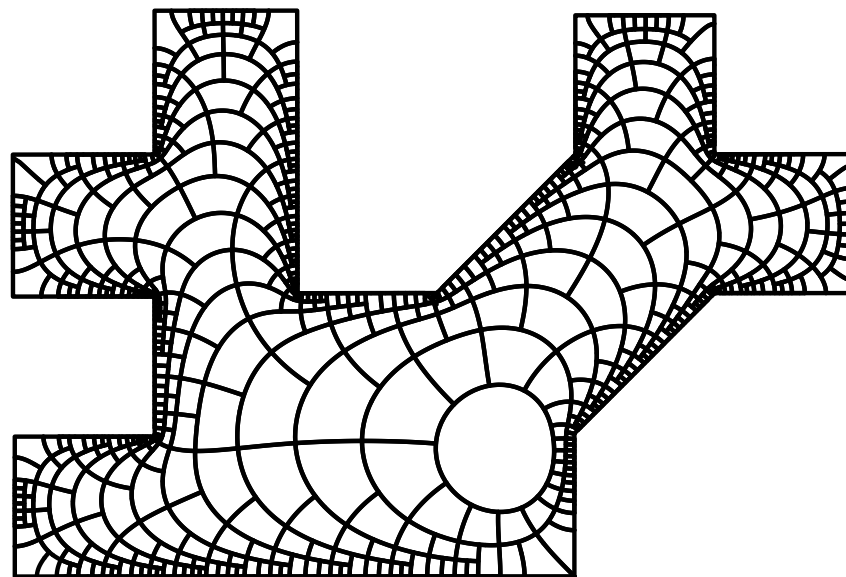
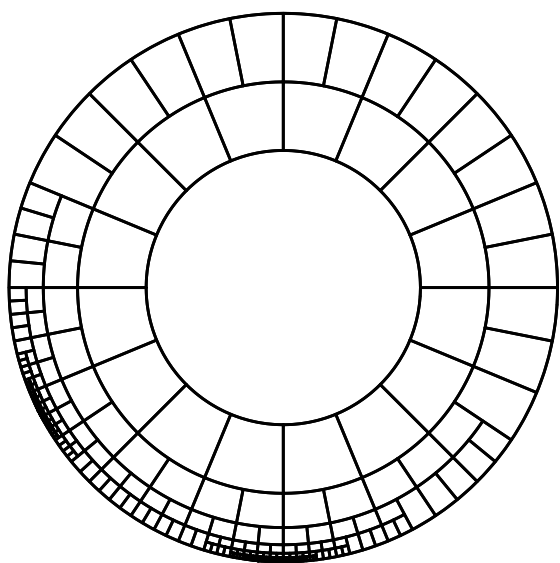


MAT 639, Spring 2026, Stony Brook University

Topics in Real Analysis: Harmonic Measure

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Harmonic Measure, Chapter IV

Extremal Distance

following text by John Garnett and Don Marshall

Goals for Chapter IV:

- Define extremal length and conformal modulus
- Basic properties
- Uniqueness and extremal metrics (summarize)
- The $\int dx/\theta(x)$ estimate

Section IV.1: Definition and Examples

A **path family** in a domain Ω is a non-empty set Γ of countable unions of rectifiable arcs in Ω .

An element $\gamma \in \Gamma$ is called a **curve** even though γ may not be connected and may have many self-intersections. The **Euclidean length** of the path family Γ is

$$\inf_{\gamma \in \Gamma} \int_{\gamma} ds, \quad (1.1)$$

A **metric** is a non-negative Borel measurable function ρ on Ω such that the **area**,

$$A(\Omega, \rho) \equiv \int_{\Omega} \rho^2 dx dy,$$

satisfies $0 < A(\Omega, \rho) < \infty$.

When ρ is a metric and Γ is a path family, we define the **ρ -length** of Γ by

$$L(\Gamma, \rho) = \inf_{\gamma \in \Gamma} \int_{\gamma} \rho |dz|,$$

and the **extremal length** of Γ by

$$\lambda_{\Omega}(\Gamma) = \sup_{\rho} \frac{L(\Gamma, \rho)^2}{A(\Omega, \rho)}. \quad (1.3)$$

A conformal mapping φ transforms the metric ρ on Ω into the metric $\rho(\varphi^{-1}(z))|(\varphi^{-1})'(z)|$ on $\varphi(\Omega)$.

Therefore $\lambda_{\varphi(\Omega)}(\varphi(\Gamma)) = \lambda_{\Omega}(\Gamma)$ and the extremal length (1.3) is conformally invariant.

An **extremal metric** for Γ is a metric ρ which attains the supremum (1.3).

In many important cases there exists an extremal metric of the form $|\varphi'(z)|$ for some conformal mapping φ , and in those cases (1.3) is just the square of (1.2).

The advantage of the more general expression (1.3) is that every metric ρ provides a lower bound for $\lambda_\Omega(\Gamma)$. Indeed, we often use metrics not of the form $|\varphi'(z)|$.

Because of its homogeneity, the ratio (1.3) is unchanged if the metric ρ is multiplied by a positive constant. Thus we can normalize the metric to satisfy $A(\Omega, \rho) = 1$ so that

$$\lambda_{\Omega}(\Gamma) = \sup\{L^2(\Gamma, \rho) : A(\Omega, \rho) = 1\},$$

or we can normalize the metric by $L(\Gamma, \rho) = 1$ so that

$$\lambda_{\Omega}(\Gamma)^{-1} = \inf\{A(\Omega, \rho) : L(\Gamma, \rho) = 1\}.$$

The quantity $\lambda_{\Omega}(\Gamma)^{-1}$ is called the **modulus** of Γ .

