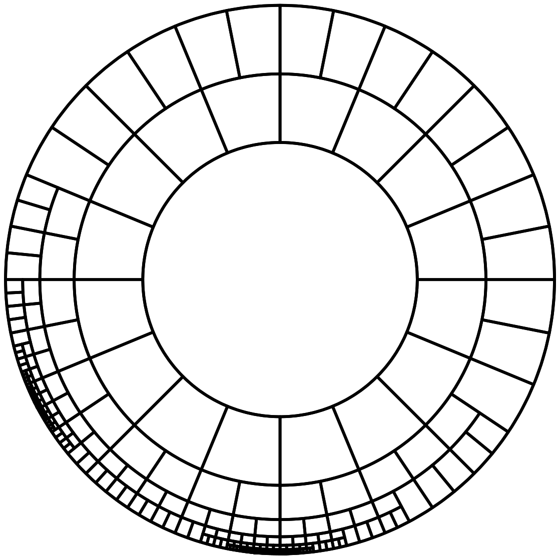
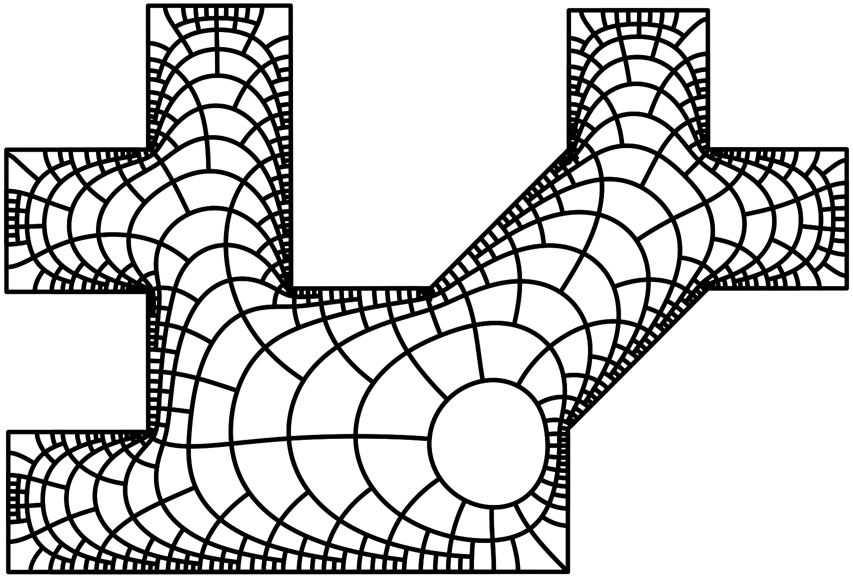
MAT 639, Spring 2026, Stony Brook University

Topics in Real Analysis: Harmonic Measure

Christopher Bishop





Harmonic Measure, Chapter VI

Simply Connected Domains, Part I

following text by John Garnett and Don Marshall

Sections for Chapter VI:

- The F. and M. Riesz Theorem
- Privalov's Theorem and Plessner's Theorem
- Accesible Points
- Cone Points and McMillan's Therorem
- Compression and Expansion (Makarov dim ≤ 1)
- Pommerenke's extension

Appendix A: Hardy Spaces

If $0 < p < \infty$, then by definition an analytic function $f(z)$ on the unit disc $\mathbb{D}$ is in the **Hardy space** H^p if

$$\sup_{0<r<1} \int |f(re^{i\theta})|^p d\theta = ||f||_{H^p}^p < \infty. \tag{A.1}$$

For $p = \infty$ the Hardy space H^∞ is defined to be the space of bounded analytic functions on $\mathbb{D}$ with norm $||f||_\infty = \sup_{\mathbb{D}} |f(z)|$.

Write $f_r(z) = f(rz)$ and suppose $0 < r < s < 1$. Then $||f_r||_\infty \leq ||f_s||_\infty$. If $p < \infty$, then

$$|f_r(z)|^p \leq \int_0^{2\pi} P_{rz/s}|f_s(e^{i\theta})|^p d\theta$$

because the function $|f(z)|^p$ is subharmonic. It then follows from Fubini's theorem that $||f_r||_p$ is an increasing function of r whenever $0 < p \leq \infty$.

Theorem A.1: *Assume* $1 \le p < \infty$ *and let* $f \in H^p$*. Then*

(a) The function $f(z)$ *has a nontangential limit* $f(e^{i\theta})$ *almost everywhere* $d\theta$ *and*

$$\lim_{r \to 1} \int_0^{2\pi} \bigl| f(re^{i\theta}) - f(e^{i\theta}) \bigr|^p d\theta = 0. \tag{A.2}$$

(b) If $f \not\equiv 0$*, then*

$$\int \log \bigl| f(e^{i\theta}) \bigr| d\theta > -\infty. \tag{A.3}$$

The case $p > 1$ of part (a) follows from Exercise I.11 of Chapter I, and the significance of Theorem A.1 lies in the case $p = 1$ and the inequality (A.3).

From part (a) it follows that

$$f(z) = \lim_{r \to 1} f(rz) = \int P_z(\theta) f(e^{i\theta}) d\theta,$$

and since $||f_r||_p$ is increasing in r, we obtain

$$||f||_{H^p}^p = ||f||_{L^p(\partial\mathbb{D})}^p$$

for all $f \in H^p$.

Theorem A.1 is also true for $f \in H^p$ for all $p < 1$. See Exercise A.1.

Part (b) has the following important corollary.

Corollary A.2: *If* $f \in H^1$ *and* $f \not\equiv 0$, *then* $|f(e^{i\theta})| > 0$ *almost everywhere.*

The proof of Theorem A.1 will use Theorem A.3 below, which is called the Hardy-Littlewood maximal theorem for H^p functions:

Theorem A.3: *Let* $\alpha > 1$ *be fixed, let* $0 < p \leq \infty$, *and let* $f \in H^p$. *Then the nontangential maximal function*

$$f_\alpha{}^*(\zeta) = \sup_{\Gamma_\alpha(\zeta)} |f(z)|$$

satisfies

$$||f_\alpha{}^*||_p \leq A_\alpha ||f||_{H^p}, \tag{A.4}$$

where the constant A_α *depends only on* α.

We first derive Theorem A.1 from Theorem A.3 and then return to the proof of Theorem A.3.

Proof of Theorem A.1(a)

First suppose $p = 1$. By (A.1) and the Banach–Alaoglu theorem, there are $r_n \to 1$ so that $f(r_n e^{i\theta})d\theta$ converges weak-star to a finite complex measure μ on $\partial\mathbb{D}$. Then

$$f(z) = \lim_{n\to\infty} f(r_n z) = \lim_{n\to\infty} \int P_z(\theta) f(r_n e^{i\theta}) d\theta = \int P_z(\theta) d\mu(\theta).$$

Write

$$d\mu = f d\theta + d\nu,$$

where $f(e^{i\theta}) \in L^1(d\theta)$ and ν is singular to Lebesgue measure. Then

$$f(z) = \int P_z(\theta) f(e^{i\theta}) d\theta + \int P_z(\theta) d\nu(\theta),$$

and by Fatou's theorem $\int P_z(\theta) f(e^{i\theta}) d\theta$ has nontangential limit $f(e^{i\theta})$ a.e.

We will prove that

$$\lim_{\Gamma_\alpha(\zeta)\ni z\to\zeta}\int P_z(\theta)d\nu = 0 \tag{A.5}$$

almost everywhere. That will mean $f(z)$ has nontangential limit $f(e^{i\theta})$ almost everywhere, and (A.2) will then follow from (A.4) and dominated convergence.

To prove (A.5) we can assume $\nu > 0$.

Fix $\alpha > 1$ and $\epsilon > 0$. Take a compact set $K \subset \partial\mathbb{D}$ so that $|K| < \epsilon$ and $\nu(\partial\mathbb{D} \setminus K) < \epsilon$, and write $\nu_1 = \chi_{_K}\nu$ and $\nu_2 = \nu - \nu_1$.

Then $\lim_{z\to\zeta} \int P_z(\theta) d\nu_1(\theta) = 0$ for all $\zeta \in \partial\mathbb{D} \setminus K$.

The proofs of Lemmas I.2.2 and I.2.4 show that

$$\left|\left\{\zeta \in \partial\mathbb{D} : \sup_{\Gamma_\alpha(\zeta)} \int P_z(\theta) d\nu_2(\theta) > \lambda\right\}\right| \leq \frac{3+6\alpha}{\lambda} \int d\nu_2 \leq \frac{(3+6\alpha)\epsilon}{\lambda}.$$

Taking $\lambda = \epsilon^{1/2}$ then gives

$$\begin{aligned} \left| \left\{ \zeta \in \partial\mathbb{D} : \limsup_{\Gamma_\alpha(\zeta) \ni z \to \zeta} \int P_z(\theta) d\nu(\theta) \geq \epsilon^{1/2} \right\} \right| &\leq |K| + (3 + 6\alpha)\epsilon^{1/2} \\ &\leq \epsilon + (3 + 6\alpha)\epsilon^{1/2}, \end{aligned}$$

so that (A.5) holds almost everywhere.

