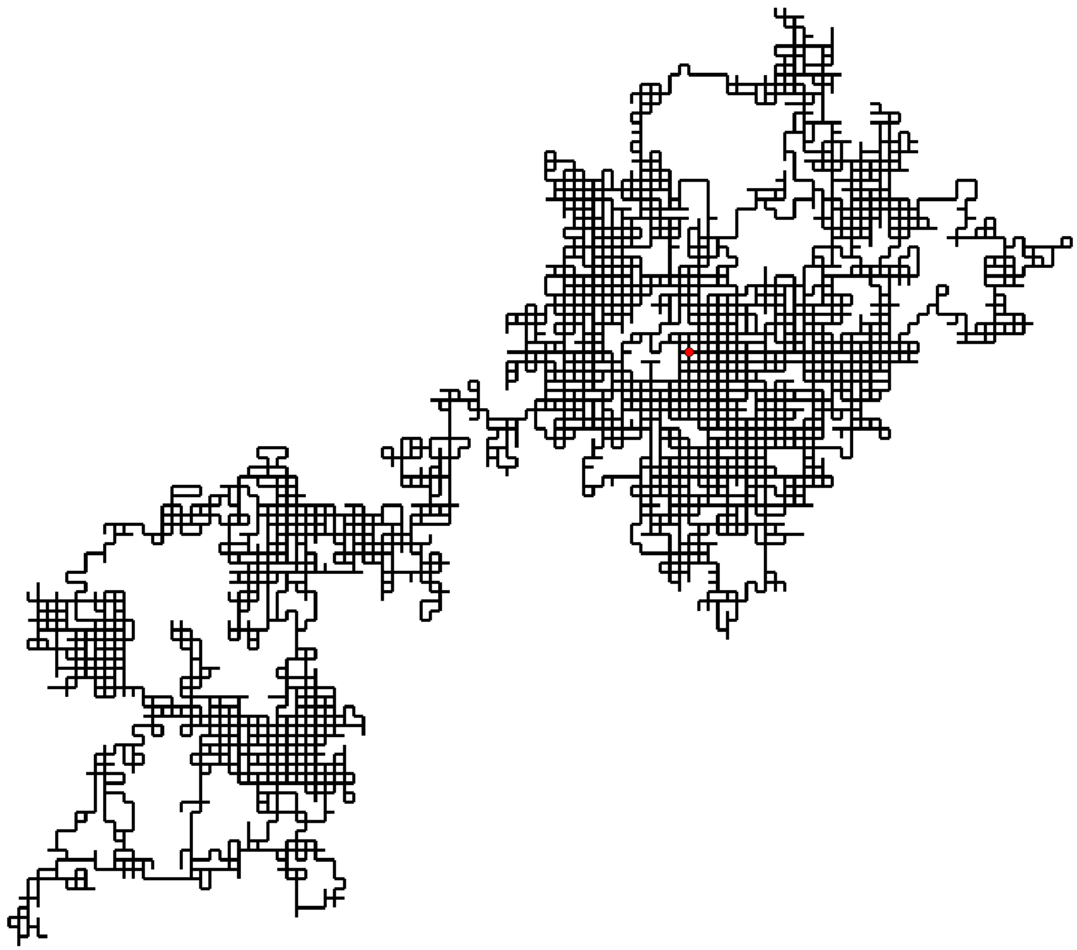




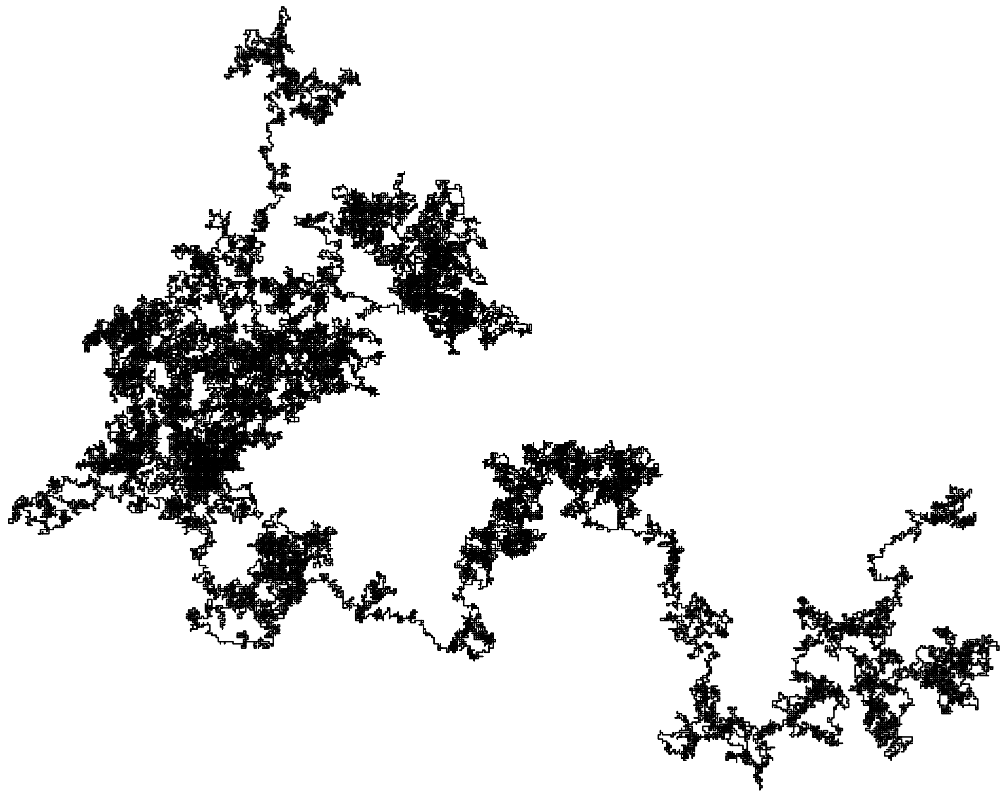
200 step random walk.



1000 step random walk.

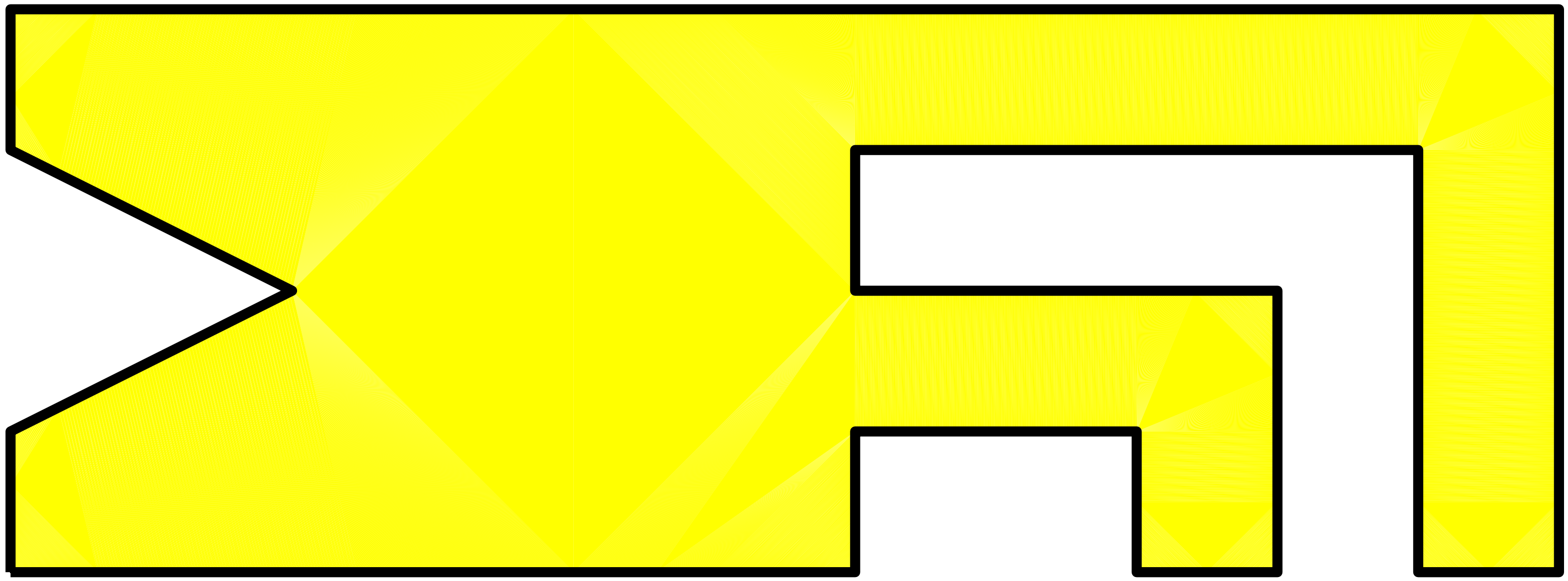


10,000 step random walk.



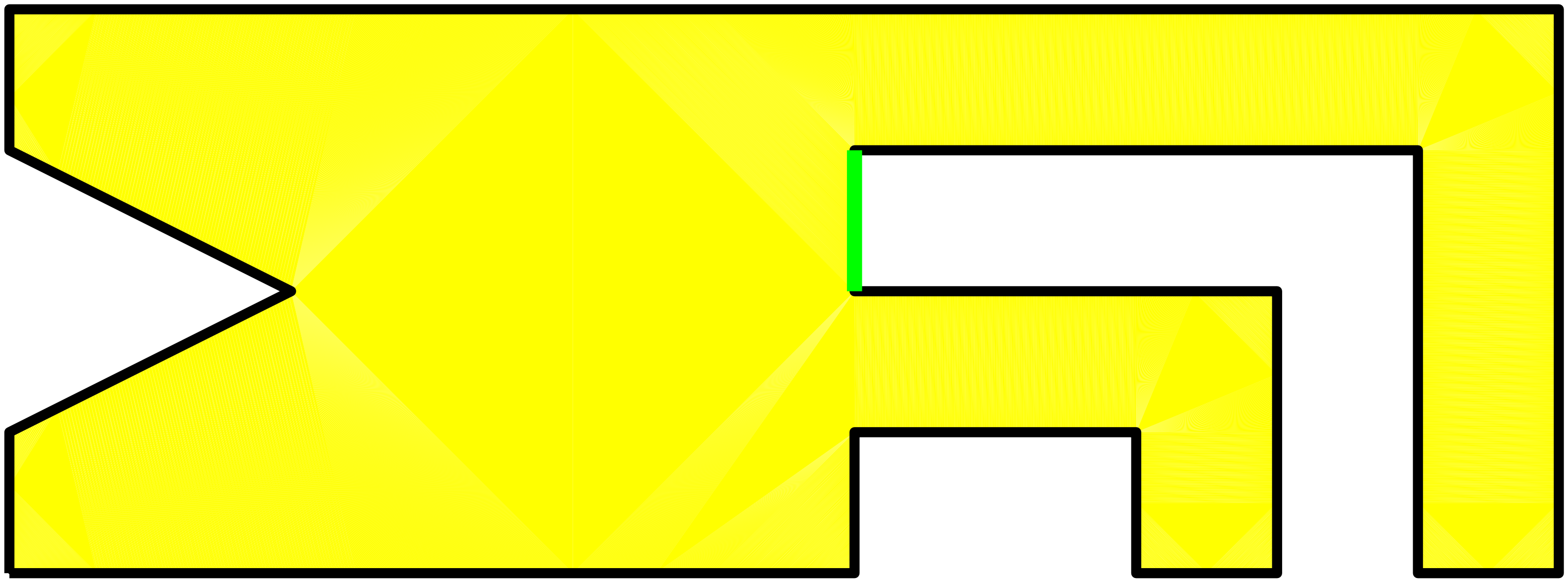
100,000 step random walk.

Harmonic measure = hitting distribution of Brownian motion



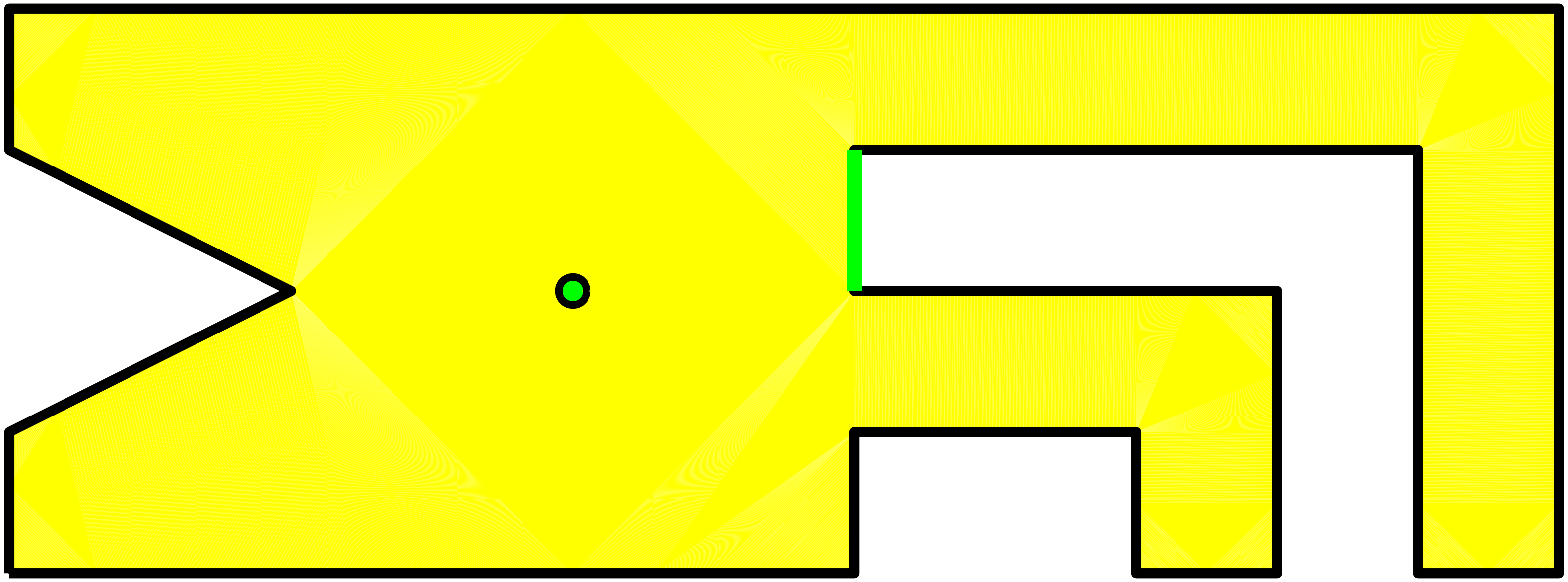
Suppose Ω is a planar Jordan domain.

Harmonic measure = hitting distribution of Brownian motion



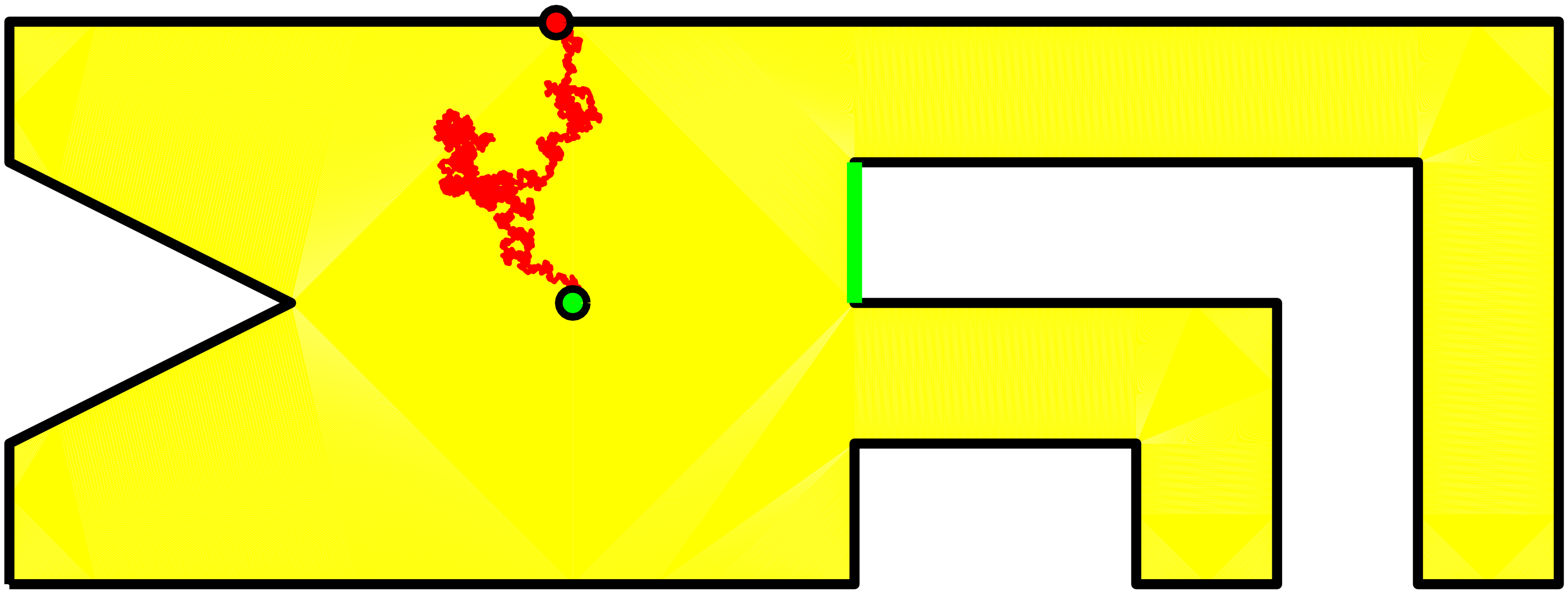
Let E be a subset of the boundary, $\partial\Omega$.

Harmonic measure = hitting distribution of Brownian motion



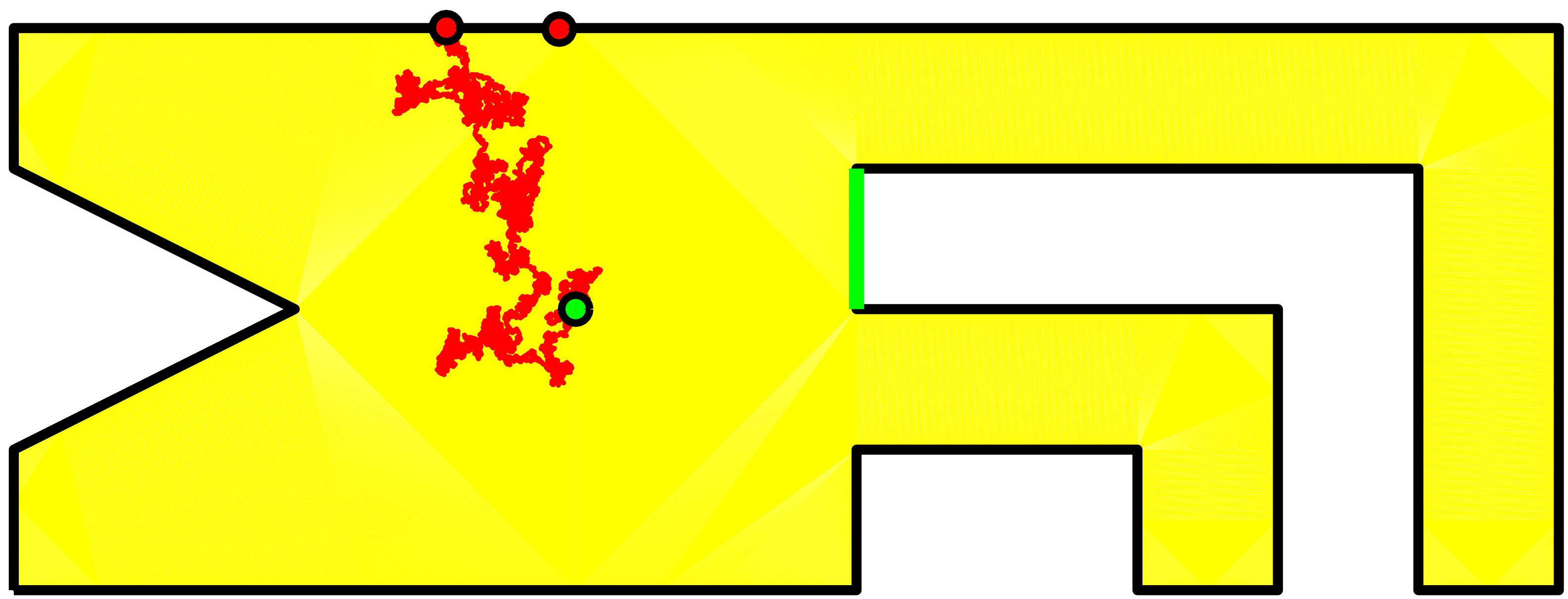
Choose an interior point $z \in \Omega$.

Harmonic measure = hitting distribution of Brownian motion



$\omega(z, E, \Omega)$ = probability a particle started at z first hits $\partial\Omega$ in E.