When $E \subset \bar{\Omega}$ and $F \subset \bar{\Omega}$ the **extremal distance** $d_{\Omega}(E, F)$ from E to F is defined by

$$d_{\Omega}(E, F) = \lambda_{\Omega}(\Gamma),$$

where Γ is the family of connected arcs in Ω that join E and F .

The **conjugate extremal distance**, $d_{\Omega}^*(E, F)$, is defined to be the extremal length of the family Γ^* of curves that separate E from F .

To compute $d_{\Omega}^*(E, F)$ we allow a curve $\gamma^* \in \Gamma^*$ to be any finite union (not necessarily connected) of arcs and closed curves in Ω such that E and F lie in the boundaries of distinct components of $\Omega \setminus \gamma$.

Example 1.1 (rectangle): *If $R = \{(x, y) : 0 < x < \ell \text{ and } 0 < y < h\}$ is a rectangle vertical sides E and F is*

$$d_R(E, F) = \frac{\ell}{h}. \quad (1.4)$$

Proof: Let Γ be the family of connected arcs in \mathbb{R} joining E and F and let ρ be a metric on \mathbb{R} . Then

$$L^2(\Gamma, \rho) \leq \left(\int_0^\ell \rho(x + iy) dx \right)^2 \leq \ell \int_0^\ell \rho^2(x + iy) dx \quad (1.5)$$

by the Cauchy–Schwarz inequality.

Integrating (1.5) with respect to y gives

$$L^2(\Gamma, \rho)h \leq \ell A(\mathbb{R}, \rho)$$

and hence

$$d_{\mathbb{R}}(E, F) = \sup_{\rho} \frac{L^2(\Gamma, \rho)}{A(\mathbb{R}, \rho)} \leq \frac{\ell}{h}. \quad (1.6)$$

Equality holds when $\rho = 1$, and that proves (1.4). \square

By the condition for equality in the Cauchy–Schwarz inequality, (1.5) is an equality if and only if ρ is constant almost everywhere $dx dy$.

Therefore every extremal metric is constant almost everywhere.

The conjugate extremal distance $d_{\mathbb{R}}^*(E, F)$ is the extremal distance between the two horizontal sides of \mathbb{R} , and thus $d_{\mathbb{R}}^*(E, F) = \frac{h}{\ell}$.

More generally, let Ω be a Jordan domain, let $E = [\zeta_1, \zeta_2]$ and $F = [\zeta_3, \zeta_4]$ be two disjoint subarcs of $\partial\Omega$, where $\zeta_1, \zeta_2, \zeta_3$, and ζ_4 are listed in the counterclockwise ordering of $\partial\Omega$.

There exist a unique $\ell > 0$ and a conformal mapping φ from Ω to a rectangle $R = \{(x, y) : 0 < x < \ell, 0 < y < 1\}$ so that $\zeta_1, \zeta_2, \zeta_3$, and ζ_4 are mapped respectively to the corners $i, 0, \ell$, and $\ell + i$.

To prove the rectangle exists, map Ω to a half-plane and apply a Schwarz–Christoffel map.

To prove ℓ is unique use equality (1.4) and the conformal invariance of extremal lengths or apply Schwarz reflection to the map between two rectangles. By conformal invariance

$$\ell = d_{\Omega}(E, F)$$

and the extremal metrics for $d_{\Omega}(E, F)$ are the constant multiples of $|\varphi'|$.

Thus (1.3) is the square of (1.2) in this case. For the rectangle we also have

$$d_{\Omega}(E, F) d_{\Omega}^*(E, F) = \ell \cdot \frac{1}{\ell} = 1. \quad (1.7)$$

Very useful fact:

Every metric gives a lower bound for $d_{\Omega}(E, F)$.

By Equality (1.7) every metric also gives a lower bound for $d_{\Omega}^*(E, F)$, and when (1.7) holds every lower bound for $d_{\Omega}^*(E, F)$ provides an upper bound for $d_{\Omega}(E, F)$.

Thus for extremal distance we can avoid Cauchy-Schwarz type arguments and give both upper and lower bounds by guessing good metrics.

Example 1.2 (annulus): *The extremal distance between the two boundary circles C_r and C_R of the annulus $A = \{z : r < |z| < R\}$ is*

$$d_A(C_r, C_R) = \frac{1}{2\pi} \log \left(\frac{R}{r} \right).$$

Proof: Since (1.3) is a supremum, we can assume $0 < r < R < \infty$. Let Γ be the family of connected arcs in A joining C_r to C_R and let ρ be a metric on A . Then by Cauchy–Schwarz,

$$L^2(\Gamma, \rho) \leq \left(\int_r^R \rho(te^{i\theta}) dt \right)^2 \leq \log\left(\frac{R}{r}\right) \int_r^R \rho(te^{i\theta})^2 t dt. \quad (1.8)$$

Integrating (1.8) with respect to θ gives

$$2\pi L^2(\Gamma, \rho) \leq \log\left(\frac{R}{r}\right) A(A, \rho)$$

and consequently

$$d_A(C_r, C_R) = \sup_{\rho} \frac{L^2(\Gamma, \rho)}{A(A, \rho)} \leq \frac{1}{2\pi} \log\left(\frac{R}{r}\right). \quad (1.9)$$

We will get equality holds in (1.8) if and only if $\rho(z) = c/|z|$ a.e., where c is a positive constant, and for such ρ we also get equality in (1.9). Thus $d_A(C_r, C_R) = \frac{1}{2\pi} \log\left(\frac{R}{r}\right)$. \square

A **ring domain** is a doubly connected plane Ω . By Exercise 7 of Chapter II every ring domain is conformally equivalent to an annulus of the form $A = \{z : r < |z| < R\}$.