If $1 < p < \infty$, part (a) is easier.

There are $r_n \to 1$ so that $f_{r_n}(\theta) = f(r_n e^{i\theta})$ converges weakly in L^p to some function $f \in L^p$. Then

$$f(z) = \lim_{n\to\infty} f(r_n z) = \int P_z(\theta) f(e^{i\theta}) d\theta$$

because $P_z \in L^q, q = \frac{p}{p-1}$, and then Fatou's theorem, (A.4), and dominated convergence yield (A.2). $\square$

Proof of Theorem A.1(b):

Part (b) follows from part (a).

We may suppose $f(0) \neq 0$, by dividing f by z^n in case $f(0) = 0$. Then for $r < 1$ the familiar Jensen formula gives

$$\begin{aligned} \log|f(0)| &\leq \int \log|f(re^{i\theta})|\frac{d\theta}{2\pi} \\ &= \int \log^+|f(re^{i\theta})|\frac{d\theta}{2\pi} - \int \log^-|f(re^{i\theta})|\frac{d\theta}{2\pi}, \end{aligned}$$

where $\log^- x = (\log(1/x))^+ = \max(-\log x, 0)$.

Jensen Formula:

$$\log|f(0)| + \sum_{|z_j|<r} \log \frac{r}{|z_j|} = \frac{1}{2\pi} \int \log|f(re^{i\theta}| d\theta$$

where $\{z_j\}$ are zeros of f (with multiplicity).

By dominated convergence and part (a),

$$\lim_{r\to 1}\int \log^+|f(re^{i\theta})|d\theta = \int \log^+|f(e^{i\theta})|d\theta,$$

while by Fatou's lemma,

$$\int \log^-|f(e^{i\theta})|d\theta \le \liminf_{r\to 1}\int \log^-|f(re^{i\theta})|d\theta.$$

Therefore we have the important inequality

$$\log|f(0)| \le \int \log|f(e^{i\theta})|\frac{d\theta}{2\pi}, \tag{A.6}$$

which implies (A.3). ☐

Proof of Theorem A.3.

The proof will be divided into two cases, $p \geq 2$ and $0 < p < 2$.

In the first case, $p \geq 2$, f is the Poisson integral of an L^p function $f(e^{i\theta})$ so that by Lemma I.2.2,

$$f_\alpha{}^*(e^{i\theta}) \leq (1 + 2\alpha) M f(e^{i\theta}), \tag{A.7}$$

where Mf is the Hardy-Littlewood maximal function of $f(e^{i\theta})$.

That means (A.4) is a consequence of the basic inequality

$$||Mf||_p \leq C_p||f||_p, \qquad 1 < p \leq \infty, \tag{A.8}$$

which we will establish in a moment. In (A.8) the constants C_p depend on p, but $C_p \leq 2\sqrt{6}$ for $p \geq 2$.

Except for its range of p, (A.8) is just (A.4) with f_α^* replaced by Mf, and for this reason (A.8) is also known as the Hardy-Littlewood maximal theorem.

The second case, $0 < p < 2$, will be reduced to the first case by exploiting the subharmonicity of $\log\bigl|f(z)\bigr|$.

Case I: $p \geq 2$. We prove (A.8). Let $f \in L^p$. For $\lambda > 0$ set

$$m(\lambda) = \bigl|\{Mf > \lambda\}\bigr|.$$

The function $m(\lambda)$ is called the **distribution function** of Mf.

By Fubini's theorem,

$$\int (Mf)^p d\theta = \int_0^{2\pi} \Bigl\{\int_0^{Mf(e^{i\theta})} p\lambda^{p-1} d\lambda\Bigr\} d\theta = \int_0^{\infty} p\lambda^{p-1} m(\lambda) d\lambda. \tag{A.9}$$

We also have

$$m(\lambda) \leq \frac{3||f||_1}{\lambda}, \tag{A.10}$$

from Lemma I.2.4. Now we use an important trick, due to Marcinkiewicz, to improve inequality (A.10). Set

$$f_1 = f\chi_{\{|f|>\frac{\lambda}{2}\}},$$

and

$$f_2 = f\chi_{\{|f|\leq\frac{\lambda}{2}\}}.$$

Then $Mf \leq M(f_1) + M(f_2)$, while $M(f_2) \leq ||f_2||_\infty \leq \frac{\lambda}{2}$, so that by (A.10),

$$m(\lambda) \leq \left|\left\{M(f_1) > \frac{\lambda}{2}\right\}\right| \leq \frac{6}{\lambda}\int_{\{|f|>\frac{\lambda}{2}\}} |f| d\theta.$$

Then from (A.9) and Fubini's theorem we get

$$\begin{aligned}||Mf||_p^p &\leq 6\int_0^\infty p\lambda^{p-2}\Big(\int_{\{|f|>\frac{\lambda}{2}\}}|f|d\theta\Big)d\lambda \\ &= 6\int\big|f(e^{i\theta})\big|\int_0^{2|f(e^{i\theta})|}p\lambda^{p-2}d\lambda d\theta \\ &= \frac{3p2^p}{p-1}\int|f|^p d\theta.\end{aligned}$$

That proves (A.8), and because $\big(\frac{3p}{p-1}\big)^{\frac{1}{p}}$ is decreasing we also have the bound $C_p \leq 2\sqrt{6}$ for $p \geq 2$.

Together (A.8) and (A.7) prove (A.4) when $p \geq 2$ with $A_\alpha \leq 2\sqrt{6}(1+2\alpha)$.

Case II. $0 < p \leq 2$. We can assume $f \not\equiv 0$. Take $0 < r < 1$ so that f has no zeros on $\{|z| = r\}$ and set

$$U_r(z) = \frac{1}{2\pi} \int P_z(\theta) \log |f(re^{i\theta})| d\theta.$$

Then $\log |f(rz)| \leq U_r(z)$ on $\mathbb{D}$. Since U_r is real and harmonic on $\mathbb{D}$, there is a unique harmonic $\tilde{U}_r$ such that $U_r + i\tilde{U}_r$ is analytic and $\tilde{U}_r(0) = 0$. Write

$$g_r(z) = e^{(p/2)(U_r + i\tilde{U}_r)}.$$

Then $|f(rz)|^p \leq |g_r(z)|^2$ and

$$\int |g_r|^2 d\theta = \int |f(re^{i\theta})|^p d\theta \leq ||f||_p^p. \tag{A.11}$$

Let $r \to 1$, and let g be a weak limit of g_r in L^2. Then $g \in H^2$ because Poisson kernels are in L^2 and $||g||_{H^2}^{2/p} \leq ||f||_{H^p}$ by (A.11). Moreover,

$$|f(z)| = \lim_{r\to 1} |f(rz)| \leq |g(z)|^{2/p}.$$

Therefore $(f_\alpha^*)^p \leq (g_\alpha^*)^2 \in L^1$, and

$$||f_\alpha^*||_p \leq ||g_\alpha^*||_2^{2/p}.$$

Thus (A.4) holds for $0 < p < 2$ with same bound as for $p = 2$. $\square$

Section VI.1: The F. and M. Riesz Theorems

Frigyes Riesz (1880-1956)

Marcel Riesz (1886-1969)

Theorem 1.1 (The F. and M. Riesz Theorem): *Let* Ω *be a domain such that* $\partial\Omega = \Gamma$ *is a Jordan curve and let* φ *be a conformal map from* $\mathbb{D}$ *onto* Ω*. Then the curve* Γ *is rectifiable if and only if* $\varphi' \in H^1$*. If* $\varphi' \in H^1$*, then*

$$||\varphi'||_{H^1} = \ell(\Gamma) = \Lambda_1(\Gamma). \tag{1.1}$$

Proof: By Carathéodory's theorem, φ extends to a homeomorphism from $\overline{\mathbb{D}}$ to $\overline{\Omega}$.

