2-DIMENSIONAL ALMOST AREA MINIMIZING CURRENTS

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1. Introduction

The Plateau's problem investigates those surfaces of least area spanning a given contour. It is one of the most classical problems in the calculus of variations, it lies at the crossroad of several branches of mathematics and it has generated a large amount of mathematical theory in the last one hundred years. Although its original formulation is restricted to 2-dimensional surfaces spanning a given curve γ in the 3-dimensional space, in modern mathematics it is customary to consider a much more general setting, where the ambient space is a Riemannian manifold and the dimension of the surface is arbitrary. Such generalization has not only an intrinsic mathematical beauty, but it has also proved very fruitful. In fact even the condition of being area minimizing might be relaxed in several ways.

The very formulation of the Plateau's problem has proved to be a quite challenging mathematical question. In particular, how general are the surfaces that one should consider? What is the correct concept of "spanning" and the correct concept of "m-dimensional volume" that one should use? The author believes that there are no final answers to these two questions: many different significant ones have been given in the history of our subject and, depending upon the context, the features of one formulation might be considered more important than those of the others.

In this paper we focus on a point of view which is rather popular for its numerous geometric applications. Let Σ be a fixed smooth oriented manifold and Γ a smooth m-dimensional oriented submanifold of Σ. On Γ we can integrate compactly supported forms and this action gives a natural correspondence between smooth submanifolds and linear functionals on the space of smooth (compactly supported) forms $\mathcal{D}^m(\Sigma)$. Following a pioneering idea of De Rham (cf. [28]) we define the m-dimensional *currents* as linear functionals on $\mathcal{D}^m(\Sigma)$ satisfying a suitable continuity property. The action of a current T on a form ω is then given by $T(\omega)$ and we can introduce naturally a concept of boundary "enforcing" Stokes' theorem: $\partial T(\omega) := T(d\omega)$.

Note however that De Rham's theory allows real multiplicities, or more generally oriented surfaces with variable densities, which in several situations is certainly not desirable. We could try to remedy to this shortcoming by considering only the subspace of currents which can be represented as classical chains with integer multiplicities and then look at its closure in the appropriate topology. This point of view was taken in the celebrated paper [30] by Federer and Fleming and the corresponding objects, called *integral currents*, provide an ideal framework for the general *oriented* Plateau's problem. In fact the work of Federer and Fleming is the natural generalization of a previous theory developed by De Giorgi in codimension 1, following a pioneering idea of Caccioppoli, cf. [9, 11, 12, 14].

The fundamental paper of Federer and Fleming addresses some of the basic questions that a satisfactory variational theory is expected to handle. In particular they give a rather general and powerful existence theorem for the Plateau's problem and very flexible approximation results. Since then a lot of efforts have been dedicated to "regularity questions". More precisely, how regular is a minimizer in the Federer-Fleming theory? The problem is especially intriguing in all those cases where it is known that minimizers *are not* necessarily regular: of course in these cases it is highly desirable to have a good description of the singular set.

Thanks to the efforts of several outstanding mathematicians a rather far-reaching (and satisfactory) regularity theory was achieved in the seventies in codimension 1 (see for instance [34]). This theory has been digested by the subsequent generations of scholars working in differential geometry and PDEs, leading ultimately to many breakthroughs in different problems in geometry, PDEs and mathematical physics. Indeed the codimension 1 case is considerably easier than the higher codimension: the reason is that integral m-dimensional cycles in $\mathbb{R}^{m+1}$ are in fact (countable integral combinations of) boundaries of sets.

In the higher codimension case the most important conclusion of the regularity theory can be attributed to the monumental work of a single person, F. J. Almgren Jr., [4], originally a typewritten manuscript of more than 1700 pages. Unlike the codimension one case, only a relatively small portion of the theory of Almgren has been truly understood. In a recent series of papers Emanuele Spadaro and the author have given a new, much shorter, account of Almgren's regularity program, relying on the several advances in geometric measure theory of the last two decades and on some new ideas, cf. [19, 20, 23, 21, 22] and the survey papers [44, 16, 17].

In a nutshell Almgren's main theorem asserts that the set of interior singularities of an m-dimensional Federer-Fleming solution of the Plateau's problem is a closed set of Hausdorff dimension at most $m-2$. By a classical theorem of Federer it is known that if the ambient space has dimension stricty larger than $m+1$, then Almgren's result is optimal: indeed holorphic subvarieties of $\mathbb{C}^n$ are always area minimizing integral currents. Of special interest is then the case of 2-dimensional area minimizing currents: holomorphic curves in $\mathbb{C}^n$ might have isolated singularities, where they "branch". In [10] Chang, a student of Almgren, built upon Almgren's work to show that for general 2-dimensional area minimizing integral currents the set of interior singular points is discrete and moreover in a neighborhood of each singularity the current in question behaves essentially as a branching holomorphic curve.

In [10] however, the author does not give the details of a rather important portion of the proof, the construction of the so-called branched center manifold. Such proof needed a suitable modification of the most complicated part of Almgren's theory (the construction of a non-branched center manifold, which occupies more than half of Almgren's original monograph). In a series of joint works with Emanuele Spadaro and Luca Spolaor we have given a complete proof of the existence of such center manifold. In addition we have extended the regularity result of Chang to two interesting classes of 2-dimensional currents which are not area minimizing, but could be considered as some sort of "perturbed cases": these are semicalibrated currents and spherical cross-sections of 3-dimensional area minimizing cones. In this note we will describe the theorems and the main ideas of the proofs contained in the corresponding papers [27, 26, 24, 25].

2. The main regularity theorem

From now on we assume that the reader is sufficiently familiar with the classical results of the Federer-Fleming theory of integral currents, namely compactness theorems, boundary rectifiability, homotopy formulae, deformation lemma and isoperimetric inequalities. We will also assume familiarity with the preliminary facts in the regularity theory of stationary and area minimizing currents, namely

- The monotonicity formula for stationary currents;
- The precompactness of sequences of area minimizing currents under a uniform control of their mass;
- The existence of tangent cones.

This material is covered in classical textbooks such as [29] and [43]. However for a gentle introduction to the topic the reader may consult the first sections of the survey [16], whereas who is already familiar with some of the relevant concepts may consult the first sections of [17] for the notation and terminology which we will use in this note.

Our results will concern three special classes of 2-dimensional integral currents:

(a) area minimizing in Riemannian manifolds Σ. As it is customary in the regularity theory, we will assume Σ to be at least of class C^2 and to be a submanifold of the Euclidean space. Note that by Nash's isometric embedding theorem, this does not mean any loss of generality.

Thus from now on Σ is a fixed C^2 submanifold of dimension $2+\bar{n}$ of the Euclidean space $\mathbb{R}^{2+n}$. As usual a 2-dimensional (integral) current T in $\mathbb{R}^{2+n}$ whose support is contained in Σ is area minimizing in Σ if $\mathbf{M}(T + \partial S) \geq \mathbf{M}(T)$ for every 3-dimensional (integral) current S wth $\mathrm{spt}(S) \subset \Sigma$.

(b) Semicalibrated currents in a Riemannian manifold Σ. Even in this case we assume that Σ is a C^2 embedded submanifold of $\mathbb{R}^{2+n}$. For the meaning of the terms "semicalibrations" and "semicalibrated" we refer to Section 2.1 below.

(c) Spherical cross sections of area minimizing 3-dimensional cones. In this case the 2-dimensional current T is assumed to be supported in some sphere $\partial \mathbf{B}_R(p) \subset \mathbb{R}^{2+n}$ of radius R and center p: T is then a spherical cross section of an area minimizing cone if $T \mathbin{⨳} p$ is area minimizing in $\mathbb{R}^{2+n}$.

We warn the reader that in case (c) we are *not* assuming that T is a cycle: ∂T is not necessarily 0.

Occasionally we will state some intermediate theorems for the more general case of m-dimensional currents T, since the proof for $m \geq 2$ does not need any substantial change.

In this case we keep the notation $\bar{n}$ and n for the codimension of the current T in the Riemannian manifold Σ and in the ambient euclidean space. Of course the dimensions of the latter objects are then $m+\bar{n}$ and $m+n$.

Note that a spherical cross section T as in (c) is *not always* area minimizing in the sphere that contains it. In fact it is well-known that the round spheres contain no nontrivial area minimizing cycles, whereas on the other hand there is an abundance of cycles T supported in $\partial \mathbf{B}_1(0)$ for which the cone $0 \times\!\!\!\times T$ is area minimizing. Now, if T were a regular submanifold, then it would be at least area minimizing in any sufficiently small ball. This is however a trivial property of *any* smooth minimal surface, even if highly unstable: the price to pay for such generality is that the scale at which such minimizing property "kicks in" is that at which the surfaces is sufficiently close to its tangent space. Thus, although being a spherical cross section of an area minimizing euclidean cone is a rather strong assumption, there is no obvious reduction of the regularity theory for (c) to that for (a), because in our case both classes of currents do allow for singularities.

2.1. **Calibrations and semicalibrations.** We illustrate here the simple, yet elegant and fruitful principle behind calibrations and calibrated geometries. Recall first the notion of comass of a form

Definition 2.1 (Comass, cf. [29, Section 1.8])**.** Let $\omega \in \mathcal{D}^m(\Sigma)$ with Σ Riemannian manifold. Then the comass of ω is the norm

$$\|\omega\|_c := \max\left\{\langle \omega(p), v_1 \wedge \ldots \wedge v_m\rangle : |v_1 \wedge \ldots \wedge v_m| = 1, v_i \in T_p\Sigma, p \in \Sigma\right\} .$$

Calibrations are a particular subclass of closed forms.

Definition 2.2 (Calibrations, cf. [35])**.** A *calibration* ω is a closed m-form on a Riemannian manifold Σ such that $\|\omega\|_c \leq 1$. An integer rectifiable current T is said to be calibrated by a calibration ω if $\langle \omega_p, \vec{T}(p)\rangle = 1$ for $\|T\|$-a.e. p.

A semicalibration is a form ω satisfying the requirements above except for the closedness. The corresponding T such that $\langle \omega_p, \vec{T}(p)\rangle = 1$ for $\|T\|$-a.e. p are then called semicalibrated by ω.

Observe in particular that the inequality $\mathbf{M}(T) \geq T(\omega)$ holds for every semicalibration ω and for every current T. Moreover the equality sign holds if and only if T is semicalibrated by ω. The following is then a trivial fact.

Lemma 2.3. *If T is calibrated by a calibration ω, then T is an area minimizing current.*

Proof. Assume $\dim(T) = m$ and let S be an $(m+1)$-dimensional integral current. Then

$$\mathbf{M}(T) = T(\omega) = T(\omega) + S(d\omega) = (T + \partial S)(\omega) \leq \mathbf{M}(T + \partial S)\,. \tag{1}$$

□

Of course, if ω were just a semicalibration, then (1) could be replaced by

$$\mathbf{M}(T) \leq \mathbf{M}(T + \partial S) + \|d\omega\|_0 \mathbf{M}(S) \qquad \forall S \text{ integral with } \mathrm{spt}(S) \subset \Sigma. \tag{2}$$

Assume now that $\bar{\mathbf{B}}_{r_0}(p) \cap \Sigma$ is diffeomorphic to the closed unit ball of $\mathbb{R}^{m+\bar{n}}$ (which is certainly true for a sufficiently small r_0) and let T' satisfy

- $\mathrm{spt}(T') \subset \mathbf{B}_r(p) \cap \Sigma$ for some $r \leq r_0$;
- $\partial(T' + T \llcorner \mathbf{B}_r(p)) = 0$.

Then, by the isoperimetric inequality there is a current S supported in $\mathbf{B}_r(p)$ such that $\partial S = T' + T \llcorner \mathbf{B}_r(p)$ and $\mathbf{M}(S) \leq C\big(\|T\|(\mathbf{B}_r(p)) + \|T'\|(\mathbf{B}_r(p))\big)^{1+1/m}$, where the constant C depends only on Σ (and in particular might be assumed independent of $r < r_0$). Hence the inequality (2) implies

$$\|T\|(\mathbf{B}_r(p)) \leq \|T'\|(\mathbf{B}_r(p)) + C\|d\omega\|_0\big(\|T\|(\mathbf{B}_r(p)) + \|T'\|(\mathbf{B}_r(p))\big)^{1+1/m} .$$

So, although semicalibrated currents are not area minimizing, they could be considered as "almost minimizing".

2.2. **The main regularity theorem.** We are now ready to state the main regularity theorem contained in the papers [27, 26, 24, 25]. We first introduce the relevant concepts of singular and regular sets for an integral current T.

Definition 2.4. Given an integer rectifiable current T, we denote by $\mathrm{Reg}(T)$ the subset of $\mathrm{spt}(T) \setminus \mathrm{spt}(\partial T)$ consisting of those points x for which there is a neighborhood U such that $T \llcorner U$ is a (constant multiple of) a regular submanifold. Correspondingly, $\mathrm{Sing}(T)$ is the set $\mathrm{spt}(T) \setminus (\mathrm{spt}(\partial T) \cup \mathrm{Reg}(T))$.

Observe that $\mathrm{Reg}(T)$ is relatively open in $\mathrm{spt}(T)\setminus\mathrm{spt}(\partial T)$ and thus $\mathrm{Sing}(T)$ is relatively closed.

Theorem 2.5. *Let Σ be a C^{3,ε_0} submanifold of $\mathbb{R}^{2+n}$ and ω a C^{2,ε_0} semicalibration, where $\varepsilon_0 > 0$. If T is a 2-dimensional current satisfying any of the following requirements, then $\mathrm{Sing}(T)$ consists of isolated points:*

(a) *T is area minimizing in Σ;*
(b) *$\mathrm{spt}(T) \subset \Sigma$ and T is semicalibrated by ω;*
(c) *$\Sigma = \partial\mathbf{B}_R(p)$, $\mathrm{spt}(T) \subset \Sigma$ and $T \times\!\!\!\!\times\, p$ is area minimizing in $\mathbb{R}^{2+n}$.*

Clearly Chang's result is covered by case (a). An alternative proof of Chang's theorem has been found by Rivière and Tian in [41] for the special case of J-holomorphic curves. Later on the approach of Rivière and Tian has been generalized by Bellettini and Rivière in [7] to handle the first and (prior to the papers [27, 26, 24, 25]) only case which is not covered by [10], namely that of special Legendrian cycles in $\mathbb{S}^5$ (see also [8] for a further generalization). Observe that these currents form a special subclass of both (b) and (c). Indeed these cycles arise as spherical cross-sections of 3-dimensional special Lagrangian cones: as such they are then spherical cross sections of area minimizing cones but they are also semicalibrated by a specific smooth form on $\mathbb{S}^5$.

As already mentioned, the regularity theory of [27, 26, 24, 25] proves in fact much more, since it gives also a rather precise description of the asymptotic behavior of the current T at each singular point. However the corresponding statement would become much more involved and we prefer to analyze it while describing the main steps of the proof of Theorem 2.5.

3. Preliminary considerations and almost minimizing properties

A first important elementary remark is that in the semicalibrated case we do not loose any generality if we consider Σ to be the ambient Euclidean space. More precisely, it is simple to prove the following lemma (cf. [27, Lemma 1.1]).

Lemma 3.1. *Let $k \in \mathbb{N} \setminus \{0\}$, $\varepsilon_0 \in [0,1]$, $\Sigma \subset \mathbb{R}^{m+n}$ be a C^{k+1,ε_0} $m+\bar{n}$-dimensional submanifold, $V \subset \mathbb{R}^{m+n}$ an open subset and ω a C^{k,ε_0} m-form on $V \cap \Sigma$. If T is a cycle in $V \cap \Sigma$ semicalibrated by ω, then T is semicalibrated in V by a C^{k,ε_0} form $\tilde{\omega}$.*

Thanks to such lemma, in the case (b) of Theorem 2.5 we can assume that the ambient submanifold is in fact Σ itself. In particular we can formulate Theorem 2.5 in the following simplified version:

Theorem 3.2. *Let Σ be a C^{3,ε_0} submanifold of $\mathbb{R}^{2+n}$ and ω a C^{2,ε_0} semicalibration, where $\varepsilon_0 > 0$. If T is a 2-dimensional current satisfying any of the following requirements, then* $\mathrm{Sing}(T)$ *consists of isolated points:*

(a) *T is area minimizing in Σ;*
(b) $\mathrm{spt}(T) \subset \Sigma = \mathbb{R}^{2+n}$ *and T is semicalibrated by ω;*
(c) $\Sigma = \partial \mathbf{B}_R(p)$, $\mathrm{spt}(T) \subset \Sigma$ *and $T ⨳ p$ is area minimizing in $\mathbb{R}^{2+n}$.*

Observe that in both cases (a) and (c) of Theorem 3.2 the current T is stationary in Σ (resp. in $\partial \mathbf{B}_R(p)$) and has, therefore, bounded generalized mean curvature in $\mathbb{R}^{2+n}$. It turns out that T has bounded generalized mean curvature in case (b) as well. Additionally, even in the case (c) we have an "almost minimizing property" which resembles formally the one described in the previous section for semicalibrated currents. More precisely, relatively simple computations in [27] show the following proposition, which in fact is not limited to the 2-dimensional case

Proposition 3.3 (Cf. [27, Proposition 1.2])**.** *Let T be an m-dimensional current as in Theorem 3.2(b) or (c) (where we replace R^{2+n} with $\mathbb{R}^{m+n}$). Then there is a constant $\mathbf{\Omega}$ such that*

$$\mathbf{M}(T) \leq \mathbf{M}(T + \partial S) + \mathbf{\Omega}\, \mathbf{M}(S) \qquad \forall S \in \mathbf{I}_{m+1}(\mathbb{R}^{m+n}) \quad \textit{with compact support.} \tag{3}$$

More precisely, $\mathbf{\Omega} \leq \|d\omega\|_0$ in case (b) and $\mathbf{\Omega} \leq (m+1)R^{-1}$ in case (c).

Moreover, if $\chi \in C^\infty_c(\mathbb{R}^{m+n} \setminus \mathrm{spt}(\partial T), \mathbb{R}^{m+n})$, we have

$$\delta T(\chi) = T(d\omega \,\lrcorner\, \chi) \qquad \textit{in case (b)}, \tag{4}$$

$$\delta T(\chi) = \int m R^{-1}\, x \cdot \chi(x)\, d\|T\|(x) \qquad \textit{in case (c)}. \tag{5}$$

As already mentioned, the monotonicity formula holds for area minimizing currents. Thanks to the pioneering work of Allard [2], the monotonicity formula holds for currents (in fact more generally for varifolds) with bounded generalized mean curvature. In turn this fact can be combined with Proposition 3.3 to show that in all cases of Theorem 3.2 the current T has a "classical" almost minimizing property.

Definition 3.4. An m-dimensional integer rectifiable current T in $\mathbb{R}^{m+n}$ is *almost (area) minimizing* if for every $x \notin \operatorname{spt}(\partial T)$ there are constants $C_0, r_0, \alpha_0 > 0$ such that

$$\|T\|(\mathbf{B}_r(x)) \leq \|T + \partial S\|(\mathbf{B}_r(x)) + C_0\, r^{m+\alpha_0} \tag{6}$$

for all $0 < r < r_0$ and for all integral $(m+1)$-dimensional currents S supported in $\mathbf{B}_r(x)$.

Proposition 3.5 (Cf. [27, Proposition 0.4])**.** *Any m-dimensional current T as in (a), (b) and (c) of Theorem 3.2 is almost minimizing in the sense of Definition 3.4. Indeed the exponent α_0 can be taken equal to 1 and the constant C_0 is proportional to the L^∞ norm of the second fundamental form of Σ (in case (a)) and to $\mathbf{\Omega}$ as in Proposition 3.3 (in the cases (b) and (c)).*

4. The uniqueness of tangent cones

As already observed, in all the three cases of Theorem 3.2 the current T satisfies (a suitable perturbation of) the classical monotonicity formula. More generally, it is possible to derive a suitable monotonicity formula for almost minimizing currents as in Definition 3.4. It is also rather straightforward to see that, under the very same assumption, all tangent cones at a point $p \in \operatorname{spt}(T) \setminus \operatorname{spt}(\partial T)$ must necessarily be area minimizing currents in the euclidean space. Under the assumption that T is 2-dimensional, it is then well known that such tangent cones must necessarily be of the form $\sum_i Q_i [\![\pi_i]\!]$, where the π_i's are finitely many distinct 2-dimensional planes, each pair of which intersects *only* at the origin, cf. [31].

A remarkable theorem of White, [46], asserts that such tangent cones are unique (at each point p) if the original 2-dimensional current T is area minimizing in the ambient Euclidean space. More precisely, consider the currents $(\iota_{x,r})_\sharp T$, where the map $\iota_{x,r}$ is given by $\mathbb{R}^{2+n} \ni y \mapsto \frac{y-x}{r} \in \mathbb{R}^{2+n}$. Recall that an area minimizing cone S is an integral area minimizing current such that $(\iota_{0,r})_\sharp S = S$ for every $r > 0$. Then White's Theorem guarantees that, if T is 2-dimensional and area minimizing in $\mathbb{R}^{2+n}$, then at each point $p \in \operatorname{spt}(T)\setminus\operatorname{spt}(\partial T)$ the limit of $T_{p,r}$ as $r \downarrow 0$ exists and is a cone S, which as discussed above takes the special form $\sum_i Q_i [\![\pi_i]\!]$, for finitely many positive integers Q_i and 2-dimensional oriented planes π_i's, each pair of which intersects only at the origin.

The uniqueness of tangent cones for 2-dimensional area minimizing currents in Riemannian manifolds (case (a) of Theorem 3.2) was proved by Chang in [10]. The same statement for semicalibrated integral 2-dimensional cycles (case (b) of Theorem 3.2) has been shown more recently by Pumberger and Rivière in [37]. As far as we know the result for spherical cross sections of 3-dimensional area minimizing cones was open so far. In both [10] and [37] the proofs followed from inspecting White's epiperimetric inequality, the key point of [46], and showing that his argument can be suitably modified to handle the respective cases of interest.

In [27] we have instead noticed that for general 2-dimensional currents the almost minimizing property of Definition 3.4 implies an epiperimetric-type inequality which can be reduced to White's statement via a simple compactness argument. This gives therefore a rather short and direct proof of the uniqueness of tangent cones in all possible "perturbative cases", together with a suitable rate of convergence which is in turn a fundamental starting point of the proof of Theorem 3.2. In this section we explain these consequences and give a rough outline of the contents of [27] and [46].

