MAT 639, Spring 2026, Stony Brook University

Topics in Real Analysis: Harmonic Measure

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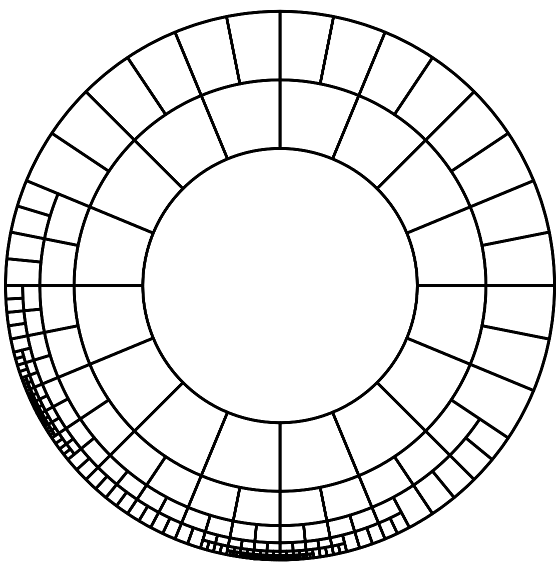

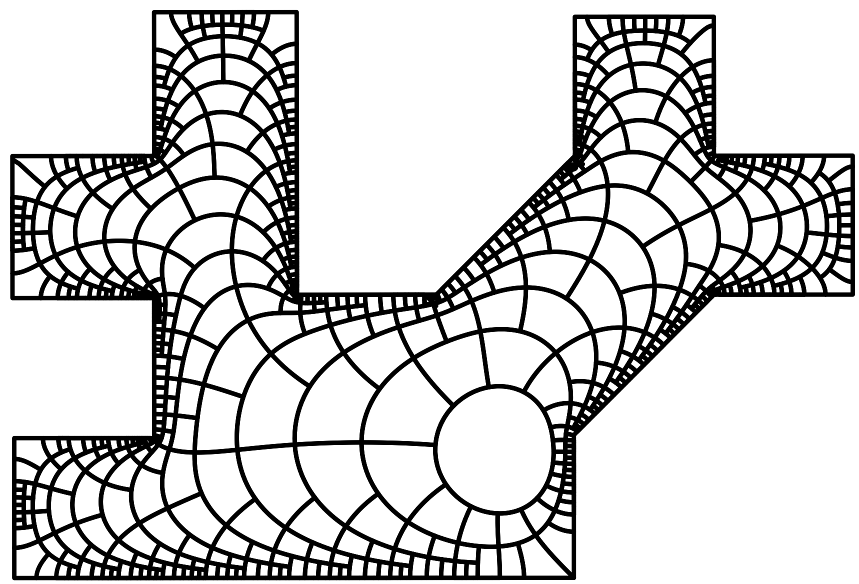

Harmonic Measure, Chapter VIII

Simply Connected Domains, Part II

following text by John Garnett and Don Marshall

Sections for Chapter VIII:

- The Law of the Iterated Logarithm for Bloch Functions
- Harmonic Measure and Hausdorff Dimension

Section VIII.1: The Law of the Iterated Logarithm for Bloch Functions

Recall that a **Bloch function** is a holomorphic function on $\mathbb{D}$ that is Lipschitz from the hyperbolic metric on $\mathbb{D}$ to the Euclidean metric on $\mathbb{C}$.

This is equivalent to

$$||g||_{\mathcal{B}} = \sup_{z \in \mathbb{D}} |g'(z)|(1 - |z|^2) < \infty. \tag{1.1}$$

This implies that

$$|g(z) - g(0)| \leq C \log \frac{1}{1 - |z|}.$$

This is the best growth estimate that is true on **all** radii.

But a better one is true on **almost all** radii.

Theorem 1.1 (Makarov): *There is a constant $C > 0$ such that if $g(z)$ is a Bloch function on $\mathbb{D}$, then for a.e. $\zeta \in \partial\mathbb{D}$*

$$\limsup_{r\to 1} \frac{|g(r\zeta)|}{\sqrt{\log\frac{1}{1-r}\log\log\log\frac{1}{1-r}}} \le C\|g\|_{\mathcal{B}}. \tag{1.1}$$

Nicolai G. Makarov (1955-)

Consider the Bloch function

$$g(z) = \sum_{n=1}^{\infty} z^{2^n}. \tag{1.2}$$

The series (1.2) behaves much like a sum of independent identically distributed random variables, and Salem and Zygmund proved in 1950 that its partial sums $S_n(z) = \sum_{k=0}^{n} z^{2^k}$ satisfy

$$\limsup \frac{|S_n(\zeta)|}{\sqrt{n \log\log n}} = 1 \tag{1.3}$$

for a.e. $\zeta \in \partial\mathbb{D}$. But if $g(z)$ is defined by (1.2), then (1.3) is equivalent to

$$\limsup_{r\to 1} \frac{|g(r\zeta)|}{\sqrt{\log \frac{1}{1-r} \log\log\log \frac{1}{1-r}}} = 1,$$

which is an instance of (1.1).