First assume $\varphi' \in H^1$. Let $\{0 = \theta_0 < \theta_1 < . \cdots < \theta_n = 2\pi\}$ be any partition of $[0, 2\pi]$. Then

$$\begin{aligned}
\sum_{j=1}^{n} \left|\varphi(e^{i\theta_j}) - \varphi(e^{i\theta_{j-1}})\right| &= \lim_{r\to 1} \sum_{j=1}^{n} \left|\varphi(re^{i\theta_j}) - \varphi(re^{i\theta_{j-1}})\right| \\
&= \lim_{r\to 1} \sum_{j=1}^{n} \left|\int_{\theta_{j-1}}^{\theta_j} \varphi'(re^{i\theta}) ire^{i\theta} d\theta\right| \\
&\leq ||\varphi'||_{H^1}. \qquad (1.2)
\end{aligned}$$

But $\ell(\Gamma)$ is the supremum, over all partitions, of the left side of (1.2). Therefore Γ is rectifiable and

$$\ell(\Gamma) \leq ||\varphi'||_{H^1}.$$

Conversely, assume Γ is rectifiable. Then given $r < 1$, choose a partition $\{\theta_0 < \theta_1 < . \cdots < \theta_n\}$ of $[0, 2\pi]$, so that

$$\sum_{j=1}^{n} \Big|\varphi(re^{i\theta_j}) - \varphi(re^{i\theta_{j-1}})\Big| \geq \ell(\Gamma_r) - \epsilon,$$

where $\Gamma_r = \varphi(\{|z| = r\})$. Write

$$\psi(z) = \sum_{j=1}^{n} \Big|\varphi(ze^{i\theta_j}) - \varphi(ze^{i\theta_{j-1}})\Big|.$$

Then ψ is subharmonic on $\mathbb{D}$ and by Carathéodory's theorem ψ is continuous on $\overline{\mathbb{D}}$, so that $\sup_{\mathbb{D}} \psi(z) = \sup_\theta \psi(e^{i\theta}) \leq \ell(\Gamma)$. Thus

$$\int |\varphi'(re^{i\theta})| d\theta = \ell(\Gamma_r) \leq \psi(r) + \epsilon \leq \ell(\Gamma) + \epsilon.$$

Therefore $\varphi' \in H^1$ and the equality (1.1) holds. $\square$

Theorem 1.2: *Let* Ω *be a simply connected plane domain such that* $\Gamma = \partial\Omega$ *is a rectifiable Jordan curve and let* $\varphi : \mathbb{D} \to \Omega$ *be conformal. Then* $\varphi' \in L^1(\partial\mathbb{D})$*. For any Borel set* $E \subset \partial\mathbb{D}$*,*

$$\Lambda_1(\varphi(E)) = \int_E |\varphi'| d\theta,$$

and for any Borel set $A \subset \partial\Omega$*,*

$$\omega(A) = 0 \Longleftrightarrow \Lambda_1(A) = 0. \tag{1.4}$$

Proof: Assume that $\Gamma = \partial\Omega$ is a rectifiable Jordan curve and let $A \subset \Gamma$ be a Borel set. Write $A = \varphi(E)$ and $z_0 = \varphi(0)$.

Then by Carathéodory's theorem

$$\omega(z_0, A, \Omega) = \frac{1}{2\pi}|E|.$$

When A is an arc, Theorem A.1 from Appendix A and the proof of (1.1) yield

$$\Lambda_1(A) = \Lambda_1(\varphi(E)) = \lim_{r\to 1}\int_E |\varphi'(re^{i\theta})|d\theta = \int_E |\varphi'(e^{i\theta})|d\theta, \tag{1.3}$$

because $\varphi' \in H^1$, and if (1.3) holds for arcs then (1.3) also holds for all Borel sets $A \subset \partial\Omega$.

Consequently

$$\omega(A) = 0 \Longrightarrow \Lambda_1(A) = 0.$$

Conversely,

$$\Lambda_1(A) = 0 \Longrightarrow \omega(A) = 0,$$

because by Corollary A.2,

$$\Big|\{\theta : |\varphi'(\theta)| = 0\}\Big| = 0.$$

Thus when Γ is rectifiable, harmonic measure for Ω and linear measure on Γ are mutually absolutely continuous,

$$\omega \ll \Lambda_1 \ll \omega. \quad \square$$

Corollary 1.3 *Let* Ω *be a domain such that* $\partial\Omega = \Gamma$ *is a rectifiable Jordan curve and let* φ *be a conformal map from* $\mathbb{D}$ *onto* Ω*. Then* φ *has a non-zero angular derivative at* ζ *and* Γ *has a tangent at* $\varphi(\zeta)$*, both for Lebesgue almost every* $\zeta \in \partial\mathbb{D}$ *and for* Λ_1 *almost every* $\varphi(\zeta) \in \Gamma$

For the proof, see the textbook.

Section VI.2: Privalov's Theorem and Plessner's Theorem

Ivan Ivanovich Privalov (1891-1941)

Abraham Ezechiel Plessner (1900-1961)

Let $\zeta \in \partial\mathbb{D}$, let $\alpha > 1$ and let $0 < h < 1$. The **truncated cone** Γ_α^h is

$$\Gamma_\alpha^h(\zeta) = \{z : |z - \zeta| < \alpha(1 - |z|) < \alpha h\}.$$

Definition V.5.1: We say that $\partial\Omega$ has an **inner tangent with inner normal** $e^{i\theta}$ at $\zeta \in \partial\Omega$ if for every $\beta \in (0, \pi/2)$ there is an $\epsilon = \epsilon(\beta) > 0$ so that

$$\Gamma_{\beta}^{\epsilon}(\zeta, \theta) \subset \Omega. \tag{5.1}$$

When $\theta = \pi/2$ we say $\partial\Omega$ has **vertical inner normal**.

If $\partial\Omega$ has an inner tangent at ζ and if

$$\lim_{\Gamma_{\beta}^{\epsilon}(\zeta) \ni z \to \zeta} f(z) = A,$$

we say the function f has **nontangential limit** A at ζ.

Definition V.5.2: Suppose $F : \Omega \to \Omega'$ is a conformal mapping and suppose $\partial\Omega$ has an inner tangent at $\zeta \in \partial\Omega$. We say F is **conformal at** ζ if F has a non- tangential limit

$$F(\zeta) \equiv \lim_{\Gamma^{\epsilon}_{\beta}(\zeta) \ni z \to \zeta} F(z)$$

and if the limit

$$A_\zeta = \lim_{\Gamma^{\epsilon}_{\beta}(\zeta) \ni z \to \zeta} \arg \frac{F(z) - F(\zeta)}{z - \zeta} \tag{5.2}$$

exists for every $\beta \in (0, \pi/2)$.

Definition V.5.3: Suppose $F : \Omega \to \Omega'$ is a conformal map defined on Ω and suppose $\partial\Omega$ has an inner tangent at $\zeta \in \partial\Omega$. We say F has **angular derivative** $F'(\zeta)$ if for all $\beta \in (0, \pi/2)$ the two limits

$$F(\zeta) \equiv \lim_{\Gamma_\beta^\epsilon(\zeta) \ni z \to \zeta} F(z)$$

and

$$F'(\zeta) \equiv \lim_{\Gamma_\beta^\epsilon(\zeta) \ni z \to \zeta} \frac{F(z) - F(\zeta)}{z - \zeta} \tag{5.3}$$

exist and are finite.

It is clear that if F has a *non-zero* angular derivative at ζ then F is conformal at ζ and $A_\zeta = \arg F'(\zeta)$.

Defn 2.1: *The function u is* **nontangentially bounded** *at ζ if there exist $M < \infty, \alpha > 1,$ and $0 < h < 1$ such that*

$$|u(z)| \leq M \text{ on } \Gamma_\alpha^h(\zeta).$$

Notice that the definition of nontangential boundedness only requires that $u(z)$ be bounded on a single truncated cone, whereas the definition of nontangential convergence requires that $u(z)$ has a limit through every cone $\Gamma_\alpha(\zeta)$.

Theorem 2.2 (Privalov): *Suppose* $E \subset \partial\mathbb{D}$ *and suppose* $u(z)$ *is nontangentially bounded at each* $\zeta \in E$*. Then* $u(z)$ *has a nontangential limit at almost every* $\zeta \in E$*.*

Proof: Let $\varepsilon > 0$. By a little measure theory there exists a compact set $K \subset E$ with $|E \setminus K| < \varepsilon$ and there exist constants α, h, and M such that $|u| < M$ on

$$U = \bigcup\nolimits_{\zeta \in K} \Gamma_{\alpha}^{h}(\zeta).$$

To prove the theorem it is enough to show u has a nontangential limit almost everywhere on K.

Let U_i be one component of U and let φ_i be a conformal map of $\mathbb{D}$ onto U_i. Then $u \circ \varphi_i$ is a bounded harmonic function on $\mathbb{D}$ and by Fatou's theorem, $u \circ \varphi_i$ has a finite nontangential limit almost everywhere on $\partial\mathbb{D}$.

Because α and h are fixed, ∂U_i is a rectifiable Jordan curve. See Figure.

For Lebesgue almost every $\zeta \in K \cap \partial U_i$, φ_i is conformal at ${\varphi_i}^{-1}(\zeta)$ and $u \circ \varphi_i$ has a finite nontangential limit at ${\varphi_i}^{-1}(\zeta)$, both by the

But when φ_i is conformal at ${\varphi_i}^{-1}(\zeta)$, $u \circ \varphi_i$ has a nontangential limit at ${\varphi_i}^{-1}(\zeta)$ if and only if u has a nontangential limit at ζ.

Since $K = \bigcup_i K \cap \partial U_i$, we conclude that u has a finite nontangential limit at almost every $\zeta \in K$. $\square$

Many proofs are simple variations on the previous argument.