4.1. **Uniqueness and irreducibility.** The main theorem in [27] is then the following

Theorem 4.1 (Cf. [27, Theorem 0.2])**.** *Assume T is a 2-dimensional integral almost minimizer in $\mathbb{R}^{n+2}$ in the sense of Definition 3.4. Then for every $p \in \mathrm{spt}(T) \setminus \mathrm{spt}(\partial T)$ there is a 2-dimensional integral area minimizing cone T_p with $\partial T_p = 0$ such that $T_{p,r} \to T_p$ (in the sense of currents) as $r \downarrow 0$.*

In fact the proof of Theorem 4.1 gives several additional pieces of information which turn out to be rather useful: the most notable consequence is a rate of converge of $T_{p,r}$ towards the tangent cone. To state things more precisely, we introduce a suitable flat distance on the space of integral currents. More precisely, for any given m-dimensional integral current R in $\mathbb{R}^{m+n}$ (for which from now on we will use the abbreviation $R \in \mathbf{I}_m(\mathbb{R}^{m+n})$), we set

$$\mathcal{F}(R) := \inf\{\mathbf{M}(Z) + \mathbf{M}(W) : Z \in \mathbf{I}_m, W \in \mathbf{I}_{m+1}, Z + \partial W = R\}\,.$$

Theorem 4.2 (Uniqueness of tangent cones for almost minimizers, cf. [27, Theorem 3.1])**.** *Let $T \in \mathbf{I}_2(\mathbb{R}^{n+2})$ be an almost minimizer and $p \in \mathrm{spt}(T)\setminus\mathrm{spt}(\partial T)$. Then there is a $\gamma_0 > 0$, J 2-dim. distinct planes π_i, each pair of which intersect only at 0, and J integers n_i such that, if we set $S := \sum_i n_i \llbracket \pi_i \rrbracket$, then*

$$\mathcal{F}\big((T_{p,r} - S) \llcorner \mathbf{B}_1\big) \leq C_{11}\, r^{\gamma_0}, \tag{7}$$

$$\mathrm{dist}\big(\mathrm{spt}(T \llcorner \mathbf{B}_r(p)), \mathrm{spt}(S)\big) \leq C_{11}\, r^{1+\gamma_0}. \tag{8}$$

Moreover, there are $\bar r > 0$ and $J \geq 1$ currents $T^j \in \mathbf{I}_2(\mathbf{B}_{\bar r}(x))$ such that

(i) *$\partial T^j \llcorner \mathbf{B}_{\bar r}(p) = 0$ and each T^j is an almost minimizer;*
(ii) *$T \llcorner \mathbf{B}_{\bar r}(p) = \sum_j T^j$ and $\mathrm{spt}(T_j) \cap \mathrm{spt}(T_i) = \{p\}$ for every $i \neq j$;*
(iii) *$n_j \llbracket \pi_j \rrbracket$ is the unique tangent cone to each T^j at p.*

Precise bounds for γ_0 can be written in terms of the exponent α_0 appearing in the almost minimizing condition and of the density $\Theta(T, x)$ (which in turn equals $\sum_j n_j$). A rather elementary consequence of Theorem 4.2(ii) is that each "piece" T_j is in fact area minimizing, semicalibrated or a spherical cross section of an area minimizing cone whenever the original T is. This suggests the introduction of a suitable "irreducibility concept", which simplifies certain technical aspects in both the statements and the proofs of our main theorems

Definition 4.3 (Irreducibility, cf. [24, Definition 1.2(iii)])**.** A current T is irreducible in any neighborhood U of $p \in \mathrm{spt}(T) \setminus \mathrm{spt}(\partial T)$ with respect to the point p if it is not possible to find S, Z non-zero integer rectifiable currents in U with $\partial S = \partial Z = 0$ (in U), $T = S + Z$ and $\mathrm{spt}(S) \cap \mathrm{spt}(Z) = \{p\}$.

It is rather easy to see that any almost-minimizing current can be decomposed into irreducible pieces and in fact for the purpose of the regularity theory we can assume, in all our cases, that the current in question is irreducible in any neighborhood of p with respect to p. More precisely, Theorem 3.2 can be reduced to the following statement:

Theorem 4.4. *Let Σ be a C^{3,ε_0} submanifold of $\mathbb{R}^{2+n}$ and ω a C^{2,ε_0} semicalibration, where $\varepsilon_0 > 0$. Let T be a 2-dimensional integral current and $p \in \mathrm{Sing}(T)$ a point with respect to which T is irreducible in any neighborhood. If in addition one of the following three conditions holds, then p is necessarily an isolated singularity:*

(a) *T is area minimizing in Σ;*
(b) *$\mathrm{spt}(T) \subset \Sigma = \mathbb{R}^{n+2}$ and T is semicalibrated by ω;*
(c) *$\mathrm{spt}(T) \subset \Sigma = \partial \mathbf{B}_R(p)$ and $p ⋇ T$ is area minimizing in $\mathbb{R}^{2+n}$.*

An obvious consequence of the assumptions of Theorem 4.4 is that the tangent cones S as in Theorem 4.2 is necessarily the multiple of a single plane.

Corollary 4.5. *Assume T and p are as in Theorem 4.4. Then the unique tangent cone S to T at p has the form $Q [\![\pi_0]\!]$ for some 2-dimensional plane π_0 and some $Q \in \mathbb{N} \setminus \{0\}$.*

4.2. **White's epiperimetric inequality.** As already mentioned, the key ingredient in the proof of Theorem 4.2 is a suitable generalization of White's epiperimetric inequality [46]. To explain the concept of "epiperimetric inequality", consider an area minimizing integral current T without boundary in $\partial \mathbf{B}_1(0)$ and let Z be its spherical slice: $Z := \partial(T \llcorner \mathbf{B}_1(0))$. The cone $0 ⋇ Z$ over Z is then a competitor for the minimality property of T and we have the obvious estimate

$$\|T\|(\mathbf{B}_1(0)) \leq \|0 ⋇ Z\|(\mathbf{B}_1(0)) \,. \tag{9}$$

The latter inequality leads then directly to the well-known monotonicity formula. Assume next that T is in the proximity of a cone S, for instance a tangent cone T_0 to T at 0, which therefore has less mass then T. The following would then be an improvement of the crude estimate (9) if $\varepsilon > 0$

$$\|T\|(\mathbf{B}_1(0)) \leq (1-\varepsilon)\|0 ⋇ Z\|(\mathbf{B}_1(0)) + \varepsilon\|T_0\|(\mathbf{B}_1(0)) \ .$$

A key observation of Reifenberg in his pioneering work [38] is that such an improvement would then imply a power-law (in r) convergence of $T_{0,r}$ to T_0 (and thus also the uniqueness of the tangent cone to T at 0).

It is known from the work of Adams and Simon, see [1], that we cannot expect an epiperimetric inequality to hold for general area minimizing currents. However, such an inequality always holds in the 2-dimensional case: this is essentially the content of White's work [46], which we summarize in the following proposition (although the literal statement is not present in [46]).

Lemma 4.6. *Let $S \in \mathbf{I}_2(\mathbb{R}^{n+2})$ be an area minimizing cone. There exists a constant $\varepsilon_{13} > 0$ with the following property. If $R := \partial(S \llcorner \mathbf{B}_1)$ and $Z \in \mathbf{I}_1(\partial \mathbf{B}_1)$ is a cycle with*

(i) $\mathcal{F}(Z - R) < \varepsilon_{13}$,
(ii) $\mathbf{M}(Z) - \mathbf{M}(R) < \varepsilon_{13}$,
(iii) $\mathrm{dist}\big(\mathrm{spt}(Z), \mathrm{spt}(R)\big) < \varepsilon_{13}$,

then there exists $H \in \mathbf{I}_2(\mathbf{B}_1)$ such that $\partial H = Z$ and

$$\|H\|(\mathbf{B}_1) \leq (1-\varepsilon_{13})\|0 \times\!\!\!\times Z\|(\mathbf{B}_1) + \varepsilon_{13}\|S\|(\mathbf{B}_1)\,.$$

The paper [27] contains indeed a derivation of Lemma 4.6 which differs from that of White in [46] in a minor technical point. First following [46] the lemma is reduced to the case where Z is a curve "winding" Q times in the proximity of a circle R: the main idea of [46] is then to first parametrize such a curve over $[0, 2\pi Q]$ as $(\cos\theta, \sin\theta, f(\sin\theta)) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^n$ for a $2\pi Q$ periodic f: if the system of coordinates is chosen appropriately, a harmonic-like extension of f yields the desired surface (current) H (in fact such harmonic extension is what we describe in Section 8.2 as a centered harmonic extension of a multiple valued map). In particular it suffices to choose the system of coordinates so that the Fourier modes $v\cos\theta + w\sin\theta$ in the Fourier decomposition of f are not too large compared to the whole f. White in [46] achieves the existence of such coordinates via an implicit function theorem, whereas in [27] we point out that it suffices to solve an elementary minimum problem.

The main contribution of [27] is instead the observation that an inequality as in Lemma 4.6 can be derived in the almost minimizing case *directly* from Lemma 4.6, via a standard contradiction argument. More precisely we have

Proposition 4.7 (Cf. [27, Proposition 3.4])**.** *Let $S \in \mathbf{I}_2(\mathbb{R}^{n+2})$ be an area minimizing cone. For every $C_{12} > 0$ there exists a constant $\varepsilon_{11} > 0$, depending only on the constants C_{01} and α_0 of Definition 3.4 and upon S, with the following property. Assume that $T \in \mathbf{I}_2(\mathbb{R}^{n+2})$ is an almost minimizer with $0 \in \mathrm{spt}(T)$ and set $T_\rho := (\iota_{0,\rho})_\sharp T$. If r is a positive number with*

- *$0 < 2\,r < \min\{2^{-1}\mathrm{dist}(0, \mathrm{spt}(\partial T)), 2\varepsilon_{11}\}$,*
- *$\mathcal{F}\big((T_{2r} - S)\llcorner \mathbf{B}_1\big) < 2\,\varepsilon_{11}$, $\|T\|(\mathbf{B}_{2r}) \leq C_{12} r^2$*
- *and $\partial(T \llcorner \mathbf{B}_r) \in \mathbf{I}_1(\mathbb{R}^{n+2})$,*

then

$$\|T_r\|(\mathbf{B}_1) \leq (1-\varepsilon_{12})\,\|0 \times\!\!\!\times \partial(T_r \llcorner \mathbf{B}_1)\|(\mathbf{B}_1) + \varepsilon_{12}\|S\|(\mathbf{B}_1) + \bar{c}\, r^{\alpha_0}. \tag{10}$$

$\bar c$ depends only on C_{01}, α_0 and $\Theta(0, S)$ and $\varepsilon_{12} > 0$ is any number smaller than some $\bar\varepsilon > 0$, which also depends on C_{01}, α_0 and $\Theta(0, S)$. Moreover $\bar c$ depends linearly on C_{01}. In particular, if T is as in Theorem 4.4, then $\alpha_0 = 1$ and: $\bar c$ depends linearly on $\mathbf{A} := \|A_\Sigma\|_\infty$ in case (a), it depends linearly on $\mathbf{\Omega} := \|d\omega\|_\infty$ in case (b) and it equals $C_0 R^{-1}$ for some geometric constant C_0 in case (c).

From this proposition it is then relatively straightforward to infer Theorem 4.2 (again the proof is a minor adaptation of [46]).

5. The classical ε-regularity theorem

The first breakthrough in the regularity theory of area minimizing currents is due to De Giorgi: he realized in his fundamental work [13] that in codimension 1 the existence of one flat tangent plane at p is enough to conclude that p is a regular point. His theorem was then extended to any codimension by Almgren in [3] (see also [42]) under an important assumption on the density which we will discuss extensively in a moment (indeed the latter theorem can be suitably generalized to Hilbert spaces, cf. [6]). In fact Almgren's statement covers many more geometric functionals, which satisfy an appropriate ellipticity assumption. In the framework of minimal (i.e. only *stationary*) surfaces the most important generalization of De Giorgi's ε-regularity theorem is due to Allard in [2] (cf. also [43, Chapter 4] and [15]): his theorem, valid for a far reaching generalization of classical stationary surfaces (namely integer rectifiable varifolds with sufficiently summable generalized mean curvature) is the starting point of a variety of applications of the minimal surface theory to geometric and topological problems.

We will recall the De Giorgi-Almgren ε-regularity theorem in all dimensions and codimensions, after introducing the key parameter of "flatness"

Definition 5.1 (Spherical excess)**.** Let T be an integer rectifiable m-dimensional current and π be an m-dimensional plane, oriented by the unit simple m-vector $\vec{\pi}$. The (spherical) excess of T in the ball $\mathbf{B}_\rho(p)$ with respect to π is the quantity

$$\mathbf{E}(T, \mathbf{B}_\rho(p), \pi) := \frac{1}{\omega_m \rho^m} \int_{\mathbf{B}_\rho(p)} |\vec{T}(x) - \vec{\pi}|^2 \, d\|T\|(x) \,. \tag{11}$$

The *excess* in $\mathbf{B}_\rho(p)$ is

$$\mathbf{E}(T, \mathbf{B}_\rho(p)) := \min\{\mathbf{E}(T, \mathbf{B}_\rho(p), \pi) : \pi \text{ is an oriented } m\text{-plane}\} \,. \tag{12}$$

If π achieves the minimum in the right hand side of (12) we then say that π *optimizes* the excess.

Since we will often deal with m-dimensional balls in m-dimensional planes π, we introduce here the notation $B_r(p, \pi)$ for the set $\mathbf{B}_r(p) \cap (p + \pi)$.

Theorem 5.2 (ε-regularity)**.** *Let T be an m-dimensional integer rectifiable area minimizing current in a C^2 submanifold Σ of dimension $m + \bar{n}$. There are constants $\alpha > 0$, $\varepsilon > 0$ and C, depending only upon m and $\bar{n}$, such that the following holds. Assume that for some $\rho > 0$ and some m-dimensional plane π we have*

(a) $\partial T \llcorner \mathbf{B}_{2\rho}(p) = 0$;
(b) $\Theta(T, p) = Q$ *and* $\Theta = Q$ $\|T\|$*-a.e. on* $\mathbf{B}_{2\rho}(p)$*, for some positive integer* Q;
(c) $\|T\|(\mathbf{B}_{2\rho}(p)) \leq (Q\omega_m + \varepsilon)(2\rho)^m$;
(d) $E := \mathbf{E}(T, \mathbf{B}_{2\rho}(p), \pi) < \varepsilon$ *and* $\rho \mathbf{A} := \rho \max_{\Sigma \cap \mathbf{B}_{2\rho}(p)} |A_\Sigma| < \varepsilon$.

Then $T \llcorner \mathbf{B}_\rho(p) = Q [\![\Gamma]\!]$ *for a surface* Γ *which is the graph of a suitable* $C^{1,\alpha}$ *function* $u : B_r(p, \pi) \to \pi^\perp$*. Moreover* $[Du]_{0,\alpha} \leq C(E^{1/2} + \rho \mathbf{A})\rho^{-\alpha}$*.*

The essential point in the proof of Theorem 5.2 is that, under the above assumptions, the current T is close to the graph of an harmonic function v and hence the excess in a ball is close to the L^2 mean oscillation of the gradient of Dv: since the latter quantity has nice decay properties for harmonic functions, it is possible to show that the spherical excess of T decays suitably. We do not discuss this point further and we instead refer the reader to the surveys [16, 17], where a rather detailed description is given. We only point out that the exponent α can be made arbitrarily close to 1, namely that, for any fixed positive $\delta > 0$, if ε is chosen suitably small then under the Assumptions of Theorem 5.2 we have

$$\mathbf{E}(T, \mathbf{B}_\rho(p)) \leq 2^{-2+2\delta}\mathbf{E}(T, \mathbf{B}_{2\rho}(p)) \,. \tag{13}$$

The Hölder exponent of the conclusion in Theorem 5.2 will then equal $1 - \delta$.

5.1. **The ε-regularity theory in codimension 1.** It is rather simple to see that the conditions (a), (c) and (d) will be met at a sufficiently small radius ρ as soon as $p \in \mathrm{spt}(T) \setminus \mathrm{spt}(\partial T)$ and there is at least *one* flat tangent cone at p. It seems therefore that we are in a wonderful position in the case of 2-dimensional area minimizing currents where we have reduced our attention to points where the tangent cone is flat and even unique. However condition (b) makes the story much harder and indeed discriminates severely between the codimension 1 case ($\bar{n} = 1$) and the higher codimensions. The structure of integral currents T in codimension 1, more precisely the fact that they can be regarded, away from $\mathrm{spt}(\partial T)$, as boundaries of *sets*, allows to show, via a coarea-type formula and elementary PDE considerations, that the condition (b) is in fact redundant, see [17] for a thorough discussion. More precisely we have the following

Corollary 5.3. *If T is an area minimizing current of dimension m in a C^2 submanifold Σ of dimension $m + 1$, then any point p at which there is a flat tangent cone is a regular point.*

In fact, for the sake of our future discussions we stress on an equivalent way to state the ε-regularity theory in codimension 1, underlying that "singularities persist in the limit": we will see later that this persistence can be seen as the major difference between the codimension 1 and the higher codimension.

Proposition 5.4 (Persistence of singularities in codimension 1)**.** *Let Σ_k be a sequence of C^2 submanifolds of $\mathbb{R}^{m+n}$ of dimension $m + \bar{n}$ which converge in C^2 to Σ and let T_k be a sequence of integer rectifiable area minimizing currents in Σ_k of dimension m with $\sup_k \mathbf{M}(T_k) < \infty$. Assume that $\partial T_k = 0$ on some open set Ω and that $T_k \llcorner \Omega \to T$. If $p_k \in \mathrm{Sing}(T_k)$ and $p_k \to p \in \Omega$, then $p \in \mathrm{Sing}(T)$.*

In the next section we will discuss why the conclusion of the ε-regularity theorem fails in higher codimension if we drop assumption (b) and we will in fact see that both Corollary 5.3 and Proposition 5.4 fail in codimension higher than 1.

6. Holomorphic curves and branching points

We start by recalling that holomorphic subvarieties of $\mathbb{C}^{k+j}$, namely zeros of holomorphic maps $u : \mathbb{C}^{k+j} \to \mathbb{C}^j$ (k and j being, respectively, the complex dimension and codimension of the variety) can be given a natural orientation. In what follows we identify $\mathbb{C}^{k+j}$ with $\mathbb{R}^{2k+2j}$ in the usual way: if $z_1, \ldots, z_{k+j}$ are complex coordinates and $x_j = \mathrm{Re}\, z_j$, $y_j = \mathrm{Im}\, z_j$, we let $x_1, y_1, \ldots, x_{k+j}, y_{k+j}$ be the standard coordinates of $\mathbb{R}^{2k+2j}$. Recall then that an holomorphic subvariety Γ of $\mathbb{C}^{k+j}$ of complex dimension k is a (real analytic) submanifold of $\mathbb{R}^{2k+2j} \setminus \mathrm{Sing}(\Gamma)$ of (real) dimension $m = 2k$, where $\mathrm{Sing}(\Gamma)$ is an holomorphic subvariety of complex dimension $k-1$.

Furthermore, at each point $p \in \Gamma \setminus \mathrm{Sing}(\Gamma)$, the (real) tangent $2k$-dim. plane $T_p\Gamma$ can be identified with a complex k-dimensional plane of $\mathbb{C}^n$. If $v_1, \ldots, v_k$ is a complex basis of $T_p\Gamma$, we can then define a canonical orientation for $T_p\Gamma$ using the simple $2k$-vector

$$\mathrm{Re}\, v_1 \wedge \mathrm{Im}\, v_1 \wedge \ldots \wedge \mathrm{Re}\, v_k \wedge \mathrm{Im}\, v_k\,.$$

This allows us to define the current $[\![\Gamma]\!]$ by integrating forms over the oriented submanifold $\Gamma \setminus \mathrm{Sing}(\Gamma)$. It is also easy to check that $\partial\, [\![\Gamma]\!] = 0$, the reason being that the "singular set" $\mathrm{Sing}(\Gamma)$ is a set of (locally) finite $\mathcal{H}^{2k-2}$ measure.

The discussion can be "localized" to holomorphic subvarieties in open subsets Ω of $\mathbb{C}^{k+j}$ (and more generally in complex hermitian manifolds). Note also that, if Ω' is a bounded open subset of the domain Ω where Γ is defined, then $[\![\Gamma]\!]$ has finite mass in Ω' and it is thus an integer rectifiable current. The following fundamental observation is due to Federer and is based on a classical computation of Wirtinger ([47]).

Theorem 6.1 (Federer, cf. [29, Section 5.4.19]). *Consider standard coordinates $z_\ell = x_\ell + iy_\ell$ in $\mathbb{C}^{k+j}$, let ω be the standard Kähler form*

$$\omega := dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + \ldots + dx_{k+j} \wedge dy_{k+j}$$

and denote by ν its k-th exterior power divided by $k!$:

$$\nu = \frac{1}{k!} \underbrace{\omega \wedge \ldots \wedge \omega}_{k\ \text{times}}\,.$$

Then ν is a calibration in $\mathbb{R}^{2k+2j}$ and it calibrates holomorphic subvarities of $\mathbb{C}^{k+j}$.

Hence, for every finite collection $\Gamma_1, \ldots \Gamma_N$ of holomorphic subvarities of complex dimension k in $\Omega \subset \mathbb{C}^{k+j} = \mathbb{R}^{2k+2j}$ and for every finite collection $k_1, \ldots, k_N$ of positive integers, the current $T := k_1\, [\![\Gamma_1]\!] + \ldots + k_N\, [\![\Gamma_N]\!]$ is area minimizing in Ω.

Indeed the above theorem holds in general Kähler manifolds, cf. [29, 5.4.19]. In the rest of this section we will use holomorphic curves to gain an inutition on the many difficulties that one has to face to prove Theorem 4.4.

6.1. **Singularities disappear.** Theorem 6.1 has very deep consequences in the regularity theory for area minimizing currents in codimension higher than 1. Holomorphic subvarieties give easy counterexamples to Corollary 5.3 when $\bar{n} > 1$: assumption (b) in Theorem 5.2 is absolutely crucial in this case. As a byproduct even Proposition 5.4 fails and singularities might disappear in the limit when we deal with sequences of area minimizing currents in codimension higher than 1: one core difficulty in the proof of Theorem 4.4 is precisely this phenomenon of "disappearance of singularities". We illustrate these points with three explicit examples.

Example 6.2. Let $\delta > 0$ be a small number and consider the holomorphic curve

$$\Gamma_\delta := \{(z, w) \in \mathbb{C}^2 : z^2 = \delta w\}$$

and the plane

$$\pi := \{(z, w) \in \mathbb{C}^2 : z = 0\} . \tag{14}$$

There is no neighborhood of 0 where Γ_δ is the graph of a function $z = f(w)$, in spite of the fact that $\mathbf{E}(\llbracket \Gamma_\delta \rrbracket, \mathbf{B}_1(0), \pi)$ converges to 0 as $\delta \downarrow 0$. In fact the conclusion of Theorem 5.2 does not apply: although each Γ_δ is smooth and it is graphical in $\mathbf{B}_\rho(0)$ for any ρ, there is no uniform control of the $C^{1,\alpha}$ norm of the graph in terms of the excess. Observe that the Γ_δ do not satisfy the condition (c) in Theorem 5.2, although they satisfy (a), (b) and (d). This is however a minor problem: by the monotonicity formula, we do know that (c) holds for a sufficiently small ball $\mathbf{B}_\rho(0)$: at that scale we will be able to apply Theorem 5.2 and conclude regularity and graphicality with respect to the plane π.

Example 6.3. The following holomorphic curve is instead much more problematic:

$$\Gamma := \{(z, w) \in \mathbb{C}^2 : z^2 = w^3\} .$$

The origin belongs to $\mathrm{Sing}(\llbracket \Gamma \rrbracket)$. On the other hand:

- The unique tangent cone at 0 is given by $2 \llbracket \pi \rrbracket$ for π as in (14).
- The density of $\llbracket \Gamma \rrbracket$ equals 2 at 0;
- $$\lim_{r \downarrow 0} \mathbf{E}(\llbracket \Gamma \rrbracket, \mathbf{B}_r(0), \pi) = 0 .$$

Therefore:

- Corollary 5.3 is false for 2-dimensional area minimizing currents in $\mathbb{R}^4$: Γ is singular at the origin in spite of the existence of a flat tangent cone there.
- Again Theorem 5.2 does not apply in any ball $\mathbf{B}_{2\rho}(0)$. Note however that the only missing assumption is (b): the density $\Theta(\llbracket \Gamma \rrbracket, p)$ equals 1 at every point $p \in \Gamma \setminus \{0\}$ and equals 2 at $p = 0$.
- Proposition 5.4 fails for 2-dimensional area minimizing currents in $\mathbb{R}^4$. Indeed 0 is a singular point for $\llbracket \Gamma \rrbracket_{0,r}$ for every positive r. On the other hand $\llbracket \Gamma \rrbracket_{0,r} \to 2 \llbracket \pi \rrbracket$ and thus 0 is *not* a singular point of the limit: the singularity "has disappeared".

Example 6.4. Consider next the holomorphic curve

$$\Xi := \{(z, w) \in \mathbb{C}^2 : (z - w^2)^2 = w^{2015}\} .$$

All the considerations valid for the holomorphic curve Γ of Example 6.3 are also valid for Ξ. Ξ does not add much for the moment to our discussion, but it will play a crucial role later: observe that 0 is a singular point in spite of the fact that Ξ is an almost imperceptible perturbation of the smooth current $2 \llbracket \{z = w^2\} \rrbracket$. In fact consider the variant

$$\Xi := \{(z, w) \in \mathbb{C}^2 : (z - w^2 h(w))^2 = w^{2015}\} .$$

If we want to capture the singular behavior $\{z^2 = w^{2015}\}$ via an anisotropic scaling of type $w \mapsto \frac{w}{r}$, $z \mapsto \frac{z}{r^{2015/2}}$, then we should find a way of removing all the terms $a_k w^k$ with $k \leq 1006$ in the Taylor expansion of h at 0.

