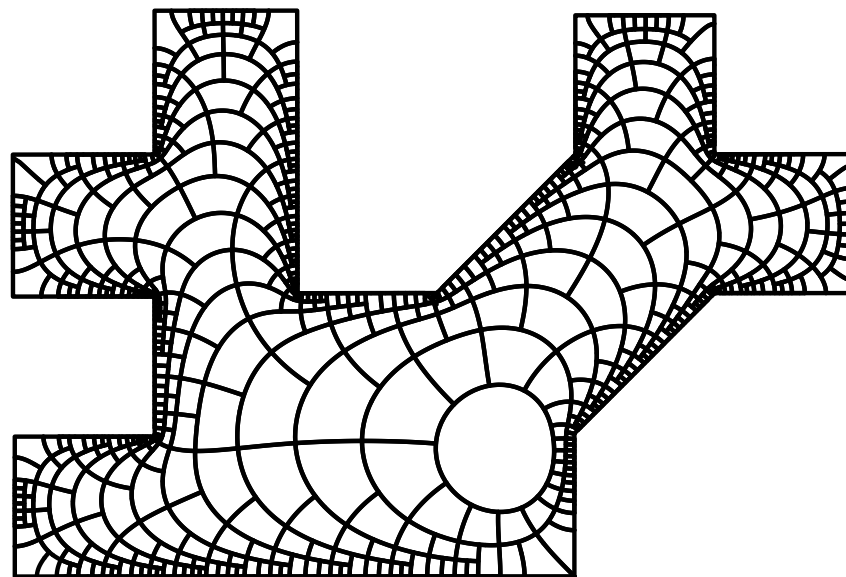
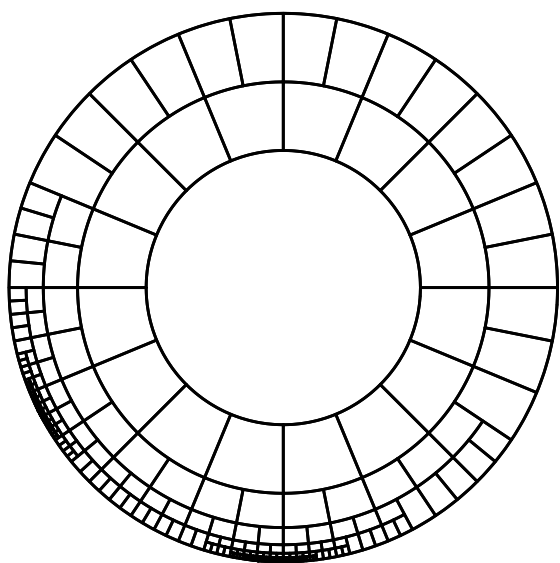


MAT 639, Spring 2026, Stony Brook University

Topics in Real Analysis: Harmonic Measure

Christopher Bishop



Harmonic Measure, Chapter V

Applications and Reverse Inequalities

following text by John Garnett and Don Marshall

Sections for Chapter IV:

- Asymptotic Values of Entire Functions
- Lower Bounds
- Reduced Extremal Distance
- Teichmüller's Modulsatz
- Boundary conformality and Angular Derivatives
- Conditions More Geometric

Section V.1: Asymptotic Values of Entire Functions

An entire function $f(z)$ has **asymptotic value** a if there is a Jordan arc Γ tending to ∞ such that

$$\lim_{\Gamma \ni z \rightarrow \infty} f(z) = a.$$

For example

$$f(z) = \int_0^z e^{-w^n} dw$$

has n distinct asymptotic values, one along each of the curves $\arg z = 2\pi j/n$, $j = 1, \dots, n$.

Write

$$M(r) = \sup_{|z|=r} |f(z)|$$

for the maximum of $|f(z)|$ on $\{|z| = r\}$.