Raphaël Salem (1898-1963)

Proof of Theorem 1.1: This proof is due to Lennart Carleson (unpublished) and independently to Pommerenke. We give Pommerenke's version, which yields the best known constant $C = 1$ in (1.1).

We may assume $g(0) = 0$ and $||g||_{\mathcal{B}} = 1$. Let $p \geq 0$ be an integer and consider the integral means

$$I_p(r) = \frac{1}{2\pi} \int_0^{2\pi} |g(re^{i\theta})|^{2p} d\theta.$$

The proof begins with the identity

$$\frac{d}{dr}(rI_p'(r)) = \frac{4p^2 r}{2\pi} \int_0^{2\pi} |g(re^{i\theta})|^{2p-2} |g'(re^{i\theta})|^2 d\theta, \tag{1.6}$$

which is known as **Hardy's identity**. It can be proved by examining the Fourier series of $|g|^p$ or by differentiating $I_p(r)$ directly.

Proof of Hardy's equality:

Suppose g is holomorphic on $\mathbb{D}$ and set $h = g^p$. Then h is holomorphic so

$$h(z) = \sum_{n=1}^{\infty} a_n z_n.$$

Then

$$I_p(r) = \frac{1}{2}\int_0^{2\pi} |g(re^{i\theta})|^{2p} d\theta = \frac{1}{2}\int_0^{2\pi} |h(re^{i\theta})|^2 d\theta = \|h\|_2^2 = \sum |a_n|^2 r^{2n}.$$

Therefore

$$\begin{aligned}
I_p'(r) &= \sum 2n|a_n|^2 r^{2n-1} \\
rI_p'(r) &= \sum 2n|a_n|^2 r^{2n} \\
(rI_p'(r)' &= \sum (2n)^2|a_n|^2 r^{2n-1} \\
&= r\sum (2n)^2|a_n|^2 r^{2n-2} \\
&= 4r\|h'\|_2^2 = 4r\|(g^p)'\|_2^2 \\
&= 4r\|pg^{p-1}g'\|_2^2 \\
&= \frac{4rp^2}{2\pi}\int_0^{2\pi} |g^{p-1}|^2|g'|^2 d\theta. \quad \square
\end{aligned}$$

A corollary of Hardy's identity is the inequality

$$I_p(r) \le p!\Big(\log\frac{1}{1-r^2}\Big)^p \le p!\Big(\log\frac{1}{1-r}\Big)^p, \tag{1.7}$$

valid for $||g||_{\mathcal{B}} = 1$ and for integral $p \ge 0$. Indeed, (1.7) is trivial when $p = 0$, and to prove (1.7) when $p \ge 1$ we use (1.6) and induction to get

$$\begin{aligned}\frac{d}{dr}(rI_p'(r)) &\le \frac{4p^2 r}{(1-r^2)^2} I_{p-1}(r) \\ &\le \frac{4pp!r}{(1-r^2)^2}\Big(\log\frac{1}{1-r^2}\Big)^{p-1} \\ &\le p!\frac{d}{dr}\Big(r\frac{d}{dr}\Big(\log\frac{1}{1-r^2}\Big)^p\Big),\end{aligned}$$

and since $I_p(0) = 0$, an integration then yields (1.7).

Apply the Hardy-Littlewood maximal theorem to the function $|g(re^{i\theta})|^p \in L^2$.

