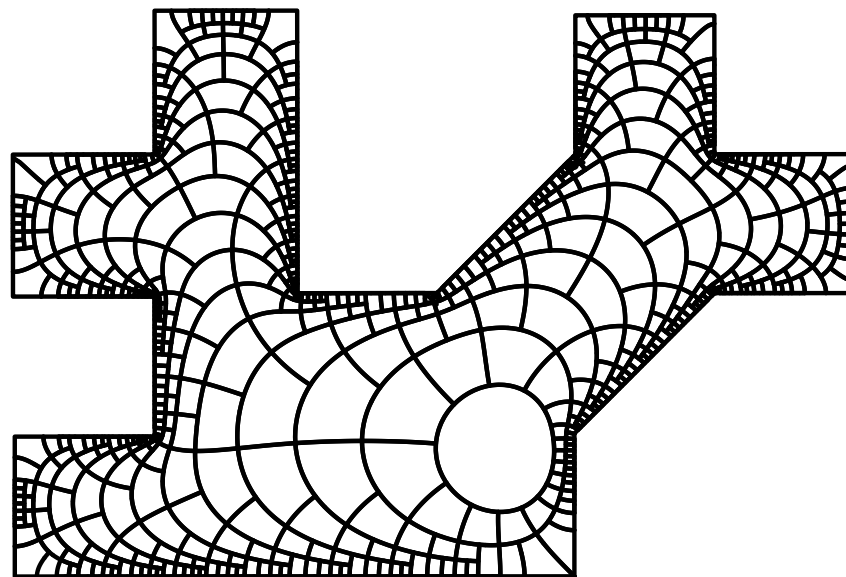
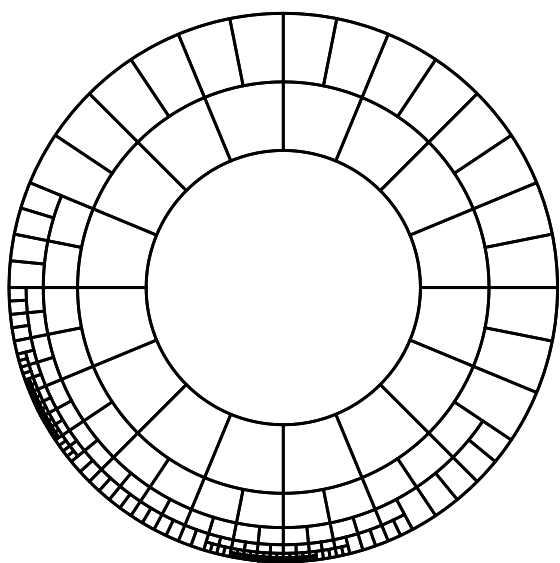


MAT 639, Spring 2026, Stony Brook University

Topics in Real Analysis: Harmonic Measure

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Harmonic Measure, Chapter VII

Bloch Functions and Quasicircles

following text by John Garnett and Don Marshall

Sections for Chapter VII:

- Bloch Functions
- Bloch Functions and Univalent Functions
- Quasicircles
- Chord-Arc Curves and the A^∞ Condition
- BMO domains

Section VII.1: Bloch functions

A **Bloch function** is a holomorphic function on \mathbb{D} that is Lipschitz from the hyperbolic metric on \mathbb{D} to the Euclidean metric on \mathbb{C} .

This is equivalent to

$$\|g\|_{\mathcal{B}} = \sup_{z \in \mathbb{D}} |g'(z)|(1 - |z|^2) < \infty. \quad (1.1)$$

We write \mathcal{B} for the space of Bloch functions and we call $\|g\|_{\mathcal{B}}$ the **Bloch norm**

Actually $\|g\|_{\mathcal{B}}$ is only a semi-norm, but $\|g\|_{\mathcal{B}} = 0$ only if g is constant.



André Bloch (1893-1948)

His work was done from a criminal lunatic asylum, where he was confined for the murder of his brother, aunt and uncle in 1917.

Invariance: Because

$$\left. \frac{d}{dz} g\left(\frac{z+z_0}{1+\bar{z}_0 z}\right) \right|_{z=0} = g'(z_0)(1-|z_0|^2), \quad (1.2)$$

we have

$$\|g\|_{\mathcal{B}} = \sup_{T \in \mathcal{M}} |(g \circ T)'(0)|, \quad (1.3)$$

where \mathcal{M} is the set of conformal self maps of \mathbb{D} :

$$T(z) = \lambda \frac{z+a}{1+\bar{a}z}, \quad a \in \mathbb{D}, |\lambda| = 1.$$

Thus the Bloch space \mathcal{B} is invariant under \mathcal{M} :

$$\|g \circ T\|_{\mathcal{B}} = \|g\|_{\mathcal{B}}, \quad T \in \mathcal{M}. \quad (1.4)$$

If g is bounded, then $g \in \mathcal{B}$ by (1.2).

In fact by (II.3.4) and (II.3.5), $g \in \mathcal{B}$ if only $\operatorname{Re}(g)$ is bounded.

The function

$$g(z) = \sum_{n=1}^{\infty} z^{2^n} \tag{1.5}$$

is a good example of an unbounded Bloch function in no H^p space.

The **spherical distance** $d(p^*, q^*)$ between $p^*, q^* \in \mathbb{S}^2$ is the arc-length of the shortest curve on the sphere containing p^* and q^* . The quantity $f^\#$ is called the spherical derivative because

$$f^\#(z) = \lim_{w \rightarrow z} \frac{d(f(z)^*, f(w)^*)}{|z - w|},$$

where $f(z)^*$ denotes the stereographic projection of $f(z)$.

(Theorem 10.10 of Marshall's book), Marty's Theorem: *A family \mathcal{F} of meromorphic functions on a region $\Omega \subset \mathbb{C}$ is normal in the chordal metric if and only if $\mathcal{F}^\# = \{f^\# : f \in \mathcal{F}\}$ is locally bounded.*

Defn: A function $g(z)$ on \mathbb{D} is **normal** if $\{g \circ T : T \in \mathcal{M}\}$ is a normal family, in the extended sense that $f \equiv \infty$ is a permitted limit.

Theorem 1.1: *Every Bloch function is a normal function.*

Proof: By Marty's theorem $g(z)$ is normal if and only if

$$\sup_{T \in \mathcal{M}} \frac{|(g \circ T)'(z)|}{1 + |g \circ T(z)|^2}$$

is bounded on each compact subset of \mathbb{D} .

Since \mathcal{M} is transitive, it follows from (1.2) that g is normal if and only if

$$\sup_{\mathbb{D}} \frac{|g'(z)|(1 - |z|^2)}{1 + |g(z)|^2} < \infty. \quad \square$$

Theorem 1.2: *Let $g(z)$ be analytic on \mathbb{D} . Then $g \in \mathcal{B}$ if and only if g is Lipschitz continuous as a map from the hyperbolic metric on \mathbb{D} to the Euclidean metric on \mathbb{C} . Furthermore,*

$$\|g\|_{\mathcal{B}} = \sup_{z, w \in \mathbb{D}} \frac{|g(z) - g(w)|}{\rho(z, w)}.$$

Proof: If

$$|g(z_1) - g(z_2)| \leq B \int_{z_1}^{z_2} \frac{1}{1 - |z|^2} |dz|,$$

then a differentiation shows $\|g\|_{\mathcal{B}} \leq B$.

Conversely, if $g \in \mathcal{B}$ then we get

$$|g(z_1) - g(z_2)| \leq \|g\|_{\mathcal{B}} \rho(z_1, z_2) \tag{1.6}$$

by integrating along a hyperbolic geodesic. \square

Recall from Chapter II, the Zygmund class Z^* of continuous f on $\partial\mathbb{D}$ such that

$$\sup_{t>0} \frac{\|f(\theta + t) + f(\theta - t) - 2f(\theta)\|_\infty}{t} < \infty.$$

The Zygmund class has norm

$$\|f\|_{Z^*} = \|f\|_\infty + \sup_{t>0} \frac{\|f(\theta + t) + f(\theta - t) - 2f(\theta)\|_\infty}{t}.$$

Theorem 1.3: *Let $g(z)$ be analytic on \mathbb{D} and let $G(z)$ be the primitive*

$$G(z) = \int_0^z g(w)dw.$$

Then $g \in \mathcal{B}$ if and only if G is continuous on $\overline{\mathbb{D}}$ and $G(e^{i\theta})$ is a Zygmund function, $G(e^{i\theta}) \in Z^$. When $g \in \mathcal{B}$, there is a constant C independent of g such that*

$$\frac{1}{C}(\|g\|_{\mathcal{B}} + |g(0)|) \leq \|G\|_{Z^*} \leq C(\|g\|_{\mathcal{B}} + |g(0)|). \quad (1.7)$$

The Bloch function g has nontangential limit at $e^{i\theta_0} \in \partial\mathbb{D}$ if and only if its primitive $G(e^{i\theta})$ is differentiable at θ_0 .