We will call "construction of a center manifold" the procedure which, roughly speaking, given a rather general current as in Theorem 4.4 and a singular point, identifies this particular smooth function which must be "removed" in order to capture the singular behavior via an anisotropic scaling.

6.2. **Main singularity models.** Much more complicated examples can be constructed iterating the structure of the previous one. More precisely, consider a sequence of integers $\{Q_1, \ldots, Q_N\} \subset \mathbb{N} \setminus \{0, 1\}$ such that

$$1 \leq p_0 < \frac{p_1}{Q_1} < \frac{p_2}{Q_1 Q_2} < \ldots < \frac{p_N}{Q_1 \cdot \ldots \cdot Q_N}$$

and $MCD(p_j, Q_1 \cdot \ldots \cdot Q_j) = 1$. We can then construct the holomorphic curve

$$\Lambda := \left\{(z, w) : ((\ldots((z - w^{p_0})^{Q_1} - w^{p_1})^{Q_2} - \ldots)^{Q_{N_1}} - w^{p_{N_1}})^{Q_N} = w^{p_N}\right\} . \tag{15}$$

If we "zoom" into the signularity and rescale anisotropically the two coordinates as $z \mapsto \frac{z}{r^{p_0}}$, $w \mapsto \frac{w}{r}$, the corresponding sequence of currents converge to

$$Q_1 \cdot \ldots \cdot Q_N \left[\!\left[\{z = w^{p_0}\}\right]\!\right] .$$

If we could first "subtract" this regular part and then zoom again, this time rescaling $z \mapsto r^{-p_1/Q_1} z$, we would see in the limit the current

$$Q_2 \cdot \ldots \cdot Q_N \left[\!\left[z^{Q_1} = w^{p_1}\}\right]\!\right] ,$$

which is a "simple" branching singularity as in Example 6.3. However this is not the end of the story, if we could "mod out" this branching at the next step and rescale the z coordinate according to $z \mapsto r^{-p_2/(Q_1 Q_2)} z$ we would see a "finer branching", that is

$$Q_3 \cdot \ldots \cdot Q_N \left[\!\left[z^{Q_1 Q_2} = w^{p_2}\}\right]\!\right] :$$

this second branching winds Q_2 times around the singularity for each "turn" which is taken by the first branching.

If we proceed further we can expand the singular behavior of Λ into a hierarchy of N layered basic "singular blocks". Note moreover that the currents $[\![\Lambda]\!]$ in (15) do not really display the most general behavior. In particular we could perturb each block w^{p_i} by multiplying it with an holomorphic function $h_i(w)$ subject to the requirement $h_i(0) \neq 0$. That is, at the first step rather then removing w^{p_0} it might be that we have to remove a polynomial of type $a_0 w^{p_0} + a_1 w^{p_0+1} + \ldots a_k w^{p_0+k}$, where k is the largest integer such that $p_0 + k < \frac{p_1}{Q_1}$. Similarly at the second step we might need to remove a polynomial of type $b_0 w^{p_1} + \ldots + b_\kappa w^{p_1+\kappa}$: this time κ is the largest integer such that $\frac{p_1+\kappa}{Q_1} \leq \frac{p_2}{Q_1 Q_2}$.

The object which must be "removed" at each stage of this analysis in order to capture the finer singular behavior is called "branched center manifold". The example just described will serve as a guiding model for illustrating the proof of Theorem 4.4.

7. Multiple valued functions

As it is rather clear from the examples of the previous section, we cannot hope that a current with small excess around a certain point p in codimension $\bar{n} > 1$ can be well approximated by a the graph of a single-valued function. We could instead hope that there is an approximation with a multivalued graph. Note also that, if the multiplicity at the point p is an integer Q, it seems reasonable to expect that the corresponding approximating map takes at most Q values.

This discussion motivates the starting idea of Almgren's monograph: in order to proceed towards a good dimensional bound for the singular set of area minimizing currents in codimension higher than 1 we need to develop an efficient theory for "multiple valued functions" minimizing a suitable generalization of the Dirichlet energy, where we can (and we will) consider the multiplicity to be a constant preassigned positive integer Q.

7.1. **The metric space of unordered Q-tuples.** The obvious model case to keep in mind is the following. Given two integers k, Q with $\mathrm{MCD}(k,Q)=1$, look at the set valued map which assigns to each point $z \in \mathbb{C}$ the set $M(z) := \{w^k : w^Q = z\} \subset \mathbb{C}$. Obviously for each z we can choose some arbitrary ordering $\{u_1(z), \ldots, u_Q(z)\}$ of the elements of the set $M(z)$. However, it is not possible to do it in such a way that the resulting "selection maps" $z \mapsto u^i(z)$ are continuous: even at the local level, this is impossible in every neighborhood of the origin.

Our example motivates the following definition. Given an integer Q we define a Q-valued map from a set $E \subset \mathbb{R}^m$ into $\mathbb{R}^n$ as a function which to each point $x \in E$ associates an unordered Q-tuple of vectors in $\mathbb{R}^n$. Following Almgren, we consider the group $\mathscr{P}_Q$ of permutations of Q elements and we let $\mathcal{A}_Q(\mathbb{R}^n)$ be the set $(\mathbb{R}^n)^Q$ modulo the equivalence relation

$$(v_1, \ldots, v_Q) \equiv (v_{\pi(1)}, \ldots, v_{\pi(Q)}) \qquad \forall \pi \in \mathscr{P}\,.$$

Hence a multiple valued map is simply a map taking values in $\mathcal{A}_Q(\mathbb{R}^n)$. There is a fairly efficient formulation of this definition which will play a pivotal role in our discussion, because the set $\mathcal{A}_Q(\mathbb{R}^n)$ can be naturally identified with a subset of the set of measures (cf. [4] and [19, Definition 0.1]).

Definition 7.1 (Unordered Q-tuples)**.** Denote by $[\![P_i]\!]$ the Dirac mass in $P_i \in \mathbb{R}^n$. Then,

$$\mathcal{A}_Q(\mathbb{R}^n) := \left\{ \sum_{i=1}^{Q} [\![P_i]\!] \;:\; P_i \in \mathbb{R}^n \text{ for every } i = 1, \ldots, Q \right\}.$$

Observe that with this definition each element of $\mathcal{A}_Q(\mathbb{R}^n)$ is in fact a 0-dimensional integral current. This set has also a natural metric structure; cf. [4] and [19, Definition 0.2] (the experts will recognize the well-known Wasserstein 2-distance, cf. [45]).

Definition 7.2. For every $T_1, T_2 \in \mathcal{A}_Q(\mathbb{R}^n)$, with $T_1 = \sum_i [\![P_i]\!]$ and $T_2 = \sum_i [\![S_i]\!]$, we set

$$\mathcal{G}(T_1, T_2) \;:=\; \min_{\sigma \in \mathscr{P}_Q} \sqrt{\sum_i \left|P_i - S_{\sigma(i)}\right|^2}\,. \tag{16}$$

Remark 7.3. Since we will often need to compute $\mathcal{G}(T, Q\,[\![0]\!])$ we introduce the special notation $|T|$ for the latter quantity. Observe, however, that $\mathcal{A}_Q(\mathbb{R}^n)$ is *not* a linear space except for the special case $Q = 1$: the map $T \to |T|$ is not a norm.

7.2. **Q-valued maps.** Using the metric structure on $\mathcal{A}_Q(\mathbb{R}^n)$ one defines obviously measurable, Lipschitz and Hölder maps from subsets of $\mathbb{R}^m$ into $\mathcal{A}_Q(\mathbb{R}^n)$. One important point to be made is about the existence of "selections". A selection for a Q-valued function u is given by Q classical single valued functions $u_1, \ldots, u_Q$ such that $u(x) = \sum_{i=1}^Q [\![u_i(x)]\!]$, cf. [19, Definition 1.1]. If the u_i are measurable, continuous, Lipschitz, etc. the selection will be called measurable, continuous, Lipschitz, etc. It is rather easy to show that a measurable selection exists for any measurable u, cf. [19, Proposition 0.4]. Incidentally, this will be used repeatedly as we write

$$\sum_i [\![u_i]\!]$$

for any given measurable Q-valued map u, tacitly assuming to have chosen some measurable selection.

However continuous maps (resp. Sobolev, Lipschitz) do not possess in general selections which are continuous (resp. Sobolev, Lipschitz): the primary examples are the maps stemming from holomorphic subvarieties already discussed at length. If, however, they do, the corresponding selection will be called *regular*. Only maps defined on 1-dimensional intervals are a notable exception, since they always have regular selections: this fact plays a crucial role when the domain of the maps is 2-dimensional and we will come back to it later.

7.3. **The generalized Dirichlet energy: geometric definitions.** If we want to approximate area minimizing currents with multiple valued functions and "linearize" the area functional in the spirit of De Giorgi, we need to define a suitable concept of Dirichlet energy. We will now show how this can be done naturally, proposing three different approaches.

Consider the model case of $Q = 2$ and assume $u : \Omega \to \mathcal{A}_2(\mathbb{R}^n)$ is a Lipschitz map. If, at some point x, $u(x) = [\![P_1]\!] + [\![P_2]\!]$ is "genuinely 2-valued", i.e. $P_1 \neq P_2$, then there exist obviously a ball $B_r(x) \subset \Omega$ and a regular (Lipschitz) selection, namely Lipschitz classical maps $u_1, u_2 : B_r(x) \to \mathbb{R}^n$ such that $u(y) = [\![u_1(y)]\!] + [\![u_2(y)]\!]$ for every $y \in B_r(x)$. On the other hand on the closed set where the values of u are "collapsed" we can find a single Lipschitz map v such that $u = 2\,[\![v]\!]$. It is easy to generalize this to the Q-valued case and to maps defined on a manifold Σ:

Lemma 7.4 (Decomposition, cf. [20, Lemma 1.1])**.** *Let $M \subset \Sigma$ be measurable and $F : M \to \mathcal{A}_Q(\mathbb{R}^n)$ Lipschitz. Then there are a countable partition of M in bounded measurable subsets M_i ($i \in \mathbb{N}$) and Lipschitz functions $f_i^j : M_i \to \mathbb{R}^n$ ($j \in \{1, \ldots, Q\}$) such that*

(a) $F|_{M_i} = \sum_{j=1}^Q [\![f_i^j]\!]$ *for every* $i \in \mathbb{N}$ *and* $\mathrm{Lip}(f_i^j) \leq \mathrm{Lip}(F)$ $\forall i, j$*;*
(b) $\forall i \in \mathbb{N}$ *and* $j, j' \in \{1, \ldots, Q\}$*, either* $f_i^j \equiv f_i^{j'}$ *or* $f_i^j(x) \neq f_i^{j'}(x)$ $\forall x \in M_i$*.*

The Dirichlet energy can be defined for Lipschitz maps F as above by

$$\mathrm{Dir}(F, \Sigma) := \sum_{i,j} \int_{M_i} |Df_i^j|^2 \,. \tag{17}$$

$W^{1,2}$ maps and their Dirichlet energy can then be defined by relaxation: assuming that Ω is open $W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ consists of those measurable maps $v : \Omega \to \mathcal{A}_Q(\mathbb{R}^n)$ for which there is a sequence of Lipschitz maps u_k converging to v a.e. and enjoying a uniform bound $\mathrm{Dir}(\Omega, u_k) \leq C$. The Dirichlet energy $\mathrm{Dir}(\Omega, v)$ is the infimum of all constants C for which there is a sequence with the properties above.

Another possible definition of the Dirichlet energy follows more closely Almgren's original idea: for Q-valued maps we can introduce a notion of differentiability in the following way

Definition 7.5 (Q-valued differential, cf. [19, Definition 1.9])**.** Let $f : \Omega \to \mathcal{A}_Q$ and $x_0 \in \Omega$. We say that f is differentiable at x_0 if there exist Q matrices L_i satisfying:

(i) $\mathcal{G}(f(x), T_{x_0} f) = o(|x - x_0|)$, where

$$T_{x_0} f(x) := \sum_i [\![L_i \cdot (x - x_0) + f_i(x_0)]\!]\,; \tag{18}$$

(ii) $L_i = L_j$ if $f_i(x_0) = f_j(x_0)$.

The Q-valued map $T_{x_0} f$ will be called the *first-order approximation* of f at x_0. The point $\sum_i [\![L_i]\!] \in \mathcal{A}_Q(\mathbb{R}^{n\times m})$ will be called the differential of f at x_0 and is denoted by $Df(x_0)$.

A Rademacher's type theorem shows that Lipschitz Q-valued maps are differentiable almost everywhere and that the Dirichlet energy defined above corresponds to the integral of $|Df| = \mathcal{G}(Df, Q\,[\![0]\!])$ (our notation is consistent, cf. Remark 7.3). The proof is in fact a straightforward corollary of Lemma 7.4 and elementary measure theory.

Proposition 7.6 (Q-valued Rademacher, cf. [19, Theorem 1.13] and [20, Lemma 1.1])**.** *Let $f : \Omega \to \mathcal{A}_Q$ be a Lipschitz function. Then, f is differentiable almost everywhere in Ω and*

$$\mathrm{Dir}(f, \Omega) = \int_\Omega |Df|^2\,,$$

where the left hand side is understood in the sense of (17). *In particular the expression in* (17) *is independent of the decomposition given by Lemma 7.4.*

Almgren's definition of Sobolev map does not follow, however, a "relaxation procedure" but uses the (biLipschitz) embedding of $\mathcal{A}_Q(\mathbb{R}^n)$ in a large Euclidean space, see below.

7.4. $W^{1,2}$ **and the generalized Dirichlet energy: metric analysis definition.** Although the definition above is certainly very natural and gives a good geometric intuition for the Dirichlet energy, it turns out that it is rather complicated to work with it, in particular if one wants to recover the usual statements of the Sobolev space theory for classical functions.

Instead, a rather efficient way to achieve such statements is to rely on a more abstract definition of Dirichlet energy and Sobolev functions, as proposed in [19]. A very general theory has been developed in the literature for Sobolev maps taking values in abstract metric spaces, following the pioneering works of Ambrosio [5] and Reshetnyak [40, 39]. The careful reader will notice, however, that there is a crucial difference between the definition of Dirichlet energy in [40] and the one given below.

Definition 7.7 (Sobolev Q-valued functions, cf. [19, Definition 0.5])**.** A measurable $f : \Omega \to \mathcal{A}_Q$ is in the Sobolev class $W^{1,p}$ $(1 \le p \le \infty)$ if there exist m functions $\varphi_j \in L^p(\Omega; \mathbb{R}^+)$ such that

(i) $x \mapsto \mathcal{G}(f(x), T) \in W^{1,p}(\Omega)$ for all $T \in \mathcal{A}_Q$;
(ii) $|\partial_j \mathcal{G}(f, T)| \le \varphi_j$ a.e. in Ω for all $T \in \mathcal{A}_Q$ and for all $j \in \{1, \dots, m\}$.

It is not difficult to show the existence of minimal functions $\tilde{\varphi}_j$ fulfilling (ii), i.e. such that, for any other φ_j satisfying (ii), $\tilde{\varphi}_j \le \varphi_j$ a.e. (cf. [19, Proposition 4.2]). Such "minimal bounds" will be denoted by $|\partial_j f|$ and we note that they are characterized by the following property (see again [19, Proposition 4.2]): for every countable dense subset $\{T_i\}_{i\in\mathbb{N}}$ of $\mathcal{A}_Q$ and for every $j = 1, \dots, m$,

$$|\partial_j f| \;=\; \sup_{i\in\mathbb{N}} |\partial_j \mathcal{G}(f, T_i)| \quad \text{almost everywhere in } \Omega. \tag{19}$$

We are now ready to give an abstract characterization of the Dirichlet energy.

Proposition 7.8 (Cf. [19, Proposition 2.17])**.** *If $u \in W^{1,2}$ is Lipschitz then*

$$|\partial_j u|^2(x) = \sum_i |Du_i(x) \cdot e_j|^2 \tag{20}$$

at a.e. point x of differentiability of u, where $\sum_i [\![Du_i(x)]\!]$ is the Q-valued differential in the sense of Definition 7.5 and $|\partial_j u|$ is as in (19)*.*

The main feature of the above proposition is that essentially all the conclusions of the usual Sobolev space theory for single valued functions can be now reduced to routine modifications of the usual arguments: among them we mention Sobolev and Morrey embeddings, compact embeddings, Poincaré inequalities, semicontinuity results, trace properties (cf. [19, Chapter 4]).

We list here some of these facts and refer to [19] for their proof.

Definition 7.9 (Trace of Sobolev Q-functions)**.** Let $\Omega \subset \mathbb{R}^m$ be a Lipschitz bounded open set and $f \in W^{1,p}(\Omega, \mathcal{A}_Q)$. A function g belonging to $L^p(\partial\Omega, \mathcal{A}_Q)$ is said to be the trace of f at $\partial\Omega$ (and we denote it by $f|_{\partial\Omega}$) if, for every $T \in \mathcal{A}_Q$, the trace of the real-valued Sobolev function $\mathcal{G}(f, T)$ coincides with $\mathcal{G}(g, T)$.

Definition 7.10 (Weak convergence)**.** Let $f_k, f \in W^{1,p}(\Omega, \mathcal{A}_Q)$. We say that f_k converges weakly to f for $k \to \infty$, (and we write $f_k \rightharpoonup f$) in $W^{1,p}(\Omega, \mathcal{A}_Q)$, if

(i) $\int \mathcal{G}(f_k, f)^p \to 0$, for $k \to \infty$;
(ii) there exists a constant C such that $\int |Df_k|^p \le C < \infty$ for every k.

Proposition 7.11 (Weak sequential closure)**.** *Let $f \in W^{1,p}(\Omega, \mathcal{A}_Q)$. Then, there is a unique function $g \in L^p(\partial\Omega, \mathcal{A}_Q)$ such that $f|_{\partial\Omega} = g$ in the sense of Definition 7.9. Moreover, $f|_{\partial\Omega} = g$ if and only if $\mathcal{G}(f, T)|_{\partial\Omega} = \mathcal{G}(g, T)|_{\partial\Omega}$ for every T in the usual sense, and the set of mappings*

$$W^{1,2}_g(\Omega, \mathcal{A}_Q) := \left\{ f \in W^{1,2}(\Omega, \mathcal{A}_Q) \,:\, f|_{\partial\Omega} = g \right\} \tag{21}$$

is sequentially weakly closed in $W^{1,2}$.

Proposition 7.12 (Sobolev Embeddings)**.** *The following embeddings hold:*

(*i*) *if* $p < m$, *then* $W^{1,p}(\Omega, \mathcal{A}_Q) \subset L^q(\Omega, \mathcal{A}_Q)$ *for every* $q \in [1, p^*]$, *and the inclusion is compact when* $q < p^*$;

(*ii*) *if* $p = m$, *then* $W^{1,p}(\Omega, \mathcal{A}_Q) \subset L^q(\Omega, \mathcal{A}_Q)$, *for every* $q \in [1, +\infty)$, *with compact inclusion;*

(*iii*) *if* $p > m$, *then* $W^{1,p}(\Omega, \mathcal{A}_Q) \subset C^{0,\alpha}(\Omega, \mathcal{A}_Q)$, *for* $\alpha = 1 - \frac{m}{p}$, *with compact inclusion.*

Proposition 7.13 (Poincaré inequality)**.** *Let* M *be a connected bounded Lipschitz open set of an* m*-dimensional Riemannian manifold and let* $p < m$. *There exists a constant* $C = C(p, m, n, Q, M)$ *with the following property: for every* $f \in W^{1,p}(M, \mathcal{A}_Q)$, *there exists a point* $\overline{f} \in \mathcal{A}_Q$ *such that*

$$\left(\int_M \mathcal{G}\big(f, \overline{f}\big)^{p^*}\right)^{\frac{1}{p^*}} \leq C \left(\int_M |Df|^p\right)^{\frac{1}{p}}. \tag{22}$$

Proposition 7.14 (Campanato-Morrey)**.** *Let* $f \in W^{1,2}(B_1, \mathcal{A}_Q)$, *with* $B_1 \subset \mathbb{R}^m$, *and* $\alpha \in (0, 1]$ *be such that*

$$\int_{B_r(y)} |Df|^2 \leq A\, r^{m-2+2\alpha} \quad \text{for every } y \in B_1 \text{ and a.e. } r \in]0, 1 - |y|[.$$

Then, for every $0 < \delta < 1$, *there is a constant* $C = C(m, n, Q, \delta)$ *with*

$$\sup_{x,y \in \overline{B_\delta}} \frac{\mathcal{G}(f(x), f(y))}{|x - y|^\alpha} =: [f]_{C^{0,\alpha}(\overline{B_\delta})} \leq C\sqrt{A}. \tag{23}$$

Lemma 7.15 (Interpolation Lemma)**.** *There is a constant* $C = C(m, n, Q)$ *with the following property. Let* $r > 0$, $g \in W^{1,2}(\partial B_r, \mathcal{A}_Q)$ *and* $f \in W^{1,2}(\partial B_{r(1-\varepsilon)}, \mathcal{A}_Q)$. *Then, there exists* $h \in W^{1,2}(B_r \setminus B_{r(1-\varepsilon)}, \mathcal{A}_Q)$ *such that* $h|_{\partial B_r} = g$, $h|_{\partial B_{r(1-\varepsilon)}} = f$ *and*

$$\begin{aligned}\mathrm{Dir}(h, B_r \setminus B_{r(1-\varepsilon)}) \leq C\,\varepsilon\, r \left[\mathrm{Dir}(g, \partial B_r) + \mathrm{Dir}(f, \partial B_{r(1-\varepsilon)})\right] + \\ + \frac{C}{\varepsilon\, r} \int_{\partial B_r} \mathcal{G}\left(g(x), f\left((1-\varepsilon)\, x\right)\right)^2 dx.\end{aligned} \tag{24}$$

7.5. **Lipschitz approximation and approximate differentiability.** An important feature of classical Sobolev maps is the existence of suitable smooth approximations. Since the space $\mathcal{A}_Q(\mathbb{R}^n)$ is itself rather singular and lacks any linear structure, the usual approximation results are indeed much more subtle. However a robust way to approximate Sobolev maps is to "truncate them" along the level sets of the Hardy-Littlewood maximal function of the modulus of their gradient. This is possible in the setting of Q-valued maps as well and will play a crucial role in the sequel.

Proposition 7.16 (Lipschitz approximation, cf. [19, Proposition 4.4])**.** *There exists a constant* $C = C(m, \Omega, Q)$ *with the following property. For every* $f \in W^{1,p}(\Omega, \mathcal{A}_Q)$ *and every* $\lambda > 0$, *there exists a* Q*-function* f_λ *such that* $\mathrm{Lip}\,(f_\lambda) \leq C\,\lambda$,

$$|E_\lambda| = \left|\left\{x \in \Omega \,:\, f(x) \neq f_\lambda(x)\right\}\right| \leq \frac{C\||Df|\|^p_{L^p}}{\lambda^p}. \tag{25}$$

$\Omega \setminus E_\lambda$ can be assumed to contain $\{x \in \Omega : M(|Df|) \leq \lambda\}$, where M is the maximal function operator.

A simple corollary of the previous proposition is that Sobolev maps are "approximate differentiable" in the following sense:

Definition 7.17 (Approximate Differentiability)**.** A Q-valued function f is approximately differentiable in x_0 if there exists a measurable subset $\tilde{\Omega} \subset \Omega$ containing x_0 such that $\tilde{\Omega}$ has density 1 at x_0 and $f|_{\tilde{\Omega}}$ is differentiable at x_0.