Harmonic measure = hitting distribution of Brownian motion



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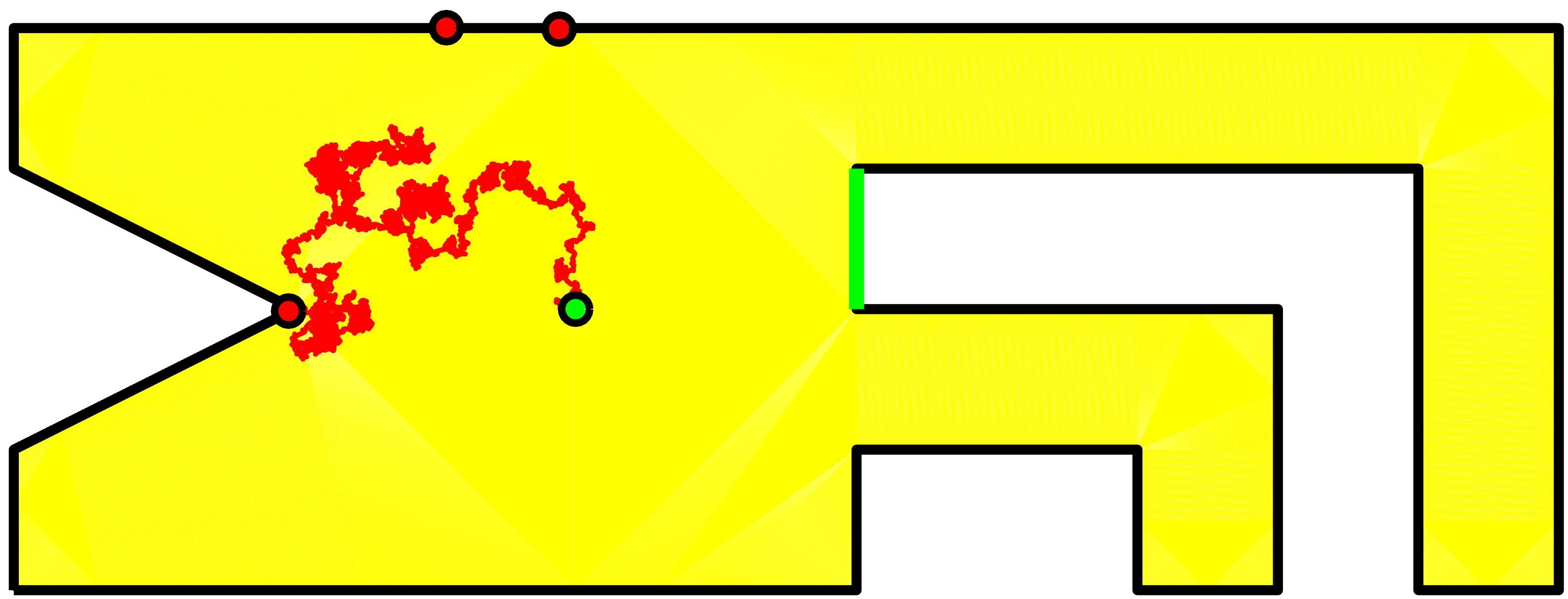

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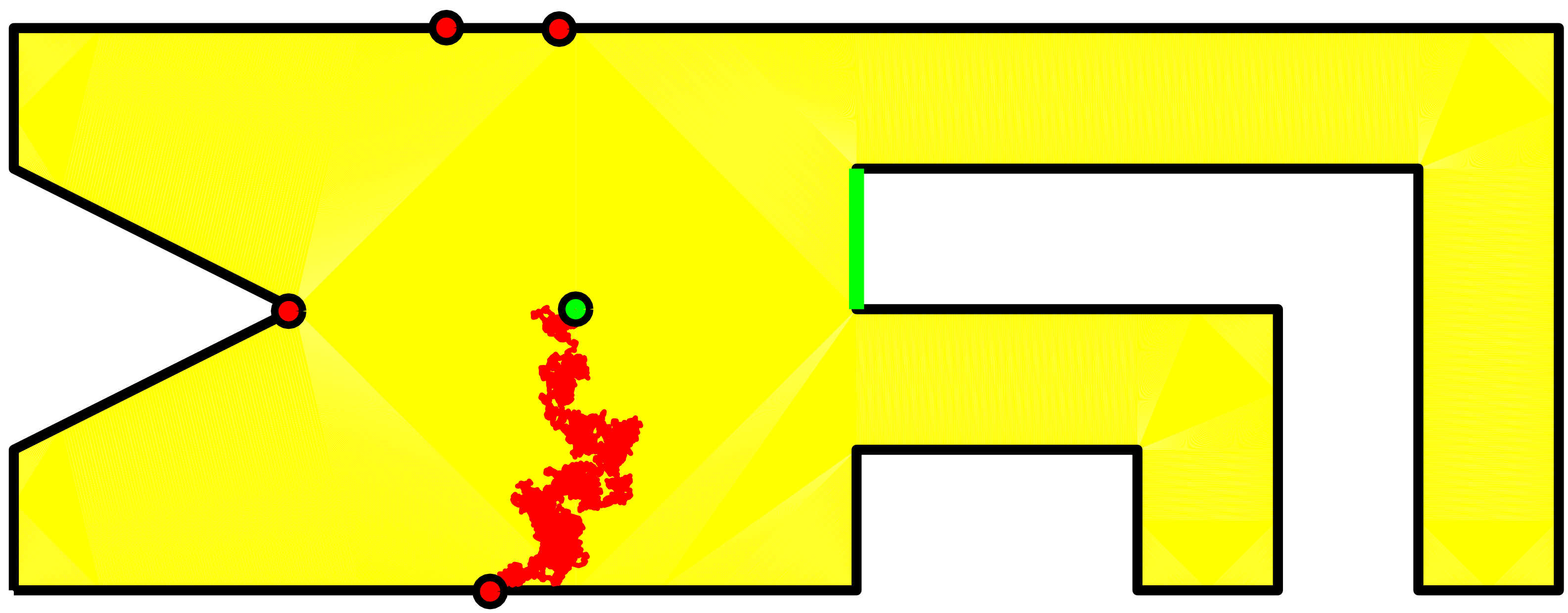

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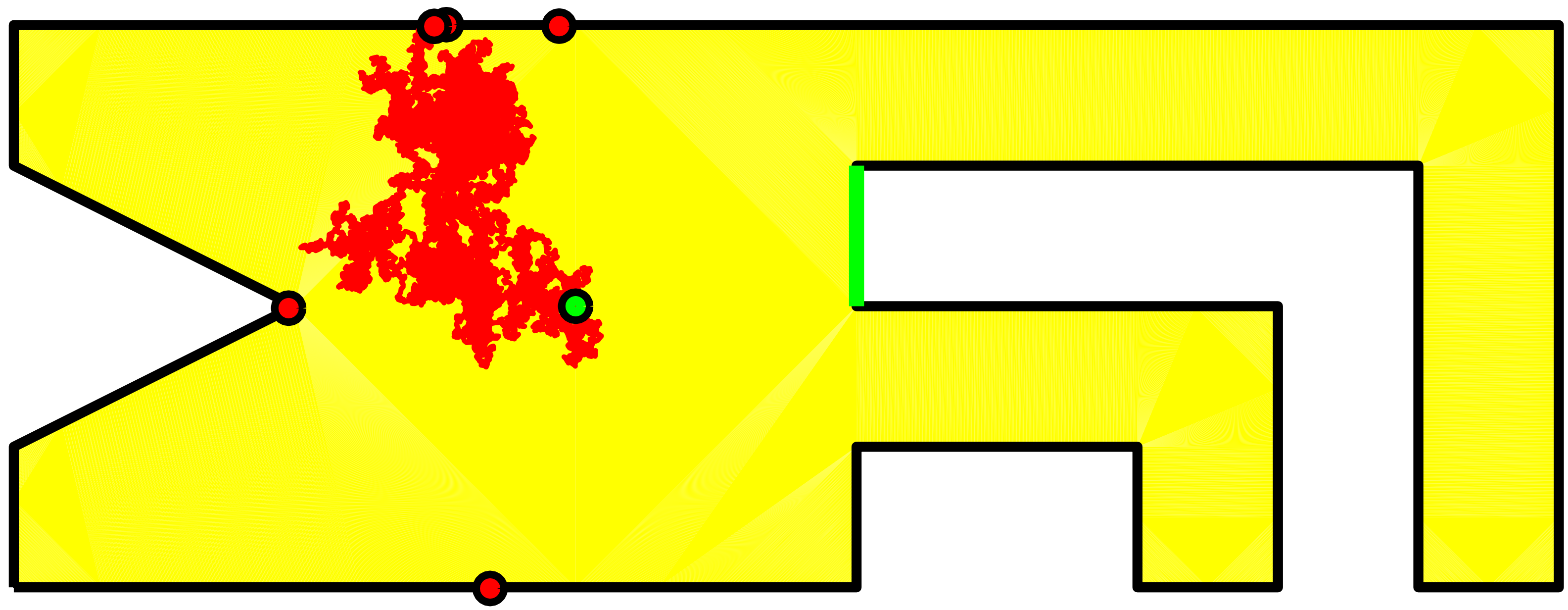

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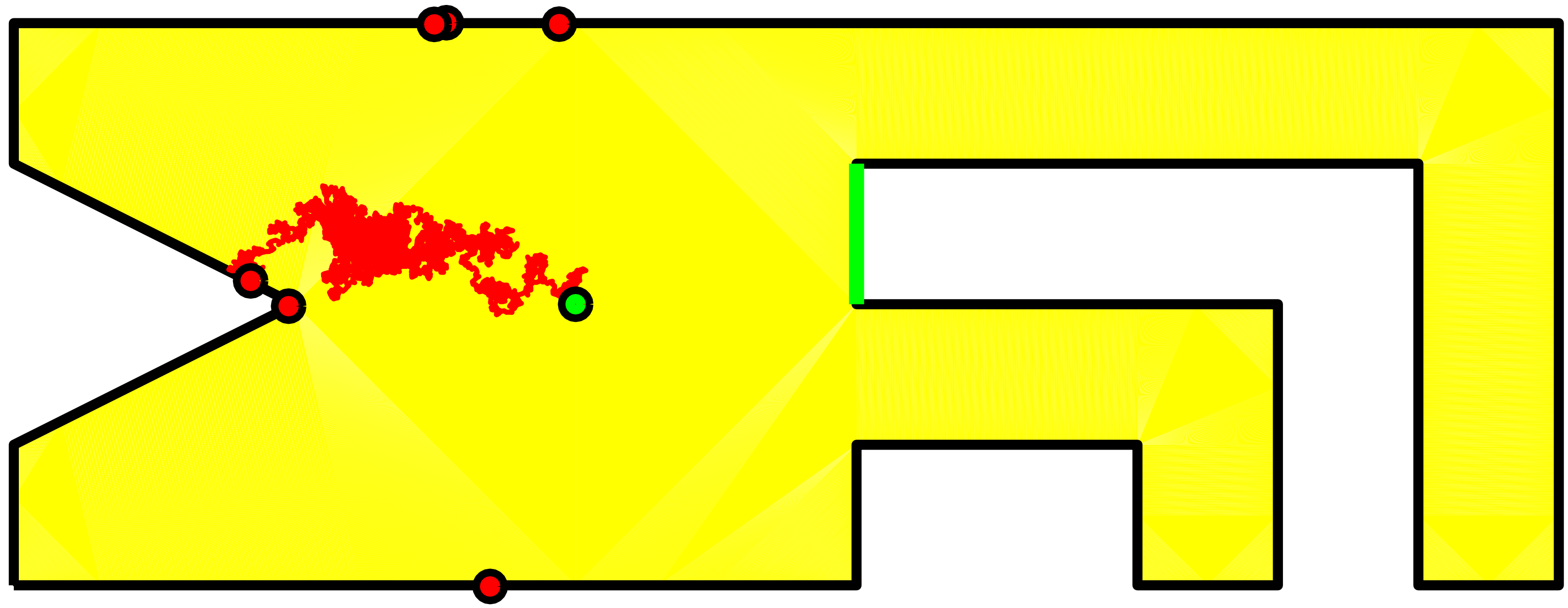

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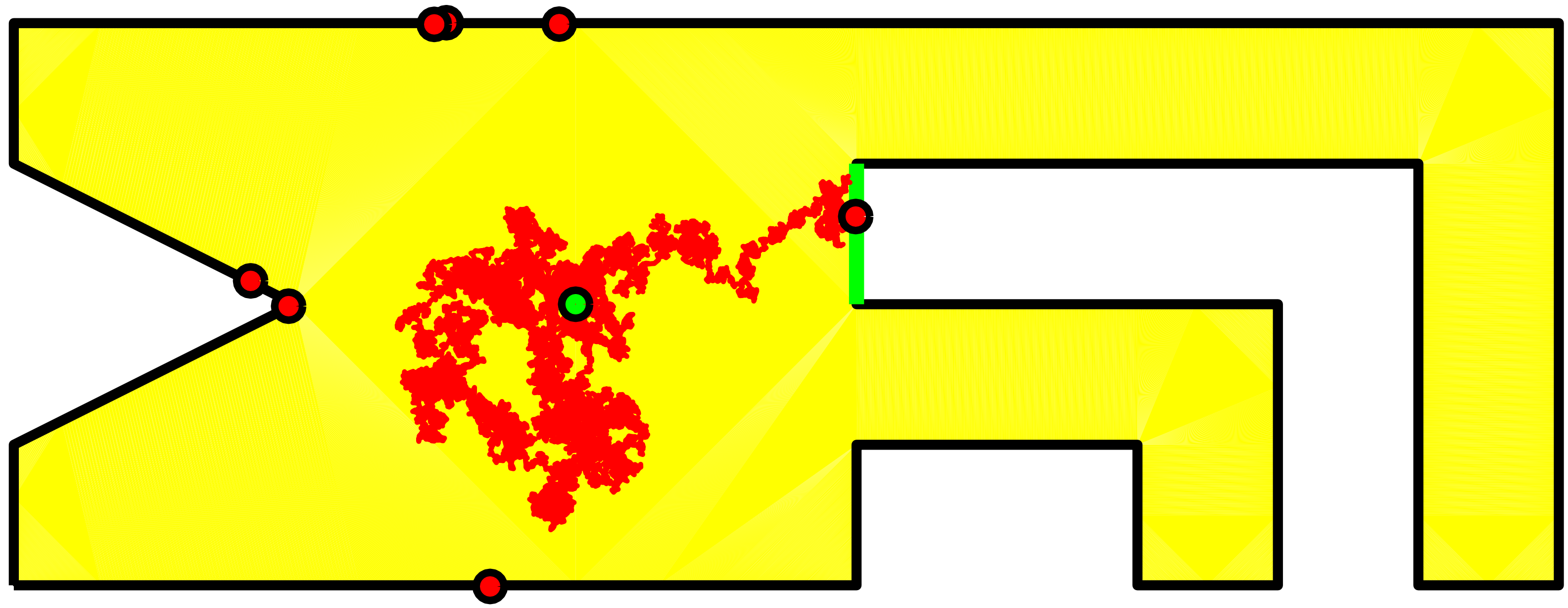

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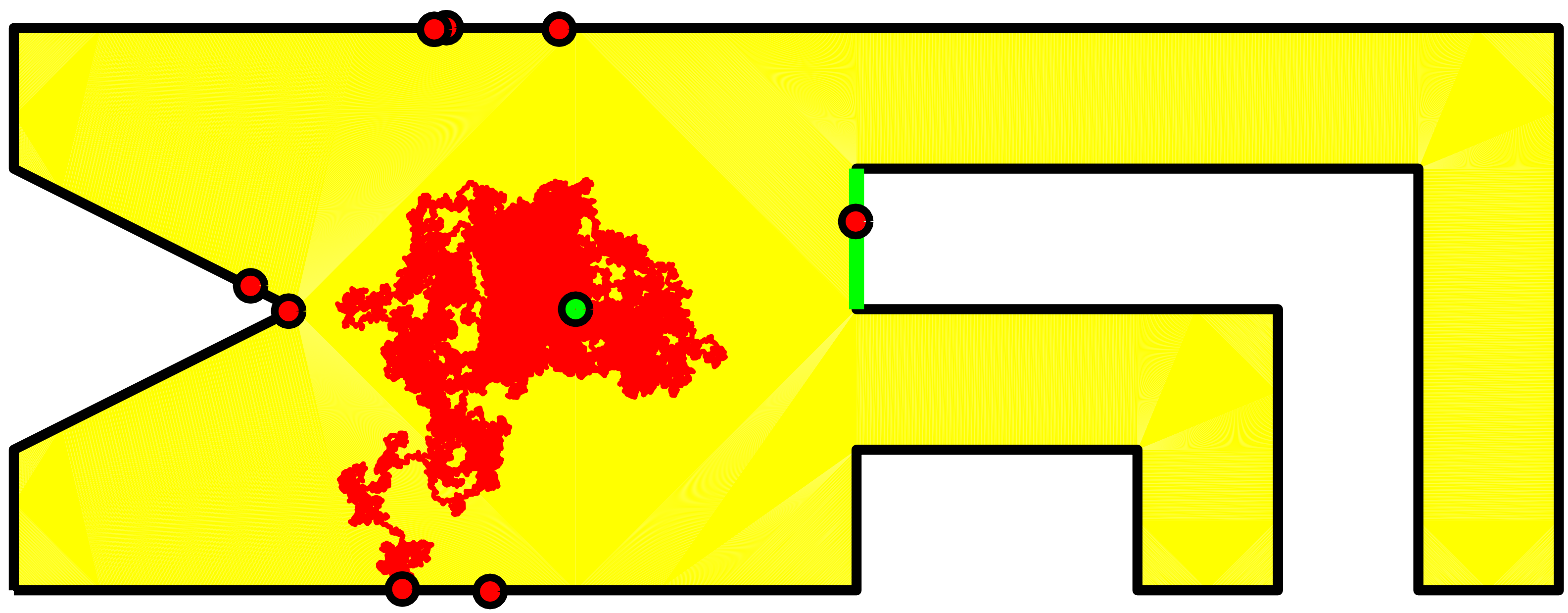

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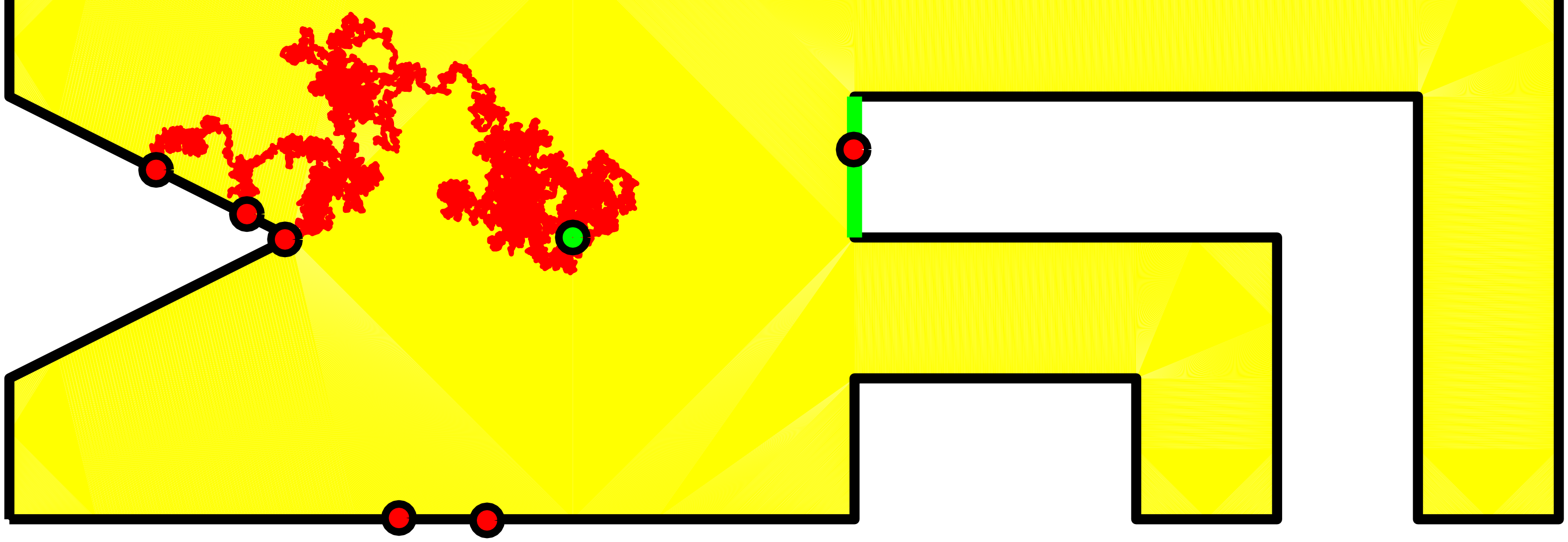

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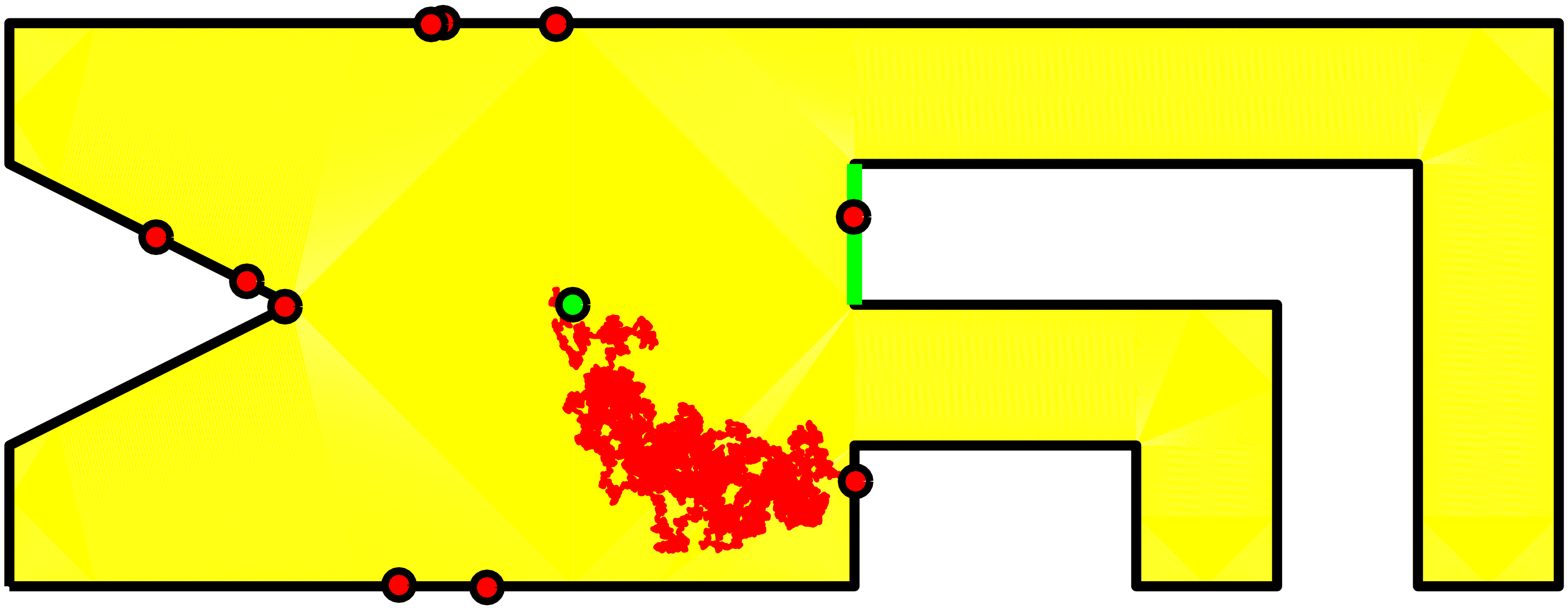
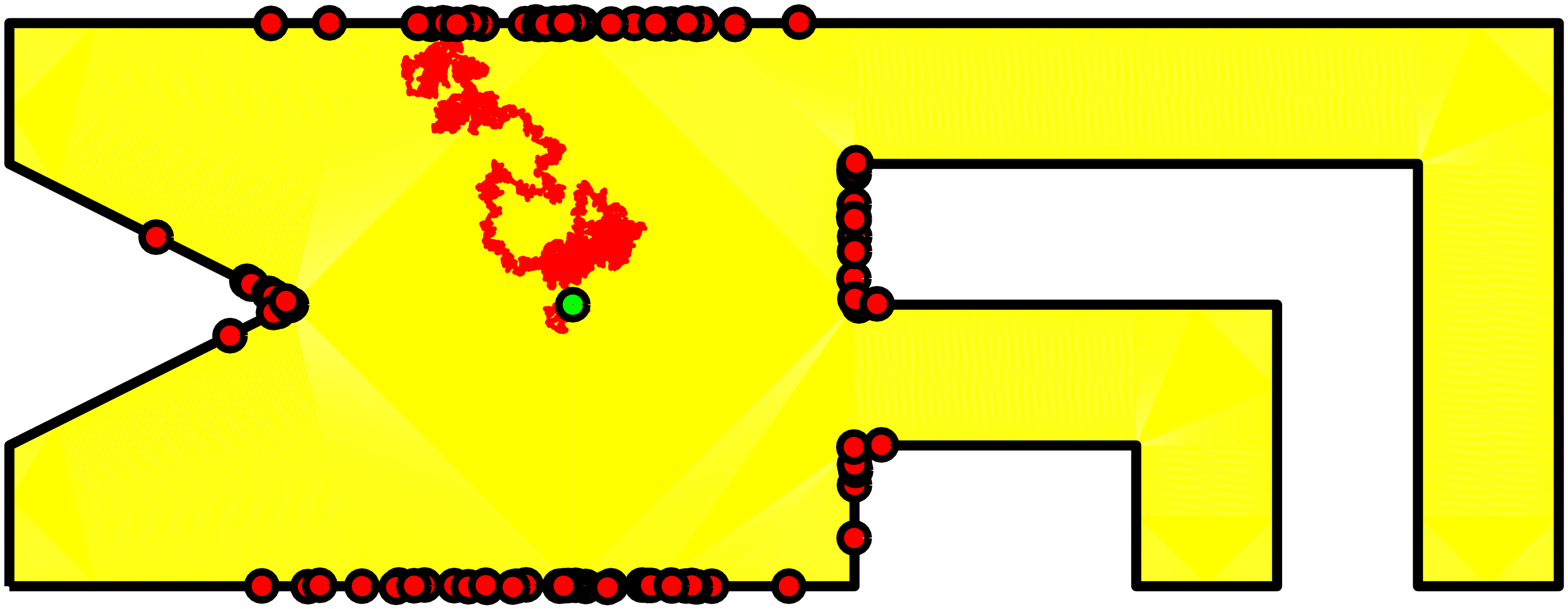


