MAT 639, Spring 2026, Stony Brook University

Topics in Real Analysis: Harmonic Measure

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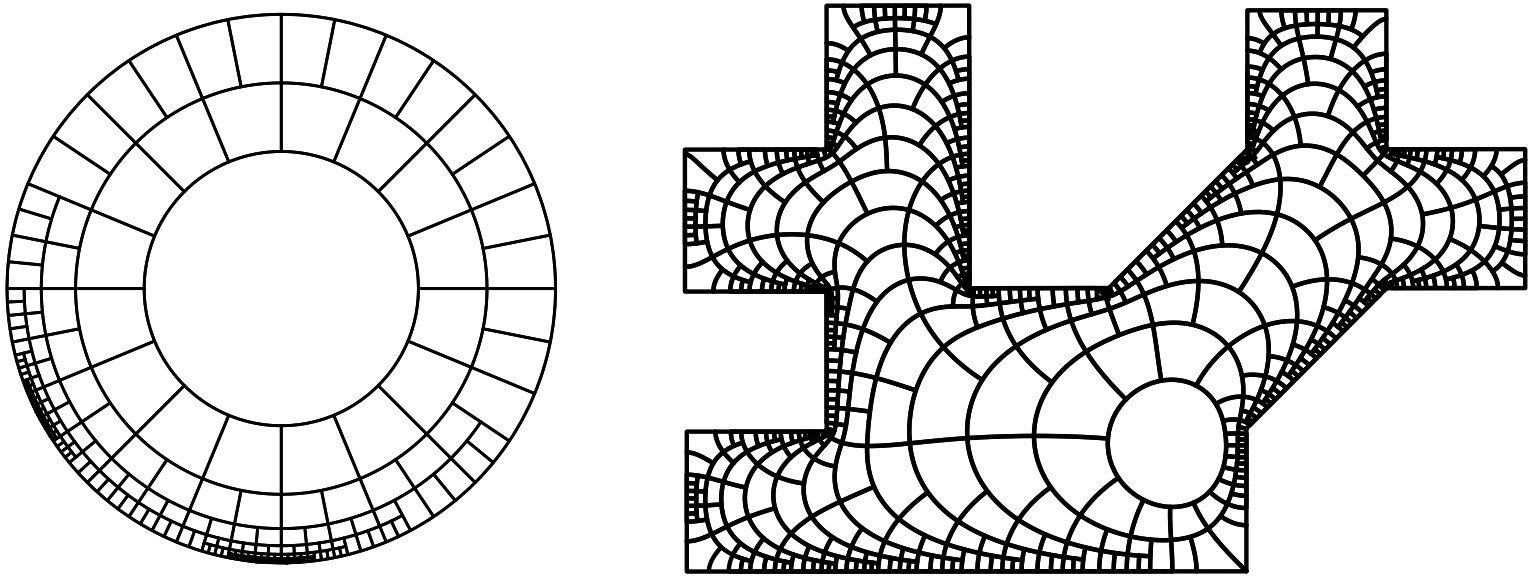


Harmonic Measure, Chapter III

Potential Theory

following text by John Garnett and Don Marshall

Goals for Chapter III:

- Solve Dirichlet problem for general planar domains
- Define logarithmic capacity
- Define and characterize regular points of boundary
- Basic estimates for harmonic measure

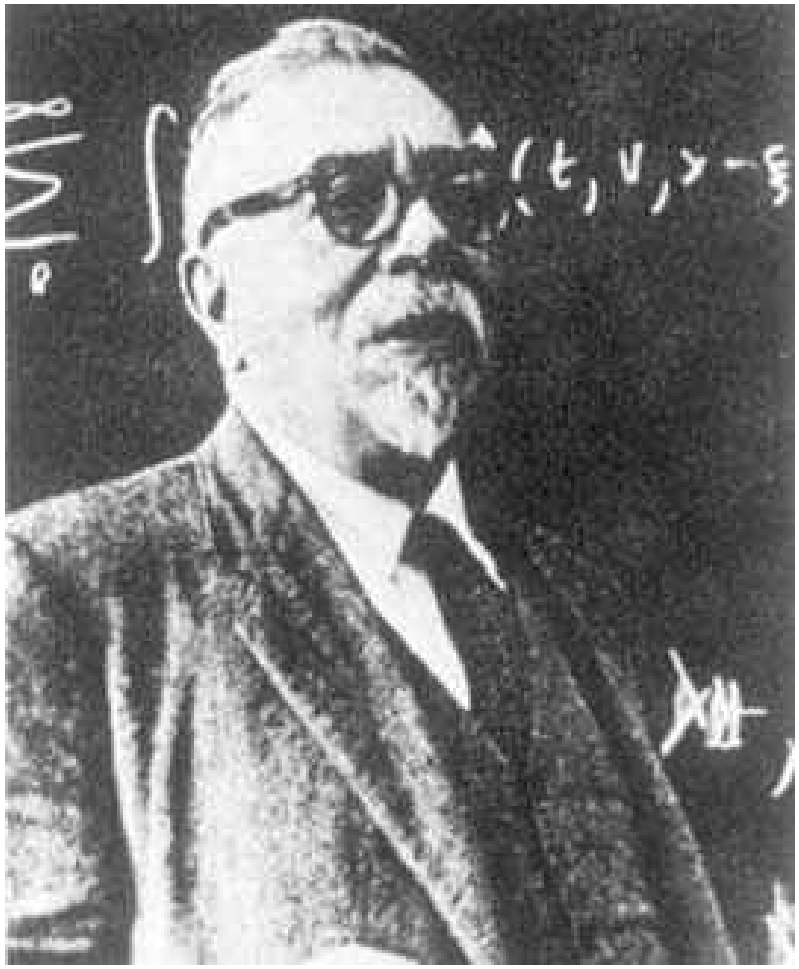
There are three traditional ways to solve the Dirichlet problem:

• The Wiener method is to approximate Ω from inside by subdomains Ω_n of the type studied in Chapter II and to show that the harmonic measures $\omega(z, E, \Omega_n)$ converge weak-star to a limit measure on $\partial\Omega$.

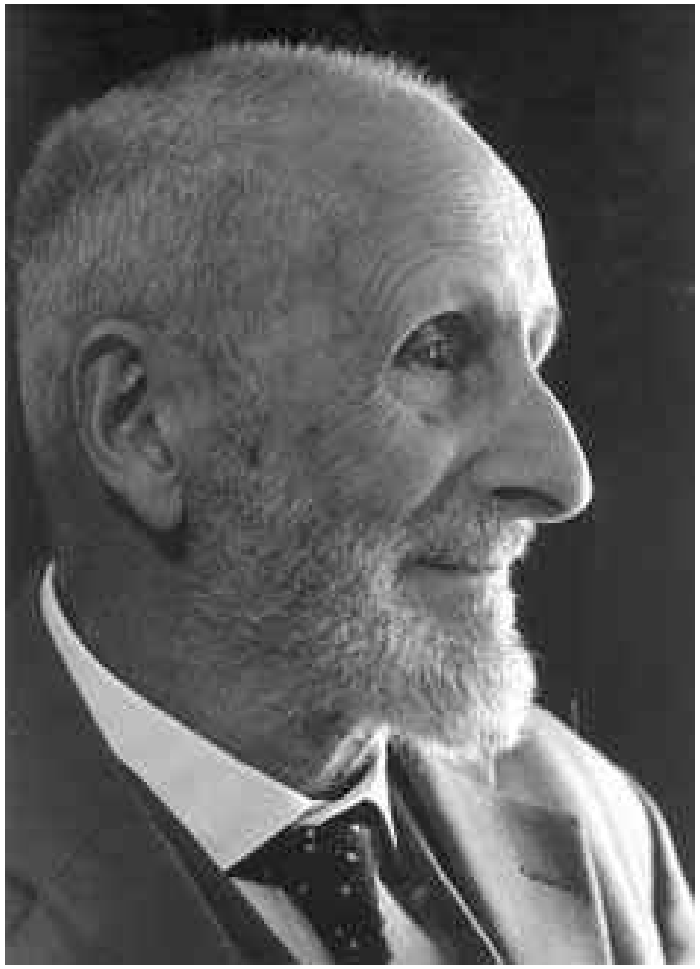
• The Perron method associates to any bounded function f on $\partial\Omega$ a harmonic function u_f on Ω. The function u_f is the upper envelop of a family of subharmonic functions constrained by f on $\partial\Omega$. Perron's method is elegant and general.

• The Brownian motion approach, originally from Kakutani [1944], identifies $\omega(z, E, \Omega)$ with the probability that a randomly moving particle, starting at z, first hits $\partial\Omega$ in the set E. This method has considerable intuitive appeal, but it leaves many theorems hard to reach.

We will follow Wiener's method.



Norbert Wiener (1894-1964)



Oskar Perron (1880-1975)

Shizuo Kakutani (1911-2004)

Section III.1: Capacity and Green's functions

Let E be a compact plane set such that $\Omega = \mathbb{C}^* \setminus E$ is a finitely connected Jordan domain. By Chapter II and a conformal mapping, we see that Ω has Green's function $g_\Omega(z, \infty)$ with pole at ∞, and if $a \notin \overline{\Omega}$

$$g_\Omega(z, \infty) = \log |z - a| + h(z, \infty),$$

where $h(z, \infty)$ is harmonic on Ω and continuous on $\partial\Omega$ and

$$h(\zeta, \infty) = -\log |\zeta - a| \qquad \zeta \in \partial\Omega.$$

The quantity

$$\gamma = \gamma(E) = h(\infty, \infty),$$

is called **Robin's constant** for E, and we have

$$g_\Omega(z, \infty) = \log|z| + \gamma + \mathrm{o}(1), \qquad \text{as } z \to \infty. \tag{1.1}$$

Define the **logarithmic capacity** of E to be $\mathrm{Cap}(E) = e^{-\gamma(E)}$.

Suppose Ω_1 and Ω_2 are finitely connected Jordan domains such that $\infty \in \Omega_j$ and set $E_j = \mathbb{C}^* \setminus \Omega_j$. Suppose $\psi : \Omega_1 \to \Omega_2$ is conformal and for $|z|$ large

$$\psi(z) = az + b_0 + \frac{b_1}{z} + \dots$$

with $a > 0$. Then $g_{\Omega_1}(z, \infty) = g_{\Omega_2}(\psi(z), \infty)$, so that by (1.1)

$$\gamma(E_1) = \gamma(E_2) + \log a \quad \text{and} \quad \mathrm{Cap}(E_2) = a\mathrm{Cap}(E_1). \tag{1.2}$$

$\mathrm{Cap}(D(x, r)) = r$, because $g_{\mathbb{C}^* \setminus \overline{\mathbb{D}}}(z, \infty) = \log |z|$.

Now let E be any compact plane set and write Ω for the component of $\mathbb{C}^* \setminus E$ such that $\infty \in \Omega$. Fix a sequence $\{\Omega_n\}$ of finitely connected domains such that

$$\infty \in \Omega_n \subset \overline{\Omega_n} \subset \Omega_{n+1} \subset \Omega,$$

such that

$$\Omega = \bigcup \Omega_n,$$

and such that $\partial\Omega_n$ consists of $C^{1+\alpha}$ Jordan curves for some $\alpha > 0$. Define $E_n = \mathbb{C} \setminus \Omega_n$. All our results will be independent of the sequence $\{\Omega_n\}$.

Because $\overline{\Omega_n} \subset \Omega_{n+1}$, the maximum principle implies that $g_{\Omega_{n+1}}(z,\infty) > g_{\Omega_n}(z,\infty)$ on Ω_n, and hence that $\gamma(E_{n+1}) > \gamma(E_n)$ and $\mathrm{Cap}(E_{n+1}) < \mathrm{Cap}(E_n)$. Define

$$\mathrm{Cap}(E) = \lim_n \mathrm{Cap}(E_n). \tag{1.3}$$

Because $g_\Omega(z,w)$ is an increasing function of Ω, an interlacing of the domains Ω_n shows that the definition (1.3) does not depend on the choice of the sequence $\{\Omega_n\}$. Note that if $\widehat{E} = \mathbb{C} \setminus \Omega$, then by definition,

$$\mathrm{Cap}(\partial E) = \mathrm{Cap}(E) = \mathrm{Cap}(\widehat{E}) = \mathrm{Cap}(\partial \widehat{E}). \tag{1.4}$$

Here $\widehat{E}$ is "polynomial hull" of $E = E$ plus bounded complementary components.