Define a **cone domain** to be a domain of the form

$$U = \bigcup_{\zeta \in K} \Gamma_\alpha(\zeta) \text{ or } U = \bigcup_{\zeta \in K} \Gamma_\alpha^h(\zeta),$$

where $K \subset \partial\mathbb{D}$ is compact, $\alpha > 1$, and $0 < h < 1$.

Theorem 2.3: *Let* $f(z)$ *be a meromorphic function on* $\mathbb{D}$ *and let* $E \subset \partial\mathbb{D}$ *have* $|E| > 0$*. If for each* $\zeta \in E$ *there is* $\alpha = \alpha(\zeta) > 1$ *such that*

$$\lim_{\Gamma_\alpha(\zeta) \ni z \to \zeta} f(z) = 0,$$

then $f(z) \equiv 0$ *in* $\mathbb{D}$*.*

Proof: Let $K \subset E$ be compact with $|K| > 0$ and take the cone domains U_i and the maps φ_i as in the proof of on $\mathbb{D}$, and by the F. and M. Riesz theorem, $f \circ \varphi_i$ has nontangential limit 0 on a set of positive measure. Then by (1.4), $f \circ \varphi_i \equiv 0$ so that $f \equiv 0$. $\square$

Nontangential approach regions are necessary in Theorems 2.2 and 2.3. Bagemihl and Seidel [1954] showed that there is a non-constant analytic function on $\mathbb{D}$ having radial limit 0 almost everywhere on $\partial\mathbb{D}$.

Definition 2.4: *Let* $f(z)$ *be a meromorphic function on* $\mathbb{D}$. *A point* $\zeta \in \partial\mathbb{D}$ *is a* **Plessner point** *for* f *if for all* $\alpha > 1$ *and all* $0 < h < 1$, $f\big(\Gamma_\alpha^h(\zeta)\big)$ is dense in $\mathbb{C}$.

Theorem 2.5 (Plessner): *Let $f(z)$ be a nonconstant meromorphic function on $\mathbb{D}$. Then there are pairwise disjoint Borel subsets N, G, and P of $\partial\mathbb{D}$ such that $\partial\mathbb{D} = N \cup G \cup P$ and*

(a) $|N| = 0$,

(b) At each $\zeta \in G$, f has a finite nontangential limit $f(\zeta)$ and $f(\zeta) \neq 0$, and

(c) Each $\zeta \in P$ is a Plessner point for f.

Proof: Let P be the set of Plessner points for f and let $E = \partial\mathbb{D} \setminus P$.

Then P and E are Borel sets, and the theorem is equivalent to the assertion that f has a finite non-zero nontangential limit at almost every $\zeta \in E$.

Let $\{w_n\}$ be a countable dense subset of $\mathbb{C}$ and set

$$\begin{aligned} E_n &= \Big\{\zeta : \text{there exist } \alpha > 1,\ 0 < h < 1, \\ &\qquad \text{and } \epsilon > 0 \text{ such that } \big|f - w_n\big| \geq \epsilon \text{ on } \Gamma_\alpha^h(\zeta)\Big\}. \end{aligned}$$

Then $E = \bigcup_n E_n$. By Privalov's theorem $1/(f(z) - w_n)$ has a nontangential limit a.e. on E_n and hence f has a nontangential limit at almost every $\zeta \in E$.

By Theorem 2.3, applied to f and to $1/f$, the limit is finite and non-zero a.e..
□

Section VI.3: Accessible Points

By Corollary V.3.6,

$$\int_0^1 \bigl|\varphi'(r\zeta)\bigr| dr < \infty \tag{3.1}$$

except on a capacity zero subset of $\partial\mathbb{D}$.

If (3.1) holds then φ has a radial limit at ζ, and it follows from Lindelöf's theorem, (Exercise II-3(d)), that φ has a nontangential limit at ζ.

Thus $\varphi(\zeta)$ is defined except on capacity zero (= Hausdorff dimension zero).

$w \in \partial\Omega$ is an **accessible point** if it is the endpoint of an open arc $\sigma \subset \Omega$.

Lemma 3.1: *If* φ *has a nontangential limit* $\varphi(\zeta)$ *at* $\zeta \in \partial\mathbb{D}$, *then* $\varphi(\zeta) \in \partial\Omega$ *and* $\varphi(\zeta)$ *is an accessible point of* $\partial\Omega$. *Conversely, every accessible* $w \in \partial\Omega$ *is a nontangential limit* $w = \varphi(\zeta)$.

Proof: Suppose φ has nontangential limit at $\zeta \in \partial\mathbb{D}$. Then clearly $\varphi(\zeta) \in \overline{\Omega}$.

But if $\varphi(\zeta) \in \Omega$ then there is $z \in \mathbb{D}$ with $\varphi(\zeta) = \varphi(z)$, and that means that φ^{-1} is not single valued in some neighborhood of $\varphi(z)$.

Therefore $\varphi(\zeta) \in \partial\Omega$ and clearly $\varphi(\zeta)$ is an accessible point.

For the converse, suppose $w \in \partial\Omega$ is the endpoint of an arc $\sigma \subset \Omega$.

The preceding argument above shows that $\lim_{\sigma \ni z \to w} |\varphi^{-1}(z)| = 1$.

Because φ has nontangential limits a.e. and $\varphi^{-1}(\sigma)$ is an arc, Theorem 2.3, applied to $\varphi(z) - w$, shows that $\lim_{\sigma \ni z \to w} \varphi^{-1}(z) = \zeta \in \partial\mathbb{D}$ exists.

It then follows from Lindelöf's theorem, Exercise II.3(d), that φ has nontangential limit w at ζ. $\square$

Let A be the set of accessible points of $\partial\Omega$.

Then A is the range of the boundary function $\varphi(\zeta)$, defined except on a set of capacity zero, and A is dense in $\partial\Omega$.

Theorem 3.2: *Let* Ω *be a simply connected domain. Then the set* A *of accessible points is an* ω*-measurable subset of* $\partial\Omega$ *and for all* $z \in \Omega$,

$$\omega(z, A, \Omega) = 1. \tag{3.2}$$

If $E \subset A$, *then*

$$\omega\big(z, E, \Omega\big) = \omega\big(\varphi^{-1}(z), \varphi^{-1}(E), \mathbb{D}\big). \tag{3.3}$$

In particular,

$$\omega(z, E, \Omega) = 0 \iff |\varphi^{-1}(E)| = 0.$$

Proof: By Egoroff's theorem and Lemma 3.1, there are closed sets $K_n \subset \partial\mathbb{D}$ such that $|\partial\mathbb{D} \setminus K_n| < 1/n$ and such that φ is continuous on the compact set $\cup_{K_n} \overline{\Gamma}_{\pi/2}(e^{i\theta})$. Hence $\varphi(K_n) \subset A$ is compact and $\omega(\cup_n \varphi(K_n)) = 1$.

Thus A is ω-measurable and (3.2) holds.

We may assume $z = \varphi(0)$.

Let $r_n \uparrow 1$, and set $\Omega_n = \varphi(\{z : |z| < r_n\})$, and let $f \in C(\overline{\Omega})$.

Then by the definition of harmonic measure, by Lemma 3.1, and by dominated convergence,

$$\int_{\partial\Omega} f(\zeta)d\omega(\zeta) = \lim_{n\to\infty} \frac{1}{2\pi} \int f(\varphi(r_n e^{i\theta}))d\theta = \frac{1}{2\pi} \int f(\varphi(e^{i\theta}))d\theta. \tag{3.4}$$

Each side of (3.3) defines a Borel measure and since Borel measures are determined by their actions on continuous functions, (3.4) also implies (3.3). □

Section VI.4: Cone Points and McMillan's Theorem

Suppose $\varphi : \mathbb{D} \to \Omega$ is conformal.

Write

$$G = \Big\{ \zeta \in \partial\mathbb{D} : \varphi \text{ has non-zero angular derivative at } \zeta \Big\}$$

and

$$B = \Big\{ \zeta \in \partial\mathbb{D} : \varphi \text{ has a nontangential limit at } \zeta, \text{ and}$$

$$\liminf_{\Gamma_\alpha(\zeta) \ni z \to \zeta} |\varphi'(z)| = 0, \text{ for all } \alpha > 1 \Big\}. \tag{4.1}$$

Theorem 4.1: *Let* φ *be a univalent function on* $\mathbb{D}$. *Then* $G \cap B = \emptyset$ *and*

$$|G \cup B| = 2\pi.$$

Proof: Because φ has a nontangential limit almost everywhere, this just Privalov's and Plessner's theorems, applied to the function φ'. $\square$

We call a point $w \in \partial\Omega$ a **cone point** of $\partial\Omega$ if w is the vertex of an open isosceles triangle $T \subset \Omega$. Every cone point is an accessible boundary point. We know that the following implications hold pointwise:

$$\begin{aligned}
&\varphi \text{ has non-zero angular derivative at } \zeta \\
&\qquad \Rightarrow \varphi \text{ is conformal at } \zeta \\
&\qquad\qquad \Rightarrow \partial\Omega \text{ has an inner tangent at } \varphi(\zeta) \\
&\qquad\qquad\qquad \Rightarrow \varphi(\zeta) \text{ is a cone point of } \partial\Omega,
\end{aligned}$$

All three converse implications fail pointwise, but Theorem 6.1 will show that these four conditions are almost everywhere equivalent.