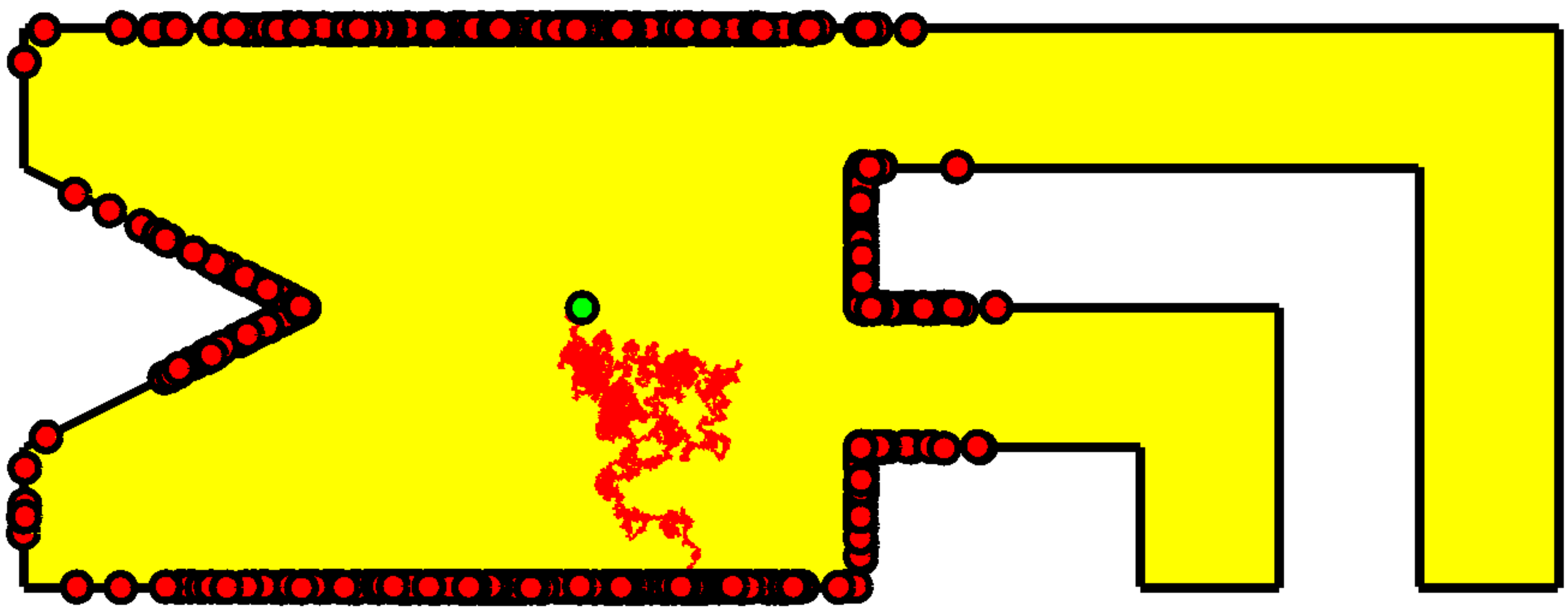
$$\omega(z, E, \Omega) \approx 1/10.$$

Harmonic measure = hitting distribution of Brownian motion



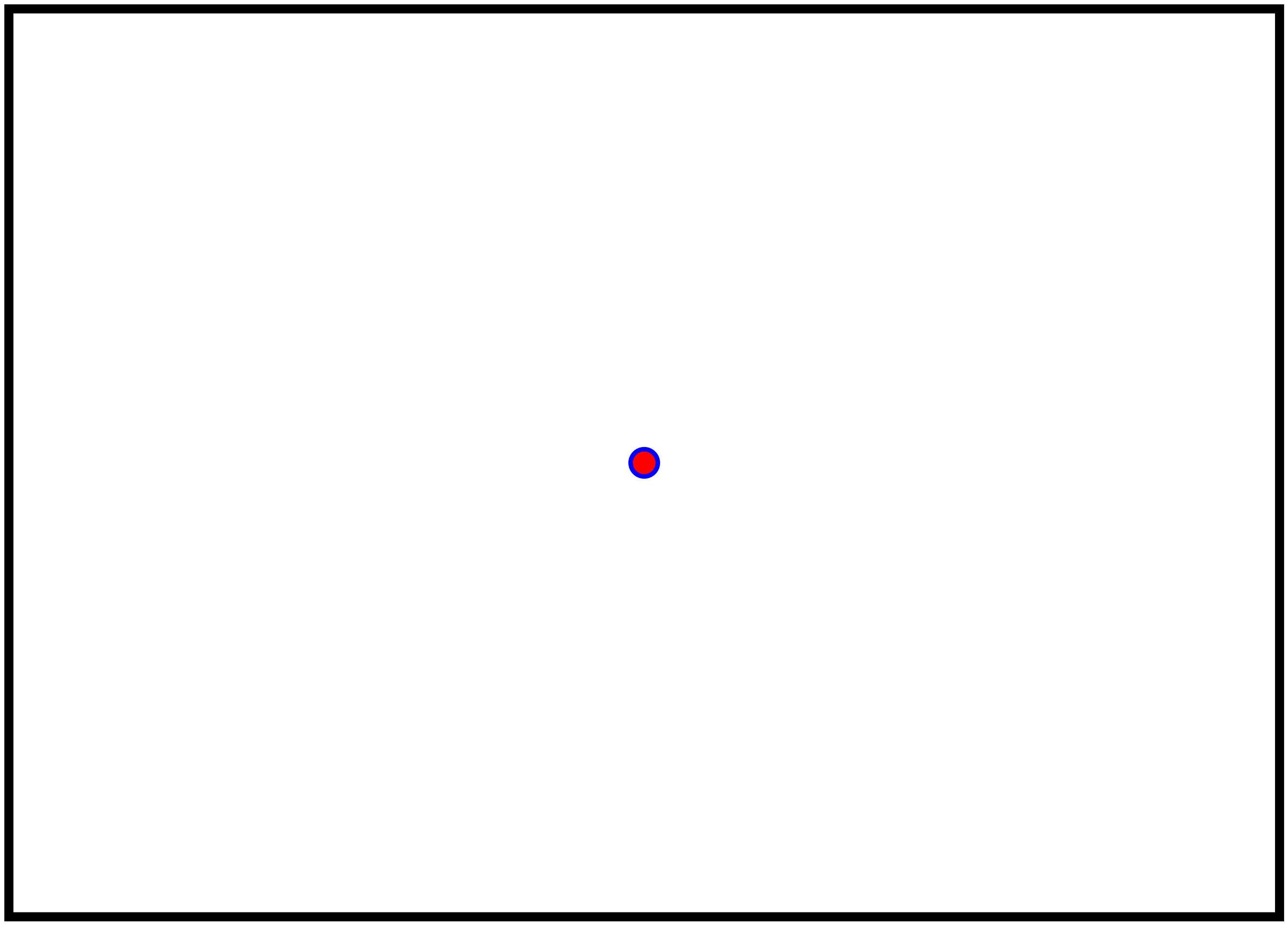
$$\omega(z, E, \Omega) \approx 13/100.$$

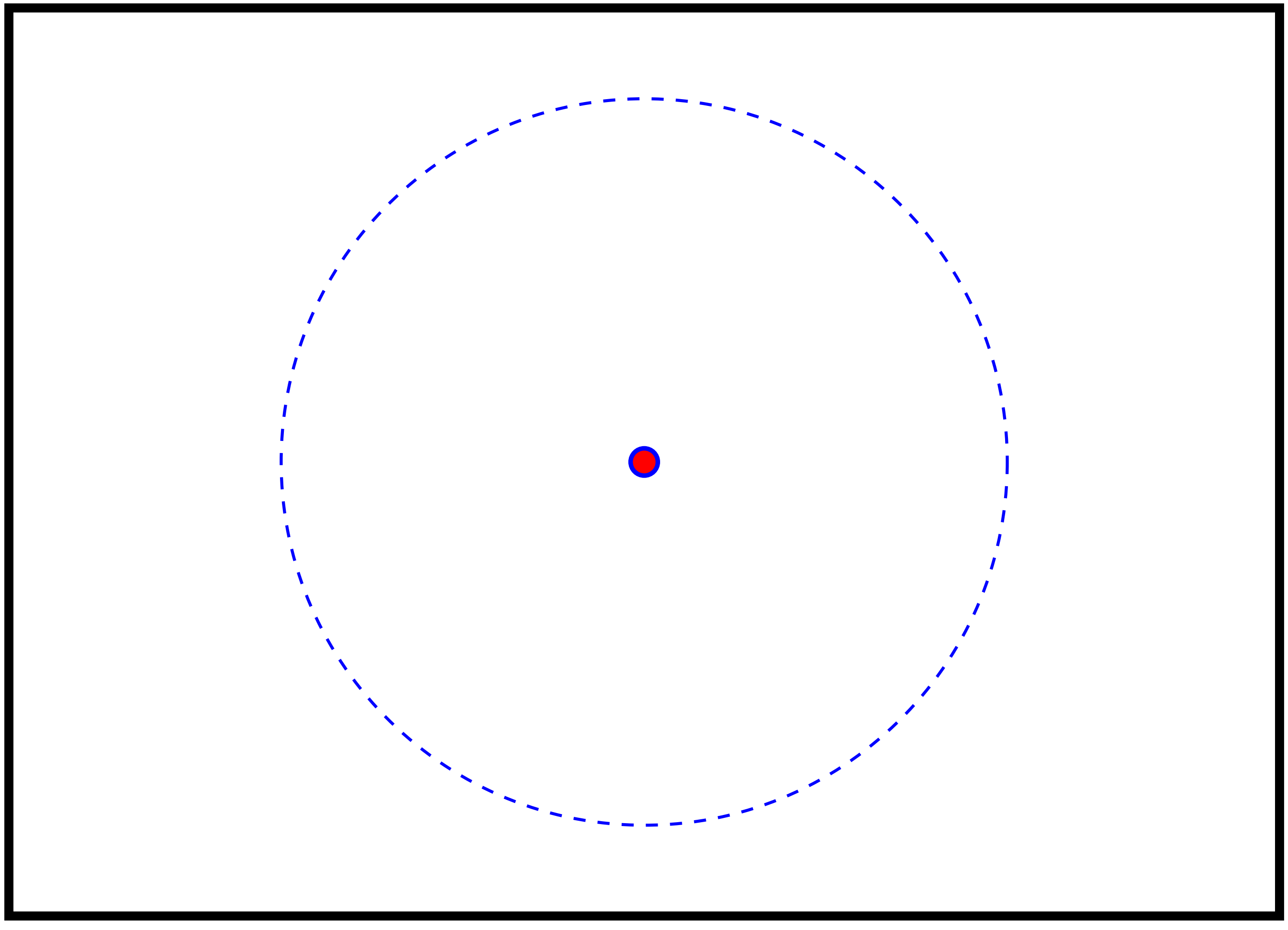
Harmonic measure = hitting distribution of Brownian motion

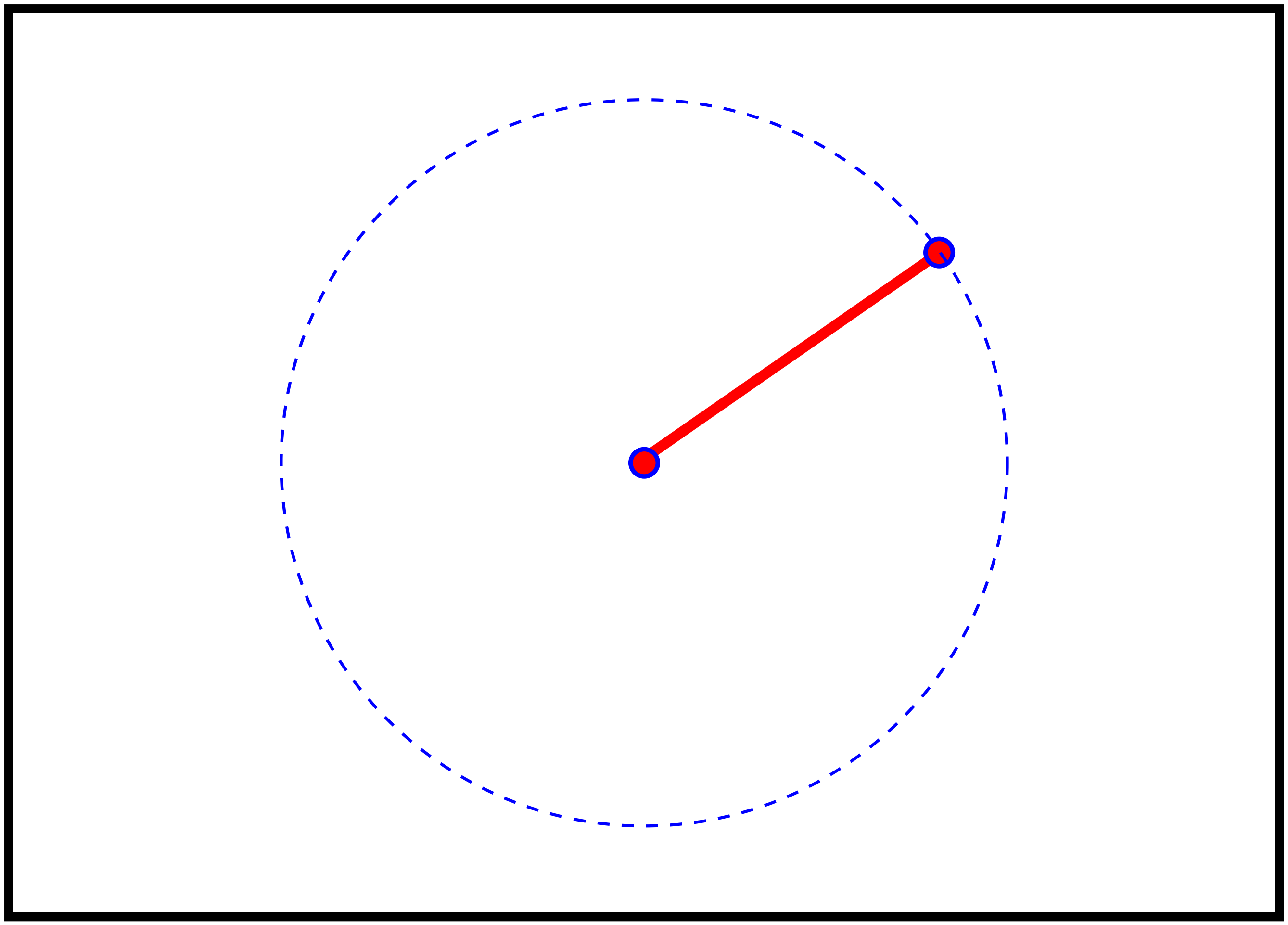


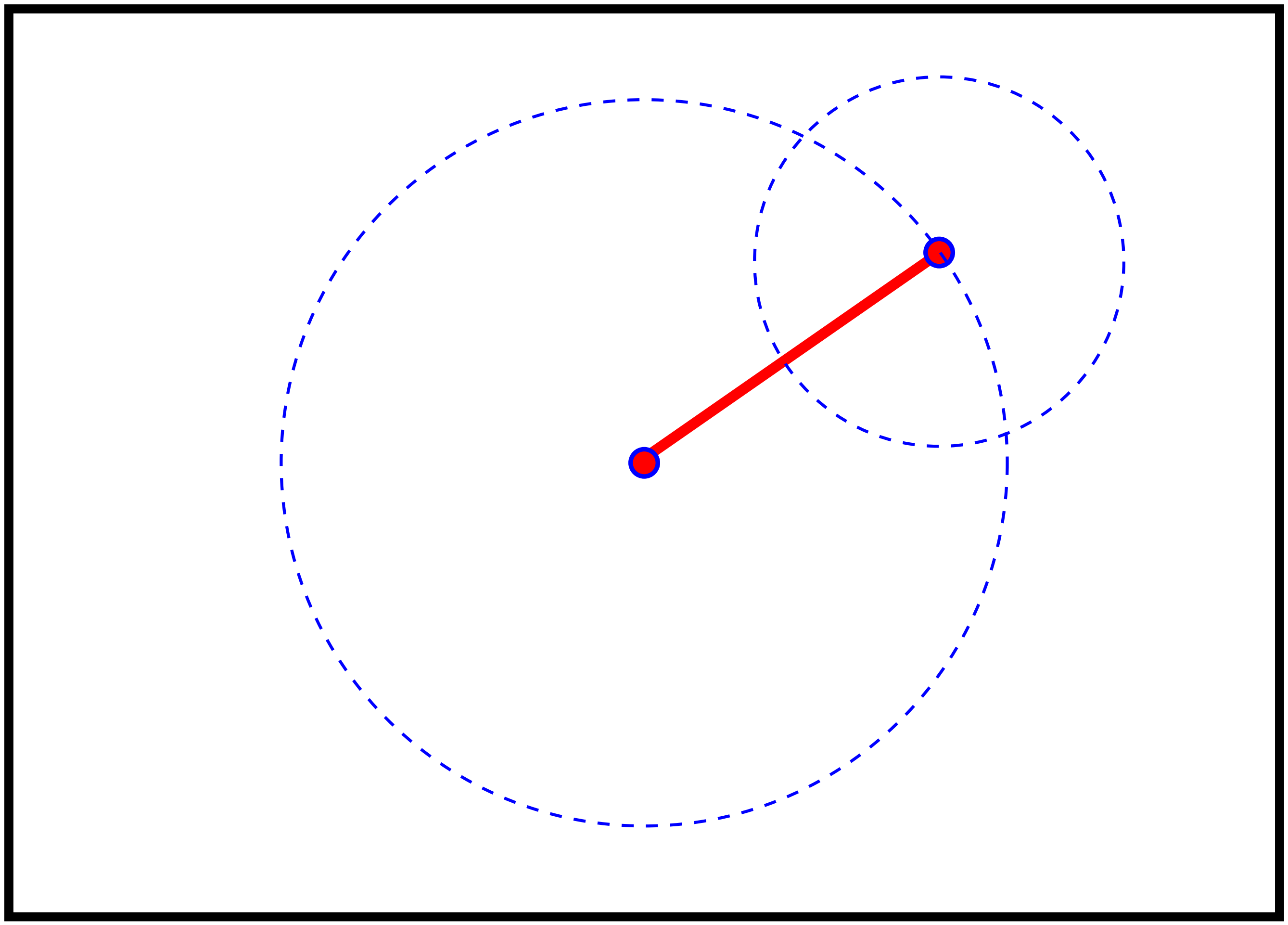
$$\omega(z, E, \Omega) \approx 126/1000.$$

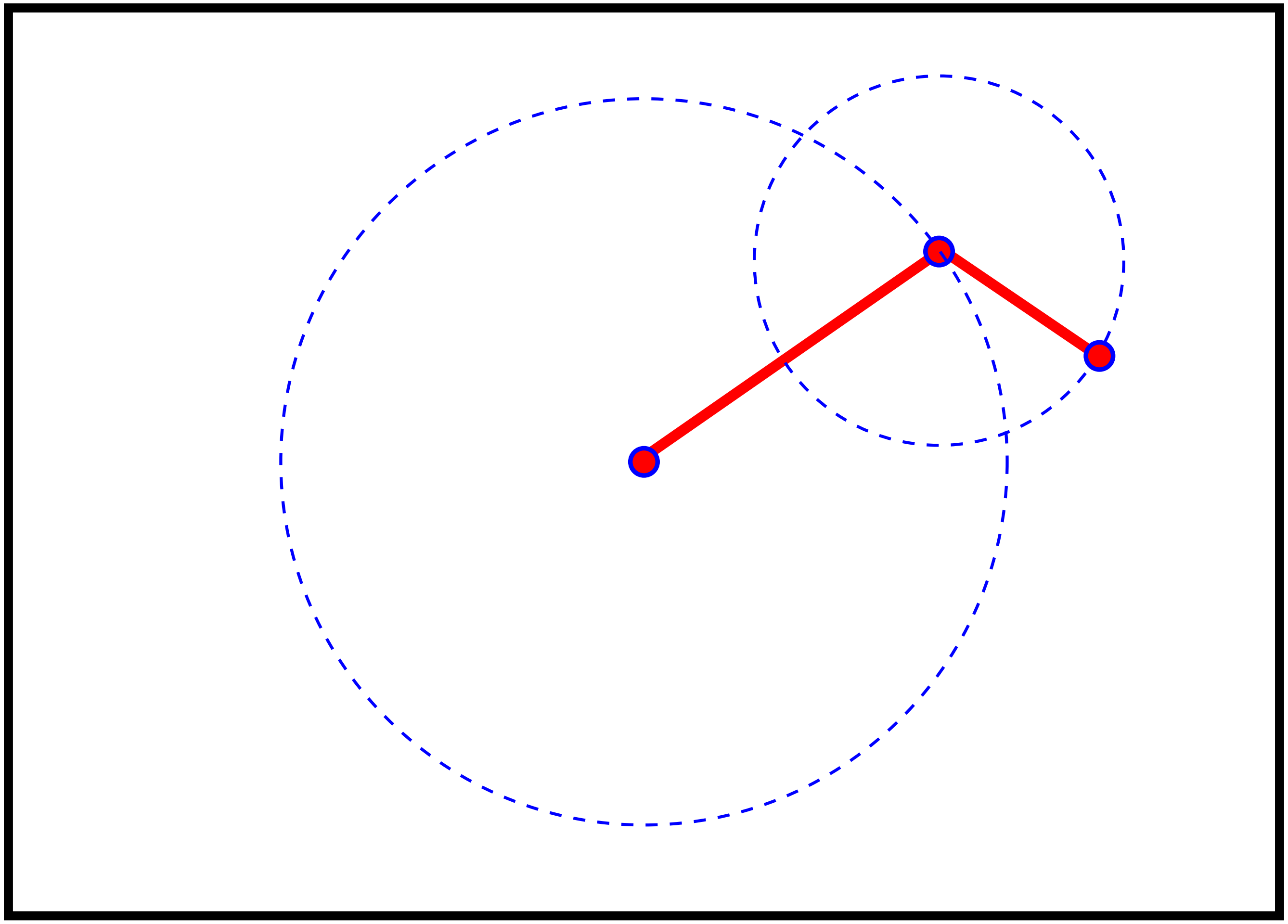
Slow to compute each path ($\simeq \epsilon^{-2}$ steps on ϵ-grid). Can we go faster?