We define the Robin's constant of E as

$$\gamma(E) \equiv \log\Big(\frac{1}{\mathrm{Cap}(E)}\Big) = \lim_n \gamma(E_n).$$

If $E \subset F$ then by (1.3)

$$\mathrm{Cap}(E) \le \mathrm{Cap}(F) \ \text{ and } \ \gamma(E) \ge \gamma(F). \tag{1.5}$$

If $\mathrm{Cap}(E) > 0$, then $\lim_n \gamma(E_n) = \gamma(E) < \infty$, and by Harnack's principle

$$g_\Omega(z,\infty) = \lim_n g_{\Omega_n}(z,\infty)$$

defines a positive harmonic function on Ω having expansion at ∞

$$g_\Omega(z,\infty) = \log|z| + \gamma(E) + \mathrm{o}(1) \tag{1.6}$$

When $z = \infty$, the symmetry (2.8) of Chapter II shows that

$$\lim_n g_{\Omega_n}(\infty, w) = \lim_n g_{\Omega_n}(w, \infty)$$

exists for all $w \in \Omega \setminus \{\infty\}$. Thus by Harnack's principle the limit

$$g_\Omega(z, w) = \lim_n g_{\Omega_n}(z, w) \tag{1.7}$$

exists for all z, $w \in \Omega$ with $z \neq w$, and $g_\Omega(z, w)$ satisfies conditions (2.2), (2.4), and (2.5) of Chapter II.

The function $g_\Omega(z, w)$ is **Green's function** for Ω **with pole at** w.

It might not be zero everywhere on the boundary (punctured disk).

Computing capacity from Laurent expansion:

Suppose E is compact and connected. Let Ω be the unbounded component of $\mathbb{C}^* \setminus E$ and let $\psi : \Omega \to \mathbb{C}^* \setminus \overline{\mathbb{D}}$ be conformal. Assume $\psi(z) = az + b_0 + \frac{b_1}{z} + \ldots, a > 0$ near ∞.

For $r > 1$, $\Omega_r = \{z : |\psi(z)| > r\}$ is bounded by an analytic Jordan curve and $g_{\Omega_r}(z, \infty) = \log |\psi(z)/r|$. Then by (1.7), $g_\Omega(z, \infty) = \log |\psi(z)|$ and by (1.2),

$$1 = \mathrm{Cap}(\overline{\mathbb{D}}) = a\mathrm{Cap}(E).$$

Example 1.1 (capacity of interval):

The Joukowski map $\frac{1}{2}(z+1/z)$ maps $\{|z|>1\}$ to $\mathbb{C}\setminus[-1,1]$, so we can compute

$$\mathrm{Cap}([-1,1]) = 1/2.$$

Consequently $\mathrm{Cap}([\alpha,\beta]) = \frac{\beta-\alpha}{4}$ for every interval in $\mathbb{R}$.

Example 1.2.a (arc of unit circle):

If $E \subset \partial\mathbb{D}$, is an arc then $\mathrm{Cap}(E) = \sin\big(|E|/4\big)$.

Assume E is centered at i. Let

$$\Omega_1 = \mathbb{C} \setminus E, \qquad \Omega_2 = \mathbb{C} \setminus \sqrt{E}$$
$$\Omega_3 = \mathbb{C} \setminus [-\sin(|E|/4), \sin(|E|/4)]$$

where $\sqrt{E}$ consists of two intervals of length $|E|/2$ centered at $\pm i$. For $j = 1, 2, 3$ let $g_j(z)$ denote the Green's function of Ω_j with pole at ∞ and let γ_j denote the Robin's constants.

Using the uniqueness of Green's functions, is not (too) hard to check that

$$g_2(z) = \frac{1}{2} g_1(z^2) \quad \text{and} \quad g_2(z) + g_2(1/z) = g_3(J(z)),$$

where $J(z) = \frac{1}{2}(z + 1/z)$. By definition, near infinity we have for $j = 1, 2, 3$

$$g_j(z) = \log|z| + \gamma_j + o(1),$$

and using symmetry we can verify that

$$g_z(0) = h(0, \infty) = h(\infty, \infty) = \gamma_2$$

where

$$g_2(z) = \log|a - z| + h(z, \infty)$$

and $h(z, \infty)$ is the harmonic extension of $\log|z - a|$ for some $a \notin \overline{\Omega}$.

Combining the equations gives by Example 1.1.

$$\gamma_1 = 2\gamma_2 \quad \text{and} \quad 2\gamma_2 = \gamma_3 - \log 2$$

$$\gamma_1 = \gamma_3 - \log 2$$

$$\mathrm{Cap}(E) = 2\mathrm{Cap}([-\sin(|E|/4), \sin(|E|/4)]) = 2 \cdot 2\sin(|E|/4)/4 = \sin(|E|)$$

Example 1.2.b: If $E \subset \partial\mathbb{D}$, then $\mathrm{Cap}(E) \geq \sin\big(|E|/4\big)$.

Because of (1.3) we may assume E is a finite union of arcs. Define

$$F(z) = \frac{1}{4}\int_E \frac{e^{i\theta}+z}{e^{i\theta}-z}d\theta,$$

and let $\omega(z) = \omega(z, E, \mathbb{D})$.

Then $\overline{F(1/\overline{z})} = -F(z)$ and $F = \frac{\pi}{2}\big(\omega + i\widetilde{\omega}\big)$ on $\mathbb{D}$. Therefore

$$-\frac{\pi}{2} \leq \mathrm{Re\,F} \leq \frac{\pi}{2}$$

on $\Omega = \mathbb{C}^* \setminus E$ and $F(0) = -F(\infty) = |E|/4$. Hence $H(z) = e^{iF(z)}$ maps Ω into the right half-plane and $H(\infty) = \overline{H(0)}$.

Now

$$G(z) = \log\left| z\frac{H(z) + \overline{H(0)}}{H(z) - H(0)} \right|$$

is superharmonic on Ω and

$$\liminf_{z \ni \Omega \to \partial\Omega} G(z) \geq 0.$$

By the maximum principle, $G(z) > 0$ in Ω so that $G(z) \geq g_{\Omega_n}(z, \infty)$ for all n. For $|z|$ large

$$G(z) = \log|z| - \log\sin\Big(\frac{|E|}{4}\Big) + \ldots,$$

so that by (1.1)

$$-\log\sin\Big(\frac{|E|}{4}\Big) \geq \gamma(\partial\Omega_n) \to \gamma(E)$$

and $\mathrm{Cap}(E) \geq \sin(|E|/4)$.

Section III.2: The Logarithmic Potential

Let μ be a finite, compactly supported signed (Borel) measure. The **logarithmic potential** of μ is the function

$$U_\mu(z) = \int \log \frac{1}{|\zeta - z|} d\mu(\zeta).$$

By Fubini's theorem, the integral U_μ is absolutely convergent for a.e. z (area).

Defn: Superharmonic

$$u(z) \geq \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta}) d\theta$$

for all r sufficiently small.

$\log 1/|z|$ is superharmonic.

Defn: Lower semicontinuous: f real-valued and $f(z) \leq \liminf_{w,\to z} f(w)$.

Alternative: $\{z : f(z) > a\}$ is open for all real a.

Let μ be a finite, compactly supported signed (Borel) measure.

The **logarithmic potential** of μ is the function

$$U_\mu(z) = \int \log \frac{1}{|\zeta - z|} d\mu(\zeta) = \log \frac{1}{|z|} * \mu = \log \frac{1}{|z|} * \mu.$$

By Fubini's theorem, the integral U_μ is absolutely convergent for a.e. z (area).

Lemma 2.1: *If $\mu > 0$, the potential U_μ is lower semicontinuous and superharmonic.*

Proof: The lemma holds because for ζ fixed, the function

$$z \to \log \frac{1}{|\zeta - z|}$$

is lower semicontinuous and superharmonic. $\square$

Suppose $\Omega = \mathbb{C}^* \setminus E$ is bounded by a finite family of disjoint $C^{1+\alpha}$ Jordan curves, $\alpha > 0$, write μ_E for the harmonic measure of ∞ relative to Ω,

$$d\mu_E = d\omega(\infty, \cdot, \Omega) = -\frac{1}{2\pi} \frac{\partial g(\zeta, \infty)}{\partial \mathrm{n}_\zeta} ds = P_\infty(\zeta) ds.$$

Theorem 2.2: *If $\Omega = \mathbb{C}^* \setminus E$ is connected and bounded by finitely many pairwise disjoint $C^{1+\alpha}$ Jordan curves, then the integral $U_{\mu_E}(z)$ is absolutely convergent at every $z \in \mathbb{C}$. The potential U_{μ_E} is continuous on $\mathbb{C}$ and*

$$g(z, \infty) = \gamma(E) - U_{\mu_E}(z), \qquad z \in \overline{\Omega}, \tag{2.1}$$

$$U_{\mu_E}(z) < \gamma(E), \qquad z \in \Omega, \tag{2.2}$$

and

$$U_{\mu_E}(z) = \gamma(E), \qquad z \in E = \mathbb{C}^* \setminus \Omega. \tag{2.3}$$

It turns out that μ_E is the unique probability measure μ on E such that U_μ is constant on E, and μ_E is called the equilibrium distribution for E.

Proof: We can assume $0 \notin \overline{\Omega}$. Clearly, the integral $U_\mu(z)$ is absolutely convergent at all $z \notin \partial\Omega$. Since

$$g(z,\ \infty) = \log|z| - \int_{\partial\Omega} \log|\zeta|\ d\omega(z,\ \zeta),$$

by setting $z = \infty$ we have $\gamma(E) = -\int_{\partial\Omega} \log|\zeta|\ d\mu_E(\zeta)$.

On the other hand, for fixed $z_0 \in \Omega$,

$$g(z, z_0) = \log \left| \frac{z}{z - z_0} \right| - \int_{\partial\Omega} \log \left| \frac{\zeta}{\zeta - z_0} \right| d\omega(z, \zeta), \tag{2.4}$$

because the right side of (2.4) satisfies the conditions (2.3), (2.4), and (2.5) from Chapter II and those conditions determine Green's function.

Then sending z to ∞ yields

$$0 < g(\infty, z_0) = \gamma(E) - \int \log \frac{1}{|\zeta - z_0|} d\mu_E(\zeta) = \gamma(E) - U_{\mu_E}(z_0).$$

For $z \in \Omega$, (2.1) is then a consequence of the symmetry, i.e., $g(\infty, z) = g(z, \infty)$.

Because $g(z, \infty) > 0$ on Ω, (2.1) implies $U_{\mu_E}(z) < \gamma(E), \qquad z \in \Omega$ which is (2.2).

For $z \notin \overline{\Omega}$, $v(\zeta) = \log(|\zeta|/|\zeta - z|)$ is harmonic on a neighborhood of $\overline{\Omega}$ and

$$\begin{aligned} 0 = v(\infty) &= \int \log|\zeta| d\mu_E(\zeta) + \int \log \frac{1}{|\zeta - z|} d\mu_E(\zeta) \\ &= -\gamma(E) + U_{\mu_E}(z). \end{aligned}$$

Therefore $U_{\mu_E}(z) = \gamma(E)$, $\qquad z \in E = \mathbb{C}^* \setminus \Omega$ (this is (2.3)) holds for $z \notin \overline{\Omega}$.

If $z \in \partial\Omega$, the lower semicontinuity of U_μ gives

$$U_\mu(z) \leq \liminf_{\Omega \ni w \to z} U_\mu(w) \leq \gamma(E) < \infty.$$

The integrand is bounded below, so the integral $U_\mu(z)$ converges absolutely. Because $\partial\Omega$ is smooth so $\mathrm{Area}(\partial\Omega) = 0$. By the superharmonicity of U_μ

$$U_\mu(z) \geq \limsup_{\delta \to 0} \int_{B(z,\delta)} U_\mu(\zeta) \frac{d\xi d\eta}{\pi\delta^2}.$$

Since $U_\mu = \gamma(E) - g(\infty, z_0)$ and g tends to zero continuously on E,

$$U_\mu(z) \geq \limsup_{\delta \to 0} \left\{ \int_{\Omega \cap B(z,\delta)} U_\mu(\zeta) \frac{d\xi d\eta}{\pi\delta^2} + \int_{B(z,\delta)\setminus\overline{\Omega}} \gamma \frac{d\xi d\eta}{\pi\delta^2} \right\} = \gamma.$$

Hence (2.1) and (2.3) hold for $z \in \partial\Omega$, and so U_μ is continuous in $\mathbb{C}$. $\square$

Let E be a compact set with $\mathrm{Cap}(E) > 0$, and let $E_n = \mathbb{C}^* \setminus \Omega_n$ be as in Section 1. Then by (1.7) and Theorem 2.2 any weak-star cluster point μ_E of the sequence $\{\mu_{E_n}\}$ satisfies both (2.1) and (2.2) on Ω. Later we will show the limit is unique.