Corollary 7.18. *Any $f \in W^{1,p}(\Omega, \mathcal{A}_Q)$ is approximately differentiable a.e.*

7.6. **Chain rule formulas: Q-valued calculus.** The latter property is very useful to extend classical computations like the chain rule to Sobolev maps. Indeed, it is rather easy to extend such formulas to Lipschitz maps using the multivalued differentiability: Proposition 7.16 can then be used to routinely justify the same formulas for general Sobolev maps.

Consider a function $f : \Omega \to \mathcal{A}_Q(\mathbb{R}^n)$. For every $\Phi : \tilde{\Omega} \to \Omega$, the right composition $f \circ \Phi$ defines a Q-valued function on $\tilde{\Omega}$. On the other hand, given a map $\Psi : \Omega \times \mathbb{R}^n \to \mathbb{R}^k$, we can consider the left composition, $x \mapsto \sum_i [\![\Psi(x, f_i(x))]\!]$, which defines a Q-valued function denoted, with a slight abuse of notation, by $\Psi(x, f)$.

Proposition 7.19 (Chain rules, cf. [19, Proposition 1.12])**.** *Let $f : \Omega \to \mathcal{A}_Q(\mathbb{R}^n)$ be differentiable at x_0.*

(*i*) *Consider $\Phi : \tilde{\Omega} \to \Omega$ such that $\Phi(y_0) = x_0$ and assume that Φ is differentiable at y_0. Then, $f \circ \Phi$ is differentiable at y_0 and*
$$D(f \circ \Phi)(y_0) = \sum_i [\![Df_i(x_0) \cdot D\Phi(y_0)]\!] . \tag{26}$$

(*ii*) *Consider $\Psi : \Omega_x \times \mathbb{R}^n_u \to \mathbb{R}^k$ such that Ψ is differentiable at $(x_0, f_i(x_0))$ for every i. Then, $\Psi(x, f)$ is differentiable at x_0 and*
$$D\Psi(x, f))(x_0) = \sum_i [\![D_u\Psi(x_0, f_i(x_0)) \cdot Df_i(x_0) + D_x\Psi(x_0, f_i(x_0))]\!] . \tag{27}$$

7.7. **Almgren's extrinsic maps.** The metric $\mathcal{G}$ on $\mathcal{A}_Q(\mathbb{R}^n)$ is "locally euclidean" at most of the points. Consider for instance the model case $Q = 2$ and a point $P = [\![P_1]\!] + [\![P_2]\!]$ with $P_1 \neq P_2$. Then, obviously, in a sufficiently small neighborhood of P, the metric space $\mathcal{A}_2(\mathbb{R}^n)$ is isometric to the Euclidean space $\mathbb{R}^{2n}$. This fails instead in any neighborhood of a point of type $P = 2\,[\![P_1]\!]$. On the other hand, if we restrict our attention to the closed subset $\{2\,[\![X]\!] : X \in \mathbb{R}^n\}$, we obtain the metric structure of $\mathbb{R}^n$.

A remarkable observation of Almgren is that $\mathcal{A}_Q(\mathbb{R}^n)$ is biLipschitz equivalent to a deformation retract of the Euclidean space (cf. [4, Section 1.3]). For a simple presentation of this fact we refer the reader to [19, Section 2.1].

Theorem 7.20 (Almgren's embedding and retraction)**.** *There exists $N = N(Q, n)$ and an injective $\boldsymbol{\xi} : \mathcal{A}_Q(\mathbb{R}^n) \to \mathbb{R}^N$ such that:*

(i) $\mathrm{Lip}(\boldsymbol{\xi}) \leq 1$;
(ii) *if* $\mathcal{Q} = \boldsymbol{\xi}(\mathcal{A}_Q)$, *then* $\mathrm{Lip}(\boldsymbol{\xi}^{-1}|_{\mathcal{Q}}) \leq C(n, Q)$.

Moreover there exists a Lipschitz map $\boldsymbol{\rho} : \mathbb{R}^N \to \mathcal{Q}$ *which is the identity on* $\mathcal{Q}$.

In fact much more can be said: the set $\mathcal{Q}$ is a cone and a polytope. On each separate face of the polytope the metric structure induced by $\mathcal{G}$ is euclidean, essentially for the reasons outlined a few paragraphs above (cf. again [4, Section 1.3] or [23, Section 6.1]). A simple, yet important, observation of White is that the map $\boldsymbol{\xi}$ can be easily constructed so that the Dirichlet energy of $\boldsymbol{\xi} \circ u$ (as clssical Euclidean map) coincides with that of u (as multivalued map) for any $u \in W^{1,2}$.

Sobolev maps were initially defined by Almgren using the map $\boldsymbol{\xi}$. With this artifact a lot of the theory outlined in the previous paragraphs can be recovered from the usual (single valued) theory using the maps $\boldsymbol{\xi}$ and $\boldsymbol{\rho}$. Presently such maps could be avoided for essentially all the arguments. However a more complicated version of the map $\boldsymbol{\rho}$ will play a rather important role at a certain point later. As already mentioned, for $Q > 1$ the space $\mathcal{A}_Q(\mathbb{R}^n)$ is not linear and we cannot regularize Q-valued maps by convolution. Nonetheless we will need a way to smooth $W^{1,2}$ maps suitably with a procedure which retains some of the basic estimates available for convolutions with a standard mollifier (in particular when computing the energy of the regularizations). A possible approach is to smooth the euclidean map $\boldsymbol{\xi} \circ u$ and then "project" it back onto $\mathcal{Q}$ using $\boldsymbol{\rho}$. However, projecting back might be rather costly in terms of the energy since the Lipschitz constant of $\boldsymbol{\rho}$ is indeed rather far from 1.

To bypass this problem, we follow Almgren and prove the existence of "almost" projections, denoted by $\boldsymbol{\rho}^\star_\delta$, which are $(1+\mu)$-Lipschitz in the δ-neighborhood of $\boldsymbol{\xi}(\mathcal{A}_Q(\mathbb{R}^n))$. These maps cannot be the identity on $\mathcal{Q}$, but they are at a uniform distance η from it. Almgren's original proof is rather complicated. In [23, Proposition 6.2] we have proposed a different proof which uses heavily Kirszbraun's extension theorem and seems to yield a better estimate of μ and η in terms of δ (in particular in the version of [23] these are suitable positive powers of δ).

Proposition 7.21 (Almost projection, cf. [23, Proposition 6.2])**.** *For every* $\bar{n}, Q \in \mathbb{N} \setminus \{0\}$ *there are geometric constants* $\delta_0, C > 0$ *with the following property. For every* $\delta \in]0, \delta_0[$ *there is* $\boldsymbol{\rho}^\star_\delta : \mathbb{R}^{N(Q,\bar{n})} \to \mathcal{Q} = \boldsymbol{\xi}(\mathcal{A}_Q(\mathbb{R}^{\bar{n}}))$ *such that* $|\boldsymbol{\rho}^\star_\delta(P) - P| \leq C\,\delta^{8^{-\bar{n}Q}}$ *for all* $P \in \mathcal{Q}$ *and, for every* $u \in W^{1,2}(\Omega, \mathbb{R}^N)$, *the following holds:*

$$\int |D(\boldsymbol{\rho}^\star_\delta \circ u)|^2 \leq \left(1 + C\,\delta^{8^{-\bar{n}Q-1}}\right) \int_{\{\mathrm{dist}(u,\mathcal{Q}) \leq \delta^{\bar{n}Q+1}\}} |Du|^2 + C \int_{\{\mathrm{dist}(u,\mathcal{Q}) > \delta^{\bar{n}Q+1}\}} |Du|^2 \,. \quad (28)$$

8. Dir-minimizers and their regularity in 2 dimensions

We are now ready to state the main results in the theory of Dir-minimizing maps. In what follows, Ω is always assumed to be a bounded open set with a sufficiently regular boundary.

Theorem 8.1 (Existence for the Dirichlet Problem, cf. [19, Theorem 0.8])**.** *Let $g \in W^{1,2}(\Omega; \mathcal{A}_Q)$. Then there exists a* Dir*-minimizing $f \in W^{1,2}(\Omega; \mathcal{A}_Q)$ such that $f|_{\partial\Omega} = g|_{\partial\Omega}$.*

Theorem 8.2 (Hölder regularity, cf. [19, Theorem 0.9])**.** *There is a positive constant $\alpha = \alpha(m, Q)$ with the following property. If $f \in W^{1,2}(\Omega; \mathcal{A}_Q)$ is* Dir*-minimizing, then $f \in C^{0,\alpha}(\Omega')$ for every $\Omega' \subset\subset \Omega \subset \mathbb{R}^m$. For two-dimensional domains, we have the explicit constant $\alpha(2, Q) = 1/Q$.*

For the second regularity theorem we need the definition of the singular set of f.

Definition 8.3 (Regular and singular points, cf. [19, Definition 0.10])**.** A Dir-minimizing f is regular at a point $x \in \Omega$ if there exists a neighborhood B of x and Q analytic functions $f_i : B \to \mathbb{R}^n$ such that

$$f(y) \;=\; \sum_i [\![f_i(y)]\!] \qquad \text{for every } y \in B \tag{29}$$

and either $f_i(y) \neq f_j(y)$ for every $y \in B$, or $f_i \equiv f_j$. The singular set $\mathrm{Sing}(f)$ is the complement of the set of regular points.

Theorem 8.4 (Estimate of the singular set, cf. [19, Theorem 0.11])**.** *Let f be* Dir*-minimizing. Then, the singular set* $\mathrm{Sing}(f)$ *is relatively closed in Ω. Moreover, if $m = 2$, then* $\mathrm{Sing}(f)$ *is at most countable, and if $m \geq 3$, then the Hausdorff dimension of* $\mathrm{Sing}(f)$ *is at most $m - 2$.*

Note in particular the striking similarity between the estimate of the size of the singular set in the case of multiple valued Dir-minimizers and in that of area minimizing currents. It will be discussed later that, even in the case of Dir-minimizers, there are singular solutions (which are no better than Hölder continuous).

Complete and self-contained proofs of these theorems can be found in [19]. The key tool for the estimate of the dimension of the singular set is the celebrated frequency function (cp. with [19, Section 3.4]), which has been indeed used in a variety of different contexts in the theory of unique continuation of elliptic partial differential equations (see for instance the papers [32], [33]). This is the central tool of our proofs as well. However, our arguments manage much more efficiently the technical intricacies of the problem and some aspects of the theory are developed in further details. For instance, we present in [19, Section 3.1] the Euler-Lagrange conditions derived from first variations in a rather general form. This is to our knowledge the first time that these conditions appear somewhere in this generality.

Largely following ideas of [10] and of White, we improved the second regularity theorem to the following optimal statement for planar maps.

Theorem 8.5 (Improved 2-dimensional estimate, cf. [19, Theorem 0.12])**.** *Let f be* Dir*-minimizing and $m = 2$. Then* $\mathrm{Sing}(f)$ *is discrete.*

This result was announced in [10]. However, to our knowledge the proof has never appeared before [19]. We will discuss the proof of Theorem 8.2 in the case $m = 2$ and that of Theorem 8.5, since these are the two facts which are really relevant for this survey. For Theorem 8.4 and Theorem 8.2 with $m \geq 3$ we refer instead to the survey article [17].

8.1. **Regular selections in 1 dimension.** If the domain is a 1-dimensional interval continuous, Hölder, Lipschitz and Sobolev multivalued maps have always correspondingly regular selections: indeed there is a linear bound relating the regularity of the selection to that of the initial map in all these cases. For the case of Sobolev and Lipschitz maps the proof is very elementary, cf. [19, Proposition 1.2]. For continuous and Hölder maps the proof turns out to be much harder, cf. [4, Proposition 1.10] and the simpler (and more general) approach of [18]. The latter approach has been extended in [36] to cover also a large class of fractional Sobolev regularity.

We record here the 1-dimensional selection theorem which will be mostly needed in this note. If $I = [a, b]$ is a closed bounded interval of $\mathbb{R}$ we will denote by $AC(I, \mathcal{A}_Q)$ the space of absolutely continuous function in its classical meaning: $f : I \to \mathcal{A}_Q$ is absolutely continuous if, for every $\varepsilon > 0$, there exists $\delta > 0$ with the following property: for every $a \leq t_1 < t_2 < ... < t_{2N} \leq b$,

$$\sum_i (t_{2i} - t_{2i-1}) < \delta \quad \text{implies} \quad \sum_i \mathcal{G}\big(f(t_{2i}), f(t_{2i-1})\big) < \varepsilon.$$

Proposition 8.6 (Cf. [19, Proposition 1.2])**.** *Let $I = [a, b] \subset \mathbb{R}$ and $f \in W^{1,p}(I, \mathcal{A}_Q)$ for some $p \in [1, \infty]$. Then,*

- (a) *$f \in AC(I, \mathcal{A}_Q)$ and, moreover, $f \in C^{0,1-\frac{1}{p}}(I, \mathcal{A}_Q)$ for $p > 1$;*
- (b) *there exists a selection $f_1, \ldots, f_Q \in W^{1,p}(I, \mathbb{R}^n)$ of f such that $|Df_i| \leq |Df|$ almost everywhere.*

Proposition 8.6 cannot be extended to maps $f \in W^{1,p}(\mathbb{S}^1, \mathcal{A}_Q)$. For instance, identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ and $\mathbb{S}^1$ with the set $\{z \in \mathbb{C} : |z| = 1\}$ and consider the map $f : \mathbb{S}^1 \to \mathcal{A}_Q(\mathbb{R}^2)$ given by $f(z) = \sum_{\zeta^2 = z} [\![\zeta]\!]$. Then, f is Lipschitz (and hence belongs to $W^{1,p}$ for every p) but it does not have a continuous selection. In the rest of the note we will often use this identification of $\mathbb{R}^2$ with complex plane and of $\mathbb{S}^1$ with the unitary complex numbers. Although Proposition 8.6 cannot be extended to maps over $\mathbb{S}^1$, we will use it to write any $f \in W^{1,p}(\mathbb{S}^1, \mathcal{A}_Q)$ into a superposition of "irreducible pieces" which wind around the origin. This decomposition will play a fundamental role throughout the rest of our discussions.

Definition 8.7. $f \in W^{1,p}(\mathbb{S}^1, \mathcal{A}_Q)$ is called *irreducible* if there is no decomposition of f into 2 simpler $W^{1,p}$ functions.

Proposition 8.8. *For every Q-function $g \in W^{1,p}(\mathbb{S}^1, \mathcal{A}_Q(\mathbb{R}^n))$, there exists a decomposition $g = \sum_{j=1}^J [\![g_j]\!]$, where each g_j is an irreducible $W^{1,p}$ map. A function g is irreducible if and only if*

- (i) *card$(\mathrm{spt}(g(z))) = Q$ for every $z \in \mathbb{S}^1$ and*
- (ii) *there exists a $W^{1,p}$ map $h : \mathbb{S}^1 \to \mathbb{R}^n$ with the property that $f(z) = \sum_{\zeta^Q = z} [\![h(\zeta)]\!]$.*

Moreover, for every irreducible g, there are exactly Q maps h fulfilling (ii).

The existence of an irreducible decomposition in the sense above is an obvious consequence of the definition of irreducible maps. The interesting part of the proposition is the characterization of the irreducible pieces, a direct corollary of Proposition 8.6.

8.2. **Centered harmonic extensions.** The decomposition of $g \in W^{1,p}(\mathbb{S}^1, \mathcal{A}_Q(\mathbb{R}^n))$ in irreducible pieces allows to define a major player in the regularity theory for 2-dimensional area minimizing currents. For convenience, we introduce the symbol $\mathbb{D}$ for the unitary disk of $\mathbb{C}$ centered at the origin (therefore, with our notation $\mathbb{S}^1 = \partial\mathbb{D}$). Let $g = \sum_{j=1}^J [\![g_j]\!]$ be a decomposition into irreducible k_j-functions as in Proposition 8.8. Consider, moreover, the $W^{1,p}$ functions $\gamma_j : \mathbb{S}^1 \to \mathbb{R}^n$ "unrolling" the g_j as in Proposition 8.8 (*ii*):

$$g_j(x) = \sum_{z^{k_j}=x} [\![\gamma_j(z)]\!] .$$

We take the harmonic extension ζ_l of γ_l in $\mathbb{D}$, and consider the k_l-valued functions f_l obtained "rolling" back the ζ_l: $f_l(x) = \sum_{z^{k_l}=x} [\![\zeta_l(z)]\!]$. If $g \in W^{1,2}$, then it is easy to check that the Q-function $\tilde{f} = \sum_{l=1}^J [\![f_l]\!]$ is in $W^{1,2}(\mathbb{D})$. From now on such an $\tilde{f}$ will be called the *centered harmonic extension* of g. We caution the reader abot one important nontrivial fact.

Remark 8.9. Centered harmonic extensions are mostly not Dir-minimizing. It is indeed easy to give examples of g for which the Dir-minimizer has more than one singularity (or for which 0 is a regular point) and any centered harmonic extension has an isolated singularity at the origin. For instance observe that, if $\mathrm{spt}(g(p))$ has maximal cardinality Q for every p and $Q \geq 2$, then its centered harmonic extension is always singular in 0.

It is rather easy to estimate the energy of the centered harmonic extension, for the simple reason that via the conformal transformations $z \mapsto z^Q$ and $z \mapsto z^{1/Q}$ (the latter defined only locally!) this can be reduced to an estimate for classical harmonic functions. In order to carry on some important computations, which will be useful also later, we introduce the following notation. If $\partial B_r(x)$ is the boundary of some disk $B_r(x) \subset \mathbb{R}^2$, we then denote by

- ν the exterior unit normal to $\partial B_r(x)$;
- τ the unit tangent vector field to $\partial B_r(x)$ orienting it counterclockwise.

Consistently, the notation $\partial_\tau f$ and $\partial_\nu f$ will be used for the multiple valued maps

$$p \mapsto \sum_i [\![Df_i(p) \cdot \tau]\!] \qquad \text{and}$$
$$p \mapsto \sum_i [\![Df_i(p) \cdot \nu]\!] ,$$

when such objects are defined (for instance if f is approximately differentiable at p).

A first crude estimate for the energy of the centered harmonic extension is the following (cf. [19, Section 3.2.]).

Proposition 8.10. *If h is a centered harmonic extension of $g : (\mathbb{S}^1) \to \mathcal{A}_Q(\mathbb{R}^n)$ to $\mathbb{D}$, then*

$$\mathrm{Dir}(h, \mathbb{D}) \leq Q \int_{\mathbb{S}^1} |\partial_\tau g|^2 . \tag{30}$$

The latter easily gives the estimate

$$\mathrm{Dir}(f, B_r) \leq Qr \int_{\partial B_r} |Df|^2 = Qr \frac{d}{dr}(\mathrm{Dir}(f, B_r)) , \tag{31}$$

for any Dir-minimizing function. Integrating the differential inequality above we conclude the decay estimate

$$\mathrm{Dir}(f, B_r) \leq C r^{1/Q} .$$

Using Proposition 7.14 we then conclude that f is $\frac{1}{2Q}$-Hölder. Indeed the inequality (31) can be improved using the equipartition of the energy (which will be explained thoroughly Section 8.4 below) to

$$\mathrm{Dir}(f, B_r) \leq \frac{Q}{2} r \frac{d}{dr}(\mathrm{Dir}(f, B_r)) , \tag{32}$$

which allows to conclude the local $\frac{1}{Q}$-Hölder continuity of f. Notably, the latter exponent is optimal!

8.3. **First variations.** There are two natural types of variations that can be used to perturb Dir-minimizing Q-valued functions. The first ones, which we call inner variations, are generated by right compositions with diffeomorphisms of the domain. The second, which we call outer variations, correspond to "left compositions". More precisely, let f be a Dir-minimizing Q-valued map.

(IV) Given $\varphi \in C^\infty_c(\Omega, \mathbb{R}^m)$, for ε sufficiently small, $x \mapsto \Phi_\varepsilon(x) = x + \varepsilon\varphi(x)$ is a diffeomorphism of Ω which leaves $\partial\Omega$ fixed. Therefore,

$$0 = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int_\Omega |D(f \circ \Phi_\varepsilon)|^2. \tag{33}$$

(OV) Given $\psi \in C^\infty(\Omega \times \mathbb{R}^n, \mathbb{R}^n)$ such that $\mathrm{spt}(\psi) \subset \Omega' \times \mathbb{R}^n$ for some $\Omega' \subset\subset \Omega$, we set $\Psi_\varepsilon(x) = \sum_i [\![f_i(x) + \varepsilon\psi(x, f_i(x))]\!]$ and derive

$$0 = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int_\Omega |D\,\Psi_\varepsilon|^2. \tag{34}$$

Using the multivalued chain rules we can turn the conditions (33) and (34) into the following identities, which we state in the case of m-dimensional domains:

Proposition 8.11 (First variations. cf. [19, Proposition 3.1])**.** *Let $f : \Omega \to \mathcal{A}_Q(\mathbb{R}^n)$ be* Dir*-minimizing. For every $\varphi \in C^\infty_c(\Omega, \mathbb{R}^m)$, we have*

$$2 \int \sum_i \big\langle Df_i : Df_i \cdot D\varphi \big\rangle - \int |Df|^2 \,\mathrm{div}\, \varphi = 0. \tag{35}$$

For every $\psi \in C^\infty(\Omega_x \times \mathbb{R}^n_u, \mathbb{R}^n)$ such that

$$\mathrm{spt}(\psi) \subset \Omega' \times \mathbb{R}^n \quad \textit{for some } \Omega' \subset\subset \Omega,$$

and

$$|D_u\psi| \leq C < \infty \quad \textit{and} \quad |\psi| + |D_x\psi| \leq C\,(1 + |u|) \ , \tag{36}$$

we have

$$\int \sum_i \left\langle Df_i(x) : D_x\psi(x, f_i(x))\right\rangle dx + \int \sum_i \left\langle Df_i(x) : D_u\psi(x, f_i(x)) \cdot Df_i(x)\right\rangle dx = 0. \tag{37}$$

8.4. **Equipartition of energy and integration by parts formulae.** (35) and (36) give particularly interesting identities when tested with functions which depend on $|x|$. The following proposition gives the relevant identities when we test with the singular functions $\varphi(y) = \mathbf{1}_{B_r(x)}(y)y$ and $\psi(x, u) = u\mathbf{1}_{B_r(x)}(y)$ (the proof follows from a standard regularization of these φ and ψ).

Proposition 8.12 (cf. [19, Proposition 3.1])**.** *Let $x \in \Omega$ and $f : \Omega \to \mathcal{A}_Q(\mathbb{R}^n)$ be* Dir*-minimizing. Then, for a.e. $0 < r < \operatorname{dist}(x, \partial\Omega)$, we have*

$$(m-2)\int_{B_r(x)} |Df|^2 = r\int_{\partial B_r(x)} |Df|^2 - 2\,r\int_{\partial B_r(x)} |\partial_\nu f|^2, \tag{38}$$

$$\int_{B_r(x)} |Df|^2 = \int_{\partial B_r(x)} \sum_i \langle \partial_\nu f_i, f_i\rangle. \tag{39}$$

Observe that in the case $m = 2$, the second identity (39) becomes the classical "equipartition of energy" of the trace of an harmonic function on any circle, namely

$$\int_{\partial B_r(x)} |\partial_\nu f|^2 = \int_{\partial B_r(x)} |\partial_\tau f|^2 = \frac{1}{2}\int_{\partial B_r(x)} |Du|^2\,. \tag{40}$$

In particular this identity allows to conclude the "improved" differential inequality (31) from the crude estimate (30).

9. Monotonicity and decay of the frequency function

We next introduce Almgren's frequency function and state his celebrated monotonicity estimate, which is a straightforward consequence of the identities (38) and (39). Recall the notation $|f|$ for the function $\mathcal{G}(f, Q\,[\![0]\!])$.

Definition 9.1 (The frequency function, cf. [19, Definition 3.13])**.** Let f be a Dir-minimizing function, $x \in \Omega$ and $0 < r < \operatorname{dist}(x, \partial\Omega)$. We define the functions

$$D_{x,f}(r) = \int_{B_r(x)} |Df|^2, \quad H_{x,f}(r) = \int_{\partial B_r} |f|^2 \quad \text{and} \quad I_{x,f}(r) = \frac{r D_{x,f}(r)}{H_{x,f}(r)}. \tag{41}$$

$I_{x,f}$ is called the *frequency function*.

