MAT 639, Spring 2026, Stony Brook University

Topics in Real Analysis: Harmonic Measure

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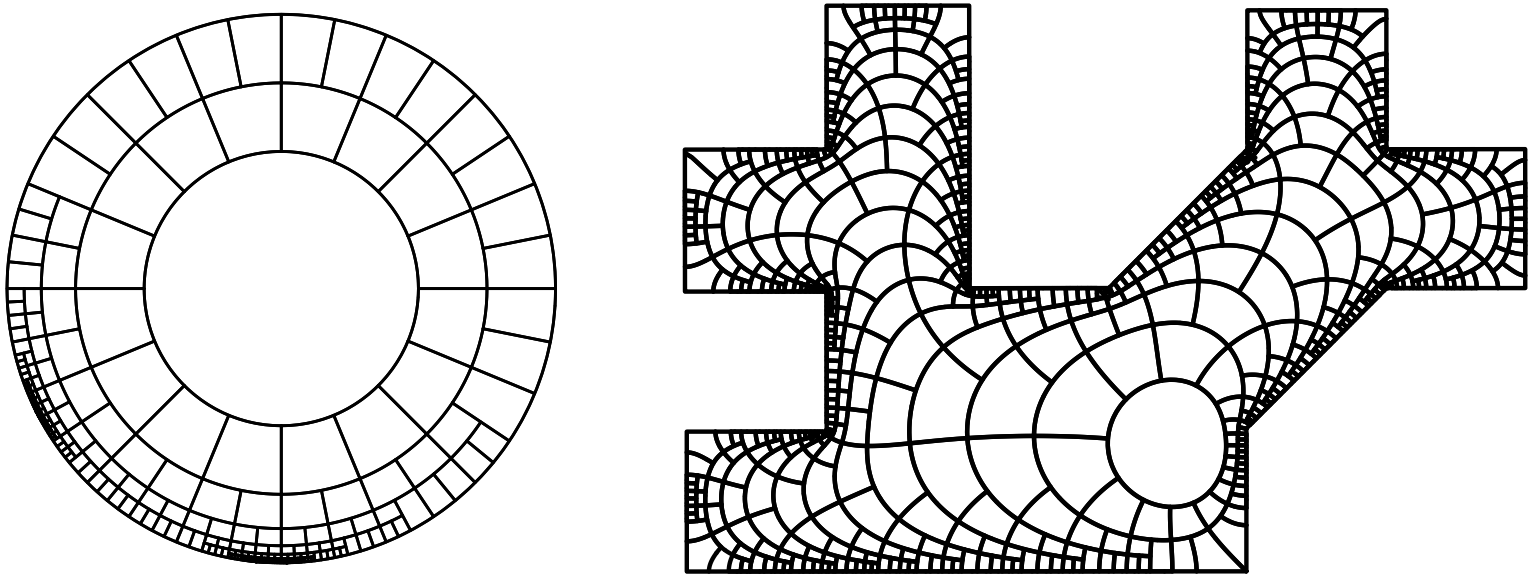


Harmonic Measure, Chapter II

Finitely Connected Domains

following text by John Garnett and Don Marshall

Goals for Chapter II:

- Solve Dirichlet problem for finitely connected domains
- Define and prove existence of Green's function
- Estimates for conjugate functions (summary)
- Boundary smoothness (summary)

Section II.1: The Schwarz Alternating Method

We say a plane domain Ω is a **finitely connected Jordan domain** if $\partial\Omega$ is a finite union of pairwise disjoint Jordan curves

$$\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_p.$$

A bounded function f on $\partial\Omega$ is **piecewise continuous** if there is a finite set $E \subset \partial\Omega$ such that f is continuous on $\partial\Omega \setminus E$ and f has left and right limits at each point of E.

By the Riemann mapping theorem, each boundary component γ of Ω can be mapped to a circle, by a map that is conformal on the side of γ containing the other boundary components.

Doing this once for each boundary component conformally maps Ω to a another domain bounded by finitely many analytic boundaries.

Koebe's circle domain theorem says every finitely connected domain can be conformally mapped to a domain bounded by circles or points. A proof is given in Karyn Lundberg's 2005 thesis.

Koebe's conjecture asks if every planar domain can be mapped to a circle domain, i.e., bounded by circles and points.

He and Schram proved this (using circle packings) for domains with countably many boundary components Fixed Points, Koebe Uniformization and Circle Packings, Annals of Math 1993

A shorter proof was later given by Schramm in the paper Transboundary extremal length. J. d'Anal. Math. 66 (1995), 307–329.

Dirichlet problem for finitely connected Jordan domains:

Theorem II.1.1: *Let Ω be a finitely connected Jordan domain, and let f be a bounded piecewise continuous function on $\partial\Omega$. Then there is a unique function $u(z) = u_f(z)$, bounded and harmonic on Ω, such that*

$$\lim_{z\to\zeta} u(z) = f(\zeta) \qquad (II.1.1)$$

Proof:

We may assume Ω is bounded. The uniqueness of u_f is immediate from Lindelöf's maximum principle,

Simply connected domains have already been treated, so we assume $p > 1$. We will solve this by iterating solutions of the simply connected case.

Take a Jordan arc σ with endpoints $a,\ b \notin \overline{\Omega}$ such that $\Omega_1 = \Omega \setminus \sigma$ is simply connected and such that $\sigma \cap \partial\Omega$ is a finite set.

Then $\varphi(z) = \sqrt{\frac{z-a}{z-b}}$ has a single valued analytic branch defined on Ω_1, and we can solve the Dirichlet problem on Ω_1 by transplanting it to the Jordan region $\varphi(\Omega_1)$.

Take a second Jordan arc σ' such that $\Omega_1' = \Omega \setminus \sigma'$ is simply connected, $\sigma' \cap \partial\Omega$ is a finite set, and $\sigma \cap \sigma' = \emptyset$.

We can also solve the Dirichlet problem on Ω_1'.

Let $E \subset \partial\Omega$ be a finite set and set $F = E \cup (\sigma \cap \partial\Omega) \cup (\sigma' \cap \partial\Omega)$. Suppose without loss of generality that $f \in C(\partial\Omega \setminus E)$ is positive and bounded.

To start, let u_1 be the solution to the Dirichlet problem on Ω_1 with boundary data

$$u_1(\zeta) = \begin{cases} f(\zeta), & \zeta \in \partial\Omega \\ \max_{\partial\Omega} f, & \zeta \in \sigma. \end{cases}$$

Then u_1 is harmonic on Ω_1 and continuous on $\overline{\Omega} \setminus F$ and u_1 matches its boundary data on $\partial\Omega_1 \setminus F$.

Next, let u_1' be the solution to the Dirichlet problem on Ω_1' with boundary data $u_1(\zeta),\ \zeta \in \partial\Omega_1'$. Then u_1' is harmonic on Ω_1' and continuous on $\overline{\Omega} \setminus F$ and u_1' matches its boundary data on $\partial\Omega_1' \setminus F$.