Section III.3: The Energy Integral

Let ν be a signed measure with compact support. If

$$\int\int \left| \log \frac{1}{|z-\zeta|} \right| \, d|\nu|(\zeta) \, d|\nu|(z) < \infty, \tag{3.1}$$

we say ν has **finite energy** and define the **energy integral** $I(\nu)$ by

$$I(\nu) = \int\int \log \frac{1}{|z-\zeta|} d\nu(\zeta) d\nu(z) = \int U_\nu(z) d\nu(z).$$

The energy integral is positive definite:

Theorem 3.1: *If* (3.1) *holds and if* $\int d\nu = 0$, *then* $I(\nu) \geq 0$. *Moreover, if* $I(\nu) = 0$, *then* $\nu \equiv 0$.

Th proof of the theorem will use the following technical lemma (a version of the Poincaré inequality).

Lemma 3.2: *Assume* $U_n(z) \in C^\infty(\mathbb{C})$ *satisfy*

$$|U_n(z)| \leq C/|z|, \tag{3.9}$$

and

$$||\nabla U_n||_2 \to 0.$$

Then

$$\iint_K |U_n|^2 dxdy \to 0, \tag{3.10}$$

for every compact set K.

Proof: Assume $K \subset [-L, L] \times [-R, R]$. Then

$$\begin{aligned}
\int_{[-L,L]} |U_n(x,y)|^2 dx &\leq 2 \int_{-L}^{L} |U_n(x,y) - U_n(x,-L)|^2 dx + 2C^2/L \\
&\leq 2 \int_{-L}^{L} \left(\int_{-R}^{y} \left| \frac{\partial U_n}{\partial y} \right| dy \right)^2 dx + 2C^2/L \\
&\leq 4L ||\nabla U_n||_2^2 + 2C^2/L,
\end{aligned}$$

and a second integration gives (3.10) when L is large compared to R. $\square$

Proof of Theorem 3.1 (energy integral is positive definite):
Write $L(z) = \log \frac{1}{|z|}$. By Green's theorem if $f \in C^\infty$ has compact support, then

$$f(z) = -\frac{1}{2\pi} \iint L(z-w)\Delta f(w) du dv. \tag{3.2}$$

First assume $d\nu = h(z)dxdy$ where $h \in C^\infty$ has compact support and satisfies

$$\iint h(z)dxdy = 0. \tag{3.3}$$

Then the convolution $U_\nu = L * h$ is also C^∞ and by (3.3),

$$|U_\nu(z)| \leq \frac{C}{|z|} \tag{3.4}$$

and

$$|\nabla U_\nu(z)| \leq \frac{C}{|z|^2} \tag{3.5}$$

for $|z|$ large.

For any $f \in C^\infty$ with compact support, Green's theorem (3.2) and Fubini's theorem give

$$\iint \Delta U_\nu f dxdy = \iint U_\nu \Delta f dxdy = -2\pi \iint h f dxdy.$$

Therefore $\Delta U_\nu = -2\pi h$.

Now by (3.4), (3.5), and Green's theorem again

$$I(\nu) = \iint U_\nu h dxdy = -\frac{1}{2\pi} \iint U_\nu \Delta U_\nu dxdy = \frac{1}{2\pi} \iint |\nabla U_\nu|^2 dxdy.$$

Thus $I(\nu) \geq 0$ in this special case.

Moreover, if $I(\nu) = 0$ then $\nabla U_\nu = 0$ and $h = -\frac{1}{2\pi}\Delta U_\nu = 0$, so $\nu = 0$.

Next we do the general case by smoothing a general measure, applying the previous case, and taking a limit.

Let $K \in C_c^\infty(\mathbb{C})$ be a smooth, compactly supported function such that

(i) K is radial, $K(z) = K(|z|)$,

(ii) $K \geq 0$, and

(iii) $\int K dxdy = 1$.

Set $K_\varepsilon(z) = \varepsilon^{-2} K(z/\varepsilon)$ and let ν_ϵ be absolutely continuous with density

$$h_\varepsilon(z) = K_\varepsilon * \nu(z) = \int K_\varepsilon(z-w) d\nu(w).$$

Then for all continuous f

$$\lim_{\epsilon \to 0} \int f d\nu_\epsilon = \int f d\nu, \tag{3.6}$$

and $h_\epsilon \in C^\infty$ has compact support and satisfies (3.3).

By Fubini's theorem and assumption (i) above,

$$I(\nu_\varepsilon) = \iint K_\varepsilon * K_\varepsilon * L(z-\zeta)d\nu(z)d\nu(\zeta),$$

where

$$K_\varepsilon * K_\varepsilon(z) = \int K_\varepsilon(z-w)K_\varepsilon(w)dudv$$

and

$$K_\varepsilon * K_\varepsilon * L(w) = \int \big(K_\varepsilon * K_\varepsilon(z)\big)L(w-z)dxdy.$$

Now because K_ε is radial, $K_\varepsilon * K_\varepsilon$ is also radial. Since $L(z)$ is superharmonic,

$$K_\varepsilon * K_\varepsilon * L(z) \leq L(z). \tag{3.7}$$

Since $L(z)$ is continuous as a map into $(-\infty, \infty]$, we have

$$K_\varepsilon * K_\varepsilon * L(z) \to L(z), \qquad \text{as } \varepsilon \to 0.$$

Since $\int d\nu = 0$, for any $\alpha > 0$ we have

$$\iint \log \frac{1}{|\alpha z - \alpha\zeta|} d\nu(\zeta) d\nu(z) = \iint \log \frac{1}{|z - \zeta|} d\nu(\zeta) d\nu(z).$$

Hence we may assume ν has support in $\{z : |z| < 1/2\}$.

Recall that we assumed

$$\int\int \left| \log \frac{1}{|z-\zeta|} \right| \, d|\nu|(\zeta) \, d|\nu|(z) < \infty. \tag{3.1}$$

Using this, (3.7) and dominated convergence, we get

$$I(\nu) = \lim_{\varepsilon \to 0} I(\nu_\varepsilon) \geq 0.$$

To finish the proof, we want to show $I(\mu) = 0$ implies $\nu = 0$.

Assume $I(\nu) = 0$ and set $U_\epsilon = U_{\nu_\epsilon}$. Then

$$\iint |\nabla U_\epsilon|^2 dxdy = I(\nu_\epsilon) \to 0. \tag{3.8}$$

We also have $U_\epsilon(z) = \mathrm{O}\left(\frac{1}{|z|}\right)$ uniformly in ϵ, because all $h_\epsilon,\ \epsilon < 1$, satisfy (3.3) and vanish outside a common compact set. Then by (3.8) and Lemma 3.2,

$$\lim_{\epsilon \to 0} \iint |U_\epsilon(z)|^2 \, dxdy = 0.$$

To prove $\nu = 0$ note that for any smooth $f \in C^\infty$ with compact support, (3.6) and Green's theorem imply

$$\int f d\nu = \lim_{\epsilon \to 0} \int f d\nu_\epsilon = \lim_{\epsilon \to 0} \frac{-1}{2\pi} \int \Delta f \ U_\epsilon dxdy = 0. \quad \square$$

Section III.4: The Equilibrium Distribution

We want to define a equilibrium measure on general sets by taking limits of measures on smooth approximating sets.

The problem is to show the limit is unique and independent of how we choose the approximations.

Let Ω be the component of $\mathbb{C}^* \setminus E$ such that $\infty \in \Omega$, and define $\widehat{E} = \mathbb{C} \setminus \Omega$. This is sometimes called the **polynomial hull** of E.

Theorem 4.1: *Assume* $\mathrm{Cap}(E) > 0$. *Then sequence* $\{\mu_n\} = \{\mu_{E_n}\}$ *converges weak-star to some* $\mu = \mu_E \in P(E)$, *and the limit does not depend on the choice of the sequence* $\{E_n\}$. *Moreover,*

$$\gamma(E) = I(\mu_E) = \inf\{I(\sigma) : \sigma \in P(E)\} \tag{4.1}$$

and $I(\sigma) > \gamma(E)$ *for all* $\sigma \in P(E)$ *with* $\sigma \neq \mu_E$.

It follows that the measure μ_E is the unique $\mu \in P(E)$ satisfying $U_\mu = \gamma(E)$ almost everywhere $d\mu$.

By definition μ_E is the **equilibrium distribution** of E, and U_{μ_E} is called the **equilibrium potential**.

Lemma 3.2: *Let E be a compact set and let $\{\mu_n\}$ be a sequence in $P(E)$. If μ_n converges weak-star to $\mu \in P(E)$ then*

$$U_\mu(z) \leq \liminf U_{\mu_n}(z), \qquad z \in \mathbb{C}, \tag{4.2}$$

and

$$I(\mu) \leq \liminf I(\mu_n). \tag{4.3}$$

Proof: Write $h(z,\zeta) = \log 1/|z-\zeta|$ and for $N > 0$ define $h_N(z,\zeta) = \min(h(z,\zeta), N)$.

Then each h_N is continuous, $h = \lim h_N$, and $h_N \le h$. Therefore

$$\begin{aligned}\int h_N(z,\zeta)d\mu(\zeta) &= \lim_n \int h_N(z,\zeta)d\mu_n(\zeta) \\ &\le \liminf_n \int h(z,\zeta)d\mu_n(\zeta) \\ &= \liminf_n U_{\mu_n}.\end{aligned}$$

Because h_N increases to h, this gives (4.2). Because $\mu_n \times \mu_n$ converges to $\mu \times \mu$ weak-star, a similar argument gives (4.3). $\square$

Proof of Theorem 4.1 (unique limits):

First we assume that $\Omega = \mathbb{C}^* \setminus E$ is bounded by finitely many pairwise disjoint $C^{1+\alpha}$ Jordan curves and prove μ_E is the unique minimizer for (4.1).

For such E, $U_{\mu_E} = \gamma(E)$ on E by (2.3), so that $I(\mu_E) = \int U_{\mu_E} d\mu_E = \gamma(E)$. Thus if $\sigma \in P(E)$, then by Fubini's theorem

$$\gamma(E) = \int U_{\mu_E} d\sigma = \int U_\sigma d\mu_E$$

But if $\sigma \neq \mu_E$, then by Theorem 3.1,

$$\begin{aligned} 0 &< I(\sigma - \mu_E) \\ &= I(\sigma) + I(\mu_E) - \int U_{\mu_E} d\sigma - \int U_\sigma d\mu_E = I(\sigma) - \gamma(E). \end{aligned}$$

Therefore (4.1) holds, and $I(\sigma) = \gamma(E)$ if and only if $\sigma = \mu_E$.

Now suppose E is any compact set. By the Banach–Alaoglu theorem there is a subsequence $\{\mu_{n_j}\}$ of $\{\mu_{E_n}\}$ that converges weak-star to some $\mu \in P(E_1)$. By (4.3),

$$I(\mu) \leq \liminf I(\mu_{n_j}) = \liminf \gamma(E_{n_j}) = \gamma(E).$$

Moreover, because μ_{E_n} has support ∂E_n, any such μ has support contained in

$$\partial \left(\bigcap_n E_n \right) = \partial \widehat{E} \subset E, \qquad (4.4).$$

If $\sigma \in P(E) \subset P(E_n)$ then, because (4.1) holds for the sets E_n,

$$I(\sigma) \geq \lim_n \gamma(E_n) = \gamma(E).$$

Therefore $I(\sigma) \geq I(\mu) = \gamma(E)$.

Suppose $\sigma \in P(E)$ satisfies $I(\sigma) = \gamma(E)$ and $\sigma \neq \mu$. By Fubini's theorem $\int U_\mu d\sigma = \int U_\sigma d\mu$, and by Theorem 3.1

$$0 < I(\sigma - \mu) = I(\sigma) + I(\mu) - 2\int U_\mu d\sigma. \tag{4.5}$$

Therefore

$$\int U_\mu d\sigma < \gamma(E). \tag{4.6}$$

noindent But if $\sigma_t = t\sigma + (1-t)\mu, \qquad 0 < t < 1$, then $\sigma_t \in P(E)$ so that

$$\gamma(E) \leq I(\sigma_t) = t^2 I(\sigma) + (1-t)^2 I(\mu) + 2t(1-t)\int U_\mu d\sigma$$

so

$$\gamma(E) \leq I(\sigma_t) \leq \gamma(E) + 2\left(\int U_\mu d\sigma - \gamma(E)\right) t + \mathrm{O}(t^2). \tag{4.7}$$

For t small (4.6) and (4.7) are in contradiction and hence μ is the unique extremal for the minimizing problem (4.1).

The uniqueness implies $\{\mu_n\}$ converges weak-star, because the sequence $\{\mu_n\}$ has a unique weak-star accumulation point.

Uniqueness also shows that the limit μ cannot depend on the choice of $\{E_n\}$.
$\square$

For any compact set E,

$$\partial\widehat{E} \subset \partial E \subset E \subset \widehat{E},$$

and by (4.4), μ_E is supported on $\partial\widehat{E}$. By (1.4) and the uniqueness in Theorem 4.1,

$$\mu_E = \mu_{\partial E} = \mu_{\widehat{E}} = \mu_{\partial\widehat{E}}.$$

The **capacity** of any Borel set A is defined as

$$\mathrm{Cap}(A) = \sup\{\mathrm{Cap}(E) : E \text{ compact}, E \subset A\}. \tag{4.8}$$

Sets of capacity zero are also called **polar sets**.