Write $K = K(\Omega) = \big\{\text{cone points for } \Omega\big\}$.

Theorem 4.2 (McMillan): *Let Ω be a bounded simply connected domain. Then K is a Borel set, with σ-finite Λ_1 measure, and when $E \subset K$,*

$$\omega(E) = 0 \Longleftrightarrow \Lambda_1(E) = 0. \tag{4.2}$$

Moreover, at almost every $w \in K$, $\partial\Omega$ has an inner tangent.

Proof: Let $\{L_n\}$ be the countable set of lines having rational slope and rational y intercept. For each n, put $w \in K_n$ if $w \in \partial\Omega \setminus L_n$ and w is the vertex of an open isosceles triangle $T_n(w) \subset \Omega$ such that

(a) $T_n(w)$ has base on the line segment $L_n \cap \{|z| \leq n\}$,

(b) $T_n(w)$ has vertex angle $\frac{\pi}{n}$, and

(c) $T_n(w)$ has height $h_n(w)$ satisfying $\frac{1}{n} \leq h_n(w) \leq n$.

Then $K = \bigcup_n K_n$, where K_n is compact, so that K is an F_σ set.

Note that $\mathrm{dist}(K_n, L_n) \geq \frac{1}{n}$. Let $\Omega_n = \bigcup_{K_n} T_n(w)$. Then $\Omega_n \subset \Omega$, and by (a), (b), and (c), Ω_n has finitely many components $\Omega_{n,j}$.

Each component $\Omega_{n,j}$ is bounded by a Jordan curve $\Gamma_{n,j}$, and except for endpoints on L_n, these Jordan curves are pairwise disjoint.

We have $K \subset \bigcup (\Gamma_{n,j} \setminus L_n)$ and K has σ-finite Λ_1 measure.

If we assume the line L_n is horizontal, then $\Gamma_{n,j}$ is the union of an arc of L_n and the graph of a Lipschitz function defined on that arc.

In particular $\Gamma_{n,j}$ is rectifiable, and Ω has an inner tangent at Λ_1 almost every point of $\Gamma_{n,j}$, and thus at Λ_1 almost every point of K.

Then to prove (4.2) we can assume $E \subset \Gamma_{n,j} \setminus L_n$.

Suppose $\Lambda_1(E) > 0$. Then by the F. and Riesz theorem, $\omega(z, E, \Omega_{n,j}) > 0$ for all $z \in \Omega_{n,j}$, and therefore $\omega(z, E, \Omega) > 0$.

Conversely, suppose $E \subset \Gamma_{n,j} \setminus L_n$ is a compact set such that $\Lambda_1(E) = 0$. We may suppose L_n is the real axis. Set $U = \bigcup_E T_n(w)$.

Translate L_n by $\frac{1}{10n}$ units in the direction of $\Omega_{n,j}$ to a parallel line L_n'. When $w \in E$, let $T_n'(w) \subset T_n(w)$ be the isosceles triangle having vertex w, base on L_n', and vertex angle $\frac{\pi}{n+1}$. Set $V = \bigcup_E T_n'(w)$.

Translate L_n by $\frac{1}{5n}$ units in the direction of $\Omega_{n,j}$ to a parallel line L_n''. Repeat this construction with L_n'' and with triangles $T_n''(w)$ now having bases on L_n'' and vertex angle $\frac{\pi}{n+2}$. Set $W = \bigcup_E T_n''(w)$.

Then W has finitely many components. We may assume E has no isolated points so that W is not a triangle. We can also assume W, and hence U and V, is connected by replacing E by a subset if necessary.

Then $W \subset V \subset U \subset \Omega_{n,j} \subset \Omega$.

Similiar to $\Gamma_{n,j}$ the domains ∂U, ∂V, and ∂W consist of a segment of L_n, L'_n, L''_n, respectively, and a Lipschitz graph over that segment.

Each component τ_k of $\partial V \setminus E$ is an arc having endpoints in E, and these endpoints are also the endpoints of a component σ_k of $\partial W \setminus E$ and a component γ_k of $\partial U \setminus E$.

$\tau_k, \sigma_k, \gamma_k$ are each polygonal arcs with at most three sides and angles bounded below.

Let $\mathcal{U}_k$ be the unbounded component of $\mathbb{C} \setminus (\tau_k \cup \sigma_k)$. Only one σ_k satisfies $\sigma_k \cap L'_n \neq \emptyset$. Then for each E

$$\inf_{\gamma_k} \omega(z, \sigma_k, \mathcal{U}_k) = \alpha > 0,$$

uniformly in k, by a normal families argument for instance.

Therefore

$$\inf_{\partial U\setminus E} \omega(z, \partial\Omega \setminus E, \Omega \setminus \overline{V}) \geq \alpha > 0 \tag{4.3}$$

and by the maximum principle

$$\inf_{\partial U\setminus E} \omega(z, \partial\Omega \setminus E, \Omega) \geq \alpha > 0.$$

But ∂U is rectifiable and $\Lambda_1(E) = 0$, so by the Riesz theorem, $\omega(z, E, U) = 0$. Thus $\omega(z, \partial\Omega \setminus E, \Omega) \geq \alpha$ on $U \supset \partial V \setminus E$ and

$$\beta = \inf_{\partial V \setminus E} \omega(z, \partial\Omega \setminus E, \Omega) \geq \alpha.$$

Therefore

$$\begin{aligned}\omega(z,\partial\Omega\setminus E,\Omega) &= \omega(z,\partial\Omega\setminus E,\Omega\setminus\overline{V}) + \int_{\partial V}\omega(\zeta,\partial\Omega\setminus E,\Omega)d\omega_{\Omega\setminus\overline{V}}(z,\zeta)\\ &\geq \omega(z,\partial\Omega\setminus E,\Omega\setminus\overline{V}) + \beta\omega(z,\partial V,\Omega\setminus\overline{V})\\ &= (1-\beta)\omega(z,\partial\Omega\setminus E,\Omega\setminus\overline{V}) + \beta,\end{aligned}$$

for all $z\in\Omega\setminus\overline{V}$. But taking $z\in\partial U\setminus E$ then gives

$$\beta\geq\alpha\geq(1-\beta)\alpha+\beta.$$

If $\beta<1$ this gives $\beta>\beta$, a contradiction. Thus $\beta=1$ and $\omega(E)=0$. $\square$

A **twist point** of $\partial\Omega$ is a point $\zeta \in \partial\Omega$ so that on $\arg(z - \zeta)$ is unbounded above and below along every curve in Ω ending at ζ.

McMillan twist point theorem: *If Ω is a Jordan domain, then a.e. (w.r.t. harmonic measure) point of $\partial\Omega$ is either a cone point or a twist point.*

This is proved in Appendix I of the textbook.

Now assume Γ is a Jordan curve and let Ω_1 and Ω_2 be the two simply connected components of $\mathbb{C}^* \setminus \Gamma$. Fix $p_j \in \Omega_j$, let $\varphi_j : \mathbb{D} \to \Omega_j$ be a conformal mapping with $\varphi_j(0) = p_j$ and write $\omega_j = \omega(p_j, \cdot, \Omega_j)$.

Set

$$\mathrm{Tn}(\Gamma) = \Big\{ w \in \Gamma : \Gamma \text{ has a tangent at } w \Big\}.$$

If $w \in \mathrm{Tn}(\Gamma)$ then by Theorem II.4.2, φ_j is conformal at ${\varphi_j}^{-1}(w)$ for $j = 1, 2$.