By inequality (1.7), $g_r^*(e^{i\theta}) = \sup_{\rho<r} |g(\rho e^{i\theta})|$ satisfies

$$\frac{1}{2\pi}\int |g_r^*(e^{i\theta})|^{2p} d\theta \le Cp! \left(\log \frac{1}{1-r}\right)^p \tag{1.8}$$

with C independent of p. Next fix $\alpha > 1$ and set

$$A_p(r) = \frac{1}{1-r} \frac{1}{\left(\log \frac{1}{1-r}\right)^{p+1}} \frac{1}{\left(\log\log \frac{1}{1-r}\right)^{\alpha}}.$$

Then

$$\int_r^1 A_p(s)ds \geq \frac{C}{p}\frac{1}{\left(\log\frac{1}{1-r}\right)^p}\frac{1}{\left(\log\log\frac{1}{1-r}\right)^\alpha}, \tag{1.9}$$

while by (1.8)

$$\begin{aligned}&\int_r^1 A_p(s)\int |g_s^*(e^{i\theta})|^{2p}d\theta ds\\ &\quad\leq Cp!\int_r^1 \frac{1}{\left(\log\log\frac{1}{1-s}\right)^\alpha}\frac{1}{\left(\log\frac{1}{1-s}\right)}\frac{ds}{1-s} \leq C_\alpha p!.\end{aligned}$$

Fubini's theorem and Chebychev's inequality then imply that the set

$$E_p = \left\{ \theta : \int_r^1 A_p(s)|g_s^*(e^{i\theta})|^{2p} ds > C_\alpha p^2 p! \right\}$$

satisfies $|E_p| \leq \frac{1}{p^2}$.

Therefore if $\theta \notin \bigcup_{p>p_0} E_p$, then by (1.9) and the definition of g_s^*,

$$\frac{|g(re^{i\theta})|}{\sqrt{\left(\log \frac{1}{1-r}\right) \log\log\log \frac{1}{1-r}}} \leq \frac{C^{\frac{-1}{2p}} C_\alpha^{\frac{1}{2p}} p^{\frac{3}{2p}} (p!)^{\frac{1}{2p}} \left(\log\log \frac{1}{1-r}\right)^{\frac{\alpha}{2p}}}{\sqrt{\log\log\log \frac{1}{1-r}}}. \tag{1.10}$$

Finally setting $p = \log\log\log \frac{1}{1-r}$ in (1.10) and using Stirling's formula, we obtain (1.1) almost everywhere with constant $C = 1$. $\square$

Section VIII.2: Harmonic Measure and Hausdorff Dimension

A real valued function $h(t)$ on an interval $(0, t_0)$ is called a **logarithmico-exponential function** or an **L-function** if $h(t)$ is defined by a finite algebraic combination of exponential functions and logarithm functions.

Hardy proved that if h and g are L-functions, then the limit

$$\lim_{t \downarrow 0} \frac{h(t)}{g(t)}$$

exists in the extended interval $[0, \infty]$.

Because the derivative of an L-function is again an L-function, it follows that every L-function is monotone on some interval $(0, t_1)$.

The idea: If $\omega \simeq \Lambda_h$ on some subset of $\partial\Omega$, then we want to cover this subset by disks D where $\omega(D(x,s)) \simeq h(s)$, Suppose $D \cap \partial\Omega$ is $\varphi(I)$ for some arc $I \subset \partial\mathbb{D}$, and $z = z_I$ is the corresponding point in $\mathbb{D}$ and $r = |z|$. By Koebe's theorem we expect $s \simeq |\varphi'(z)|(1-r)$, so

$$1 - r = |I| = \omega(D(x,s)) \simeq h(s) = h((1-r)|\varphi'(z)|)$$

$$h^{-1}(1-r) \simeq (1-r)|\varphi'(z)|$$

$$\frac{(1-r)|\varphi'(z)|}{h^{-1}(1-r)} \simeq 1.$$

Theorem 2.1 (Makarov): *Let $h > 0$ be an increasing L-function satisfying*

$$\lim_{t\to 0} \frac{h(t)}{t} = \infty. \tag{2.1}$$

Let φ be a conformal mapping from $\mathbb{D}$ onto a simply connected domain Ω and let ω denote harmonic measure for some point $w_0 \in \Omega$. Then

(a) $\omega \ll \Lambda_h \iff \liminf_{r\to 1} \dfrac{(1-r^2)|\varphi'(r\zeta)|}{h^{-1}(1-r)} > 0$ a.e. *on* $\partial\mathbb{D}$,

(b) $\omega \perp \Lambda_h \iff \liminf_{r\to 1} \dfrac{(1-r^2)|\varphi'(r\zeta)|}{h^{-1}(1-r)} = 0$ a.e. *on* $\partial\mathbb{D}$, *and*

(c) there is a set $A \subset \partial\Omega$ of σ-finite Λ_h measure such that $\omega(A) = 1$ if and only if

$$\liminf_{r\to 1} \frac{(1-r^2)|\varphi'(r\zeta)|}{h^{-1}(1-r)} < \infty \text{ a.e. } on\ \partial\mathbb{D}.$$

Theorem 2.2 (Makarov): *There is a constant $C > 0$ such that if*

$$h(t) = te^{C\sqrt{\log\frac{1}{t}\log\log\log\frac{1}{t}}},$$

then for every simply connected domain Ω,

$$\omega \ll \Lambda_h. \tag{2.2}$$

Conversely, there is $c < C$ such that if

$$h(t) = te^{c\sqrt{\log\frac{1}{t}\log\log\log\frac{1}{t}}}, \tag{2.3}$$

then there exists a Jordan domain Ω for which

$$\omega \perp \Lambda_h. \tag{2.4}$$

The optimal value of C is unknown.