Corollary 4.3: *If E is compact and* $\mathrm{Cap}(E) = 0$*, then* $I(\mu) = +\infty$ *for all* $\mu \in P(E)$*. Consequently, if A is Borel with* $\mathrm{Cap}(A) = 0$ *and if μ is a positive measure with* $I(\mu) < \infty$*, then* $\mu(A) = 0$*.*

Proof: Suppose E is compact and $\mathrm{Cap}(E) = 0$ and suppose $\mu \in P(E)$. Then $P(E) \subset P(E_n)$ and $I(\mu) \geq \gamma(E_n) \to \infty$ by Theorem 4.1.

If A is a Borel set with $\mathrm{Cap}(A) = 0$ and if σ is a positive measure with $\sigma(A) > 0$, then take $E \subset A$ compact, such that $\sigma(E) > 0$.

Then $\mu = (\chi_E/\sigma(E))\,\sigma \in P(E)$ and $I(\mu) = \infty$. Thus $I(\sigma) = \infty$. $\square$

The Hausdorff α-content of a set is the

$$\mathcal{H}_\infty^\alpha = \inf\{\sum r_j^\alpha\}$$

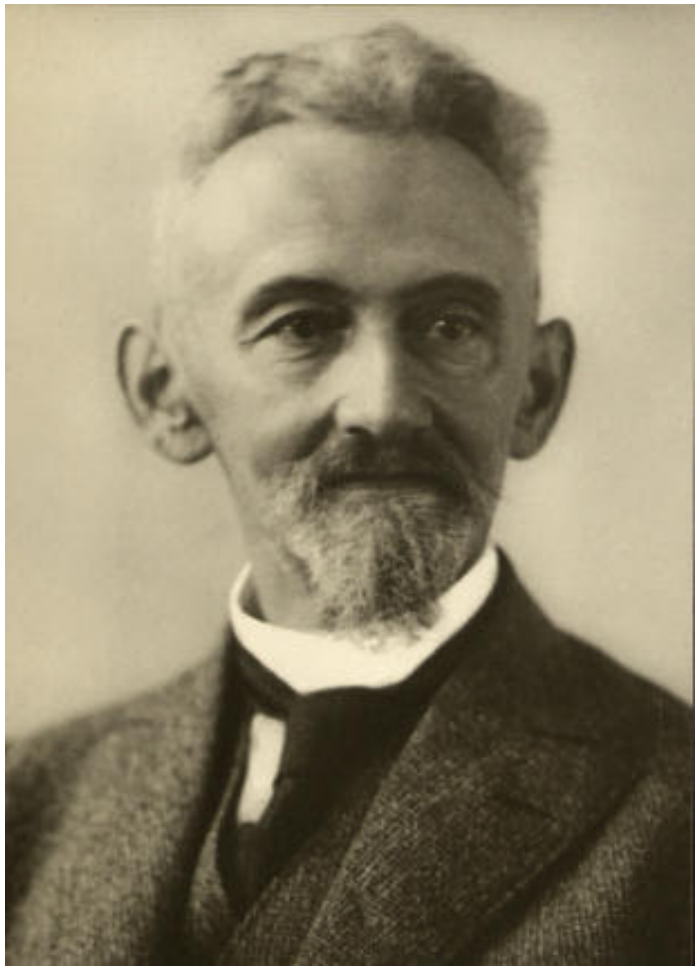
where the infimum is over collections of disks $\{z_j, r_j\}$ that cover E. The Hausdorff dimension is the infimum of α so that $\mathcal{H}_\infty^\alpha(E) = 0$.

Frostman's Lemma says that if a set has costive α-dimensional Hausdorff content, then it supports a finite positive measure μ so that for all $x \in \mathbb{C}$ and $r > 0$

$$\mu(D(z, r)) \leq r^{\alpha}.$$

It is not hard to check that such a measure had finite energy.

Thus the corollary implies that sets of zero logarithmic capacity have zero Hausdorff dimension.



Felix Hausdorff (1868-1942)

Otto Frostman (1907-1977)

Theorem 4.4: *Let E be a compact set such that* $\mathrm{Cap}(E) > 0$ *and let* μ_E *be the equilibrium distribution for E. Then*

$$U_{\mu_E}(z) \leq \gamma(E), \qquad z \in \mathbb{C} \tag{4.9}$$

$$U_{\mu_E}(z) = \gamma(E) - g_\Omega(z, \infty) < \gamma(E), \quad z \in \Omega = \mathbb{C}^* \setminus \widehat{E}, \tag{4.10}$$

$$\mathrm{Cap}\big(\{z \in \widehat{E} : U_{\mu_E}(z) \neq \gamma(E)\}\big) = 0. \tag{4.11}$$

Proof:

Since $U_{\mu_{E_n}} \leq \gamma(E_n)$ and $\gamma(E_n) \to \gamma(E)$, (4.9) follows from Lemma 4.2.

By (1.7), (2.1), and the weak-star convergence of μ_{E_n}, we conclude that $U_{\mu_E}(z) = \gamma(E) - g_\Omega(z, \infty)$ for $z \in \Omega$.

If $U_{\mu_E}(z) = \gamma(E)$ at some $z \in \Omega$, then by the maximum principle, U_{μ_E} is constant on Ω, but

$$\lim_{z \to \infty} U_{\mu_E}(z) = -\infty.$$

Therefore (4.10) holds.

To prove (4.11), let K be any compact subset of $\widehat{E}$ such that $\mathrm{Cap}(K) > 0$ and take $\sigma \in P(K)$ with $I(\sigma) < \infty$. For $0 < t < 1$, the inequality (4.7) still obtains, so that the coefficient of t in (4.7) is non-negative:

$$\int U_{\mu_E} d\sigma - \gamma(E) \geq 0.$$

But then by (4.9)

$$\int U_{\mu_E} d\sigma = \gamma(E)$$

and $U_{\mu_E}(z) = \gamma(E)$ almost everywhere with respect to σ. Thus $\mathrm{Cap}(K) = 0$ whenever $K \subset \{z \in \widehat{E} : U_{\mu_E}(z) \neq \gamma(E)\}$. $\square$

Theorem 4.5: *Suppose ψ is a contraction: $|\psi(z) - \psi(w)| \leq |z - w|$ for all $z,\ w \in E$. Then*

$$\mathrm{Cap}(\psi(E)) \leq \mathrm{Cap}(E).$$

If $E^* = \{|z| : z \in E\}$ is the circular projection of E onto $\mathbb{R}$, then

$$\mathrm{Cap}(E) \geq \mathrm{Cap}(E^*) \geq \frac{|E^*|}{4}, \tag{4.12}$$

where $|E^*|$ denotes the length of E^*.

Proof: We may assume that $\mathrm{Cap}(\psi(E)) > 0$. Let $\mu \in P(\psi(E))$ be such that $I(\mu) < \infty$. By Lemma 4.6 below there is $\nu \in P(E)$ such that $\psi_*(\nu) = \mu$, where $\psi_*(\nu)$ is defined by

$$\psi_*(\nu)(A) = \nu(\psi^{-1}(A)) \tag{4.13}$$

for all Borel $A \subset \psi(E)$. Then

$$I(\mu) = \int_E \int_E \log \frac{1}{|\psi(z) - \psi(w)|} d\nu(z) d\nu(w) \geq I(\nu),$$

and by Theorem 4.1, $\gamma(E) \leq \gamma(\psi(E))$ and $\mathrm{Cap}(E) \geq \mathrm{Cap}(\psi(E))$.

In proving (4.12) we may assume that E is a finite union of smooth curves and E^* is a finite union of intervals in $\mathbb{R}$.

Let F be the interval obtained by sliding the intervals of E^* together without altering their lengths.

Then F is obtained from E by applying two contractions, and hence $\mathrm{Cap}(E) \geq \mathrm{Cap}(E^*) \geq \mathrm{Cap}(F)$. But then by Example 1.1, $\mathrm{Cap}(F) = |F|/4 = |E^*|/4$. $\square$

Lemma 4.6: *Let E be a compact set and let ψ be a continuous map from E onto $\psi(E)$. For $\nu \in P(E)$ define $\psi_*(\nu)$ by*

$$\psi_*(\nu)(A) = \nu(\psi^{-1}(A)) \qquad (4.13).$$

Then ψ_* maps $P(E)$ onto $P(\psi(E))$.

Proof: By its definition, the map $\psi_* : P(E) \to P(\psi(E))$ is continuous, weak-star to weak-star. Given $\mu \in P(\psi(E))$, let μ_n be a sequence of atomic measures

$$\mu_n = \sum_{j=1}^{N_n} a_j^{(n)} \delta_{z_j^{(n)}},$$

where δ_z denotes the point-mass at z, such that $z_j^{(n)} \in \psi(E)$, such that $a_j^{(n)} > 0$ and $\sum a_j^{(n)} = 1$, and such that $\mu_n \longrightarrow \mu$ weak-star.

Pick any $w_j^{(n)} \in \psi^{-1}(z_j^{(n)})$, and set

$$\nu_n = \sum_{j=1}^{N_n} a_j^{(n)} \delta_{w_j^{(n)}}.$$

Then $\psi_*(\nu_n) = \mu_n$, and if a subsequence of $\{\nu_n\}$ converges weak-star to ν, then $\nu \in P(E)$ and $\psi_*(\nu) = \mu$. $\square$

Alternate proof of Koebe $\frac{1}{4}$-theorem:

Corollary 4.8 *Let $\varphi(z)$ be a univalent function on the unit disc $\mathbb{D}$. If φ is normalized by $\varphi(0) = 0$ and $\varphi'(0) = 1$, then*

$$\varphi(\mathbb{D}) \supset \{w : |w| < 1/4\}. \tag{4.15}$$

Proof: Let $a = \inf\{|w| : w \notin \varphi(\mathbb{D})\}$. Then

$$\psi(z) = \frac{1}{\varphi(\frac{1}{z})} = z + b_0 + \frac{b_1}{z} + \dots$$

is a conformal map from $\mathbb{C}^* \setminus \overline{D}$ to a domain Ω and $E = \mathbb{C}^* \setminus \Omega$ is a connected set with circular projection $E^* = [0, 1/a]$.

By Theorem 4.5 and Example 1.1, $1 = \mathrm{Cap}(E) \geq \mathrm{Cap}(E^*) = \frac{1}{4a}$, which gives (4.15). $\square$

Section III.5: Wiener's Solution to the Dirichlet Problem

Suppose $\mathrm{Cap}(\mathbb{C} \setminus \Omega) > 0$, the sequence $\{u_n(z)\}$ converges uniformly Let $\{\Omega_n\}$ be an exhaustion of Ω by smooth domains, extend f to a continuous function on $\mathbb{C}$ and let u_n be the solution of the Dirichlet problem on Ω_n with boundary values $f|_{\partial\Omega_n}$.

Theorem 5.1 (Wiener): *If* $\mathrm{Cap}(\mathbb{C} \setminus \Omega) > 0$, *the sequence* $\{u_n(z)\}$ *converges uniformly on all compact subsets of* Ω *to a harmonic function* $u(z)$, *and* $u(z)$ *does not depend on the choice of the sequence* $\{\Omega_n\}$ *nor on the choice of the extension of* $f \in C(\partial\Omega)$.

We write $u(z) = u_f(z)$ and we call $u_f(z)$ the Wiener solution to the Dirichlet problem with boundary values $f(\zeta)$.

Note that we do not claim u has boundary values f; this can fail without extra assumptions.

Proof: Suppose $\infty \in \Omega$. Let $\mu_n = \mu_{E_n} = \omega(\infty, \cdot, \Omega_n)$ where $E_n = \mathbb{C} \setminus \Omega_n$.

By Theorem 4.1, the sequence $\{\mu_n\}$ converges weak-star to μ_E, where $E = \mathbb{C} \setminus \Omega$. Hence

$$\lim_{n\to\infty} u_n(\infty) = \lim_{n\to\infty} \int_{\partial\Omega} f(\zeta) d\mu_{E_n}(\zeta) = \int_{\partial\Omega} f(\zeta) d\mu_E(\zeta). \tag{5.1}$$

The latter integral does not depend on the sequence $\{\Omega_n\}$. It also does not depend on the extension of f because the limit measure μ_E is supported on $\partial\Omega$.