In particular, $u_1' = u_1$ on $\partial\Omega \setminus F$. By the Lindelöf maximum principle we have $u_1' \leq \max_{\partial\Omega} f = u_1$ on $\sigma \cap \Omega$, and therefore $u_1' \leq u_1$ on $\overline{\Omega} \setminus F$.

Now let u_2 be the solution to the Dirichlet problem on Ω_1 with boundary data $u_1'(\zeta),\ \zeta \in \partial\Omega_1$. On $\sigma \cap \Omega$ we have $u_2 = u_1' \leq u_1$, while on $\partial\Omega \setminus F$ we have $u_2 = f = u_1$.

Therefore $u_2 \leq u_1$ on Ω_1, again by Lemma I.1.1. Consequently $u_2 \leq u_1' = u_1$ on $\sigma' \cap \Omega$, and by the maximum principle, $u_2 \leq u_1'$ on $\overline{\Omega} \setminus F$.

Continuing in this way we obtain a decreasing sequence

$$u_1 \geq u_1' \geq u_2 \geq u_2' \geq u_3 \ldots$$

of positive functions, alternately harmonic on Ω_1 and Ω_1'. By Harnack's principle, the limit

$$u(z) = \lim_{n\to\infty} u_n(z) = \lim_{n\to\infty} u_n'(z)$$

is a bounded harmonic function on Ω such that $u \leq u_1 \leq \max_{\partial\Omega} f$.

To complete the proof we show

$$\lim_{z \to \zeta} u(z) = f(\zeta) \tag{1.1}$$

whenever $\zeta \in \partial\Omega$ is a point of continuity of f.

We may assume $\zeta \notin \sigma \cup E$. Take a neighborhood V of ζ such that $W = V \cap \Omega$ is a Jordan domain and such that $W \cap (E \cup \sigma) = \emptyset$.

Let φ be a conformal map of $\mathbb{D}$ onto W. By Carathéodory's theorem $\varphi(\partial\mathbb{D}) = \partial W$, and for $w = \varphi(z) \in W$ we have

$$\begin{aligned} u_n(w) \;=\; & \int_{\varphi^{-1}(\partial\Omega)} \frac{1-|z|^2}{|e^{i\theta}-z|^2}\, f\circ\varphi(e^{i\theta})\frac{d\theta}{2\pi} \\ & + \int_{\partial\mathbb{D}\setminus\varphi^{-1}(\partial\Omega)} \frac{1-|z|^2}{|e^{i\theta}-z|^2}\, u_n\circ\varphi(e^{i\theta})\frac{d\theta}{2\pi}. \end{aligned}$$

Because $\varphi^{-1}(\partial\Omega)$ is a neighborhood of $\varphi^{-1}(\zeta)$ in $\partial\mathbb{D}$, the first integral approaches $f(\zeta)$ as $w \to \zeta$.

Because $|u_n| \le \sup_{\partial\Omega}|f|$, the second integral tends to zero, uniformly in n, as $w \to \zeta$. Therefore (1.1) holds and the theorem is proved. $\square$

If Ω is a finitely connected Jordan domain, Theorem II.1.1 shows that the map $f \to u_f(z)$ is a bounded linear functional on $C(\partial\Omega)$.

Define the **harmonic measure** of a relatively open subset $U \subset \partial\Omega$ by

$$\omega(z, U) \equiv \omega(z, U, \Omega) = \sup\{u_f(z) : f \in C(\partial\Omega),\ 0 \le f \le \chi_U\},$$

and for an arbitrary subset E of $\partial\Omega$, set

$$\omega(z, E) \equiv \omega(z, E, \Omega) = \inf\{\omega(z, U) : U \text{ open in } \partial\Omega,\ U \supset E\}.$$

This procedure, shows $\omega(z, E)$ is a Borel measure on $\partial\Omega$ such that

$$u_f(z) = \int_{\partial\Omega} f(\zeta) d\omega(z, \zeta) \tag{1.2}$$

for continuous f.

When Ω is simply connected, this definition of harmonic measure agrees with the earlier definition (I.3.3).

For every $z_1, z_2 \in \Omega$ there exists, by virtue of Harnack's inequality, a constant $c = c(z_1, z_2)$ such that

$$\frac{1}{c}\omega(z_1, E) \leq \omega(z_2, E) \leq c\omega(z_1, E), \tag{1.3}$$

The constants $c(z_1, z_2)$ remain uniformly bounded if z_1 and z_2 remain in a compact subset of Ω.

If $f \in L^1(\partial\Omega, d\omega)$, there is a sequence $\{f_n\}$ in $C(\partial\Omega)$ such that for some fixed $z_0 \in \Omega$,

$$\int |f_n(\zeta) - f(\zeta)| \, d\omega(z_0, \zeta) \to 0.$$

Write

$$u_n(z) = \int f_n(\zeta) d\omega(z, \zeta)$$

and

$$u(z) = u_f(z) = \int f(\zeta) d\omega(z, \zeta). \tag{1.4}$$

Then by (1.3) $u_n(z) \to u(z)$ for all $z \in \Omega$.

By (1.3) we also see that the harmonic functions $u_n(z)$ are uniformly bounded on compact subsets of Ω. Thus by Harnack's principle the limit function $u(z)$ is harmonic on Ω.

We call u the **solution to the Dirichlet problem for** f on Ω.

If f is bounded, then we also have

$$\sup_{z\in\Omega} |u_f(z)| \leq ||f||_{L^\infty(\Omega,d\omega)}.$$

Moreover, if f is bounded and continuous at $\zeta \in \partial\Omega$, then (1.1) (= boundary continuity) also holds at ζ. (This requires a proof, to come later.)

Section II.2: Greens functions and Poisson Kernels

Definition of Green's function: For finitely connected Jordan domain Ω, the **Green's function** with pole at $w \in \Omega$ is function $g(z, w)$ that is is continuous in $z \in \overline{\Omega} \setminus \{w\}$, and

$$g(z, w) > 0 \qquad \text{on } \Omega, \tag{2.2}$$

$$g(\zeta, w) = 0 \qquad \text{on } \partial\Omega, \tag{2.3}$$

$$z \to g(z, w) \text{ is harmonic on } \Omega \setminus \{w\}, \tag{2.4}$$

$$z \to g(z, w) - \log \frac{1}{|w - z|} \text{ is harmonic at } w. \tag{2.5}$$

The maximum principle implies such a function (if it exists) is unique.

For general domains, we will later drop the boundary continuity.

Theorem 2.1: *Let Ω be a simply connected domain bounded by a Jordan curve, let $w \in \Omega$ and let $\psi : \Omega \to D$ be a conformal map with $\psi(w) = 0$. Then*

$$g(z, w) = -\log |\psi(z)|.$$

For simply connected domains, knowing the Riemann map and knowing the Green's function are essentially the same.

