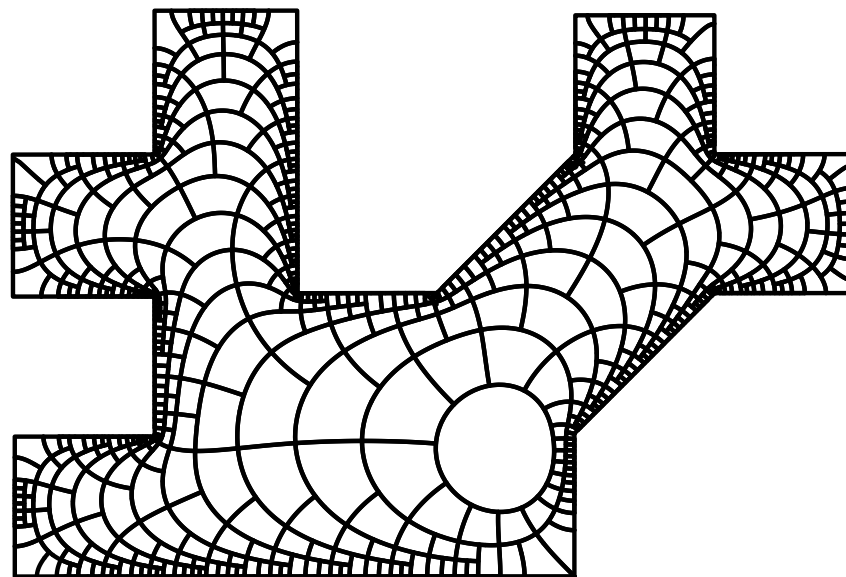
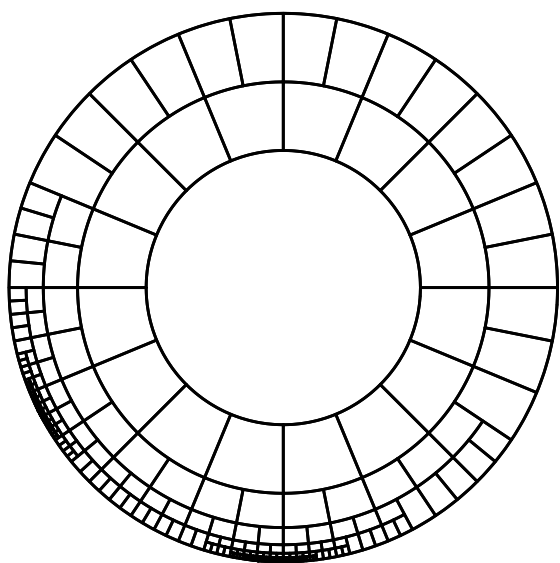


MAT 639, Spring 2026, Stony Brook University

Topics in Real Analysis: Harmonic Measure

Christopher Bishop



Harmonic Measure, Chapter 1

following text by John Garnett and Don Marshall



John Garnett (1940-)



Donald Marshall

Goals for Chapter 1:

- Construct harmonic measure and solve the Dirichlet problem in the upper half-plane and the unit disc.
- Prove the Fatou theorem on nontangential limits.
- Construct harmonic measure on domains bounded by Jordan curves, via the Riemann mapping theorem and the Carathéodory theorem.
- Review the hyperbolic metric and the elementary distortion theory for univalent functions.
- Hayman and Wu theorem on lengths of level sets.

Section I.1: Jordan Domains

Harmonic measure on the upper half-plane

Write $\mathbb{H} = \{z : \text{Im}z > 0\}$ for the upper half-plane and \mathbb{R} for the real line.

Suppose $a < b$ are real. Then the function

$$\theta = \theta(z) = \arg \left(\frac{z - b}{z - a} \right) = \text{Im} \log \left(\frac{z - b}{z - a} \right)$$

is harmonic on \mathbb{H} , and $\theta = \pi$ on (a, b) and $\theta = 0$ on $\mathbb{R} \setminus [a, b]$.

Harmonic measure on the upper half-plane

Let $E \subset \mathbb{R}$ be a finite union of open intervals and write $E = \bigcup_{j=1}^n (a_j, b_j)$ with $b_{j-1} < a_j < b_j$. Set

$$\theta_j = \theta_j(z) = \arg \left(\frac{z - b_j}{z - a_j} \right)$$

and define the **harmonic measure** of E at $z \in \mathbb{H}$ to be

$$\omega(z, E, \mathbb{H}) = \sum_{j=1}^n \frac{\theta_j}{\pi}. \tag{1.1}$$

Properties:

- (i) $0 < \omega(z, E, \mathbb{H}) < 1$ for $z \in \mathbb{H}$,
- (ii) $\omega(z, E, \mathbb{H}) \rightarrow 1$ as $z \rightarrow E$, and
- (iii) $\omega(z, E, \mathbb{H}) \rightarrow 0$ as $z \rightarrow \mathbb{R} \setminus \overline{E}$.

The function $\omega(z, E, \mathbb{H})$ is the unique harmonic function on \mathbb{H} that satisfies (i), (ii), and (iii). The uniqueness follows from Lindelöf's maximum principle.

Lemma 1.1, Lindelöf Maximum Principle: *Suppose the function $u(z)$ is harmonic and bounded above on a region Ω such that $\bar{\Omega} \neq \mathbb{C}$. Let F be a finite subset of $\partial\Omega$ and suppose*

$$\limsup_{z \rightarrow \zeta} u(z) \leq 0 \tag{1.2}$$

for all $\zeta \in \partial\Omega \setminus F$. Then $u(z) \leq 0$ on Ω .

Proof. Fix $z_0 \notin \overline{\Omega}$. Then the map $1/(z - z_0)$ transforms Ω into a bounded region, and thus we may assume Ω is bounded.

If (1.2) holds for all $\zeta \in \partial\Omega$, then the lemma is the ordinary maximum principle.

Write $F = \{\zeta_1, \dots, \zeta_N\}$, let $\epsilon > 0$, and set

$$u_\epsilon(z) = u(z) - \epsilon \sum_{j=1}^N \log \left(\frac{\text{diam}(\Omega)}{|z - \zeta_j|} \right).$$

Then u_ϵ is harmonic on Ω and $\limsup_{z \rightarrow \zeta} u_\epsilon(z) \leq 0$ for all $\zeta \in \partial\Omega$.

Therefore $u_\epsilon \leq 0$ for all ϵ , and

$$u(z) \leq \lim_{\epsilon \rightarrow 0} \epsilon \sum_{j=1}^N \log \left(\frac{\text{diam}(\Omega)}{|z - \zeta_j|} \right) = 0. \quad \square$$



Ernst Leonard Lindelöf (1870-1946)

The Dirichlet Problem:

Given a domain Ω and a function $f \in C(\partial\Omega)$, the **Dirichlet problem** for f on Ω is to find a function $u \in C(\bar{\Omega})$ such that $\Delta u = 0$ on Ω and $u|_{\partial\Omega} = f$.

The Dirichlet problem is not always solvable (hint: punctured disk).

Theorem 1.2: *If $f \in C(\mathbb{R} \cup \{\infty\})$, then there exists a unique function $u = u_f \in C(\overline{\mathbb{H}} \cup \{\infty\})$ so that u is harmonic on \mathbb{H} and $u|_{\partial\mathbb{H}} = f$.*

Proof: We can assume f is real valued and $f(\infty) = 0$.

For $\epsilon > 0$, take disjoint open intervals $I_j = (t_j, t_{j+1})$ and real constants c_j , $j = 1, \dots, n$, so that the step function

$$f_\epsilon(t) = \sum_{j=1}^n c_j \chi_{I_j}$$

satisfies

$$\|f_\epsilon - f\|_{L^\infty(\mathbb{R})} < \epsilon. \tag{1.3}$$

Set

$$u_\epsilon(z) = \sum_{j=1}^n c_j \omega(z, I_j, \mathbb{H}).$$

If $t \in \mathbb{R} \setminus \bigcup \partial I_j$, then

$$\lim_{\mathbb{H} \ni z \rightarrow t} u_\epsilon(z) = f_\epsilon(t)$$

by (ii) and (iii).