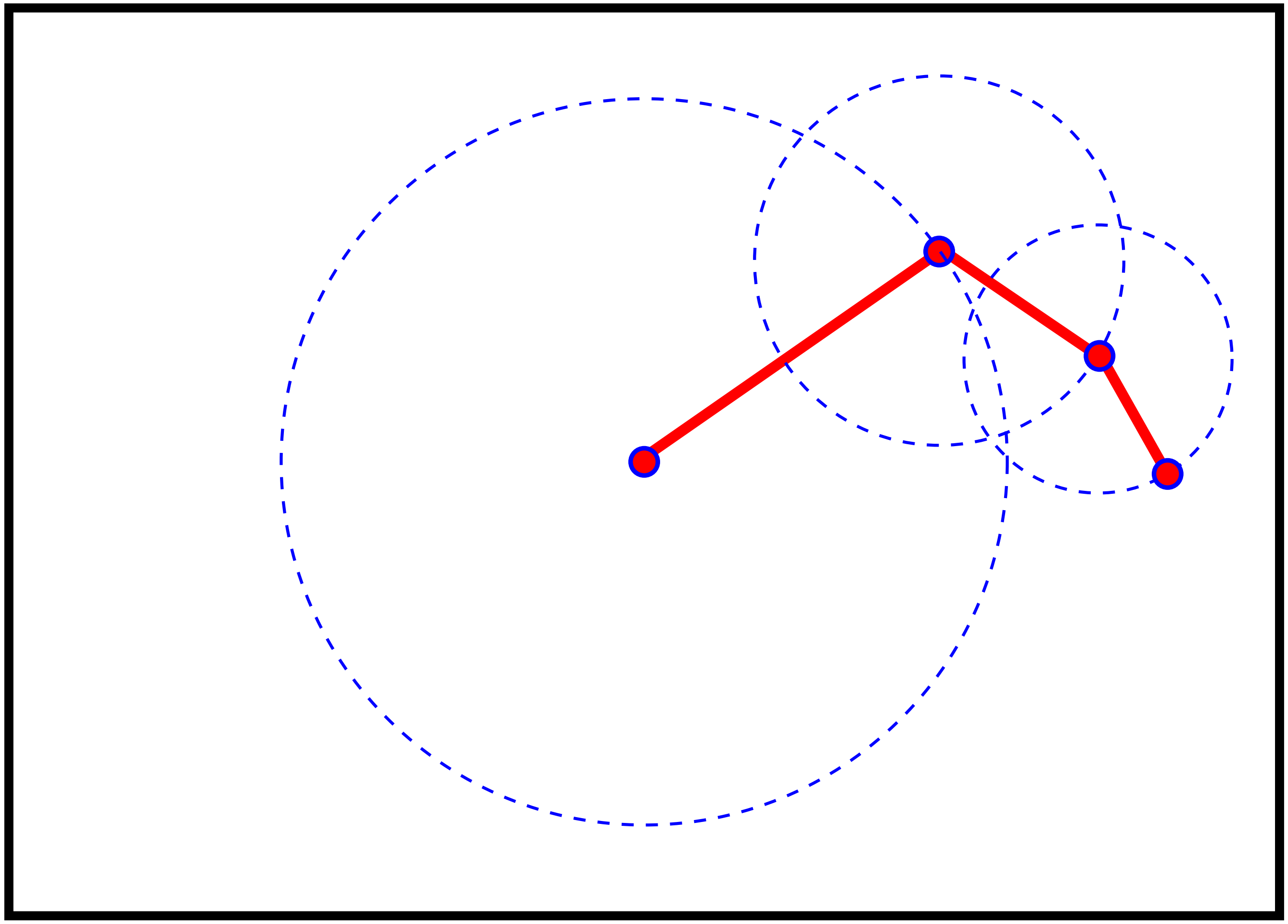


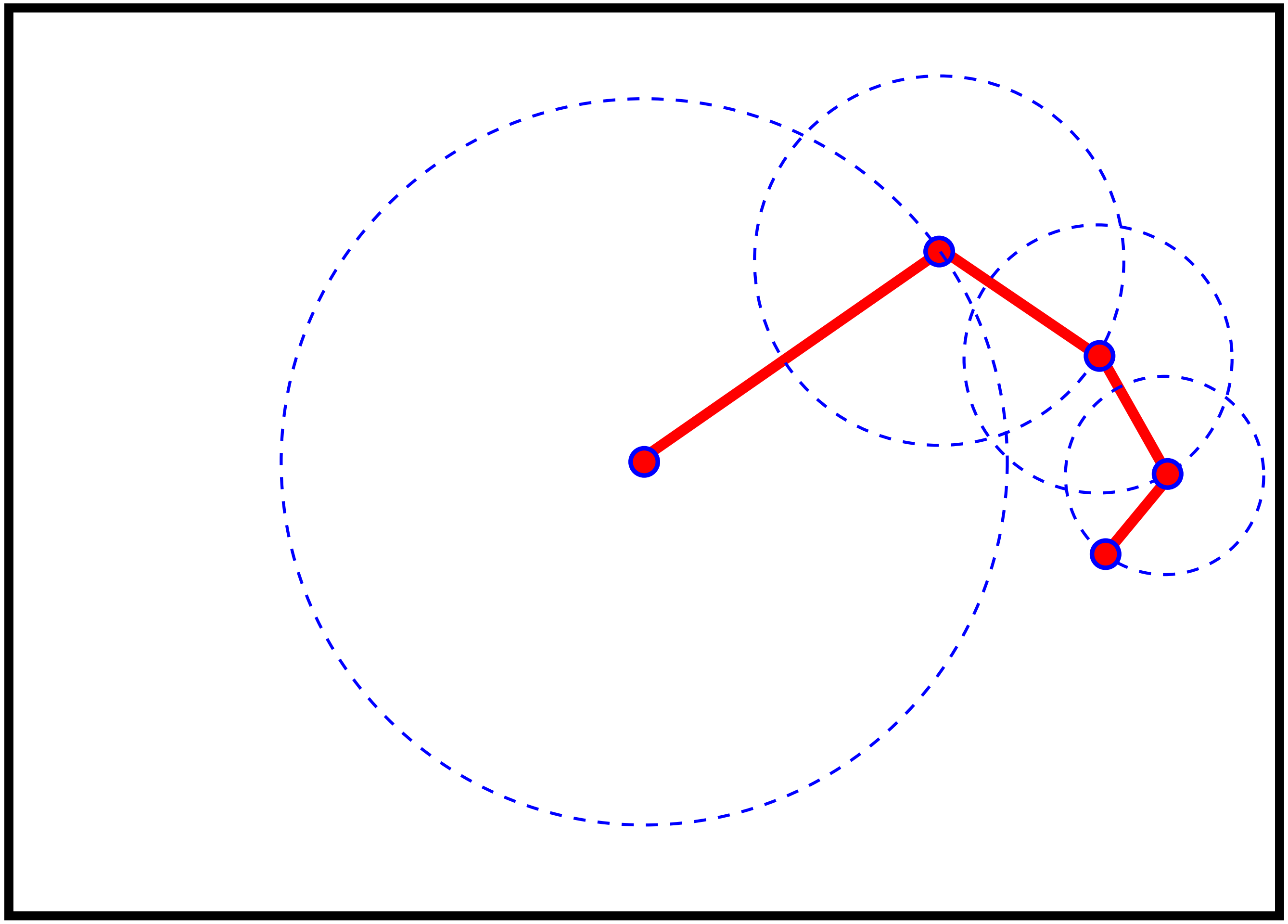


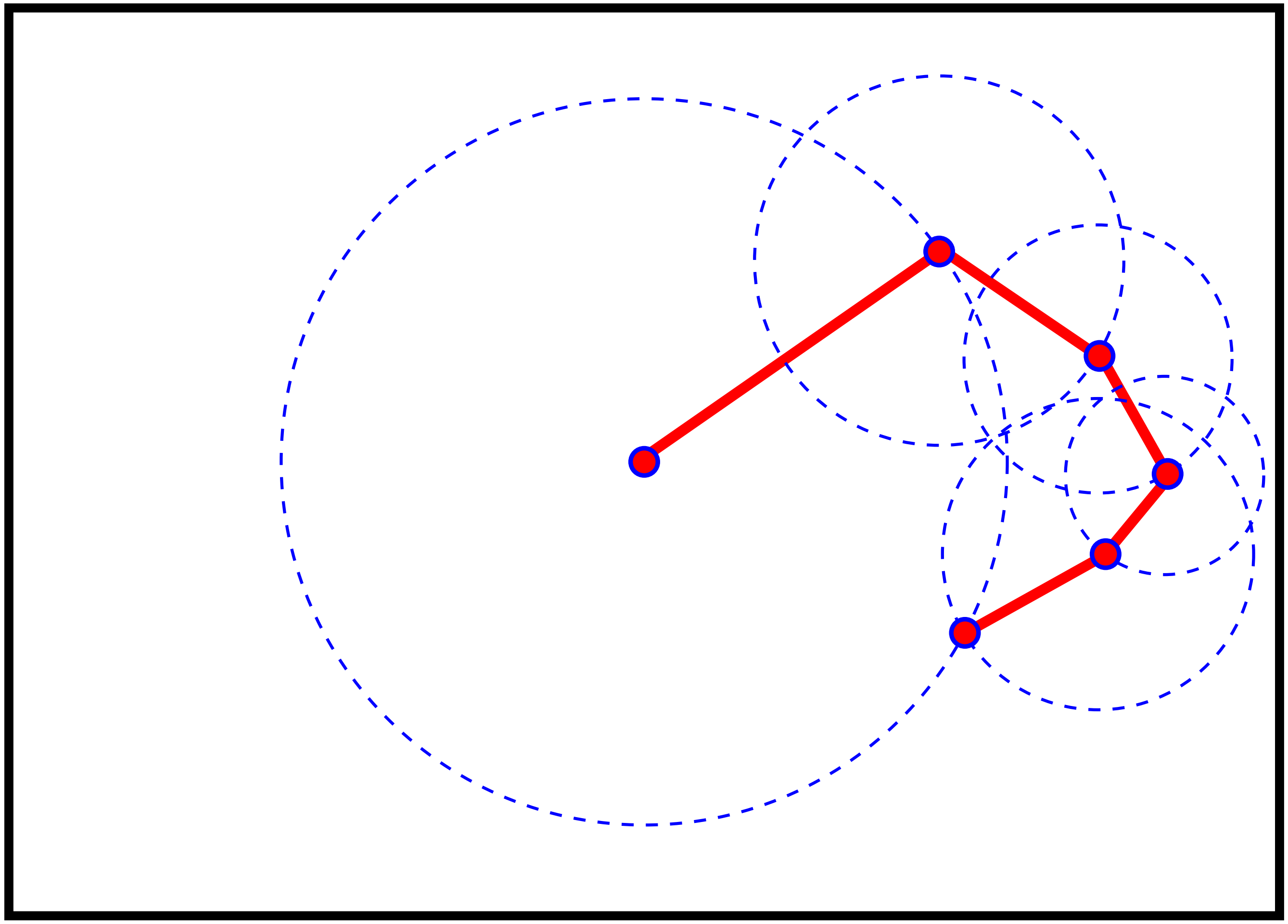


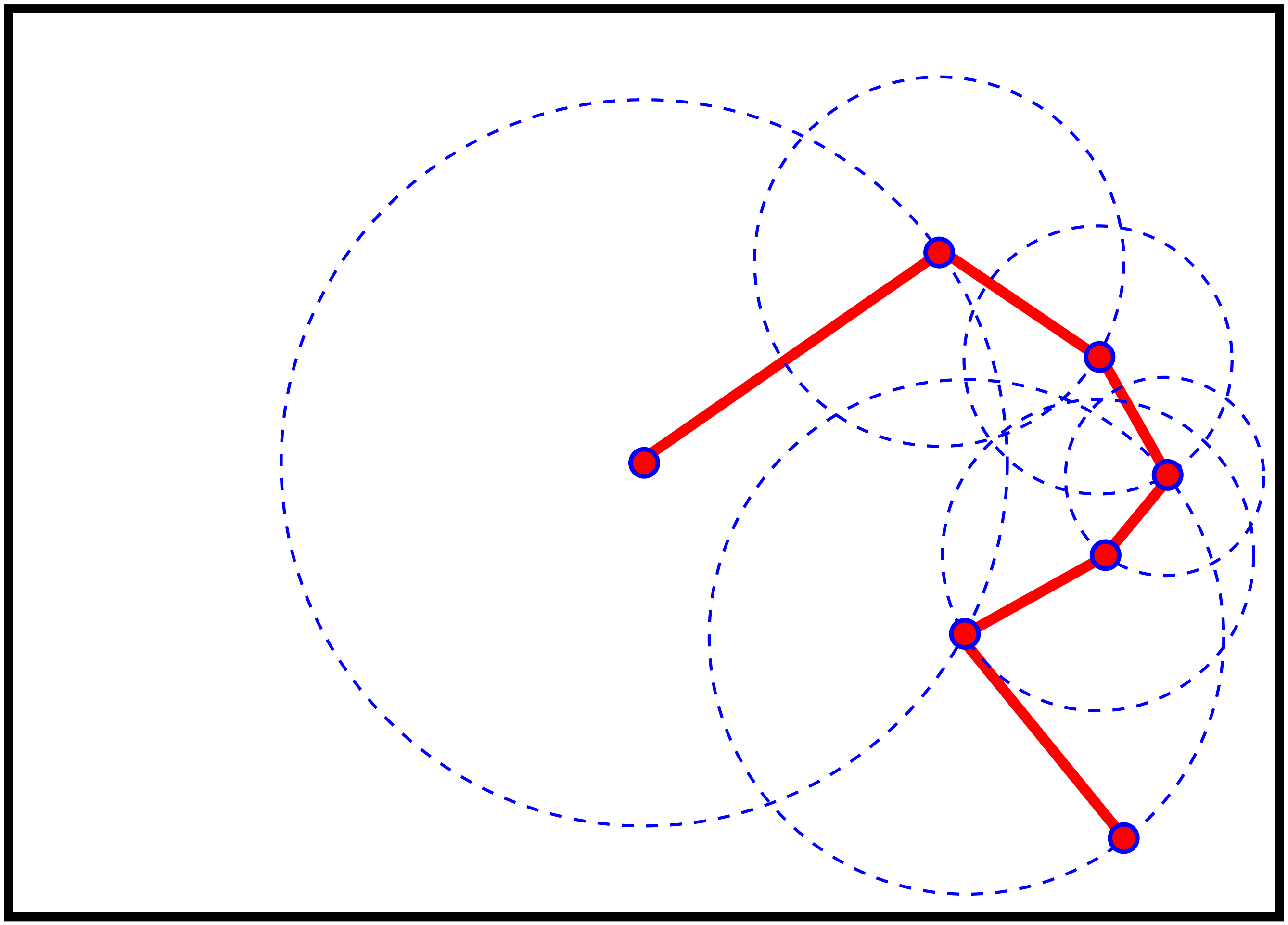


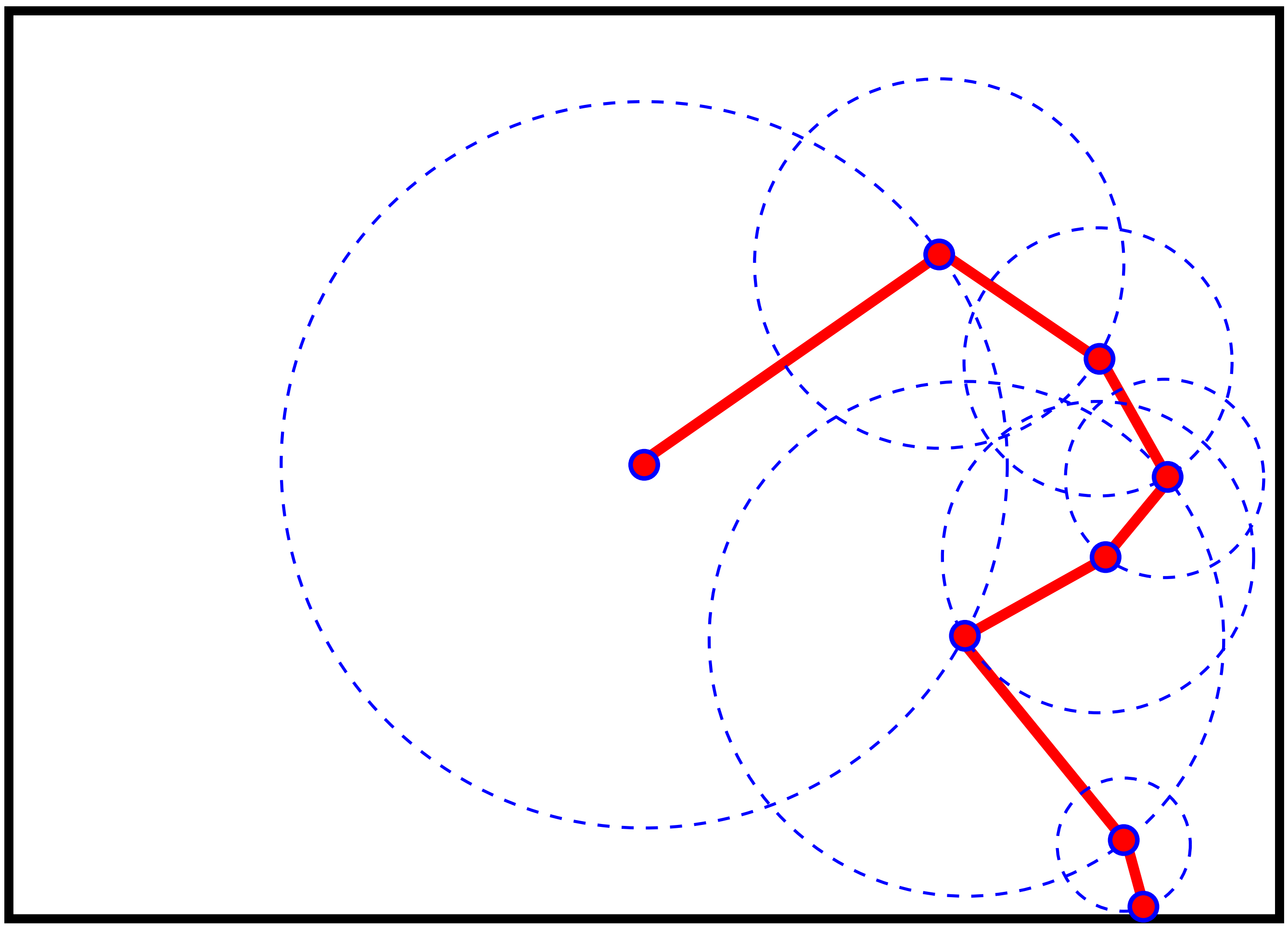






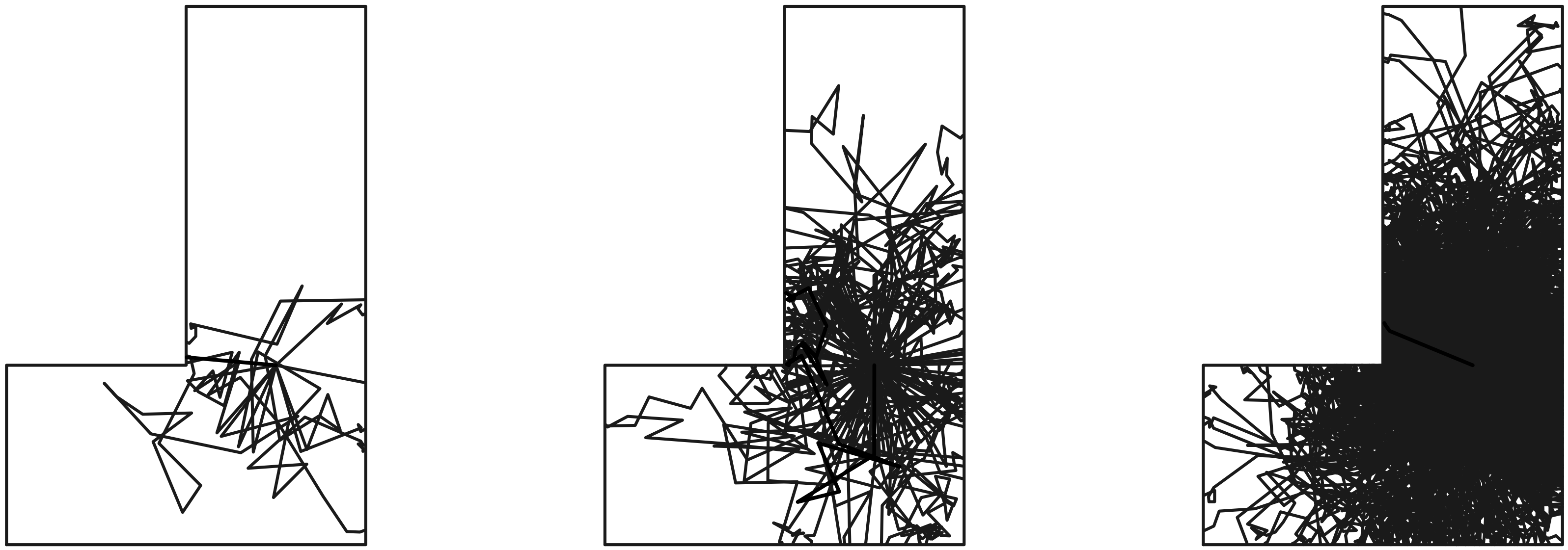






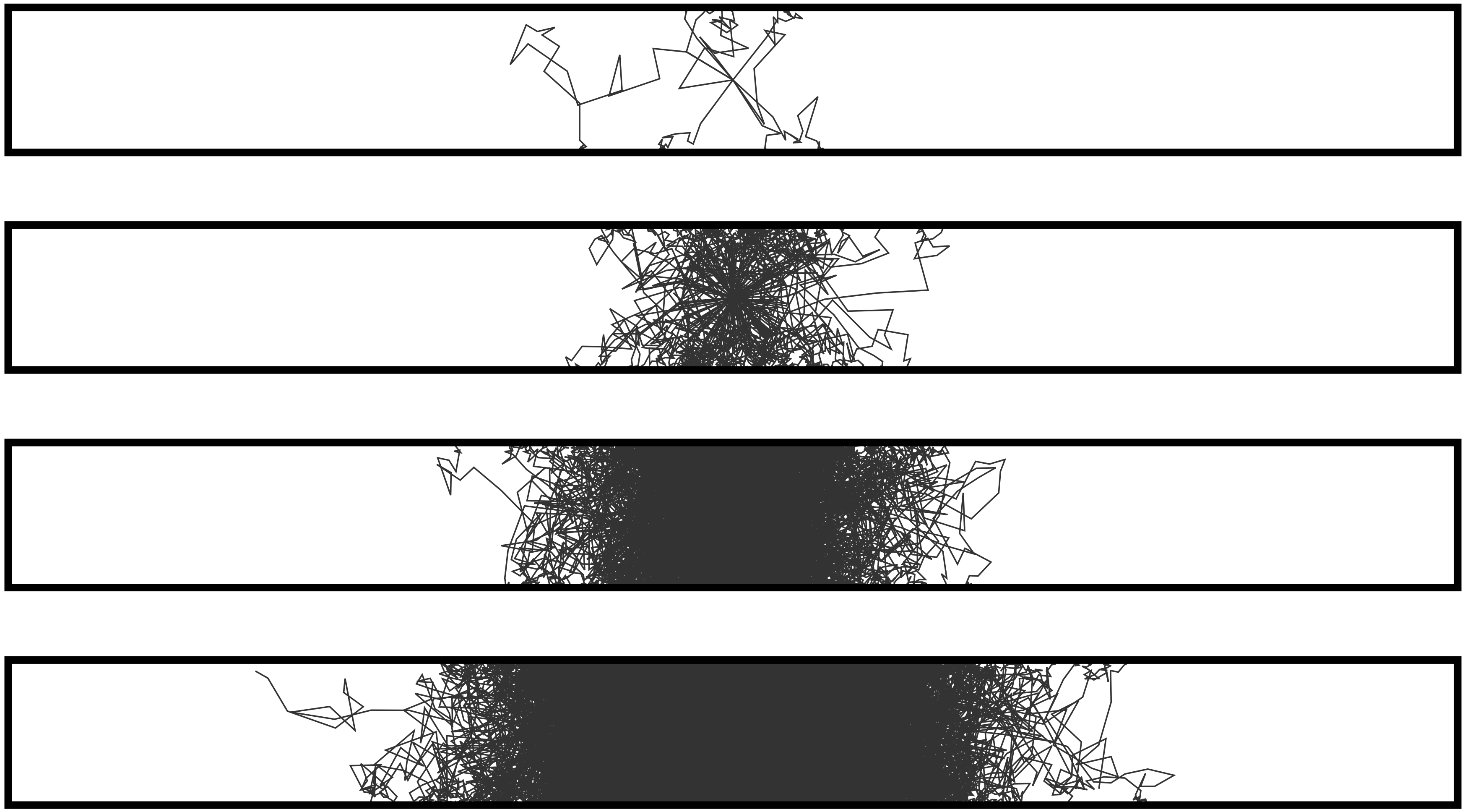
This walk will get within ϵ of boundary in about $O(\log 1/\epsilon)$ steps.