When x and f are clear from the context, we will often use the shorthand notation $D(r)$, $H(r)$ and $I(r)$.

Theorem 9.2 (Monotonicity of the frequency function, cf. [19, Theorem 3.15])**.** *Let f be* Dir*-minimizing and $x \in \Omega$. Either there exists $\varrho > 0$ such that $f|_{B_\varrho(x)} \equiv 0$ or $I_{x,f}(r)$ is an absolutely continuous nondecreasing positive function on $]0, \operatorname{dist}(x, \partial\Omega)[$. This function takes a constant value α if and only if $f(y)$ is α-homogeneous in $y - x$.*

This monotonicity is the main ingredient in the proof of Theorem 8.4. However, the "improved" regularity statement in the 2-dimensional case, namely Theorem 8.5, relies on a further feature of the frequency function, whose validity in higher dimensions is still an open question, namely the power-law decay of $I_{x,r}(f) - \lim_{\rho\to 0} I_{x,r}(f)$. In this section we discuss the effects of the monotonicity of I (for simplicity only in the case $m = 2$), the decay property and the proof of Theorem 8.5.

9.1. **Proof of the monotonicity for $m = 2$.** We start by assuming $x = 0$ and observing that

$$D'(r) = \int_{\partial B_r} |Df|^2 . \tag{42}$$

A slightly more subtle computation gives

$$\begin{aligned} H'(r) =& \frac{d}{dr}\left(r\int_{\partial B_1} |f(rx)|^2\,dx\right) = \int_{\partial B_1} |f(rx)|^2\,dx + 2r\int \sum_i \langle Df_i(rx)\cdot x, f_i(x)\rangle \\ =& \frac{H(r)}{r} + 2\int_{\partial B_1} \sum_i \langle \partial_\nu f_i, f_i\rangle . \end{aligned} \tag{43}$$

In particular we conclude that

$$\begin{aligned} I'(r) =& \frac{rD'(r)}{H(r)} + \frac{D(r)}{H(r)} - \frac{rH'(r)D(r)}{H(r)^2} = \frac{rD'(r)}{H(r)} - 2\frac{D(r)}{H(r)^2}\int_{\partial B_r} \sum_i \langle \partial_\nu f_i, f_i\rangle \\ =& \frac{rD'(r)}{H(r)} - \frac{2}{r} I(r)^2 , \end{aligned} \tag{44}$$

where in the last line we have used (39). Thus, we conclude also that

$$I'(r) = r\frac{D'(r)H(r) - 2D(r)^2}{H(r)^2} .$$

On the other hand using (39), (40) and Cauchy-Schwartz we also conclude

$$D'(r)H(r) - 2D(r)^2 = 2\int_{\partial B_r} |\partial_\nu f|^2 \int_{\partial B_r} |f|^2 - 2\left(\int \sum_i \langle \partial_\nu f_i, f_i\rangle\right)^2 \geq 0 .$$

9.2. **The first fundamental consequence of the monotonicity formula.** Theorem 9.2 has two crucial consequences, when "blowing-up" a given Dir-minimizing function. More precisely, consider a Dir-minimizing f taking $Q > 1$ values and a point p in its domain. Without loss of generality we can assume that $p = 0$. If the support of $f(0)$ contains two different points, then, by continuity, in a neighborhood U of 0 f splits into two separate functions u_1 and u_2 which are both $W^{1,2}$ and continuous. It is simple to see that both must be minimizers of the Dirichlet energy in U. 0 is then a good point, where we have reduced the complexity of the problem. For instance, if Q were 2 we would know that u_1 and u_2 are two classical (single valued) harmonic functions. The "problematic points" are then those p where $f(p) = Q\,[\![q]\!]$.

We can therefore assume that $f(0) = Q [\![q]\!]$ for some $q \in \mathbb{R}^n$. Now, according to our definition of the singular set $\mathrm{Sing}(f)$, we have two possibilities:

(a) f equals Q copies of a classical harmonic function in a neighborhood of 0;
(b) 0 is a singular point for f.

In general, an interesting object to look at is the average of the sheets of $f = \sum_i [\![f_i]\!]$, namely $\frac{1}{Q}\sum_i f_i$. For this average we fix the notation $\boldsymbol{\eta} \circ f$. It is not difficult to see that $\boldsymbol{\eta} \circ f$ is a classical harmonic function. Indeed, if we define

$$\bar{f} := \sum_i [\![f_i - \boldsymbol{\eta} \circ f]\!] \,,$$

it is immediate to see that $\mathrm{Dir}(f) = \mathrm{Dir}(\bar{f}) + Q\mathrm{Dir}(\boldsymbol{\eta} \circ f)$. In particular it is not difficult to conclude that $\bar{f}$ is also a Dir-minimizer, cf. [19, Lemma 3.23]. Looking at the latter function we can thus restate the alternative as: either $\bar{f} \equiv Q [\![0]\!]$ in a neighborhood of the origin, or 0 is a singular point for $\bar{f}$ (and thus a singular point of f!).

The discussion above leads to the consideration that, without loss of generality, we can assume $\boldsymbol{\eta} \circ f \equiv 0$. Assume further that the (more interesting!) alternative (b) above holds. Then f does not vanish identically and therefore both $D_{0,f}(r)$ and $H_{0,f}(r)$ are positive for some r. Using Theorem 8.2 it is not difficult to see that, under the assumption $f(0) = Q [\![0]\!]$, we have a uniform bound of the form

$$H_{0,f}(r) \leq C r D_{0,f}(r) \qquad\qquad \forall r \in \left]0, \tfrac{\mathrm{dist}(0,\partial\Omega)}{2}\right[\,, \tag{45}$$

where the constant C is independent of f. The obvious consequence of Theorem 9.2 is that there is also a reverse control

$$r D_{0,f}(r) \leq \bar{C} H_{0,f}(r) \tag{46}$$

although the latter constant $\bar{C}$ depends upon the point (0 in this case) and the function f. Indeed such constant approaches, for $r \downarrow 0$, the limit $I_0(f) := \lim_{\rho\downarrow 0} I_{0,\rho}(f)$, which by (45) is bounded away from 0 and by Theorem 9.2 is finite: on the other hand we have no explicit (neither universal!) upper bound: we insist that $I_0(f)$ depends upon f and the particular point (0 in this case) where we are "blowing-up".

Consider now the rescaled functions $f_{0,r}(x) := f(rx)$ and their renormalized versions

$$u_{0,r}(x) := \frac{f_{0,r}}{\mathrm{Dir}(f_{0,r}, \mathbb{D})^{1/2}} \,. \tag{47}$$

In particular the energy of $u_{0,r}$ is 1 on the disk $\mathbb{D}$. However the L^2 norm of $|u_{0,r}|$ is also under control because of (45). We then have compactness for the family $\{u_{0,r}\}_r$. Fix a map $\bar{u}$ which is the limit of any subsequence u_{0,r_k} with $r_k \downarrow 0$. Such a map will be called, from now on, a "tangent function". It is not difficult to see that a sequence of minimizers with such uniform controls converge *strongly* in $W^{1,2}$ in any compact subset: namely the Dirichlet energy of the limiting function is the limit of the Dirichlet energy of the corresponding functions on any subdomain Ω which is compactly contained in $\mathbb{D}$, cf. [19, Proposition 3.20]. However the minimizing property alone does not guarantee strong convergence on the *whole* domain $\mathbb{D}$.

To understand the latter statement, consider for instance the planar (single valued!) harmonic functions

$$f_k(x_1, x_2) = \mathrm{Re}\,(x_1 + ix_2)^k$$

and their normalizations

$$u_k := f_k/\mathrm{Dir}(f_k, B_1(0))\,.$$

It is very elementary to see that u_k converges to 0 in $\mathbb{D}$: in fact most of the Dirichlet energy of u_k lies in a thin layer around the boundary $\partial\mathbb{D}$. For k large the layer becomes thinner and thinner and all the energy is "pushed" towards the boundary $\partial\mathbb{D}$. On the other hand it is easy to see that the ratio

$$\frac{D_{0,u_k}(1)}{H_{0,u_k}(1)} = \frac{1}{H_{0,u_k}(1)}$$

explodes, namely that the L^2 norm of u_k on $\partial\mathbb{D}$ converges to 0.

This highlights the first important consequence of the frequency function: the "reverse Poincaré" inequality (46) excludes that the energy of $u_{0,r}$ concentrates towards the boundary. Any limit $\bar{u}$ of a sequence u_{0,r_k} (i.e. any tangent function) must therefore have energy equal to 1. Since Theorem 8.2 guarantees uniform convergence, we also conclude that $\bar{u}(0) = Q\,[\![0]\!]$. Moreover, $\boldsymbol{\eta}\circ\bar{u} \equiv 0$ because $\boldsymbol{\eta}\circ u_{0,r} \equiv 0$.

Thus 0 must be a singular point of u as well: the only way $\bar{u}$ could be regular around 0 would be to take the value $Q\,[\![0]\!]$ identically in a neighborhood of 0. However notice that $I_{0,\bar{u}}(r) = I_{0,f}(0) =: \alpha$ for every r. But then Theorem 9.2 implies that $\bar{u}$ is α-homogeneous, and if $\bar{u}$ would vanish in a neighborhood of 0, then it would vanish on the entire disk $\mathbb{D}$, contradicting the fact that the Dirichlet energy of u is indeed 1.

The conclusion is that the singularity has *persisted* in the limit. Recalling that our main concern in proving Theorem 4.4 was the disappearance of singular points along sequences of converging currents, the reader will understand why the monotonicity of the frequency function is such an exciting discovery. It must also be noticed that the monotonicity of the frequency function was unknown even for classical single valued harmonic functions before [4]: the shear observation that Almgren was able to discover a new fundamental fact for classical harmonic functions around 1970 gives in my opinion the true measure of his genius.

9.3. **The second fundamental consequence of the monotonicity formula.** The second fundamental consequence of the monotonicity of the frequency function is that $I_{0,\bar{u}}(r)$ is indeed constant in r and equals $\alpha := I_{0,f}(0)$, which, as already noticed, gives that $\bar{u}$ is α-homogeneous. In particular when the domain is 2-dimensional, it is not difficult to classify all α-homogeneous Dir-minimizers and to show that their only singularity is at the origin, cf. [19, Proposition 5.1],

Lemma 9.3. *If $\bar{u} : \mathbb{R}^2 \to \mathcal{A}_Q(\mathbb{R}^n)$ is locally* Dir*-minimizing, α-homogeneous and $\boldsymbol{\eta}\circ\bar{u} = 0$, the either $u \equiv Q\,[\![0]\!]$ or 0 is the only point at which $\bar{u}$ takes the value $Q\,[\![0]\!]$.*

Let us look at this statement a bit more closely. The homogeneity implies that $\bar{u}$ takes the form

$$\bar{u}(x) = |x|^\alpha g\left(\frac{x}{|x|}\right) \qquad \text{for } x \neq 0,$$

where $g : \mathbb{S}^1 \to \mathcal{A}_Q(\mathbb{R}^n)$ is then the trace of $\bar{u}$ at $\mathbb{S}^1 = \partial\mathbb{D}$. Assume for simplicity that $Q = 2$. We then want to show that 0 is the only point at which $\bar{u}$ takes the value $2\,[\![0]\!]$, unless $\bar{u}$ is the trivial function identically equal to $2\,[\![0]\!]$. If $\mathrm{spt}(g(p))$ consists of 2 distinct points for each $p \in \mathbb{S}^1$, then we are done. If not, then there is a point p_0 where $g(p_0) = 2\,[\![0]\!]$ (recall that $\boldsymbol{\eta}\circ\bar{u} \equiv 0$!). On the other hand, unless $\bar{u}$ is trivial, there must be a point p where $g(p) = [\![P_1]\!] + [\![P_2]\!]$ for some $P_1 \neq P_2$. Now notice that the presence of the "double point" p_0 and the "simple point" p imply the existence of two distinct "regular representations" for g:

(a) There are two distinct $W^{1,2}$ functions $g_1, g_2 : \mathbb{S}^1 \to \mathbb{R}^n$ such that $g(z) = [\![g_1(z)]\!] + [\![g_2(z)]\!]$ $\forall z \in \mathbb{S}^1$;
(b1) There is a $W^{1,2}$ function $\gamma : \mathbb{S}^1 \to \mathbb{R}^n$ such that $g(z) = \sum_{w^2=z} \gamma(w)$, where again we are identifiying points $z, w \in \mathbb{R}^2$ with complex numbers.

This has a rather interesting consequence. We pass to polar coordinates on $\mathbb{R}^2$ and by the Dir-minimality of $\bar{u}$ we conclude that

(a1) The maps $h_i(\theta,\rho) := \rho^\alpha g_i(\theta)$ are both harmonic functions, and indeed $h_1 = -h_2$;
(b1) The map $\zeta(\theta,\rho) := \rho^{2\alpha}\gamma(\theta)$ is an harmonic function and $\zeta(\theta,\rho) = -\zeta(\theta+2\pi,\rho)$;
(c1) For each (θ,ρ) we must have

$$f(\theta,\rho) = [\![g_1(\theta,\rho)]\!] + [\![g_2(\theta,\rho)]\!] = \left[\!\!\left[\zeta(\tfrac{\theta}{2},\rho^{1/2})\right]\!\!\right] + \left[\!\!\left[\zeta(\tfrac{\theta}{2}+\pi,\rho^{1/2})\right]\!\!\right] .$$

However, it is rather easy to see that (a1), (b1) and (c1) are not compatible, except for the trivial case $f \equiv 2\,[\![0]\!]$.

In fact the analysis above can be refined to give a classification-type result for tangent functions $\bar{u}$.

Proposition 9.4 (Cf. [19, Proposition 5.1])**.** *Let $f : \mathbb{D} \to \mathcal{A}_Q(\mathbb{R}^n)$ be a nontrivial, α-homogeneous function which is* Dir*-minimizing. Assume in addition that $\boldsymbol{\eta}\circ f = 0$. Then,*

(a) $\alpha = \frac{n^*}{Q^*} \in \mathbb{Q}$, *with* $\mathrm{MCD}\,(n^*, Q^*) = 1$*;*
(b) there exist injective ($\mathbb{R}$*-)linear maps $L_j : \mathbb{C} \to \mathbb{R}^n$ and $k_j \in \mathbb{N}$ such that*

$$f(x) = k_0\,[\![0]\!] + \sum_{j=1}^{J} k_j \sum_{z^{Q^*}=x} \left[\!\left[L_j \cdot z^{n^*}\right]\!\right] =: k_0\,[\![0]\!] + \sum_{j=1}^{J} k_j\,[\![f_j(x)]\!]\,. \tag{48}$$

Moreover, $J \geq 1$ and $k_j \geq 1$ for all $j \geq 1$. If $Q^* = 1$, either $J \geq 2$ or $k_0 > 0$.
(c) For any $i \neq j$ and any $x \neq 0$, the supports of $f_i(x)$ and $f_j(x)$ are disjoint.

9.4. **Proof of Theorem 8.4 for** $m = 2$**.** With Lemma 9.3 at hand, the proof of Theorem 8.4 for $m = 2$ is rather easy. Take a Dir-minimizing Q-valued function f and assume without loss of generality that its domain Ω is connected. We claim now that, either $f = Q\,[\![\boldsymbol{\eta}\circ f]\!]$ on the disk, or

$$\Sigma_Q := \{x\in\Omega : f(x) = Q\,[\![\boldsymbol{\eta}\circ f(x)]\!]$$

is discrete. Without loss of generality we can assume that $\boldsymbol{\eta}\circ f \equiv 0$, since as already observed we can subtract the average from every sheet. In this case the claim reduces to showing that the set $\Sigma_Q := \{x : f(x) = Q\,[\![0]\!]\}$ is discrete. Fix a point $p\in\Sigma_Q$: we will indeed show that if p is the limit of a sequence $\{p_j\}\subset\Sigma_Q\setminus\{p\}$, then f must necessarily vanish identically in a neighborhood of p: then the original claim would be a simple consequence of the connectedness of Ω.

Fix thus such a p and without loss of generality assume it is the origin. If f does not vanish identically on any $B_r(0)$, then obviously $D(r) > 0$ for every r and we can apply Theorem 9.2 and the discussion of the previous sections. In particular, if we fix a sequence of radii $r_j := 2|p_j|$ and consider the corresponding blow-up sequence $\{u_{0,r_j}\}$, we conclude that such a sequence converges to a nontrivial tangent function $\bar u$: nontrivial in the sense that the energy must be 1 on $\mathbb{D}$ and thus $\bar u$ cannot vanish identically. Hence, by Lemma 9.3, the origin is the only point p such that $\bar u(p) = 0$. On the other hand, up to extraction of a subsequence, we can assume that p_j/r_j converges to some $\bar p$ with $|\bar p| = \frac{1}{2}$. But then, since the u_{0,r_j} converge uniformly to $\bar u$ on compact sets and $u_{0,r_j}(p_j) = Q\,[\![0]\!]$, we must have $u(\bar p) = Q\,[\![0]\!]$, which is a contradiction.

At this point the proof of Theorem 8.4 in the case $m = 2$ is essentially complete. Indeed, by the Hölder continuity of Dir-minimizers, in $\Omega\setminus\Sigma_Q$ f splits into simpler Dir-minimizing functions and we can prove the claim of Theorem 8.4 by induction over Q. For instance, the argument above suffices to show that the singularities of a planar 2-valued function are isolated, since on the open set $\Omega\setminus\Sigma_2$ the function is locally the superposition of two classical harmonic functions which assume nowhere the same value. So let $Q = 3$. In this case, for each point $p\in\Omega\setminus\Sigma_3$ there is a neighborhood U_p where $u = u' + u''$ where u'' is 2-valued, u' is 1-valued and $\mathrm{spt}(u'(q))\cap\mathrm{spt}(u''(q)) = \emptyset$ for every $q\in U_p$. We thus conclude that the singular set of u is discrete in $\Omega\setminus\Sigma_3$ and hence countable. We then conclude similarly by induction over Q.

However, as soon as $Q\geq 3$ the latter argument does not imply the discreteness of $\mathrm{Sing}(f)$. In particular it does not exclude the following model situation:

- Σ_3 consists of a single point, say 0.
- There is a sequence of points $\{p_j\}\subset\Omega\setminus\{0\}$ such that $p_j\to p$ and the cardinality of $\mathrm{spt}(u(p_j))$ equals 2 for every j. Namely $u(p_j) = 2\,[\![P_j]\!] + [\![P'_j]\!]$ for some $P_j\neq P'_j$;
- $\mathrm{spt}(u(p))$ has cardinality 3 for every $p\notin\{0\}\cup\{p_j : j\in\mathbb{N}\}$, namely $\mathrm{Sing}(f) = \{0\}\cup\{p_j : j\in\mathbb{N}\}$.

In order to prove Theorem 8.5 we need to exclude that such a distribution of singularities might occur for a Dir-minimizer, namely that "singularities with lower multiplicities" cannot accumulate towards "a singularity with higher multiplicity".

9.5. **The decay of the frequency function and uniqueness of tangent functions.** The reason why Theorem 8.4 can be improved to Theorem 8.5 in 2 dimension is that it that the frequency function converges to its limit in a power-law fashion. More precisely we have the following

Proposition 9.5 (Cf. [19, Proposition 5.2]). *Let $f \in W^{1,2}(\mathbb{D}, \mathcal{A}_Q)$ be* Dir*-minimizing, with* $\mathrm{Dir}(f, \mathbb{D}) > 0$ *and set $\alpha = I_{0,f}(0) = I(0)$. Then, there exist constants $\gamma > 0$, $C > 0$, $H_0 > 0$ and $D_0 > 0$ such that, for every $0 < r \leq 1$,*

$$0 \leq I(r) - \alpha \leq C\, r^\gamma, \tag{49}$$

$$0 \leq \frac{H(r)}{r^{2\alpha+1}} - H_0 \leq C\, r^\gamma \quad \textit{and} \quad 0 \leq \frac{D(r)}{r^{2\alpha}} - D_0 \leq C\, r^\gamma. \tag{50}$$

Although the proof of this result follows computations similar to those of [10], such a statement for Dir-minimizer has appeared in [19] for the first time. A simple corollary of (49) and (50) is the uniqueness of tangent functions.

Theorem 9.6 (Cf. [19, Theorem 5.3]). *Let $f : \mathbb{D} \to \mathcal{A}_Q(\mathbb{R}^n)$ be a* Dir*-minimizing Q-valued functions, with* $\mathrm{Dir}(f, \mathbb{D}) > 0$ *and $f(0) = Q\,[\![0]\!]$. Then, there exists a unique tangent map $\bar{u}$ to f at 0 (i.e. the maps $u_{0,\rho}$ defined in (47) converge locally uniformly to $\bar{u}$).*

The reason for the decay in Proposition 9.5 plays a primary role in the proof of Theorem 4.4 as well and it is achieved integrating the following differential inequality

$$I'(r) \geq \frac{\gamma}{r}\left(I(r) - \alpha\right). \tag{51}$$

Recalling (44) the latter is equivalent to

$$r D'(r) \geq (2\gamma + 4\alpha) D(r) - \frac{2\alpha(\alpha+\gamma}{r} H(r)\,.$$

By a simple scaling argument we can assume $r = 1$. Moreover, by (40) we are left with the inequality

$$(\gamma + 2\alpha) D(1) \leq \alpha(\alpha+\gamma) H(1) + \int_{\mathbb{S}^1} |\partial_\tau f|^2\,. \tag{52}$$

In order to show the latter inequality we will use the centered harmonic extension. For simplicity we assume that $f|_{\mathbb{S}^1}$ is irreducible. So, for some integer smaller or equal to Q, which by abuse of notation we keep denoting Q,

$$f(\theta, 1) = \sum_{j=1}^{Q} \left[\!\left[\gamma\left(\tfrac{\theta + 2\pi j}{Q}\right) \right]\!\right]$$

where we are using polar coordinates (θ, r) and $\gamma : \mathbb{S}^1 \to \mathbb{R}^n$ is a $W^{1,2}$ (classical!) function.

We might expand γ in Fourier series and give the corresponding representation for its harmonic extension:

$$\gamma(\theta) = \frac{a_0}{2} + \sum_{\ell=1}^{\infty} (a_\ell \cos \ell\theta + b_\ell \sin \ell\theta)\,,$$

$$\zeta(\theta,\rho) = \frac{a_0}{2} + \sum_{\ell=1}^{\infty} \rho^\ell (a_\ell \cos \ell\theta + b_\ell \sin \ell\theta)\,.$$

The centered harmonic extension of f to $\mathbb{D}$ is then the map

$$g(\theta,\rho) = \frac{a_0}{2} + \sum_{j=1}^{Q} \left[\!\left[\zeta\left(\tfrac{\theta+2\pi j}{Q}, \rho^{1/Q}\right)\right]\!\right]$$

and straightforward computations give

$$D(1) \leq \mathrm{Dir}(g,\mathbb{D}) = \pi \sum_\ell \ell(a_\ell^2 + b_\ell^2) \tag{53}$$

$$H(1) = Q\pi\frac{a_0^2}{2} + Q\pi \sum_\ell (a_\ell^2 + b_\ell^2) \tag{54}$$

$$\int_{\mathbb{S}}^1 |\partial_\tau f|^2 = \frac{\pi}{Q} \sum_\ell \ell^2 (a_\ell^2 + b_\ell^2)\,. \tag{55}$$

It then follows that, having fixed $\alpha > 0$ and $Q \in \mathbb{N}$, the inequality (52) certainly holds if $\gamma > 0$ satisfies the inequality

$$(2\alpha + \gamma)\, \ell \leq \frac{\ell^2}{Q} + \alpha\,(\alpha+\gamma)\,Q, \quad \text{for every } \ell \in \mathbb{N}\setminus\{0\}\ .$$

The latter can be rewritten as

$$\gamma\, Q\,(\ell - \alpha\, Q) \leq (\ell - \alpha\, Q)^2 \qquad \text{for every } \ell \in \mathbb{N}\setminus\{0\}. \tag{56}$$

However, since ℓ varies among the positive natural numbers, the following real

$$\gamma_0 = \min\{\ell - \alpha Q : \ell > \alpha Q, \ell \in \mathbb{N}\}$$

is positive and (56) is verified whenever $\gamma < \gamma_0/Q$.