Therefore the module of the family Γ of curves in Ω separating the two components of $\partial\Omega$ is $\lambda_\Omega(\Gamma)^{-1} = \frac{1}{2\pi} \log\left(\frac{R}{r}\right)$, and an extremal metric is $\frac{|\psi'(z)|}{|\psi(z)|}$ where $\psi : \Omega \rightarrow A$ is a conformal map.

The quantity $\lambda_\Omega(\Gamma)^{-1}$ is also called the **module** of the ring domain Ω and denoted by

$$\lambda_\Omega(\Gamma)^{-1} = \text{mod}(\Omega) = \frac{1}{2\pi} \log\left(\frac{R}{r}\right).$$

Section IV.2: Uniqueness of Extremal Metrics

It is an open problem to determine when a path family Γ has an extremal metric. However, if an extremal metric does exist, then up to a multiplicative constant it is unique area almost everywhere.

Theorem 2.1: *Let Γ be a path family on Ω and let ρ_1 and ρ_2 be metrics on Ω satisfying*

$$\frac{L^2(\Gamma, \rho_1)}{A(\Omega, \rho_1)} = \frac{L^2(\Gamma, \rho_2)}{A(\Omega, \rho_2)}.$$

Let $\rho_3 = c_1\rho_1 + c_2\rho_2$, where $c_j = \frac{1}{2}A(\Omega, \rho_j)^{-\frac{1}{2}}$. Then

$$\frac{L^2(\Gamma, \rho_3)}{A(\Omega, \rho_3)} \geq \frac{L^2(\Gamma, \rho_1)}{A(\Omega, \rho_1)}. \quad (2.1)$$

If equality holds in (2.1) then $c_1\rho_1 = c_2\rho_2$ a.e. $dxdy$. In particular, if ρ_1 and ρ_2 are extremal metrics for Γ , normalized by $A(\Omega, \rho_1) = A(\Omega, \rho_2)$, then $\rho_1 = \rho_2$ a.e. $dxdy$.

Proof: Without loss of generality, we may assume $A(\Omega, \rho_1) = A(\Omega, \rho_2) = 1$. Then $L(\Gamma, \rho_1) = L(\Gamma, \rho_2)$ and by the definition of $L(\Gamma, \rho)$,

$$L(\Gamma, \rho_3) \geq \frac{1}{2} \left(L(\Gamma, \rho_1) + L(\Gamma, \rho_2) \right) = L(\Gamma, \rho_1).$$

On the other hand, by the Cauchy–Schwarz inequality,

$$A(\Omega, \rho_3) = \frac{1}{4}A(\Omega, \rho_1) + \frac{1}{4}A(\Omega, \rho_2) + \frac{1}{2} \int_{\Omega} \rho_1 \rho_2 dx dy \leq 1. \quad (2.2)$$

Therefore (2.1) holds. If equality occurs in (2.1), then $A(\Omega, \rho_3) = 1$ and equality occurs in (2.2), so that $\rho_1 = \rho_2$ almost everywhere $dx dy$, again by the condition for equality in the Cauchy–Schwarz inequality.

In particular, if ρ_1 and ρ_2 are extremal metrics, then equality holds in (2.1) and $\rho_1 = \rho_2$ almost everywhere. \square

Section IV.3: Four Rules for Extremal Length

1. The extension rule *Let $\Omega \subset \Omega'$ be domains and let Γ be a path family in Ω . Then*

$$\lambda_{\Omega'}(\Gamma) = \lambda_{\Omega}(\Gamma). \quad (3.1)$$

Moreover, if Γ' is a path family in Ω' such that every $\gamma' \in \Gamma'$ contains some $\gamma \in \Gamma$, then

$$\lambda_{\Omega'}(\Gamma') \geq \lambda_{\Omega}(\Gamma). \quad (3.2)$$

Equality (3.1) says that the extremal length depends on the path family Γ and not the domain Ω , and for this reason we often write $\lambda(\Gamma)$ for $\lambda_{\Omega}(\Gamma)$.

The extension rule shows that $d_{\Omega}(E, F)$ is decreased if any of the sets E, F , or Ω is increased and $d_{\Omega}(E, F) = d_{\Omega \setminus (E \cup F)}(E, F)$ when E and F are closed.

Proof: The curves in the family Γ' are longer and fewer than the curves in the family Γ . Therefore $\lambda_{\Omega'}(\Gamma') \geq \lambda_{\Omega'}(\Gamma)$ and inequality (3.2) will follow from equality (3.1).

Let ρ' be any metric on Ω' and set $\rho = \rho'|_{\Omega}$. Then $L(\Gamma, \rho) = L(\Gamma, \rho')$ and $A(\Omega, \rho) \leq A(\Omega', \rho')$. Taking the supremum over ρ' yields $\lambda_{\Omega'}(\Gamma) \leq \lambda_{\Omega}(\Gamma)$.

Now let ρ be any metric on Ω and define the metric ρ' on Ω' by $\rho' = \rho\chi_{\Omega}$. Then $L(\Gamma, \rho) = L(\Gamma, \rho')$ and $A(\Omega, \rho) = A(\Omega', \rho')$.

Taking the supremum over ρ , we get $\lambda_{\Omega'}(\Gamma) \geq \lambda_{\Omega}(\Gamma)$, which proves (3.1). \square

2. The serial rule: *Let Γ_1 and Γ_2 be path families contained in disjoint open sets Ω_1 and Ω_2 respectively, and let Γ be a path family contained in a domain $\Omega \supset \Omega_1 \cup \Omega_2$. If each $\gamma \in \Gamma$ contains some $\gamma_1 \in \Gamma_1$ and some $\gamma_2 \in \Gamma_2$, then*

$$\lambda(\Gamma) \geq \lambda(\Gamma_1) + \lambda(\Gamma_2).$$

Proof: If either $\lambda(\Gamma_1)$ or $\lambda(\Gamma_2)$ is 0 or ∞ , the result follows from the extension rule.