Therefore by (1.3) and Lemma 1.1,

$$\sup_{\mathbb{H}} |u_{\epsilon_1}(z) - u_{\epsilon_2}(z)| < \epsilon_1 + \epsilon_2.$$

Consequently the limit $u(z) \equiv \lim_{\epsilon \rightarrow 0} u_\epsilon(z)$ exists, and the limit $u(z)$ is harmonic on \mathbb{H} and satisfies

$$\sup_{\mathbb{H}} |u(z) - u_\epsilon(z)| \leq 2\epsilon.$$

We claim that for all $t \in \mathbb{R}$,

$$\limsup_{z \rightarrow t} |u_\epsilon(z) - f(t)| \leq \epsilon. \tag{1.4}$$

It is clear that (1.4) holds when $t \notin \bigcup \partial I_j$. To verify (1.4) at the endpoint $t_{j+1} \in \partial I_j \cap \partial I_{j+1}$, notice that by (ii), (iii), and Lemma 1.1,

$$\begin{aligned} \sup_{\mathbb{H}} \left| c_j \omega(z, I_j, \mathbb{H}) + c_{j+1} \omega(z, I_{j+1}, \mathbb{H}) - \left(\frac{c_j + c_{j+1}}{2} \right) \omega(z, I_j \cup I_{j+1}, \mathbb{H}) \right| \\ \leq \left| \frac{c_j - c_{j+1}}{2} \right|, \end{aligned}$$

while

$$\lim_{z \rightarrow t_{j+1}} \left(\frac{c_j + c_{j+1}}{2} \right) \omega(z, I_j \cup I_{j+1}, \mathbb{H}) = \frac{c_j + c_{j+1}}{2}.$$

Hence all limit values of $u_\epsilon(z)$ at t_{j+1} lie in the closed interval with endpoints c_j and c_{j+1} , and then (1.3) yields (1.4) for the endpoint t_{j+1} .

Now let $t \in \mathbb{R}$. By (1.4)

$$\limsup_{z \rightarrow t} |u(z) - f(t)| \leq \sup_{z \in \mathbb{H}} |u(z) - u_\epsilon(z)| + \limsup_{z \rightarrow t} |u_\epsilon(z) - f(t)| \leq 3\epsilon.$$

The same estimate holds if $t = \infty$. Therefore u extends to be continuous on $\overline{\mathbb{H}}$ and $u|_{\partial\mathbb{H}} = f$. Uniqueness follows from the maximum principle. \square

The Poisson Kernel:

For $a < b$, elementary calculus gives

$$\begin{aligned}\omega(x + iy, (a, b), \mathbb{H}) &= \frac{1}{\pi} \left(\tan^{-1} \left(\frac{x - a}{y} \right) - \tan^{-1} \left(\frac{x - b}{y} \right) \right) \\ &= \int_a^b \frac{y}{(t - x)^2 + y^2} \frac{dt}{\pi}\end{aligned}$$

If $E \subset \mathbb{R}$ is measurable, the **harmonic measure** of E at $z \in \mathbb{H}$ is

$$\omega(z, E, \mathbb{H}) = \int_E \frac{y}{(t - x)^2 + y^2} \frac{dt}{\pi}. \quad (1.5)$$

When E is finitely many arc, this is the same as Definition (1.1).

For $z = x + iy \in \mathbb{H}$, the density

$$P_z(t) = \frac{1}{\pi} \frac{y}{(x-t)^2 + y^2}$$

is called the **Poisson kernel** for \mathbb{H} .

If $f \in C(\mathbb{R} \cup \{\infty\})$, Theorem 1.2 shows that

$$u_f(z) = \int_{\mathbb{R}} f(t) P_z(t) dt,$$

and so u_f called the **Poisson integral** of f .

Harnack's inequality:

Note that $\omega(z, E, \Omega)$ is a harmonic function in its first variable z and a probability measure in its second variable E .

If $z_1, z_2 \in \mathbb{H}$ then

$$0 < C^{-1} \leq \frac{\omega(z_1, E, \mathbb{H})}{\omega(z_2, E, \mathbb{H})} \leq C < \infty,$$

where C depends on z_1 and z_2 but not on E .

This is **Harnack's inequality**. It is proven by comparing the kernels in (1.5).



Karl Gustav Axel von Harnack (1851-1888)

Harmonic measure on the disk:

Now let \mathbb{D} be the unit disc $\{z : |z| < 1\}$ and let E be a finite union of open arcs on $\partial\mathbb{D}$. Then we define the **harmonic measure** of E at z in \mathbb{D} to be

$$\omega(z, E, \mathbb{D}) \equiv \omega(\varphi(z), \varphi(E), \mathbb{H}), \quad (1.6)$$

where φ is any conformal map of \mathbb{D} onto \mathbb{H} with $\varphi(z) = 0$.

This harmonic function satisfies conditions analogous to (i), (ii), and (iii), so that by Lemma 1.1 the definition (1.6) does not depend on the choice of φ .

It follows by the change of variables $\varphi(z) = i(1+z)/(1-z)$ that

$$\omega(z, E, \mathbb{D}) = \int_E \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \frac{d\theta}{2\pi}.$$

Harmonic measure of arcs on \mathbb{T} can also be defined using subtended angles.

Harmonic measure of arcs on \mathbb{T} can also be defined using subtended angles.

Dirichlet Problem on the disk:

Theorem 1.3 *Let $f(e^{i\theta})$ be an integrable function on $\partial\mathbb{D}$ and set*

$$u(z) = u_f(z) = \int_0^{2\pi} f(e^{i\theta}) \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \frac{d\theta}{2\pi}. \quad (1.7)$$

Then $u(z)$ is harmonic on \mathbb{D} . If f is continuous at $e^{i\theta_0} \in \partial\mathbb{D}$, then

$$\lim_{\mathbb{D} \ni z \rightarrow e^{i\theta_0}} u(z) = f(e^{i\theta_0}). \quad (1.8)$$

Dirichlet Problem on the disk:

(1.8) also holds if the f is changed on a measure zero subset of $\partial\mathbb{D} \setminus \{e^{i\theta_0}\}$.

The function $u = u_f$ is called the **Poisson integral** of f and the kernel

$$P_z(\theta) = \frac{1}{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2}$$

is the **Poisson kernel** for the disc. If $f \in C(\partial\mathbb{D})$ then

$$U(z) = \begin{cases} u_f(z), & z \in \mathbb{D} \\ f(z), & z \in \partial\mathbb{D} \end{cases}$$

is the solution of the **Dirichlet problem** for f on \mathbb{D} .

In the special case when $f(e^{i\theta})$ is continuous, Theorem 1.3 follows from Theorem 1.2 and a change of variables. Conversely, Theorem 1.3 shows that Theorem 1.2 can be extended to $f \in L^1(dt/(1+t^2))$, again by changing variables.

Proof of Theorem 1.3 (Poisson kernel solves Dirichlet Prob):

We may suppose f is real valued. From the identity

$$\operatorname{Re}\left(\frac{e^{i\theta} + z}{e^{i\theta} - z}\right) = 2\pi P_z(\theta),$$

we see that u is the real part of the analytic function

$$\int_0^{2\pi} f(e^{i\theta}) \frac{e^{i\theta} + z}{e^{i\theta} - z} \frac{d\theta}{2\pi},$$

and therefore that u is a harmonic function.

Suppose f is continuous at $e^{i\theta_0}$ and let $\epsilon > 0$. Then

$$|f(e^{i\theta}) - f(e^{i\theta_0})| < \epsilon$$

on an interval $I = (\theta_1, \theta_2)$ containing θ_0 . Setting

$$u_\epsilon(z) = \int_{[0, 2\pi] \setminus I} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} f(e^{i\theta}) \frac{d\theta}{2\pi} + f(e^{i\theta_0}) \omega(z, I, \mathbb{D}),$$

we have

$$|u(z) - u_\epsilon(z)| = \left| \int_I \frac{1 - |z|^2}{|e^{i\theta} - z|^2} (f(e^{i\theta}) - f(e^{i\theta_0})) \frac{d\theta}{2\pi} \right| \leq \epsilon \omega(z, I, \mathbb{D}) \leq \epsilon.$$

However, $\lim_{z \rightarrow e^{i\theta_0}} u_\epsilon(z) = f(e^{i\theta_0})$ by the definition. Therefore

$$\limsup_{z \rightarrow e^{i\theta_0}} |u(z) - f(e^{i\theta_0})| < \epsilon,$$

and (1.8) holds when f is continuous at $e^{i\theta_0}$. \square

Section I.2: Fatou's Theorem and Maximal Functions



Pierre Fatou (1878-1929)

Suppose f is in a normed space $(X, \|\cdot\|)$ on a measure space.