Proof: Suppose $g \in \mathcal{B}$. Then (1.6) gives, for $0 < r < s < 1$,

$$\begin{aligned} |G(se^{i\theta}) - G(re^{i\theta})| &= \left| \int_r^s g(te^{i\theta})e^{i\theta} dt \right| \leq |g(re^{i\theta})|(s - r) \\ &\quad + \frac{\|g\|_{\mathcal{B}}}{2} \int_r^s \log\left(\frac{1+t}{1-t}\right) dt, \end{aligned} \tag{1.8}$$

and hence G extends continuously to $\overline{\mathbb{D}}$. Then (1.1) and Theorem II.3.4 give (1.7).

The proof of (1.8) also gives the estimate

$$\begin{aligned} \int_r^1 |g(se^{i\theta}) - g(re^{i\theta})| ds &\leq \|g\|_{\mathcal{B}} \int_r^1 \rho(r, s) ds \\ &= \|g\|_{\mathcal{B}} \log\left(\frac{2}{1+r}\right) \\ &\leq \|g\|_{\mathcal{B}}(1-r). \end{aligned} \tag{1.9}$$

Now suppose $\|g\|_{\mathcal{B}} \leq 1$ and suppose g has nontangential limit 0 at $e^{i\theta}$. Then (1.9) implies that $\frac{dG}{d\theta} = 0$.

Indeed, let ϵ be small and take $\alpha > 1$ so that for all small t we have

$$(1 - \epsilon t)e^{i(\theta+t)} \in \Gamma_{\alpha}(e^{i\theta}). \quad (1.10)$$

Set $r = 1 - \varepsilon t$. Then

$$\begin{aligned} G(e^{i(\theta+t)}) - G(e^{i\theta}) &= \lim_{s \rightarrow 1} \int_{se^{i\theta}}^{se^{i(\theta+t)}} g(z) dz \\ &= \int_{re^{i\theta}}^{re^{i(\theta+t)}} g(z) dz - e^{i\theta} \int_r^1 g(se^{i\theta}) ds \\ &\quad + (1 - r)e^{i(\theta+t)} g(re^{i(\theta+t)}) \\ &\quad + e^{i(\theta+t)} \int_r^1 \left(g(se^{i(\theta+t)}) - g(re^{i(\theta+t)}) \right) ds. \quad (1.11) \end{aligned}$$

By (1.10) the first three terms on the far right-hand side of (1.11) are each $o(t)$, because $g(z) \rightarrow 0$ when $\Gamma_\alpha(e^{i\theta}) \ni z \rightarrow e^{i\theta}$, and by (1.9) the last term in (1.11) is bounded by

$$\|g\|_{\mathcal{B}} \log\left(\frac{2}{1+r}\right) = O(\epsilon t).$$

Hence $G(e^{i(\theta+t)}) - G(e^{i\theta}) = o(t)$ and $G'(e^{i\theta}) = 0$.

The converse assertion, that G' has a nontangential limit wherever $\frac{dG}{d\theta}$ exists, is a simple property of the Poisson kernel. \square

Section VII.2: Bloch Functions and Univalent Functions

Notation: Fix a pair of functions $\varphi(z)$ and $g(z)$ analytic on \mathbb{D} such that

$$g(z) = \log(\varphi'(z)), \quad \varphi(0) = 0, \quad (2.1)$$

or equivalently

$$\varphi(z) = \int_0^z e^{g(w)} dw.$$

Theorem 2.1: *If φ is univalent on \mathbb{D} , then $g \in \mathcal{B}$ and $\|g\|_{\mathcal{B}} \leq 6$.*

Conversely, there is $\beta < 1$ such that if $\|g\|_{\mathcal{B}} < \beta$, then φ is univalent and $\varphi(\mathbb{D})$ is bounded by a Jordan curve.

Proof that $g \in \mathcal{B}$:

Fix $z \in \mathbb{D}$ and write

$$\psi(w) = \frac{\varphi\left(\frac{w+z}{1+\bar{z}w}\right) - \varphi(z)}{\varphi'(z)(1 - |z|^2)}.$$

Then ψ is a normalized univalent function, $\psi(0) = 0$ and $\psi'(0) = 1$, and by Theorem I.4.1, $|\psi''(0)| \leq 4$.

Then because

$$\psi''(0) = \frac{\varphi''(z)}{\varphi'(z)}(1 - |z|^2) - 2\bar{z} = g'(z)(1 - |z|^2) - 2\bar{z},$$

it follows that

$$|g'(z)|(1 - |z|^2) \leq 6. \quad \square$$

The converse is harder. We start with a lemma. Recall

$$\varphi(z) = \int_0^z e^{g(w)} dw.$$

Let I be an arc on $\partial\mathbb{D}$ with $|I| \leq \pi$. Let C_I be the circle that intersects $\partial\mathbb{D}$ orthogonally at the endpoints of I , and let z_I be the point on C_I closest to 0.

Lemma 2.2: *If $\|g\|_{\mathcal{B}} \leq \beta < 2$, then at each $\zeta \in \partial\mathbb{D}$, $\varphi(\zeta) = \lim_{\mathbb{D} \ni z \rightarrow \zeta} \varphi(z)$ exists and*

$$|\varphi(\zeta) - \varphi(z_I) - (\zeta - z_I)\varphi'(z_I)| \leq \frac{2\beta}{2 - \beta} |\varphi'(z_I)| |I| \quad (2.2)$$

for all $\zeta \in I$. Furthermore, φ extends continuously to $\overline{\mathbb{D}}$ and $\varphi \in C^{1-\beta/2}(\overline{\mathbb{D}})$.

Proof: Let $J = \partial\mathbb{D} \cap \{\operatorname{Re}(w) \geq 0\}$ and let T be Möbius satisfy $T(J) = I$.

Then

$$T(w) = \frac{z_I}{|z_I|} \frac{w + |z_I|}{1 + |z_I|w}.$$

Note that when $\operatorname{Re}(w) \geq 0$, by the chain rule

$$|T'(w)| = \left| \frac{1 - |z_I|^2}{(1 + |z_I|w)^2} \right| \leq 1 - |z_I|^2 \leq |I| \quad (2.3)$$

Suppose for a moment that the limit $\varphi(\zeta)$ exists. Then

$$\begin{aligned}\varphi(\zeta) - \varphi(z_I) - \varphi'(z_I)(\zeta - z_I) \\ &= \varphi'(z_I) \int_{z_I}^{\zeta} (e^{g(z)-g(z_I)} - 1) dz \\ &= \varphi'(z_I) \int_0^{T^{-1}(\zeta)} (e^{g(T(w))-g(T(0))} - 1) T'(w) dw.\end{aligned}$$

The inequality $|e^z - 1| \leq e^{|z|} - 1$ is trivial by Taylor series. With (2.3) and the hypothesis $\|g\|_{\mathcal{B}} \leq \beta$, it yields

$$\begin{aligned}
& \left| \varphi(\zeta) - \varphi(z_I) - \varphi'(z_I)(\zeta - z_I) \right| \\
&= \varphi'(z_I) \int_0^{T^{-1}(\zeta)} (e^{g(T(w)) - g(T(0))} - 1) T'(w) dw \\
&\leq |\varphi'(z_I)| |I| \int_0^{T^{-1}(\zeta)} \left(\left(\frac{1 + |w|}{1 - |w|} \right)^{\frac{\beta}{2}} - 1 \right) d|w| \\
&\leq 2|\varphi'(z_I)| |I| \int_0^1 t^{-\beta/2} dx \\
&\leq \frac{2\beta}{2 - \beta} |\varphi'(z_I)| |I|.
\end{aligned}$$

which is (2.2).