Otherwise, choose metrics ρ_1 in Ω_1 and ρ_2 in Ω_2 , normalized by the conditions $L(\Gamma_1, \rho_1) = A(\Omega_1, \rho_1)$ and $L(\Gamma_2, \rho_2) = A(\Omega_2, \rho_2)$.

Then the metric $\rho = \rho_1\chi_{\Omega_1} + \rho_2\chi_{\Omega_2}$ on Ω satisfies $L(\Gamma, \rho) \geq L(\Gamma_1, \rho_1) + L(\Gamma_2, \rho_2)$ and $A(\Omega, \rho) = A(\Omega_1, \rho_1) + A(\Omega_2, \rho_2) = L(\Gamma_1, \rho_1) + L(\Gamma_2, \rho_2)$. Thus

$$\frac{L^2(\Gamma, \rho)}{A(\Omega, \rho)} \geq L(\Gamma_1, \rho_1) + L(\Gamma_2, \rho_2) = \frac{L^2(\Gamma_1, \rho_1)}{A(\Omega_1, \rho_1)} + \frac{L^2(\Gamma_2, \rho_2)}{A(\Omega_2, \rho_2)}.$$

Taking the supremum over all ρ_1 and ρ_2 , we obtain $\lambda(\Gamma) \geq \lambda(\Gamma_1) + \lambda(\Gamma_2)$. \square

3. The parallel rule *Let Γ_1 and Γ_2 be path families contained in disjoint open sets Ω_1 and Ω_2 respectively. If Γ is a path family in $\Omega \supset \Omega_1 \cup \Omega_2$ such that every $\gamma \in \Gamma_1 \cup \Gamma_2$ contains some $\gamma' \in \Gamma$, then*

$$\frac{1}{\lambda(\Gamma)} \geq \frac{1}{\lambda(\Gamma_1)} + \frac{1}{\lambda(\Gamma_2)}. \quad (3.3)$$

Proof: If $\lambda(\Gamma) > 0$, let ρ be a metric on Ω , normalized by $L(\Gamma, \rho) = 1$.

Then $L(\Gamma_1, \rho) \geq 1$ and $L(\Gamma_2, \rho) \geq 1$ and

$$A(\Omega, \rho) \geq A(\Omega_1, \rho) + A(\Omega_2, \rho) \geq \frac{1}{\lambda(\Gamma_1)} + \frac{1}{\lambda(\Gamma_2)}.$$

Taking the supremum (1.3) over all ρ , we obtain (2.3). \square

4. The symmetry rule Let $T : \Omega \rightarrow \Omega$ satisfy $T \circ T(z) = z$ and suppose that either $T(z)$ or $\overline{T(z)}$ is analytic. If Γ is a path family in Ω such that $T(\Gamma) = \Gamma$, then

$$\lambda(\Gamma) = \sup \left\{ \frac{L^2(\Gamma, \rho)}{A(\Omega, \rho)} : \rho = (\rho \circ T) |J_T| \right\}, \quad (3.4)$$

where $|J_T| = |T'|$ when T is analytic and $|J_T| = |\overline{T}'|$ when $\overline{T(z)}$ is analytic.

Proof: If $\rho_1 = (\rho \circ T)|J_T|$ then

$$\int_{\gamma} \rho |dz| = \int_{T^{-1}(\gamma)} \rho_1 |dz|.$$

Because T maps Γ onto Γ , $L(\Gamma, \rho) = L(\Gamma, \rho_1)$, and because T is one-to-one, $A(\Omega, \rho) = A(\Omega, \rho_1)$. Then by Theorem 2.1, $\rho_2 = \frac{1}{2}(\rho + \rho_1)$ satisfies

$$\frac{L^2(\Gamma, \rho_2)}{A(\Omega, \rho_2)} \geq \frac{L^2(\Gamma, \rho)}{A(\Omega, \rho)}.$$

But since $T \circ T(z) = z$, we have $\rho_2 \circ T|J_T| = \rho_2$, and hence (3.4) holds. \square

There is an analogy between extremal distance and electrical resistance.

Suppose a conductor is a uniformly thin sheet of material in the shape of a Jordan domain Ω . Put a copper coating on two arcs of $\partial\Omega$ and attach each copper arc to a terminal of the battery.

Then the electrical resistance of the conductor is the extremal distance between the two intervals, the magnitude of the electric field is the extremal metric, the “field lines” are the shortest curves in the extremal metric, and the “equipotential lines” are the shortest curves for the conjugate extremal distance.

Rules for extremal length correspond to analogous rules governing electric circuits.

Section IV.4: Extremal Metrics for Extremal Distance

Let

$$R = \{(x, y) : 0 < x < 1, 0 < y < h\}$$

be a rectangle with sides parallel to the axes, let $\Omega = R \setminus \bigcup L_j$ where $\{L_j\}$ is a finite family of horizontal line segments in R , and let E and F be the vertical sides of the rectangle R .

Then

$$d_{\Omega}(E, F) = d_R(E, F)$$

by the argument used in Example 1.1. Moreover, every extremal metric for $d_{\Omega}(E, F)$ is constant almost everywhere.

The same holds for the conjugate extremal distance

$$d_{\Omega}^*(E, F) = d_{\mathbb{R}}^*(E, F)$$

because by definition the “curves” that separate E from F need not be connected.

Next suppose there is a conformal mapping $\varphi : \Omega \rightarrow \mathbb{R} \setminus \bigcup L_j$, where \mathbb{R} and L_j are as above. such that φ is continuous on $\overline{\Omega}$, $\varphi(E)$ is the left vertical side of \mathbb{R} , and $\varphi(F)$ is the right vertical side of \mathbb{R} .