Proof of Thm 2.2 from Thm 2.1:

Let Ω be any simply connected domain, let φ be a conformal map from $\mathbb{D}$ onto Ω and let $g = \log(\varphi')$.

Then by Theorem VII.2.1, $||g||_{\mathcal{B}} \leq 6$, and by Theorem 1.1 there is a constant $C > 0$ such that

$$\limsup_{r\to 1} \frac{|\mathrm{Re}(g(r\zeta))|}{\sqrt{\log\left(\frac{1}{1-r}\right) \log\log\log\left(\frac{1}{1-r}\right)}} \leq C \tag{2.5}$$

almost everywhere on $\partial\mathbb{D}$.

For $C > 0$ define
$$\varphi_C(t) = te^{C\sqrt{\log\frac{1}{t}\log\log\log\frac{1}{t}}}.$$

Define h_1 be setting
$$h_1^{-1}(t) = \varphi_{-C}(t) = te^{-C\sqrt{\log\frac{1}{t}\log\log\log\frac{1}{t}}}.$$

Then by (2.5),
$$\liminf_{r\to 1} \frac{(1-r)|\varphi'(r\zeta)|}{h_1^{-1}(1-r)} \geq 1$$

almost everywhere, and by Theorem 2.1(a), $\omega \ll \Lambda_{h_1}$.

We claim that

$$h_{-C}(t) = o(h_C(t)) = \mathrm{o}\left(te^{C\sqrt{\log\frac{1}{t}\log\log\log\frac{1}{t}}}\right) = o(\varphi_C(t))$$

as $t \to 0$. If so, then (2.2) follows.

To prove this, set $y = h_1^{-1}(t)$ (hence $t = h_1(y)$). Note that $y = o(t)$. Thus

$$\begin{aligned}\log y &= \log h_1^{-1}(t) = \log t - C\log\sqrt{\log\frac{1}{t}\log\log\log\frac{1}{t}} \\ &\gg \log t - C\log\sqrt{\log\frac{1}{y}\log\log\log\frac{1}{y}}.\end{aligned}$$

Hence

$$\log \varphi_C(y) = \log y + C \log \sqrt{\log \frac{1}{y} \log \log \log \frac{1}{y}} \gg \log t = \log h_1(y)$$

so

$$h_1(y) = o(\varphi_C(y))$$

which is the claim. This completes the proof of (2.2).

Conversely, let Ω be any Jordan domain that satisfies condition (c) of Theorem VII.2.4. The snowflake is one example.

Let φ be the conformal mapping from $\mathbb{D}$ to Ω and let $g = \log(\varphi')$. Then by Theorem VII.2.4 and Theorem 1.2, there is $c_3 > 0$ such that

$$\limsup_{r \to 1} \frac{-\mathrm{Re}(g(r\zeta))}{\sqrt{\log\left(\frac{1}{1-r}\right) \log\log\log\left(\frac{1}{1-r}\right)}} \geq c_3 > 0,$$

almost everywhere on $\partial\mathbb{D}$.

If $c < c_3$, take c' with $c < c' < c_3$ and set

$$h_2^{-1}(t) = te^{-c'\sqrt{\log \frac{1}{t} \log\log\log \frac{1}{t}}}.$$

Then

$$\liminf_{r \to 1} \frac{(1-r)|\varphi'(r\zeta)|}{h_2^{-1}(1-r)} = 0$$

almost everywhere, so that by (b) of Theorem 2.1, $\omega \perp \Lambda_{h_2}$.

If (2.3) holds, then for small t $h(t) \leq h_2(t)$ and thus (2.4) holds for $c < c_3$. $\square$

Theorem 2.1 will follow from the following lemma.