Takes about ϵ^{-2} steps on an ϵ-grid.



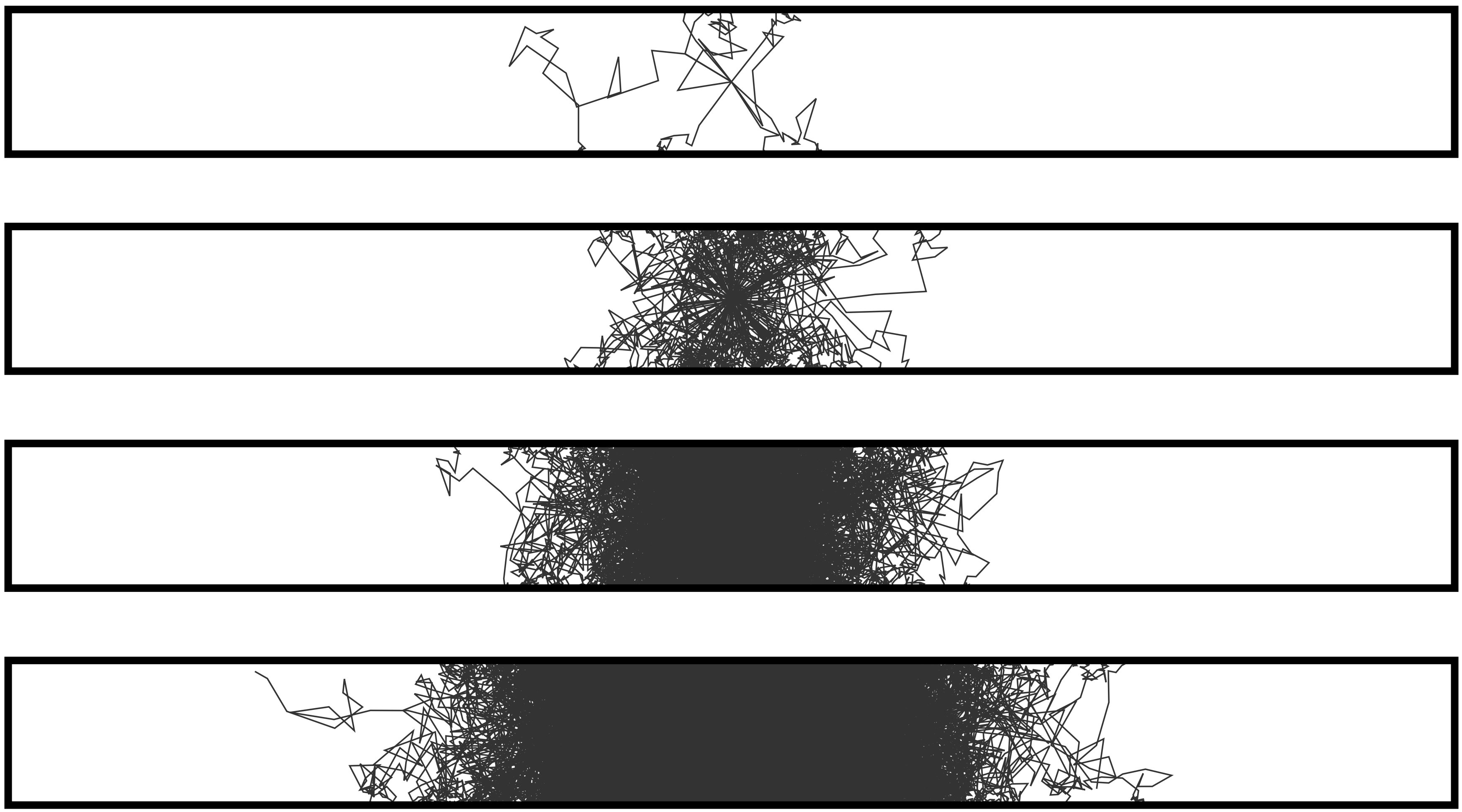
10, 100 and 1000 steps of walk inside a polygon. This illustrates the difficulty Brownian motion has in penetrating narrow corridors.

This is called the crowding phenomena (name to be explained later).



10, 100, 1000 and 10000 walks inside a 1×10 polygon.

Illustrates exponential difficulty of penetrating corridors.

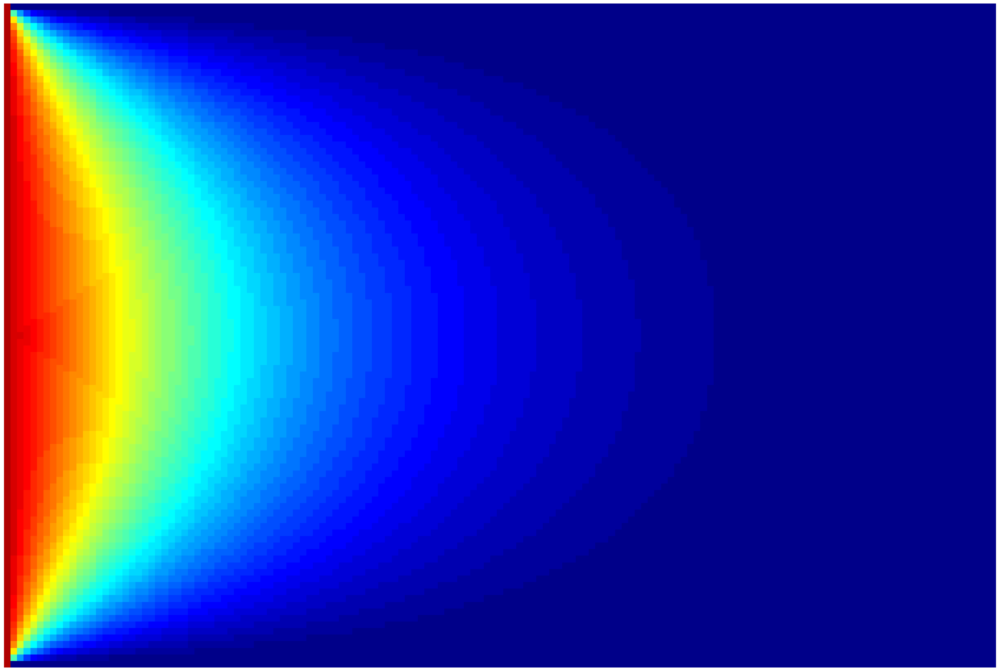


Harmonic measure of short sides of $1 \times R$ rectangle is $\approx \exp(-\pi R/2)$.

Harmonic measure at z is average of harmonic measures on small circle.

Harmonic measure satisfies the mean value property.

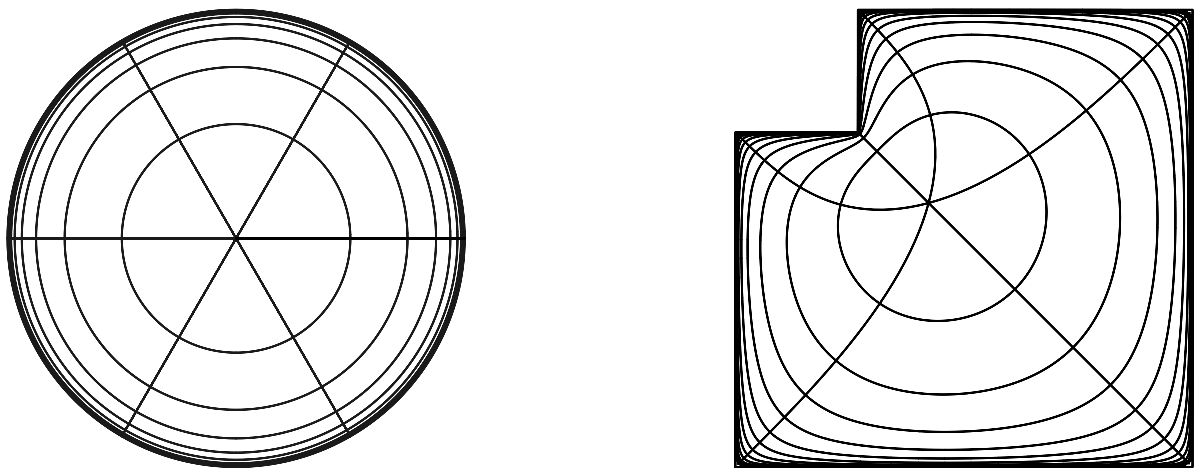
$\Rightarrow$ Harmonic measure is a harmonic function of z.



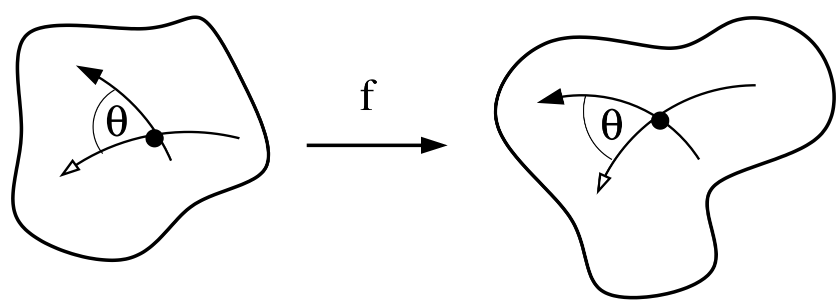
Harmonic measure of $E \subset \partial\Omega$ is the harmonic function of base point with boundary values 1 on E and 0 elsewhere.

Harmonic functions are conformally invariant.

Riemann Mapping Theorem: If $\Omega \subsetneq \mathbb{R}^2$ is simply connected, then there is a conformal map $f : \mathbb{D} \to \Omega$.



Conformal = angle preserving



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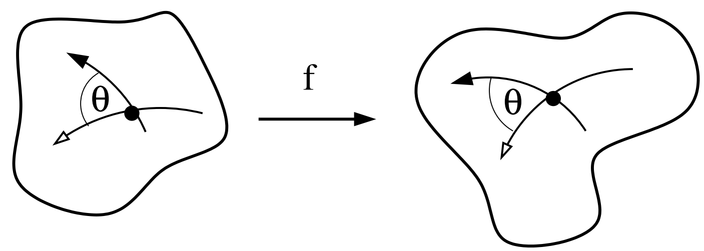
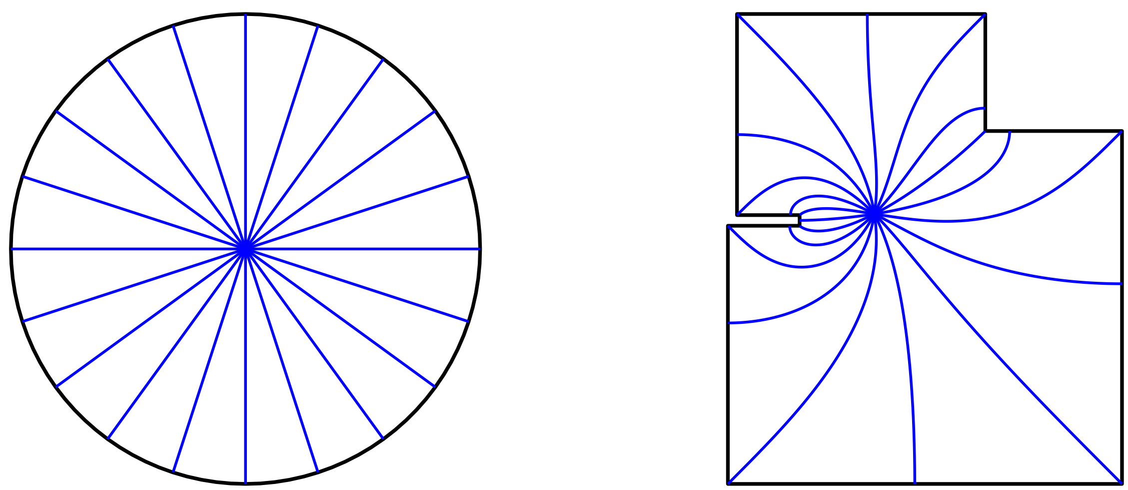


Brownian motion is conformally invariant, so normalized length measure maps to harmonic measure. Fastest way to compute harmonic measure.



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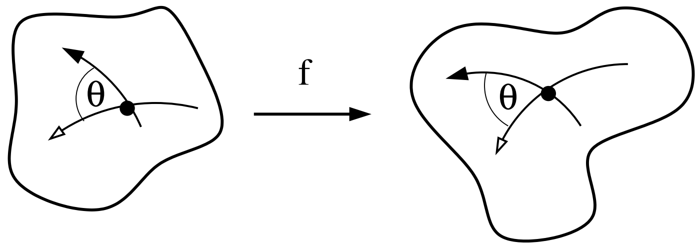

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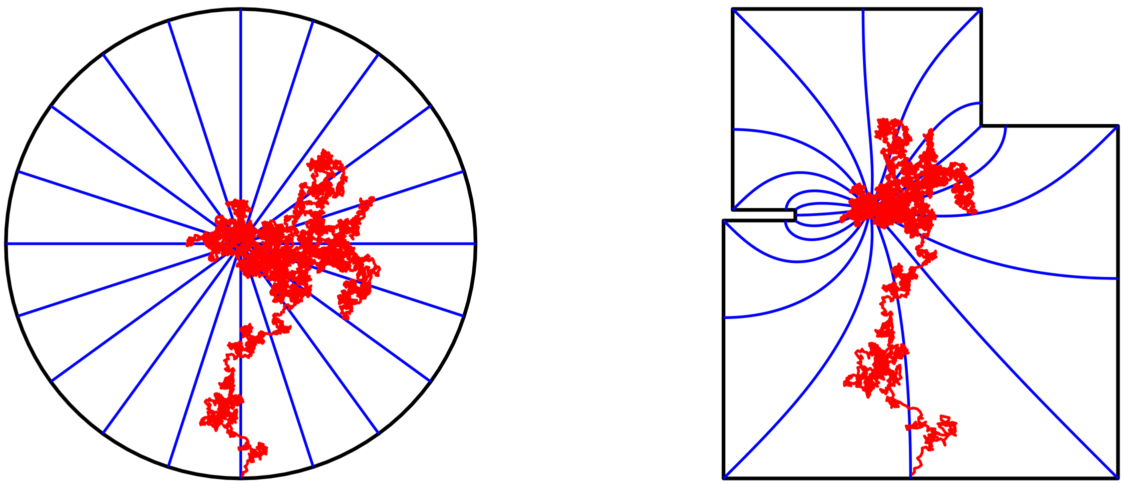
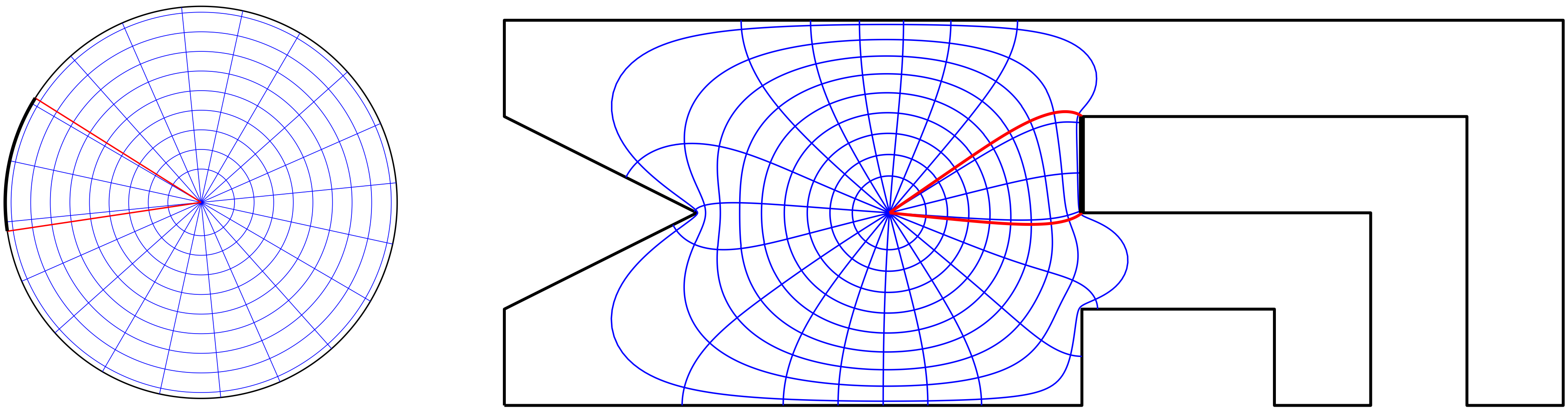
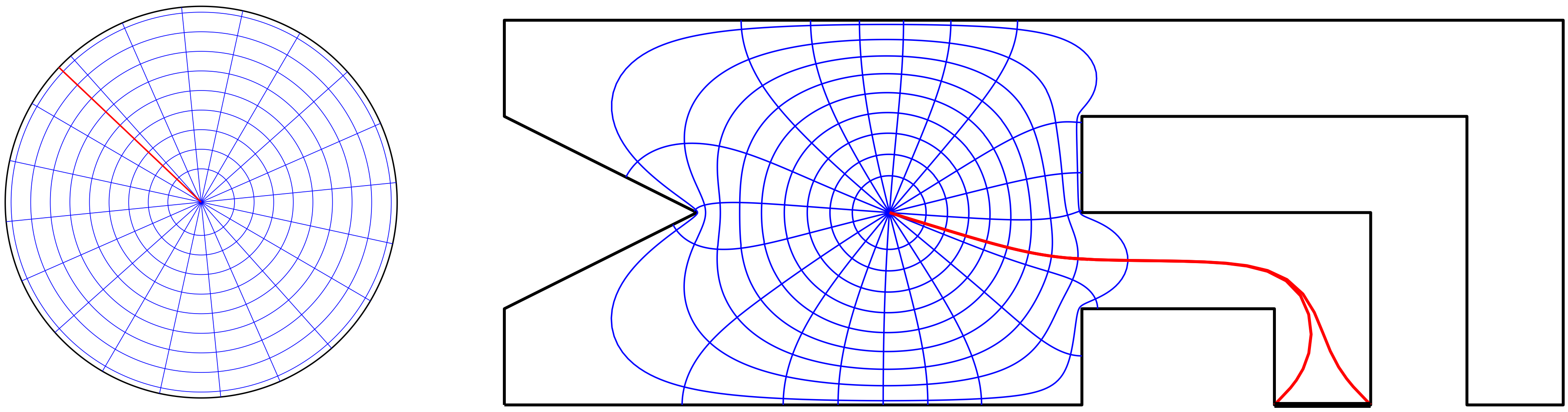




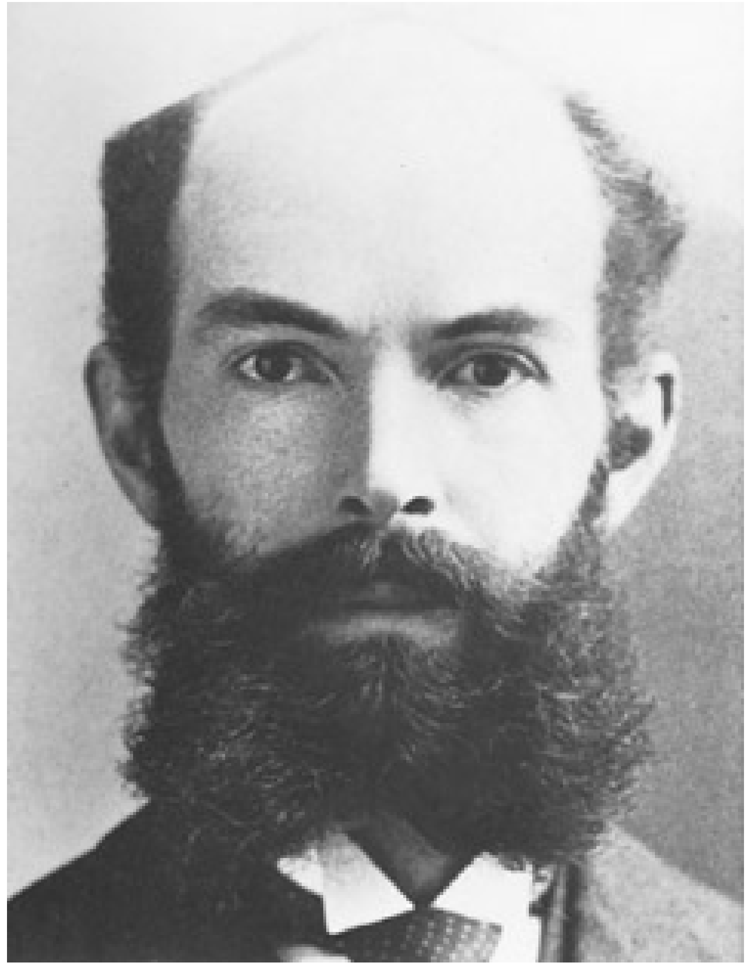
harmonic measure ≈ 0.1128027



harmonic measure $\approx 1.22155 \times 10^{-6}$

Georg Friedrich Bernhard Riemann

Stated RMT in 1851



William Fogg Osgood

First proof of RMT, Trans. AMS, vol. 1, 1900

Harvard 1866, Math Faculty 1890-1933, Chair 1918-22

Schwarz-Christoffel formula for maps to polygons (1867):

$$f(z) = A + C \int^z \prod_{k=1}^{n} (1 - \frac{w}{z_k})^{\alpha_k - 1} dw,$$

Christoffel

Schwarz

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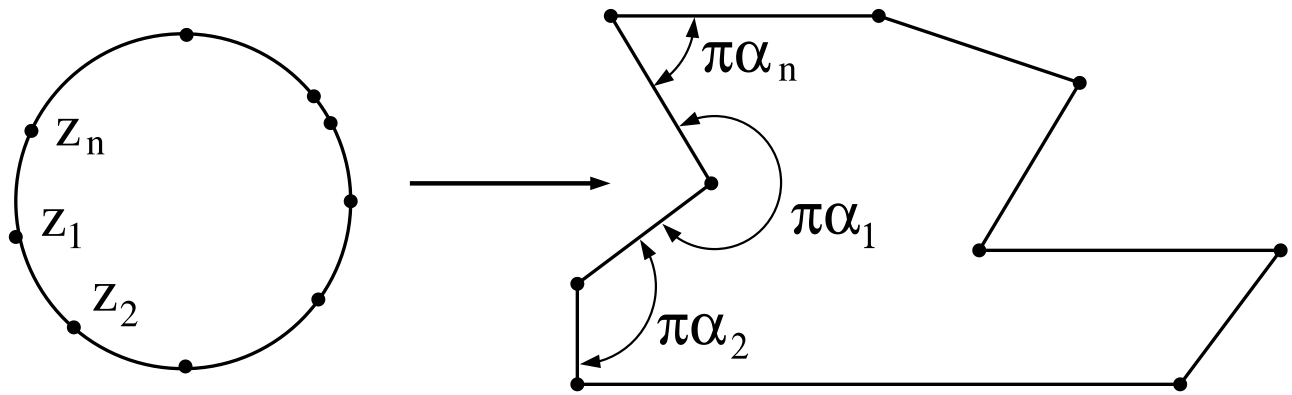

α's known. z's unknown (= **SC-parameters** = **pre-vertices**)

Finding SC-parameters = Finding harmonic measure of edges

Numerical conformal mapping:

- Koebe
- Theodorsen
- Fornberg
- Wegman
- Gaier
- Symm
- Kerzman-Stein
- Integral equations via fast multipole, Rokhlin
- Circle packing, Sullivan, Rodin, Stephenson
- `CRDT`, Driscoll and Vavasis
- `SCToolbox`, Trefethen, Driscoll
- `ZIPPER`, Marshall

Problem: given n-gon, how fast can we compute the SC-parameters?