Since (1.6) and (2.1) give

$$|\varphi'(z_I)| \leq e^{|g(0)|} \left(\frac{2}{1 - |z_I|} \right)^{\beta/2},$$

Theorem II.3.2 shows that φ extends continuously to $\overline{\mathbb{D}}$, $\varphi \in C^{1-\beta/2}(\overline{\mathbb{D}})$, and

$$|\varphi(\zeta_1) - \varphi(\zeta_2)| \leq \frac{C}{2 - \beta} |z_1 - z_2|^{1-\beta/2}. \quad \square$$

Finish proof of Theorem 2.1:

Let $\zeta_1, \zeta_2 \in \partial\mathbb{D}$ and let I be the shorter arc of ∂D having endpoints ζ_1 and ζ_2 . Then by (2.2) and triangle inequalities,

$$\begin{aligned} |\varphi(\zeta_1) - \varphi(\zeta_2)| &\geq |\varphi'(z_I)| |\zeta_1 - \zeta_2| - \frac{4\beta}{2-\beta} |\varphi'(z_I)| |I| \\ &\geq \left(1 - \frac{2\pi\beta}{2-\beta}\right) |\varphi'(z_I)| |\zeta_1 - \zeta_2|, \end{aligned} \tag{2.4}$$

because $|I| \leq \frac{\pi}{2} |\zeta_1 - \zeta_2|$. Thus when $\frac{2\pi\beta}{2-\beta} < 1$, φ is a one-to-one map from $\partial\mathbb{D}$ onto a Jordan curve, and it then follows by the argument principle that φ is univalent on \mathbb{D} . \square

A Jordan curve Γ is a **quasicircle** if there is a constant A such that

$$\frac{|w_1 - w| + |w - w_2|}{|w_1 - w_2|} \leq A, \quad (2.5)$$

whenever $w \in \Gamma$ lies on that subarc of Γ having endpoints w_1 and w_2 and smaller diameter. We call (2.5) the Ahlfors condition.

Any C^1 curve or Lipschitz curve is a quasicircle and the von Koch snowflake is a non-rectifiable quasicircle. A **quasidisk** is a domain bounded by a quasicircle.

Theorem 2.3: *There is $\beta_0 < 1$ such that if $\|g\|_{\mathcal{B}} < \beta < \beta_0$, then $\varphi(\partial\mathbb{D})$ is a quasicircle with constant*

$$A \leq A(\beta) \leq \sqrt{2}(1 + C_1\beta). \quad (2.6)$$

Proof: Take $w_j = \varphi(\zeta_j)$ and let $w = \varphi(\zeta)$ lie on the smaller diameter subarc $\varphi(I)$ of $\partial\mathbb{D}$ with endpoints ζ_1 and ζ_2 . Replacing φ by $\varphi \circ T$ for some $T \in \mathcal{M}$ if necessary, we can assume that $|I| \leq \pi$. Then by (2.2),

$$|\varphi(\zeta_j) - \varphi(\zeta) - (\zeta_j - \zeta)\varphi'(z_I)| \leq \frac{4\beta}{2 - \beta} |\varphi'(z_I)| |I|,$$

so that

$$\begin{aligned} & |w_1 - w| + |w - w_2| \\ & \leq |\varphi'(z_I)| (|\zeta_1 - \zeta| + |\zeta - \zeta_2|) + \frac{8\beta}{2 - \beta} |\varphi'(z_I)| |I| \\ & \leq \left(1 + \frac{4\pi\beta}{2 - \beta}\right) |\varphi'(z_I)| (|\zeta_1 - \zeta| + |\zeta - \zeta_2|). \end{aligned}$$

Then by (2.4) we obtain (2.5) with the estimate

$$A \leq A(\beta) = \left(\frac{2 + (4\pi - 1)\beta}{2 - (2\pi + 1)\beta} \right) \sqrt{2}.$$

Therefore $\varphi(\partial\mathbb{D})$ is a quasicircle and (2.6) holds provided $\beta < \frac{2}{(2\pi+1)}$ \square .

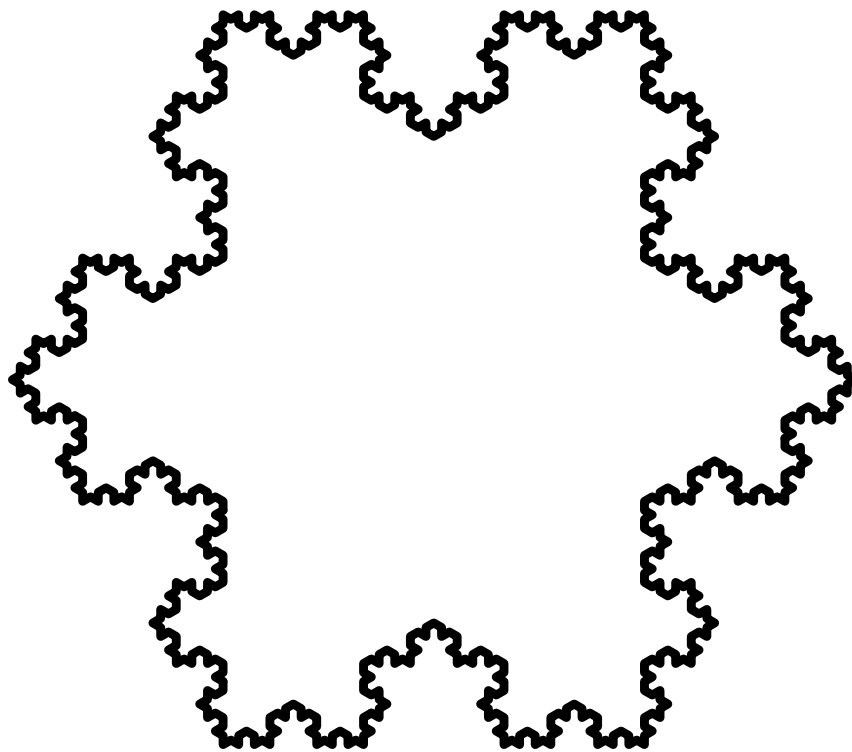
Example 2.5: *There exists a Jordan domain Ω such that the conformal mapping $\varphi : \mathbb{D} \rightarrow \Omega$ has a non-zero angular derivative at no point.*

Proof 1: Let

$$\log(\varphi') = \epsilon \sum z^{2^n}.$$

If ϵ is small, then φ maps \mathbb{D} to a quasidisk; and by the remark following Theorem 1.3, φ has an angular derivative at no $\zeta \in \partial\mathbb{D}$. \square

Proof 2: Let Ω be the interior of the von Koch snowflake. The conformal map from \mathbb{D} to the interior domain Ω has a non-zero angular derivative at no point of $\partial\Omega$. Indeed if there is an inner tangent at ζ , then ζ belongs to Γ_n for some n .



If ζ is a vertex of Γ_n , then Ω either contains a truncated cone with opening $4\pi/3$ or contains no truncated cone with opening greater than $\pi/3$ at ζ .

If ζ is not a vertex then Ω contains the union of a half disc and an equilateral triangle where the disc, centered at ζ , can be arbitrarily small and the size of the equilateral triangle is comparable to the diameter of the disc. See Figure VII.5.

By the easy half of Ostrowski's Theorem V.5.5, Ω does not have a nonzero angular derivative at ζ . In fact the conformal map of the unit disc onto Ω is not conformal at any point of $\partial\mathbb{D}$. \square



Neils Fabian Helge von Koch (1870-1924)

Section VII.3: Quasicircles

Let Ω and Ω' be domains in the extended plane \mathbb{C}^* , let $f : \Omega \rightarrow \Omega'$ be an orientation preserving homeomorphism, and let $K \geq 1$. Then we say f is a **K -quasiconformal mapping** if

(a) f is absolutely continuous on almost every horizontal or vertical line segment in Ω , and

(b) the (area almost everywhere defined) derivatives

$$f_z = \frac{f_x - if_y}{2} \text{ and } f_{\bar{z}} = \frac{f_x + if_y}{2}$$

satisfy

$$|f_{\bar{z}}| \leq \frac{K - 1}{K + 1} |f_z| \tag{3.1}$$

almost everywhere on Ω .

When (a) holds we say f is *ACL*, for **absolutely continuous on lines**.

The smallest K for which (3.1) holds is called the **dilatation** of f on Ω .

Let $f : \Omega \rightarrow \Omega'$ be K -quasiconformal, let $z \in \Omega \setminus \{\infty, f^{-1}(\infty)\}$, and suppose $B(z, r) \subset \Omega$. Define

$$m(z, r) = \inf_{|w-z|=r} |f(w) - f(z)|,$$

$$M(z, r) = \sup_{|w-z|=r} |f(w) - f(z)|,$$

$$H(z) = \limsup_{r \rightarrow 0} \frac{M(z, r)}{m(z, r)}.$$

When f is *ACL* it follows from (3.1) that

$$H(z) \leq K \tag{3.2}$$

almost everywhere on Ω . Conversely, f is K -quasiconformal if f is an orientation preserving *ACL* homeomorphism for which (3.2) holds almost everywhere.

The equivalence of (3.1) and (3.2) is easy to prove when f is an orientation preserving diffeomorphism, but the general case requires the differentiation theorem of Gehring and Lehto [1959].

Heinonen and Koskela [1995] derived (3.1) from a weaker form of (3.2).