Theorem 1.1: *If the non-constant entire function $f(z)$ has n distinct finite asymptotic values, then*

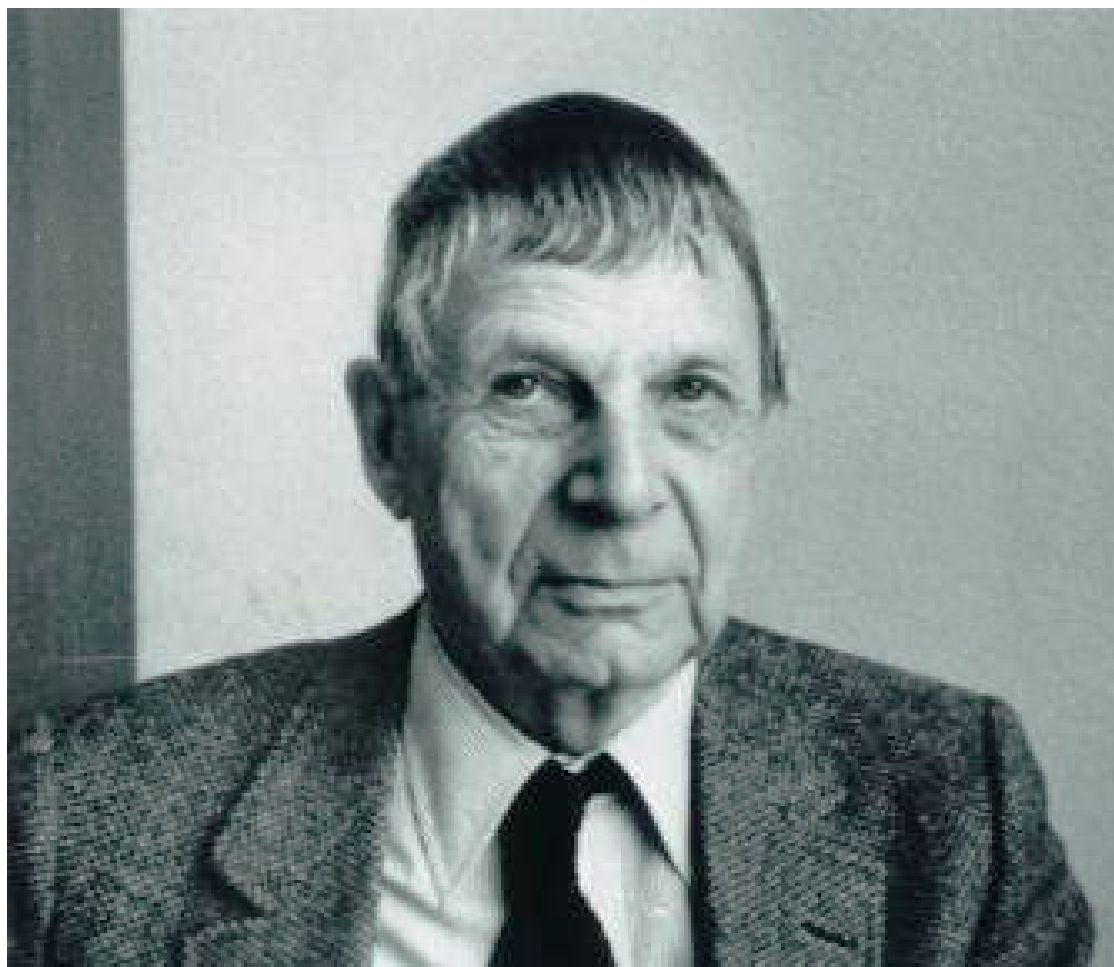
$$\liminf_{r \rightarrow \infty} \left(\frac{\log M(r)}{r^{n/2}} \right) > 0. \quad (1.1)$$

This was called the Denjoy conjecture until it was proven by Ahlfors in 1930.

This was among the results for which Ahlfors was awarded a Fields Medal in 1936.

This was one of the first two Fields medals. The other went to Jesse Douglas for work on the Plateau problem.

Amusing story: during WWII Ahlfors had to pawn his Fields medal to get money to flee Finland for Sweden. He later redeemed the medal. It may be the only Fields medal ever pawned.