For $z \in \Omega$ and $z \neq \infty$, let T be a Möbius transformation such that $Tz = \infty$ and note that

$$\mathrm{Cap}(\partial\Omega) > 0 \Longleftrightarrow \mathrm{Cap}(T(\partial\Omega)) > 0. \tag{5.2}$$

Indeed, if $\mathrm{Cap}(\partial\Omega) > 0$, then by (1.7), $g_\Omega(w, T^{-1}(\infty))$ exists and (5.2) is implied by

$$\begin{aligned} g_\Omega(w, T^{-1}(\infty)) &= \lim_n g_{\Omega_n}(w, T^{-1}(\infty)) \\ &= \lim_n g_{T(\Omega_n)}(Tw, \infty) = g_{T(\Omega)}(Tw, \infty), \end{aligned}$$

The argument for $z = \infty$ shows $\lim_{n\to\infty} u_n(z) = u(z)$ exists for all $z \in \Omega$ and the limit $u(z)$ is independent of $\{\Omega_n\}$ and the extension of f.

Because the functions $\{u_n(z)\}$ are uniformly bounded and harmonic, the convergence $u_n(z) \to u(z)$ is uniform on compact subsets of Ω and the limit $u(z)$ is harmonic in Ω. $\square$

Theorem 5.1 says

$$\omega(z, \cdot\,, \Omega) = \lim_{n\to\infty} \omega(z, \cdot\,, \Omega_n) \tag{5.3}$$

exists and is independent of the choice of $\{\Omega_n\}$. Then for any $f \in C(\partial\Omega)$,

$$u_f(z) = \int_{\partial\Omega} f(\zeta) d\omega(z, \zeta, \Omega),$$

where $u_f(z)$ is the Wiener solution to the Dirichlet problem for $f(\zeta)$.

By definition, the limit measure $\omega(z, \cdot, \Omega)$ is the **harmonic measure** for z relative to Ω.

This definition of harmonic measure agrees with the definition in Chapter II for finitely connected Jordan domains.

Mutual absolute continuity.

$\omega(z, A, \Omega)$ is a harmonic function of $z \in \Omega$ and $0 \leq \omega(z, A, \Omega) \leq 1$. By Harnack's inequality,

$$\omega(z_1, E, \Omega) \leq C_{z_1, z_2} \cdot \omega(z_2, E, \Omega)$$

for every pair $z_1,\ z_2$ of points in Ω, where the constant $C_{z_1, z_2} > 0$ depends only on $z_1, z_2,$ and Ω.

Thus the measures $\omega(z, \cdot, \Omega)$, $z \in \Omega$, are mutually absolutely continuous.

Equilibrium measure = harmonic measure from infinity.

If $z = \infty$, then (5.1) shows that $\omega(\infty, \cdot\ , \Omega) = \mu_{\partial\Omega}$. If $z \in \Omega$ and if T is a Möbius transformation with $T(z) = \infty$, then for every Borel set A,

$$\omega(z, A, \Omega) = \mu_{T(\partial\Omega)}(T(A)).$$

By (5.2) and Corollary 4.3

$$\mathrm{Cap}(A) = 0 \implies \omega(z, A, \Omega) = 0 \tag{5.4}$$

for every Ω with $\mathrm{Cap}(\mathbb{C} \setminus \Omega) > 0$ and every $z \in \Omega$.

Polar sets have no harmonic measure.

If $\mathrm{Cap}(\mathbb{C}^* \setminus \Omega) = 0$ then, as we shall see in Section 8, each bounded or positive harmonic function on Ω is constant.

Therefore we cannot define harmonic measure or solve the Dirichlet problem when $\mathrm{Cap}(\mathbb{C}^* \setminus \Omega) = 0$.

Harmonic measure has no atoms.

Proposition 5.3: *Let* $\{\Omega_n\}$ *be a smooth exhaustion of* Ω. *If* $E = \mathbb{C} \setminus \Omega$ *satisfies* $\mathrm{Cap}(E) > 0$, *then for any* $\zeta \in \partial\Omega$, *and any* $\epsilon > 0$,

$$\lim_{\delta \to 0} \sup_{z \in \Omega_n \setminus B_\epsilon(\zeta)} \omega(z, \partial\Omega_n \cap B_\delta(\zeta), \Omega_n) = 0 \tag{5.5}$$

uniformly in n.

Proof: Without loss of generality, assume $\infty \in \Omega$ and $\zeta = 0$.

Let $J = E \cap \{z : 1/M \leq |z| \leq M\}$ with M so that $\mathrm{Cap}(J) > 0$. For Ω_n as before, set $J_n = \{z : 1/M \leq |z| \leq M\} \setminus \Omega_n$.

Let μ_n be the equilibrium distribution of J_n and let γ_n be Robin's constant for J_n. Then

$$U_n(z) \equiv \log \frac{1}{|z|} - U_{\mu_n}(z) = \int_{J_n} \log \left| \frac{z - \zeta}{z} \right| d\mu_n(\zeta)$$

is harmonic on $\mathbb{C}^* \setminus (J_n \cup \{0\})$.

For $\delta < 1/M$ we have

$$U_n(z) \geq \log\left(\frac{1}{M\delta} - 1\right), \ z \in \partial B_\delta(0) = -\log M + \log(\frac{1}{\delta} - M),$$

and

$$U_n(z) \geq \log\frac{1}{M} - \gamma_n, \ z \in \mathbb{C}.$$

These imply that

$$U_n(x) + \log M + \gamma_n \geq 0$$

everywhere, and that

$$U_n(x) + \log M + \gamma_n \geq 1$$

on the boundary of $B_\delta(0)$.

For $\delta < \epsilon$, these inequalities and the maximum principle give

$$\omega(z, \partial\Omega_n \cap B_\delta(0), \Omega_n) \leq \frac{U_n(z) + \log M + \gamma_n}{\log(\frac{1}{\delta} - M) + \gamma_n}$$

on $\{|z| > \delta\} \cap \Omega_n$. Hence if $|z| \geq \epsilon \geq \delta$,

$$\omega(z, \partial\Omega_n \cap B_\delta(0), \Omega_n) \leq \frac{\log\left(1 + \frac{M}{\epsilon}\right) + \log M + \gamma_n}{\log\left(\frac{1}{\delta} - M\right) + \gamma_n}.$$

But because M is fixed and $J \subset J_n \subset B_M(0)$, the quantities γ_n are bounded above and below, and therefore (5.5) holds. $\square$

Suppose $0 \notin \overline{\Omega}$. Then by Theorem 5.1 and the definition of $g(z, \infty)$ for finitely connected domains,

$$h(z, \infty) = g(z, \infty) - \log |z|$$

coincides with the solution to the Dirichlet problem in Ω with boundary value $-\log |z|$. The following proposition gives a similar result when $\overline{\Omega} = \mathbb{C}^*$.

Proposition 5.3: *Let $E \subset \{z : |z| < R\}$ be a compact set such that $\mathrm{Cap}(E) > 0$ and let $\Omega = \mathbb{C}^* \setminus E$. Then U_{μ_E} is the solution to the Dirichlet problem on $\Omega^R = \Omega \cap B(0, R)$ with boundary data*

$$f(\zeta) = \begin{cases} \gamma(E) & \text{for } \zeta \in \partial\Omega, \\ U_{\mu_E}(\zeta) & \text{for } |\zeta| = R, \end{cases}$$

and $g_\Omega(\cdot, \infty)$ is the solution to the Dirichlet problem on Ω^R with boundary data $\gamma(E) - f(\zeta)$.

For the proof, see the textbook.

Section III.6: Regular Points

Next we study when the Wiener solution has the correct boundary values.

A point $\zeta \in \partial\Omega$ is a **regular point** for Ω if

$$\lim_{\Omega \ni z \to \zeta} u_f(z) = f(\zeta) \tag{6.1}$$

for all $f \in C(\partial\Omega)$.

Equivalently, $\zeta \in \partial\Omega$ is a regular point if

$$\lim_{\Omega \ni z \to \zeta} \omega(z, \cdot, \Omega) = \delta_\zeta$$

in the weak-star topology, where δ_ζ is the point mass at ζ.

A boundary point this is not regular is called an **irregular point**.

In $\Omega = \mathbb{D} \setminus \{0\}$, the origin is irregular.

We shall prove that "most" boundary points are regular.

Theorem 6.1 (Barriers): *The point $\zeta \in \partial\Omega$ is a regular point for Ω if and only if there is $\rho > 0$ such that for every $\varepsilon > 0$, there exists a function V_ε, harmonic in $\Omega \cap \{z : |z - \zeta| < \rho\}$, satisfying*

$$V_\varepsilon(z) \geq 0, \tag{6.2}$$

$$\limsup_{z \to \zeta} V_\varepsilon(z) \leq \varepsilon, \tag{6.3}$$

$$V_\varepsilon(z) > 1/2 \quad \textit{on} \quad \{z : \varepsilon < |z - \zeta| \leq \rho\} \cap \Omega. \tag{6.4}$$

Theorem 6.1 implies that the regularity of ζ is a local question; it depends only on the behavior of Ω in some neighborhood of ζ.

A **barrier** is a harmonic function V on Ω such that

$$\lim_{z \to \zeta} V(z) = 0,$$

and

$$\liminf_{z \to \alpha} V(z) > 0$$

for $\alpha \in \partial\Omega \setminus \{\zeta\}$. If V is a barrier, then there are constants $C_\epsilon > 0$ such that the functions $V_\varepsilon = C_\varepsilon V$ satisfy (6.2), (6.3), and (6.4).

Proof: Let ζ be a regular point. Fix $\varepsilon > 0$. For $0 < \delta < \varepsilon$ choose $f \in C(\partial\Omega)$ such that

$$0 \leq f \leq 1,$$
$$\{z \in \partial\Omega : f(z) = 0\} = \{\zeta\},$$

and

$$f(z) = 1, \text{ when } |z - \zeta| > \delta.$$

Then clearly $V_\varepsilon(z) = u_f(z)$ satisfies (6.2) and (6.3), and when δ is small enough (6.4) also holds by Proposition 5.2.

Conversely, suppose the functions $V_\varepsilon(z)$ exist.

Let $f \in C(\mathbb{C}^*)$ satisfy

$$0 \leq f \leq 1/2 \text{ and } f^{-1}(0) = \zeta \tag{6.5}$$

and let Ω_n be as above. If $\delta > 0$, then for ε sufficiently small,

$$f(z) \leq V_\varepsilon(z) + \delta$$

on $\{z : |z - \zeta| < \rho\} \cap \partial\Omega_n$.

Let u_n solve the Dirichlet problem on the subdomain Ω_n for the boundary value f. Then

$$u_n(z) \leq \sup f \leq 1/2 \leq V_\varepsilon(z)$$

on $\Omega_n \cap \{z : |z - \zeta| = \rho\}$, and by the maximum principle

$$u_n(z) \leq V_\varepsilon(z) + \delta$$

on $\Omega_n \cap \{|z - \zeta| < \rho\}$. Thus

$$0 \leq u_f(z) \leq V_\varepsilon(z) + \delta$$

on $\Omega \cap \{|z - \zeta| < \rho\}$. Because δ is arbitrary, (6.3) now yields

$$\lim_{z \to \zeta} u_f(z) = 0.$$

Because every real $g \in C(\mathbb{C}^*)$ is a constant plus a linear combination of two functions satisfying (6.5), it follows that ζ is a regular point.

Corollary 6.2: *Let $\zeta \in \partial\Omega$. If the connected component of $\partial\Omega$ containing ζ consists of more than one point, then ζ is a regular point for Ω. In particular, if Ω is simply connected, then every point of $\partial\Omega$ is a regular point.*

Proof: For $\epsilon > 0$, let E_ϵ be the component of $\partial\Omega \cap \overline{B(\zeta, \epsilon/2)}$ containing ζ. Then $E_\epsilon \neq \{\zeta\}$.

Let ψ_ϵ be a Riemann map of $\mathbb{C}^* \setminus E_\epsilon$ onto $\mathbb{D}$ with $\psi_\epsilon(\infty) = 0$. Then $|\psi_\epsilon(z)| \to 1$ as $\mathbb{C}^* \setminus E_\epsilon \ni z \to E_\epsilon$ and

$$V_\epsilon(z) \equiv \frac{\log \left| \frac{1}{\psi_\epsilon(z)} \right|}{\inf\limits_{\epsilon<|z|<1} \log \left| \frac{1}{\psi_\epsilon(z)} \right|}$$

satisfies (6.2), (6.3), and (6.4) with $\rho = 1$. $\square$

Theorem 6.3: *Assume* $E = \mathbb{C}^* \setminus \Omega$ *is a compact plane set with* $\mathrm{Cap}(E) > 0$ *and let* μ_E *be the equilibrium distribution for* E*. Then for* $\zeta \in \partial\Omega$ *the following conditions are equivalent:*

$$\zeta \quad \textit{is a regular point}, \tag{6.6}$$

$$\lim_{\Omega \ni z \to \zeta} g_\Omega(z, \infty) = 0, \tag{6.7}$$

$$U_{\mu_E}(\zeta) = \gamma(E). \tag{6.8}$$

Proof: (6.6) $\Rightarrow$ (6.7).