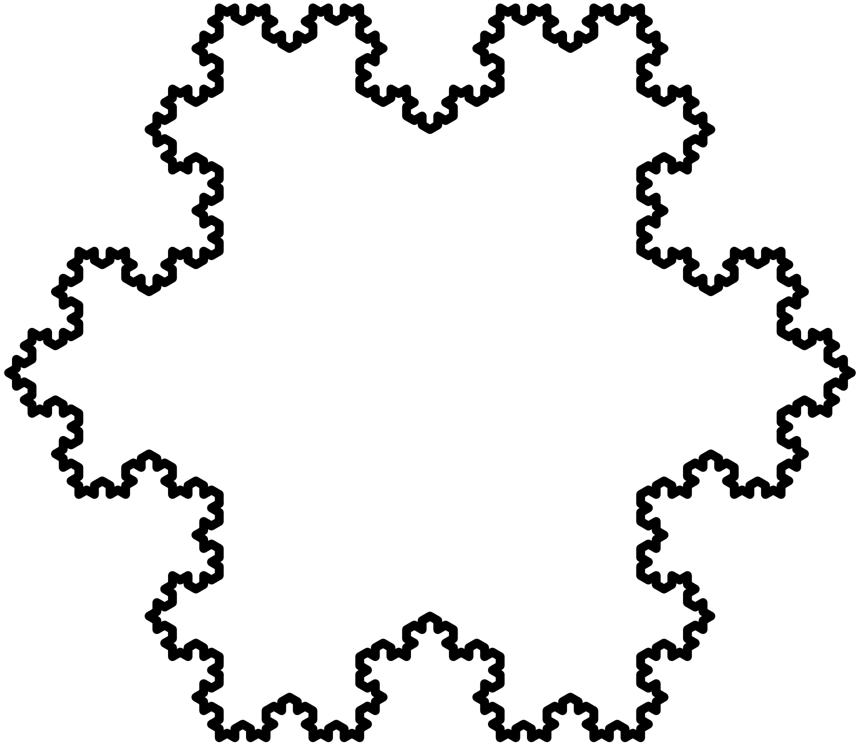
Consequently, by Theorem 4.2 the three measures ω_1, ω_2, and Λ_1 are in the same measure class on $\mathrm{Tn} = \mathrm{Tn}(\Gamma)$,

$$\chi_{\mathrm{Tn}}\omega_1 \ll \chi_{\mathrm{Tn}}\Lambda_1 \ll \chi_{\mathrm{Tn}}\omega_2 \ll \chi_{\mathrm{Tn}}\omega_1.$$

Theorem 6.3 below will show, without using the twist point theorem, that on $\Gamma \setminus \mathrm{Tn}(\Gamma)$, $\omega_1 \perp \omega_2$.

Write $\mathrm{Twi}(\Gamma)$ for the set of twist points of Γ.

Example 4.3 (von Koch Snowflake):



Section VI.5: Compression and Expansion

$\Lambda_\alpha = \alpha$-Hausdorff measure.

$$\Lambda_\alpha(E) = \lim_{\delta \searrow 0} \inf\{\sum_j r_j^\alpha : E \subset \cup_j D(x_j, r_j), r_j < \delta\}$$

$$\dim(E) = \inf\{s : \lambda_s(E) = 0\}.$$

If $h : [0, \infty) \to [0, \infty)$ is a homeomorphism, we call it a gauge function and define

$$\Lambda_h(E) = \lim_{\delta \searrow 0} \inf\{\sum_j h(r_j) : E \subset \cup_j D(x_j, r_j), r_j < \delta\}.$$

For two measures, $\mu \ll \nu$ if $\nu(E) = 0 \Rightarrow \mu(E) = 0$.

For two measures, $\mu \perp \nu$ if for some set E, $\nu(E) = 0$ and $\mu(E^c) = 0$.

Let Ω be a simply connected domain, let ω be harmonic measure for some fixed $z_0 \in \Omega$ and let $\varphi : \mathbb{D} \to \Omega$ be a conformal mapping.

If $\Lambda_\alpha(\partial\Omega) > 0$ and $\omega \perp \Lambda_\alpha$, then the conformal map φ **compresses** a set $E \subset \partial\mathbb{D}$ having full harmonic measure into a set $\varphi(E)$ having $\Lambda_\alpha(\varphi(E)) = 0$.

On the other hand, if $\omega \ll \Lambda_\alpha$, then φ cannot compress a set E with $|E| > 0$ into a set with $\Lambda_\alpha(\varphi(E)) = 0$.

If $\partial\Omega$ is a rectifiable Jordan curve then by the F. and M. $\omega \ll \Lambda_1 \ll \omega$.

In 1936 Lavrentiev made the more precise lower estimate

$$\omega(E) \leq \frac{C \log \Lambda_1(\partial\Omega)}{1 + |\log \Lambda_1(E)|}, \tag{5.1}$$

when $\partial\Omega$ is rectifiable, $E \subset \partial\Omega$, and $\operatorname{dist}(z_0, \partial\Omega) \geq 1$.

In 1994 Bishop and Jones proved a version that only requires E to lie on a rectifiable curve Γ

$$\frac{\omega(E)}{\log \omega E)} \leq \frac{C \log^+ \Lambda_1(\Gamma)}{1 + |\log \Lambda_1(E)|}, \tag{5.1}$$

Mikhail Alekseevich Lavrentev (1900–1980)

Peter W. Jones (1952–)

Math Reviews of Jones's papers

In 1936 Lavrentiev constructed a Jordan domain for which $\omega \not\ll \Lambda_1$.

In 1973 McMillan and Piranian constructed a Jordan domain such that

$$\omega \perp \Lambda_1.$$

Kaufman and Wu (1982) built, for any measure function of the form

$$h(t) = t\exp(|\log t|^{\alpha}),\ 0 < \alpha < \frac{1}{2},$$

a Jordan domain for which $\omega \perp \Lambda_h$.

Thus φ can compress E with $|E| = 2\pi$ into $\varphi(E)$ where $\Lambda_h(\varphi(E)) = 0$.

On the other hand, φ cannot compress harmonic measure very much.

Beurling's projection theorem (Corollary III.9.3) implies $\omega \ll \Lambda_{\frac{1}{2}}$.

In 1973 Carleson improved this to $\omega \ll \Lambda_\beta$ for some β, $\frac{1}{2} < \beta < 1$.

In 1985 Makarov gave an surprising improvement of this.

Theorem 5.1 (Makarov): *Let ω be harmonic measure for a simply connected domain Ω and let $0 < \alpha < 1$. Then*

$$\omega \ll \Lambda_\alpha.$$

Theorem 5.2 (Makarov): *Let ω be harmonic measure for a simply connected domain Ω and let $h(t)$ be a measure function such that*

$$\lim_{t \to 0} \frac{h(t)}{t} = 0.$$

Then ω is singular to Λ_h, $\omega \perp \Lambda_h$.

Thus harmonic measure on simply connected domains has dimension 1.

Later, Jones and Wollf proved it has dimension ≤ 1 for any planar domain.

We will prove Thm 5.2 now, and Thm 5.1 in the next chapter.

Nikolai Georgievich Makarov

Tom Wolff (1954-2000)

Defn: Let $E \subset \partial\mathbb{D}$ and let $S = \{z_k\}$ be a sequence in $\mathbb{D}$. We say S is **nontangentially dense a.e.** on E if there is $\alpha > 1$ such that for almost every $\zeta \in E$,

$$\zeta \in \overline{S \cap \Gamma_\alpha(\zeta)}.$$

Lemma 5.3: *Assume* $\{z_k\}$ *is nontangentially dense a.e. on* $E \subset \partial\mathbb{D}$ *and let* φ *be the conformal map from* $\mathbb{D}$ *onto a simply connected domain* Ω*. Then set* $w_k = \varphi(z_k), r_k = \operatorname{dist}(w_k, \partial\Omega), B_k = B(w_k, 2r_k),$ *and*

$$V = \partial\Omega \cap \big(\bigcup B_k\big) \cap A,$$

where A *is the set of accessible boundary points. Then*

$$|E \setminus \varphi^{-1}(V)| = 0. \tag{5.2}$$

Proof: Suppose (5.2) is false and let W_k be the component of $B_k \cap \Omega$ such that $w_k \in W_k$.

Since $\partial\Omega$ is connected,

$$\omega(w_k, V, \Omega) \geq \omega(w_k, \partial\Omega \cap B_k, W_k) \geq c,$$

by the Beurling projection theorem (Exercise III.10).

Consequently $u(z) = \omega(\varphi(z), V, \Omega)$ satisfies $u(z_k) \geq c$.

But u has nontangential limits a.e. and almost every $\zeta \in E$ is the nontangential limit of a subsequence of $\{z_k\}$, so that

$$\lim_{\Gamma_\alpha(\zeta)\ni z\to\zeta} u(z) \geq c$$

almost everywhere on E.

On the other hand, by Theorem 3.2 $u(z) = \omega(z, \varphi^{-1}(V), \mathbb{D})$, so that $u(z)$ has nontangential limit $\chi_{\varphi^{-1}(V)}$ almost everywhere on $\partial\mathbb{D}$. Consequently

$$\chi_{\varphi^{-1}(V)} \geq c$$

almost everywhere on E, and (5.2) follows. $\square$

Proof of Theorem 5.2: Fix $\epsilon > 0$. When $z \in \mathbb{D}$ and $|z| > \frac{1}{2}$, define

$$I(z) = \big\{\zeta \in \partial\mathbb{D} : |z - \zeta| < 2(1 - |z|)\big\}.$$

Then $1 - |z|^2 \le c|I(z)|$ and $\zeta \in I(z)$ if and only if $z \in \Gamma_2(\zeta)$.