For a bounded, finitely connected Jordan domain Ω and a fixed $w \in \Omega$, let $h(z, w)$ be the solution to the Dirichlet problem for the boundary value $f(\zeta) = \log|\zeta - w| \in C(\partial\Omega)$. The Green's function with pole w is then

$$g(z, w) = \log \frac{1}{|z - w|} + h(z, w). \tag{2.1}$$

When Ω is unbounded we fix $a \notin \overline{\Omega}$. For $w \neq \infty$, we let $h(z, w)$ solve the Dirichlet problem on Ω for $f(\zeta) = \log\left|\frac{\zeta - w}{\zeta - a}\right|$, and define

$$g(z, w) = \log \left|\frac{z - a}{z - w}\right| + h(z, w).$$

For $w = \infty$, we instead use $f(\zeta) = \log\left|\frac{1}{\zeta - a}\right|$ to define $h(z, \infty)$ and set $g(z, \infty) = \log|z - a| + h(z, \infty)$.

Suppose φ is a conformal mapping from one finitely connected Jordan domain Ω onto another finitely connected Jordan domain Ω'. Then $\varphi(z) \to \partial\Omega'$ whenever $z \to \partial\Omega$, because $\varphi : \Omega \to \Omega'$ is a homeomorphism.

It follows that

$$g_{\Omega'}(\varphi(z), \varphi(w)) = g_{\Omega}(z, w) \tag{2.6}$$

Connection of Green's function to Brownian motion:

If a Brownian motion is started at $w \in \Omega$, the the expected amount of time it spends in a set $E \subset \Omega$ before hitting $\partial\Omega$ is

$$\int_E g(z, w) dx dy.$$

We will not use this formally, but it often gives a good intuition about the behavior of Green's function.

If $\Omega = f(\mathbb{D})$ where $f(z) = \sum a_n a^n$ is conformal, then the expected time for Brownian motion to leave Ω starting from 0 is $\frac{1}{2}\sum |a_n|^2$.

This is due to Markowski (2011); see also Lemma 7.2 of [Bishop-Peres, 2017].

Lemma 2.2: *Let Ω be a finitely connected Jordan domain. Then there exists a finitely connected Jordan domain Ω^* such that $\partial\Omega^*$ consists of finitely many pairwise disjoint analytic Jordan curves and there exists a conformal map from Ω onto Ω^* which extends to be a homeomorphism from $\overline{\Omega}$ to $\overline{\Omega^*}$.*

Lemma 2.4: *Suppose Ω is a finitely connected Jordan domain and suppose $\gamma \subset \partial\Omega$ is an analytic arc. Let $u(z)$ be a harmonic function in Ω. If*

$$\lim_{z\to\zeta} u(z) = 0$$

for all $\zeta \in \gamma$, then there is an open set $W \supset \gamma \cup \Omega$ such that u extends to be harmonic on W. If also $u(z) > 0$ in Ω, then

$$\frac{\partial u}{\partial \mathrm{n}}(\zeta) < 0 \tag{2.10}$$

for all $\zeta \in \gamma$.

Proof follows from Schwarz reflection principle.

Symmetry of Green's Function:

Theorem 2.3: *Let Ω be a finitely connected Jordan domain and suppose that $z_1,\ z_2 \in \Omega$. Then*

$$g(z_1, z_2) = g(z_2, z_1). \tag{2.8}$$

Proof: By Lemma 2.2 we may assume $\partial\Omega$ consists of analytic Jordan curves. When $\partial\Omega$ consists of analytic curves, there is a neighborhood V of $\partial\Omega$ to which $z \to g(z,w)$ has a harmonic extension.

Hence $g(z,w)$ is C^∞ on some neighborhood V of $\partial\Omega$ and we can use Green's theorem in the form

$$\iint_{\mathcal{U}} (u\Delta v - v\Delta u)dxdy = \int_{\partial\mathcal{U}} \left(u\frac{\partial v}{\partial \mathrm{n}} - v\frac{\partial u}{\partial \mathrm{n}} \right) ds,$$

where n is the unit normal vector pointing out from the domain $\mathcal{U}$.

Fix distinct $z_1, z_2 \in \Omega$. We apply Green's theorem on the domain

$$\Omega_\varepsilon = \Omega \setminus \Big(\{|z - z_1| \le \varepsilon\} \cup \{|z - z_2| \le \varepsilon\} \Big),$$

when ε is small, with $u(z) = g(z, z_1)$ and $v(z) = g(z, z_2)$. Because $u = v = 0$ on $\partial\Omega$,

$$\int_{\partial\Omega} \left(u \frac{\partial v}{\partial \mathrm{n}} - v \frac{\partial u}{\partial \mathrm{n}} \right) ds = 0,$$

and because u and v are harmonic on Ω_ε, the area integral in Green's theorem vanishes.

We conclude that

$$\begin{aligned}
&\varepsilon \int_0^{2\pi} g(z_1 + \varepsilon e^{i\theta}, z_1) \frac{\partial g}{\partial r}(z_1 + \varepsilon e^{i\theta}, z_2) \frac{d\theta}{2\pi} \\
&- \varepsilon \int_0^{2\pi} g(z_1 + \varepsilon e^{i\theta}, z_2) \frac{\partial g}{\partial r}(z_1 + \varepsilon e^{i\theta}, z_1) \frac{d\theta}{2\pi} \\
&= \varepsilon \int_0^{2\pi} (g(z_2 + \varepsilon e^{i\theta}, z_2) \frac{\partial g}{\partial r}(z_2 + \varepsilon e^{i\theta}, z_1) \frac{d\theta}{2\pi} \qquad (2.9) \\
&- \varepsilon \int_0^{2\pi} g(z_2 + \varepsilon e^{i\theta}, z_1) \frac{\partial g}{\partial r}(z_2 + \varepsilon e^{i\theta}, z_2) \frac{d\theta}{2\pi}.
\end{aligned}$$

For ϵ small, $g(z_j + \varepsilon e^{i\theta}, z_j) \leq 2\log(1/\varepsilon)$ and for $k \neq j$, $g(z, z_j)$ has bounded derivatives near z_k.

That means the first and third integrals in (2.9) approach 0 as $\varepsilon \to 0$. By (2.1),

$$-\frac{\partial g}{\partial r}(z_j + \varepsilon e^{i\theta}, z_j) = \frac{\partial \log \varepsilon}{\partial \varepsilon} + \mathrm{O}(1),$$

so that, as $\varepsilon \to 0$, the second integral tends to $g(z_1, z_2)$ and the fourth integral tends to $g(z_2, z_1)$. Therefore (2.8) holds. $\square$

Harmonic measure = normal derivative of Green's function:

Theorem 2.5: *Assume $\partial\Omega$ consists of finitely many pairwise disjoint analytic Jordan curves, and let $z \in \Omega$. Then Green's function $g(\zeta, z)$ extends to be harmonic (and hence real analytic) on a neighborhood of $\partial\Omega$ and*

$$\frac{-\partial g(\zeta, z)}{\partial \mathrm{n}_\zeta} > 0 \tag{2.11}$$

on $\partial\Omega$, where n_ζ is the unit outer normal at $\zeta \in \partial\Omega$. If $u \in C(\overline{\Omega})$ is harmonic on Ω, then

$$u(z) = \int_{\partial\Omega} \frac{-\partial g(\zeta, z)}{\partial \mathrm{n}_\zeta} u(\zeta) \frac{ds(\zeta)}{2\pi}. \tag{2.12}$$

Because of (2.12),

$$P_z(\zeta) = -\frac{1}{2\pi} \frac{\partial g(\zeta, z)}{\partial \mathrm{n}_\zeta}$$

is called the Poisson kernel for Ω.