Take $R > 0$ such that $E \subset B(0, R)$. Then by Proposition 5.3, $g_\Omega(\cdot, \infty)$ is the solution to the Dirichlet problem in $\Omega^R = \Omega \cap B(0, R)$, for the boundary data

$$\begin{cases} 0 & \text{for } z \in \partial\Omega, \\ g_\Omega(z, \infty) & \text{for } |z| = R. \end{cases}$$

If $\zeta \in \partial\Omega$ is a regular point, then by Theorem 6.1, ζ is also a regular point for Ω^R and hence (6.7) holds.

(6.7) $\Rightarrow$ (6.8).

Now suppose (6.7) holds. Then by (4.10)

$$U_{\mu_E}(z) = \gamma(E) - g_\Omega(z, \infty) \to \gamma(E), \tag{6.9}$$

as $z \in \Omega \to E$. Because $\log 1/|z|$ is superharmonic, we have

$$U_{\mu_E}(\zeta) \geq \frac{1}{\pi r^2} \iint_{B(\zeta,r)} U_{\mu_E}(z) dx dy,$$

while by Theorem 4.4, for all z we have

$$U_{\mu_E}(z) \leq \gamma(E) \tag{6.10}$$

Therefore

$$0 \leq \gamma(E) - U_{\mu_E}(\zeta) \leq \limsup_{r\to 0} \frac{1}{\pi r^2} \iint_{B(\zeta,r)\cap\Omega} (\gamma(E) - U_{\mu_E}(z))dxdy$$
$$+ \limsup_{r\to 0} \frac{1}{\pi r^2} \iint_{B(\zeta,r)\setminus\Omega} (\gamma(E) - U_{\mu_E}(z))dxdy.$$

By (6.9) and (6.10) the first limit is 0, while by the remark after Corollary 4.3 and Theorem 4.4,

$$\mathrm{Area}(\{\mathrm{z} \notin \Omega : \mathrm{U}_{\mu_\mathrm{E}}(\mathrm{z}) \neq \gamma(\mathrm{E})\}) = 0,$$

so that the second limit is also 0. Hence (6.8) holds.

(6.8) $\Rightarrow$ (6.6).

Set $E_\varepsilon = E \cap B\big(\zeta, \epsilon^2/4\big)$. If ζ and $\{z : |z-\zeta| = \epsilon^2/4\}$ lie in the same component of $E_\epsilon \cup \{z : |z-\zeta| = \epsilon^2/4\}$, then the complement in $\mathbb{C}^*$ of that component is a simply connected domain $\Omega_\epsilon \supset \Omega \cap B\big(\zeta, \epsilon^2/4\big)$, and $\zeta \in \partial\Omega_\epsilon$.

By Corollary 6.2, ζ is regular for Ω_ϵ, and then by Theorem 6.1, ζ is regular for Ω. Thus we may assume there is a (closed) C^2 Jordan curve $\sigma_\epsilon \subset \Omega \cap B\big(\zeta, \epsilon^2/4\big)$ such that ζ is in the bounded component W_ϵ of $\mathbb{C} \setminus \sigma_\epsilon$.

Let μ_ϵ be the equilibrium distribution for E_ϵ, and let U_ϵ be its logarithmic potential. We will verify conditions (6.2), (6.3), and (6.4) for the functions

$$V_\varepsilon(z) = 1 - \frac{U_\varepsilon(z)}{\gamma(E_\varepsilon)}.$$

and this will show that ζ is regular.

First note that $e^{-\gamma(E_\varepsilon)} = \mathrm{Cap}(E_\varepsilon) > 0$, because if not, then by Corollary 4.3, $\mu_E(B(\zeta, \epsilon^2/4)) = 0$.

But then ζ would be in the unbounded component of $V = \mathbb{C} \setminus \mathrm{supp}(\mu_E)$, where $\mathrm{supp}(\mu_E)$ is the closed support of μ_E. Then (4.10) and the maximum principle would give $U_{\mu_E}(\zeta) < \gamma(E)$, contrary to the assumption (6.8).

By Theorem 4.4, (6.2) holds for V_ϵ. Note that if $|z-\zeta|>\varepsilon$, then

$$U_\varepsilon(z)=\int_{E_\varepsilon}\log\frac{1}{|z-w|}d\mu_{E_\varepsilon}(w)\le\log\frac{2}{\epsilon}.$$

Since $\gamma(E_\varepsilon)\ge\log\frac{4}{\varepsilon^2}$, we get $V_\varepsilon(z)>1/2$ in $|z-\zeta|\ge\varepsilon$, and (6.4) holds for V_ϵ.

Choose $M_\varepsilon>0$ so that when $z\in\sigma_\epsilon$,

$$\gamma(E_\varepsilon)-U_\varepsilon(z)\le M_\varepsilon\big(\gamma(E)-U_{\mu_E}(z)\big). \tag{6.11}$$

By Proposition 5.3, each side of (6.11) is the solution to a Dirichlet problem in $W_\epsilon\cap\Omega$ with boundary data 0 on $\partial\Omega\cap W_\epsilon$, and hence the inequality (6.11) continues to hold in $W_\epsilon\cap\Omega$. Sending $z\to\zeta$ then yields (6.3). Then ζ is a regular point by Theorem 6.1. $\square$

Corollary 6.4 (Kellogg's theorem): *Let Ω be any domain in $\mathbb{C}^*$. Then*

$$\mathrm{Cap}\big(\{\zeta \in \partial\Omega : \zeta \text{ is irregular}\}\big) = 0.$$

Proof: We may assume $\infty \notin \Omega$. Then the corollary is immediate from Theorem 4.4 and Theorem 6.3. $\square$

Corollary 6.5: *Let* Ω *be any domain in* $\mathbb{C}^*$ *with* $\mathrm{Cap}(\mathbb{C}^* \setminus \Omega) > 0$. *Then*

$$\omega(z, \{\zeta \in \partial\Omega : \zeta \text{ is irregular}\}, \Omega) = 0$$

for all $z \in \Omega$.

Proof By (6.8), the set of irregular points is a Borel set, so the corollary follows from (5.4) and Corollary 6.4. $\square$

Section III.7: Wiener Series

Let $A_n = A_n(\zeta)$ be the ring $A_n = \{z : 2^{-n} \leq |z - \zeta| \leq 2^{-n+1}\}$, and recall the Robin's constant

$$\gamma(E) = \log\Big(1/\mathrm{Cap}(E)\Big).$$

Theorem 7.1 (Wiener): *Let Ω be a domain in $\mathbb{C}^*$ and let $\zeta \in \partial\Omega$. Then the following are equivalent:*

$$\zeta \textit{ is a regular point}, \tag{7.1}$$

$$\sum_{n=1}^{\infty} \frac{n}{\gamma(A_n \setminus \Omega)} = \infty, \tag{7.2}$$

$$\sum_{n=1}^{\infty} \frac{n}{\gamma(A_n \cap \partial\Omega)} = \infty. \tag{7.3}$$

For proof, see textbook.

Application: If Ω is the complement of the Cantor middle third set $E = \partial\Omega$, then every point of E is regular.

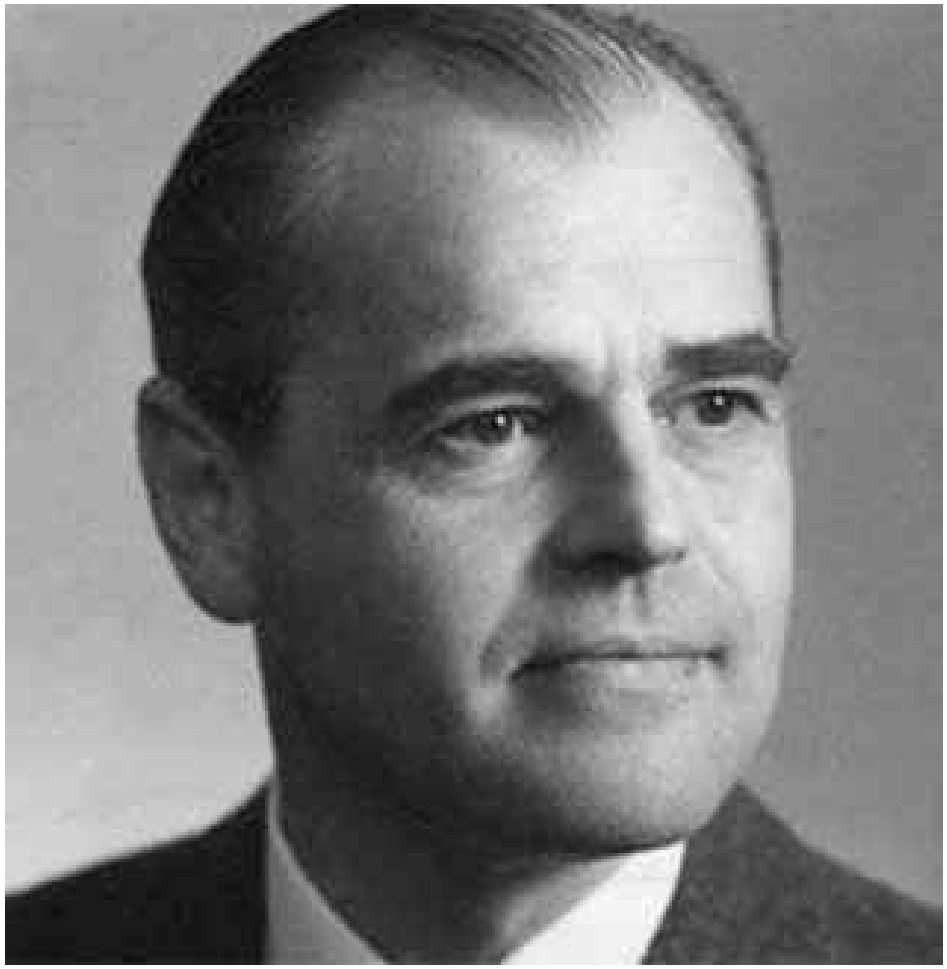
Corollary (Beurling): *Let E be a compact set. Assume $0 \in E$ and set $E^* = \{|z| : z \in E\}$. If*

$$\int_{E^*} \frac{dr}{r} = \infty, \tag{7.9}$$

then 0 is a regular point for $\Omega = \mathbb{C}^* \setminus E$.

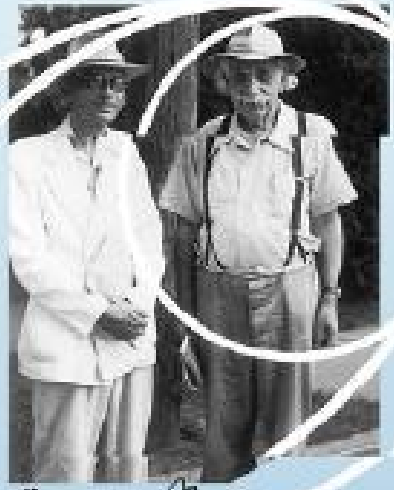
Proof: Set $E_n = E \cap A_n$ and ${E_n}^* = \{|z| : z \in E_n\}$. Then $\mathrm{Cap}(E_n) \geq |E_n^*|/4$ by Theorem 4.5, and consequently we have $\sum 2^n \mathrm{Cap}(E_n) = \infty$ by (7.9).

But $2^n \mathrm{Cap}(E_n) \leq Cn/\gamma(E_n)$, because $\mathrm{Cap}(E_n) \leq 2^{-n+1}$ and the function $x \log 1/x$ is increasing on $[0, 1/e]$. Therefore the series (7.3) also diverges and 0 is a regular point. $\square$



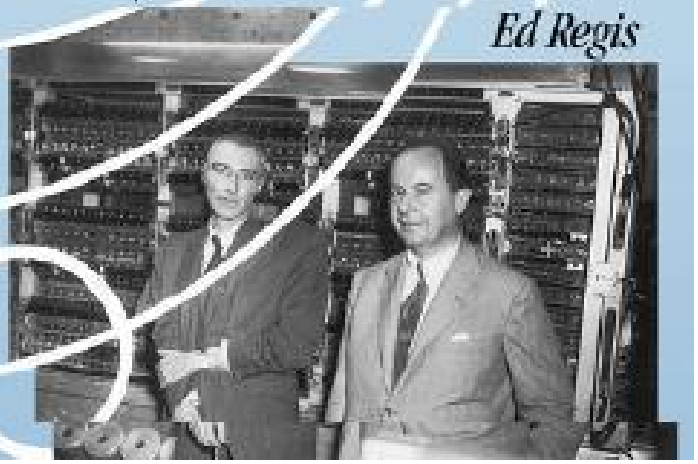
Arne Beurling (1905-1986)

Who Got Einstein's Office?