Suppose we are given an associated sequence $\{f_n\}$. We want to prove $f_n \rightarrow t$ almost everywhere for all $f \in X$.

(1) Show this is true for some dense set of g 's.

(2) Define $Mf(x) = \sup_n |f_n(x)|$.

(3) Prove $|\{x : Mf(x) > \lambda\}| \leq C\|f\|/\lambda$.

Choose g close to f so convergence occurs. Then

$$\begin{aligned} |\{x : |f_n(x) - f(x)| > 3\lambda\}| &\leq |\{x : |f_n(x) - g_n(x)| > \lambda\}| \\ &\quad + |\{x : |g_n(x) - g(x)| > \lambda\}| \\ &\quad + |\{x : |g(x) - f(x)| > \lambda\}| \\ &\leq |\{x : M(f - g)(x) > \lambda\}| \\ &\quad + |\{x : |g_n(x) - g(x)| > \lambda\}| \\ &\quad + |\{x : |g(x) - f(x)| > \lambda\}| \end{aligned}$$

We can make all three terms small if $\|f - g\|$ is small and n is large.

We want to apply this idea when f is in L^p on the unit circle, and f_n is harmonic extension restricted to circles of radius < 1 .

A bounded harmonic function on \mathbb{D} can be discontinuous at all boundary points.

For example, take harmonic extension of indicator function of a Cantor set of positive length on the unit circle $\mathbb{T} = \partial\mathbb{D} = \{|z| = 1\}$.

Radial limits and non-tangential limits:

For $\zeta \in \partial\mathbb{D}$ and $\alpha > 1$, we define the **cone**

$$\Gamma_\alpha(\zeta) = \{z : |z - \zeta| < \alpha(1 - |z|)\}.$$

The cone $\Gamma_\alpha(\zeta)$ is asymptotic to a sector with vertex ζ and angle $2 \sec^{-1}(\alpha)$ that is symmetric about the radius $[0, \zeta]$. The cones $\Gamma_\alpha(\zeta)$ expand as α increases.

We say u has a non-tangential limit at $\zeta \in \mathbb{T}$ if it has a limit in every cone at ζ .

If this happens, u automatically has a radial limit, i.e., $\lim_{r \nearrow 1} u(r\zeta)$ exists.

Consider $u(z) = \exp((z + 1)/(z - 1))$.

- u is continuous on boundary except at 1.
- $|u| = 1$ on whole circle except at 1.
- u has nontangential limit 0 at 1.

Non-tangential maximal function:

$$u_{\alpha}^{*}(\zeta) = \sup_{\Gamma_{\alpha}(\zeta)} |u(z)|.$$

If u has a finite nontangential limit at ζ , then $u_{\alpha}^{*}(\zeta) < \infty$ for every $\alpha > 1$.

We are interested in converse: if maximal function is bounded on a subset of \mathbb{T} , then u has a non-tangential limit almost everywhere on this set.

Of course, for general functions, boundedness does not imply a limit exists.

Fatou's Theorem:

Theorem 2.1: *Let $f(e^{i\theta}) \in L^1(\partial\mathbb{D})$ and let $u(z)$ be the Poisson integral of f . Then at almost every $\zeta = e^{i\theta} \in \partial\mathbb{D}$,*

$$\lim_{\Gamma_\alpha(\zeta) \ni z \rightarrow \zeta} u(z) = f(\zeta) \quad (2.2)$$

for all $\alpha > 1$. Moreover, for each $\alpha > 1$

$$\left| \{ \zeta \in \partial\mathbb{D} : u_\alpha^*(\zeta) > \lambda \} \right| \leq \frac{3 + 6\alpha}{\lambda} \|f\|_1. \quad (2.3)$$

Inequality (2.3) says the operator $L^1(\partial\mathbb{D}) \ni f \rightarrow u_\alpha^*$ is **weak-type 1-1**. It follows from (2.2) that $u_\alpha^*(\zeta) < \infty$ almost everywhere, but (2.3) is a sharper, quantitative result.

The proof follows the plan described earlier. We know that continuous functions are dense in L^1 and the boundary limits exist for continuous functions.

Proof: First we assume (2.3) and show (2.3) implies (2.2).

Fix α temporarily. We may assume f is real valued. Set

$$W_f(\zeta) = \limsup_{\Gamma_\alpha \ni z \rightarrow \zeta} |u_f(z) - f(\zeta)|.$$

Then $W_f(\zeta) \leq u_\alpha^*(\zeta) + |f(\zeta)|$. Chebyshev's inequality gives

$$|\{\zeta : |f(\zeta)| > \lambda\}| \leq \frac{\|f\|_1}{\lambda},$$

so that by (2.3),

$$|\{\zeta : W_f(\zeta) > \lambda\}| \leq |\{\zeta : u_\alpha^*(\zeta) > \lambda/2\}| + |\{\zeta : |f(\zeta)| > \lambda/2\}|$$

hence

$$|\{\zeta : W_f(\zeta) > \lambda\}| \leq \frac{8 + 12\alpha}{\lambda} \|f\|_1. \quad (2.4)$$

Fix $\epsilon > 0$ and let $g \in C(\partial\mathbb{D})$ be such that $\|f - g\|_1 \leq \epsilon^2$. Now $W_g(\zeta) = 0$ by Theorem 1.3, and hence

$$W_f(\zeta) = W_{f-g}(\zeta).$$

Applying (2.4) to $f - g$ then gives

$$|\{\zeta : W_f(\zeta) > \epsilon\}| \leq \frac{(8 + 12\alpha)\epsilon^2}{\epsilon} = (8 + 12\alpha)\epsilon.$$

Therefore, for any fixed α , (2.2) holds almost everywhere.

Because the cones Γ_α increase with α , it follows that (2.2) holds for every $\alpha > 1$, except for ζ in a set of measure zero.

To prove (2.3) we will dominate the nontangential maximal function with a second, simpler maximal function.

Let $f \in L^1(\partial\mathbb{D})$ and write

$$Mf(\zeta) = \sup_{I \ni \zeta} \frac{1}{|I|} \int_I |f| d\theta$$

for the maximal average of $|f|$ over subarcs $I \subset \partial\mathbb{D}$ that contain ζ . The function Mf is called the Hardy-Littlewood Maximal function.



G.H. Hardy

Hardy's A Mathematician's Apology



John E. Littlewood

Littlewood's Miscellany

Hardy-Littlewood maximal function dominates the non-tangential maximal function.

Lemma 2.2: *Let $u(z)$ be the Poisson integral of $f \in L^1(\partial\mathbb{D})$ and let $\alpha > 1$.*

Then

$$u_{\alpha}^*(\zeta) \leq (1 + 2\alpha)Mf(\zeta). \quad (2.5)$$

Proof: Assume $\zeta = 1$. Fix z so that $\theta_0 = \arg z$ has $|\theta_0| \leq \pi$. Define

$$\begin{aligned} P_z^*(\theta) &= \sup\{P_z(\varphi) : |\theta| \leq |\varphi| \leq \pi\} \\ &= \begin{cases} \frac{1}{2\pi} \frac{1+|z|}{1-|z|}, & |\theta| \leq |\theta_0| \\ \max(P_z(\theta), P_z(-\theta)), & |\theta_0| < |\theta| \leq \pi. \end{cases} \end{aligned}$$

The function P_z^* satisfies

- (i) $P_z^*(\theta)$ is an even function of $\theta \in [-\pi, \pi]$,
- (ii) $P_z^*(\theta)$ is decreasing on $[0, \pi]$, and
- (iii) $P_z^*(\theta) \geq P_z(\theta)$.

The even function P_z^* is the smallest decreasing majorant of P_z on $[0, \pi]$.