Lars Ahlfors (1907–1996)

Lars Ahlfors – At the Summit of Mathematics, by Olli Lehto

We will need:

Lindelöf's theorem (Exercise II.3(d)): *Suppose $f(z)$ is bounded and analytic on the disc \mathbb{D} . Let γ be an arc in \mathbb{D} terminating at a point $\zeta \in \partial\mathbb{D}$, and assume*

$$\lim_{\gamma \ni z \rightarrow \zeta} f(z) = a.$$

Then $f(z)$ has nontangential limit a at ζ .

Proof: Assume f has n distinct asymptotic values a_j taken along n distinct curves Γ_j .

Moving each Γ_j slightly, we may assume that Γ_j is a simple polygonal arc whose vertices tend only to ∞ . Because the asymptotic values a_j are distinct, we may further move the Γ_j so that $\Gamma_j \cap \Gamma_k = \{0\}$ for $j \neq k$, and so that $0 \in \Gamma_1$ when $n = 1$.

Then the Γ_j divide the plane into n Jordan domains G_1, G_2, \dots, G_n , and when $n > 1$ each G_j is bounded by two distinct Γ_k .

On each domain G_j the function $f(z)$ must be unbounded. When $n = 1$, this holds because f is not constant. When $n > 1$, it follows from Lindelöf's theorem (Exercise II.3(d)), applied to $f \circ \varphi$ where φ is a conformal map from \mathbb{D} onto G_j .

We can assume $|f(z)| \leq 1$ on $\bigcup \Gamma_j$, and by the maximum principle we can choose R_0 large so that for each $j = 1, \dots, n$ there is a $z_j \in G_j$ with $|z_j| = R_0$ and $|f(z_j)| > 1$. Let $R > R_0$.

Write

$$\omega_j(R) = \omega(z_j, \{|z| = R\}, G_j \cap \{|z| < R\}),$$

and write $\Theta_j(r)$ for the angular measure of $G_j \cap \{|z| = r\}$.

Then by the maximum principle

$$\log|f(z_j)| \leq \omega_j(R)\log M(R)$$

and hence by (IV.6.4),

$$\log|f(z_j)| \leq \frac{8}{\pi} \exp\left(-\pi \int_{R_0}^R \frac{dr}{r\Theta_j(r)}\right) \log M(R).$$

(When $n = 1$, G_1 can be mapped to a Jordan domain using \sqrt{z} and (IV.6.4) can then be applied.)

By our assumptions on the Γ_j ,

$$\sum_{j=1}^n \Theta_j(r) = 2\pi,$$

and by the Cauchy–Schwarz inequality,

$$\left(\sum_{j=1}^n \frac{1}{\Theta_j(r)} \right) \left(\sum_{j=1}^n \Theta_j(r) \right) \geq n^2,$$

so that

$$\sum_{j=1}^n \frac{1}{\Theta_j(r)} \geq \frac{n^2}{2\pi}.$$

Hence

$$\frac{1}{n} \sum_{j=1}^n \int_{R_0}^R \frac{dr}{r\Theta_j(r)} \geq \frac{n}{2\pi} \log \left(\frac{R}{R_0} \right),$$

and there is $j = j(R)$ such that

$$\exp \left(-\pi \int_{R_0}^R \frac{dr}{r\Theta_j(r)} \right) \leq \left(\frac{R_0}{R} \right)^{n/2}.$$

We conclude that

$$0 < \min_j \log |f(z_j)| \leq \frac{8}{\pi} R_0^{\frac{n}{2}} \left(\frac{\log M(R)}{R^{\frac{n}{2}}} \right),$$

from which (1.1) follows. \square

Section V.3: Reduced Extremal Distance

Let Ω be a finitely connected Jordan domain, let $z_0 \in \Omega$, and let E be a finite union of closed subarcs of $\partial\Omega$. Delete a small disc,

$$B_\epsilon = B_\epsilon(z_0) = \{z : |z - z_0| < \epsilon\}$$

from Ω and compute the extremal distance $d_{\Omega \setminus B_\epsilon}(\partial B_\epsilon, E)$.