Lemma 2.3: *Let φ be the conformal mapping from $\mathbb{D}$ to a simply connected domain Ω and let $E \subset \partial\mathbb{D}$ be a Borel set.*

(a) If

$$\liminf \frac{(1-r)|\varphi'(r\zeta)|}{h^{-1}(1-r)} \leq A$$

a.e. on E, then there exists $E_1 \subset E$ such that $|E \setminus E_1| = 0$ and

$$\Lambda_h(\varphi(E_1)) \leq 4A|E|.$$

(b) There is $c_1 > 0$, not depending on E, such that if

$$\liminf \frac{(1-r)|\varphi'(r\zeta)|}{h^{-1}(1-r)} > B > 0 \tag{2.6}$$

a.e. on E, then

$$\Lambda_h(\varphi(E)) \geq c_1 \frac{B}{(1+B)^{\frac{1}{2}}}|E| > 0.$$

Proof of Lemma 2.3(a):

The proof of (a) is almost the same as the proof of Theorem VI.5.2. We may assume $A > 0$. Fix $\alpha = \max\{2, \frac{1}{2A}\}$ and define

$$I(z) = \{\zeta \in \partial\mathbb{D} : |z - \zeta| < \alpha(1 - |z|)\}.$$

Let $\{z_n\}$ be a sequence in $\mathbb{D}$ such that

$$(1 - |z_n|)|\varphi'(z_n)| \leq Ah^{-1}(1 - |z_n|),$$

and such that $\{z_n\}$ is nontangentially dense on E.

Fix $\delta > 0$. By the Vitali covering lemma there is a subsequence $\{z_k\}$ of $\{z_n\}$ so that the intervals $I(z_k)$ are pairwise disjoint,

$$(1 - |z_k|)|\varphi'(z_k)| < \frac{\delta}{2\alpha},$$

and

$$|E \setminus \bigcup I(z_k)| = 0.$$

Take $w_k = \varphi(z_k)$, $r_k = \alpha \cdot \operatorname{dist}(w_k, \partial\Omega)$, $B_k = B(w_k, r_k)$, and

$$V_\delta = \partial\Omega \cap \left(\bigcup B_k\right).$$

Then $r_k \leq 2\alpha(1-|z_k|)|\varphi'(z_k)| < \delta$ and

$$h(r_k) \leq h\Big(2\alpha A h^{-1}(1-|z_k|)\Big).$$

If $A \leq \frac{1}{4}$, then

$$h(r_k) \leq (1-|z_k|),$$

and if $A > \frac{1}{4}$, then since $\frac{h(t)}{t}$ is decreasing by (2.1),

$$h(r_k) \leq 4A(1-|z_k|).$$

In either case we obtain

$$\sum h(r_k) \leq 4A \sum |I(z_k)| \leq 4A|E|.$$

Therefore $V = \bigcap V_{1/m}$ satisfies $\Lambda_h(V) \leq 4A|E|$.

On the other hand, because $\alpha \geq 2$, Lemma VI.5.3 implies that $|E \backslash \varphi^{-1}(V)| = 0$, and thus (a) holds for $E_1 = E \cap \varphi^{-1}(V)$. $\square$

For the proof of Lemma 2.3(b) we need three additional lemmas.

Lemma 2.4: *Let* φ *be the conformal mapping from* $\mathbb{D}$ *onto a simply connected domain* Ω*, and let* γ *be a crosscut of* Ω *with endpoints* $w_1, w_2 \in \partial\Omega$*. Set* $\zeta_j = \varphi^{-1}(w_j)$*,* $j = 1, 2$*. If* σ *is the geodesic in* $\mathbb{D}$ *connecting* ζ_1 *to* ζ_2*, let* $z_\sigma \in \sigma$ *satisfy* $|z_\sigma| = \inf_\sigma |z|$*. Then*

$$\operatorname{diam}(\gamma) \geq c(1 - |z_\sigma|)|\varphi'(z_\sigma)|, \tag{2.7}$$

for some absolute constant $c > 0$*.*

Proof: Applying a Möbius transformation to the disc and linear map to Ω, we may suppose that $\sigma = (-1, 1)$, $z_\sigma = 0$, $\varphi(0) = 0$ and $\varphi'(0) = 1$.

We need to prove $\mathrm{diam}(\gamma) \geq c > 0$. Let $B_r = \{|z| \leq r\}$. By the Koebe one-quarter theorem, $B_{\frac{1}{4}} \subset \Omega$.
If $B(w_1, \mathrm{diam}(\gamma)) \cap B_{\frac{1}{5}} \neq \emptyset$, then $\mathrm{diam}(\gamma) \geq \frac{1}{20}$.