Juha Heinonen (1960-2007)



Pekka Koskela

If f is a K -quasiconformal, then f^{-1} is also K -quasiconformal, and f is a 1-quasiconformal mapping if and only if f is a conformal mapping.

Let f be a K -quasiconformal on a domain Ω , let Γ be a path family in Ω , and write $f(\Gamma) = \{f(\gamma) : \gamma \in \Gamma\}$. Then (3.1) implies

$$\frac{1}{K}\lambda_{\Omega}(\Gamma) \leq \lambda_{f(\Omega)}(f(\Gamma)) \leq K\lambda_{\Omega}(\Gamma). \quad (3.3)$$

Therefore a K -quasiconformal mapping can only increase or decrease an extremal distance by the factor K ,

$$\frac{1}{K}d_{\Omega}(E, F) \leq d_{f(\Omega)}(f(E), f(F)) \leq Kd_{\Omega}(E, F),$$

whenever f is K -quasiconformal on a neighborhood of $\bar{\Omega}$. It follows that a K -quasiconformal mapping can only increase or decrease the module of a ring domain $W \subset \bar{\Omega}$ by the same factor K ,

$$\frac{1}{K}\text{mod}(f(W)) \leq \text{mod}(W) \leq K\text{mod}(f(W)). \quad (3.4)$$

Conversely, if f is an orientation preserving homeomorphism defined on an open set U and if (3.4) holds for every ring $W \subset U$, then f is K -quasiconformal on U .

It is not hard to prove that an orientation preserving diffeomorphism is quasiconformal if and only if it satisfies (3.4), but in the general case the argument is much deeper.

Proposition 3.1: *If $f : \mathbb{C} \rightarrow \mathbb{C}$ is K -quasiconformal, then for all $z \in \mathbb{C}$ and all $r > 0$,*

$$M(z, r) \leq e^{8K} m(z, r).$$

In other words, if $|z_1 - z| \leq |z_2 - z|$, then

$$|f(z_1) - f(z)| \leq e^{8K} |f(z_2) - f(z)|.$$

Proof is left as an exercise in the textbook.

If $f : \mathbb{C} \rightarrow \mathbb{C}$ is K -quasiconformal and if $f(\partial\mathbb{D}) = \partial\mathbb{D}$, then (3.4) implies there is a constant $M = M(K)$ such that

$$\frac{1}{M} \leq \frac{|f(e^{i(\theta+t)}) - f(e^{i\theta})|}{|f(e^{i\theta}) - f(e^{i(\theta-t)})|} \leq M, \quad (3.5)$$

for all θ and all $t < \pi$. The converse is also true.

Homeomorphisms satisfying (3.5) are called **quasisymmetric**.

Theorem 31.2: *Let f be an orientation preserving homeomorphism of $\partial\mathbb{D}$ to $\partial\mathbb{D}$. Then f is the restriction to $\partial\mathbb{D}$ of a quasiconformal homeomorphism of \mathbb{C} to \mathbb{C} if and only if (3.5) holds. If (3.5) holds with constant M then the extension satisfies (3.1) with $K \leq K(M)$, and if f is K -quasiconformal and $f(\partial\mathbb{D}) = \partial\mathbb{D}$, then (3.5) holds with $M \leq M(K)$.*

Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be K -quasiconformal. It then follows easily from (3.4) that $\Gamma = f(\partial\mathbb{D})$ is a quasicircle with constant $A \leq A(K)$. The converse holds also.

Theorem 3.3: *Let Γ be a Jordan curve in the plane. Then Γ satisfies the Ahlfors condition if and only if*

$$\Gamma = f(\partial\mathbb{D}), \text{ where } f : \mathbb{C} \rightarrow \mathbb{C} \text{ is } K\text{-quasiconformal} . \quad (3.6)$$

If (2.5) holds with constant A , then (3.6) holds with $K \leq K(A)$, and if (3.6) holds with constant K , then (2.5) holds with $A \leq A(K)$.

Proof is left as an exercise.

Corollary 3.4: *Let Ω be a quasidisc and let $\varphi : \mathbb{D} \rightarrow \Omega$ be a conformal mapping. Then φ has an extension to a quasiconformal mapping from \mathbb{C} onto \mathbb{C} .*

Proof: Let f be a quasiconformal mapping with $\Omega = f(\mathbb{D})$ and $f(0) = \varphi(0)$. Set $F = f^{-1} \circ \varphi$. Then F is a homeomorphism of $\overline{\mathbb{D}}$ and F is quasiconformal on \mathbb{D} . Extend F to $\mathbb{C} \setminus \overline{\mathbb{D}}$ by reflection

$$\overline{F(z)} = \frac{1}{F(\frac{1}{\bar{z}})}.$$

Then F is a quasiconformal self map of \mathbb{C} and $f \circ F$ is the desired quasiconformal extension of φ . \square

A positive measure μ on a Jordan curve Γ is a **doubling measure** if there is a constant C such that

$$\mu(I) \leq C\mu(J)$$

whenever I and J are adjacent subarcs of Γ with $\text{diam}(I) \leq \text{diam}(J)$.

Interchanging I and J , we see that μ is doubling if and only if

$$\text{diam}(I) \leq C'\text{diam}(J)$$

whenever I and J are adjacent arcs with $\mu(I) \leq \mu(J)$.

If $f : \partial\mathbb{D} \rightarrow \partial\mathbb{D}$ is a homeomorphism, then f satisfies the Beurling-Ahlfors condition (3.5) (= quasisymmetric) iff $\mu(E) = |f(E)|$ is a doubling measure on $\partial\mathbb{D}$.

Kahane gave (Example 2.6) a doubling measure on \mathbb{R} singular to Lebesgue measure.

Theorem 3.5 (Jerison-Kenig 1982): *Suppose Γ is a Jordan curve in the plane and suppose Ω_1 and Ω_2 are the two components of the complement $\mathbb{C}^* \setminus \Gamma$. Let $z_j \in \Omega_j$ and let $\omega_j(E) = \omega(z_j, E, \Omega_j)$. Then Γ is a quasicircle if and only if both ω_1 and ω_2 are doubling measures on Γ .*



David Jerison



Carlos Kenig (1953-)

Proof (doubling \Rightarrow quasicircle): Assume ω_1 and ω_2 are doubling measures and assume $z_2 = \infty \in \Omega_2$. Let $\varphi_1 : \mathbb{D} \rightarrow \Omega_1$ and $\varphi_2 : \mathbb{C}^* \setminus \overline{\mathbb{D}} \rightarrow \Omega_2$ be conformal mappings such that $\varphi_2(\infty) = \infty$ and $\varphi_1(0) = z_1$, and define the **welding map** $h : \partial\mathbb{D} \rightarrow \partial\mathbb{D}$ by

$$h = \varphi_2^{-1} \circ \varphi_1.$$

Let I and J be adjacent arcs on $\partial\mathbb{D}$ with $|I| = |J| \leq \pi$. Because ω_1 is doubling,

$$\text{diam}(\varphi_1(J)) \leq C_1 \text{diam}(\varphi_1(I)).$$

Then because ω_2 is doubling,

$$\frac{|\varphi_2^{-1}(\varphi_1(J))|}{2\pi} = \omega_2(\varphi_1(J)) \leq C_2 C_1 \omega_2(\varphi_1(I)) = \frac{|\varphi_2^{-1}(\varphi_1(I))|}{2\pi}.$$

Consequently

$$\frac{|h(J)|}{|h(I)|} \leq C_1 C_2,$$

and after interchanging I and J we see that h is quasisymmetric (satisfies (3.5)).

Now let $H : \mathbb{C} \rightarrow \mathbb{C}$ denote the quasiconformal extension of h given by Corollary 3.4 and define

$$\Phi_1(z) = \begin{cases} \varphi_1(z), & \text{if } |z| \leq 1, \\ \varphi_2(H(z)), & \text{if } |z| > 1. \end{cases}$$

Then Φ_1 is a quasiconformal map such that $\Phi_1(\partial\mathbb{D}) = \Gamma$, and Γ is a quasicircle.

□

Proof (quasicircle \Rightarrow doubling): Assume Γ is a quasicircle and let $\varphi : \mathbb{D} \rightarrow \Omega_1$. Let I_1 and I_2 be adjacent arcs of Γ with $\text{diam}(I_1) = \text{diam}(I_2) \leq \text{diam}(\Gamma)/4$. Let I_3 be an arc adjacent to I_2 with $\text{diam}(I_3) = \text{diam}(I_2)$ and $I_3 \cap I_1 = \emptyset$, and let w_j, w_{j+1} be the endpoints of $I_j, j = 1, 2, 3$.