If $\beta < \epsilon$, then by the serial rule and Example IV.1.2,

$$\begin{aligned} d_{\Omega \setminus B_\beta}(\partial B_\beta, E) &\geq d_{\Omega \setminus B_\epsilon}(\partial B_\epsilon, E) + d_{B_\epsilon \setminus B_\beta}(\partial B_\beta, \partial B_\epsilon) \\ &= d_{\Omega \setminus B_\epsilon}(\partial B_\epsilon, E) + \frac{1}{2\pi} \log(\epsilon/\beta). \end{aligned} \tag{3.1}$$

Hence

$$h(\epsilon) = d_{\Omega \setminus B_\epsilon}(\partial B_\epsilon, E) + \frac{1}{2\pi} \log \epsilon$$

is a decreasing function of ϵ .

Assume $h(\epsilon)$ is bounded above. Then the three limits

$$\lim_{\epsilon \rightarrow 0} d_{\Omega \setminus B_\epsilon}(\partial B_\epsilon, E) + \frac{1}{2\pi} \log \epsilon,$$
$$\lim_{\epsilon \rightarrow 0} d_{\Omega \setminus B_\epsilon}(\partial B_\epsilon, \partial\Omega) + \frac{1}{2\pi} \log \epsilon,$$

and

$$\delta(z_0, E) = \delta_\Omega(z_0, E) = \lim_{\epsilon \rightarrow 0} (d_{\Omega \setminus B_\epsilon}(\partial B_\epsilon, E) - d_{\Omega \setminus B_\epsilon}(\partial B_\epsilon, \partial\Omega))$$

all exist and are finite.

The **reduced extremal distance** is defined to be the third limit $\delta(z_0, E)$.

To see that $h(\epsilon)$ is bounded above, choose a component E_1 of E and let ψ be a conformal map of $\mathbb{C}^* \setminus E_1$ to \mathbb{D} such that $\psi(z_0) = 0$.

Then $\psi(B_\epsilon) \supset B(0, a\epsilon)$ for some $a > 0$, so that by conformal invariance and the extension rule,

$$\begin{aligned} h(\epsilon) &\leq d_{\Omega \setminus B_\epsilon}(\partial B_\epsilon, E_1) + \frac{1}{2\pi} \log \epsilon \\ &\leq d_{\mathbb{D} \setminus B(0, a\epsilon)}(\partial B(0, a\epsilon), \partial \mathbb{D}) + \frac{1}{2\pi} \log \epsilon = \frac{1}{2\pi} \log \frac{1}{a}, \end{aligned}$$

and $h(\epsilon)$ is bounded.

Lemma 3.1: *Reduced extremal distance $\delta(z_0, E)$ is conformally invariant.*

Proof Suppose φ is a conformal map defined on Ω . We may suppose that $z_0 = 0$ and $\varphi(z_0) = 0$. Because φ is conformal at 0, there exist $\epsilon_0 > 0$ and $K > 0$ such that for $\alpha = \epsilon|\varphi'(0)| - K\epsilon^2$ and $\beta = \epsilon|\varphi'(0)| + K\epsilon^2$,

$$B_\alpha \subset \varphi(B_\epsilon) \subset B_\beta, \text{ when } \epsilon < \epsilon_0.$$

Because $k(\alpha) = d_{\varphi(\Omega) \setminus B_\alpha}(\partial B_\alpha, \varphi(E)) + \frac{1}{2\pi} \log \alpha$ is decreasing and bounded above and because $\lim_{\epsilon \rightarrow 0} \log(\beta/\alpha) = 0$, we have

$$\lim_{\epsilon \rightarrow 0} (d_{\varphi(\Omega) \setminus B_\alpha}(\partial B_\alpha, \varphi(E)) - d_{\varphi(\Omega) \setminus B_\beta}(\partial B_\beta, \varphi(E))) = \lim_{\epsilon \rightarrow 0} (k(\alpha) - k(\beta)) = 0.$$