Then by the conformal invariance of extremal length and the preceding observation,

$$d_{\Omega}(E, F) = d_{\varphi(\Omega)}(\varphi(E), \varphi(F)) = 1/h = 1/d_{\Omega}^*(E, F),$$

and an extremal metric for $d_{\Omega}(E, F)$ is $\rho_0(z) = |\varphi'(z)|$.

Theorem 4.1: *Let Ω be a Jordan domain and let E and F be finite unions of closed subarcs of $\partial\Omega$. Assume $E \cap F = \emptyset$. Then there is a rectangle R having sides parallel to the axes and a conformal map φ of Ω onto the rectangle R with a finite number of horizontal line segments removed such that $\varphi \in C(\overline{\Omega})$ and $\varphi(E)$ and $\varphi(F)$ are the vertical sides of the rectangle if and only if there is an arc $\sigma \subset \partial\Omega$ such that*

$$E \subset \sigma \quad \text{and} \quad F \cap \sigma = \emptyset. \quad (4.1)$$

In this case, the extremal distance from E to F is the ratio of the length to height of this rectangle, the conjugate extremal distance satisfies

$$d_{\Omega}^*(E, F) = 1/d_{\Omega}(E, F),$$

and the extremal metrics on Ω for $d_{\Omega}(E, F)$ are the positive constant multiples of $\rho_0(z) = |\varphi'(z)|$.

Proof: Assume φ is a conformal map of Ω onto the rectangle \mathbb{R} with a finite number of horizontal line segments removed such that $\varphi(E)$ and $\varphi(F)$ are the vertical sides of \mathbb{R} .

Let γ be a curve in $\varphi(\Omega)$ connecting the top and bottom edges of the rectangle. The curve γ divides $\varphi(\Omega)$ into two regions U_1 and U_2 such that $\varphi(E) \subset \partial U_1$ and $\varphi(F) \subset \partial U_2$.

Then $\varphi^{-1}(\gamma)$ is a curve in Ω with two endpoints on $\partial\Omega$. Because Ω is a Jordan curve, these endpoints divide $\partial\Omega$ into two arcs σ_1 and σ_2 with $E \subset \sigma_1$ and $F \subset \sigma_2$.

Conversely, assume there is an arc σ such that (4.1) holds.

By the conformal invariance of extremal distance we may suppose that Ω is the unit disc \mathbb{D} and that $E \cup F \subset \partial\mathbb{D}$.

Let $\Omega_d = \mathbb{C}^* \setminus (E \cup F)$ and let $\omega(z) = \omega(z, F, \Omega_d)$ be the harmonic measure of F in Ω_d .

Let $\tilde{\omega}(z)$ be the harmonic conjugate of ω in \mathbb{D} .

We claim that $\varphi = \omega + i\tilde{\omega}$ is the conformal map promised by the theorem.

Because ω is harmonic in Ω_d , φ extends to be analytic across $\partial\mathbb{D} \setminus (E \cup F)$. By the symmetry of Ω_d and ω ,

$$\frac{\partial\tilde{\omega}}{\partial\theta} = \frac{\partial\omega}{\partial r} = 0 \quad (4.2)$$

on $\partial\mathbb{D} \setminus (E \cup F)$.

Let $\zeta \in E^\circ$, where E° is the relative interior of E in $\partial\mathbb{D}$.

Because $\omega = 0$ on E , we can by the Schwarz reflection principle extend φ to be analytic in a neighborhood W of ζ . The extension satisfies $\operatorname{Re}\varphi > 0$ on $W \cap \mathbb{D}$ and $\operatorname{Re}\varphi < 0$ on $W \setminus \overline{\mathbb{D}}$. This implies $\varphi'(\zeta) \neq 0$ and $\partial\omega/\partial r \leq 0$ by Lemma II.2.4.

Then because $\partial\omega/\partial\theta = 0$, we obtain

$$\frac{\partial\tilde{\omega}}{\partial\theta} = \frac{\partial\omega}{\partial r} < 0 \quad (4.3)$$

at ζ , and hence on all of E° . Applying the same argument to $1 - \varphi$ on F° we obtain

$$\frac{\partial\tilde{\omega}}{\partial\theta} = \frac{\partial\omega}{\partial r} > 0 \quad (4.4)$$

on F° .

Let ζ be an endpoint of $E \cup F$ and let W be a neighborhood of ζ so that $U = W \setminus \gamma$ is simply connected, where γ is the component of $E \cup F$ containing ζ .

Then ω is continuous on W and $\varphi = \omega + i\tilde{\omega}$ is analytic in U . By mapping $\mathbb{C}^* \setminus \gamma$ to the upper half-plane and using Schwarz reflection again, we see that φ extends to be continuous on $\mathbb{D} \cup \{\zeta\}$.

We conclude that φ is continuous on $\overline{\mathbb{D}}$.

Follow $\varphi(z)$ as z traverses $\partial\mathbb{D}$ counterclockwise.

On each component of E , $\omega = 0$ and $\tilde{\omega}$ is strictly decreasing, by (4.3).

On each component of F , $\omega = 1$ and $\tilde{\omega}$ is strictly increasing, by (4.4).

On any component γ_k of $\partial\mathbb{D} \setminus (E \cup F)$, $0 < \omega < 1$ and $\tilde{\omega}$ is constant, by (4.2).

If both endpoints of γ_k are in E , then the curve $\varphi(\gamma_k)$ traces a horizontal line segment $L_k \subset \{z : 0 \leq \operatorname{Re}z \leq 1\}$, beginning and ending on $\{z : \operatorname{Re}z = 0\}$.