Proof: Fix $z \in \Omega$. By Lemma 2.4, $g(\zeta, z)$ extends to be harmonic (and real analytic) on some neighborhood of $\partial\Omega$ and then (2.11) follows from (2.10).

To prove (2.12) we first assume that u is C^∞ on a neighborhood of $\partial\Omega$. We then apply Green's theorem on $\Omega_\varepsilon = \Omega \setminus \{w : |w - z| < \varepsilon\}$ with ε small and $v(w) = g(w, z)$. Because $\Delta_w\, g(w, z) = \Delta u = 0$ on Ω_ε and because $g = 0$ on $\partial\Omega$, Green's theorem collapses into

$$\int_{\partial\Omega} -\frac{\partial g(\zeta, z)}{\partial \mathrm{n}_\zeta} u(\zeta) \frac{ds(\zeta)}{2\pi} = \varepsilon \int_0^{2\pi} g(z + \varepsilon e^{i\theta}, z) \frac{\partial u(z + \varepsilon e^{i\theta})}{\partial r} \frac{d\theta}{2\pi}$$

$$- \epsilon \int_0^{2\pi} u(z + \varepsilon e^{i\theta}) \frac{\partial g(z + \varepsilon e^{i\theta}, z)}{\partial r} \frac{d\theta}{2\pi}.$$

Since $g(z+\varepsilon e^{i\theta}, z) \le 2\log 1/\varepsilon$ for small ε, and since u is C^∞, we have

$$\lim_{\varepsilon\to 0} \varepsilon \int_0^{2\pi} g(z+\varepsilon e^{i\theta}, z) \frac{\partial u(z+\varepsilon e^{i\theta})}{\partial r} \frac{d\theta}{2\pi} = 0.$$

(2.1) yields

$$\frac{\partial g(z+\varepsilon e^{i\theta}, z)}{\partial r} = -\frac{\partial \log \varepsilon}{\partial \varepsilon} + \mathrm{O}(1),$$

while $u(z+\varepsilon e^{i\theta}) = u(z) + \mathrm{O}(\varepsilon)$. Hence

$$-\varepsilon \int_0^{2\pi} u(z+\varepsilon e^{i\theta}) \frac{\partial g}{\partial r}(z+\varepsilon e^{i\theta}, z) d\theta = u(z) + \mathrm{O}(\varepsilon),$$

and that gives (2.12) when u is C^∞ on a neighborhood of $\partial\Omega$.

To prove (2.12) in general, set $\Omega^\delta = \{w \in \Omega : \ g(w,z) > \delta\}$ where δ is small. By uniqueness, Ω^δ has Green's function $g_\delta(w,z) = g(w,z) - \delta$. Therefore

$$\frac{\partial g_\delta}{\partial \mathrm{n}_\zeta}(\zeta, z) = \frac{\partial g}{\partial \mathrm{n}_\zeta}(\zeta, z)$$

on $\partial\Omega^\delta$.

In a neighborhood of a point $\zeta_0 \in \partial\Omega$, the function $\varphi = g + i\widetilde{g}$ is a conformal map and

$$\frac{1}{2\pi}\int_{N\cap\partial\Omega^\delta} \frac{-\partial g(\zeta,z)}{\partial \mathrm{n}_\zeta} u(\zeta)ds = \int_{\{Rez=\delta\}\cap\varphi(N)} u\circ\varphi^{-1}ds,$$

which converges to

$$\int_{\{Rez=0\}\cap\varphi(N)} u\circ\varphi^{-1}ds = \frac{1}{2\pi}\int_{\partial\Omega} \frac{-\partial g(\zeta,z)}{\partial \mathrm{n}_\zeta} u(\zeta)ds.$$

Therefore

$$\frac{1}{2\pi}\int_{\partial\Omega^\delta}\frac{-\partial g(\zeta,z)}{\partial \mathrm{n}_\zeta}u(\zeta)ds \to \frac{1}{2\pi}\int_{\partial\Omega}\frac{-\partial g(\zeta,z)}{\partial \mathrm{n}_\zeta}u(\zeta)ds \tag{2.13}$$

as $\delta \to 0$.

But (2.12) holds for u on Ω^δ because u is C^∞ on a neighborhood of $\partial\Omega_\delta$, and hence (2.13) yields (2.12) for Ω. $\square$

Corollary 2.6: *If $\partial\Omega$ consists of finitely many pairwise disjoint analytic Jordan curves and if $z \in \Omega$, then*

$$d\omega(z,\zeta) = -\frac{\partial g(z,\zeta)}{\partial \mathrm{n}_\zeta}\frac{ds(\zeta)}{2\pi}. \tag{2.14}$$

In other words, harmonic measure for $z \in \Omega$ is absolutely continuous to arc length on $\partial\Omega$, the density

$$\frac{d\omega}{ds} = -\frac{1}{2\pi}\frac{\partial g(z,\zeta)}{\partial \mathrm{n}_\zeta} = P_z(\zeta)$$

is real analytic on $\partial\Omega$, and

$$c_1 < \frac{d\omega}{ds} < c_2 \tag{2.15}$$

for positive constants c_1 and c_2.

Proof: The identity (2.14) follows from (1.2) and (2.12) and the fact that Borel measures are determined by their actions on continuous functions. The inequalities (2.15) are immediate from (2.11). $\square$

Theorem 2.7: *Assume $\partial\Omega$ consists of finitely many pairwise disjoint analytic Jordan curves and for $\zeta \in \partial\Omega$ and $\alpha > 1$ define*

$$\Gamma_\alpha(\zeta) = \big\{ z \in \Omega : |z - \zeta| < \alpha \operatorname{dist}(z, \partial\Omega) \big\}.$$

If $u(z)$ is bounded and harmonic on Ω, then for ds almost all $\zeta \in \partial\Omega$

$$\lim_{\Gamma_\alpha(\zeta) \ni z \to \zeta} u(z) = f(\zeta) \tag{2.16}$$

exists,

$$u(z) = \int_{\partial\Omega} P_z(\zeta) f(\zeta) ds(\zeta), \tag{2.17}$$

and

$$\sup_{\Omega} |u(z)| = ||f||_{L^\infty}. \tag{2.18}$$

Conversely, if f is a bounded Borel function on $\partial\Omega$, then (2.17) defines a bounded harmonic function $u(z)$ on Ω such that (2.18) holds and (2.16) holds ds almost everywhere. Moreover, if f is continuous at $\zeta_0 \in \partial\Omega$, then

$$\lim_{z \to \zeta_0} u(z) = f(\zeta_0). \tag{2.19}$$

This establishes an isometry between the space of bounded harmonic functions on Ω and $L^\infty(\partial\Omega, ds)$ when $\partial\Omega$ consists of analytic curves.

It can fail sometimes, e.g., a disk with a slit removed.