We conclude that

$$\begin{aligned}\delta(0, E) &= \lim_{\epsilon \rightarrow 0} (d_{\Omega}(\partial B_{\epsilon}, E) - d_{\Omega}(\partial B_{\epsilon}, \partial\Omega)) \\ &= \lim_{\epsilon \rightarrow 0} (d_{\varphi(\Omega)}(\varphi(\partial B_{\epsilon}), \varphi(E)) - d_{\varphi(\Omega)}(\varphi(\partial B_{\epsilon}), \partial\varphi(\Omega))) \\ &= \lim_{\epsilon \rightarrow 0} (d_{\varphi(\Omega)}(\partial B_{\alpha}, \varphi(E)) - d_{\varphi(\Omega)}(\partial B_{\alpha}, \partial\varphi(\Omega))) \\ &= \delta(0, \varphi(E)),\end{aligned}$$

and $\delta(z_0, E)$ is a conformal invariant. \square

Theorem 3.2 (Ahlfors–Beurling): *If E is a finite union of closed arcs in $\partial\mathbb{D}$, then*

$$\delta_{\mathbb{D}}(0, E) = \delta_{\overline{\mathbb{D}}^c}(\infty, E) = \gamma_E/\pi,$$

where γ_E is Robin's constant for E .

Proof: Set $\Omega_d = \mathbb{C}^* \setminus E$ and $G(z) = g(z) + g(1/\bar{z})$, where $g(z)$ is Green's function in Ω_d with pole at ∞ .

Note that $g(z) - g(1/\bar{z}) - \log |z|$ is harmonic in Ω_d and zero on $\partial\Omega_d$ and hence $g(1/\bar{z}) = g(z) - \log |z|$ in Ω_d .

Thus $G(z) = 2g(z) - \log |z|$ and $g(0) = \lim_{|z| \rightarrow \infty} g(z) - \log |z| = \gamma_E$.

Since G is symmetric about $\partial\mathbb{D}$, $\partial G/\partial r = 0$ on $\partial\mathbb{D} \setminus E$ and $G = 0$ on E .

If \tilde{g} is a harmonic conjugate of g in \mathbb{D} , then $f(z) = ze^{-2(g+i\tilde{g})}$ is analytic in \mathbb{D} and $-\log |f| = G$.

By repeating the proof of Theorem IV.4.1, but working with the symmetric function G instead of with ω , we see that f extends to be continuous on $\overline{\mathbb{D}}$, that $|f| = 1$ on E , and that $\partial(\arg f)/\partial\theta = 0$ on $\partial\mathbb{D} \setminus E$.

The argument principle shows that f is a conformal map of \mathbb{D} onto the disc with radial slits removed, because f has exactly one zero in \mathbb{D} . Furthermore, $\partial\mathbb{D} = f(E)$ and $f(0) = 0$.

The extremal distance from $F_\epsilon \equiv \{z : |f(z)| = \epsilon\}$ to E is then $\frac{1}{2\pi} \log(1/\epsilon)$.

Because $g(0) = \gamma_E$, we have $\epsilon = |z|e^{-2g(z)} \sim |z|e^{-2\gamma_E}$ on F_ϵ and hence F_ϵ is close to the circle of radius $\epsilon e^{2\gamma_E}$.

But then the proof of Lemma 3.1 yields

$$\begin{aligned}\delta_{\mathbb{D}}(0, E) &= \lim_{\epsilon \rightarrow 0} d_{\mathbb{D}}(F_{\epsilon}, E) - d_{\mathbb{D}}(F_{\epsilon}, \partial\mathbb{D}) \\ &= \lim_{\epsilon \rightarrow 0} \left(\frac{1}{2\pi} \log \frac{1}{\epsilon} - \frac{1}{2\pi} \log \frac{1}{\epsilon e^{2\gamma_E}} \right) \\ &= \gamma_E / \pi. \quad \square\end{aligned}$$

Corollary 3.3: *If Ω is a Jordan domain and if E is a finite union of closed arcs on $\partial\Omega$, then*

$$\omega(z_0, E) \leq e^{-\pi\delta(z_0, E)}.$$

If E is a single arc, then

$$\omega(z_0, E) = \frac{2}{\pi} \sin^{-1}(e^{-\pi\delta(z_0, E)}) \geq \frac{2}{\pi} e^{-\pi\delta(z_0, E)}.$$

Proof: Theorem 3.2 includes an estimate for the harmonic measure of $E \subset \partial\mathbb{D}$, because by Example III.1.2,

$$e^{-\gamma_E} \geq \sin \frac{|E|}{4} \geq \frac{|E|}{2\pi} = \omega(0, E). \quad (3.2)$$