Then by the Ahlfors condition (2.5),

$$\text{dist}(I_1, I_3) \leq |w_2 - w_3| \leq A \text{diam}(I_2) = A \text{diam}(I_1) = A \text{diam}(I_3),$$

so that by Exercise 6(b) of Chapter IV and a rescaling, The extremal distance between I_1 and I_3 is bounded:

$$d_{\mathbb{C}}(I_1, I_3) \leq A'.$$

Set $J_k = \varphi^{-1}(I_k)$. We may suppose that $\omega(0, \cup_k J_k, \mathbb{D}) \leq 1/2$. Then by Corollary 3.4 and (3.4), $d_{\mathbb{C}}(J_1, J_3) \leq KA'$.

Hence $\text{diam}(J_2) \leq C \text{diam}(J_1)$, for otherwise an annulus of large modulus separates J_1 and J_3 . Thus ω_1 is doubling. The proof for ω_2 is of course the same. \square

The proof of doubling by Jerison and Kenig is a different argument that applies to higher dimensional generalizations of quasidisks known as **nontangentially accessible domains**.

A one-sided version of the definition of quasicircle yields domains called John domains. Let Ω be a simply connected domain such that $\infty \notin \partial\Omega$. Then Ω is called a **John domain** if there is $z_0 \in \Omega$ and $c > 0$ such that for every $z_1 \in \Omega$ there is an arc $\sigma \in \Omega$ joining z_0 to z_1 such that

$$\text{dist}(z, \partial\Omega) \geq c|z - z_1|$$

See Exercise VII.13 for several properties of John Domains.



Fritz John (1910-1994)

Section VII.4: Chord-Arc Curves and the A^∞ Condition

A compact set K is called Ahlfors regular (or Ahlfors-David regular) if there is a constant A such that

$$\Lambda_1(K \cap B(z, r)) \leq Ar$$

for all $z \in K$. A Jordan curve Γ is called a **chord-arc curve** or a Lavrentiev curve if Γ is an Ahlfors regular quasicircle.

Thus Γ is a chord-arc curve if and only if Γ is rectifiable and there is a constant A' such that for all $w_1, w_2 \in \Gamma$,

$$\frac{\ell(w_1, w_2)}{|w_1 - w_2|} \leq A', \tag{4.1}$$

where $\ell(w_1, w_2)$ is the length of the shorter arc of Γ joining w_1 to w_2 .

A Jordan curve Γ is called a **Lipschitz curve** if

$$\Gamma = \{e^{F(\theta)+i\theta} : 0 \leq \theta \leq 2\pi\},$$

where $F(\theta + 2\pi) = F(\theta)$ and $|F(\theta + t) - F(\theta)| \leq A|t|$ for all t .

The smallest constant A is called the **Lipschitz constant** for Γ . It is clear that every Lipschitz curve is a chord-arc curve.

Let μ and ν be continuous positive measures on a curve Γ .

We say μ and ν are **A^∞ -equivalent** if there exists $\epsilon < 1$ and $\eta < 1$ such that whenever I is a subarc of Γ and $E \subset I$,

$$\frac{\mu(E)}{\mu(I)} \leq \epsilon \implies \frac{\nu(E)}{\nu(I)} \leq \eta. \quad (4.2)$$

To see that (4.2) defines an equivalence relation, simply replace E by $I \setminus E$. It is not hard to show that $\nu \ll \mu \ll \nu$ if μ and ν are A^∞ -equivalent.

If Γ is a chord-arc curve and if μ is to arc length on Γ , then μ is a doubling measure. However, not every doubling measure on the unit circle is A^∞ -equivalent to arc length.

Example 2.6 exhibits a doubling measure singular to Lebesgue measure.

A stopping time argument given in Exercise VII.14 shows that A^∞ is equivalent to this stronger looking condition: There exist $\alpha > 0$ and $C > 0$ such that for all arcs $I \subset \Gamma$ and all $E \subset I$,

$$C^{-1} \left(\frac{\nu(E)}{\nu(I)} \right)^{\frac{1}{\alpha}} \leq \frac{\mu(E)}{\mu(I)} \leq C \left(\frac{\nu(E)}{\nu(I)} \right)^\alpha.$$

When $\mu = k(\theta)d\theta$ and $\nu = d\theta$ are A^∞ -equivalent, we say k is an **A^∞ -weight** and write $k(\theta) \in A^\infty$.

Now let Γ be a rectifiable curve and let $\mu = k(z)ds$ be a positive finite measure on Γ . We say $k(z)$ satisfies a **reverse Hölder condition** if there exist $\delta > 0$ and $C > 0$ such that whenever $I \subset \Gamma$ is a subarc having length $\ell(I)$,

$$\left(\frac{1}{\ell(I)} \int_I k(z)^{1+\delta} ds \right)^{\frac{1}{1+\delta}} \leq C \frac{1}{\ell(I)} \int_I k(z) ds.$$

When $p < \infty$ we say $k(z)$ is an **A^p -weight** if there is $C > 0$ such that for all arcs $I \subset \Gamma$,

$$\frac{1}{\ell(I)} \int_I k(z) ds \left(\frac{1}{\ell(I)} \int_I \left(\frac{1}{k(z)} \right)^{\frac{1}{p-1}} ds \right)^{p-1} \leq C. \quad (4.3)$$

The A^p -condition (4.3) is important because of the famous theorems of Muckenhoupt and of Hunt-Muckenhoupt-Wheeden telling us that when μ is a measure on the unit circle and $1 < p < \infty$, the Hardy-Littlewood maximal operator $f \rightarrow Mf$ or the conjugation operator $f \rightarrow \tilde{f}$ is bounded on $L^p(\mu)$ if and only if $\mu = kdx$ where k is an A^p -weight.

Proposition 4.1: *Let Γ be a rectifiable curve and let $\mu = k(z)ds$ be a finite positive measure on Γ . Then the following conditions are equivalent:*

- (a) μ and arc length measure ds are A^∞ -equivalent,*
- (b) $k(z)$ satisfies a reverse Hölder condition, and*
- (c) There is p , $1 < p < \infty$, such that $k(z)$ is an A^p -weight.*

There exist $\delta(\epsilon, \eta)$, $p(\epsilon, \eta)$, and $C = C(\epsilon, \eta)$ such that when (a) holds with constants ϵ and η , (b) holds for C and all $\delta < \delta(\epsilon, \eta)$ and (c) holds for C and all $p \geq p(\epsilon, \eta)$.

The textbook gives citations for the proof.

Theorem 4.3 will show that every chord-arc curve satisfies the conditions of Proposition 4.1. The easier Theorem 4.2 shows that Lipschitz curves satisfy a strong form of condition (c).

Theorem 4.2: *Let Ω be a domain bounded by a Lipschitz curve Γ (as above), let $\varphi : \mathbb{D} \rightarrow \Omega$ be a conformal mapping, and let $\omega(E) = \omega(0, E, \Omega)$. Then there is a $p < 2$, depending only on the Lipschitz constant of Γ , such that $|\varphi'|$ and $1/|\varphi'|$ are A^p -weights on $\partial\mathbb{D}$ and $k = \frac{d\omega}{ds}$ is an $A^{\frac{1}{2-p}}$ -weight on Γ .*

Proof: Let $\varphi : \mathbb{D} \rightarrow \Omega$ with $\varphi(0) = 0$. Then almost everywhere the magnitude of the angle between the tangent vector to Γ and the radius vector is

$$\left| \arg \left(\frac{e^{i\theta} \varphi'(e^{i\theta})}{\varphi(e^{i\theta})} \right) \right| \leq |\arg(1 + iA)| = \alpha < \frac{\pi}{2},$$

and so by Zygmund's theorem, Theorem II.3.1,

$$\begin{aligned} & \sup_{a \in \mathbb{D}} \left(\int \left| \frac{z\varphi'}{\varphi} \right|^\lambda P_a d\theta \right)^{\frac{1}{\lambda}} \left(\int \left| \frac{\varphi}{z\varphi'} \right| P_a d\theta \right) \\ &= \sup_{T \in \mathcal{M}} \left(\int \left| \frac{z\varphi'}{\varphi} \circ T \right|^\lambda dt \right)^{\frac{1}{\lambda}} \left(\int \left| \frac{\varphi}{z\varphi'} \circ T \right| dt \right) < \infty, \end{aligned}$$

for all $\lambda < \pi/(2\alpha)$.