Eccentricity and Genius at the Institute for Advanced Study

Ed Regis



Corollary 7.4: *Assume* $\zeta \in E = \partial\Omega$. *If*

$$\limsup_{r\to 0} \frac{\mathrm{Cap}\big(E \cap B(\zeta, r)\big)}{r} > 0, \tag{7.11}$$

or if

$$\limsup_{r\to 0} \frac{\log(1/r)}{\gamma(E \cap B(\zeta, r))} > 0, \tag{7.12}$$

then ζ *is a regular point of* $\partial\Omega$.

Proof: It is clear that (7.11) implies (7.12) and that (7.12) implies

$$\limsup_{n\to\infty} \frac{n}{\gamma(E \cap B(\zeta, 2^{-n}))} > 0.$$

Then by Lemma 7.5 below

$$\limsup_{n\to\infty} \sum_{n}^{\infty} \frac{k}{\gamma(E \cap A_k)} > 0,$$

and (7.3) again diverges. $\square$

Lemma 7.5 (subadditivity): *Suppose $E_1, \ldots, E_n$ are disjoint Borel subsets of $B(0, 1/2)$. Then*

$$\frac{1}{\gamma(\bigcup E_k)} \leq \sum_{k=1}^{n} \frac{1}{\gamma(E_k)}.$$

Proof: By the definition of capacity, we can suppose each E_k is bounded by finitely many pairwise disjoint C^2 Jordan curves.

Let μ be the equilibrium distribution for $E = \bigcup E_k$, let μ_k be the equilibrium distribution for E_k, let $\sigma = \sum \mu_k/\gamma(E_k)$ and let U_μ, U_{μ_k}, and U_σ be their potentials.

Then because $U_\mu = \gamma(E)$ on E and $\mathrm{supp}(\sigma) \subset E$, by Fubini's theorem we have

$$\left(\sum \frac{1}{\gamma(E_k)}\right)\gamma(E) = \int U_\mu d\sigma = \int U_\sigma d\mu.$$

Because $U_{\mu_k} \geq 0$ on E (it has diameter ≤ 1) and $U_{\mu_k} = \gamma(E_k)$ on E_k, we have $U_\sigma \geq 1$ on $\mathrm{supp}(\mu)$.

Consequently

$$\int U_\sigma d\mu \geq 1$$

and the lemma follows. $\square$

Section III.8: Polar Sets and Sets of Harmonic Measure Zero

Theorem 8.1: *Suppose Ω is a domain such that $\mathrm{Cap}(\mathbb{C}^* \setminus \Omega) > 0$, and suppose A is a Borel subset of $\partial\Omega$ such that*

$$\omega(z_0, A, \Omega) = 0, \tag{8.1}$$

holds for some $z_0 \in \Omega$. Let $u(z)$ be a subharmonic function in Ω. If u is bounded above in Ω and if

$$\limsup_{z \to \zeta} u(z) \leq m \tag{8.2}$$

for all $\zeta \in \partial\Omega \setminus A$, then

$$u(z) \leq m$$

for all $z \in \Omega$.

Proof: On $\overline{\Omega}$, define

$$V(\zeta) = \begin{cases} u(\zeta), & \zeta \in \Omega, \\ \limsup\limits_{\Omega \ni z \to \zeta} u(z), & \zeta \in \partial\Omega. \end{cases}$$

Then by hypothesis, V is bounded above, and V is upper semicontinuous on $\overline{\Omega}$ because subharmonic functions are upper semicontinuous. By (8.1) and (8.2),

$$\int_{\partial\Omega} V(\zeta) d\omega(z, \zeta, \Omega) \leq m \tag{8.3}$$

for every $z \in \Omega$.

Let $\{\Omega_n\}$ be the subdomains from before. Then for any $z \in \Omega$,

$$\omega(z, \cdot, \Omega) = \lim_{n \to \infty} \omega(z, \cdot, \Omega_n),$$

with respect to weak-star convergence of measures on $\overline{\Omega}$.

Let $V_k(\zeta)$ be a sequence of continuous functions that decreases to $V(\zeta)$ on $\overline{\Omega}$, and let $u_{k,n}(z)$ be the harmonic extension of V_k to Ω_n.

Because u is upper semicontinuous on $\overline{\Omega_n} \subset \Omega$,

$$\limsup_{\Omega_n \ni z \to \zeta} \big(u(z) - u_{k,n}(z)\big) = u(\zeta) - V_k(\zeta) \leq 0$$

at every $\alpha \in \partial\Omega_n$. Hence by the ordinary maximum principle,

$$u(z) \leq u_{k,n}(z) = \int_{\partial\Omega_n} V_k(\zeta) d\omega(z, \zeta, \Omega_n)$$

for $z \in \Omega_n$. Then by the weak-star convergence

$$u(z) \leq \int_{\partial\Omega} V_k(\zeta) d\omega(z, \zeta, \Omega), \tag{8.4}$$

so that by monotone convergence

$$u(z) \leq \int_{\partial\Omega} V(\zeta) d\omega(z, \zeta, \Omega), \tag{8.5}$$

$z \in \Omega$. Then (8.3) and (8.5) imply that $u(z) \leq m$ in $\Omega = \bigcup \Omega_n$. $\square$

Theorem 8.2: *Suppose A is a Borel set. Then A is a polar set if and only if*

$$\omega(z, A \cap \partial\Omega, \Omega) = 0 \tag{8.6}$$

for every domain Ω such that $\mathrm{Cap}(\mathbb{C}^* \setminus \Omega) > 0$ *and for every $z \in \Omega$.*

Proof: By (4.8) and the regularity of harmonic measure, we may assume that the set A is compact. If $\mathrm{Cap}(A) = 0$, then (8.6) follows from (5.4).

Conversely, if $\mathrm{Cap}(A) > 0$ let μ_A be the equilibrium distribution for A and let Ω be the unbounded component of $\mathbb{C} \setminus A$. Then $\mathrm{Cap}(\mathbb{C}^* \setminus \Omega) = \mathrm{Cap}(A) > 0$ and $\omega(\infty, A \cap \partial\Omega, \Omega) = \mu_A(A) = 1$. $\square$

Corollary 8.3: *Suppose Ω is a domain such that* $\mathrm{Cap}(\mathbb{C}^* \setminus \Omega) > 0$, *and suppose u is subharmonic and bounded above in Ω. If*

$$\limsup_{z \to \zeta} u(z) \leq m$$

for all $\zeta \in \partial\Omega \setminus A$, where A is a Borel set and $\mathrm{Cap}(A) = 0$, *then*

$$\sup_{\Omega} u(z) \leq m.$$

Proof: Immediate from Theorems 8.1 and 8.2. $\square$

Corollary 8.4: *Suppose E is a compact polar set. If $u(z)$ is subharmonic and bounded above in $\mathbb{C}^* \setminus E$, then u is constant. If $\Omega \supset E$ is a Jordan domain then every bounded harmonic function on $\Omega \setminus E$ extends to be harmonic on E. Conversely, if $\mathrm{Cap}(E) > 0$, if Ω is the unbounded component of $\mathbb{C}^* \setminus E$, and if A is a compact subset of E such that $0 < \mu_E(A) < 1$, then $u(z) = \omega(z, A, \Omega)$ is a non-constant bounded harmonic function on Ω.*

If E is a compact polar set then by Corollary 8.4 every positive harmonic function in $\mathbb{C}^* \setminus E$ is constant and every bounded harmonic function in $\mathbb{C}^* \setminus E$ is constant.

Proof: Let $z_0 \in \mathbb{C}^* \setminus E$, $\delta < \operatorname{dist}(z_0, E)$, and $\Omega = \mathbb{C}^* \setminus (E \cup \overline{B}(z_0, \delta))$. Then $\operatorname{Cap}(\mathbb{C}^* \setminus \Omega) > 0$ and by Corollary 8.3,

$$\sup_{\Omega} u(z) \leq \sup_{\partial B(z_0,\delta)} u(z).$$

Letting $\delta \to 0$ we obtain

$$\sup_{\mathbb{C}^* \setminus E} u(z) \leq u(z_0).$$

Because z_0 is arbitrary in $\mathbb{C}^* \setminus E$, u is constant.

To prove the second statement, we shrink Ω so that u is continuous on $\partial\Omega$. Let v be the solution to the Dirichlet problem on Ω for the function u. Then by Corollary 8.3, $u = v$ and v is the harmonic extension of u to Ω. The converse assertion is clear. $\square$

Section III.9: Estimates for Harmonic Measure

Theorem 9.1: *Suppose* $E \subset \{z : 0 < \delta < |z| < r < 1\}$ *and set*

$$\gamma(E) = -\log\left(\mathrm{Cap}(E)\right).$$

Then there are constants $c_1(r)$ *and* $c_2(\delta, r)$ *such that*

$$\frac{c_1(r)}{\gamma(E)} \leq \omega(0, E, \mathbb{D} \setminus E) \leq \frac{c_2(\delta, r)}{\gamma(E)}. \tag{9.1}$$

Proof: Let

$$g(z,\zeta) = \log\left|\frac{1-\overline{z}\zeta}{z-\zeta}\right|$$

vskip.1in be Green's function on $\mathbb{D}$ and let μ_E be the equilibrium distribution for E. Then the Green potential

$$V(z) = \int_{\partial E} g(z,\zeta)d\mu_E(\zeta)$$

is continuous on $\overline{\mathbb{D}\setminus E}$ and harmonic on $\mathbb{D}\setminus E$.

Also, $\log\frac{1}{r} \le V(0) \le \log\frac{1}{\delta}$, $V=0$ on $\partial\mathbb{D}$, and for $z\in E$, by (2.3) we have

$$\gamma(E)+\log(1-r^2) \le V(z) \le \gamma(E)+\log(1+r^2)$$

Hence

$$\frac{\log\left(\frac{1}{r}\right)}{\gamma(E)+\log(1+r^2)} \le \omega(0,E,\mathbb{D}\setminus E) \le \frac{\log\left(\frac{1}{\delta}\right)}{\gamma(E)+\log(1-r^2)}, \tag{9.2}$$

provided that $\gamma(E)+\log(1-r^2)>0$.

This gives the left-hand inequality in (9.1) with $c_1(r) = (\log r)^2/\log(r + 1/r)$, because $\gamma(E) \geq \log(1/r)$.

If $\gamma(E) \geq 2\log 1/(1-r^2)$ the upper bound in (9.2) gives the upper bound in (9.1) with constant $2\log(1/\delta)$, and if $\gamma(E) \leq 2\log 1/(1-r^2)$, the upper bound in (9.1) follows from the trivial estimate

$$\omega(0, E, \mathbb{D} \setminus E) \leq 1 \leq \frac{2\log\frac{1}{1-r^2}}{\gamma(E)}. \quad \square$$

Theorem 9.2 (Beurling): *If* $E \subset \overline{\mathbb{D}} \setminus \{0\}$, *and* $E^* = \{|z| : z \in E\}$ *is the circular projection of* E, *then*

$$\omega(z, E, \mathbb{D} \setminus E) \geq \omega(-|z|, E^*, \mathbb{D} \setminus E^*).$$

Proof: By Green's theorem

$$\omega(z, E, \mathbb{D} \setminus E) = \omega(z) = \frac{1}{2\pi} \int_{\partial E} g(z, \zeta) \frac{\partial \omega(\zeta)}{\partial \mathrm{n}_\zeta} ds$$
$$= \int_{\partial E} g(z, \zeta) d\sigma(\zeta), (9.3)$$

in which the normal n_ζ points out of $\mathbb{D} \setminus E$, so that $\sigma \geq 0$.