When E is a single arc, equality holds in the first inequality in (3.2). \square

Corollary 3.4 (Pfluger): *If $\Omega = \mathbb{D}$, if E is a finite union of closed arcs in $\partial\mathbb{D}$, and if ∂B_ϵ is the circle centered at 0 of radius ϵ , then*

$$\frac{\gamma_E}{\pi} + \frac{\log 1/\epsilon}{2\pi} + \frac{\log(1 - \epsilon)}{\pi} \leq d_{\mathbb{D}}(\partial B_\epsilon, E) \leq \frac{\gamma_E}{\pi} + \frac{\log 1/\epsilon}{2\pi}.$$

Proof: By (3.1) the function

$$h(\epsilon) = d_{\mathbb{D}}(\partial B_{\epsilon}, E) - d_{\mathbb{D}}(\partial B_{\epsilon}, \partial \mathbb{D}) = d_{\mathbb{D}}(\partial B_{\epsilon}, E) + \frac{1}{2\pi} \log \epsilon$$

increases to γ_E/π as $\epsilon \rightarrow 0$. Thus

$$d_{\mathbb{D}}(\partial B_{\epsilon}, E) \leq (\gamma_E/\pi) - \frac{1}{2\pi} \log \epsilon = (\gamma_E/\pi) + \frac{1}{2\pi} \log \frac{1}{\epsilon}.$$

This proves the right-hand inequality.

By Theorem III.2.2 , we have

$$\gamma_E - g(z) = U_\mu(z) = \int_E \log \frac{1}{|z - \zeta|} d\mu(\zeta),$$

where μ is a probability measure and $g(z) = g_\Omega(z, \infty)$.

If $f = ze^{-2(g+i\tilde{g})}$ as in the proof of Theorem 3.2, then for $z \in \partial B_\epsilon$

$$\log |f| = -2g(z) + \log \epsilon \leq -2\gamma_E - 2 \log(1 - \epsilon) + \log \epsilon.$$

Thus if $\log 1/r = 2\gamma_E + 2 \log(1 - \epsilon) - \log \epsilon$,

$$\begin{aligned} d_{\mathbb{D}}(\partial B_\epsilon, E) &\geq d_{f(\mathbb{D})}(\partial B_r, \partial \mathbb{D}) = d_{\mathbb{D}}(\partial B_r, \partial \mathbb{D}) \\ &= \frac{1}{2\pi}(2\gamma_E + 2 \log(1 - \epsilon) - \log \epsilon). \quad \square \end{aligned}$$

Fix $z_0 \in \Omega$ and let

$$\text{dist}_\Omega(z_0, w) = \inf \left\{ \int_\gamma ds : \gamma \text{ is a curve in } \Omega \text{ and } z_0, w \in \bar{\gamma} \right\}$$

be the **Euclidean distance in Ω** from z_0 to w .

Theorem 3.5: *Let $\psi : \mathbb{D} \rightarrow \Omega$ be a normalized univalent function, $\psi(0) = 0$ and $\psi'(0) = 1$, and let $E \subset \{\zeta \in \partial\Omega : \text{dist}_\Omega(0, \zeta) \geq R\}$. Then*

$$\delta_\Omega(0, E) \geq \frac{1}{2\pi} \log R. \quad (3.3)$$

Equality holds in (3.3) if Ω is a disc of radius R , with radial slits removed.

Proof: We may assume that $\partial\Omega$ is an analytic curve.

Since $\psi(0) = 0$ and $|\psi'(0)| = 1$,

$$\delta_{\Omega}(0, E) = \lim_{\epsilon \rightarrow 0} \left(d_{\Omega}(\partial B_{\epsilon}, E) - \frac{1}{2\pi} \log \frac{1}{\epsilon} \right).$$

By the extension rule, we have may assume $E = \{z \in \bar{\Omega} : \text{dist}_{\Omega}(0, z) \geq R\}$.