9.6. **Proof of Theorem 8.5.** In this section we will sketch the proof of Theorem 8.5, which will serve as prototype for the approach to Theorem 4.4. We fix therefore a Dir-minimizing Q-valued map $f : \Omega \to \mathcal{A}_Q(\mathbb{R}^n)$, where $\Omega \subset \mathbb{R}^2$ is open, and a point x_0 which is singular for f. Without loss of generality we can assume that

(a) $x_0 = 0$ and $f(0) = Q\,[\![0]\!]$;
(b) f is irreducible in any neighborhood U of 0, namely it is not possible to write $f|_U$ as $f_1 + f_2$ for 2 Q_i-valued maps f_i with $Q_1 + Q_2 = Q$ and $\mathrm{spt}(f_1(x)) \cap \mathrm{spt}(f_2(x)) = \emptyset$ $\forall x \in U \setminus \{0\}$.

For what concerns (b) note that if such a decomposition existed, then necessarily f_1 and f_2 would be Dir-minimizing and thus we could just restrict to show that for each f_i the singularity at 0 is isolated (or 0 is a regular point). Since $\max\{Q_1, Q_2\} < Q$, this process must end somewhere and we are left with an irreducible Dir-minimizer.

Note next that, as in the proof of Theorem 8.4, we can subtract the average of the sheets to f and assume that $\boldsymbol{\eta} \circ f \equiv 0$. We then proceed with our blow-up approach, namely we define the maps $u_{0,r}$ as in (47) and from the discussion of the previous subsection we infer the following conclusions:

(i) $u_{0,r}$ converges (locally uniformly on $\mathbb{R}^2$) to a unique tangent function $\bar{u}$;
(ii) $\bar{u}$ is Dir-minimizing and $\boldsymbol{\eta} \circ \bar{u} \equiv 0$;
(iii) $\bar{u}$ is α-homogeneous, where $\alpha = \lim_{r\downarrow 0} I(r)$ is a positive real number.

Now, it turns out, from (i) and (ii) that

$$\mathcal{G}(f(x), \bar{u}(x)) = o(|x|^\alpha)\,. \tag{57}$$

From the latter decay property and (iii), it follows easily that $\bar{u}$ must be itself irreducible in any neighborhood of the origin, otherwise (b) would be violated. From Proposition 9.4, it follows immediately that $\bar{u}$ must necessarily have the following representation. First of all $\alpha = n^\star/Q^\star$ for some integers $n^\star$, $Q^\star$ with $\mathrm{MCD}(n^\star, Q^\star)$. Next, $Q^\star > 1$ and it is a divisor of Q. Moreover, if we set $Q_1 = Q/Q^\star$, then there is an injective linear map $L : \mathbb{C} \to \mathbb{R}^n$ so that

$$\bar{u}(w) = \sum_{z: z^{Q^\star} = w} Q_1 \left[\!\left[L \cdot z^{n^\star} \right]\!\right]\,. \tag{58}$$

Note that any branch of $z \mapsto z^{Q^\star}$ is conformal. Thus it is not difficult to see that (57) and (58) together imply the existence of an $r > 0$ and of a Dir-minimizing $\bar{Q}$-valued map $g : B_{r^{Q^\star}} \to \mathcal{A}_{Q_1}(\mathbb{R}^n)$ such that

$$f(w) = \sum_{z: z^{Q^\star} = w} g(z)\,. \tag{59}$$

So, our task is reduced to prove that either 0 is a regular point of g (which would happen if and only if g is identically equal to $Q_1 [\![\boldsymbol{\eta} \circ g]\!]$) or that 0 is an isolated singularity of g.

If 0 is a singular point of g, we can obviously repeat the entire process above, i.e. subtract the average, find a nontrivial tangent map and reduce further g in a neighborhood of 0 to a representation of the type (59) namely

$$g(w) = \sum_{z: z^{\tilde{Q}} = w} g_2(z)\,,$$

where g_2 is Q_2-valued, $Q_2 \tilde{Q} = Q_1$ and $\tilde{Q} > 1$.

However each time that we apply this analysis the number of sheets Q_i of the new function decreases. So obviously the procedure must stop after a finite number of iterations with a final function for which 0 is a regular point. In turn this implies that 0 is an isolated singularity for our starting map f.

10. Preliminaries, notation and model singularity for Theorem 4.4

We are now ready to start describing the proof of Theorem 4.4. The underlying theme is that we would like to set up a recursive blow-up procedure as in Section 9.6 above: the precise statement will be given in Section 11. In this section we collect first a series of simplifying assumptions that can be made and which are useful to reduce several technicalities. We then introduce some devices which will be of fundamental importance in setting up the recursive step, whose precise statement is given in Theorem 11.4.

For the notation concerning submanifolds $\Sigma \subset \mathbb{R}^{2+n}$ we refer to [23, Section 1]. As in the previous sections, with $\mathbf{B}_r(p)$ and $B_r(x)$ we denote, respectively, the open ball with radius r and center p in $\mathbb{R}^{2+n}$ and the open ball with radius r and center x in $\mathbb{R}^2$. $\mathbf{C}_r(p)$ and $\mathbf{C}_r(x)$ will always denote the cylinder $B_r(x) \times \mathbb{R}^n$, where $p = (x, y) \in \mathbb{R}^2 \times \mathbb{R}^n$. We will often need to consider cylinders whose bases are parallel to other 2-dimensional planes, as well as balls in m-dimensional affine planes. We then introduce the notation $B_r(p, \pi)$ for $\mathbf{B}_r(p) \cap (p + \pi)$ and $\mathbf{C}_r(p, \pi)$ for $B_r(p, \pi) + \pi^\perp$. e_i will denote the unit vectors in the standard basis, π_0 the (oriented) plane $\mathbb{R}^2 \times \{0\}$ and $\vec{\pi}_0$ the 2-vector $e_1 \wedge e_2$ orienting it. Given an m-dimensional plane π, we denote by $\mathbf{p}_\pi$ and $\mathbf{p}_\pi^\perp$ the orthogonal projections onto, respectively, π and its orthogonal complement $\pi^\perp$. Since π is used recurrently for 2-dimensional planes, the 2-dimensional area of the unit circle in $\mathbb{R}^2$ will be denoted by ω_2.

10.1. **Multivalued push-forwards, graphs, the area formula and the Taylor expansion of the mass.** One first technical detail that we have to tackle concerns the currents which are naturally induced by multivalued maps. Assume therefore to have fixed a Lipschitz map $F : \mathbb{R}^m \supset \Omega \to \mathcal{A}_Q(\mathbb{R}^{m+n})$ on a bounded open set Ω. Consider the regions M_i and the functions f_i^j of Lemma 7.4. Through them we can define the "multivalued" pushforward

$$\mathbf{T}_F := \sum_i (f_i^j))_\sharp \llbracket M_i \rrbracket .$$

In a similar fashion we can define multivalued pushforwards when the domain Ω is a Riemannian manifold with finite volume. Moreover the current naturally carried by the graph of a multivalued function u can be defined using the pushforward through the map $x \mapsto \sum_i \llbracket (x, u_i(x)) \rrbracket$. Such current will be denoted by $\mathbf{G}_u$, whereas for the set-theoretic objects we will use the notation $\mathrm{Im}(F)$ and $\mathrm{Gr}(u)$.

The currents introduced above are well-defined (namely they are independent of the decomposition chosen in Lemma 7.4) and in fact the assumption of boundedness of Ω and finiteness of the volume of M can be removed if F satisfies a suitable "properness" assumption, cf. [20, Definition 1.2 & Definition 1.3]. Moreover, the usual formulas and conclusions valid in the classical-valued setting holds in the multivalued case as well. We record here some important conclusions.

Lemma 10.1 (Bilipschitz invariance, cf. [20, Lemma 1.8])**.** *Let $F : \Sigma \to \mathcal{A}_Q(\mathbb{R}^n)$ be a Lipschitz and proper map, $\Phi : \Sigma' \to \Sigma$ a bilipschitz homeomorphism and $G := F \circ \Phi$. Then, $\mathbf{T}_F = \mathbf{T}_G$.*

Lemma 10.2 (Q-valued area formula, cf. [20, Lemma 1.9])**.** *Let $\Sigma \subset \mathbb{R}^N$ be a Lipschitz oriented submanifold, $M \subset \Sigma$ a measurable subset and $F : M \to \mathcal{A}_Q(\mathbb{R}^n)$ a proper Lipschitz* *map. For any bounded Borel function $h : \mathbb{R}^n \to [0, \infty[$, we have*

$$\int h(p)\, d\|\mathbf{T}_F\|(p) \leq \int_M \sum_j h(F^j(x))\, \mathbf{J}F^j(x)\, d\mathcal{H}^m(x)\,, \tag{60}$$

where

$$\mathbf{J}F^j(x) = \left|DF^j(x)_\sharp \vec{e}\right| = \sqrt{\det((DF^j(x))^T \cdot DF^j(x))}$$

Equality holds in (60) *if there is a set $M' \subset M$ of full measure for which*

$$\langle DF^j(x)_\sharp \vec{e}(x), DF^i(y)_\sharp \vec{e}(y)\rangle \geq 0 \qquad \forall x, y \in M' \text{ and } i, j \text{ with } F^i(x) = F^j(y)\,. \tag{61}$$

If (61) *holds the formula is valid also for bounded* real-*valued Borel h with compact support.*

Corollary 10.3 (Area formula for Q-graphs, cf. [20, Corollary 1.11])**.** *Let $\Sigma = \mathbb{R}^m$, $M \subset \mathbb{R}^m$ and $f : \Sigma \to \mathcal{A}_Q(\mathbb{R}^n)$ be a proper Lipschitz map. Then, for any bounded compactly supported Borel $h : \mathbb{R}^{m+n} \to \mathbb{R}$, we have*

$$\int h(p)\, d\|\mathbf{G}_f\|(p) = \int_M \sum_i h(x, f_i(x)) \Big(1 + \sum_{k=1}^m \sum_{A \in M^k(Df^i)} (\det A)^2\Big)^{\frac{1}{2}} dx. \tag{62}$$

Theorem 10.4 (Boundary of the push-forward, cf. [20, Theorem 2.1])**.** *Let Σ be a Lipschitz submanifold of $\mathbb{R}^N$ with Lipschitz boundary, $F : \Sigma \to \mathcal{A}_Q(\mathbb{R}^n)$ a proper Lipschitz function and $f = F|_{\partial\Sigma}$. Then, $\partial \mathbf{T}_F = \mathbf{T}_f$.*

One crucial point in our discussions is the Taylor expansion of the mass of a multivalued graph.

Corollary 10.5 (Expansion of $\mathbf{M}(\mathbf{G}_f)$, cf. [20, Corollary 3.3])**.** *Assume $\Omega \subset \mathbb{R}^m$ is an open set with bounded measure and $f : \Omega \to \mathcal{A}_Q(\mathbb{R}^n)$ a Lipschitz map with* $\mathrm{Lip}(f) \leq \bar{c}$*. Then,*

$$\mathbf{M}(\mathbf{G}_f) = Q|\Omega| + \frac{1}{2}\int_\Omega |Df|^2 + \int_\Omega \sum_i \bar{R}_4(Df_i)\,, \tag{63}$$

where $\bar{R}_4 \in C^1$ satisfies $|\bar{R}_4(D)| = |D|^3 \bar{L}(D)$ for $\bar{L}$ with $\mathrm{Lip}(\bar{L}) \leq C$ *and $\bar{L}(0) = 0$.*

10.2. **Preliminary assumptions.** In the rest of these notes we will always make the following

Assumption 10.6. T is an integral current of dimension 2 with bounded support and it satisfies one of the three conditions (a), (b) or (c) in Theorem 4.4. Moreover

- In case (a), $\Sigma \subset \mathbb{R}^{2+n}$ is a C^{3,ε_0} submanifold of dimension $2 + \bar{n} = 2 + n - l$, which is the graph of an entire function $\Psi : \mathbb{R}^{2+\bar{n}} \to \mathbb{R}^l$ and satisfies the bounds
$$\|D\Psi\|_0 \leq c_0 \quad \text{and} \quad \mathbf{A} := \|A_\Sigma\|_0 \leq c_0, \tag{64}$$
where c_0 is a positive (small) dimensional constant and $\varepsilon_0 \in]0, 1[$.
- In case (b) we assume that $\Sigma = \mathbb{R}^{2+n}$ and that the semicalibrating form ω is C^{2,ε_0}.

- In case (c) we assume that T is supported in $\Sigma = \partial \mathbf{B}_R(p_0)$ for some p_0 with $|p_0| = R$, so that $0 \in \partial \mathbf{B}_R(p_0)$. We assume also that $T_0 \partial \mathbf{B}_R(p_0)$ is $\mathbb{R}^{2+n-1}$ (namely $p_0 = (0, \ldots, 0, \pm|p_0|)$ and we let $\Psi : \mathbb{R}^{2+n-1} \to \mathbb{R}$ be a smooth extension to the whole space of the function which describes Σ in $\mathbf{B}_2(0)$. We assume then that (64) holds, which is equivalent to the requirement that R^{-1} be sufficiently small.

In other words:

(i) In case (b) we remove the ambient Riemannian manifold Σ thanks to Lemma 3.1.
(ii) In the cases (a) and (c) we assume that the ambient Riemannian manifold Σ is very flat and globally graphical. Since out theorem has a local nature and is invariant under rescalings, it is clear that the latter assumption is totally harmless.

Next we focus our attention on a point in the support of T, which we assume to be a singular point: the goal is thus to prove that this point is isolated. By the discussion in Section 4 we can then impose also the next conditions.

Assumption 10.7. In addition to Assumption 10.6 we assume the following:

(i) $\partial T \llcorner \mathbf{C}_2(0, \pi_0) = 0$;
(ii) $0 \in \mathrm{spt}(T)$ and the tangent cone at 0 is given by $\Theta(T, 0) [\![\pi_0]\!]$ where $\Theta(T, 0) \in \mathbb{N} \setminus \{0\}$;
(iii) T is irreducible in any neighborhood U of 0 in the following sense: it is not possible to find S, Z non-zero integer rectifiable currents in U with $\partial S = \partial Z = 0$ (in U), $T = S + Z$ and $\mathrm{spt}(S) \cap \mathrm{spt}(Z) = \{0\}$.

10.3. **Branching model.** We next introduce the object with which we will model the "most basic" singular behavior of a 2-dimensional area minimizing current: for each positive natural number Q we will denote by $\mathfrak{B}_{Q,\rho}$ the flat Riemann surface which is a disk with a conical singularity, in the origin, of angle $2\pi Q$ and radius $\rho > 0$. More precisely we have

Definition 10.8. $\mathfrak{B}_{Q,\rho}$ is topologically an open 2-dimensional disk, which we identify with the topological space $\{(z, w) \in \mathbb{C}^2 : w^Q = z, |z| < \rho\}$. For each $(z_0, w_0) \neq 0$ in $\mathfrak{B}_{Q,\rho}$ we consider the connected component $\mathfrak{D}(z_0, w_0)$ of $\mathfrak{B}_{Q,\rho} \cap \{(z, w) : |z - z_0| < |z_0|/2\}$ which contains (z_0, w_0). We then consider the smooth manifold given by the atlas

$$\{(\mathfrak{D}(z, w)), (x_1, x_2)) : (z, w) \in \mathfrak{B}_{Q,\rho} \setminus \{0\}\} ,$$

where (x_1, x_2) is the function which gives the real and imaginary part of the first complex coordinate of a generic point of $\mathfrak{B}_{Q,\rho}$. On such smooth manifold we consider the following flat Riemannian metric: on each $\mathfrak{D}(z, w)$ with the chart (x_1, x_2) the metric tensor is the usual euclidean one $dx_1^2 + dx_2^2$. Such metric will be called the *canonical flat metric* and denoted by e_Q.

When $Q = 1$ we can extend smoothly the metric tensor to the origin and we obtain the usual euclidean 2-dimensional disk. For $Q > 1$ the metric tensor does not extend smoothly to 0, but we can nonetheless complete the induced geodesic distance on $\mathfrak{B}_{Q,\rho}$ in a neighborhood of 0: for $(z, w) \neq 0$ the distance to the origin will then correspond to $|z|$. The resulting metric space is a well-known object in the literature, namely a flat Riemann surface with an isolated conical singularity at the origin (see for instance [48]). Note that for each z_0 and $0 < r \leq \min\{|z_0|, \rho - |z_0|\}$ the set $\mathfrak{B}_{Q,\rho} \cap \{|z - z_0| < r\}$ consists then of Q nonintersecting 2-dimensional disks, each of which is a geodesic ball of $\mathfrak{B}_{Q,\rho}$ with radius r and center (z_0, w_i) for some $w_i \in \mathbb{C}$ with $w_i^Q = z_0$. We then denote each of them by $B_r(z_0, w_i)$ and treat it as a standard disk in the euclidean 2-dimensional plane (which is correct from the metric point of view). We use however the same notation for the distance disk $B_r(0)$, namely for the set $\{(z, w) : |z| < r\}$, although the latter is *not isometric* to the standard euclidean disk. Since this might create some ambiguity, we will use the specification $\mathbb{R}^2 \supset B_r(0)$ when referring to the standard disk in $\mathbb{R}^2$.

10.4. **Admissible Q-branchings.** When one of (or both) the parameters Q and ρ are clear from the context, the corresponding subscript (or both) will be omitted. We will always treat each point of $\mathfrak{B}$ as an element of $\mathbb{C}^2$, mostly using z and w for the horizontal and vertical complex coordinates. Often $\mathbb{C}$ will be identified with $\mathbb{R}^2$ and thus the coordinate z will be treated as a two-dimensional real vector, avoiding the more cumbersome notation (x_1, x_2).

Definition 10.9 (Q-branchings)**.** Let $\alpha \in]0, 1[$, $b > 1$, $Q \in \mathbb{N} \setminus \{0\}$ and $n \in \mathbb{N} \setminus \{0\}$. An admissible α-smooth and b-separated Q-branching in $\mathbb{R}^{2+n}$ (shortly a Q-branching) is the graph

$$\mathrm{Gr}(u) := \{(z, u(z, w)) : (z, w) \in \mathfrak{B}_{Q,2\rho}\} \subset \mathbb{R}^{2+n} \tag{65}$$

of a map $u : \mathfrak{B}_{Q,2\rho} \to \mathbb{R}^n$ satisfying the following assumptions. For some constants $C_i > 0$ we have

- u is continuous, $u \in C^{3,\alpha}$ on $\mathfrak{B}_{Q,2\rho} \setminus \{0\}$ and $u(0) = 0$;
- $|D^j u(z, w)| \leq C_i |z|^{1-j+\alpha}$ $\forall (z, w) \neq 0$ and $j \in \{0, 1, 2, 3\}$;
- $[D^3 u]_{\alpha, B_r(z,w)} \leq C_i |z|^{-2}$ for every $(z, w) \neq 0$ with $|z| = 2r$;
- If $Q > 1$, then there is a positive constant $c_s \in]0, 1[$ such that

$$\min\{|u(z, w) - u(z, w')| : w \neq w'\} \geq 4c_s |z|^b \qquad \text{for all } (z, w) \neq 0. \tag{66}$$

The map $\mathbf{\Phi}(z, w) := (z, u(z, w))$ will be called the *graphical parametrization* of the Q-branching.

Any Q-branching as in the Definition above is an immersed disk in $\mathbb{R}^{2+n}$ and can be given a natural structure as integer rectifiable current, which will be denoted by $\mathbf{G}_u$. For $Q = 1$ a map u as in Definition 10.9 is a (single valued) $C^{1,\alpha}$ map $u : \mathbb{R}^2 \supset B_{2\rho}(0) \to \mathbb{R}^n$. Although the term branching is not appropriate in this case, the advantage of our setup is that $Q = 1$ will not be a special case in the induction statement of Theorem 11.4 below. Observe that for $Q > 1$ the map u can be thought as a Q-valued map $u : \mathbb{R}^2 \supset B_{2\rho}(0) \to \mathcal{A}_Q(\mathbb{R}^n)$, setting $u(z) = \sum_{(z,w_i) \in \mathfrak{B}} [\![u(z, w_i)]\!]$ for $z \neq 0$ and $u(0) = Q\,[\![0]\!]$. The notation $\mathrm{Gr}(u)$ and $\mathbf{G}_u$ is then coherent with the notation introduced above for the (set-theoretic and measure-theoretic) graphs of Q-valued maps.

11. The inductive statement

Before coming to the key inductive statement, we need to introduce some more terminology.

Definition 11.1 (Horned Neighborhood)**.** Let $\mathrm{Gr}(u)$ be a b-separated Q-branching. For every $a > b$ we define the *horned neighborhood* $\mathbf{V}_{u,a}$ of $\mathrm{Gr}(u)$ to be

$$\mathbf{V}_{u,a} := \{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^n : \exists (x,w) \in \mathfrak{B}_{Q,2\rho} \text{ with } |y - u(x,w)| < c_s|x|^a\}\,, \tag{67}$$

where c_s is the constant in (66).

Definition 11.2 (Excess)**.** Given an m-dimensional current T in $\mathbb{R}^{m+n}$ with finite mass, its *excess* in the ball $\mathbf{B}_r(x)$ and in the cylinder $\mathbf{C}_r(p,\pi')$ with respect to the m-plane π are

$$\mathbf{E}(T, \mathbf{B}_r(p), \pi) := (2\omega_m\, r^m)^{-1} \int_{\mathbf{B}_r(p)} |\vec{T} - \vec{\pi}|^2\, d\|T\| \tag{68}$$

$$\mathbf{E}(T, \mathbf{C}_r(p,\pi'), \pi) := (2\omega_m\, r^m)^{-1} \int_{\mathbf{C}_r(p,\pi')} |\vec{T} - \vec{\pi}|^2\, d\|T\|\,. \tag{69}$$

For cylinders we omit the third entry when $\pi = \pi'$, i.e. $\mathbf{E}(T, \mathbf{C}_r(p,\pi)) := \mathbf{E}(T, \mathbf{C}_r(p,\pi), \pi)$. In order to define the spherical excess we consider T as in Assumption 10.6 and we say that π *optimizes the excess* of T in a ball $\mathbf{B}_r(x)$ if

- In case (b)
$$\mathbf{E}(T, \mathbf{B}_r(x)) := \min_\tau \mathbf{E}(T, \mathbf{B}_r(x), \tau) = \mathbf{E}(T, \mathbf{B}_r(x), \pi); \tag{70}$$
- In case (a) and (c) $\pi \subset T_x\Sigma$ and
$$\mathbf{E}(T, \mathbf{B}_r(x)) := \min_{\tau \subset T_x\Sigma} \mathbf{E}(T, \mathbf{B}_r(x), \tau) = \mathbf{E}(T, \mathbf{B}_r(x), \pi)\,. \tag{71}$$

11.1. **Inductive assumption and inductive statement.** Our main induction assumption is then the following

Assumption 11.3 (Inductive Assumption)**.** T is as in Assumption 10.6 and 10.7. For some constants $\bar{Q} \in \mathbb{N} \setminus \{0\}$ and $0 < \alpha < \frac{1}{2Q}$ there is an α-admissible $\bar{Q}$-branching $\mathrm{Gr}(u)$ with $u : \mathfrak{B}_{\bar{Q},2} \to \mathbb{R}^n$ such that

- (Sep) If $\bar{Q} > 1$, u is b-separated for some $b > 1$; a choice of some $b > 1$ is fixed also in the case $\bar{Q} = 1$, although in this case the separation condition is empty.
- (Hor) $\mathrm{spt}(T) \subset \mathbf{V}_{u,a} \cup \{0\}$ for some $a > b$;
- (Dec) There exist $\gamma > 0$ and a $C_i > 0$ with the following property. Let $p = (x_0, y_0) \in \mathrm{spt}(T) \cap \mathbf{C}_{\sqrt{2}}(0)$ and $4d := |x_0| > 0$, let V be the connected component of $\mathbf{V}_{u,a} \cap \{(x,y) : |x - x_0| < d\}$ containing p and let $\pi(p)$ be the plane tangent to $\mathrm{Gr}(u)$ at the only point of the form $(x_0, u(x_0, w_i))$ which is contained in V. Then
$$\mathbf{E}(T \llcorner V, \mathbf{B}_\sigma(p), \pi(p)) \le C_i^2 d^{2\gamma-2} \sigma^2 \qquad \forall \sigma \in \left[\tfrac{1}{2} d^{(b+1)/2}, d\right]\,. \tag{72}$$

The main inductive step is then the following theorem, where we denote by $T_{p,r}$ the rescaled current $(\iota_{p,r})_\sharp T$, through the map $\iota_{p,r}(q) := (q-p)/r$.