Theorem: Can compute ϵ-conformal map onto n-gon in time $C_\epsilon \cdot n$.

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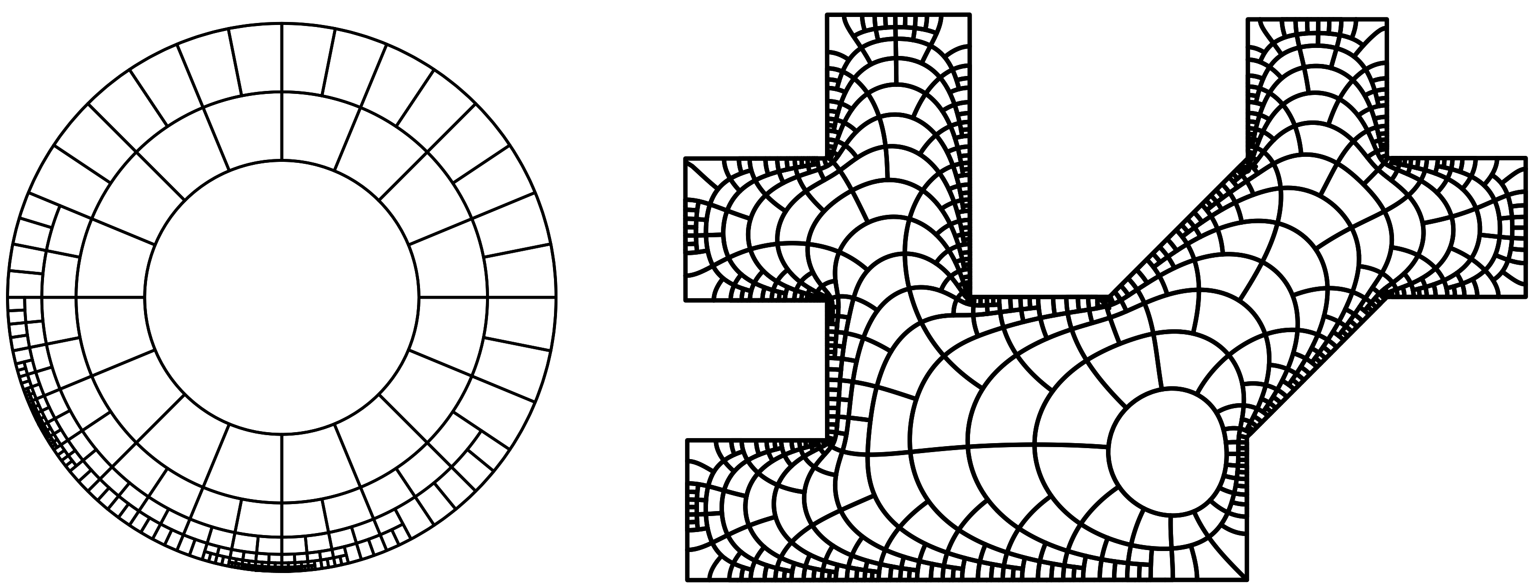
ϵ-conformal $= 1 + \epsilon$ quasiconformal.

$C_\epsilon = O(\log \frac{1}{\epsilon} \log \log \frac{1}{\epsilon})$.

Data held as $O(n)$ Laurent series of length $p = \log \frac{1}{\epsilon}$.

Bottleneck is doing $O(1)$ FFTs per vertex of polygon.



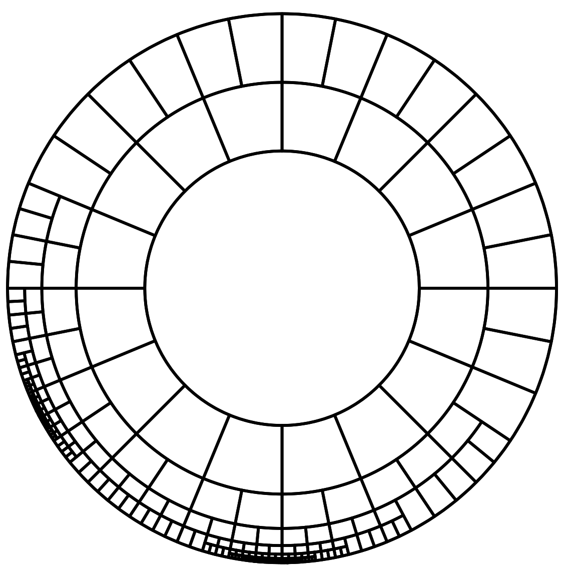
Some basic properties of harmonic measure:

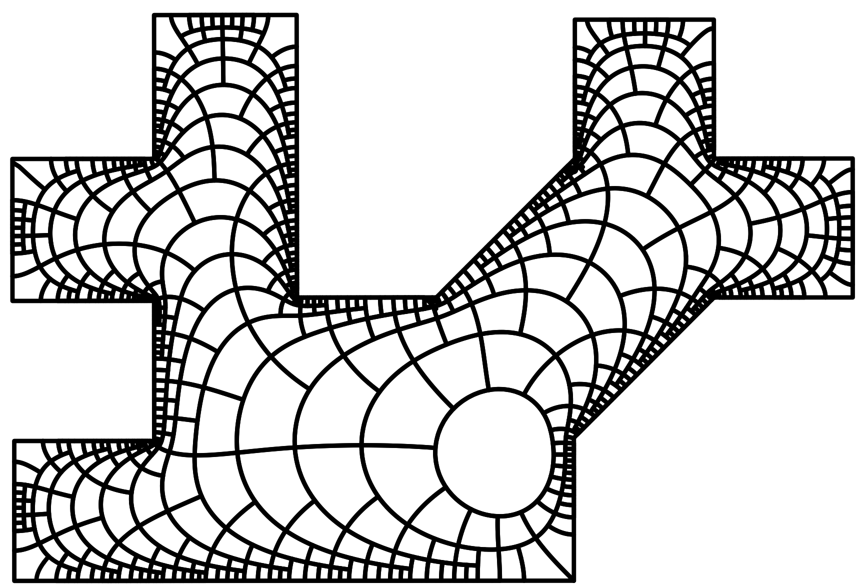
$\omega(z, E, \Omega)$ is the harmonic measure of $E \subset \partial\Omega$.

ω is harmonic in z and $0 \le \omega \le 1$

Harnack $\Rightarrow$ different base points give comparable measures.

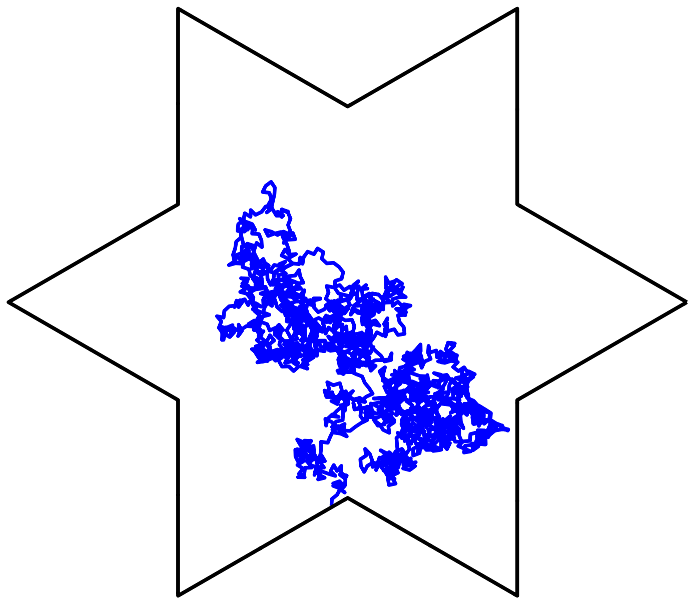
(If base points are on same side of curve.)





Thm (F & M Riesz 1916):

For rectifiable boundaries, $\omega(E) = 0$ iff E has zero length.



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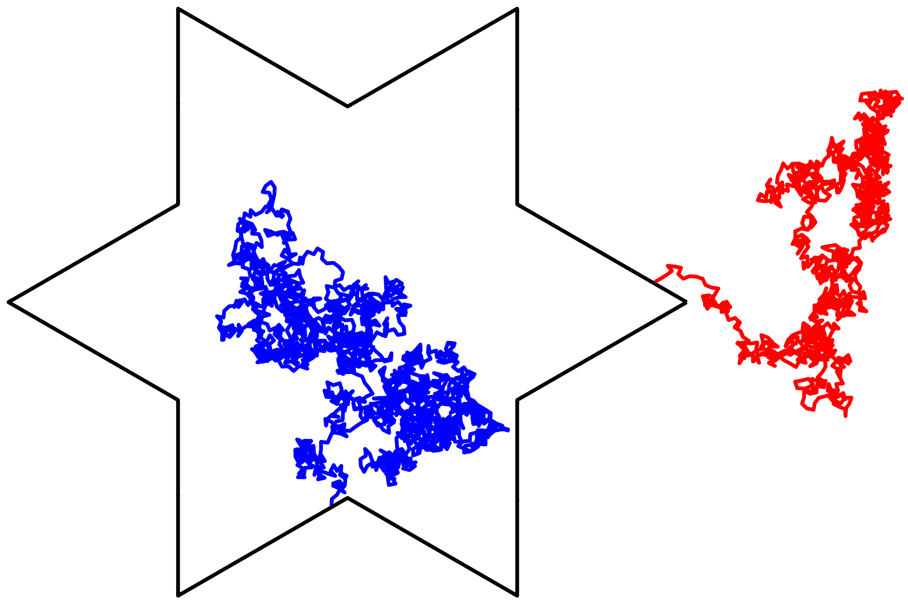
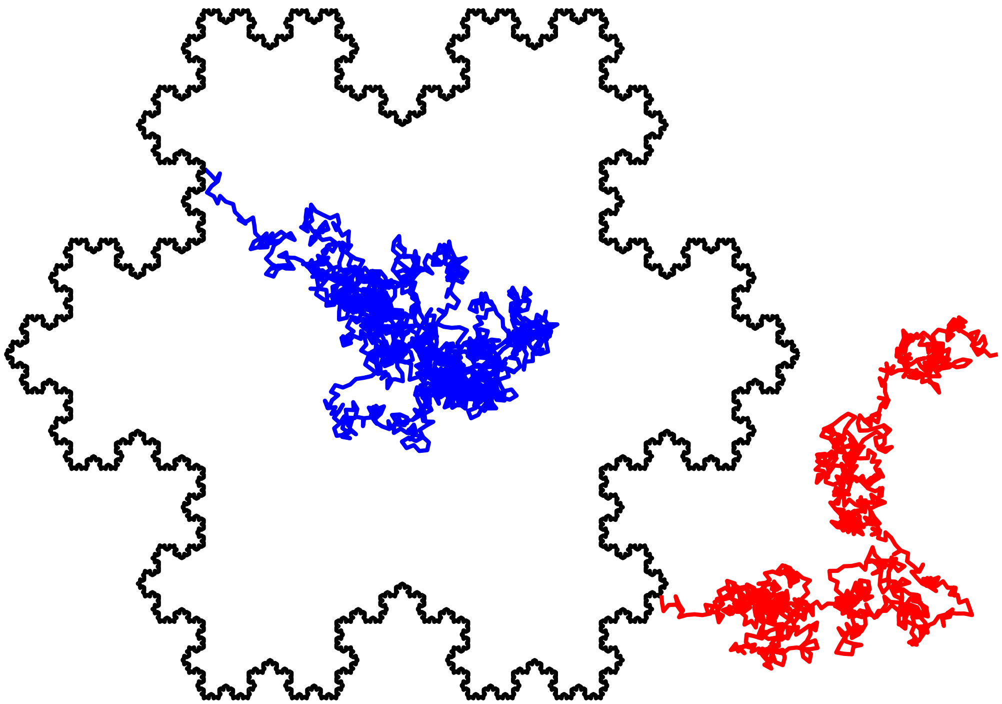


$\Rightarrow$ "Inside" and "outside" harmonic measures have same null sets.

$\Rightarrow$ ω_1, ω_2 are mutually absolutely continuous.

Recent deep generalizations to $\mathbb{R}^n$ by Tolsa and others.

For a fractal curve, inside and outside harmonic measures are singular.



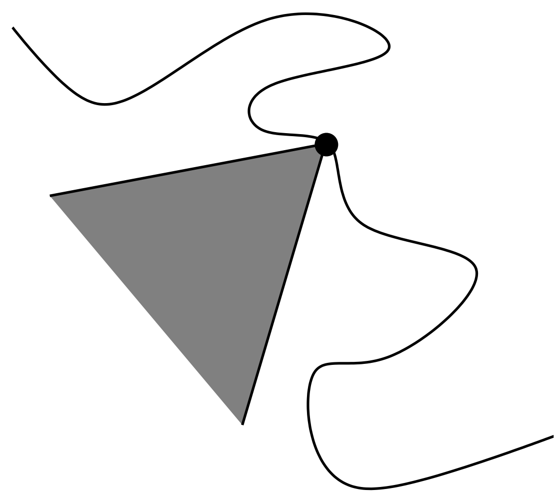
$\omega_1 \perp \omega_2$ iff tangents points have zero length.

For simply connected domains Makarov proved:

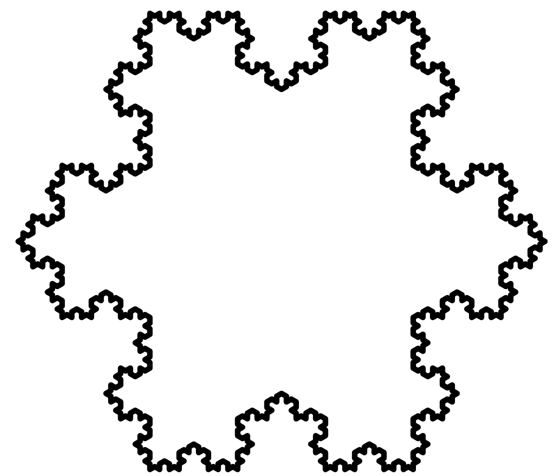
- Harmonic measure gives full mass to some set of dimension 1.
- Harmonic measure gives zero mass to sets of dimension < 1.
- Computed sharp guage function (LIL) and where $\omega \ll \Lambda_1$ or $\omega \perp \Lambda_1$.

F. and M. Riesz (1916): $\Lambda_1 \ll \omega \ll \Lambda_1$ on cone points.

Makarov (1984): $\limsup_{r \to 1} \frac{\omega(D(x,r))}{r} = \infty$, ω-a.e. off cone points.



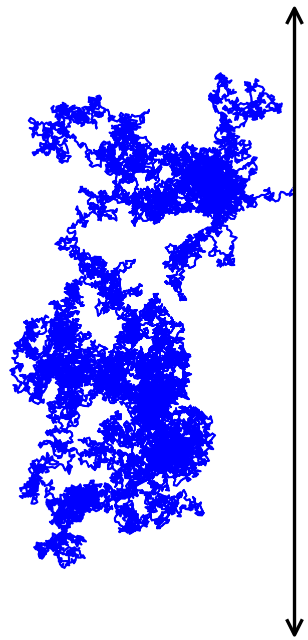
A cone point



Cone points have zero length

For which curves is $\omega_1 = \omega_2$?

For which curves is $\omega_1 = \omega_2$? True for lines:



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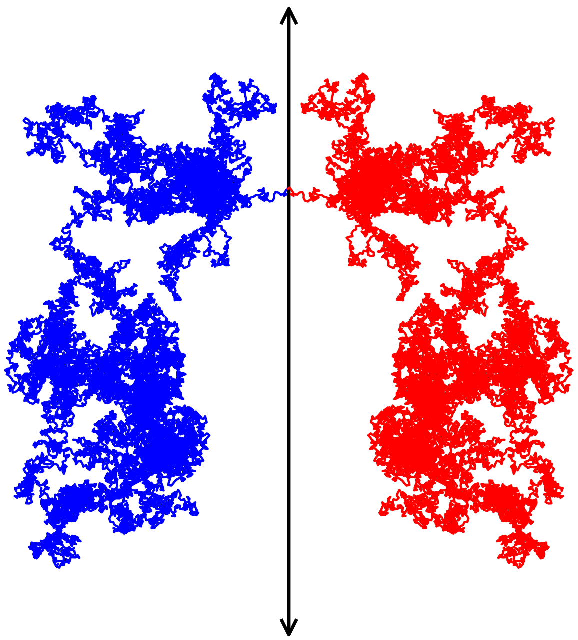
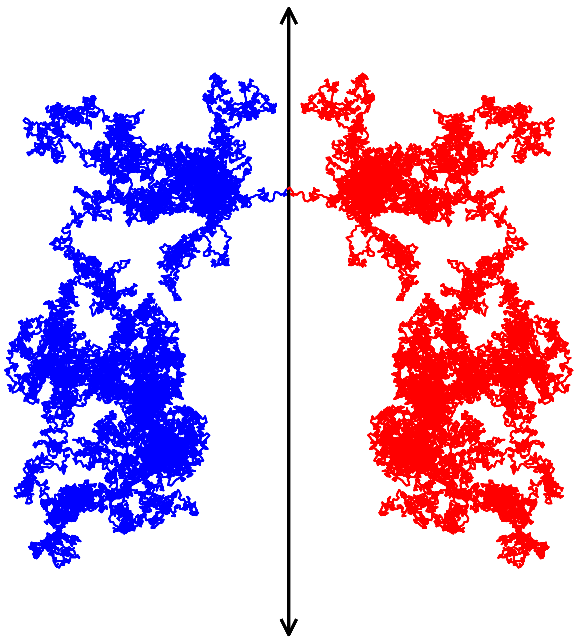


Also for circles (= lines conformally).

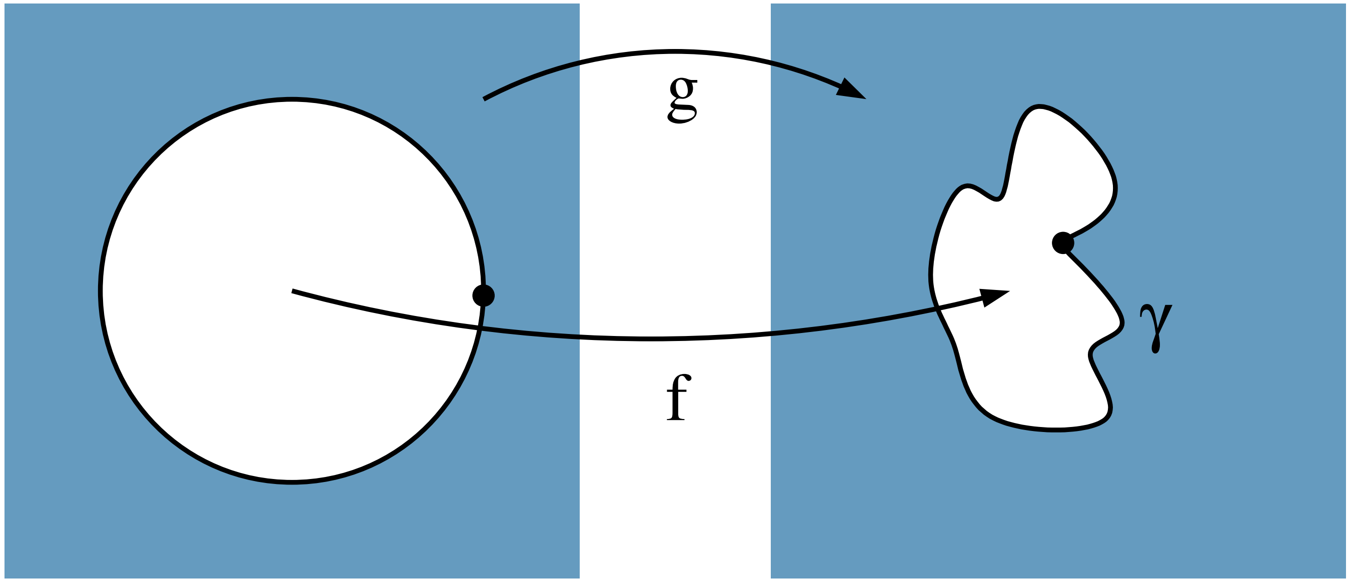
Converse? If $\omega_1 = \omega_2$ must Γ be a circle/line?

For which curves is $\omega_1 = \omega_2$? True for lines:



Also for circles (= lines conformally).