We may assume $f(e^{i\theta}) \geq 0$, so that

$$\int f(e^{i\theta})P_z(\theta)d\theta \leq \int f(e^{i\theta})P_z^*(\theta)d\theta.$$

Then properties (i) and (ii) imply

$$\int f(e^{i\theta})P_z^*(\theta)d\theta \leq \|P_z^*\|_1 Mf(1) \tag{2.6}$$

because P_z^* is the increasing limit of a sequence of functions of the form

$$\sum c_j \left(\frac{1}{2\theta_j} \chi_{(-\theta_j, \theta_j)}(\theta) \right)$$

with $c_j \geq 0$ and $\sum c_j \leq \|P_z^*\|_1$.

Now we claim that when $z \in \Gamma_\alpha(1)$,

$$\|P_z^*\|_1 \leq (1 + 2\alpha). \quad (2.7)$$

Note that (iii), (2.6), and (2.7) imply (2.5). To prove (2.7) we first assume $-\pi/2 \leq \theta_0 = \arg z \leq \pi/2$.

Then by the law of sines,

$$\frac{|\theta_0|}{1 - |z|} \leq \alpha \frac{|\theta_0|}{|1 - z|} \leq \frac{\pi\alpha |\sin \theta_0|}{2 |1 - z|} = \frac{\pi\alpha |\sin \beta|}{2 \cdot 1} \leq \frac{\pi\alpha}{2},$$

where $\beta = \arg(z - 1)/z$ is explained by Figure I.2.

If $\pi/2 \leq |\theta_0| \leq \pi$ and $z \in \Gamma_\alpha(1)$, then $|1 - z| \geq 1$ and

$$\frac{|\theta_0|}{1 - |z|} \leq \alpha \frac{|\theta_0|}{|1 - z|} \leq \pi\alpha.$$

Hence (see Figure I.3)

$$\|P_z^*\|_1 = 2 \int_{|\theta_0|}^{\pi} P_z(\theta) d\theta + \frac{2|\theta_0|}{2\pi} \frac{1 + |z|}{1 - |z|} \leq (1 + 2\alpha).$$

This proves (2.7) and therefore Lemma 2.2. \square

The following two results are proven in Real Analysis I (MAT 532).

Lemma 2.3: *Let μ be a positive Borel measure on $\partial\mathbb{D}$ and let $\{I_j\}$ be a finite sequence of open intervals in $\partial\mathbb{D}$. Then $\{I_j\}$ contains a pairwise disjoint sub-family $\{J_k\}$ such that*

$$\sum \mu(J_k) \geq \frac{1}{3} \mu\left(\bigcup I_j\right). \quad (2.9)$$

Lemma 2.4: *The operator $f \rightarrow M(f)$ is weak-type 1-1: If $f \in L^1$, then*

$$\left| \{\zeta \in \partial\mathbb{D} : Mf(\zeta) > \lambda\} \right| \leq \frac{3\|f\|_1}{\lambda}. \quad (2.8)$$

Together with our earlier estimates, these prove Fatou's Theorem. \square

Corollary 2.6: *If u is a bounded harmonic function on \mathbb{D} , then for every $\alpha > 1$ and for almost every $\zeta = e^{i\theta} \in \partial\mathbb{D}$,*

$$f(\zeta) = \lim_{\Gamma_\alpha(\zeta) \ni z \rightarrow \zeta} u(z)$$

exists, $u(z)$ is the Poisson integral of f , and $\|f\|_\infty = \sup_{z \in \mathbb{D}} |u(z)|$.

Proof: Let $r_n \rightarrow 1$ and let $f_n(e^{i\theta}) = u(r_n e^{i\theta})$. By the Banach–Alaoglu theorem, the sequence $\{f_n\}$ has a weak-star cluster point $f \in L^\infty(\partial\mathbb{D})$ satisfying $\|f\|_\infty \leq \limsup \|f_n\|_\infty \leq \sup_{\mathbb{D}} |u(z)|$.

Since $u(r_n z)$ is the Poisson integral of f_n , and Poisson kernels are in L^1 , u must be the Poisson integral of f . But then $|u(z)| \leq \|f\|_\infty$. The corollary now follows from Fatou’s theorem. \square

In particular, for any measurable $E \subset \partial\mathbb{D}$, there exists a unique bounded harmonic function $u(z)$ on \mathbb{D} such that $u(z)$ has nontangential limit χ_E almost everywhere. This is the harmonic measure of E , $u(z) = \omega(z, E, \mathbb{D})$.

Remarks:

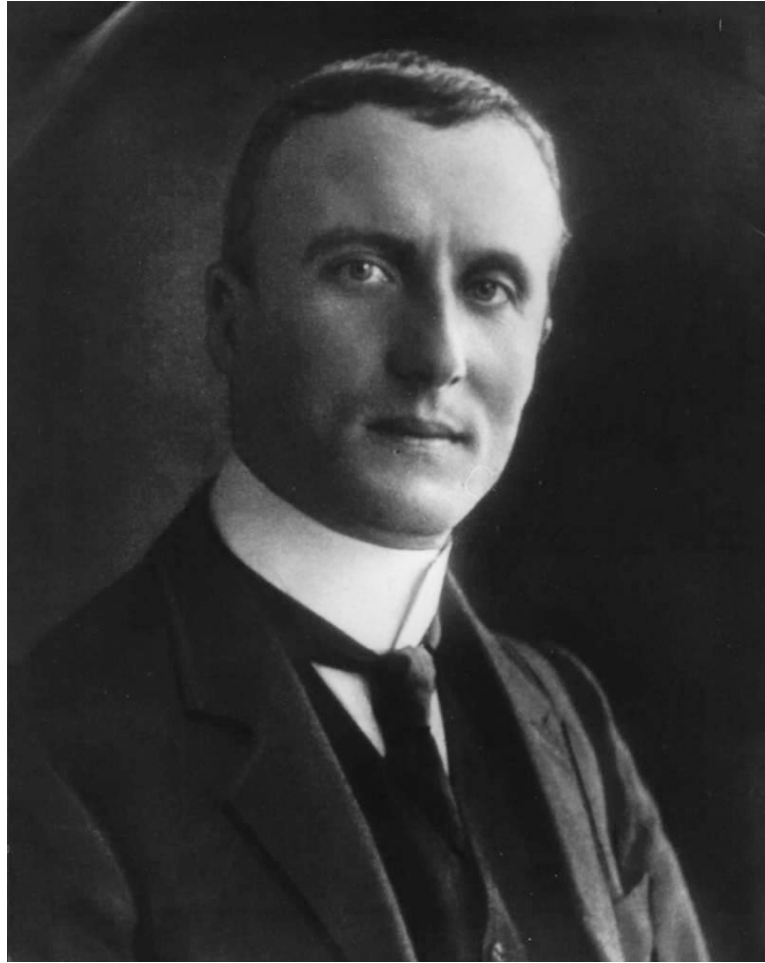
A special case is the conformal map φ from disk onto a bounded domain Ω .

Then φ has radial limits almost everywhere on the circle.

We can use these to map Lebesgue measure on \mathbb{T} to harmonic measure on $\partial\Omega$.

Beurling proved that univalent maps have radial limits except on a set of zero logarithmic capacity. This is much smaller than zero Lebesgue measure. More about this in Chapter III and V.

Section I.3: Carathéodory's Theorem



Constantin Carathéodory (1873-1950)

Theorem I.3.1: *Let φ be a conformal mapping from the unit disc \mathbb{D} onto a Jordan domain Ω . Then φ has a continuous extension to $\overline{\mathbb{D}}$, and the extension is a one-to-one map from $\overline{\mathbb{D}}$ onto $\overline{\Omega}$.*

This is actually due to Carathéodory's student, Marie Torhorst, in her 1918 PhD thesis. See [On Prime Ends and Local Connectivity](#), by Lasse Rempe.



Marie Torhorst (right, 1888-1989) and her sister Adelheid

Proof: We may assume Ω is bounded. Fix $\zeta \in \partial\mathbb{D}$. First we show φ has a continuous extension at ζ . Let $0 < \delta < 1$, write

$$B(\zeta, \delta) = \{z : |z - \zeta| < \delta\}$$

and set $\gamma_\delta = \mathbb{D} \cap \underline{B}(\zeta, \delta)$.