If I is an interval in $\partial\mathbb{D}$, take $a \in \mathbb{D}$ such that $a/|a|$ is the center of I and $1 - |a| = |I|/2$. Then $P_a \geq C/|I|$ on I and since $|\varphi(z)/z|$ is bounded above and below on $\partial\mathbb{D}$,

$$\sup_I \left(\frac{1}{|I|} \int_I |\varphi'|^\lambda dt \right)^{\frac{1}{\lambda}} \left(\frac{1}{|I|} \int_I \left| \frac{1}{\varphi'} \right| dt \right) < \infty.$$

This shows that $|1/\varphi'| \in A^p$ with $p = 1 + 1/\lambda > 1 + 2\alpha/\pi$ and a similar argument shows that $|\varphi'| \in A^p$.

Because $k = 1/|\varphi' \circ \varphi^{-1}|$, we obtain for $J = \varphi(I)$

$$\begin{aligned}
& \left(\frac{1}{\ell(J)} \int_J k ds \right) \left(\frac{1}{\ell(J)} \int_J \left(\frac{1}{k} \right)^{\frac{1}{q-1}} ds \right)^{q-1} \\
&= \frac{\left(\frac{1}{|I|} \int_I |\varphi'|^{\frac{q}{q-1}} d\theta \right)^{q-1}}{\left(\frac{1}{|I|} \int_I |\varphi'| d\theta \right)^q} \\
&\leq \left[\left(\frac{1}{|I|} \int_I \frac{1}{|\varphi'|} d\theta \right) \left(\frac{1}{|I|} \int_I |\varphi'|^{\frac{q}{q-1}} d\theta \right)^{\frac{q}{q-1}} \right]^q < \infty,
\end{aligned}$$

and $k \in A^q$ for $q > \pi/(\pi - 2\alpha)$. \square

Theorem 4.3 (Lavrentiev): *Let Ω be the bounded domain bounded by a chord-arc curve Γ , let $z_0 \in \Omega$, and let $\omega(E) = \omega(z_0, E, \Omega)$.*

(a) Then ω and arc length on Γ are A^∞ -equivalent, with constants depending only on the chord-arc constant of Γ and $\text{dist}(z_0, \partial\Omega)/\text{diam}(\Omega)$.

(b) Moreover, if $\varphi : \mathbb{D} \rightarrow \Omega$ is a conformal mapping with $\varphi(0) = z_0$, then arc length and harmonic measure for z_0 on the curves $\varphi(\{|z| = r\})$, $0 < r < 1$ are A^∞ -equivalent with constants independent of r .

Proof: To prove (a), let I be a small arc on $\partial\mathbb{D}$, form the Carleson box

$$Q = \{z = t\zeta : 1 - |I| < t < 1; \zeta \in I\},$$

and let z_Q be its center. Given that φ has a quasiconformal extension to \mathbb{C} ,

Proposition 3.1 tells us there is $r > 0$ such that

$$B(\varphi(z_Q), r) \subset \varphi(Q) \subset B(\varphi(z_Q), Cr), \quad (4.4)$$

where C depends only on the quasiconformality of the extension of φ . Since Γ is a chord-arc curve,

$$\ell(\varphi(I)) \leq A \text{diam}(\varphi(I)) \leq A'r.$$

Let $J = (1 - |I|)I$ be the top edge of Q .

By the Koebe estimates (I.4.13),

$$\ell(\varphi(J)) = \int_J |\varphi'(z)| ds \leq 4 \text{dist}(\varphi(J), \Gamma) \leq A'r.$$

Now let $L = (\zeta, \zeta_0)$, $\zeta_0 = (1 - |I|)\zeta$, be one of the edges in $\underline{Q} \setminus (I \cup J)$ of Q . Let $c = e^{8K}$, where K is the dilatation of the quasiconformal extension of φ from \mathbb{C} to \mathbb{C} , and define $\zeta_k \in L$ to be the point of smallest modulus such that

$$|\varphi(\zeta_k) - \varphi(\zeta)| = c^{-k} |\varphi(\zeta_0) - \varphi(\zeta)|.$$

If $2^N \geq c$, then

$$|\varphi(\zeta_k) - \varphi(\zeta)| \leq 2^N |\varphi(\zeta_{k+1}) - \varphi(\zeta)|,$$

and because φ^{-1} also has a K -quasiconformal extension, Proposition 3.1 and induction yield

$$|\zeta_k - \zeta| \leq (2c)^N |\zeta_{k+1} - \zeta|. \tag{4.5}$$

Write $L_k = [\zeta_{k+1}, \zeta_k]$. Then by Proposition 3.1 and the definition of ζ_{k+1} ,

$$\text{dist}(\varphi(L_k), \Gamma) \leq c|\varphi(\zeta_k) - \varphi(\zeta)| = c^{-k+1}|\varphi(\zeta_0) - \varphi(\zeta)|,$$

so that by the Koebe estimates

$$\ell(\varphi(L_k)) \leq 4c^{-k+1}|\varphi(\zeta_0) - \varphi(\zeta)| \log\left(\frac{|\zeta_k - \zeta|}{|\zeta_{k+1} - \zeta|}\right).$$

With (4.5) we then obtain

$$\ell(\varphi(L)) \leq |\varphi(\zeta_0) - \varphi(\zeta)| \sum_k 4c^{-k+1} N \log(2c) \leq Cr.$$

Thus we have established

$$\ell(\partial\varphi(Q)) \leq C'r. \tag{4.6}$$

With (4.6) and (4.4) we can apply (VI.5.1) to the domain $\varphi(Q)$ and the point $\varphi(z_Q)$, and the bounds will not depend on Q . Let $E \subset \varphi(I)$. Given $\epsilon > 0$ there exists $\delta > 0$ such that if $\frac{\ell(E)}{\ell(\varphi(I))} < \delta$ then by (VI.5.1)

$$\frac{\omega(z_Q, \varphi^{-1}(E), Q)}{\omega(z_Q, I, Q)} = \frac{\omega(\varphi(z_Q), E, \varphi(Q))}{\omega(\varphi(z_Q), \varphi(I), \varphi(Q))} < \epsilon. \quad (4.7)$$

But if (4.7) holds with suitably small ϵ , then

$$\frac{\omega(z_0, E, \Omega)}{\omega(z_0, \varphi(I), \Omega)} = \frac{|\varphi^{-1}(E)|}{|I|} < 1/2,$$

and we conclude that ω and ds are A^∞ -equivalent on $\partial\Omega$.

Proof of (b): By Lemma 4.4 below, the function

$$I(z) = \log \int_{\alpha}^{\beta} |\varphi'(ze^{i\theta})| d\theta + \log \frac{|z|/A'}{|\varphi(ze^{i\beta}) - \varphi(ze^{i\alpha})|}$$

is subharmonic on \mathbb{D} for $A' > 0$.

Because Γ is chord-arc, we can choose A' so that $I \leq 0$ on $\partial\mathbb{D}$.

Then by the maximum principle, $I(r) \leq 0$, and therefore $\varphi(\{|z| = r\})$ is a chord-arc curve with a constant independent of r .

Lemma 4.4: *If $f(z)$ is analytic and nowhere zero on \mathbb{D} and if $0 \leq \alpha < \beta < 2\pi$ then*

$$J(z) = \log \int_{\alpha}^{\beta} |f(ze^{i\theta})| d\theta$$

is subharmonic on \mathbb{D} .

Proof: Write $f = g^2$. Uniformly on compact sets, the function $J(z)$ is a limit of functions of the form

$$K(z) = \log \sum_{j=1}^n |g_j|^2,$$

and a calculation with Cauchy–Schwarz shows that

$$\frac{\partial^2}{\partial \bar{z} \partial z} K(z) \geq 0.$$

Corollary 4.5: *Let Ω be a bounded domain bounded by a chord-arc curve and let $\varphi : \mathbb{D} \rightarrow \Omega$ be a conformal map. Then*

(a) Ω is a Smirnov domain, and

(b) There is $q > 0$ such that $\frac{1}{\varphi'} \in H^q$.

The index q in (b) depends only on the chord-arc constant of $\partial\Omega$ and on the dilatation of the quasiconformal extension of φ .

Proof of (b): By Theorem 4.3, the measures $d\theta$ and $|\varphi'(re^{i\theta})|d\theta$ are A^∞ -equivalent, with constants independent of r . Thus by part (c) of Proposition 4.1, there is $q = \frac{1}{p-1}$, depending only on the constants of $\partial\Omega$, such that

$$\sup_r \int \left| \frac{1}{\varphi'(re^{i\theta})} \right|^q d\theta < \infty$$

and (b) holds. \square

Proof of (a): Recall that Ω is a Smirnov domain if and only if

$$\log |\varphi'(0)| = \int_{\partial\mathbb{D}} \log |\varphi'(e^{i\theta})| d\theta.$$

Because $\varphi' \in H^1$, we have

$$\log |\varphi'(0)| \leq \int_{\partial\mathbb{D}} \log |\varphi'(e^{i\theta})| d\theta.$$

But $(\varphi')^{-q} \in H^1$ by (b), so that we also have

$$\log |\varphi'(0)| \geq \int_{\partial\mathbb{D}} \log |\varphi'(e^{i\theta})| d\theta.$$

Therefore Ω is a Smirnov domain and (a) holds. \square

Theorem 4.6: *Let Γ be a Jordan curve and let Ω be the bounded component of $\mathbb{C} \setminus \Gamma$. Then Γ is a chord-arc curve if and only if the following three conditions all hold:*

(i) Γ is a quasicircle,

(ii) Arclength on Γ is A^∞ -equivalent to harmonic measure for any $z \in \Omega$,

(iii) Ω is a Smirnov domain.