Domains for which this isometry holds are called **Poissonian domains** and were characterized in [Bishop, 1991]

Proof: A simple localization argument gives the existence of the nontangential limit f. If I is an open arc on $\partial\Omega$, there exists a neighborhood $V \supset I$ such that $V \cap \partial\Omega = I$ and $V \cap \Omega$ is simply connected, and there exists a conformal mapping ψ defined on V such that $\psi(V \cap \Omega) = \mathbb{D}$ and $\psi(I)$ is an arc on $\partial\mathbb{D}$.

It follows that ψ maps conical approach regions at $\zeta \in V \cap \partial\Omega$ into cones at $\psi(\zeta)$:

$$\psi\big(V \cap \Gamma_\alpha(\zeta) \cap B_\delta(\zeta)\big) \subset \Gamma_{\beta(\alpha)}(\psi(\zeta)),$$

where $B_\delta(\zeta) = \{z : |z - \zeta| < \delta = \delta(\zeta)\}$.

Then if $u(z)$ is a bounded harmonic function on Ω, we can apply Fatou's theorem to $u \circ (\psi^{-1})$ to obtain (2.16) ds almost everywhere on $V \cap \partial\Omega$.

The proof of (2.17) is exactly the same as the proof of (2.12) except that the dominated convergence theorem is applied in (2.13). By (2.16) we have $||f||_\infty \leq \sup_\Omega |u(z)|$, and because $P_z \geq 0$ and $\int_{\partial\Omega} P_z ds = 1$, we have $\sup_\Omega |u(z)| \leq ||f||_\infty$. Hence (2.16) and (2.17) imply (2.18).

To prove the converse, let $f \in L^\infty(\partial\Omega, ds)$. Then the discussion following (1.4) shows (2.17) defines a bounded harmonic function $u(z)$ on Ω and $\sup_\Omega |u(z)| \leq ||f||_\infty$. Therefore by (2.16), u has almost everywhere a nontangential limit, which we will temporarily call F, and u is the Poisson integral of F.

What we must prove is that $F = f$ almost everywhere. Again let V be a neighborhood of an open arc $I = V \cap \partial\Omega$ such that $V \cap \Omega$ is simply connected.

For $h \in L^\infty(\partial\Omega, ds)$, define

$$v_h(z) = \int_{\partial\Omega} P_z(\zeta) h(\zeta) ds(\zeta) - \int_I P_z(\zeta, V) h(\zeta) ds(\zeta),$$

where $P_z(\zeta, V)$ is the Poisson kernel for $z \in V \cap \Omega$.

If $h \in C(\partial\Omega)$ then by (1.2), Theorem 1.1, (2.12), and (3.2) of Chapter I, $\lim_{z\to\zeta} v_h(z) = 0$ for all $\zeta \in I$, and hence by Lemma 2.4, v_h extends to be harmonic in a neighborhood W of I which does not depend on h.

Thus by (3.5) below, if J is a compact subset of I and if $\epsilon > 0$ then there is a neighborhood N of J, depending only on $||h||_\infty$ and ϵ, so that $|v_h| < \epsilon$ in N.

Now take $h_n \in C(\partial\Omega)$ so that h_n converges to f in L^1, and $||h_n||_\infty \leq ||f||_\infty$. For each $z \in V \cap N$, $v_{h_n}(z)$ converges to $v_f(z)$ and so $|v_f(z)| < \epsilon$. Because $\epsilon > 0$ was arbitrary, we conclude that $v_f(z) \to 0$ as $z \to \zeta \in J$. But by Theorem I.1.3,

$$F(\zeta) - f(\zeta) = \lim_{\Gamma_{\alpha(\zeta)} \ni z \to \zeta} v_f(z) = 0$$

almost everywhere in J. Consequently $F = f$ almost everywhere and (2.16) and (2.18) hold for all $f \in L^\infty(\partial\Omega, ds)$.

Finally, if f is continuous at $\zeta_0 \in I$ then $\int_I P_z(\zeta, V) f(\zeta) ds(\zeta)$ is continuous at ζ_0 by (3.2) of Chapter I and so by the continuity of v_f, (2.19) holds. $\square$

The converse can be proved another way. Using the real analyticity of $g(w,z)$, one can refine the proof of Lemma I.2.2 and show that

$$\sup_{\Gamma_\alpha(\zeta)} |u(z)| \leq C(\alpha,\Omega) M_s f(\zeta), \tag{2.20}$$

where u is the Poisson integral (2.17) and where the maximal function $M_s f(\zeta)$ is the supremum of the averages of f over arcs $\gamma \subset \partial\Omega$ with $\zeta \in \gamma$:

$$M_s f(\zeta) = \sup_{\gamma \ni \zeta} \frac{1}{\ell(\gamma)} \int_\gamma |f| ds.$$

A variation on the covering lemma shows that M_s is weak-type 1-1, and an approximation, as in the proof of Theorem I.2.1, then yields (2.16) for the Poisson integral of f. This is the argument that must be used in the Euclidean spaces $\mathbb{R}^d$, with $d \geq 3$, and in other situations.

Question 1. If $\partial\Omega$ has some degree of differentiability and if $f \in C(\partial\Omega)$ also has some degree of differentiability along $\partial\Omega$, how smooth is the solution $u_f(z)$, as z approaches $\partial\Omega$?

Question 2. What smoothness conditions on $\partial\Omega$, weaker than real analyticity, will ensure that $\partial g(z,\zeta)/\partial \mathrm{n}_\zeta$ exists on $\partial\Omega$ and that (2.14) and (2.15) still hold?

These two questions are equivalent. Their answers will depend on Kellogg's theorem about the boundary behavior of conformal mappings. The proof of Kellogg's theorem in turn depends on the estimates for conjugate functions in the next section.

Section II.3: Conjugate Functions

Let $f \in L^1(\partial\mathbb{D})$ be real. For convenience we write $f(\theta)$ for $f(e^{i\theta})$.

If $u(z)$ is the Poisson integral of f on $\mathbb{D}$, then $u(z)$ is harmonic and real and there exists a unique harmonic function $\tilde{u}(z)$ such that $\tilde{u}(0) = 0$ and

$$F = u + i\tilde{u}$$

is analytic on $\mathbb{D}$.

The function $\tilde{u}$ is called the **conjugate function** or **harmonic conjugate** of u.

The nontangential limit

$$\widetilde{f}(\theta) = \lim_{\Gamma_\alpha(e^{i\theta})\ni z\to e^{i\theta}} \widetilde{u}(z) \tag{3.1}$$

exists almost everywhere, and this has an easy proof from Fatou's theorem: We may assume $f \geq 0$, so that

$$G(z) = e^{-(u(z)+i\tilde{u}(z))}$$

is bounded and analytic on $\mathbb{D}$. By Corollary I.2.5, G has a nontangential limit $G(e^{i\theta})$ almost everywhere.

Since $|G(e^{i\theta})| = e^{-f(\theta)}$ and $f \in L^1$, $|G(e^{i\theta})| > 0$ almost everywhere. At such $e^{i\theta}$, G is continuous and non-zero on the cone $K = \overline{\Gamma_\alpha(e^{i\theta})}$.

Consequently $\log G = -(u + i\tilde{u})$ has a continuous extension to $K \cap \{z : |G(z) - G(e^{i\theta})| < \frac{1}{2}|G(e^{i\theta}|\}$ and the limit (3.1) exists at $e^{i\theta}$.