Consider the circular projection σ^* of σ,

$$\sigma^*(A) = \sigma(\{z \in E : |z| \in A\}),$$

and its Green potential

$$V(z) = \int_{(\partial E)^*} g(z, |\zeta|) d\sigma^*.$$

Because

$$g(-|z|, |\zeta|) \leq g(z, \zeta) \leq g(|z|, |\zeta|),$$

we have

$$V(-|z|) \leq \omega(z, E, \mathbb{D} \setminus E) \leq V(|z|).$$

Let $z_n \to z \in E$. By the right-hand inequality,

$$\liminf V(|z_n|) \geq 1,$$

which implies $\omega(z, E^*, \mathbb{D} \setminus E^*) \leq V(z)$. With the left-hand inequality we get

$$\omega(-|z|, E^*, \mathbb{D} \setminus E^*) \leq V(-|z|) \leq \omega(z),$$

which proves the theorem. $\square$

Corollary 9.3: *Let Ω be a simply connected domain, let $z \in \Omega$ and let $\zeta \in \partial\Omega$. Then for r sufficiently small,*

$$\omega(z, B(\zeta, r) \cap \partial\Omega, \Omega) \leq Cr^{\frac{1}{2}}.$$

Proof: We may assume $\zeta = 0$ and $z = -1$. We project $\partial\Omega \setminus B(0,r)$ to the positive real axis and apply Beurling's theorem in $\mathbb{C}^* \setminus B(0,r)$ to get

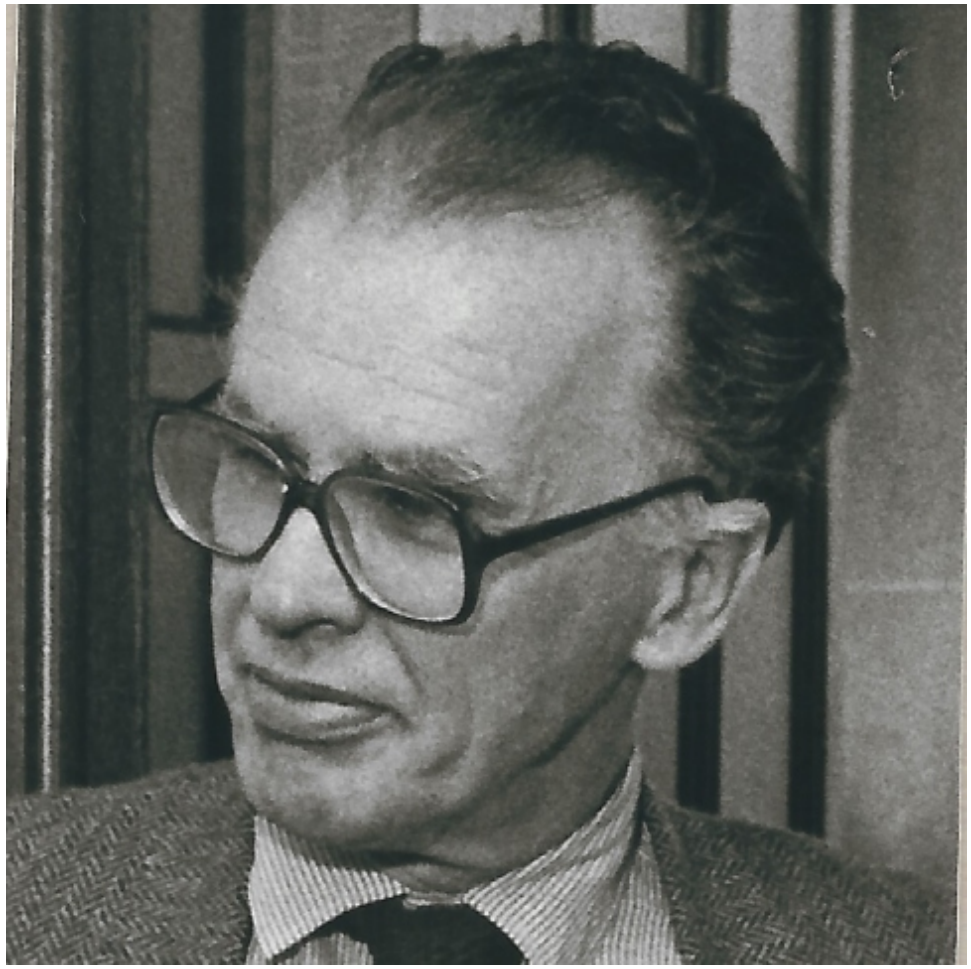
$$\begin{aligned}
\omega(z, B(0,r) \cap \partial\Omega, \Omega) &\leq \omega\left(-1, B(0,r), \mathbb{C} \setminus (B(0,r) \cup [r,\infty))\right) \\
&= \frac{2}{\pi} \arg\left(\frac{i - \sqrt{r}}{i + \sqrt{r}}\right) \\
&= \frac{4}{\pi} \tan^{-1} \sqrt{r}. \quad \square
\end{aligned}$$

A dyadic **Carleson square** in the upper half-plane is a set of the form

$$Q = \{j2^{-k} \leq x \leq (j+1)2^{-k}; 0 < y \leq 2^{-k}\},$$

and write $\ell(Q) = 2^{-k}$. Fix $z_0 = i$ and for convenience assume E is compact and

$$E \subset \big\{0 \leq x \leq 1,\ 0 < y \leq 1/2\big\}. \tag{9.4}$$



Lennart Carleson (1928-)

Let $\mathcal{A} = \mathcal{A}_E$ be the set of positive measures σ on E such that

(i) σ is a Carleson measure with constant 1, and or every dyadic Carleson box Q,

$$\sigma(Q) \leq \ell(Q), \tag{9.5}$$

(ii) the top half $T(Q) = Q \cap \{y \geq \ell(Q)/2\}$ of every dyadic Carleson box Q satisfies

$$\frac{\sigma(T(Q))}{\ell(Q)} \leq \frac{1}{1 + \gamma(E \cap T(Q)) - \gamma(T(Q))} = \frac{1}{1 + \log\left(\frac{\text{Cap } T(Q)}{\text{Cap } E \cap T(Q)}\right)} \tag{9.6}$$

Theorem 9.4: *Assume E satisfies* (9.4)*. Then there are constants C_1 and C_2 such that*

$$C_1\omega(i, E, \mathbb{H} \setminus E) \leq \sup\{\sigma(E) : \sigma \in \mathcal{A}_E\} \leq C_2\omega(i, E, \mathbb{H} \setminus E). \tag{9.7}$$

Special Case 1:

Beurling's projection theorem (with a larger constant) is a special case:

Suppose $\sigma_0(A) = \omega(0, A \cap E^*, \mathbb{D} \setminus E^*)$ and let σ be any positive measure on E that projects circularly onto σ_0. Then an elementary comparison gives

$$\sigma(Q) \leq \sigma_0(Q^*) \leq C\ell(Q^*)$$

for some constant C, so that $C^{-1}\sigma$ satisfies (9.5).

To check (9.6) for $C\sigma$ notice that by (4.12) the right side of (9.6) becomes smaller when E is replaced by E^* and then apply the right-hand inequality in (9.2) with $\delta = 1 - \ell(Q)$ and $r = (1 + \delta)/2$.

Special Case 2:

When $E \subset \mathbb{H}$ we still write $E^* = \{|z| : z \in E\}$ for the circular projection of E, but for $z = x + iy \in \mathbb{H}$ we define $z^* = -|x| + iy$. The estimate

$$\omega(z, E, \mathbb{H} \setminus E) \geq \frac{2}{3}\omega(z^*, E^*, \mathbb{H})$$

is known as **Hall's lemma**. With a different constant, the most important special case of Hall's Lemma, when $E \subset \{\arg z < \pi/4\}$ and $z = z^*$ also follows easily Theorem 9.4.

Proof of Theorem 9.4: The proof uses the Green's function for the upper half-plane:

$$g(z,\zeta) = \log\left|\frac{z-\overline{\zeta}}{z-\zeta}\right|,$$

and $g(z,\zeta)$ satisfies

$$g(z,\zeta) = \log\frac{\eta}{|z-\zeta|} + \mathrm{O}(1), \text{ if } \frac{|z-\zeta|}{|z-\overline{\zeta}|} \le c < 1, \tag{9.8}$$

and

$$g(z,\zeta) \sim \left|\frac{z-\overline{\zeta}}{z-\zeta}\right|^2 - 1 = \frac{4y\eta}{|z-\zeta|^2}, \text{ if } |z-\zeta| > \frac{\eta}{4}. \tag{9.9}$$

The **Green potential** of a measure μ on $\mathbb{H}$ is

$$G_\mu(z) = \int_{\mathbb{H}} g(z,\zeta)d\mu(\zeta).$$

Let μ_Q denote the (logarithmic) equilibrium distribution of $E \cap T(Q)$. Then by (9.8), for $z \in E \cap T(Q)$ we have

$$G_{\mu_Q}(z) = \int g(z,\zeta)d\mu_Q(\zeta) \sim 1 + \gamma(E \cap T(Q)) - \gamma(T(Q)).$$

Thus if we define

$$\nu_Q = \frac{\mu_Q}{1 + \gamma(E \cap T(Q)) - \gamma(T(Q))},$$

then for some constant C

$$C^{-1} \leq \int_{E \cap T(Q)} g(z,\zeta)d\nu_Q \leq C \tag{9.10}$$

on $E \cap T(Q)$.

To prove the left half of (9.7) we may assume $E \subset \{y \geq 2^{-N}\}$.

We inductively define a measure $\nu = \nu_N$ on E as follows: Set $\nu_0 = 0$.

Assume that ν_k is defined and has support in $E \cap \{\zeta : \eta \leq 2^{-N+k}\}$. Let

$$Q \in \mathcal{D}_k = \{Q \text{ dyadic} : \ell(Q) = 2^{-N+k+1} \text{ and } \mathrm{Cap}(E \cap T(Q)) > 0\}.$$

If $\inf_{E \cap T(Q)} G_{\nu_k} < 1$, take $\beta_Q > 0$ such that

$$\inf_{E \cap T(Q)} (G_{\nu_k}(z) + \beta_Q G_{\nu_Q}(z)) = 1. \tag{9.11}$$

If $\inf_{E\cap T(Q)} G_{\nu_k} \geq 1$, take $\beta_Q = 0$. Then define

$$\nu_{k+1} = \nu_k + \sum_{\mathcal{D}_{k+1}} \beta_Q \, \nu_Q.$$

Finally take $\nu = \nu_N$, and $\sigma = \eta\nu$. Then by construction $G_\nu \geq 1$ on E and $G_\nu = 0$ on $\mathbb{R}$. But by (9.4) and (9.9),

$$G_\nu(i) \leq C \int_E \eta d\nu(\zeta) = C\sigma(E).$$

Therefore by the maximum principle

$$\omega(i, E, \mathbb{H} \setminus E) \leq G_\nu(i) \leq C\sigma(E),$$

and if we can show that $c\sigma \in \mathcal{A}_E$ for some constant c, then the left-hand inequality in (9.7) will hold with $C_1 = (cC)^{-1}$.

To verify (9.5) for $c\sigma$ let Q be a dyadic box of side 2^{-N+k+1} and first assume $\beta_Q > 0$. Then by (9.11) there exists $z \in E \cap T(Q)$ such that

$$
\begin{aligned}
1 &= G_{\nu_k}(z) + \beta_Q G_{\nu_Q}(z) \\
\geq \int_Q g(z,\zeta)\frac{d\sigma}{\eta} & \\
&\geq C\frac{\sigma(T(Q))}{\ell(Q)} + C\int_{Q\setminus T(Q)} \frac{y}{|z-\zeta|^2} d\sigma \\
&\geq C\frac{\sigma(Q)}{\ell(Q)},
\end{aligned}
$$

and therefore (9.5) holds for Q.

But if $\beta_Q = 0$ then $\sigma(T(Q)) = 0$ and

$$\sigma(Q) = \sum \{\sigma(Q') : Q' \subset Q,\ \beta_{Q'} > 0,\ Q' \ \text{ maximal}\},$$

so that (9.5) also holds for Q.

To verify (9.6) for the dyadic box Q observe that by (9.10) and (9.11)

$$\frac{\sigma(T(Q))}{\ell(Q)} \leq \nu(T(Q)) \leq \frac{\beta_Q}{1 + \gamma(E \cap T(Q)) - \gamma(T(Q))} \leq C.$$

This proves the left side of (9.7).

To prove the right-hand part of (9.7) let $\sigma \in \mathcal{A}_E$ and set

$$\sigma' = \sum_{Q \text{ dyadic}} \sigma(T(Q))\mu_Q \text{ and } \nu = \frac{\sigma'}{\eta}.$$

Then G_ν is harmonic on $\mathbb{H} \setminus E$ and

$$G_\nu = 0 \tag{9.12}$$

on $\mathbb{R}$ since E is compact. By (9.4) and (9.9)

$$G_\nu(i) \sim \int_E \eta d\nu = \sigma(E).$$

Now let $z \in E$ and fix a dyadic box Q with $z \in T(Q)$. Then by (9.5) and (9.9)

$$\int_{|\zeta - z| \geq \ell(Q)/4} g(z,\zeta) d\nu(\zeta) \leq Cy \int_{|\zeta - z| \geq \ell(Q)/4} \frac{d\sigma'(\zeta)}{|z-\zeta|^2}$$

$$\leq Cy \sum_{n=0}^{\infty} \frac{\sigma'(\{|\zeta - z| \leq 2^{n-2}\ell(Q)\})}{2^{2n}\ell(Q)^2} \leq C,$$

and by (9.6) and (9.8)

$$\int_{|\zeta - z| < \ell(Q)/4} g(z,\zeta) d\nu(\zeta) \leq C$$

for some constant C. Therefore $G_\nu(z) \leq C$ on E and by (9.12)

$$\omega(i, E, \mathbb{H} \setminus E) \geq C^{-1} G_\nu(i) \geq C\sigma(E).$$

That proves the right-hand inequality in (9.7). $\square$