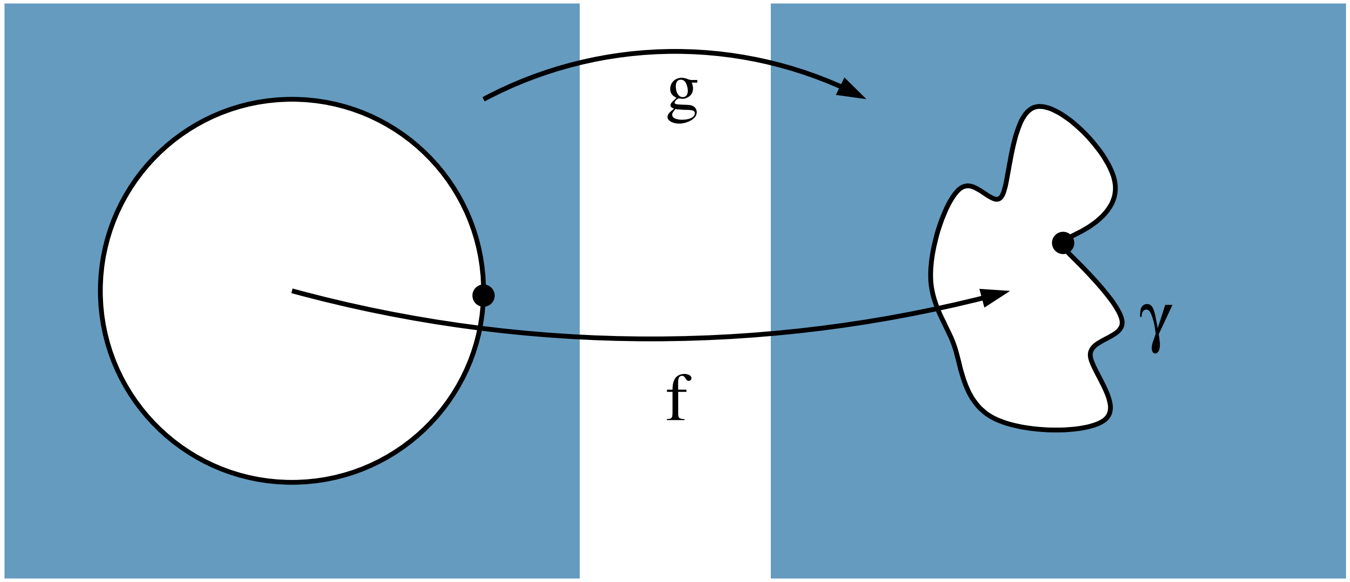
Converse? If $\omega_1 = \omega_2$ must Γ be a circle/line? **Yes**



Suppose $\omega_1 = \omega_2$ for a curve γ.

Conformally map two sides of circle to two sides of γ so $f(1) = g(1)$.

$\omega_1 = \omega_2$ implies maps agree on whole boundary.



Suppose $\omega_1 = \omega_2$ for a curve γ.

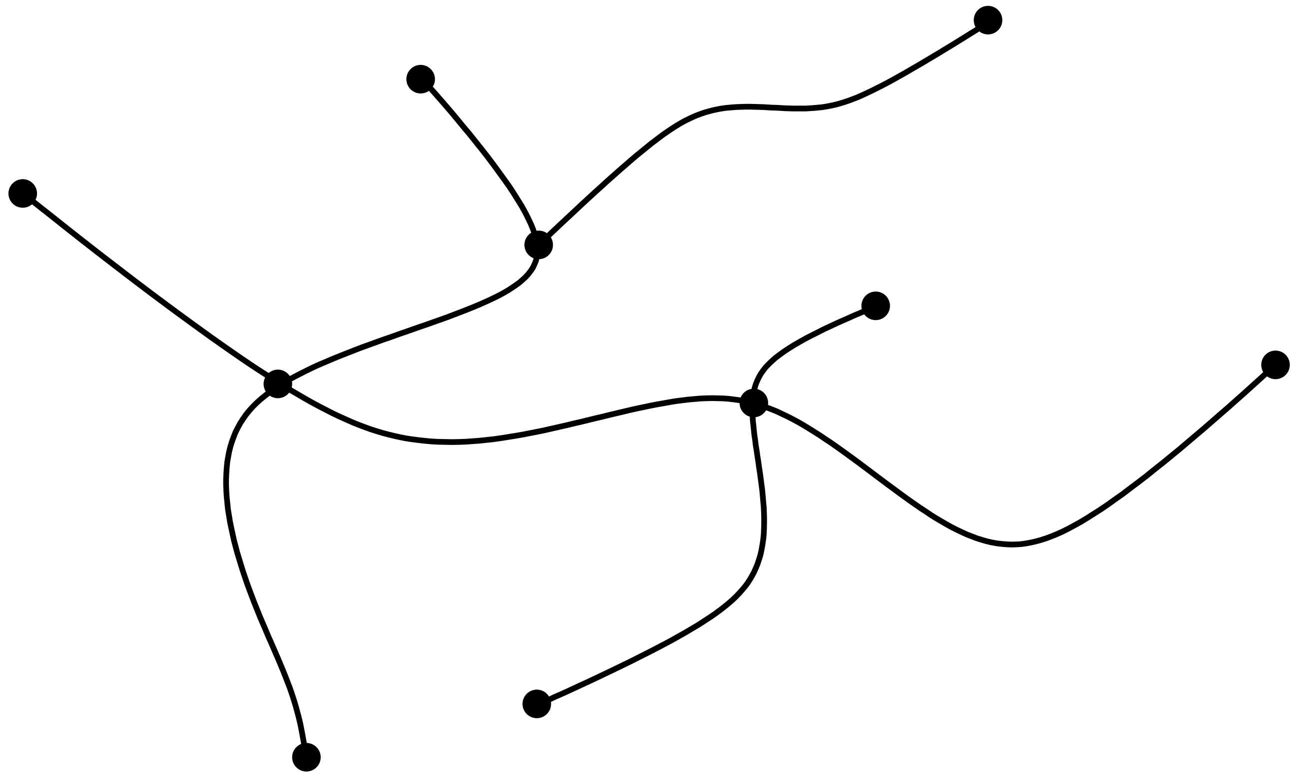
Conformally map two sides of circle to two sides of γ so $f(1) = g(1)$.

$\omega_1 = \omega_2$ implies maps agree on whole boundary.

So f, g define homeomorphism h of plane holomorphic off circle.

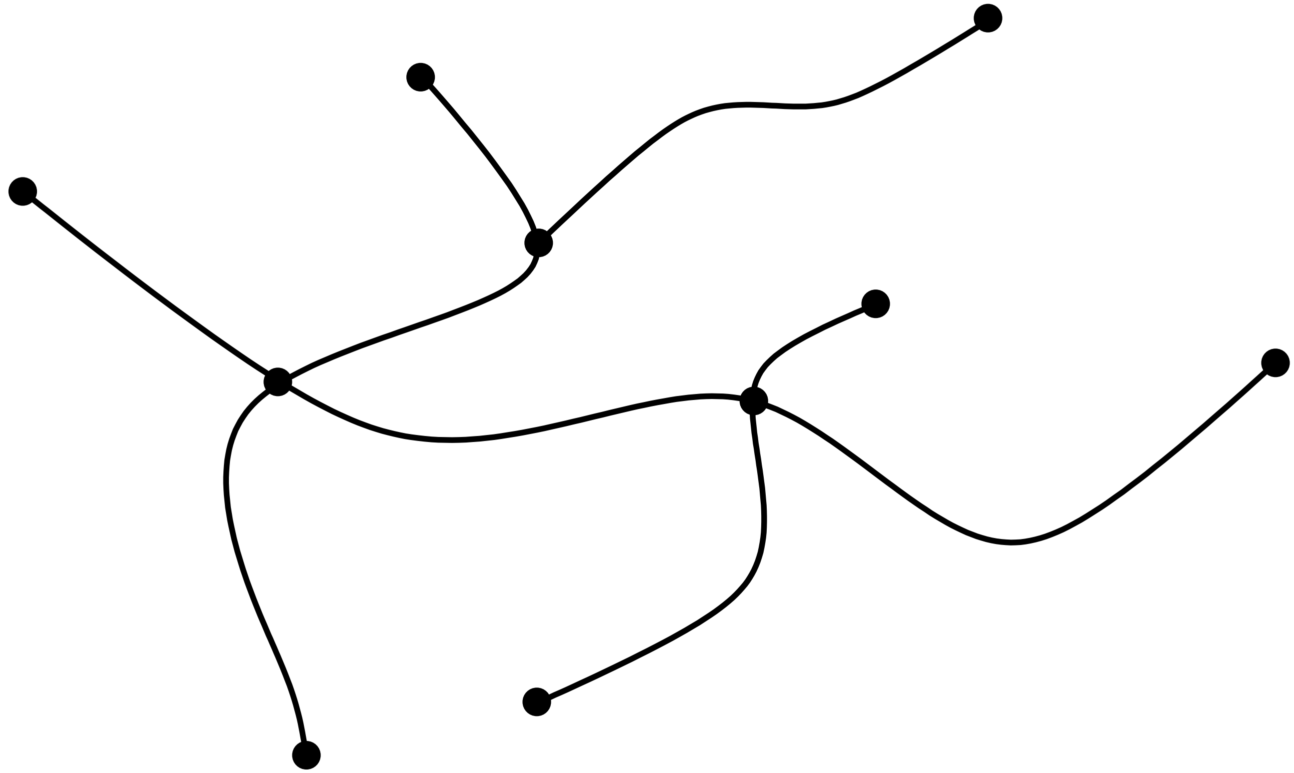
Then h is entire by Morera's theorem.

Entire and 1-1 implies h is linear (Liouville's thm), so γ is a circle.



A planar graph is a finite set of points connected by non-crossing edges.

It is a tree if there are no closed loops.



A planar tree is **conformally balanced** if

- every edge has equal harmonic measure from ∞
- edge subsets have same measure from both sides

This is also called a "**true tree**" (true form of a tree).

A line segment is an example.

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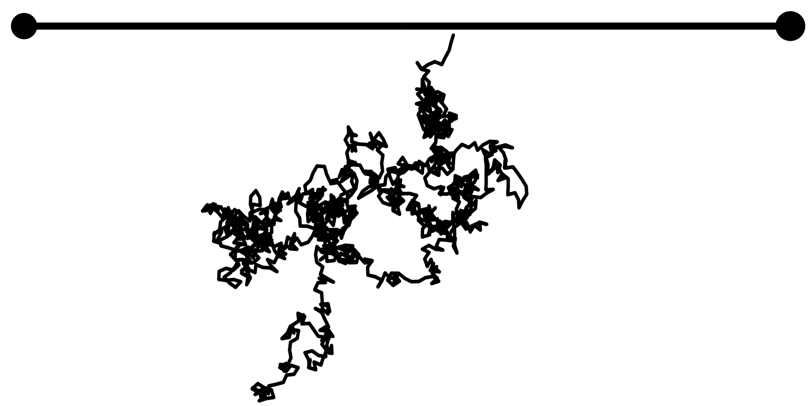
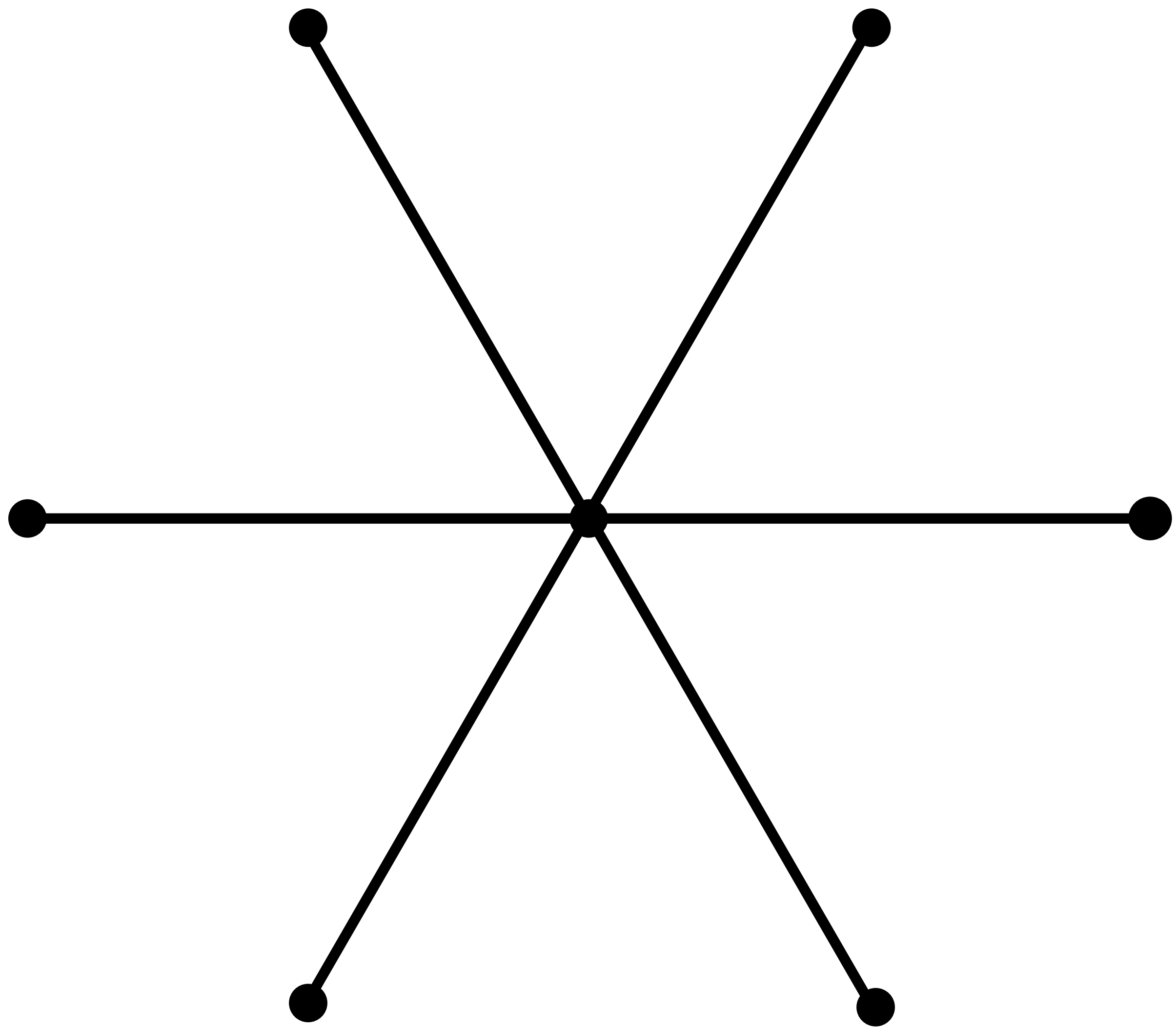
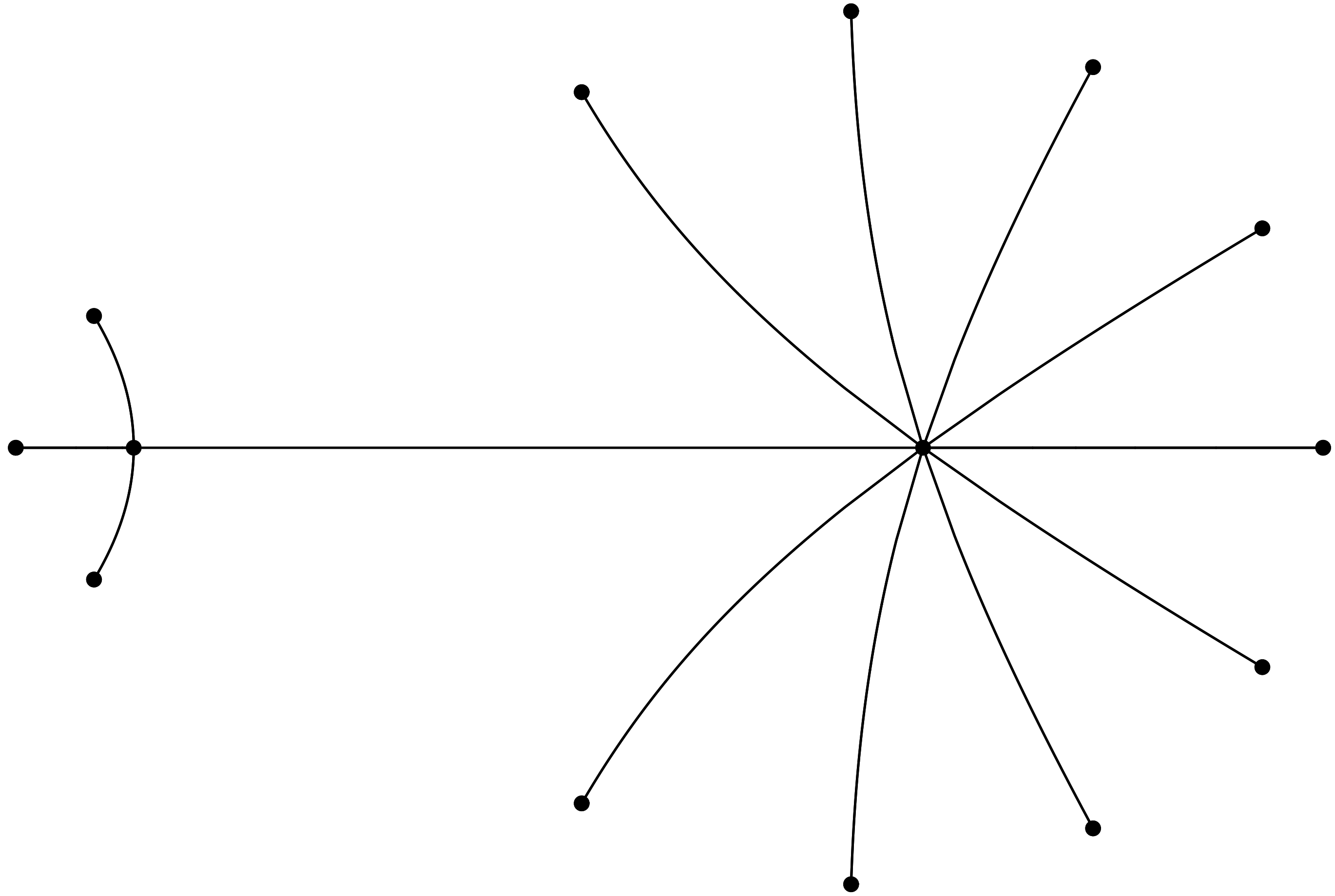
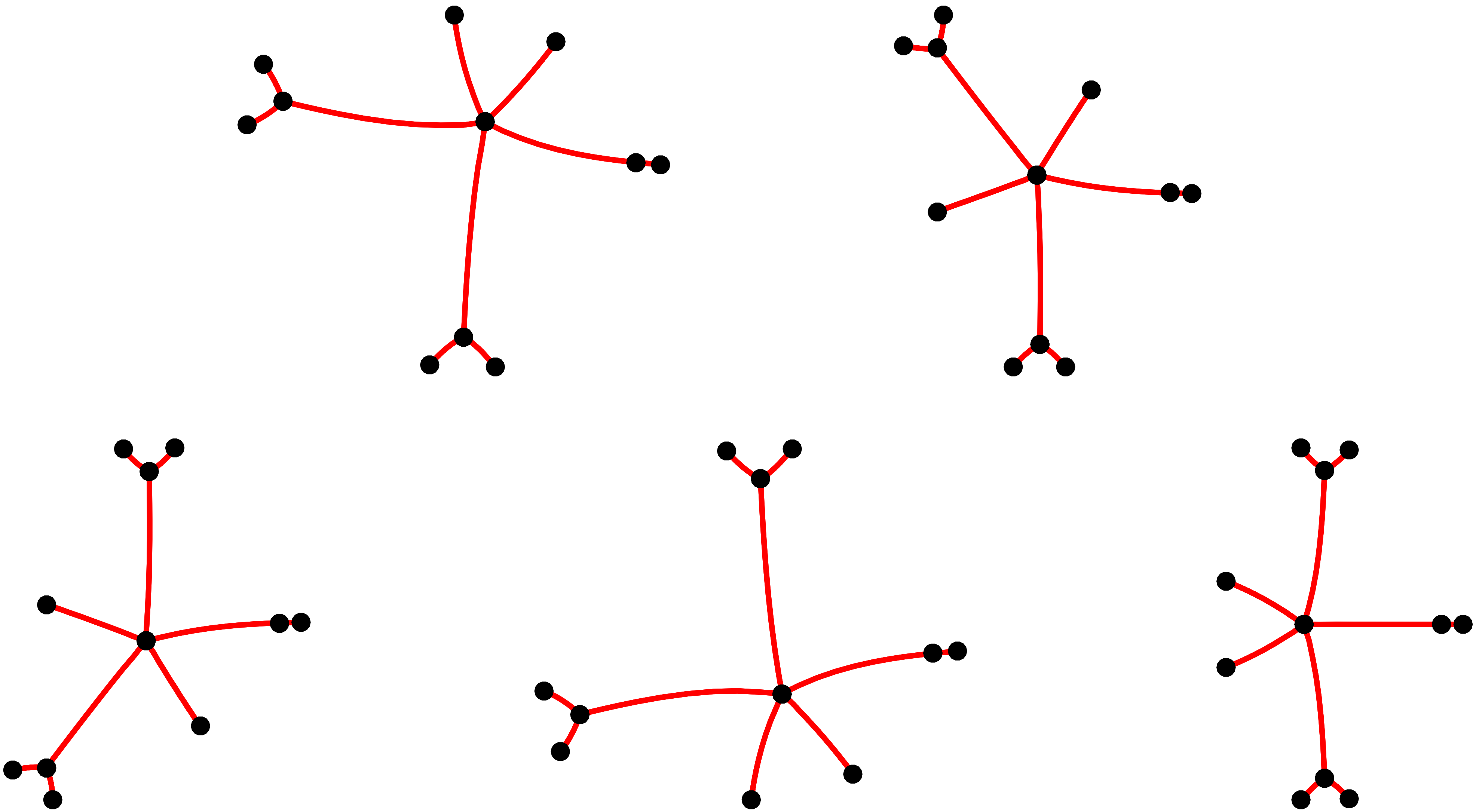