On the other hand, if $B(w_1, \mathrm{diam}(\gamma)) \cap B_{\frac{1}{5}} = \emptyset$, then $\varphi^{-1}(\gamma)$ separates $B_{\frac{1}{24}} \subset \varphi^{-1}(B_{\frac{1}{6}})$ from a semicircle, say $T^+ = \partial\mathbb{D} \cap \{\mathrm{Im} z > 0\}$. Thus

$$d_{\mathbb{D}}(B_{\frac{1}{24}}, T^+) \geq d_{\mathbb{C}}(B_{\frac{1}{6}}, B(\zeta_1, \mathrm{diam}(\gamma)) \geq \log\Big(\frac{C}{\mathrm{diam}(\gamma)}\Big),$$

and (2.7) holds. $\square$

Lemma 2.5 (Carleson): *Let Ω be a simply connected domain. Fix $w_0 \in \Omega$ and let $\zeta_0 \in \partial\Omega$. For $0 < r < |w_0 - \zeta_0|/2$ set $D = B(\zeta_0, r)$ and $\tilde{D} = B(\zeta_0, 2r)$, and for any $M > 0$ take*

$$k_0 = k_0(M) = \left[1 + \frac{M}{\pi}\right].$$

Then there exist $r_0 = r_0(M) > 0$ such that if $r < r_0$, then $\Omega \cap \partial\tilde{D}$ contains

$$N \leq \frac{2\pi}{\log 2} k_0 \log \frac{1}{r}$$

crosscuts $\gamma_1, \ \ldots \ , \gamma_N$ of Ω, such that each γ_j separates w_0 from a continuum $\beta_j \subset \partial\Omega$ and

$$\omega(D \cap \partial\Omega \setminus \bigcup_{j=1}^{N} \beta_j) < r^M, \tag{2.8}$$

where $\omega(E) = \omega(w_0, E, \Omega)$.

Proof: Fix one curve $\sigma \subset \Omega \setminus \overline{\tilde{D}}$ connecting w_0 to $\partial\Omega$.

Let $\{\gamma_j\}$ be the set of component arcs of $\Omega \cap \partial\tilde{D}$ having the property that there exists a curve $P \subset \Omega$ connecting w_0 to $w_j \in \Omega \cap D$ such that P first crosses $\Omega \cap \partial\tilde{D}$ through γ_j.

By Exercise I.13, $\Omega \setminus \gamma_j$ consists of two simply connected components; take U_j to be that component with $w_0 \notin U_j$. Because Ω is simply connected and $\partial\Omega \setminus \tilde{D} \neq \emptyset$, $U_j \cap U_k = \emptyset$ whenever $\gamma_j \neq \gamma_k$.

Also by Exercise I.13, $\beta_j = (\partial U_j) \setminus \gamma_j \subset \partial\Omega$ is a continuum and γ_j separates β_j from w_0.

Now $U_j \bigcap \partial D$ is a union of arcs $\tau_{j,k}$, and by Exercise I.13, each $\tau_{j,k}$ separates w_0 from a continuum $\alpha_{j,k} \subset \beta_j$. Let $\alpha_j = \bigcup_k \alpha_{j,k}$ and let Γ_j be the family of paths in Ω joining σ to α_j.

Then by Theorem IV 5.3 and Exercise VI.7,

$$\omega_j = \omega(\alpha_j) \leq \frac{8}{\pi} e^{-\pi\lambda(\Gamma_j)}, \tag{2.9}$$

even if Ω is not a Jordan domain or α_j is not connected. Let Γ'_j be the set of paths in U_j joining $\partial\widetilde{D}$ to $\cup_k \tau_{j,k}$.

Then every path in Γ_j contains a path from Γ'_j, so that by the extension rule,

$$\lambda(\Gamma'_j) \leq \lambda(\Gamma_j).$$

By the parallel rule,

$$\sum_j \frac{1}{\lambda(\Gamma'_j)} \leq \frac{1}{d_{\tilde{D}\setminus\overline{D}}\big(\partial\tilde{D}, \partial D\big)} = \frac{2\pi}{\log 2}.$$

Hence for $k = 1, 2, \dots$,

$$\#\left\{ j : \frac{1}{\lambda(\Gamma'_j)} \geq \frac{1}{k \log \frac{1}{r}} \right\} \leq \frac{2\pi}{\log 2} k \log \frac{1}{r}$$

and

$$\#\left\{ j : \omega_j \geq \frac{8}{\pi} r^{k\pi} \right\} \leq \frac{2\pi}{\log 2} k \log \frac{1}{r}. \tag{2.10}$$