Then $\varphi(\gamma_\delta)$ is a Jordan arc having length

$$L(\delta) = \int_{\gamma_\delta} |\varphi'(z)| ds.$$

By the Cauchy–Schwarz inequality

$$L^2(\delta) \leq \pi\delta \int_{\gamma_\delta} |\varphi'(z)|^2 ds,$$

so that for $\rho < 1$,

$$\int_0^\rho \frac{L^2(\delta)}{\delta} d\delta \leq \pi \int \int_{\mathbb{D} \cap B(\zeta, \rho)} |\varphi'(z)|^2 dx dy$$

so

$$\int_0^\rho \frac{L^2(\delta)}{\delta} d\delta \leq \pi \text{Area}(\varphi(\mathbb{D} \cap B(\zeta, \rho))) < \infty. \quad (3.5)$$

Therefore there is a sequence $\delta_n \downarrow 0$ such that $L(\delta_n) \rightarrow 0$. When $L(\delta_n) < \infty$, the curve $\varphi(\gamma_{\delta_n})$ has endpoints $\alpha_n, \beta_n \in \bar{\Omega}$, and both of these endpoints must lie on $\Gamma = \partial\Omega$.

Indeed, if $\alpha_n \in \Omega$, then some point near α_n has two distinct preimages in \mathbb{D} because φ maps \mathbb{D} onto Ω , and that is impossible because φ is one-to-one.

Furthermore,

$$|\alpha_n - \beta_n| \leq L(\delta_n) \rightarrow 0. \tag{3.6}$$

Let σ_n be that closed subarc of Γ having endpoints α_n and β_n and having smaller diameter. Then (3.6) implies $\text{diam}(\sigma_n) \rightarrow 0$, because Γ is homeomorphic to the circle.

By the Jordan curve theorem the curve $\sigma_n \cup \varphi(\gamma_{\delta_n})$ divides the plane into two regions, and one of these regions, say U_n , is bounded. Then $U_n \subset \Omega$, because $\mathbb{C}^* \setminus \overline{\Omega}$ is arc-wise connected. Since

$$\text{diam}(\partial U_n) = \text{diam}(\sigma_n \cup \varphi(\gamma_{\delta_n})) \rightarrow 0,$$

we conclude that

$$\text{diam}(U_n) \rightarrow 0. \tag{3.7}$$

Set $D_n = \mathbb{D} \cap \{z : |z - \zeta| < \delta_n\}$. We claim that for large n , $\varphi(D_n) = U_n$. If not, then by connectedness $\varphi(\mathbb{D} \setminus \overline{D_n}) = U_n$ and

$$\text{diam}(U_n) \geq \text{diam}(\varphi(B(0, 1/2))) > 0,$$

which contradicts (3.7). i

Therefore $\text{diam}(\varphi(D_n)) \rightarrow 0$ and $\bigcap \overline{\varphi(D_n)}$ consists of a single point, because $\varphi(D_{n+1}) \subset \varphi(D_n)$.

That means φ has a continuous extension to $\{\zeta\} \cup \mathbb{D}$. It is an exercise to show that the union over ζ of these extensions defines a continuous map on $\overline{\mathbb{D}}$.

Let φ also denote the extension $\varphi : \overline{\mathbb{D}} \rightarrow \overline{\Omega}$. Since $\varphi(\mathbb{D}) = \Omega$, φ maps $\overline{\mathbb{D}}$ onto $\overline{\Omega}$. To show φ is one-to-one, suppose $\varphi(\zeta_1) = \varphi(\zeta_2)$ but $\zeta_1 \neq \zeta_2$.

The argument used to show $\alpha_n \in \Gamma$ also shows that $\varphi(\partial\mathbb{D}) = \Gamma$, and so we can assume $\zeta_j \in \partial\mathbb{D}$, $j = 1, 2$. The Jordan curve

$$\{\varphi(r\zeta_1) : 0 \leq r \leq 1\} \cup \{\varphi(r\zeta_2) : 0 \leq r \leq 1\}$$

bounds a domain $W \subset \Omega$, and then $\varphi^{-1}(W)$ is one of the two components of

$$\mathbb{D} \setminus \left(\{r\zeta_1 : 0 \leq r \leq 1\} \cup \{r\zeta_2 : 0 \leq r \leq 1\} \right).$$

But since $\varphi(\partial\mathbb{D}) \subset \Gamma$,

$$\varphi(\partial\mathbb{D} \cap \partial\varphi^{-1}(W)) \subset \partial W \cap \partial\Omega = \{\varphi(\zeta_1)\}$$

and φ is constant on an arc of $\partial\mathbb{D}$. It follows that φ is constant, either by Schwarz reflection principle or by the Jensen formula, and this contradiction shows $\varphi(\zeta_1) \neq \varphi(\zeta_2)$. \square

Dirichlet problem on Jordan domains:

Consider a Jordan domain Ω with boundary $\Gamma = \partial\Omega$. Let $\varphi : \mathbb{D} \rightarrow \Omega$ be conformal. Let f be Borel function on Γ such that $f \circ \varphi$ is integrable on $\partial\mathbb{D}$.

If $w = \varphi^{-1}(z)$, then

$$u(z) \equiv u_f(z) = \int_0^{2\pi} f \circ \varphi(e^{i\theta}) \frac{1 - |w|^2}{|e^{i\theta} - w|^2} \frac{d\theta}{2\pi} \quad (3.1)$$

is harmonic on Ω , and by Theorem 3.1 and Theorem 1.3,

$$\lim_{\Omega \ni z \rightarrow \zeta} u(z) = f(\zeta) \quad (3.2)$$

whenever $\varphi^{-1}(\zeta) \in \partial\mathbb{D}$ is a point of continuity of $f \circ \varphi$.

If f is continuous, then $u_f(z)$ solves the **Dirichlet problem** for f on Ω .

If f is a bounded Borel function on Γ , then $f \circ \varphi$ is Borel and the integral (3.1) is defined.

For any Borel set $E \subset \Gamma$ we use (3.1) with $f = \chi_E$ to define the **harmonic measure** of E relative to Ω by:

$$\omega(z, E, \Omega) = \omega(w, \varphi^{-1}(E), \mathbb{D}) = \int_{\varphi^{-1}(E)} \frac{1 - |w|^2}{|e^{i\theta} - w|^2} \frac{d\theta}{2\pi}. \quad (3.3)$$

Then $E \rightarrow \omega(z, E)$ is a Borel measure on $\partial\Omega$, and (3.1) can be rewritten as

$$u(z) = \int_{\partial\Omega} f(\zeta) d\omega(z, \zeta). \quad (3.4)$$

Equations (3.3) and (3.4) do not depend on the choice of φ , because for every conformal self map T of \mathbb{D} ,

$$\omega(T(w), T(\varphi^{-1}(E)), \mathbb{D}) = \omega(w, \varphi^{-1}(E), \mathbb{D}).$$

When f is a bounded Borel function on $\partial\Omega$, (3.4) and Fatou's theorem give

$$\sup_{z \in \Omega} |u(z)| = \|f\|_{L^\infty(\omega)}.$$

Moreover, Corollary 2.5 shows that every bounded harmonic function on Ω can be expressed in the form (3.4).

Conclusion: for a Jordan domain questions about harmonic measure are equivalent to questions about the boundary behavior of conformal mappings.

Section I.4: Distortion and the Hyperbolic Metric

Let \mathbb{D} be the open unit disc. The **hyperbolic distance** from $z_1 \in \mathbb{D}$ to $z_2 \in \mathbb{D}$ is

$$\rho(z_1, z_2) = \rho_{\mathbb{D}}(z_1, z_2) = \inf \int_{z_1}^{z_2} \frac{|dz|}{1 - |z|^2}, \quad (4.1)$$

where the infimum is taken over all arcs in \mathbb{D} connecting z_1 to z_2 .

Let \mathcal{M} denote the set of conformal self maps of \mathbb{D} :

$$T(z) = \lambda \frac{z - a}{1 - \bar{a}z}, \quad a \in \mathbb{D}, \quad |\lambda| = 1.$$

When $T \in \mathcal{M}$, we have

$$\frac{|T'(z)|}{1 - |T(z)|^2} = \frac{1}{1 - |z|^2},$$

and thus the hyperbolic distance is conformally invariant,

$$\rho(T(z_1), T(z_2)) = \rho(z_1, z_2), \quad T \in \mathcal{M}. \quad (4.2)$$

The **hyperbolic metric** is the infinitesimal form $|dz|/(1 - |z|^2)$ of the hyperbolic distance.