By another application of the extension rule, we may suppose that $\text{dist}_{\Omega}(0, \zeta) \leq R$ for all $\zeta \in \partial\Omega$ and $E = \{\zeta \in \partial\Omega : \text{dist}_{\Omega}(0, \zeta) = R\}$.

Define a metric ρ on Ω by

$$\rho(z) = \frac{1}{\text{dist}_{\Omega}(0, z)}.$$

If $\gamma : [0, L) \rightarrow \Omega$ is a curve parameterized by arc length with $\gamma(0) = 0$, then

$$|\gamma(s)| \leq \text{dist}_\Omega(0, \gamma(s)) \leq s.$$

Any curve σ connecting ∂B_ϵ to E in $\Omega \setminus B_\epsilon$ can be extended to a curve γ connecting 0 to E by adding a line segment of length ϵ . Thus

$$\int_\sigma \rho(z) ds = \int_{|\gamma| \geq \epsilon} \rho(z) ds \geq \int_\epsilon^R \frac{1}{s} ds = \log \frac{R}{\epsilon}$$

and

$$\iint_{\Omega \setminus B_\epsilon} \rho^2(z) dx dy \leq \iint_{\epsilon < |w| < R} \frac{1}{|w|^2} dA(w) = 2\pi \log \frac{R}{\epsilon},$$

where $B_\epsilon = \{z : |z| < \epsilon\}$.

Therefore

$$d_{\Omega}(\partial B_{\epsilon}, E) = d_{\Omega \setminus B_{\epsilon}}(\partial B_{\epsilon}, E) \geq \frac{1}{2\pi} \log \frac{R}{\epsilon}$$

and

$$\delta(0, E) = \lim_{\epsilon \rightarrow 0} \left(d_{\Omega}(\partial B_{\epsilon}, E) - \frac{1}{2\pi} \log \frac{1}{\epsilon} \right) \geq \frac{1}{2\pi} \log R. \quad (3.4)$$

By Theorem IV.4.2, equality holds in (3.4) when E is the circle of radius R and Ω is the disc of radius R with radial slits removed. \square

Gehring–Hayman Inequality (Exercise III.16): *Let $\varphi : \mathbb{D} \rightarrow \Omega$ be univalent and assume φ extends continuously to the open arc $\sigma = \partial\mathbb{D} \cap \{\operatorname{Im} z > 0\}$. Then*

$$\ell(\varphi((-1, 1))) \leq K\ell(\varphi(\sigma)),$$

with constant K not depending on φ .

Proof: Set $I_j = [1 - 2^{-j}, 1 - 2^{-j-1}]$ and $\sigma_j = \sigma \cap \left(B(1, 2^{-j}) \setminus \overline{B(1, 2^{-j-1})} \right)$, for $j = 0, 1, \dots$

Then clearly

$$\inf_{z \in I_j} \omega(z, \sigma_j, \mathbb{D}) \geq c_1 > 0. \quad (\text{E.8})$$

We may assume $l_j = l(\varphi(\sigma_j)) < \infty$.

Suppose for the moment that for all j we have

$$\sup_{I_j} |\varphi'(z)| \leq c_2 2^j l_j \tag{E.9}$$

Then summing over j , and doing the same for $-I_j$ we get

$$\int_{-1}^1 |\varphi'(x)| dx \leq c_2 l(\varphi(\sigma)),$$

which gives the Gehring–Hayman theorem, assuming (E.9).

By Theorem I.4.3 (Koebe's estimate), equation (E.9) holds if and only if

$$\text{dist}(\varphi(z), \partial\Omega) \leq c_3 \ell_j \tag{E.10}$$

for some constant c_3 and for all $z \in I_j$.