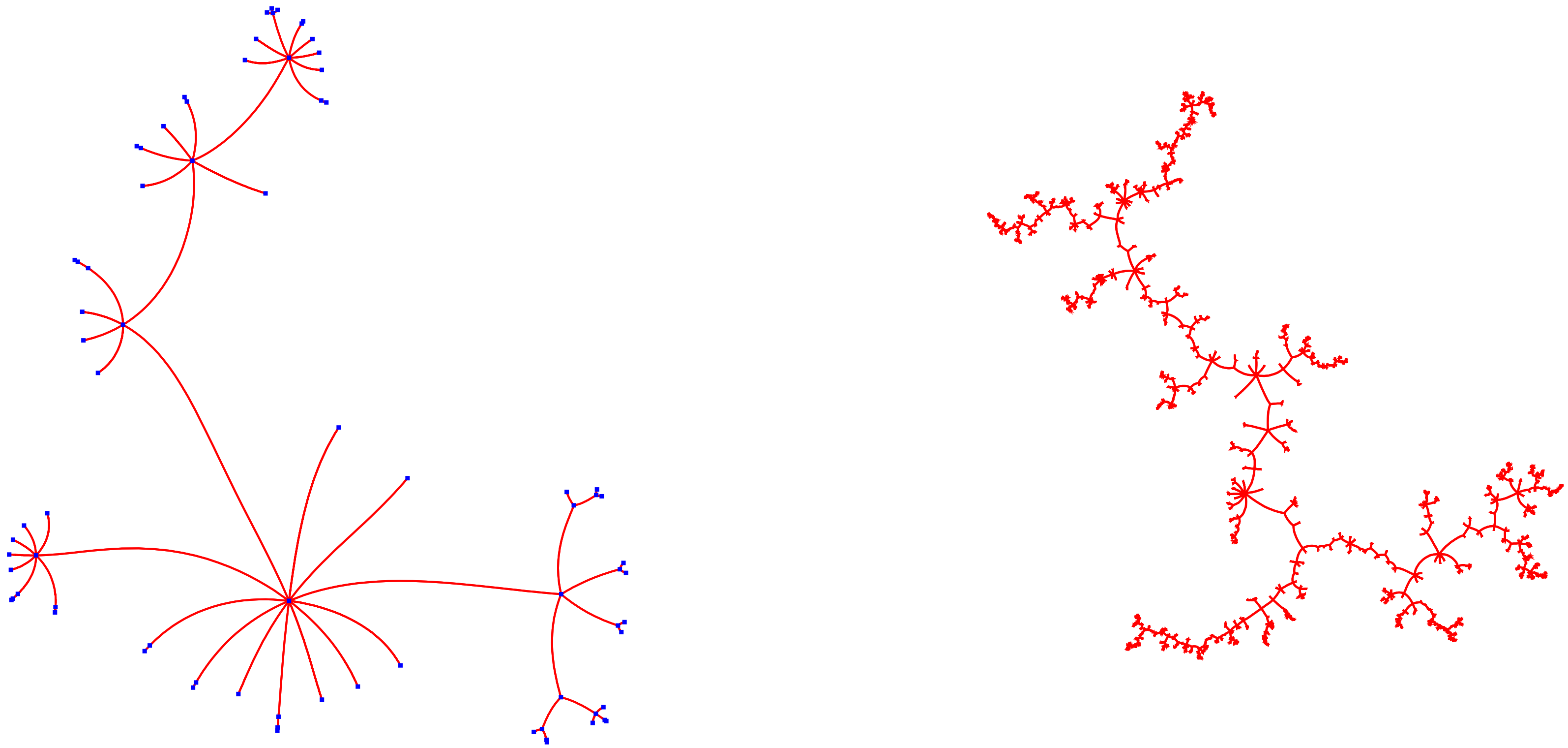


Trivially true by symmetry



Non-obvious true tree



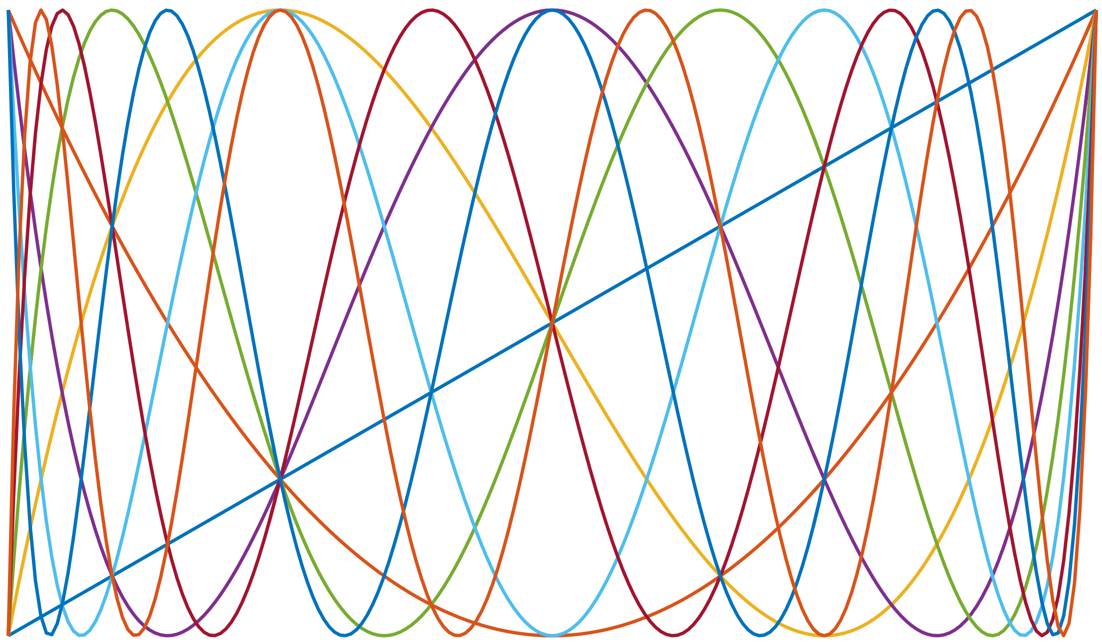


Some random true trees by Don Marshall and Steffen Rohde

Definition of critical value: if p = polynomial, then

$$\mathrm{CV}(p) = \{p(z) : p'(z) = 0\} = \text{critical values}$$

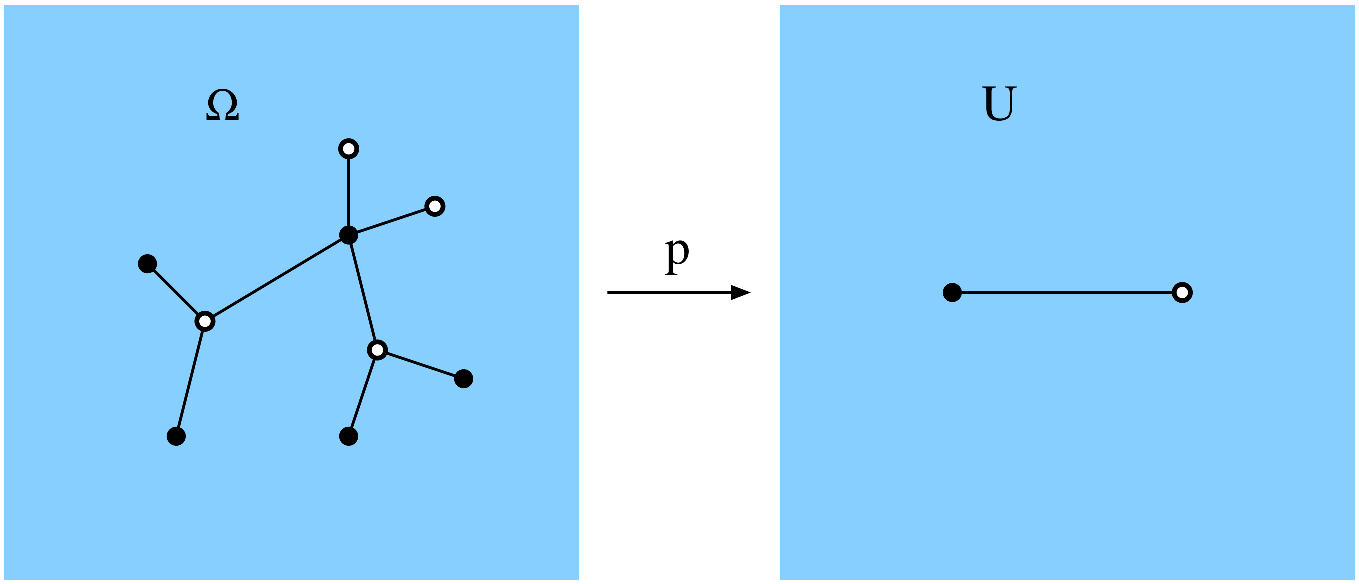
If $\mathrm{CV}(p) = \pm 1$, p is called **generalized Chebyshev** or **Shabat**.



10 classical Chebyshev polynomials

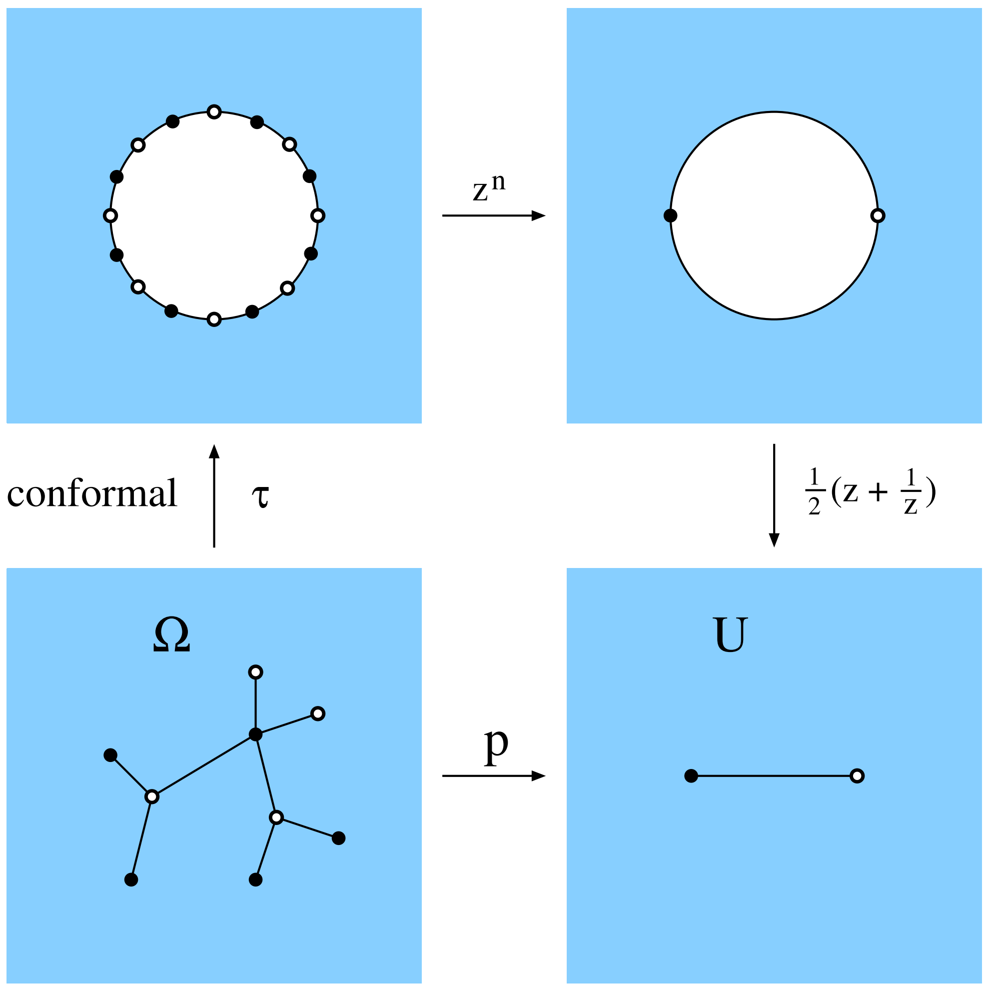
Balanced trees $\leftrightarrow$ Shabat polynomials

Fact: T is balanced iff $T = p^{-1}([-1,1])$, $p =$ Shabat.



$\Omega = \mathbb{C} \setminus T$ $\qquad$ $U = \mathbb{C} \setminus [-1,1]$

T conformally balanced $\Leftrightarrow$ p Shabat.



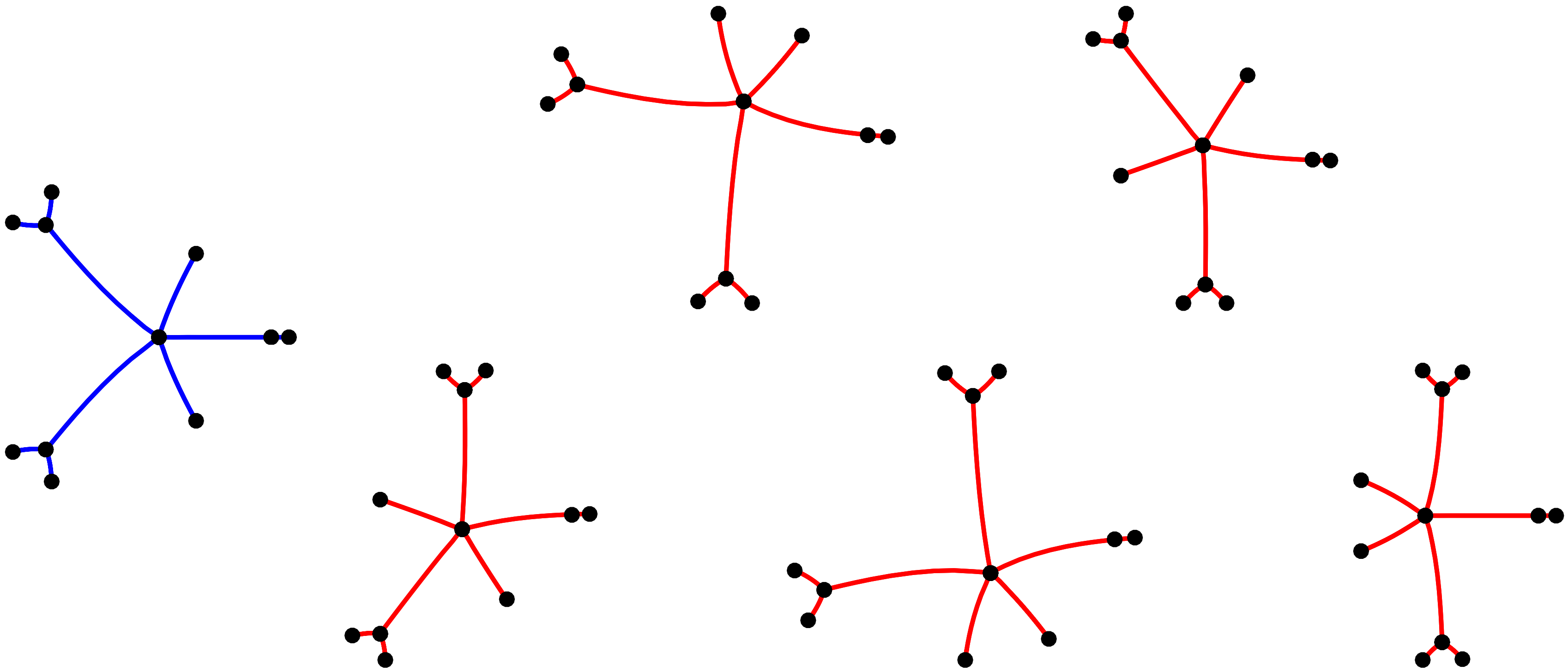
p is entire and n-to-1 $\Leftrightarrow$ p = polynomial.

$\mathrm{CV}(p) \notin U \Leftrightarrow p : \Omega \to U$ is covering map.

Algebraic aside:

True trees are examples of Grothendieck's *dessins d'enfants* on sphere.

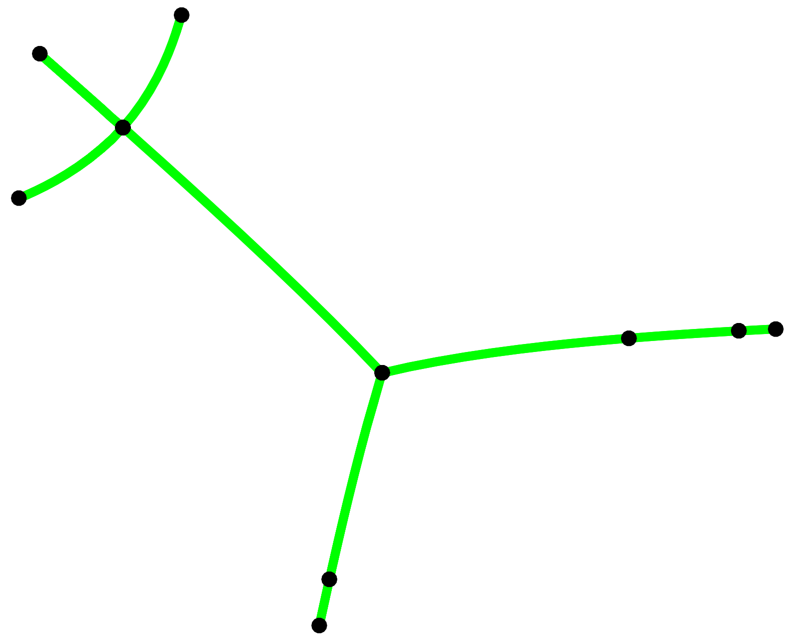
Normalized polynomials are algebraic, so planar trees correspond to number fields. Absolute Galois group acts on trees, but orbits unknown.



Six graphs of type 5 1 1 1 1 1 - 3 3 2 1 1, two orbits.

Even computing number field from tree is difficult.

Kochetkov (2009, 2014): did all trees with 9 and 10 edges.



For example, the polynomial for this 9-edge tree is

$$p(z) = z^4(z^2 + az + b)^2(z - 1),$$

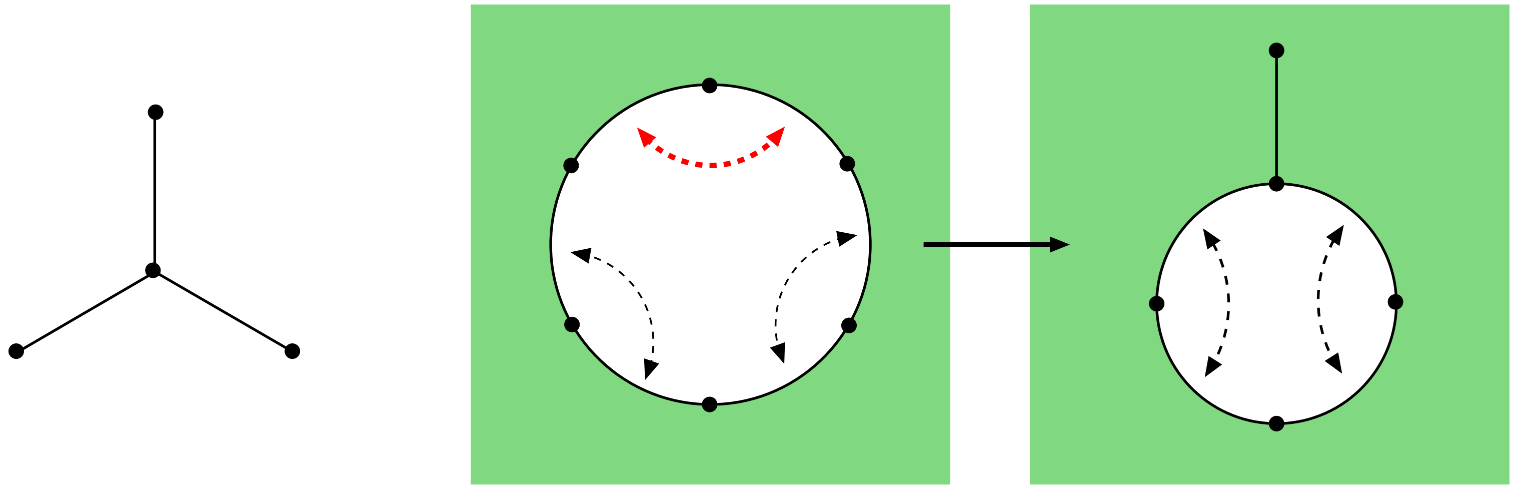
where a is a root of ...

$$
\begin{aligned}
0 = \quad & 126105021875\, a^{15} + 873367351500\, a^{14} \\
& +2340460381665\, a^{13} + 2877817869766\, a^{12} \\
& +3181427453757\, a^{11} - 68622755391456\, a^{10} \\
& -680918281137097\, a^{9} - 2851406436711330\, a^{8} \\
& -7139130404618520\, a^{7} - 12051656256571792\, a^{6} \\
& -14350515598839120\, a^{5} - 12058311779508768\, a^{4} \\
& -6916678783373312\, a^{3} - 2556853615656960\, a^{2} \\
& -561846360735744\, a - 65703906377728
\end{aligned}
$$

This is **not** the most complicated formula in Kochetkov's paper.

However, true form can be drawn without knowing the polynomial.

Don Marshall's `ZIPPER` uses conformal mapping to draw true trees.



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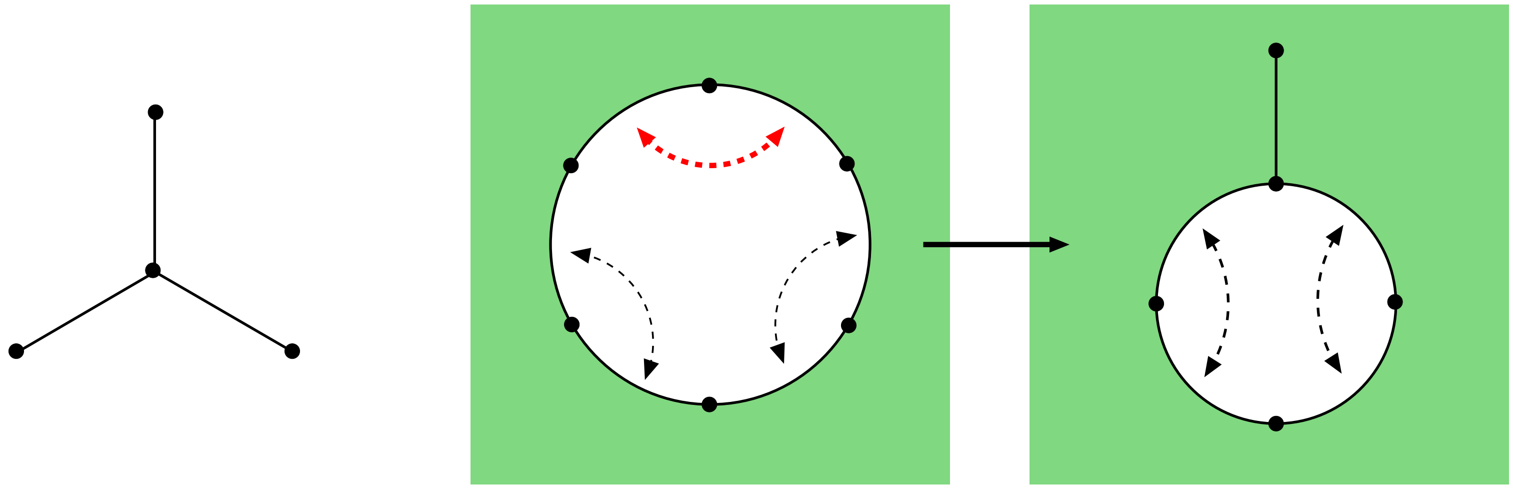

Marshall and Rohde approximated all true trees with ≤ 14 edges.

They can compute vertices to 1000's of digits of accuracy.

Test if $\alpha \in \mathbb{C}$ is algebraic by seeking integer relationships between $1, \alpha, \alpha^2, \ldots$ using lattice reduction or PSLQ algorithm.

This semester I hope to cover the following chapters from "Harmonic Measure" by Garnett and Marshall.

- Chapter 1: Jordan domains
- Chapter 2: Finitely Connected Domains (summarize)
- Chapter 3: Potential Theory
- Chapter 4: Extremal Distance
- Chapter 5: Applications and Reverse Inequalities (summarize)
- Chapter 6: Simply Connected Domains, Part One (Riesz, Makarov)
- Chapter 7: Bloch Functions and Quasicircles
- Chapter 8: Simply Connected Domains, Part Two (Makarov's LIL)
- Chapter 9: Infinitely Connected Domains (summarize only)
- Chapter 10: Rectifiability and Quadratic Expressions (summarize only)