Taking $T(z) = \frac{z-z_1}{1-\bar{z}_1z}$ gives $\rho(z_1, z_2) = \rho(0, T(z_2)) = \int_0^{T(z_2)} \frac{|dz|}{1-|z|^2}$

Therefore the hyperbolically shortest arc from 0 to $T(z_2)$ is the radius $[0, T(z_2)]$, and its hyperbolic length is

$$\rho(0, T(z_2)) = \frac{1}{2} \log \left(\frac{1 + |T(z_2)|}{1 - |T(z_2)|} \right).$$

In general

$$\rho(z_1, z_2) = \frac{1}{2} \log \left(\frac{1 + \left| \frac{z_2 - z_1}{1 - \bar{z}_1 z_2} \right|}{1 - \left| \frac{z_2 - z_1}{1 - \bar{z}_1 z_2} \right|} \right). \quad (4.3)$$

By (4.3) we have

$$\left| \frac{z_2 - z_1}{1 - \bar{z}_1 z_2} \right| = \frac{e^{2\rho(z_1, z_2)} - 1}{e^{2\rho(z_1, z_2)} + 1} = \tanh \rho(z_1, z_2).$$

Write

$$t = t(d) = \tanh(d) = \frac{e^{2d} - 1}{e^{2d} + 1}.$$

Then the hyperbolic ball $B = \{z : \rho(z, a) < d\}$ is the Euclidean disc

$$\left\{ z : \left| \frac{z - a}{1 - \bar{a}z} \right| < t \right\},$$

and a calculation shows that B has Euclidean radius

$$r(a, d) = \frac{t(1 - |a|^2)}{1 - t^2|a|^2} \tag{4.4}$$

and Euclidean distance to $\partial\mathbb{D}$

$$\text{dist}(B, \partial\mathbb{D}) = \left(\frac{1 - t}{1 + |a|t} \right) (1 - |a|). \tag{4.5}$$

$\psi(z)$ is a **univalent function** in \mathbb{D} if ψ is analytic and one-to-one on \mathbb{D} .

After dilating, translating, and rotating the domain $\psi(\mathbb{D})$, ψ is **normalized** by $\psi(0) = 0$ and $\psi'(0) = 1$, so that

$$\psi(z) = z + a_2 z^2 + \cdots . \quad (4.6)$$

Pick's theorem: *If $f : \mathbb{D} \rightarrow \mathbb{D}$ is analytic, then*

$$\left| \frac{f(z) - f(w)}{1 - \overline{f(w)}f(z)} \right| \leq \left| \frac{z - w}{1 - \overline{w}z} \right|.$$

Equivalently,

$$\rho(f(z), f(w)) \leq \rho(z, w).$$

In other words, analytic maps from \mathbb{D} to \mathbb{D} are Lipschitz with respect to the hyperbolic metric.

This follows from the proof of the Schwarz lemma, using $(S \circ f)/T$ for suitable Möbius maps S and T .

It follows that

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}.$$

and equality holds iff f is Möbius.

If f is univalent, then the $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ valued function

$$F(z) = \frac{1}{f(z)} = \frac{1}{z} - \frac{a_2}{2}z + \cdots = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n \quad (4.11)$$

is also univalent in \mathbb{D} .

Lemma 4.2 (Area Theorem): *If the univalent function $F(z)$ satisfies (4.11) then*

$$\sum_{n=1}^{\infty} n|b_n|^2 \leq 1. \quad (4.12)$$

Proof: For $r < 1$, the Jordan curve $\Gamma_r = \{F(re^{i\theta}) : 0 \leq \theta \leq 2\pi\}$ encloses an area $A(r)$, and by Green's theorem,

$$A(r) = \frac{-i}{2} \int_{\Gamma_r} w d\bar{w} = \frac{-i}{2} \int_0^{2\pi} F(re^{i\theta}) \frac{\partial \bar{F}}{\partial \theta}(re^{i\theta}) d\theta.$$

Therefore by (4.11) and Fourier series,

$$A(r) = \pi \left(\frac{1}{r^2} - \sum_{n=1}^{\infty} n |b_n|^2 r^{2n} \right)$$

and

$$1 - \sum_{n=1}^{\infty} n |b_n|^2 = \lim_{r \rightarrow 1} \frac{A(r)}{\pi} \geq 0,$$

which yields (4.12). □

Theorem 4,1 (Koebe one-quarter theorem): Assume $\psi(z)$ is a univalent function on \mathbb{D} . If $\psi(z)$ has the form (4.6) then

$$|a_2| \leq 2 \tag{4.8}$$

and

$$\text{dist}(0, \partial\psi(\mathbb{D})) \geq \frac{1}{4}. \tag{4.9}$$

Proof of Koebe (1/4)-Theorem:

We claim (4.8) implies (4.9). Suppose $w \notin \psi(\mathbb{D})$. Then (4.8) holds for the univalent function

$$g(z) = \frac{w\psi(z)}{w - \psi(z)} = z + \left(a_2 + \frac{1}{w}\right)z^2 + \cdots,$$

so that

$$\left|a_2 + \frac{1}{w}\right| \leq 2, \tag{4.10}$$

and together (4.10) and (4.8) give $|w| \geq 1/4$.

To prove (4.8), consider the odd function

$$f(z) = z\sqrt{\frac{\psi(z^2)}{z^2}} = z + \frac{a_2}{2}z^3 + \dots.$$

Then f is univalent because ψ is univalent, and we apply the Area Theorem to

$$F(z) = \frac{1}{f(z)} = \frac{1}{z} - \frac{a_2}{2}z + \dots = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n. \quad (4.11)$$

Applying (4.12) to $F = 1/f$ gives $|b_1| \leq 1$ and $b_1 = -a_2/2$ gives $|a_2| \leq 2$. \square



Paul Koebe (1882-1945)

Theorem 4.3 (Koebe's estimate): *Let $\varphi(z)$ be a conformal mapping from the unit disc \mathbb{D} onto a simply connected domain Ω . Then for all $z \in \mathbb{D}$*

$$\frac{1}{4}|\varphi'(z)|(1 - |z|^2) \leq \text{dist}(\varphi(z), \partial\Omega) \leq |\varphi'(z)|(1 - |z|^2). \quad (4.13)$$

Proof: First we prove the left-hand inequality.

Fix $z_0 \in \mathbb{D}$. Then the univalent function

$$\psi(z) = \frac{\varphi\left(\frac{z+z_0}{1+\bar{z}_0z}\right) - \varphi(z_0)}{\varphi'(z_0)(1 - |z_0|^2)}$$

satisfies $\psi(0) = 0$ and $\psi'(0) = 1$.

Hence if $w \notin \varphi(\mathbb{D})$, then

$$\left| \frac{w - \varphi(z_0)}{\varphi'(z_0)(1 - |z_0|^2)} \right| \geq \frac{1}{4}$$

by (4.9), and this gives the left-hand inequality in (4.13).

To prove the right-hand inequality, fix $z \in \mathbb{D}$, take

$$f(w) = \varphi^{-1}\left(\varphi(z) + \text{dist}(\varphi(z), \partial\Omega)w\right),$$

and apply the Schwarz lemma at $w = 0$ to the function

$$g(w) = \frac{f(w) - z}{1 - \bar{z}f(w)}. \quad \square$$

Corollary 4.2 (Invariant form of Koebe estimate): *Let ψ be a conformal mapping from a simply connected domain Ω_1 onto a simply connected domain Ω_2 , and let $\psi(z_0) = w_0$. Then*

$$\frac{|\psi'(z_0)|}{4} \leq \frac{\text{dist}(w_0, \partial\Omega_2)}{\text{dist}(z_0, \partial\Omega_1)} \leq 4|\psi'(z_0)| \quad (4.14)$$

Proof: Applying (4.13) to $\varphi(z) = \psi(z_0 + \text{dist}(z_0, \partial\Omega_1)z)$ gives the left-hand inequality and the same argument with ψ^{-1} gives the right-hand inequality. \square

Hyperbolic and Quasihyperbolic distance:

In a simply connected domain $\Omega \neq \mathbb{C}$, the **hyperbolic distance** is defined by

$$\rho_{\Omega}(w_1, w_2) = \rho_{\mathbb{D}}(z_1, z_2)$$

when $w_j = \varphi(z_j)$. By (4.2), this does not depend on choice of φ .