If u is harmonic and if $|u| < \pi/2$, then

$$\varphi(z) = \int_0^z e^{i(u+i\tilde{u})(\zeta)} d\zeta$$

is a conformal map from $\mathbb{D}$ to a simply connected domain and $u = \arg \varphi'$.

Indeed, if $a \neq b \in \mathbb{D}$, then

$$\varphi(b) - \varphi(a) = (b-a) \int_0^1 \varphi'(a + t[b-a]) dt \neq 0,$$

because $\mathrm{Re}\,(\varphi') > 0$.

When f is bounded, or even continuous, it can happen that $\tilde{f}$ is not bounded. For an example, let $u + i\tilde{u}$ be the conformal map of $\mathbb{D}$ onto the region

$$\{0 < x < 1/(1 + |y|)\}.$$

Then u is continuous on $\overline{\mathbb{D}}$ by Caratheodory's theorem, but $\tilde{u}$ is not bounded.

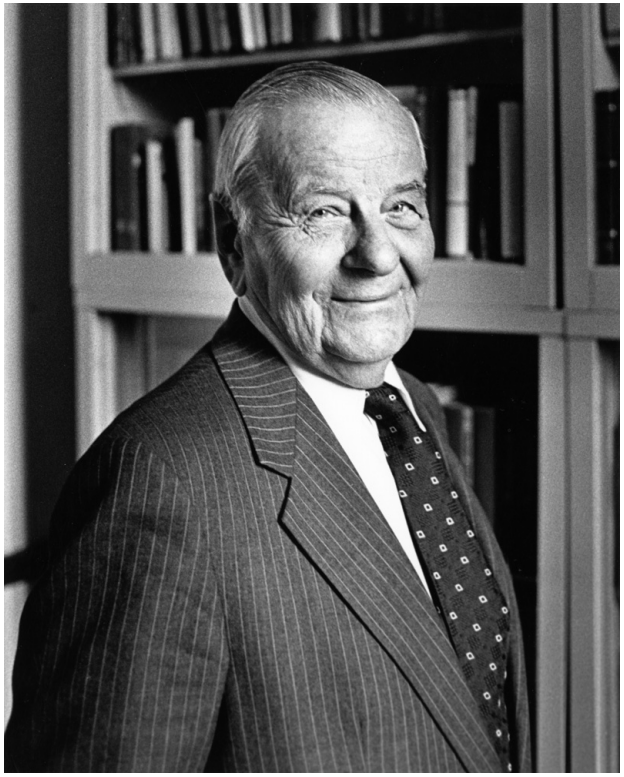
Theorem 3.1 (Zygmund): *Let $f \in L^\infty(\partial\mathbb{D})$ be real with $||f||_\infty \leq 1$.*
(a) For $0 < \lambda < \pi/2$ there is a constant C_λ, depending only on λ, such that

$$\int e^{\lambda|\tilde{f}(\theta)|}\frac{d\theta}{2\pi} \leq C_\lambda.$$

(b) If $f \in C(\partial\mathbb{D})$, then for all $\lambda < \infty$

$$\sup_{0<r<1}\int e^{\lambda|\tilde{u}(re^{i\theta})|}\frac{d\theta}{2\pi} < \infty.$$

For the proof, see the textbook.



Antoni Zygmund, (1900-1992)

"Surely, Antoni Zygmund's "Trigonometric series" has been, and continues to be, one of the most influential books in the history of mathematical analysis. " – Robert Fefferman

Let $0 < \alpha < 1$. The **Lipschitz class** C^α is

$$C^\alpha = \left\{ f \in L^\infty(\partial\mathbb{D}) : \sup_{t>0} \frac{\|f(\theta+t) - f(\theta)\|_\infty}{t^\alpha} < \infty \right\}.$$

Every $f \in C^\alpha$ agrees almost everywhere with a function continuous on $\partial\mathbb{D}$.

The class C^α is given the **Lipschitz norm**

$$\|f\|_{C^\alpha} = \|f\|_\infty + \sup_{t>0} \frac{\|f(\theta+t) - f(\theta)\|_\infty}{t^\alpha}. \tag{3.2}$$

In the Poisson integral formula

$$u(z) = \mathrm{Re} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} f(t) \frac{dt}{2\pi},$$

the kernel $(e^{it} + z)/(e^{it} - z)$ is analytic in $z \in \mathbb{D}$ and real at $z = 0$. Then the uniqueness of $\tilde{u}$ shows

$$u(z) + i\tilde{u}(z) = F(z) = \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} f(t) \frac{dt}{2\pi}. \tag{3.3}$$

The analytic function $F(z)$ in (3.3) is called the **Herglotz integral** of f.

Write

$$\nabla u = \Big(\frac{\partial u}{\partial x},\ \frac{\partial u}{\partial y}\Big)$$

when u is differentiable on an open plane set.

If u is harmonic and bounded on $\mathbb{D}$ then u is the Poisson integral of some $f \in L^\infty(\partial\mathbb{D})$ and by (3.3) and the Cauchy–Riemann equations,

$$|\nabla u(z)| = \big|F'(z)\big| = \Big|\int_{-\pi}^{\pi} \frac{e^{it} f(t)}{(e^{it} - z)^2} \frac{dt}{\pi}\Big|. \tag{3.4}$$

Therefore (as in Exercise I.5(e))

$$|\nabla u(z)| \le \Big(\int_{-\pi}^{\pi} \frac{1}{|e^{it} - z|^2} \frac{dt}{\pi}\Big) \|f\|_\infty \le 2(1 - |z|)^{-1} \|f\|_\infty. \tag{3.5}$$

The following consequence of (3.5) is important: If $u(z)$ is harmonic and $|u(z)| \leq M$ on $B(z,R)$, then

$$\sup_{B(z,\frac{R}{2})} |\nabla u| \leq \frac{4M}{R}. \tag{3.6}$$

To prove (3.6) simply apply (3.5) to $U(w) = u(z + Rw)$.

Theorem 3.2: *Let $0 < \alpha < 1$, let $f \in L^\infty(\partial\mathbb{D})$ be real, and let $u(z)$ be the Poisson integral of f. Then the following conditions are equivalent:*

(a) $f \in C^\alpha$.

(b) $\tilde{f} \in C^\alpha$.

(c) $|\nabla u(z)| = \mathrm{O}\big((1-|z|)^{\alpha-1}\big)$.

(d) $u \in C^\alpha(\overline{\mathbb{D}})$*; that is, for all* $z_1, z_2 \in \mathbb{D}$,

$$|u(z_1) - u(z_2)| = \mathrm{O}\big(|z_1 - z_2|^\alpha\big).$$

Moreover there is a constant C_1, independent of α, such that

$$||\tilde{f}||_{C^\alpha} \le \frac{C_1}{\alpha(1-\alpha)}||f||_{C^\alpha}, \tag{3.7}$$

$$|\nabla u(z)| \le \frac{C_1}{(1-\alpha)}(1-|z|)^{\alpha-1}||f||_{C^\alpha}, \tag{3.8}$$

and

$$\sup_{z_1 \neq z_2} \frac{|u(z_1) - u(z_2)|}{|z_1 - z_2|^\alpha} \le \frac{C_1}{\alpha} \sup_{|z|<1} \left\{(1-|z|)^{1-\alpha}|\nabla u(z)|\right\}. \tag{3.9}$$

For the proof, see the textbook.