Then since $k_0 = \left[1 + \frac{M}{\pi}\right]$,

$$\sum_{\omega_j \le \frac{8}{\pi} r^{\pi k_0}} \omega_j \le \sum_{k=k_0}^{\infty} \Big(\frac{2\pi}{\log 2} \log \frac{1}{r}\Big) \frac{8}{\pi} (k+1) r^{\pi k} \le r^M$$

if $r < r_0$ and if r_0 is small. Let

$$N = \#\Big\{ j : \omega_j \ge \frac{8}{\pi} r^{\pi k_0} \Big\}.$$

Then $N \le k_0 \frac{2\pi}{\log 2} \log \frac{1}{r}$ by (2.10). Select $\gamma_1, \dots, \gamma_N$ to have the largest ω_j. Then

$$\omega(D \cap \partial\Omega \setminus \bigcup_1^N \beta_j) \le \sum_{\omega_j \le \frac{8}{\pi} r^{\pi k_0}} \omega_j \le r^M. \quad \square$$

Lemma 2.6: *Assume the L-function $h(t)$ satisfies (2.1) and*

$$Ct \le h(t) \le ct \exp\big(\log(1/t)\big)^{2/3}. \tag{2.11}$$

Then there is $C_2 > 0$ such that if $0 < t < C_2$ then

$$\frac{B}{(1+B)^{\frac{1}{2}}} h(t/B) \le h(t), \tag{2.12}$$

and there is $C_3 < \infty$ such that

$$\log(1/t) h\Big(\frac{t}{\log(1/t)}\Big) \le C_3 h(t). \tag{2.13}$$

Proof: By (2.1), $h(t)/t$ is decreasing and (2.12) holds if $B \leq 1$. Assume $B > 1$. Set $x = 1/t$ and $g(x) = \log\left(\frac{h(t)}{t}\right)$. Then by (2.11)

$$C'' \leq g(x) \leq (\log x)^{2/3}$$

for large x. Then since $g(x)$ and $(\log x)^{2/3}$ are both L-functions, Theorem 19 of [1954] gives

$$g'(x) \leq c' \frac{2}{3x} (\log x)^{-1/3}$$

for x large and integration then yields (2.12) and (2.13). $\square$

Proof of Lemma 2.3(b): We may assume that $h(t)$ satisfies (2.11).

Indeed, since (2.11) holds for every measure function of the form

$$h_1(t) = te^{C\sqrt{\log\frac{1}{t}\log\log\log\frac{1}{t}}}$$

we will have established Theorem 2.2 if we prove (b) under the additional assumption (2.11).

On the other hand, if (2.11) fails, then by Hardy's theorem on L-functions,

$$h_1(t) = \mathrm{o}(h(t)).$$

But then we can use Theorem 2.2 (which would have been proved for the measure function h_1) to obtain $\Lambda_{h_1}(\varphi(E)) > 0$ and $\Lambda_h(\varphi(E)) = \infty$ whenever $E \subset \partial\mathbb{D}$ has $|E| > 0$. Therefore part (b) of Lemma 2.3 will also hold when (2.11) fails.

Now assume (2.6) holds for E with $|E| > 0$. Fix $\delta > 2$.

By Koebe's theorem there is a constant $c > 0$ so that for $\zeta \in E$

$$\liminf_{z \in \Gamma_\delta(\zeta)} \frac{(1 - |z|)|\varphi'(z)|}{h^{-1}(1 - |z|)} > cB > 0.$$

Let $\Gamma_\delta^{\frac{1}{n}}(\zeta) = \Gamma_\delta(\zeta) \cap \{z : |z| > 1 - \frac{1}{n}\}$.

For sufficiently large n, the closed set

$$E_n = \{\zeta \in E : \inf_{z \in \Gamma_\delta^{\frac{1}{n}}(\zeta)} \frac{(1 - |z|)|\varphi'(z)|}{h^{-1}(1 - |z|)} \geq (cB + \frac{1}{n})\} \subset E$$

satisfies $|E| < 2|E_n|$, and we may replace E by E_n and B by $B' = cB + \frac{1}{n}$.

By defintion, we can cover $\varphi(E)$ by discs D_ν of radius $r_\nu < r_0$ such that

$$\sum h(r_\nu) \leq 2\Lambda_h(\varphi(E)). \tag{2.14}$$

By choosing r_0 sufficiently small we can assume that

$$2\pi\omega(\partial\widetilde{D}_\nu) < \frac{1}{n},$$

$$\operatorname{dist}(\varphi(0), \varphi(E)) > 4r_0,$$

and $h(r_0)/r_0 > B$, since $h(t)/t$ is decreasing.