The **quasihyperbolic distance** from $w_1 \in \Omega$ to $w_2 \in \Omega$ is

$$Q_{\Omega}(w_1, w_2) = \inf \int_{w_1}^{w_2} \frac{|dw|}{\text{dist}(w, \partial\Omega)},$$

in which the infimum is taken over all arcs in Ω joining w_1 to w_2 .

Since (4.13) can be written as

$$\frac{|dz|}{1 - |z|^2} \leq \frac{|dw|}{\text{dist}(w, \partial\Omega)} \leq \frac{4|dz|}{1 - |z|^2},$$

where $w = \varphi(z)$, we have

$$\rho_\Omega(w_1, w_2) \leq Q_\Omega(w_1, w_2) \leq 4\rho_\Omega(w_1, w_2). \quad (4.15)$$

Hence, for simply connected domains, the hyperbolic distance is “approximately” the Euclidean distance divided by the distance to the boundary.

This fails for multiply connected domains. In that case, the hyperbolic metric is defined using a covering map (given by the uniformization theorem instead of the Riemann mapping theorem) and there are different estimates relating the hyperbolic and Euclidean metrics.

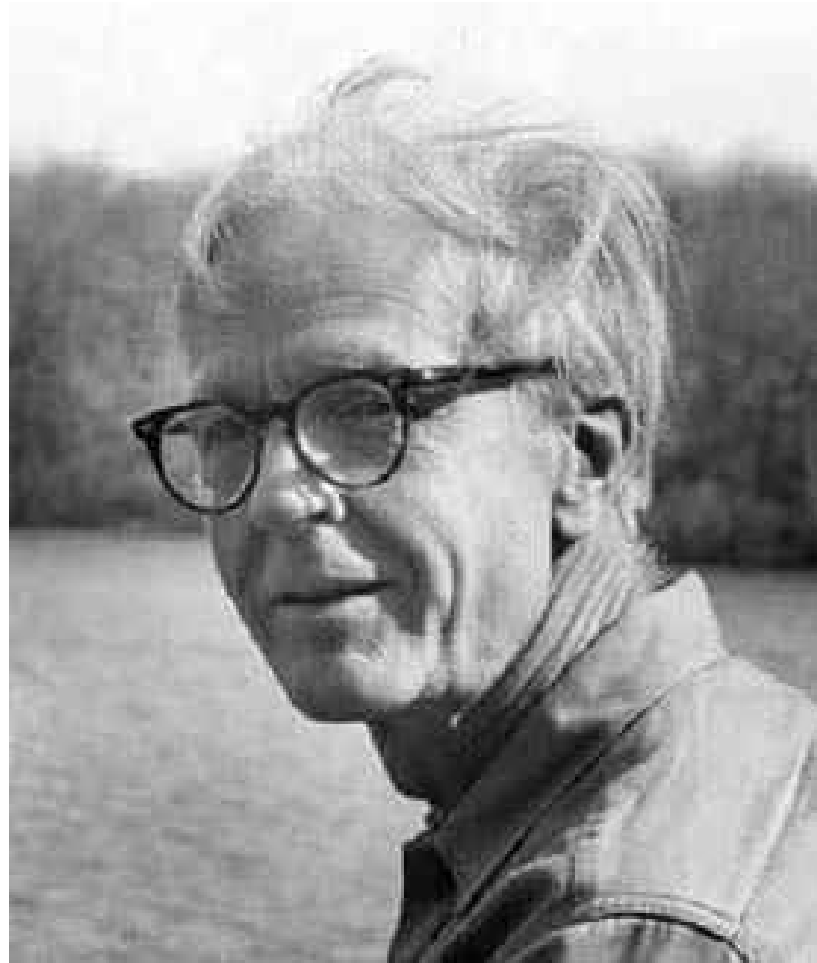
Whitney squares:

Suppose Ω is a proper open subset of \mathbb{C} . Then there exist closed squares $\{S_j\}$, having pairwise disjoint interiors and sides parallel to the axes, such that

- (i) S_j has sidelength $\ell(S_j) = 2^{-n_j}$
- (ii) $\Omega = \bigcup S_j$, and
- (iii) $\text{diam}(S_j) \leq \text{dist}(S_j, \partial\Omega) < 4 \text{diam}(S_j)$.

The squares $\{S_j\}$ are called **Whitney squares**.

For each $x \in \Omega$ let Q_x be the maximal dyadic square containing x so that $\text{diam}(Q_x) \leq \frac{1}{2}\text{dist}(Q_x, \partial\Omega)$. By nested property of dyadic squares, these form a Whitney decomposition.



Hassler Whitney (1907-1989)

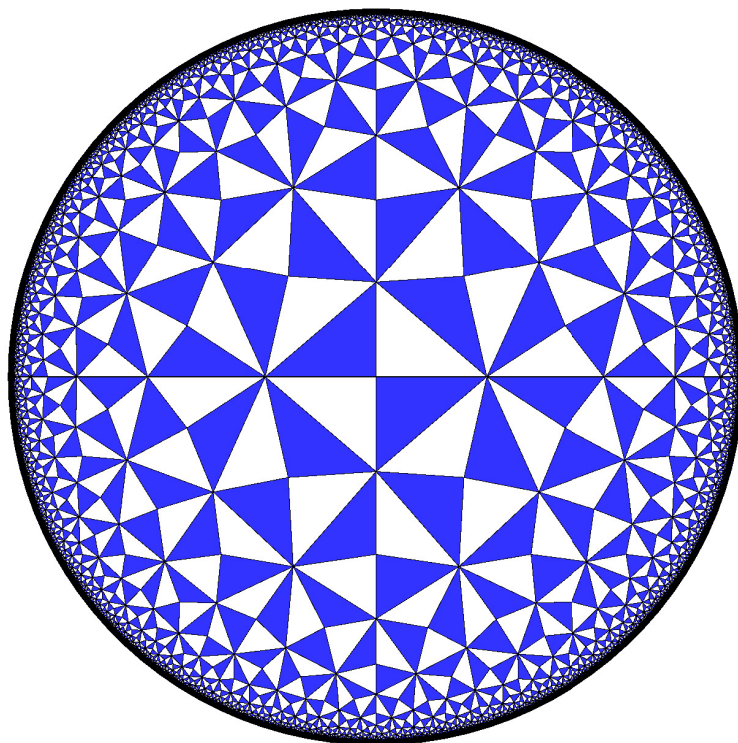
Never a Dull Moment (biography by Keith Kendig)

Whitney squares are approximately unit sized in the hyperbolic and quasi-hyperbolic metric.

Hyperbolic areas and lengths of a set X can often be estimated by counting the number of Whitney squares that X hits.

On the unit disk, we also have tessellations that are invariant under groups of hyperbolic isometries.

These are a fancier version of Whitney squares. Harder to compute, especially in a general (non-disk) domain.



Theorem 4.5 (The Distortion Theorem): *Let $\psi(z)$ be a univalent function satisfying $\psi(0) = 0$ and $\psi'(0) = 1$. Then*

$$\frac{|z|}{(1 + |z|)^2} \leq |\psi(z)| \leq \frac{|z|}{(1 - |z|)^2}, \quad (4.16)$$

and

$$\frac{1 - |z|}{(1 + |z|)^3} \leq |\psi'(z)| \leq \frac{1 + |z|}{(1 - |z|)^3}. \quad (4.17)$$

Moreover,

$$\frac{1 - |z|}{|z|(1 + |z|)} \leq \frac{|\psi'(z)|}{|\psi(z)|} \leq \frac{1 + |z|}{|z|(1 - |z|)}. \quad (4.18)$$

Inequality (4.16) is known as the **growth theorem**, while (4.17) is called the **distortion theorem**.

For the proof, see the textbook.

Section I.5: The Hayman-Wu Theorem



Walter Hayman (1926-2020)



Jang-Mei Wu



The Miracle Group

Left to Right: Sun-Yung Alice Chang, Jang-Mei Wu, Mei Liang,
Wen-Ching Winnie Li, Shou-Jen Hu, Fan Chung.