If both endpoints are in F , then $\varphi(\gamma_k)$ traces a horizontal segment $L_k \subset \{z : 0 \leq \operatorname{Re}z \leq 1\}$, beginning and ending on $\{z : \operatorname{Re}z = 1\}$.

There are only two components γ_k having endpoints on both E and F ; on these $\tilde{\omega}$ is constant and $\omega = \operatorname{Re}\varphi$ goes from 0 to 1 or from 1 to 0.

Thus $\varphi(\partial\mathbb{D})$ contains the boundary of a rectangle R and the contour $\varphi(\partial\mathbb{D})$ has winding number 1 about each point in $R \setminus \bigcup L_k$ and winding number 0 about each point in $\mathbb{C} \setminus \bar{R}$.

The argument principle then shows that

$$\varphi(\mathbb{D}) = R \setminus \bigcup L_k$$

and that φ is one-to-one.

The statements about the extremal distance and the extremal metrics now follow from Theorem 2.1 and the observation made at the beginning of this section.

□

Theorem 4.2: *Suppose Ω is bounded by two disjoint Jordan curves Γ_1, Γ_2 and suppose E and F are finite unions of closed arcs with $E \subset \Gamma_1$ and $F \subset \Gamma_2$. Then there is a conformal map ψ of Ω onto an annulus with finitely many radial slits removed such that ψ is continuous on $\bar{\Omega}$ and the images of E and F are the two boundary circles. The extremal metrics for the extremal distance between E and F are constant multiples of $|\psi'(z)|/|\psi(z)|$ and*

$$d_{\Omega}(E, F) = \frac{1}{d_{\Omega}^*(E, F)} = \frac{1}{2\pi} \log R,$$

where $R > 1$ is the ratio of the radii of the boundary circles.

For the proof, see the textbook.

Section IV.5: Extremal Distance and Harmonic Measure

Lemma 5.1: *If $E_L = \{z \in \partial R_L : |\operatorname{Re} z| = L\}$ is the union of the vertical edges of R_L , then*

$$e^{-\frac{\pi}{2}L} \leq \omega(0, E_L, R_L) \leq \frac{8}{\pi} e^{-\frac{\pi}{2}L}. \quad (5.1)$$

The estimates in (5.1) are sharp, because

$$\lim_{L \rightarrow \infty} \omega(0, E_L, R_L) e^{\frac{\pi}{2}L} = \frac{8}{\pi} \text{ and } \lim_{L \rightarrow 0} \omega(0, E_L, R_L) e^{\frac{\pi}{2}L} = 1. \quad (5.2)$$

Proof: Let S_L be the infinite strip $\{z : \operatorname{Re}z > -L \text{ and } |\operatorname{Im}z| < 1\}$ and let $\omega_L(z)$ be the harmonic measure of the left edge $\partial S_L \cap \{\operatorname{Re}z = -L\}$ in S_L .

Using the conformal map $z \rightarrow e^{\pi z/2}$ and some elementary geometry, we obtain

$$\omega_L(z) = \frac{2}{\pi} \arg \left(\frac{e^{\frac{\pi}{2}z} - (-ie^{-\frac{\pi}{2}L})}{e^{\frac{\pi}{2}z} - ie^{-\frac{\pi}{2}L}} \right). \quad (5.3)$$

In particular, $\omega_L(0) = \frac{4}{\pi} \tan^{-1}(e^{-\frac{\pi}{2}L})$.

By (5.3) $\omega_L(z) \leq \omega_L(L)$ on $\operatorname{Re}z = L$.

Thus for $z \in \partial R_L$,

$$\omega(z, E, R_L) \leq \omega_L(z) + \omega_L(-z) \leq (1 + \omega_L(L))\omega(z, E, R_L).$$

By the maximum principle, this inequality persists at $z = 0$, and so

$$\frac{2\omega_L(0)}{1 + \omega_L(L)} \leq \omega(0, E, R_L) \leq 2\omega_L(0).$$

Because

$$\omega_L(L) = \omega_{2L}(0) = \frac{4}{\pi} \tan^{-1}(e^{-\pi L}),$$

we obtain

$$\frac{\frac{8}{\pi} \tan^{-1}(e^{-\frac{\pi}{2}L})}{1 + \frac{4}{\pi} \tan^{-1}(e^{-\pi L})} \leq \omega(0, E, R_L) \leq \frac{8}{\pi} \tan^{-1}(e^{-\frac{\pi}{2}L}). \quad (5.4)$$

The estimates (5.1) now follow from the elementary inequalities

$$\frac{\pi}{4}t \leq \tan^{-1}(t) \leq \min\left(t, \frac{\pi}{4}\right),$$

valid for $0 \leq t \leq 1$, and both equalities in (5.2) are immediate from (5.4). \square

Now let Ω be a Jordan domain, let E be an arc on $\partial\Omega$ and let $z_0 \in \Omega$.

Consider all Jordan arcs $\sigma \subset \Omega$ that join z_0 to $\partial\Omega \setminus E$, and define

$$\lambda(z_0, E) = \sup_{\sigma} d_{\Omega \setminus \sigma}(\sigma, E),$$

where the supremum is taken over all such Jordan arcs.

Both quantities $\lambda(z_0, E)$ and $\omega(z_0, E, \Omega)$ are conformally invariant but $\omega(z, E, \Omega)$ is strictly increasing in E , while $\lambda(z_0, E)$ is strictly decreasing in E . This means there is a function connecting λ to ω .