Theorem 11.4 (Inductive statement, cf. [25, Theorem 1.8])**.** *Let T be as in Assumption 11.3 for some $\bar{Q} = Q_0$. Then,*

(a) *either T is, in a neighborhood of 0, a Q multiple of a $\bar{Q}$-branching $\mathrm{Gr}(v)$;*
(b) *or there are $r > 0$ and $Q_1 > Q_0$ such that $T_{0,r}$ satisfies Assumption 11.3 with $\bar{Q} = Q_1$.*

Theorem 4.4 follows then easily from Theorem 11.4 and Section 4. In fact, by Theorem 4.2 T satisfies Assumption 11.3 with $\bar{Q} = 1$: it suffices to chose $u \equiv 0$ as admissible smooth branching. If T were not regular in any punctured neighborhood of 0, we could then apply Theorem 11.4 inductively to find a sequence of rescalings T_{0,ρ_j} with $\rho_j \downarrow 0$ which satisfy Assumption 11.3 with $\bar{Q} = Q_j$ for some strictly increasing sequence of integers. It is however elementary that the density $\Theta(0,T)$ bounds Q_j from above, which is a contradiction. Note the similarity with concluding argument of Section 9.6.

12. The overall approach to Theorem 11.4

From now on we fix T satisfying Assumption 11.3. Observe that, without loss of generality, we are always free to rescale homothetically our current T with a factor larger than 1 and ignore whatever portion falls outside $\mathbf{C}_2(0)$. We will do this several times, with factors which will be assumed to be sufficiently large. Hence, if we can prove that something holds in a sufficiently small neighborhood of 0, then we can assume, withouth loss of generality, that it holds in $\mathbf{C}_2$. For this reason we can assume that the constants C_i in Definition 10.9 and Assumption 11.3 are as small as we want. In turns this implies that there is a well-defined orthogonal projection $\mathbf{P} : \mathbf{V}_{u,a} \cap \mathbf{C}_1 \to \mathrm{Gr}(u) \cap \mathbf{C}_2$, which is a $C^{2,\alpha}$ map.

By the constancy theorem, $(\mathbf{P}_\sharp(T \llcorner \mathbf{C}_1)) \llcorner \mathbf{C}_{1/2}$ coincides with the current $Q\mathbf{G}_u \llcorner \mathbf{C}_{1/2}$ (again, we are assuming C_i in Definition 10.9 sufficiently small), where $Q \in \mathbb{Z}$. If Q were 0, condition (Dec) in Assumption 11.3 and a simple covering argument would imply that $\|T\|(\mathbf{C}_{1/2}(0)) \leq C_0 C_i^2$, where C_0 is a geometric constant. In particular, when C_i is sufficiently small, this would violate, by the monotonicity formula, the assumption $0 \in \mathrm{spt}(T)$. Thus $Q \neq 0$. On the other hand condition (Dec) in Assumption 11.3 implies also that Q must be positive (again, provided C_i is smaller than a geometric constant).

Now, recall from [27, Theorem 3.1] that the density $\Theta(p,T)$ is a positive integer at any $p \in \mathrm{spt}(T) \setminus \mathrm{spt}(\partial T)$. Moreover, the rescaled currents $T_{0,r}$ converge to $\Theta(0,T) [\![\pi_0]\!]$. It is easy to see that the rescaled currents $(\mathbf{G}_u)_{0,r}$ converge to $\bar{Q} [\![\pi_0]\!]$ and that $(\mathbf{P}_\sharp T)_{0,r}$ converges to $\Theta(0,T) [\![\pi_0]\!]$. We then conclude that $\Theta(0,T) = \bar{Q}Q$.

We summarize these conclusions in the following lemma, where we also claim an additional important bound on the density of T outside 0, which will be proved later.

Lemma 12.1 (Cf. [24, Lemma 2.1])**.** *Let T and u be as in Assumption 11.3 for some $\bar{Q}$ and sufficiently small C_i. Then the nearest point projection $\mathbf{P} : \mathbf{V}_{u,a} \cap \mathbf{C}_1 \to \mathrm{Gr}(u)$ is a well-defined $C^{2,\alpha}$ map. In addition there is $Q \in \mathbb{N} \setminus \{0\}$ such that $\Theta(0,T) = Q\bar{Q}$ and the unique tangent cone to T at 0 is $Q\bar{Q} [\![\pi_0]\!]$. Finally, after possibly rescaling T, $\Theta(p,T) \leq Q$ for every $p \in \mathbf{C}_2 \setminus \{0\}$ and, for every $x \in B_2(0)$, each connected component of $(\{x\} \times \mathbb{R}^n) \cap \mathbf{V}_{u,a}$ contains at least one point of $\mathrm{spt}(T)$.*

Since we will assume during the rest of the paper that the above discussion applies, we summarize the relevant conclusions in the following

Assumption 12.2. T satisfies Assumption 11.3 for some $\bar{Q}$ and with C_i sufficiently small. $Q \geq 1$ is an integer, $\Theta(0,T) = Q\bar{Q}$ and $\Theta(p,T) \leq Q$ for all $p \in \mathbf{C}_2 \setminus \{0\}$.

The overall plan to prove Theorem 11.4 is then the following:

(CM) We construct first a branched center manifold, i.e. a second admissible smooth branching $\boldsymbol{\varphi}$ on $\mathfrak{B}_{\bar{Q}}$, and a corresponding Q-valued map N defined on the normal bundle of $\mathrm{Gr}(\boldsymbol{\varphi})$, which approximates T with a very high degree of accuracy (in particular more accurately than u) and whose average $\boldsymbol{\eta} \circ N$ is very small;

(BU) Assuming that alternative (a) in Theorem 11.4 does not hold, we study the asymptotic behavior of N around 0 and use it to build a new admissible smooth branching v on some $\mathfrak{B}_{k\bar{Q}}$ where $k \geq 2$ is a factor of Q: this map will then be the one sought in alternative (b) of Theorem 11.4 and a suitable rescaling of T will lie in a horned neighborhood of its graph.

The first part of the program is achieved in [24], whereas the second part is completed in [25]. Note that, when $Q = 1$, from (BU) we will conclude that alternative (a) necessarily holds: this will be a simple corollary of the general case, but we observe that it could also be proved resorting to the classical Allard's regularity theorem. For poth portions of the proof of Theorem 11.4, namely for (CM) and (B) described above, we can formulate a single main statement, for which we refer to Theorem 13.3 and Theorem 14.1 below. The inductive step, namely Theorem 11.4, is then a direct consequence of these two "pieces", although some nontrivial argument is still needed, cf. [25, Section 2.3] for the details.

13. The center manifold

In order to carry on the plan outlined in the previous subsection, it is convenient to use parametrizations of Q-branchings which are not graphical but instead satisfy a suitable conformality property.

If we remove the origin, any admissible Q-branching is a Riemannian submanifold of $\mathbb{R}^{2+n} \setminus \{0\}$: this gives a Riemannian tensor $g := \mathbf{\Phi}^\sharp e$ (where e denotes the euclidean metric on $\mathbb{R}^{2+n}$) on the punctured disk $\mathfrak{B}_{Q,2\rho} \setminus \{0\}$. Note that in (z,w) the difference between the metric tensor g and the canonical flat metric e_Q can be estimated by (a constant times) $|z|^{2\alpha}$: thus, as it happens for the canonical flat metric e_Q, when $Q > 1$ it is not possible to extend the metric g to the origin. However, using well-known arguments in differential geometry, we can find a conformal map from $\mathfrak{B}_{Q,r}$ onto (a neighborhood of 0 in) $\mathrm{Gr}(u)$ which maps the conical singularity of $\mathfrak{B}_{Q,r}$ in the conical singularity of the Q-branching $\mathrm{Gr}(u)$. In fact, we need the following accurate estimates for such a map.

Proposition 13.1 (Conformal parametrization, cf. [24, Proposition 2.4])**.** *Given an admissible b-separated α-smooth Q-branching $\mathrm{Gr}(u)$ with $\alpha < 1/(2Q)$ there exist a constant $C_0(Q,\alpha) > 0$, a radius $r > 0$ and functions $\mathbf{\Psi}\colon \mathfrak{B}_{Q,r} \to \mathrm{Gr}(u)$ and $\lambda\colon \mathfrak{B}_{Q,r} \to \mathbb{R}_+$ such that*

(i) *$\mathbf{\Psi}$ is a homeomorphism of $\mathfrak{B}_{Q,r}$ with a neighborhood of 0 in $\mathrm{Gr}(u)$;*

(ii) $\mathbf{\Psi} \in C^{3,\alpha}(\mathfrak{B}_{Q,r} \setminus \{0\})$, *with the estimates*

$$|D^l\big(\mathbf{\Psi}(z,w) - (z,0)\big)| \leq C_0 C_i |z|^{1+\alpha-l} \qquad \textit{for } l = 0, \ldots, 3,\ z \neq 0\,, \tag{73}$$

$$[D^3\mathbf{\Psi}]_{\alpha, B_r(z,w)} \leq C_0 C_i |z|^{-2} \qquad \textit{for } z \neq 0 \textit{ and } r = |z|/2\,; \tag{74}$$

(iii) $\mathbf{\Psi}$ *is a conformal map with conformal factor* λ, *namely, if we denote by* e *the ambient euclidean metric in* $\mathbb{R}^{2+n}$ *and by* e_Q *the canonical euclidean metric of* $\mathfrak{B}_{Q,r}$,

$$g := \mathbf{\Psi}^\sharp e = \lambda\, e_Q \qquad \textit{on } \mathfrak{B}_{Q,r} \setminus \{0\}. \tag{75}$$

(iv) *The conformal factor* λ *satisfies*

$$|D^l(\lambda - 1)(z,w)| \leq C_0 C_i |z|^{2\alpha - l} \qquad \textit{for } l = 0, 1, \ldots, 2 \tag{76}$$

$$[D^2\lambda]_{\alpha, B_r(z,w)} \leq C_0 C_i |z|^{\alpha-2} \qquad \textit{for } z \neq 0 \textit{ and } r = |z|/2\,. \tag{77}$$

A proof of Proposition 13.1 is given in the appendix of [24].

Definition 13.2. A map $\mathbf{\Psi}$ as in Proposition 13.1 will be called a *conformal parametrization* of an admissible Q-branching.

We are finally ready to state the theorem which identifies our center manifold.

Theorem 13.3 (Center Manifold Approximation, cf. [24, Theorem 2.6])**.** *Let* T *be as in Assumptions 11.3 and 12.2. Then there exist* $\eta_0, \gamma_0, r_0, C > 0$, $b > 1$, *an admissible* b*-separated* γ_0*-smooth* $\bar{Q}$*-branching* $\mathcal{M}$, *a corresponding conformal parametrization* $\mathbf{\Psi} : \mathfrak{B}_{\bar{Q},2} \to \mathcal{M}$ *and a* Q*-valued map* $\mathcal{N} : \mathfrak{B}_{\bar{Q},2} \to \mathcal{A}_Q(\mathbb{R}^{2+n})$ *with the following properties:*

(i) $\bar{Q}Q = \Theta(T, 0)$ *and*

$$|D(\mathbf{\Psi}(z,w) - (z,0))| \leq C \boldsymbol{m}_0^{1/2} |z|^{\gamma_0} \tag{78}$$

$$|D^2\mathbf{\Psi}(z,w)| + |z|^{-1}|D^3\mathbf{\Psi}(z,w)| \leq C \boldsymbol{m}_0^{1/2} |z|^{\gamma_0 - 1}\,; \tag{79}$$

in particular, if we denote by $A_{\mathcal{M}}$ *the second fundamental form of* $\mathcal{M} \setminus \{0\}$,

$$|A_{\mathcal{M}}(\mathbf{\Psi}(z,w))| + |z|^{-1}|D_{\mathcal{M}} A_{\mathcal{M}}(\mathbf{\Psi}(z,w))| \leq C \boldsymbol{m}_0^{1/2} |z|^{\gamma_0 - 1}\,.$$

(ii) $\mathcal{N}_i(z,w)$ *is orthogonal to the tangent plane, at* $\mathbf{\Psi}(z,w)$, *to* $\mathcal{M}$.

(iii) *If we define* $S := T_{0,r_0}$, *then* $\operatorname{spt}(S) \cap \mathbf{C}_1 \setminus \{0\}$ *is contained in a suitable horned neighborhood of the* $\bar{Q}$*-branching, where the orthogonal projection* $\mathbf{P}$ *onto it is well-defined. Moreover, for every* $r \in]0,1[$ *we have*

$$\|\mathcal{N}|_{B_r}\|_0 + \sup_{p \in \operatorname{spt}(S) \cap \mathbf{P}^{-1}(\mathbf{\Psi}(B_r))} |p - \mathbf{P}(p)| \leq C \boldsymbol{m}_0^{1/4} r^{1+\gamma_0/2}\,. \tag{80}$$

(iv) *If we define*

$$\mathbf{D}(r) := \int_{B_r} |D\mathcal{N}|^2 \quad \textit{and} \quad \mathbf{H}(r) := \int_{\partial B_r} |\mathcal{N}|^2\,,$$

$$\mathbf{F}(r) := \int_0^r \frac{\mathbf{H}(t)}{t^{2-\gamma_0}}\, dt \quad \textit{and} \quad \mathbf{\Lambda}(r) := \mathbf{D}(r) + \mathbf{F}(r)\,,$$

then the following estimates hold for every $r \in]0,1[$*:*

$$\mathrm{Lip}(\mathcal{N}|_{B_r}) \le C \min\{\mathbf{\Lambda}^{\eta_0}(r), \boldsymbol{m}_0^{\eta_0} r^{\eta_0}\} \tag{81}$$

$$\boldsymbol{m}_0^{\eta_0} \int_{B_r} |z|^{\gamma_0 - 1} |\boldsymbol{\eta} \circ \mathcal{N}(z,w)| \le C\, \mathbf{\Lambda}^{\eta_0}(r)\, \mathbf{D}(r) + C\, \mathbf{F}(r)\,. \tag{82}$$

(v) *Finally, if we set*

$$\mathcal{F}(z,w) := \sum_i [\![\mathbf{\Psi}(z,w) + \mathcal{N}_i(z,w)]\!]\ ,$$

then

$$\|S - \mathbf{T}_{\mathcal{F}}\| \big(\mathbf{P}^{-1}(\mathbf{\Psi}(B_r))\big) \le C\, \mathbf{\Lambda}^{\eta_0}(r)\, \mathbf{D}(r) + C\, \mathbf{F}(r)\,. \tag{83}$$

14. The blow up

We are now ready for the second main piece of the proof of Theorem 11.4, which is the analysis of the asymptotic behaviour of $\mathcal{N}$ around the origin.

In order to state it, we agree to define $W^{1,2}$ functions on $\mathfrak{B}$ in the following fashion: removing the origin 0 from $\mathfrak{B}$ we have a C^3_{loc} (flat) Riemannian manifold embedded in $\mathbb{R}^4$ and we can define $W^{1,2}$ maps on it following [19]. Alternatively we can use the conformal parametrization $\mathbf{W} : \mathbb{R}^2 = \mathbb{C} \to \mathfrak{B}_{\bar Q}$ given by $\mathbf{W}(z) = (z^{\bar Q}, z)$ and agree that $u \in W^{1,2}(\mathfrak{B}_{\bar Q})$ if $u \circ \mathbf{W}$ is in $W^{1,2}(\mathbb{R}^2)$. Since discrete sets have zero 2-capacity, it is immediate to verify that these two definitions are equivalent.

In a similar fashion, we will ignore the origin when integrating by parts Lipschitz vector fields, treating $\mathfrak{B}_{\bar Q}$ as a C^1 Riemannian manifold. It is straightforward to show that our assumption is correct, for instance removing a disk of radius ε centered at the origin, integrating by parts and then letting $\varepsilon \downarrow 0$.

Theorem 14.1 (Blowup Analysis, cf. [25, Theorem 2.8])**.** *Under the assumptions of Theorem 13.3, the following dichotomy holds:*

(i) *either there exists* $s > 0$ *such that* $\mathcal{N}|_{B_s} \equiv Q\,[\![0]\!]$*;*

(ii) *or there exist constants* $I_0 > 1$, $a_0, \bar r, C > 0$ *and an* I_0*-homogeneous nontrivial Dir-minimizing function* $g : \mathfrak{B}_{\bar Q} \to \mathcal{A}_Q(\mathbb{R}^{2+n})$ *such that*
   - $\boldsymbol{\eta} \circ g \equiv 0$,
   - $g = \sum_i [\![(0, \bar g_i, 0)]\!]$*, where* $\bar g_i(x) \in \mathbb{R}^{\bar n}$ *and* $(0, \bar g_i(x), 0) \in \mathbb{R}^2 \times \mathbb{R}^{\bar n} \times \mathbb{R}^l$,
   - *and the following estimates hold:*

$$\mathcal{G}\big(\mathcal{N}(z,w), g(z,w)\big) \le C |z|^{I_0 + a_0} \qquad \forall\, (z,w) \in \mathfrak{B}_Q,\ |z| < \bar r, \tag{84}$$

$$\int_{B_{r+\rho} \setminus B_{r-\rho}} |D\mathcal{N}|^2 \le C\, r^{2I_0 + a_0} + C\, r^{2I_0 - 1}\, \rho \qquad \forall\, 4\,\rho \le r < 1, \tag{85}$$

$$\mathbf{H}(r) \le C\, r\, \mathbf{D}(r) \qquad \forall\, r < 1. \tag{86}$$

Remark 14.2. Note that, when $\bar Q = \Theta(T, 0)$, we necessarily have $Q = 1$ and the second alternative is excluded. In particular we conclude that T coincides with $[\![\mathcal{M}]\!]$ in a neighborhood of 0 and thus it is a regular submanifold in a *punctured neighborhood* of 0.

Remark 14.3. By a simple dyadic argument it follows from (85) and (86) that

$$\int_{B_r} |D\mathcal{N}|^2 \leq C\, r^{2I_0} \quad \text{and} \quad \mathbf{F}(r) \leq C\, r^{2I_0+\gamma_0} \quad \forall\, r < 1. \tag{87}$$

At this point, having split the proof of Theorem 4.4 into the two Theorems 13.3 and 14.1, we will dedicate the remaining sections to an informal description of their proofs.

15. An informal description of the construction of $\mathcal{M}$ and $\mathcal{N}$

15.1. **The nonbranched center manifold.** Let us look again at the proof of Theorem 8.4 given in Section 9.2. If we understand the case of a Dir-minimizing functions f as a model, the role of the center manifold is in fact taken by the average of the sheets $\boldsymbol{\eta}\circ f$: thanks to some reminiscence of the linear structure of classical harmonic functions, $\boldsymbol{\eta}\circ f$ is harmonic and we can subtract it from all the sheets to gain a "well-balanced" Dir minimizer $\bar f := \sum_i [\![f_i - \boldsymbol{\eta}\circ f]\!]$. Hence, the graph of $\boldsymbol{\eta}\circ f$ should guide us to understand the role of the center manifold $\mathcal{M}$ of Theorem 13.3 and the map $\bar f$ should be substituted, in some sense, by $\mathcal{N}$.

For simplicity let us look at the "starting situation" of our inductive proof of Theorem 4.4 and discuss the construction of the very first center manifold $\mathcal{M}$. More precisely consider Theorem 11.4 in the case $\bar Q = 1$. In this case $\mathcal{M}$ is parametrized by $\mathfrak{B}_{1,2}$ and will therefore be a $C^{1,\alpha}$ graph over the reference plane $\mathbb{R}^2\times\{0\} = \pi_0$. Our starting assumption is then that the excess $\mathbf{E}(T,\mathbf{C}_r)$ decays like a power of r.

One first attempt would be to first approximate the current T with a Lipschitz graph and then take the average of such approximation to build $\mathcal{M}$. Although it is possible to approximate T rather accurately with the graph $\mathbf{G}_f$ of a Lipschitz map (as we will see in the next section), the graph of the average $\boldsymbol{\eta}\circ f$ will not be a satisfactory center manifold $\mathcal{M}$ for at least two fundamental reasons.

- (R) First of all $\mathcal{M}$ is, a-priori, not regular enough: even if the current T were already a Q-valued graph, the nonlinearity of the problem does not give any partial differential equation for the average of the sheets. We will need therefore an appropriate regularization mechanism.
- (L) Secondly, the approximation $\mathcal{N}$ should be accurate at every scale where the current T is very flat and its sheets are very close, otherwise the estimate (82) could not hold (observe that the left hand side is linear in $\mathcal{N}$, whereas the right and side is essentially quadratic).

If the first issue can be considered a minor one, since we have several tools to regularize classical maps, the second issue is more serious and it is clear that we would like to "localize" our Lipschitz approximation. Namely, for any point p we would like to identify a first scale at which the current has either some well separated sheets or it is not very flat: at that scale we would like to take a good Lipschitz approximation f and regularize its average. The graph of such regularization would then be a local model for the center manifold and we would thus like to patch all these local models together in a smooth (more specifically C^3) submanifold. Now it is clear that (R) and (L) together pose a lot of difficulties: on the one hand we might be forced to regularize functions at small scales, on the other hand we still hope that our "patchwork" $\mathcal{M}$ stays C^3.

An efficient way to perform the desired localization procedure is to use a Whitney type decomposition of the reference plane $\mathbb{R}^2 \times \{0\}$ and introduce corresponding stopping time conditions for the refining of cubes, cf. [17] and Section 17.2 below for a thorough explanation. Assuming to have constructed $\mathcal{M}$ in the fashion described above, we then follow the same principle as in [21] to build the approximating map $\mathcal{N}$: around a given point we first identify the "stopping scale" and hence we consider the approximation f for which $\mathcal{M}$ is essentially a perturbation of $\boldsymbol{\eta} \circ f$; at this point we "reparametrize" $\mathcal{M}$ using the curvilinear coordinates induced by the normal bundle of $\mathcal{M}$, in particular the map $\mathcal{N}$ coincides with such reparametrization in that region.

The latter reparametrization step is literally the same as in [21]: cf. [24, Section 9]. The major differences between the papers [21] and [24] are instead in the construction of $\mathcal{M}$. First of all observe that we need a suitable Lipschitz approximation in "straight coordinates" when the cylindrical excess of the current T is small. For the area minimizing case (i.e. (a) of Theorem 4.4) this is already accomplished in [23] and no additional work is needed. There are instead several adjustments to be done in the cases (b) and (c) of Theorem 4.4 to the proof of [23]. These are accomplished in the separate note [26] and will be described in the next section.

The actual construction of the branched center manifold $\mathcal{M}$ of Theorem 13.3 differs substantially from the construction of [21] even in the area minimizing case (a) and it requires a lot more work especially in the cases (b) and (c). A more detailed idea of the proof will be given in the subsequent sections: in the next paragraph we want to highlight the additional challenges that we have to face compared to [21].

15.2. **New challanges.** A first major difference between the papers [21] and [24] is that the center manifold of Theorem 13.3 must necessarily contain the origin, whereas this is not the case for the corresponding object constructed in [21]. In particular the construction of [21] would guarantee that the origin is contained in $\mathcal{M}$ only if the decay of the excess around 0 would be sufficiently close to be quadratic. However Theorem 4.2 guarantees only a small power of r and we cannot hope to do any better: for the holomorpic curve $\{(z,w) : z^Q = w^{Q+1}\}$ the decay of the excess at the origin is of the order $r^{2/Q}$.