Let φ be the conformal mapping from $\mathbb{D}$ to Ω, and set $\alpha = 2$. By Theorem 2.3,

$$\liminf_{\Gamma_\alpha(\zeta)\ni z\to\zeta} \left|\varphi'(z)\right| < \infty$$

for almost every $\zeta \in \partial\mathbb{D}$, and the sets

$$E_n = \left\{\zeta \in \partial\mathbb{D} : \liminf_{\Gamma_\alpha(\zeta)\ni z\to\zeta} \left|\varphi'(z)\right| < n\right\}$$

satisfy $E_n \subset E_{n+1}$ and $\left|\bigcup E_n\right| = 2\pi$.

Every $\zeta \in E_n$ is covered by arbitrarily small arcs $I(z)$ such that

$$\left|\varphi'(z)\right| < n, \tag{5.3}$$

and

$$1 - |z|^2 < \delta_n, \tag{5.4}$$

where we choose $\delta_n < \epsilon/n$ so small that

$$\frac{h(t)}{t} < \frac{\epsilon}{n2^{n+2}} \tag{5.5}$$

whenever $t < 4 \cdot n \cdot \delta_n$.

By the Vitali covering lemma, Exercise I.9, for each fixerd n, there is a sequence $\{z_{n,j}\}$ satisfying (5.3) and (5.4) such that

$$\Big|E_n \setminus \bigcup_j I(z_{n,j})\Big| = 0 \text{ and } \sum_j |I(z_{n,j})| \leq 2\pi. \tag{5.6}$$

Then by (5.3), (5.4), and (5.5),

$$\begin{aligned}\sum_j h\big(2|\varphi'(z_{n,j})|(1-|z_{n,j}|^2)\big) &\leq C\sum_j h(n|I_{n,j}|)\\ &\leq C\sum_j n|I_{n,j}|\epsilon/(n2^n)\end{aligned}$$

so

$$\sum_j h\big(2|\varphi'(z_{n,j})|(1-|z_{n,j}|^2)\big) \leq \frac{C\epsilon}{2^n}. \tag{5.7}$$

With ϵ fixed, we take $\{z_k\} = \bigcup_n \{z_{n,j}\}$, $w_k = \varphi(z_k)$, $r_k = \operatorname{dist}(w_k, \partial\Omega)$, $B_k = B(w_k, 2r_k)$, and

$$V_\epsilon = \partial\Omega \cap (\cup B_k)).$$

Then by (5.4) and Theorem I.4.3,

$$2r_k \leq 2|\varphi'(z_k)|(1 - |z_k|^2) \leq 4\epsilon,$$

so that (5.7) yields

$$\sum_k h(2r_k) \leq C\epsilon.$$

Consequently, $V = \bigcap_m^\infty V_{\frac{1}{m}}$ satisfies $\Lambda_h(V) = 0$.

But by (5.6), $\{z_k\}$ is nontangentially dense a.e. in $\partial\mathbb{D}$.

Therefore $\omega(\,\cdot\,, V_{\frac{1}{m}}, \Omega) = 1$ by Lemma 5.3 and $\omega(\,\cdot\,, V, \Omega) = 1$. $\square$

Theorem, Besicovitch (1956): Suppose that A is an analytic set, f is a gauge function, and A is not σ-finite for Λ_f. Then there is a gauge function g such that $g(t) = o(f(t))$ and A is not σ-finite for Λ_g.

Besicovitch, A. S. (1956). "On the definition of tangents to sets of infinite linear measure". *Proc. Cambridge Philos. Soc.* 52, 20–29

Extending Borel's Conjecture From Measure to Dimension Theodore A. Slaman.

We could deduce harmonic measure gives full mass to a σ-finite set, if Makarov's proof applied to a single set of full measure, independent of h.

Section VI.6: Pommerenke's Extension

Recall the notation

$$G = \left\{ \zeta \in \partial\mathbb{D} : \varphi \text{ has non-zero angular derivative at } \zeta \right\}$$

$$\begin{aligned} B = \Big\{ \zeta \in \partial\mathbb{D} : \varphi &\text{ has a nontangential limit at } \zeta, \text{ and} \\ &\liminf_{\Gamma_\alpha(\zeta) \ni z \to \zeta} |\varphi'(z)| = 0, \text{ for all } \alpha > 1 \Big\}. \end{aligned} \tag{4.1}$$

$$K = \left\{ \text{cone points for } \Omega \right\}.$$

We know $G \cap B = \emptyset, |G \cup B| = 2\pi$ and $\varphi(G) \subset K$.

Theorem 6.1 (Pommerenke): *Let Ω be a simply connected domain and let $\varphi : \mathbb{D} \to \Omega$. Then there is a subset $S \subset \varphi(B) \setminus K$ such that*

$$\Lambda_1(S) = 0. \tag{6.1}$$

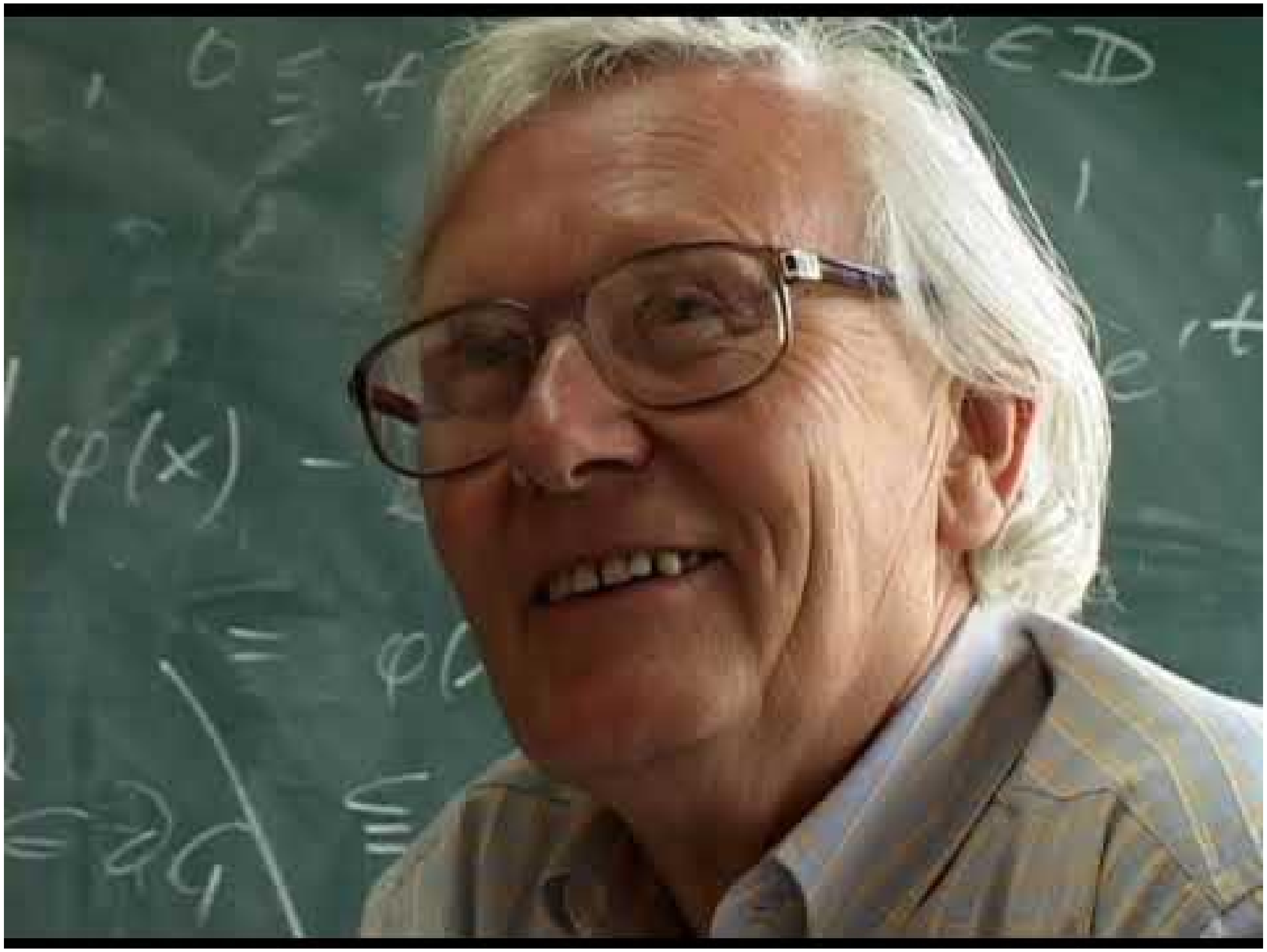
and

$$\omega(S \cup \varphi(G)) = 1. \tag{6.2}$$

In particular, the set $P = \varphi(G) \cup S$ has $\sigma-$finite Λ_1 measure and $\omega(P) = 1$. Consequently,

$$\Lambda_1(K \setminus \varphi(G)) = \omega(K \setminus \varphi(G)) = 0.$$

By McMillian's twist point theorem, ω a.e. $w \in S$ is a twist point of $\partial\Omega$.



Christian Pommerenenke (1933-2024)

Proof: We follow the proof of Theorem 5.2.