Theorem 5.1 (The Hayman–Wu Theorem, 1981): *Let φ be a conformal mapping from \mathbb{D} to a simply connected domain Ω and let L be any line. Then*

$$\text{len}(\varphi^{-1}(L \cap \Omega)) \leq 4\pi. \quad (5.1)$$

Hayman and Wu gave the first proof of (5.1) with 4π replaced by some large unknown constant.

In 1992 Øyama obtained the constant 4π . and in 1993 showed best value is $\geq \pi^2$.

In 2002 Rohde proved that the best constant in (5.1) is strictly smaller than 4π .

The sharp constant in (5.1) is not known.

For the proof it will be convenient to replace the hyperbolic metric $\rho(z_1, z_2)$ by the **pseudohyperbolic metric**, defined in \mathbb{D} by

$$\delta_{\mathbb{D}}(z_1, z_2) = \left| \frac{z_1 - z_2}{1 - \bar{z}_1 z_2} \right| = \tanh \rho(z_1, z_2),$$

and in Ω by

$$\delta_{\Omega}(w_1, w_2) = \delta_{\mathbb{D}}(\varphi^{-1}(w_1), \varphi^{-1}(w_2)).$$

We follow Øyama's proof, as modified by Rohde.

Proof: Assume that φ is analytic and one-to-one in a neighborhood of $\overline{\mathbb{D}}$ and that $L = \mathbb{R}$.

Let L_k denote the components of $\Omega \cap L$ and let Ω_k be that component of

$$\Omega \cap \overline{\Omega} = \Omega \cap \{\bar{z} : z \in \Omega\}$$

such that $L_k \subset \Omega_k$. Then Ω_k is a Jordan domain symmetric about \mathbb{R} . When $k \neq j$, $\partial\Omega_k \cap \partial\Omega_j \subset \mathbb{R}$ and $\partial\Omega_k \cap \partial\Omega_j$ contains at most one point, because Ω is simply connected.

By symmetry there is a conformal mapping $\psi_k : \Omega_k \rightarrow -i\mathbb{H}$ (= right half-plane) such that $\psi_k(L_k) = \mathbb{R}^+$ and ψ_k extends continuously to $\overline{\Omega_k}$.

Recall $\varphi : \mathbb{D} \rightarrow \Omega$.

For $\zeta \in \partial\varphi^{-1}(\Omega_k) \cap \partial\mathbb{D}$, set

$$\alpha = \varphi(\zeta), \quad x = |\psi_k(\alpha)|, \quad \beta = \psi_k^{-1}(x), \quad \text{and} \quad z = \varphi^{-1}(\beta).$$

Then the composition

$$\Phi \equiv \varphi^{-1} \circ \psi_k^{-1}(|\psi_k \circ \varphi|)$$

is a smooth map of $\varphi^{-1}(\bigcup \partial\Omega_k \cap \partial\Omega \setminus P) \subset \partial\mathbb{D}$ onto $\varphi^{-1}(\bigcup L_k) \setminus P'$ where P and P' are finite sets.

To prove Theorem 5.1, it suffices to show that

$$|\nabla\Phi| \leq 2. \tag{5.2}$$

To prove (5.2), suppose that $I = (\zeta, \zeta')$ is an open interval in $\varphi^{-1}(\partial\Omega_k) \cap \partial\mathbb{D}$.

Set

$$\alpha' = \varphi(\zeta'), \quad x' = |\psi_k(\alpha')|, \quad \beta' = \psi_k^{-1}(x'), \quad \text{and} \quad z' = \varphi^{-1}(\beta').$$

Then by Pick's theorem (see Exercise 14(a))

$$\delta_{\mathbb{D}}(\Phi(\zeta), \Phi(\zeta')) = \delta_{\Omega}(\beta, \beta') \leq \delta_{\Omega_k}(\beta, \beta') = \delta_{-i\mathbb{H}}(x, x') = \left| \frac{x - x'}{x + x'} \right|$$

We claim that

$$\left| \frac{x - x'}{x + x'} \right| \approx \omega(x, \psi_k(\varphi(I)), -i\mathbb{H}).$$

Let $J = \psi_k(\varphi(I))$. The harmonic measure of J with respect to x is θ/π where θ is the angle subtended by J at x .

Consider the triangle with vertices at x and the endpoints of $J = \{ix, ix'\}$. The angle at ix is $\pi/4$ and by the Law of Sines for $x \approx x'$ we have

$$\frac{\sin \theta}{|x - x'|} = \frac{\sin \pi/4}{|x - ix'|} \approx \frac{1}{2x} \approx \frac{1}{x + x'}$$

$$\frac{|x - x'|}{|x - x'|} \approx \sin \theta \approx \theta = \pi \cdot \omega(x, \psi_k(\varphi(I)), -i\mathbb{H}).$$

By Lindelöf's maximum principle

$$\omega(x, \psi_k(\varphi(I)), -i\mathbb{H}) = \omega(z, I, \varphi^{-1}(\Omega_k)) \leq \omega(\Phi(\zeta), I, \mathbb{D}).$$

Letting $\zeta' \rightarrow \zeta$ we obtain the inequality

$$\begin{aligned}
\frac{|\nabla\Phi|}{1 - |\Phi(\zeta)|^2} &= \lim_{\zeta' \rightarrow \zeta} \frac{|\Phi(\zeta) - \Phi(\zeta')|}{|\zeta - \zeta'| \cdot (1 - |\Phi(\zeta)|^2)} = \lim_{\zeta' \rightarrow \zeta} \frac{|\Phi(\zeta) - \Phi(\zeta')|}{|\zeta - \zeta'| (1 - \overline{\Phi(\zeta)}\Phi(\zeta'))} \\
&= \lim_{\zeta' \rightarrow \zeta} \frac{\delta_{\mathbb{D}}(\Phi(\zeta), \Phi(\zeta'))}{|\zeta - \zeta'|} \leq \pi \cdot \lim_{\zeta' \rightarrow \zeta} \frac{\omega(\Phi(\zeta), I, \mathbb{D})}{|\zeta - \zeta'|} = \pi \cdot \lim_{\zeta' \rightarrow \zeta} \frac{|I|P_{\Phi(\zeta)}(\zeta)}{|\zeta - \zeta'|} \\
&= \pi \cdot \lim_{\zeta' \rightarrow \zeta} \frac{|\zeta - \zeta'| (1/2\pi) (1 - |\Phi(\zeta)|^2) / |\zeta - \Phi(\zeta)|^2}{|\zeta - \zeta'|} \\
&= \frac{1}{2} \cdot \frac{1 - |\Phi(\zeta)|^2}{|\zeta - \Phi(\zeta)|^2}.
\end{aligned}$$

Thus we have shown

$$\frac{|\nabla\Phi|}{1 - |\Phi(\zeta)|^2} \leq \frac{1}{2} \cdot \frac{1 - |\Phi(\zeta)|^2}{|\zeta - \Phi(\zeta)|^2}, \quad (5.3)$$

or

$$|\nabla\Phi| \leq \frac{1}{2} \cdot \frac{(1 - |\Phi(\zeta)|^2)^2}{|\zeta - \Phi(\zeta)|^2} = \frac{1}{2} \cdot \frac{(1 - |\Phi(\zeta)|)^2(1 + |\Phi(\zeta)|)^2}{|\zeta - \Phi(\zeta)|^2},$$

Since $(1 - |\Phi(\zeta)|) \leq |\zeta - \Phi(\zeta)|$ we have

$$|\nabla\Phi| \leq \frac{1}{2} \cdot (1 + |\Phi(\zeta)|)^2 \leq \frac{1}{2} \cdot 2^2 = 2,$$

as desired. \square

Generalized Hayman-Wu Theorem:

A compact set Γ is *Ahlfors regular* if there is a constant M so that for every disk $B(z, r)$ we have

$$\Lambda_1(\Gamma \cap B(z, r)) \leq Mr$$

Theorem X.11.1 (Bishop-Jones,1990): *Let Γ be a compact connected set. Then Γ is Ahlfors regular iff there is a constant $C(\Gamma)$ such that*

$$\Lambda_1(\varphi^{-1}(\Gamma \cap \Omega)) \leq C(\Gamma) \tag{11.1}$$

for every simply connected domain Ω and every conformal mapping φ from \mathbb{D} onto Ω .

Lines are obviously Ahlfors regular, so Hayman-Wu theorem is a special case.