Let c_3 be large and let β_j be the arc length midpoint of $\varphi(\sigma_j)$. Then

$$B_j = B(\beta_j, c_3 \ell_j) \supset \varphi(\sigma_j).$$

If (E.10) fails at z_j , then $\varphi(z_j) \notin B_j$. But using the inversion $Tz = \frac{\ell_j}{z - \beta_j}$ and the continuum $E = \mathbb{D} \cap T(\partial\Omega)$, we see by (E.8) and the Beurling projection theorem that (E.10) will hold at z_j if c_3 is sufficiently large. \square

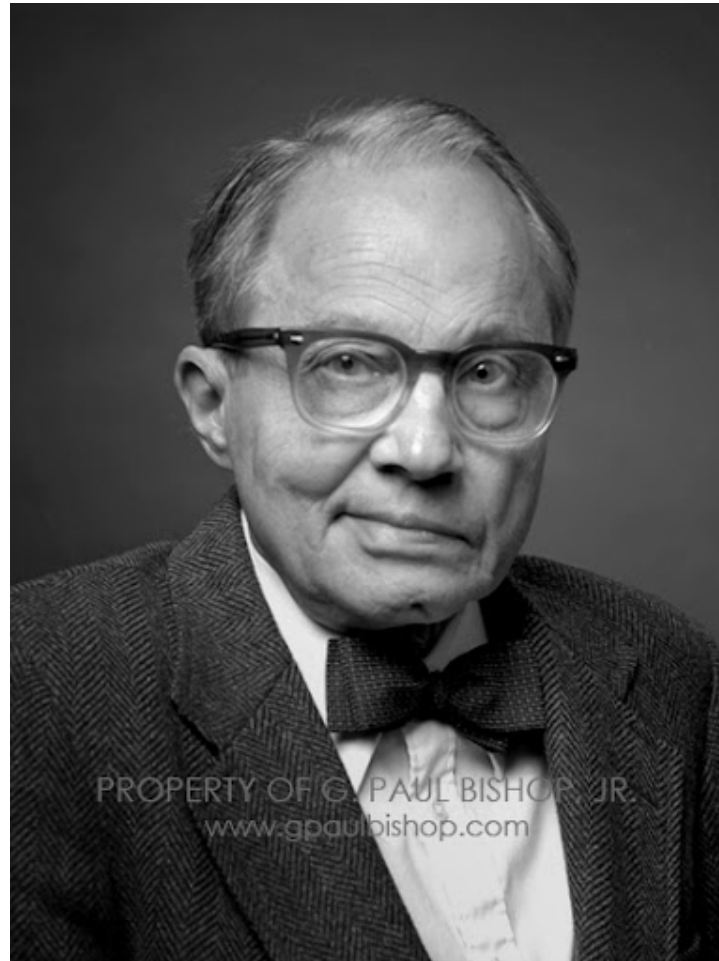
Corollary (Exercise III.16(b)): *If there is an arc $\gamma \subset \mathbb{D}$, joining 0 to 1 such that*

$$\ell(\varphi(\gamma)) = \int_{\gamma} |\varphi'(z)| ds = \ell < \infty,$$

then

$$\ell(\varphi((0, 1))) = \int_0^1 |\varphi'(x)| dx \leq K\ell. \quad (\text{E.11})$$

Proof: Write $I_z = \partial\mathbb{D} \cap \{|\zeta - z| < 2(1 - |z|)\}$, for $z \in \mathbb{D}$. If φ extends continuously to I_z then $\ell(\varphi(I_z)) \geq K' \text{dist}(\varphi(z), \partial\Omega)$. \square



Fred Gehring (1925-2012)



Walter Hayman (1926-2020)

Corollary 3.6 (Balogh-Bonk): *Let $\psi(z)$ be a normalized univalent function, $\psi(0) = 0$ and $\psi'(0) = 1$, in \mathbb{D} and let $R > 0$. Then there is a constant C , independent of ψ and R , such that*

$$\mathfrak{m}\left\{\zeta \in \partial\mathbb{D} : \int_0^1 |\psi'(r\zeta)| dr > \lambda\right\} \leq C\lambda^{-1/2}.$$

Proof Set $\Omega = \psi(\mathbb{D})$. By an approximation, we may suppose that $\partial\Omega$ is an analytic Jordan curve.

By the Gehring-Hayman inequality (Exercise III.16),

$$\int_0^1 |\psi'(r\zeta)| dr \leq K \operatorname{dist}_\Omega(0, \psi(\zeta))$$

for some constant K independent of ψ and ζ . Corollary 3.6 now follows from Theorems 3.2 and 3.5 with $R = \lambda/K$ and $C = \sqrt{K}$. \square