Again by the Vitali covering lemma, there are $\{z_{n,j}\}$ such that

$$\begin{gathered} |\varphi'(z_{n,j})| < 2^{-n-3}, \\ \big|B \setminus \bigcup_j I(z_{n,j})\big| = 0, \end{gathered} \tag{6.3}$$

and

$$\sum_j |I(z_{n,j})| \le 2\pi.$$

Now take $w_{n,j} = \varphi(z_{n,j}), r_{n,j} = \operatorname{dist}(w_{n,j}, \partial\Omega)$, $B_{n,j} = B(w_{n,j}, 2r_{n,j})$, and

$$V_n = \partial\Omega \cap (\bigcup_j B_{n,j}).$$

Then as before, (6.3) yields

$$\sum_j 2r_{n,j} \leq \sum_j 2|\varphi'(z_{n,j})|(1 - |z_{n,j}|^2) \leq c\pi 2^{-n},$$

and

$$V = \bigcap_k \bigcup_{n \geq k} V_n$$

has $\Lambda_1(V) = 0$. Set

$$S = V \cap \varphi(B).$$

Then (6.1) holds for S.

By Beurling's projection theorem, $\omega(w_{n,j}, V_n, \Omega) \geq c$, and

$$\bigcup_{n\geq N} \{z_{n,j}\}$$

is nontangentially dense on B. Therefore by Lemma 5.3 (non-tang. accessible pts have full measure)

$$\Big|B \setminus \bigcup_{n\geq k} \varphi^{-1}(A \cap V_n)\Big| = 0$$

for any k, where A is the set of accessible points. Hence

$$\Big|B \setminus \varphi^{-1}(S)\Big| = 0,$$

and (6.2) then follows from Theorem 4.1 ($G \cup B$ has full measure).

Since $\Lambda_1(S) = 0$, if we replace S by $S \setminus K$ then (6.1) and (6.2) still hold by Theorem 4.2 and $\omega(K \setminus \varphi(G)) = 0$. By Theorem 4.2, $\Lambda_1(K \setminus \varphi(G)) = 0$. $\square$

Corollary 6.2: *Let* Ω *be a simply connected domain and let* $S \subset \partial\Omega$ *be the set given in Theorem* 6.1. *Then*

$$\omega \ll \Lambda_1 \Longleftrightarrow \omega(S) = 0 \Longleftrightarrow |G| = 2\pi,$$

and

$$\omega \perp \Lambda_1 \Longleftrightarrow |G| = 0.$$

Theorem 6.3 (Bishop): *Suppose* Γ *is a Jordan curve, and let* Ω_1 *and* Ω_2 *be the two components of the complement* $\mathbb{C}^* \setminus \Gamma$. *Let* E *be a Borel subset of* Γ *such that* $\omega(z_j, E, \Omega) > 0$ *for* $z_j \in \Omega_j$, $j = 1, 2$, *and let* $\omega_j|_E$ *be the restriction of* $\omega_j = \omega(z_j, \cdot, \Omega_j)$ *to* E. *Then*

$$\omega_1|_E \perp \omega_2|_E$$

if and only if

$$\Lambda_1(\mathrm{Tn}(\Gamma) \cap E) = 0,$$

where $\mathrm{Tn}(\Gamma)$ *is the set of tangent points of* Γ.

Proof:

We start with the easy direction.

Assume $\Lambda_1(\mathrm{Tn}(\Gamma) \cap E) > 0$. On Tn (Γ), $\omega_j \ll \Lambda_1 \ll \omega_j$ by Theorem 4.2, so that for $j = 1, 2$, $\omega_j|_E(\mathrm{Tn}(\Gamma)) > 0$ and

$$\omega_1|_E \not\perp \omega_2|_E.$$

For the other direction, assume $\omega_1|_E \not\perp \omega_2|_E$. We want to prove that E intersects the tangent set in positive length (equivalently, positive harmonic measure for either side).

We may assume $\operatorname{dist}(z_j, \Gamma) \geq 1$. Let $w \in \Gamma$. For $0 < t < 1$, let $J_j(t)$ be any arc in $\Omega_j \cap \{z : |z - w| = t\}$ such that J_j separates z_j from w, and write

$$t\theta_j(t) = \ell(J_j(t)).$$

Then by Theorem IV.6.2,

$$\omega_j(B(w,r)) \leq \frac{8}{\pi} \exp\left(-\pi \int_r^1 \frac{dt}{t\theta_j(t)} \right).$$

Because $\left(\frac{1}{\theta_1} + \frac{1}{\theta_2}\right)(\theta_1 + \theta_2) \geq 4$, (by Cauchy–Schwarz) and $\theta_1 + \theta_2 \leq 2\pi$, we have

$$\frac{1}{\theta_1} + \frac{1}{\theta_2} \geq \frac{2}{\pi}.$$

Therefore

$$\omega_1(B(w,r))\omega_2(B(w,r)) \leq \frac{64}{\pi^2} \exp\left(- \int_r^1 \frac{2dt}{t} \right) = \frac{64}{\pi^2} r^2. \tag{6.4}$$

or

$$\frac{\pi}{8}\sqrt{\omega_1(B(w,r))\omega_2(B(w,r))} \leq r.$$

By assumption, there is a compact set $E_N \subset E$ such that $\omega_j(E_N) > 0$ and

$$\frac{\omega_1(S)}{N} \leq \omega_2(S) \leq N\omega_1(S)$$

for all Borel $S \subset E_N$.

Cover $S \subset E_N$ by balls $B(w_k, r_k)$, $w_k \in \Gamma$, and $r_k < 1$. Then by (6.4),

$$\sum r_k \geq \frac{\pi}{8\sqrt{N}}\omega_j(S),$$

and hence

$$\Lambda_1(S) \geq \frac{\pi}{4\sqrt{N}}\omega_j(S). \tag{6.5}$$

Let φ_j be a conformal map of $\mathbb{D}$ onto Ω_j, let $G_j \subset \partial\mathbb{D}$ be the set where φ_j has non-zero angular derivative and let $S = S_j \subset B$ be the set of zero length and full harmonic measure in B given by Theorem 6.1 for Ω_j. Then $\omega_1(S \cap E_N) = 0$ by (6.1) and (6.5).

Therefore

$$\omega_1\big(\varphi_1(G_1) \cap \varphi_2(G_2) \cap E_N\big) > 0$$

so that by Theorem 4.2,

$$\Lambda_1\big(\varphi_1(G_1) \cap \varphi_2(G_2) \cap E_N\big) > 0,$$

which proves the theorem because

$$\varphi_1(G_1) \cap \varphi_2(G_2) \subset \mathrm{Tn}(\Gamma). \quad \square$$

My original proof of this is given on pages 5 to 18 of my PhD thesis

Let $w \in \Gamma$, take $\theta_j(t)$ as in that proof, and set

$$\epsilon(w,t) = \max\{|\pi - \theta_j(t)| : j = 1, 2\}.$$

Corollary 6.4 (Bishop, 1987): *If* Γ *is a Jordan curve, then at* Λ_1 *almost every tangent point* w *of* Γ*,*

$$\int_0^1 \epsilon^2(w,t)\frac{dt}{t} < \infty. \tag{6.7}$$

Converse was known as Carleson's ϵ^2-conjecture, and was eventually proven by Jaye, Tolsa and Villa in 2021 *Annals of Mathematics* paper, using a partial result of Bishop and Jones.

Proof: Since

$$\epsilon(w,t) = \max\{|\pi - \theta_j(t)| : j = 1,2\}$$

by Taylor series

$$\frac{1}{\theta_1} + \frac{1}{\theta_2} \geq \frac{2}{\pi} + \frac{2}{\pi}\Big(\frac{\epsilon(w,t)}{\pi}\Big)^2.$$

The proof of (6.4) yields

$$\frac{\omega_1(B(w,r))}{r}\frac{\omega_2(B(w,r))}{r} \leq \frac{64}{\pi^2}\exp\Big(-\frac{2}{\pi^2}\int_r^1 \epsilon^2(w,t)\frac{dt}{t}\Big). \tag{6.6}$$

But by Theorem 4.2 the left side of (6.6) is bounded below at Λ_1 (or ω_j) almost every point of Tn(Γ). Thus the integeral is bounded at these points. □

Example (Exercise VI.9): Suppose Γ is the graph of a real-valued function f and consider the domain Ω_1 and Ω_2 above and below Γ.

- both harmonic measures are absolutely continuous to length measure.

- If f is nowhere diffentiable, then $\omega_1 \perp \omega_2$.

- For $f(x) = \sum 2^{-n} \cos(2^n x)$ the function is nowhere differentiable but the graph Γ has tangents ω_j almost everywhere, hence on a set of positive linear measure. (Uses fact the Γ is a quasicircle.)

Example (Exercise VI.10):

- There is a Jordan curve so that harmonic measure on one side is supported on the cone points and for the other side is supported on the twist points.