The equivalence (a) $\Longleftrightarrow$ (b) was first proved by Privalov [1916] who worked directly with the imaginary part of the integral (3.3); (a) $\Longleftrightarrow$ (c) was proved by Hardy and Littlewood [1931].

Corollary 3.3: *Assume k is a non-negative integer and assume $0 < \alpha < 1$. Let $f \in C(\partial\mathbb{D})$ be real and let $F(z) = u(z) + i\tilde{u}(z)$ be the Herglotz integral of f. Then $F \in C^{k+\alpha}(\overline{\mathbb{D}})$ if and only if $f \in C^{k+\alpha}(\partial\mathbb{D})$.*

Theorem 3.2 fails when $\alpha = 1$. The harmonic conjugate of a continuously differentiable function on $\partial\mathbb{D}$ need not have a continuous derivative, and the conjugate of a **Lipschitz function**, that is, a function satisfying

$$\big|f(\theta + t) - f(\theta)\big| \leq M|t|,$$

need not be a Lipschitz function.

For the same reason, Corollary 3.3 is also false when $\alpha = 0$ and k is a non-negative integer.

However, conjugation does preserve the Zygmund class Z^* of continuous f on $\partial\mathbb{D}$ such that

$$\sup_{t>0} \frac{\|f(\theta+t)+f(\theta-t)-2f(\theta)\|_\infty}{t} < \infty.$$

The Zygmund class has norm

$$\|f\|_{Z^*} = \|f\|_\infty + \sup_{t>0} \frac{\|f(\theta+t)+f(\theta-t)-2f(\theta)\|_\infty}{t}.$$

When $f \in Z^*$, we say f is a Zygmund function.

Define

$$\big|\nabla_2 u(z)\big| = \big|\nabla u_x\big| = \Big(|u_{xx}(z)|^2 + |u_{yx}(z)|^2\Big)^{\frac{1}{2}}$$

$$= \Big|F''(z)\Big| = \Big|\frac{2}{\pi}\int_{-\pi}^{\pi} \frac{e^{it} f(t)}{(e^{it} - z)^3} dt\Big|,$$

where $F = u + i\tilde{u}$ is the Herglotz integral of $f \in L^1(\partial\mathbb{D})$.

If $f \in C^\alpha$, then (3.8) and (3.6), applied to u_x in the disc $B(z, \frac{1-|z|}{2})$, give us

$$\big|\nabla_2 u(z)\big| = \mathrm{O}\big((1-|z|)^{\alpha-2}\big). \tag{3.10}$$

Conversely, if (3.10) holds, then integrating F'' along radii shows that $\big|\nabla u(z)\big| = \mathrm{O}\big((1-|z|)^{\alpha-1}\big)$, and $f \in C^\alpha$. Thus (3.10) provides yet another characterization of C^α functions, in terms of second derivatives.

Theorem 3.4: *Let $f \in L^\infty(\partial\mathbb{D})$ be real and let $u(z)$ be the Poisson integral of f. Then the following are equivalent:*

(a) $f \in Z^*$.

(b) $\tilde{f} \in Z^*$.

(c) $\big|\nabla_2 u(z)\big| = \mathrm{O}\big((1-|z|)^{-1}\big)$.

There is a constant C such that

$$\big\|\tilde{f}\big\|_{Z^*} \le C\big\|f\big\|_{Z^*} \tag{3.11}$$

and

$$\frac{1}{C}\big\|f\big\|_{Z^*} \le ||f||_\infty + \sup_{z\in\mathbb{D}}\big\{(1-|z|)\big|\nabla_2 u(z)\big|\big\} \le C\big\|f\big\|_{Z^*}. \tag{3.12}$$

Foe the proof, see the textbook.

Notice that by (3.12) and (3.10), $Z^* \subset C^\alpha$ for all $\alpha < 1$.

In particular, if $f \in Z^*$, then f and $\tilde{f}$ are continuous. On the other hand, if f is Lipschitz, then clearly $f \in Z^*$.

Thus we have the increasing scale of spaces

$$C^1 \subset \text{Lipschitz} \subset Z^* \subset C^\alpha \subset C,$$

$\alpha < 1$, where C^1 denotes the space of continuously differentiable functions and C is the space of continuous functions.

Section II.4: Boundary Smoothness

Theorem 4.1: *Let Ω_1 and Ω_2 be Jordan domains such that $\Omega_1 \subset \Omega_2$ and let $\gamma \subset \partial\Omega_1 \cap \partial\Omega_2$ be an open subarc. Let φ_j be a conformal map of $\mathbb{D}$ onto Ω_j. Then $\psi = \varphi_2^{-1} \circ \varphi_1$ has an analytic continuation across $\varphi_1^{-1}(\gamma)$, and $\psi' \neq 0$ on $\varphi_1^{-1}(\gamma)$.*

Proof: The analytic function $\psi = \varphi_2^{-1} \circ \varphi_1$ from $\mathbb{D}$ into $\mathbb{D}$ has a continuous and unimodular extension to the arc $\varphi_1^{-1}(\gamma)$. By Schwarz reflection ψ has an analytic and one-to-one extension to a neighborhood of $\varphi_1^{-1}(\gamma)$ in $\mathbb{C}$ and hence $\varphi' \neq 0$ on ${\varphi_1}^{-1}(\gamma)$. $\square$

Let Γ be an arc parameterized as $\{\zeta(t) : a < t < b\}$. We say Γ has a **tangent** at $\zeta_0 = \zeta(t_0)$ if

$$\lim_{t \downarrow t_0} \frac{\zeta(t) - \zeta_0}{|\zeta(t) - \zeta_0|} = e^{i\tau} \tag{4.1}$$

and

$$\lim_{t \uparrow t_0} \frac{\zeta(t) - \zeta_0}{|\zeta(t) - \zeta_0|} = -e^{i\tau}. \tag{4.2}$$

If (4.1) and (4.2) hold then Γ has unit tangent vector $e^{i\tau}$ at ζ_0. Except for reversals of orientation, the existence of a tangent at ζ_0 and its value $e^{i\tau}$ do not depend on the choice of the parameterization $t \to \zeta(t)$.

We say Γ has a **continuous tangent** if Γ has a tangent at each $\zeta \in \Gamma$ and if $e^{i\tau(\zeta)}$ is continuous on Γ.