Proof: We have already seen that (i), (ii), and (iii) hold when Γ is a chord-arc curve. Now assume (i), (ii), and (iii) hold.

Let $\varphi : \mathbb{D} \rightarrow \Omega$, let $I \subset \partial\mathbb{D}$ be an arc such that $\text{diam}(\varphi(I)) \leq \text{diam}(\Gamma)/2$, let

$$Q = \{z = t\zeta : 1 - |I| < t < 1; \zeta \in I\},$$

and let z_Q be its center. Because ∂Q is a quasicircle and φ has a quasiconformal extension, there exists $r > 0$ and $C > 0$ such that

$$B(\varphi(z_Q), r) \subset \varphi(Q) \subset B(\varphi(z_Q), Cr)$$

and the endpoints w_1 and w_2 of $\varphi(I)$ satisfy

$$|w_1 - w_2| \geq \frac{r}{C}.$$

By (ii) and Proposition 4.1, $|\varphi'|$ satisfies the A^p -condition

$$\frac{1}{|I|} \int_I |\varphi'| d\theta \leq C \left(\frac{1}{|I|} \int_I \left| \frac{1}{\varphi'} \right|^{\frac{1}{p-1}} d\theta \right)^{1-p}$$

for some $p > 1$. But by Jensen's inequality,

$$\left(\frac{1}{|I|} \int_I \left| \frac{1}{\varphi'} \right|^{\frac{1}{p-1}} d\theta \right)^{1-p} \leq \exp \left(\frac{1}{|I|} \int_I \log |\varphi'| d\theta \right)$$

so that

$$\begin{aligned} \frac{1}{|I|} \int_I |\varphi'| d\theta &\leq C \exp \left(\frac{1}{|I|} \int_I \log |\varphi'| d\theta \right) \\ &\leq C \exp \left(\int_{\partial\mathbb{D}} \log |\varphi'(e^{i\theta})| P_{z_Q}(\theta) d\theta \right). \end{aligned} \tag{4.8}$$

But by (iii) and (2.12),

$$\log |\varphi'(z_Q)| = \int_{\partial\mathbb{D}} \log |\varphi'(e^{i\theta})| P_{z_Q}(\theta) d\theta,$$

so that

$$\frac{1}{|I|} \int_I |\varphi'| d\theta \leq C |\varphi'(z_Q)| \leq \frac{2Cr}{|I|},$$

by Koebe's estimate. Therefore

$$\ell(\varphi(I)) = \int_I |\varphi'| d\theta \leq 2C^2 |w_1 - w_2|$$

and Γ is a chord-arc curve. \square

Theorem 4.7: *Let Γ be a quasicircle, let Ω be the bounded component of $\mathbb{C} \setminus \Gamma$, and let $\varphi : \mathbb{D} \rightarrow \Omega$ be a conformal mapping. Then Γ is a chord-arc curve if and only if arc length and harmonic measure for $\varphi(0)$ are A^∞ -equivalent on $\varphi(\{|z| = r\})$, $0 < r < 1$, with constants that are independent of r .*

Proof: If Γ is a chord-arc curve then by Theorem 4.3 arc length and harmonic measure on $\varphi(\{|z| = r\})$ are A^∞ -equivalent uniformly in r .

Conversely, the A^∞ -equivalence implies conditions (i), (ii), and (iii) of Theorem 4.6 hold on $\varphi(\{|z| = r\})$, uniformly in r .

Consequently the curves $\varphi(\{|z| = r\})$ are chord-arc uniformly in $r < 1$, and Γ is chord-arc. \square

Section VII.5: BMO Domains

Let $f \in L^1(\partial\mathbb{D})$. We say f has bounded mean oscillation, $f \in \text{BMO}$, if

$$\sup_I \frac{1}{|I|} \int_I |f - f_I| d\theta = \|f\|_{\text{BMO}} < \infty, \quad (5.1)$$

where I is a subarc of $\partial\mathbb{D}$ and

$$f_I = \frac{1}{|I|} \int_I f d\theta$$

is the average of f over I .

Condition (5.1) is very strong. The John-Nirenberg theorem says (see Exercise F.2), there are constants C and c , independent of f , such that

$$\sup_I \frac{|\{\theta \in I : |f(e^{i\theta}) - f_I| > \lambda\}|}{|I|} \leq C e^{-c\lambda}.$$

This implies that $f \in L^p(\partial\mathbb{D})$ for all $p < \infty$.

Condition (5.1) also implies that if the Poisson integral $f(z)$ is analytic in \mathbb{D} , then

$$|f'(z)|(1 - |z|^2) \leq C\|f\|_{\text{BMO}}.$$

In other words,

$$\|f\|_{\mathcal{B}} \leq C\|f\|_{\text{BMO}}$$

if $f(z)$ is analytic and $f(e^{i\theta}) \in \text{BMO}$. In this section we describe the simply connected domains Ω for which the Bloch function $g = \log(\varphi'(z)) \in \text{BMO}$.



Louis Nirenberg (1925-2020)

Lemma 5.1: *Let $k(\theta) \geq 0$ be a function on the unit circle $\partial\mathbb{D}$ and write $g = \log k$. Suppose $k(\theta)d\theta$ is A^∞ -equivalent to $d\theta$. Then $g = \log k \in \text{BMO}$.*

Conversely, if $g \in \text{BMO}$ then there is a constant $b > 0$ such that $e^{bg}d\theta$ and $d\theta$ are A^∞ -equivalent.

Proof: Suppose $k(\theta)d\theta$ and $d\theta$ are A^∞ -equivalent. Let $I \subset \partial\mathbb{D}$ be a subarc. By Proposition 4.1, k satisfies the A^p -condition (4.3),

$$\left(\frac{1}{|I|} \int_I \exp(g - g_I) d\theta\right) \left(\frac{1}{|I|} \int_I \exp\left(\frac{g_I - g}{p-1}\right) d\theta\right)^{p-1} \leq C \quad (5.2)$$

By Jensen's inequality

$$\frac{1}{|I|} \int_I \exp(g - g_I) d\theta \geq 1 \text{ and } \left(\frac{1}{|I|} \int_I \exp\left(\frac{g_I - g}{p-1}\right) d\theta\right)^{p-1} \geq 1.$$

Thus (5.2) holds if and only if each factor is bounded and we have

$$\exp\left(\frac{1}{|I|} \int_I (g - g_I) d\theta\right) \leq \frac{1}{|I|} \int_I \exp(g - g_I) d\theta \leq C'$$

and

$$\exp\left(\frac{1}{|I|} \int_I \frac{g_I - g}{p-1} d\theta\right) \leq \left(\frac{1}{|I|} \int_I \exp\left(\frac{g - g_I}{p-1}\right) d\theta\right)^{p-1} \leq C'.$$

Therefore

$$\sup_I \frac{1}{|I|} \int_I |g - g_I| d\theta < \infty$$

and $g \in \text{BMO}$.

Conversely if $g \in \text{BMO}$ then by the John-Nirenberg theorem (Exercise F.2) there is $b > 0$ such that

$$\sup_I \frac{1}{|I|} \int_I e^{b|g-g_I|} d\theta < \infty.$$

Then $e^{bg} \in A^2$ and $e^{bg}d\theta$ and $d\theta$ are A^∞ -equivalent by Proposition 4.1. \square

Now let $\varphi : \mathbb{D} \rightarrow \Omega$ be a conformal mapping and write $g = \log(\varphi')$. We say Ω is a **BMO domain** if $g = \log(\varphi') \in \text{BMO}$.

We say Ω is a **chord-arc domain** if Ω is bounded by a chord-arc curve.

If Ω is a chord-arc domain, then by Lemma 5.1 and Theorem 4.3, $\text{Reg}(e^{i\theta}) = \log |\varphi'(e^{i\theta})| \in \text{BMO}$, so that $g \in \text{BMO}$ by Exercise F.3.

More generally, Zinsmeister [1984] proved that every domain bounded by an Ahlfors regular Jordan curve is a BMO domain. However, not every Jordan BMO domain has an Ahlfors regular boundary. See Exercise VII.17.