Theorem 5.2: *Let Ω be a Jordan domain, let E be a subarc of $\partial\Omega$ and let $z_0 \in \Omega$. Then*

$$e^{-\pi\lambda(z_0, E)} \leq \omega(z_0, E, \Omega) \leq \frac{8}{\pi} e^{-\pi\lambda(z_0, E)}.$$

Moreover

$$\lim_{\lambda \rightarrow \infty} \omega(z_0, E, \Omega) e^{\pi\lambda(z_0, E)} = \frac{8}{\pi} \text{ and } \lim_{\lambda \rightarrow 0} \omega(z_0, E, \Omega) e^{\pi\lambda(z_0, E)} = 1.$$

By Theorem 5.2 every choice of the arc σ and every choice of the metric ρ give an upper bound for $\omega(z_0, E, \Omega)$, because $\lambda(z_0, E)$ and the extremal distance $d_{\Omega \setminus \sigma}(\sigma, E)$ are both suprema. This idea of varying σ is due to Beurling.

Proof: By conformal invariance we may suppose that $\Omega = \mathbb{D}$, that $z_0 = 0$ and that E is an arc on $\partial\mathbb{D}$.

Let E_1 and E_2 be the two disjoint arcs on $\partial\mathbb{D}$ given by $E_1 \cup E_2 = \{e^{i\theta} : e^{2i\theta} \in E\}$. Because the arcs E_1 and E_2 are symmetric about 0, there is a conformal map f of \mathbb{D} onto a rectangle R such that R has center 0 and sides parallel to the axes, such that $f(0) = 0$ and such that $\tilde{E}_j = f(E_j)$, $j = 1, 2$, are the vertical sides of R .

Let σ be any arc in \mathbb{D} connecting 0 to $\partial\mathbb{D} \setminus E$ and let $\tilde{\sigma} = \{z \in R : (f^{-1}(z))^2 \in \sigma\}$. The map \sqrt{z} is conformal on $\mathbb{D} \setminus \sigma$ and the two branches of $f(\sqrt{z})$ map $\mathbb{D} \setminus \sigma$ conformally onto the two components of $R \setminus \tilde{\sigma}$. Hence by the serial rule and by conformal invariance,

$$d_R(\tilde{E}_1, \tilde{E}_2) \geq d(\tilde{E}_1, \tilde{\sigma}) + d(\tilde{\sigma}, \tilde{E}_2) = 2d_{\mathbb{D}}(\sigma, E).$$

Equality holds if $\tilde{\sigma}_0$ is the vertical line segment in R through 0, so that $2\lambda(0, E) = d_R(\tilde{E}_1, \tilde{E}_2)$.

Since $\omega(z^2, E, \mathbb{D}) = \omega(z, E_1 \cup E_2, \mathbb{D})$, we have

$$\omega(0, E, \mathbb{D}) = \omega(0, E_1 \cup E_2, \mathbb{D}) = \omega(0, \tilde{E}_1 \cup \tilde{E}_2, R).$$

Then by Lemma 5.1,

$$e^{-\pi\lambda(0,E)} \leq \omega(0, E, \mathbb{D}) \leq \frac{8}{\pi} e^{-\pi d_{\mathbb{D}}(\sigma, E)}.$$

□

Theorem 5.3: *Let Ω be a Jordan domain, and let E be a finite union of arcs contained in $\partial\Omega$. Then*

$$\omega(z_0, E, \Omega) \leq \frac{8}{\pi} e^{-\pi\lambda(z_0, E)}. \quad (5.5)$$

Proof We can assume $\Omega = \mathbb{D}$ and $z_0 = 0$. Let σ be an arc from 0 to $\partial\mathbb{D} \setminus E$ and set $\lambda = d_{\Omega \setminus \sigma}(\sigma, E)$. Write $\sigma_1 = \{z : z^2 \in \sigma\}$ and $\{e^{i\theta} : e^{2i\theta} \in E\} = E_1 \cup E_2$ where σ_1 separates E_1 from E_2 and $z^2(E_1) = z^2(E_2) = E$.

By Theorem 4.1, there is a conformal map φ of \mathbb{D} onto a rectangle R with horizontal slits removed so that $\tilde{E}_j = \varphi(E_j)$ are the vertical ends of R . Then as in Theorem 5.2 we obtain $d_R(\tilde{E}_1, \tilde{E}_2) \geq 2d_{\mathbb{D}}(\sigma, E)$ and consequently

$$\omega(0, E, \mathbb{D}) \leq \frac{8}{\pi} e^{-\pi\lambda}.$$

Taking the supremum over σ then gives (5.5). \square

Section IV.V: The $\int dx/\theta(x)$ estimate

This section contains one of the most important and useful methods of estimating harmonic measure.

Consider a **strip domain**:

$$\Omega = \{(x, y) : |y - m(x)| < \theta(x)/2, a < x < b\},$$

having varying width $\theta(x)$ and mid-line $y = m(x)$.

Let $a = x_0 < x_1 < \cdots < x_n = b$ and set $\sigma_j = \Omega \cap \{z : \operatorname{Re} z = x_j\}$, where $j = 0, \dots, n$. By the serial rule,

$$d_{\Omega}(\sigma_0, \sigma_n) \geq \sum_{j=1}^n d(\sigma_{j-1}, \sigma_j).$$

If $\Delta x = x_j - x_{j-1}$ is small, the region between σ_{j-1} and σ_j is approximately a thin rectangle having Δx as base, $\theta(x_j)$ as height, and $x_j + im(x_j)$ as the midpoint of its right vertical side.

Under the linear map

$$\frac{z - im(x_j)}{\theta(x_j)},$$

this rectangle is sent to a rectangle centered on \mathbb{R} with height 1 and width $\Delta x/\theta(x_j)$, so that $d(\sigma_{j-1}, \sigma_j) \approx \Delta x/\theta(x_j)$. Consider the (non-analytic) map

$$\Phi : (x, y) \longrightarrow \left(\int_a^x \frac{dt}{\theta(t)}, \frac{y - m(x)}{\theta(x)} \right) \quad (6.1)$$

from Ω onto a rectangle of height 1 and length $\int_a^b \frac{1}{\theta(t)} dt$.