Theorem 4.2: *The curve* Γ *has a tangent at* $\zeta = \varphi(e^{i\theta})$ *if and only if the limit*

$$\lim_{\mathbb{D}\ni z\to e^{i\theta}} \arg\left(\frac{\varphi(z)-\zeta}{z-e^{i\theta}}\right) \tag{4.3}$$

exists and is finite. In that case,

$$\lim_{\mathbb{D}\ni z\to e^{i\theta}} \arg\left(\frac{\varphi(z)-\zeta}{z-e^{i\theta}}\right) = \tau(\zeta) - \theta - \frac{\pi}{2} \qquad (\text{modulo } 2\pi). \tag{4.4}$$

The curve Γ *has a continuous tangent if and only if* $\arg\varphi'(z)$ *has a continuous extension to* $\overline{\mathbb{D}}$. *If* Γ *has a continuous tangent, then* $\varphi \in C^{\alpha}(\partial\mathbb{D})$ *for all* $\alpha < 1$ *and* Γ *is rectifiable.*

Let k be a non-negative integer and let $0 \leq \alpha < 1$. We say the curve Γ is of class $C^{k+\alpha}$ Γ is rectifiable and if, in the arc length parameterization

$$\Gamma = \{\gamma(s) : 0 \leq s \leq \ell(\Gamma) = \text{ length } (\Gamma)\},$$

Theorem 4.3 (Kellogg): *Let $k \geq 1$ and $0 < \alpha < 1$. Then the following conditions are equivalent:*
(a) Γ *is of class* $C^{k+\alpha}$.
(b) $\arg \varphi' \in C^{k-1+\alpha}(\partial\mathbb{D})$.
(c) $\varphi \in C^{k+\alpha}(\overline{\mathbb{D}})$ *and* $\varphi' \neq 0$ *on* $\overline{\mathbb{D}}$.

lIf $\alpha = 0$ and $k \geq 1$ then (a) does not imply (c) but (a) is still equivalent to (b).

Let l be a non-negative integer and let $0 \leq \beta < 1$. If Γ is of class $C^{k+\alpha}$ and if $f \in C(\Gamma)$ we say $f \in C^{l+\beta}(\Gamma)$ if $f(\gamma(s)) \in C^{l+\beta}$, when viewed as a function of arc length on Γ.

For proofs of the next three corollaries, see the textbook.

Corollary 4.5: *Suppose Γ is of class $C^{k+\alpha}$, where $k + \alpha > 1$, and suppose $f \in C^{l+\beta}(\Gamma)$. Set $n + \sigma = \min(k + \alpha,\ l + \beta)$, where $0 < \sigma < 1$ and n is a non-negative integer. Let φ be a conformal map of $\mathbb{D}$ onto Ω, let G be the Herglotz integral of $f \circ \varphi$, and let $F = G \circ \varphi^{-1}$. Then $F \in C^{n+\sigma}(\overline{\Omega})$.*

Corollary 4.6: *Let $\partial\Omega$ be a finite union of pairwise disjoint $C^{k+\alpha}$ Jordan curves, where $k+\alpha > 1$, and let $f \in C(\partial\Omega)$ be a $C^{l+\beta}$ function of arc length on each component of $\partial\Omega$. Set $n+\sigma = \min(k+\alpha,\ l+\beta)$, where $0 < \sigma < 1$ and n is a non-negative integer. Then $u(z) = u_f(z)$ and its first n partial derivatives extend continuously to $\overline{\Omega}$*

and

$$|D^n u(z_1) - D^n u(z_2)| \leq K|z_1 - z_2|^\sigma,$$

for all $z_1,\ z_2 \in \overline{\Omega}$, where D^n denotes any n-th partial derivative.

Corollary 4.6 answers Question 1 from the end of Section 2. The next corollary answers Question 2.

Corollary 4.7: *If $\partial\Omega$ consists of finitely many pairwise disjoint Jordan curves of class $C^{1+\alpha}$, where $\alpha > 0$, then*

$$d\omega(z,\zeta) = -\frac{\partial g(z,\zeta)}{\partial \mathrm{n}_\zeta}\frac{ds(\zeta)}{2\pi}. \tag{4.8}$$

In other words, harmonic measure for $z \in \Omega$ is absolutely continuous to arc length on $\partial\Omega$, and the density

$$\frac{d\omega}{ds} = -\frac{1}{2\pi}\frac{\partial g(\zeta,z)}{\partial \mathrm{n}_\zeta}$$

is of class $C^\alpha(\partial\Omega)$ and satisfies

$$c_1 < \frac{d\omega}{ds} < c_2 \tag{4.9}$$

for positive constants c_1 and c_2.

When $\partial\Omega$ is of class C^1, harmonic measure is absolutely continuous to arc length, but the density may not be continuous.

Exercise II.20: (Green's Theorem on Lipschitz Graphs):

A K-**Lipschitz graph** is a set of the form

$$\Gamma = \{(x, \beta(x)) : |x| \leq M\},$$

where β is a real valued Lipschitz continuous function:

$$|\beta(x_2) - \beta(x_1)| \leq K|x_2 - x_1|,$$

for some constant K.

A Lipschitz graph is rectifiable, and since β is absolutely continuous, arc length on Γ has the form

$$ds = \sqrt{1 + (\beta'(x))^2}dx,$$

and almost everywhere,

$$\mathrm{n} = \frac{(\beta'(x), -1)}{\sqrt{1 + (\beta'(x))^2}}$$

is a vector normal to Γ.

For $B > \sup_{[-M,M]} |\beta(x)|$, set

$$\Omega = \Omega_\beta = \{(x, y) : |x| < M,\ \beta(x) < y < B\}.$$

Part (a) : Prove that Green's theorem holds on Ω in this form: Suppose $u \in C^1(\Omega)$ and $v \in C^2(\Omega)$, suppose $u, \nabla u, v, \nabla v$, and Δv are bounded on Ω, and suppose u and ∇v extend continuously to $\overline{\Omega}$. Then

$$\int\int_\Omega \big(u\Delta v + \nabla u \cdot \nabla v\big) dxdy = \int_{\partial\Omega} u \frac{\partial v}{\partial \mathrm{n}} ds. \tag{E.1}$$

Hint: Approximate Ω by C^∞ subdomains as follows: Let $\chi \in C^\infty(-1,1)$ satisfy $\chi \geq 0$ and

$$\int_{-1}^{1} \chi dx = 1.$$

If $\chi_\delta(x) = \frac{1}{\delta}\chi(\frac{x}{\delta})$ and $\beta^{(\delta)}(x) = \chi_\delta * \beta(x) \in C^\infty$, $\beta^{(\delta)}(x) \to \beta(x)$ uniformly, and $\nabla\beta^{(\delta)}(x) \to \nabla\beta(x)$ almost everywhere. Show there are $\delta_n \downarrow 0$ such that $\beta_n = \delta_n + \beta^{(\delta_n)}$ satisfies $\beta_n > \beta_{n+1} > \beta$. Then derive (E.1) from Green's theorem for Ω_{β_n}.

Part (b): A finitely connected Jordan domain Ω is a **Lipschitz domain** if each $p \in \partial\Omega$ has a neighborhood V so that $V \cap \partial\Omega$ is the image of a Lipschitz graph under a linear mapping.

Prove Green's theorem (E.1) holds for the Lipschitz domain Ω under the same smoothness assumptions on u and v that we made in (a). That is, $u \in C^1(\Omega)$;and $v \in C^2(\Omega)$; $u, \nabla u, v, \nabla v$, and Δv are bounded on Ω; and u and ∇v extend continuously to $\overline{\Omega}$.