We want to show that

$$\omega(E) \leq \sum_j \omega(D_j) \leq \sum C \sum_j h(r_j) \leq C\Lambda_h(E).$$

We do this by estimating the harmonic measure of each disk D_j by considering crosscuts of Ω on ∂D_j. The crosscuts get divided into two types and we get the appropriate bounds in both cases.

The two types are given by Lemma 2.5.

Then by the $M = 1$ case of Lemma 2.5 there are subarcs $\gamma_j^{(\nu)} \subset \partial\tilde{D}_\nu$ for

$$1 \leq j \leq N(\nu) \leq \frac{2\pi}{\log 2} \log(\frac{1}{r_\nu}),$$

such that $\gamma_j^{(\nu)}$ is a crosscut of Ω separating w_0 from a continuum $\beta_j^{(\nu)} \subset \partial\Omega$ and

$$\omega(D_\nu \cap \partial\Omega \setminus \bigcup_1^{N(\nu)} \beta_j^{(\nu)}) \leq r_\nu \leq \frac{h(r_\nu)}{B}. \tag{2.15}$$

On the other hand, if $\varphi(\zeta_1)$ and $\varphi(\zeta_2)$ are the endpoints of an arc $\beta_j^{(\nu)}$, then $|\zeta_1 - \zeta_2| < 2\pi\omega(\partial\widetilde{D}_\nu) \leq \frac{1}{n}$.

By discarding some $\beta_j^{(\nu)}$ if necessary, we will suppose that $\beta_j^{(\nu)} \cap E \neq \emptyset$.

Let z_σ bethe point closest to the origin on the geodesic connecting ζ_1 and ζ_2. Then $z_\sigma \in \Gamma_\delta^{1/n}(\zeta)$ for some $\zeta \in E$, since $\delta > 2$.

By (2.6) and Lemma 2.4

$$h^{-1}(\omega(\beta_j^{(\nu)})) \leq h^{-1}(1 - |z_\sigma|) \leq \frac{c_2}{B}\,\text{diam}\gamma_j^{(\nu)},$$

so that (since γ_j are disjoint subarcs of a circle of radius r_ν)

$$\sum_{j=1}^{N(\nu)} h^{-1}(\omega(\beta_j^{(\nu)})) \leq \frac{2c_2 r_\nu}{B}.$$

Fix ν, set $t_j = h^{-1}(\omega(\beta_j^{(\nu)}))$, $t = \sum t_j \le 2c_2 r_\nu / B$, and $s = t/\log 1/t$. Then

$$\sum_{j=1}^{N(\nu)} \omega(\beta_j^{(\nu)}) = \sum_{j=1}^{N(\nu)} h(t_j) = \sum_{t_j \le s} h(t_j) + \sum_{t_j > s} h(t_j).$$

Because $N(\nu) \le \frac{2\pi}{\log 2} \log(1/t)$, (2.13) gives

$$\sum_{t_j \le s} h(t_j) \le \frac{2\pi}{\log 2} \log(1/t) h\Big(\frac{t}{\log(1/t)}\Big) \le \frac{2\pi}{\log 2} h(\frac{t}{B}) \le C_4 h\Big(\frac{2c_2 r_\nu}{B}\Big),$$

since $t = \sum t_j \le 2c_2 r_\nu / B$.

Moreover, since $h(t)/t$ is decreasing, (2.13) also gives

$$\begin{aligned}\sum_{t_j \geq s} h(t_j) &= \sum_{t_j \geq s} t_j \frac{h(t_j)}{t_j} \leq \sum_{t_j \geq s} t_j \frac{h(s)}{s} \\ &= \sum_{t_j \geq s} \frac{t_j}{t} \cdot \log(1/t) \cdot h(t/\log 1/t) \leq C_3 h(t/B) \sum_{t_j \geq s} \frac{t_j}{t} \leq C_3 h\Big(\frac{2c_2 r_\nu}{B}\Big).\end{aligned}$$

Therefore by (2.12)

$$\sum_{j=1}^{N(\nu)} \omega(\beta_j^{(\nu)}) \leq C_4 \frac{(1+B)^{\frac{1}{2}}}{B} h(r_\nu). \tag{2.16}$$

Now (2.14), (2.15), and (2.16) yield

$$|E| = \omega(\varphi(E)) \leq \sum_\nu \omega(D_\nu) \leq C_1 \frac{(1+B)^{\frac{1}{2}}}{B} \sum_\nu h(r_\nu) \leq 2C_1 \frac{(1+B)^{\frac{1}{2}}}{B} \Lambda_h(\varphi(E)),$$

which is Part (b) of the lemma. □