Theorem 5.2: *Every chord-arc domain is a BMO domain. Conversely, there is $\beta > 0$ such that if $g \in \text{BMO}$ and $\|g\|_{\text{BMO}} < \beta$, then Ω is bounded by a chord-arc curve.*

Proof: We have just seen that $g = \log(\varphi') \in \text{BMO}$ when Ω is a chord-arc domain.

Conversely, since $\|g\|_{\mathcal{B}} \leq C\|g\|_{\text{BMO}}$, Theorem 2.3 gives us a β_0 such that $\partial\Omega$ is a quasicircle if $\|g\|_{\text{BMO}} < \beta_0$.

By Lemma 5.1 there is β_1 such that if $\|g\|_{\text{BMO}} < \beta_1$, then $|\varphi'(re^{i\theta})| \in A^\infty$, uniformly in $r < 1$.

Thus by Theorem 4.7, Ω is a chord-arc domain if $\|g\|_{\text{BMO}} \leq \min(\beta_0, \beta_1)$. \square

Theorem 5.3: *The following are equivalent.*

(a) Ω is a BMO domain.

(b) *There exist $\delta > 0$ and $C > 0$ such that if $z_0 \in \Omega$ there is a subdomain $\mathcal{U} \subset \Omega$ such that*

(i) $z_0 \in \mathcal{U}$,

(ii) $\partial\mathcal{U}$ is rectifiable and $\ell(\partial\mathcal{U}) \leq C \operatorname{dist}(z_0, \partial\Omega)$, and

(iii) $\omega(z_0, \partial\Omega \cap \partial\mathcal{U}, \mathcal{U}) \geq \delta$.

(c) *There exist $\delta > 0$ and $C > 0$ such that if $z_0 \in \Omega$ there is a subdomain $\mathcal{U} \subset \Omega$ such that*

(i) $z_0 \in \mathcal{U}$ and $\operatorname{dist}(z_0, \partial\Omega) \leq C \operatorname{dist}(z_0, \partial\mathcal{U})$,

(ii) $\partial\mathcal{U}$ is chord-arc with constant $\leq C$ and $\ell(\partial\mathcal{U}) \leq C \operatorname{dist}(z_0, \partial\Omega)$,

(iii) $\ell(\partial\Omega \cap \partial\mathcal{U}) \geq \delta \operatorname{dist}(z_0, \partial\Omega)$.

A map $G : \mathbb{C} \rightarrow \mathbb{C}$ is **bilipschitz** if there is a constant A such that

$$A^{-1}|z_1 - z_2| \leq |G(z_1) - G(z_2)| \leq A|z_1 - z_2|$$

whenever $z_1 \neq z_2$.

Corollary 5.4: *The family of BMO domains is invariant under bilipschitz homeomorphisms of the plane.*

Proof: Condition (c) is invariant under bilipschitz mappings. \square

Corollary 5.5 *Let Γ be a quasicircle, and let Ω_1 and Ω_2 be the components of $C^* \setminus \Gamma$. Then Ω_1 is a BMO domain if and only if Ω_2 is a BMO domain.*

Proof: Let $F : \mathbb{C} \rightarrow \mathbb{C}$ be a quasiconformal map such that $\Omega_1 = F(\mathbb{D})$. The **quasiconformal reflection** $G : \Omega_1 \rightarrow \Omega_2$ is defined by

$$G(z) = F\left(\frac{1}{\overline{F^{-1}(z)}}\right),$$

and Ahlfors [1963] shows that G can be chosen to be bilipschitz. \square

Proof of Theorem 5.3: (a) \implies (b) \implies (c) \implies (a).

(a) \implies (b): By (a) we may assume $\varphi^{-1}(z_0) = 0$. Indeed, if $T \in \mathcal{M}$ has $T(0) = \varphi^{-1}(z_0)$ then $\|\log T'\|_{\text{BMO}} < \text{Const}$, so that by the conformal invariance of BMO (see Exercise F.3),

$$g_T(z) = \log((\varphi \circ T)'(z)) = \log \varphi'(T(z)) + \log T'(z),$$

has $\|g_T\|_{\text{BMO}} \leq \|g\|_{\text{BMO}} + \text{Const}$.

Fix $\alpha > 1$ and recall the nontangential maximal function

$$G(\zeta) = (g - g(0))_{\alpha}^*(\zeta) = \sup_{\Gamma_{\alpha}(\zeta)} |g(z) - g(0)|.$$

Let $\lambda > 0$ be large and form $E = \{\zeta : G(\zeta) \leq \lambda\}$ and $\mathcal{V} = \bigcup_E \Gamma_{\alpha}(\zeta)$. If α is sufficiently large, then $\omega(z, E, \mathbb{D}) \leq 1/2$ on $\partial\mathcal{V} \setminus E$. It follows that $\omega(0, E, \mathcal{V}) \geq \frac{|E|}{\pi} - 1 \geq \delta$ if λ is sufficiently large.

On \mathcal{V} ,

$$|\varphi'(z)| \leq e^{\lambda} |\varphi'(0)| \leq 4e^{\lambda} \text{dist}(z_0, \partial\Omega),$$

and thus (b) holds for $\mathcal{U} = \varphi(\mathcal{V})$

(b) \implies (c):

The textbook says this is another cone domain construction and leaves it as Exercise VII.19.

However, in the paper of Bishop and Jones, that this section is based on, the argument is quite involved.

(c) \implies (a):

Set $\varphi(0) = z_0$. Since \mathcal{U} in (c) has chord-arc boundary, φ has a non-zero angular derivative at almost all points of $\varphi^{-1}(\partial\mathcal{U} \cap \partial\Omega)$ and by Theorem 4.3,

$$\omega(z_0, \partial\mathcal{U} \cap \partial\Omega, \Omega) \geq \omega(z_0, \partial\mathcal{U} \cap \partial\Omega, \mathcal{U}) > \epsilon = \epsilon(C, \delta) > 0.$$

Consequently we can construct a cone domain

$$\mathcal{V} = \bigcup_E \Gamma_\alpha(\zeta) \subset \varphi^{-1}(\mathcal{U})$$

such that $|g(w) - g(0)| \leq M$ on \mathcal{V} (recall $g = \log \varphi'$) and $\omega(0, E, \mathbb{D}) \geq \epsilon$, where $E \subset \partial\mathbb{D}$ is compact.

In particular, $|g(\zeta) - g(0)| \leq M$ on E and $|E| \geq 2\pi\epsilon$.

Let I_1 be a component of $\partial\mathbb{D} \setminus E$ with center c_I and take $w_1 = (1 - \epsilon|I|)c_I$ and $z_1 = \varphi(w_1)$.

Then w_1 has bounded hyperbolic distance to \mathcal{V} , so that $|g(w_1) - g(0)| \leq CM$ since $\|g\|_{\mathcal{B}} \leq 6$. Repeating this construction with z_1 in place of z_0 , we obtain $E_1 \subset I_1$ with $\omega(w_1, E_1, \mathbb{D}) \geq \epsilon$ and $|g(\zeta) - g(w_0)| \leq 2CM$ on E_1 . Since

$$\omega(w_1, I_1 \cap E_1, \mathbb{D}) \geq \epsilon - \omega(w_1, \partial\mathbb{D} \setminus I_1, \mathbb{D}) \geq \epsilon - \frac{2}{\pi}\epsilon \geq \frac{\epsilon}{3},$$

we obtain $|I_1 \cap E_1| \geq c_2\epsilon^2|I_1|$.

We repeat this construction on all complementary intervals of E and of the newly constructed sets $E_j \cap I_j$. After n generations we arrive at a set F_n such that $|g(\zeta) - g(0)| \leq nCM$ on F_n and such that $|\partial\mathbb{D} \setminus F_n| \leq |\partial\mathbb{D} \setminus E| (1 - c_2\epsilon^2)^n$.

Thus

$$\omega(0, \{\zeta : |g(\zeta) - g(0)| > \lambda\}, \mathbb{D}) \leq c_3 e^{-c_4 \lambda},$$

where c_3 and c_4 depend only on the constants in (c).

Applying this argument to $\psi(z) = \varphi\left(\frac{z+w_0}{1+\bar{w}_0 z}\right)$, we obtain

$$\begin{aligned} \omega(w_0, \{\zeta \in \partial\mathbb{D} : |g(\zeta) - g(w_0)| > \lambda \text{ and } |\zeta - w_0| \leq 2(1 - |w_0|)\}, \mathbb{D}) \\ \leq c_5 e^{-c_4 \lambda}. \end{aligned}$$

That means that $g \in \text{BMO}$. \square