The extremal metric for $d_{\Omega}(\sigma_0, \sigma_n)$ is given by $\rho_0(z) = |\varphi'(z)| = |\nabla \operatorname{Re}\varphi(z)|$, where φ is a conformal map of Ω onto some rectangle, but the key idea here is to replace the analytic φ by Φ and use as metric

$$\rho(z) = |\nabla \operatorname{Re}\Phi| = \frac{1}{\theta(x)}, \quad z = x + iy.$$

If γ is any curve connecting σ_0 to σ_n in Ω , then

$$\int_{\gamma} \rho(z) |dz| \geq \int_a^b \frac{1}{\theta(x)} dx.$$

Moreover

$$\iint_{\Omega} \rho^2 dy dx = \int_a^b \frac{1}{\theta(x)} dx,$$

so that

$$d_{\Omega}(\sigma_0, \sigma_n) \geq \int_a^b \frac{1}{\theta(x)} dx.$$

The extremal metric for $d_\Omega(\sigma_0, \sigma_n)$ is given by $\rho_0(z) = |\varphi'(z)| = |\nabla \operatorname{Re} \varphi(z)|$, where φ is a conformal map of Ω onto some rectangle, but the key idea here is to replace the analytic φ by Φ and use as metric

$$\rho(z) = |\nabla \operatorname{Re} \Phi| = \frac{1}{\theta(x)}, \quad z = x + iy.$$

If γ is any curve connecting σ_0 to σ_n in Ω , then

$$\int_\gamma \rho(z) |dz| \geq \int_a^b \frac{1}{\theta(x)} dx.$$

Moreover

$$\iint_\Omega \rho^2 dy dx = \int_a^b \frac{1}{\theta(x)} dx,$$

so that

$$d_\Omega(\sigma_0, \sigma_n) \geq \int_a^b \frac{1}{\theta(x)} dx.$$

Theorem 6.1: *Let Ω be a Jordan domain and let $z_0 \in \Omega$. Let $b > x_0 = \operatorname{Re} z_0$ and suppose $F \subset \{z \in \partial\Omega : \operatorname{Re} z \geq b\}$. Assume that for $x_0 < x < b$, there exists $I_x \subset \Omega \cap \{\operatorname{Re} z = x\}$ separating z_0 from F and set $\theta(x) \equiv \ell(I_x)$. Then*

$$\omega(z_0, F) \leq \frac{8}{\pi} \exp \left(-\pi \int_{x_0}^b \frac{dx}{\theta(x)} \right). \quad (6.3)$$

Theorem 6.1 remains true if $\theta(x)$ is replaced by $\ell(\{z \in \Omega : \operatorname{Re} z = x\})$, because that change only increases the right side of (6.3). When $\{z \in \Omega : \operatorname{Re} z = x\}$ contains several crosscuts that separate z_0 from F , sharper versions of (6.3) are available.

Proof: Let $\tilde{\Omega}$ denote the points of Ω which are not separated from z_0 by some I_x and let \tilde{F} denote the points of $\partial\tilde{\Omega}$ which are separated from z_0 by all I_x , $x_0 < x < b$.

We can suppose that \tilde{F} is a finite union of arcs on $\partial\tilde{\Omega}$ by replacing Ω with $\varphi^{-1}(|z| < r)$ where φ is a conformal map of Ω onto \mathbb{D} . Let $\sigma \subset \{\operatorname{Re}z = x_0\}$ be a curve connecting z_0 to $\partial\tilde{\Omega}$.

By the maximum principle,

$$\omega(z_0, F, \Omega) \leq \omega(z_0, \tilde{F}, \tilde{\Omega}),$$

and then Theorem 5.3 and inequality (6.2) give (6.3). \square

Theorem G.1 (Carleman): *Let $\Omega \subset \mathbb{C}$ be a domain, let $\Omega_x = \Omega \cap \{\operatorname{Re} z = x\}$ and let $E_b = \partial\Omega \cap \{\operatorname{Re} z \geq b\}$. Suppose $|\Omega_x| \leq M < \infty$, and let $\ell(x)$ denote the length of the longest interval in Ω_x . Assume $z_0 = x_0 + iy_0 \in B(z_0, r_0) \subset \Omega$. Then for $b > x_0$*

$$\omega(z_0, E_b, \Omega) \leq \left(\frac{2\pi r_0}{9M^2} \int_{x_0}^b \exp \left\{ 2\pi \int_{x_0}^t \frac{dx}{\ell(x)} \right\} dt \right)^{-\frac{1}{2}}. \quad (\text{G.1})$$

For a proof, see Appendix G of the textbook.

Theorem 6.2: *Let Ω be a Jordan domain and let $E \subset \bar{\Omega} \cap \{|z| \geq R\}$. If $z_0 \in \Omega$ satisfies $r_0 = \max(|z_0|, \text{dist}(0, \partial\Omega)) < R$. Suppose that $J_r \subset \{z \in \Omega : |z| = r\}$ separates z_0 from E , $r_0 < r < R$, and let $r\Theta(r)$ be the length of J_r . Then*

$$\omega(z_0, E) \leq \frac{8}{\pi} \exp \left(-\pi \int_{r_0}^R \frac{dr}{r\Theta(r)} \right), \quad (6.4)$$

if $\Theta(r)$ is measurable.

Proof: Define the metric

$$\rho_A(z) = \begin{cases} \frac{1}{r^{\Theta(r)}} & \text{in } J_r, \\ & \text{in } \Omega \setminus \bigcup \{J_r : r_0 < r < R\}, \end{cases}$$

for $z \in \Omega$ and $r = |z|$ and repeat the proof of Theorem 6.1. \square