In order to guarantee that the origin is contained in $\mathcal{M}$ we need to introduce more sophisticated conditions to identify the scale at which one should perform the straight approximation f: in particular a new parameter, the distance from the origin, must be included in the stopping condition. For this reason the starting configuration of the Whitney-type recursive procedure which identifies the correct "stopping scale" must start from a decomposition of $(\mathbb{R}^2 \setminus \{0\}) \times \{0\}$ in dyadic cubes L such that their sidelength and their distance from the origin are comparable. It must also be noticed that the new parameter in the stopping time conditions must be carefully tuned with the several others in order to reach the correct estimates. Such fine tuning requires a delicate analysis.

A second obvious difference between [21] and [24] is that in Theorem 13.3 we need in fact to construct "branched" center manifolds as soon as $\bar{Q} > 1$. Although this is only a technical point, which is solved introducing a Whitney decomposition of the flat Riemann surface $\mathfrak{B}_Q \setminus \{0\}$, its solution requires care in terminology and notation.

Finally, the most important difference between [21] and [24] is in the regularization procedure mentioned in (R). In the paper [21] such regularization procedure is achieved by convolution with a smooth kernel which satisfies some momentum conditions. In [24] we have realized that the latter "ad hoc" method can in fact be replaced by a much more general and efficient PDE method: the regularization procedure for $\boldsymbol{\eta} \circ f$ consists then in solving an appropriate linear (nonhomogeneous) elliptic system of partial differential equations. Especially in case (b) of Theorem 4.4 such system differs from the Laplace equation because of additional zero and first order terms. A regularization by convolution, if possible at all, would then require a quite special choice of the kernel. The PDE approach is rather flexible, since the relevant system of partial differential equations can be identified following a suitable linearization procedure.

16. Lipschitz approximation

Based on the intuition that a "sufficiently flat" area minimizing current is close to the graph of a Dir-minimizing multivalued function, we wish now to use the theory of multi-valued Dir-minimizers to infer some interesting information upon area minimizing currents in a region where they are rather flat, i.e. the tangent planes are almost parallel to a given one (at least in an average sense). Such regions will then be cylinders $\mathbf{C}_r(x)$ where the excess is sufficiently small.

If T is a 2-dimensional integral current without boundary in $\mathbf{C}_r(0)$, a Lipschitz $u : B_r(0) \to \mathcal{A}_Q(\mathbb{R}^n)$ is an efficient approximation if $\mathbf{M}(T - \mathbf{G}_u)$ is small (compared to r^2). Since $\mathbf{G}_u$ is, in a "loose" sense a Q-fold cover of $B_r(0)$, namely $(\mathbf{p}_{\pi_0})_\sharp \mathbf{G}_u = Q\,[\![B_r(0)]\!]$, this condition must hold for a well-approximated current T as well. A fundamental step in the proof of Theorem 4.4 is then the following approximation result. Since it can be stated and proved with essentially no additional effort in the general case of m-dimensional currents, we present the general results: of course for the purpose of proving Theorem 4.4 the relevant case is $m = 2$.

First of all we start stating the approximation theorem in the case of area minimizing currents, following [23].

Assumption 16.1. $\Sigma \subset \mathbb{R}^{m+n}$ is a C^2 submanifold of dimension $m+\bar{n} = m+n-l$, which is the graph of an entire function $\Psi : \mathbb{R}^{m+\bar{n}} \to \mathbb{R}^l$ and satisfies the bounds

$$\|D\Psi\|_0 \leq c_0 \quad \text{and} \quad \mathbf{A} := \|A_\Sigma\|_0 \leq c_0, \tag{88}$$

where c_0 is a positive (small) dimensional constant. T is an integral current of dimension m with bounded support contained in Σ and which, for some open cylinder $\mathbf{C}_{4r}(x)$ (with $r \leq 1$) and some positive integer Q, satisfies

$$\mathbf{p}_\sharp T \llcorner \mathbf{C}_{4r}(x) = Q\,[\![B_{4r}(x)]\!] \quad \text{and} \quad \partial T \llcorner \mathbf{C}_{4r}(x) = 0\,, \tag{89}$$

where $\mathbf{p} : \mathbb{R}^{m+n} \to \mathbb{R}^m \times$ denotes the orthogonal projection onto the first factor.

If we denote by $\mathbf{p}_\pi^\perp$ the projection of $\mathbb{R}^{m+n}$ onto some plane π, then a quantity which will play a crucial role in the following discussion is the "height" of the current T in a given set E compared to some reference plane π, namely

$$\mathbf{h}(T,E,\pi) := \sup\{|\mathbf{p}_\pi^\perp(z-w)| : z,w \in \operatorname{spt}(T)\cap E\}\,.$$

Theorem 16.2 (Strong approximation, cf. [23, Theorem 1.4])**.** *There exist constants $C,\gamma_1,\varepsilon_1>0$ (depending on $m,n,\bar n,Q$) with the following property. Assume that T is area minimizing, satisfies Assumption 16.1 in the cylinder $\mathbf{C}_{4\,r}(x)$ and $E=\mathbf{E}(T,\mathbf{C}_{4\,r}(x))<\varepsilon_1$. Then, there is a map $f : B_r(x)\to \mathcal{A}_Q(\mathbb{R}^n)$, with $\operatorname{Gr}(f)\subset\Sigma$, and a closed set $K\subset B_r(x)$ such that*

$$\operatorname{Lip}(f)\le CE^{\gamma_1}+C\mathbf{A}r, \tag{90}$$

$$\mathbf{G}_f\llcorner(K\times\mathbb{R}^n)=T\llcorner(K\times\mathbb{R}^n)\quad\textit{and}\quad |B_r(x)\setminus K|\le C\,E^{\gamma_1}\left(E+r^2\,\mathbf{A}^2\right)\,r^m, \tag{91}$$

$$\left|\|T\|(\mathbf{C}_{\sigma\,r}(x))-Q\,\omega_m\,(\sigma\,r)^m-\tfrac12\int_{B_{\sigma\,r}(x)}|Df|^2\right|\le C\,E^{\gamma_1}\left(E+r^2\,\mathbf{A}^2\right)\,r^m\quad\forall\,0<\sigma\le 1. \tag{92}$$

If in addition $\mathbf{h}(T,\mathbf{C}_{4r}(x),\pi_0)\le r$, then

$$\operatorname{osc}(f)\le C\mathbf{h}(T,\mathbf{C}_{4r}(x),\pi_0)+C(E^{1/2}+r\,\mathbf{A})\,r\,, \tag{93}$$

where $\operatorname{osc}(f):=\sup\{|p-q| : p\in\operatorname{spt}(f(x)), q\in\operatorname{spt}(f(y)), x,y\in B_r(x)\}$.

The ideas of the proof of Theorem 16.2 are described in [17, Section 10]. In our context, we need a more general version of Theorem 16.2, where the area minimizing assumption is relaxed so to include the cases (b) and (c) in Theorem 4.4. In fact, since in case (c) the cone $T \times\!\!\!\!\times\, p$ is area minimizing, it is not difficult to see that the latter can be covered by a suitable modification of the arguments in [23]. A more delicate point is instead to include case (b).

In [26] we treat in fact a very general situation, which includes (b) as a very particular case. More precisely we prove that

Proposition 16.3. *There exist constants $M,C_{21},\beta_0,\varepsilon_{21}>0$ (depending on m,n,Q) with the following property. Assume that $T\in\mathbf{I}_m(\mathbb{R}^{m+n})$ is $\mathbf{\Omega}$-minimal (namely it satisfies (3)), that (89) holds in the cylinder $\mathbf{C}_{4r}(x)$ and $E=\mathbf{E}(T,\mathbf{C}_{4r}(x))<\varepsilon_{21}$. Then, there exist a map $f\colon B_r(x)\to\mathcal{A}_Q(\mathbb{R}^n)$ and a closed set $K\subset B_r(x)$ satisfying*

$$\operatorname{Lip}(f)\le C_{21}E^{\beta_0}+C_{21}\mathbf{\Omega}r\quad\textit{in case (a) and (c)}, \tag{94}$$

$$\mathbf{G}_f\llcorner(K\times\mathbb{R}^n)=T\llcorner(K\times\mathbb{R}^n)\quad\textit{and}\quad|B_r(x)\setminus K|\le C_{21}E^{\beta_0}\big(E+r^2\mathbf{\Omega}^2\big)r^m\,, \tag{95}$$

$$\left|\|T\|\left(\mathbf{C}_r(x)\right)-Q\omega_m r^m-\frac12\int_{B_r(x)}|Df|^2\right|\le C_{21}E^{\beta_0}\big(E+r^2\mathbf{\Omega}^2\big)r^m\,. \tag{96}$$

If in addition $\mathbf{h}(T,\mathbf{C}_{4r}(x))\le r$, then

$$\operatorname{osc}(f)\le C_{21}\mathbf{h}(T,\mathbf{C}_{4r}(x))+C_{21}(E^{1/2}+r\mathbf{\Omega})r\,. \tag{97}$$

16.1. **Truncations and homotopy lemma.** Although the general strategy of the proof of the latter Proposition is that of [23] for Theorem 16.2, several adjustments must be made when the arguments of [23] use the minimality condition. To this aim, [26] supplies suitable constructions of currents S to which the condition (3) can be applied in order to infer the necessary estimates which in [23] would be an easy consequence of the area minimizing assumption. Of course some care must be taken in order to show that the error terms which are generated are under control and the main ingredients are essentially the following ones.

The first is borrowed directly from [23], cf. [17, Section 10.3] for a thorough explanation.

Definition 16.4 (Maximal function of the excess measure, cf. [26, Proposition 1.7])**.** Given a current T as in Assumption 10.6 we introduce the "non-centered" maximal function of $\mathbf{e}_T$:

$$\mathbf{me}_T(y) := \sup_{y\in B_{s/2}(w)\subset B_{4r}(x)} \frac{\mathbf{e}_T(B_s(w))}{\omega_m\, s^m} = \sup_{y\in B_{s/2}(w)\subset B_{4r}(x)} \mathbf{E}(T, \mathbf{C}_s(w)).$$

Notice that with respect to [23, Definition 2.1], we define the Maximal function taking the supremum over balls of radius $s/2$ and not s. This is just a technicality which allows to construct the Lipschitz approximation of the next Proposition in the ball of radius $7r/2$.

Proposition 16.5 (Lipschitz approximation; cf. [23, Proposition 2.2])**.** *There exists a constant $C_{22}(m,n,Q) > 0$ with the following property. Let T be as in Proposition 16.3 in the cylinder $\mathbf{C}_{4s}(x)$. Set $E = \mathbf{E}(T, \mathbf{C}_{4r}(x))$, let $0 < \delta < 1$ be such that*

$$r_0 := 16\,\sqrt[m]{\frac{E}{\delta}} < 1,$$

and define $K := \left\{\mathbf{me}_T < \delta\right\} \cap B_{7r/2}(x)$. Then, there is $v \in \mathrm{Lip}(B_{7r/2}(x), \mathcal{A}_Q(\mathbb{R}^n))$ such that

$$\mathrm{Lip}(v) \le C_{22}\,\delta^{1/2},$$

$$\mathbf{G}_v \llcorner (K\times\mathbb{R}^n) = T\llcorner(K\times\mathbb{R}^n),$$

$$|B_s(x)\setminus K| \le \frac{10^m}{\delta}\,\mathbf{e}_T\Big(\{\mathbf{me}_T > 2^{-m}\delta\} \cap B_{s+r_0 s}(x)\Big) \quad \forall\, s \le \frac{7r}{2}\,. \tag{98}$$

When $\delta = E^{2\beta}$, the map u given by the proposition will be called, consistently with the terminology introduced in [23], *E^β-Lipschitz approximation of T in $\mathbf{C}_{7r/2}(x)$*.

The main new ingredient of [26] is then the following lemma, which is achieved using a suitable "homotopy construction", namely interpolating between the graphs of the maps f and g below.

Lemma 16.6 (Homotopy Lemma, cf. [26, Lemma 3.1])**.** *Let T be an $\mathbf{\Omega}$-almost minimizer which satisfies (89). There are positive dimensional constants ε_{22} and C_{25} such that, if $E = \mathbf{E}(T, \mathbf{C}_{4r}(x)) \le \varepsilon_{22}$, then the following holds. For every $R \in \mathbf{I}_m(\mathbf{C}_{3r}(x))$ such that $\partial R = \partial(T\llcorner\mathbf{C}_{3r}(x))$, we have*

$$\|T\|(\mathbf{C}_{3r}(x)) \le \mathbf{M}(R) + C_{25} r^{m+1}\mathbf{\Omega} E^{1/2}\,. \tag{99}$$

Moreover, let $\beta \leq \frac{1}{2m}$, $s \in]r, 2r[$, $R = \mathbf{G}_g \llcorner \mathbf{C}_s(x)$ for some Lipschitz map $g\colon B_s \to \mathcal{A}_Q(\mathbb{R}^n)$ with $\mathrm{Lip}(g) \leq 1$ and f be the E^β-approximation of T in $\mathbf{C}_{3r}$. If $f = g$ on ∂B_s and $P \in \mathbf{I}_m(\mathbb{R}^{m+n})$ is such that $\partial P = \partial((T - \mathbf{G}_f)\llcorner \mathbf{C}_s)$, then

$$\|T\|(\mathbf{C}_s(x)) \leq \mathbf{M}(\mathbf{G}_g) + \mathbf{M}(P) + C_{25}\mathbf{\Omega}\Big(E^{3/4}r^{m+1} + (\mathbf{M}(P))^{1+1/m} + \int_{B_s(x)} \mathcal{G}(f,g)\Big)\,. \tag{100}$$

17. The construction of the branched center manifold

As already mentioned, the graph of the average $\boldsymbol{\eta}\circ f$ in Proposition 16.3 provides a rather good model for the branched center manifold at scales where the excess or the height are small, but at least one of them is *not too* small. In order to identify these scales, an efficient strategy is to set up a Whitney type refining procedure of a suitable cubical subdivision of the rerefence $\mathfrak{B}_{\bar{Q},2}$. Recall that we have a T satisfying Assumption 11.3. We then start the refining procedure on some cubical subdivision of the model $\mathfrak{B}_{\bar{Q},2} \setminus \{0\}$, where where the squares become smaller and smaller as they approach the origin. For each square L we look at the map u of Assumption 11.3 restricted on L and focus on the portion of the horned neighborhood $\mathbf{V}_{u,a}$ (see again Assumption 11.3) which contains the graph of $u|_L$. We take the portion of the current T which is lying in this open set and we then look at the excess of this current in a ball which has side comparable to the side of L. If the excess and the height are both rather small, we then refine the square L in 4 squares of half the side of L. Otherwise we keep L.

This process is repeated (respecting a certain order in the selection of the squares, see the section below for the details). However, if we take this procedure literally we might end up with nearby squares whose side ratios are not controlled. In order to avoid this we also stop the subdivision of squares in any square which is adjacent to another one where the subdivision was stopped at an earlier state of the process.

At the end of this procedure we are left with a closed subset $\mathbf{\Gamma}$ of $\mathfrak{B}_{\bar{Q},2}\setminus\{0\}$ and a suitable subdivision of its complement in dyadic squares. In each square of the final subdivision we wish to select a suitable cylinder where we apply Proposition 16.3. In this region we will use the graph of the average of the sheets of the approximation as model for the center manifold, which will be written as a graph over L of a suitable map. The latter maps will be then smoothed and patched together with a partition of unity: after showing that the resulting function extends (smoothly) to $\mathbf{\Gamma}$, its graph will constitute the center manifold $\mathcal{M}$ of Theorem 13.3.

17.1. Choice of some parameters and smallness of some other constants.

As in [21] the construction of the center manifold involves several parameters. Three of them will appear as exponents of the two relevant lenghtscales (sidelength of the specific square and distance to the origin) in several estimates.

Assumption 17.1. Let T be as in Assumptions 11.3 and 12.2 and in particular recall the exponents $\bar{\alpha}, b, a$ and γ defined therein. We choose the positive exponents γ_0, β_2 and δ_1 (in the given order) so that

$$\gamma_0 < \min\{\gamma, \bar\alpha, a-b, b-\tfrac{b+1}{2}, \log_2\tfrac{6}{5}\} \tag{101}$$

$$\beta_2 < \min\{\varepsilon_0, \tfrac{\gamma_0}{4}, \tfrac{a}{b}-1, \tfrac{\bar\alpha}{2}, \tfrac{\beta_0\gamma_0}{2}\} \qquad\qquad b > \tfrac{1+b}{2}(1+\beta_2) \tag{102}$$

$$\beta_2 - 2\delta_1 \geq \tfrac{\beta_2}{3} \qquad \beta_0(2-2\delta_1) - 2\delta_1 \geq 2\beta_2 \tag{103}$$

(where β_0 is the constant of [26, Theorem 1.5] and we assume it is smaller than 1/2)

Having fixed γ_0, β_2 and δ_1 we introduce five further parameters: M_0, N_0, C_e, C_h and ε_2. We will impose several inequalities upon them, but following a very precise hierarchy, which ensures that all the conditions required in the remaining statements can be met. We will use the term "geometric" when such conditions depend only upon $\bar n, n, Q, \bar Q, \gamma_0, \beta_2$ and δ_1, whereas we keep track of their dependence on M_0, N_0, C_e and C_h using the notation $C = C(M_0), C(M_0, N_0)$ and so on. ε_2 is always the last parameter to be chosen: it will be small depending upon all the other constants, but constants will never depend upon it.

Assumption 17.2 (Hierarchy of the parameters)**.** In all the subsequent statements

- $M_0 \geq 4$ is larger than a geometric constant and N_0 is a natural number larger than $C(M_0)$; one such condition is recurrent and we state it here:
$$\sqrt{2}M_0 2^{10-N_0} \leq 1\,; \tag{104}$$
- C_e is larger than $C(M_0, N_0)$;
- C_h is larger than $C(M_0, N_0, C_e)$;
- $\varepsilon_2 > 0$ is smaller than $c(M_0, N_0, C_e, C_h) > 0$.

17.2. **Whitney decomposition of $\mathfrak{B}_{\bar Q,2}$.** From now on we will use $\mathfrak{B}$ for $\mathfrak{B}_{\bar Q,2}$, since the positive natural number $\bar Q$ is fixed for the rest of our discussions. In this section we show how to decompose $\mathfrak{B}\setminus\{0\}$ suitably. More precisely, a closed subset L of $\mathfrak{B}$ will be called a dyadic square if it is a connected component of $\mathfrak{B}\cap(H\times\mathbb{C})$ for some euclidean dyadic square $H = [a_1, a_1+2\ell]\times[a_2, a_2+2\ell] \subset \mathbb{R}^2 = \mathbb{C}$ with

- $\ell = 2^{-j}$, $j\in\mathbb{N}, j\geq 2$, and $a\in 2^{1-j}\mathbb{Z}^2$;
- $H\subset[-1,1]^2$ and $0\notin H$.

Observe that L is truly a square, both from the topological and the metric point of view. 2ℓ is the sidelength of both H and L. Note that $\mathfrak{B}\cap(H\times\mathbb{C})$ consists then of $\bar Q$ distinct squares $L_1, \ldots, L_{\bar Q}$. $z_H := a + (\ell,\ell)$ is the center of the square H. Each L lying over H will then contain a point (z_H, w_L), which is the center of L. Depending upon the context we will then use z_L rather than z_H for the first (complex) component of the center of L.

The family of all dyadic squares of $\mathfrak{B}$ defined above will be denoted by $\mathscr{C}$. We next consider, for $j\in\mathbb{N}$, the dyadic closed annuli

$$\mathcal{A}_j := \mathfrak{B}\cap\left(([-2^{-j}, 2^{-j}]^2\setminus]-2^{-j-1}, 2^{-j-1}[^2)\times\mathbb{C}\right).$$

Each dyadic square L of $\mathfrak{B}$ is then contained in exactly one annulus $\mathcal{A}_j$ and we define $\mathbf{d}(L) := 2^{-j-1}$. Moreover $\ell(L) = 2^{-j-k}$ for some $k\geq 2$. We then denote by $\mathscr{C}^{k,j}$ the family of those dyadic squares L such that $L\subset\mathcal{A}_j$ and $\ell(L) = 2^{-j-k}$. Observe that, for each $j \geq 1, k \geq 2$, $\mathscr{C}^{k,j}$ is a covering of $\mathcal{A}_j$ and that two elements of $\mathscr{C}^{k,j}$ can only intersect at their boundaries. Moreover, any element of $\mathscr{C}^{k,j}$ can intersect at most 8 other elements of $\mathscr{C}^{k,j}$. Finally, we set $\mathscr{C}^k := \bigcup_{j\geq 2} \mathscr{C}^{k,j}$. Observe now that $\mathscr{C}^k$ covers a punctured neighborhood of 0 and that if $L \in \mathscr{C}^k$, then

- L intersects at most 9 other elements $J \in \mathscr{C}^k$;
- If $L \cap J \neq \emptyset$, then $\ell(J)/2 \leq \ell(L) \leq 2\ell(L)$ and $L \cap J$ is either a vertex or a side of the smallest among the two.

More in general if the intersection of two distinct elements L and J in $\mathscr{C} = \bigcup_k \mathscr{C}^k$ has nonempty interior, then one is contained in the other: if $L \subset J$ we then say that L is a descendant of J and J an ancestor of L. If in addition $\ell(L) = \ell(J)/2$, then we say that L is a son of J and J is the father of L. When L and J intersect only at their boundaries, we then say that L and J are adjacent.

Next, for each dyadic square L we set $r_L := \sqrt{2} M_0 \ell(L)$. Note that, by our choice of N_0, we have that:

$$\text{if } L \in \mathscr{C}^{k,j} \text{ and } k \geq N_0, \text{ then } \mathbf{C}_{64 r_L}(z_L) \subset \mathbf{C}_{2^{1-j}} \setminus \mathbf{C}_{2^{-2-j}}. \tag{105}$$

In particular $\mathbf{V}_{u,a} \cap \mathbf{C}_{64 r_L}(z_L)$ consists of $\bar{Q}$ connected components and we can select the one containing $(z_L, u(z_L, w_L))$, which we will denote by $\mathbf{V}_L$. We will then denote by T_L the current $T \llcorner \mathbf{V}_L$. According to Lemma 12.1, $\mathbf{V}_L \cap \{z_L\} \times \mathbb{R}^n$ contains at least one point of $\mathrm{spt}(T)$: we select any such point and denote it by $p_L = (z_L, y_L)$. Correspondingly we will denote by $\mathbf{B}_L$ the ball $\mathbf{B}_{64 r_L}(p_L)$.

Definition 17.3. The height of a current S in a set E with respect to a plane π is given by

$$\mathbf{h}(S, E, \pi) := \sup\{|\mathbf{p}_\pi^\perp(p - q)| : p, q \in \mathrm{spt}(S) \cap E\}\,. \tag{106}$$

If $E = \mathbf{C}_r(p, \pi)$ we will then set $\mathbf{h}(S, \mathbf{C}_r(p, \pi)) := \mathbf{h}(S, \mathbf{C}_r(p, \pi), \pi)$. If $E = \mathbf{B}_r(p)$, T is as in Assumption 10.6 and $p \in \Sigma$ (in the cases (a) and (c) of Theorem 4.4), then $\mathbf{h}(T, \mathbf{B}_r(p)) := \mathbf{h}(T, \mathbf{B}_r(p), \pi)$ where π gives the minimal height among all π for which $\mathbf{E}(T, \mathbf{B}_r(p), \pi) = \mathbf{E}(T, \mathbf{B}_r(p))$ (and such that $\pi \subset T_p\Sigma$ in case (a) and (c) of Theorem 4.4). Moreover, for such π we say that it optimizes the excess and the height in $\mathbf{B}_r(p)$.

We are now ready to define the dyadic decomposition of $\mathfrak{B} \setminus 0$.

Definition 17.4 (Refining procedure)**.** We build inductively the families of squares

$$\mathscr{S} \quad \text{and} \quad \mathscr{W} = \mathscr{W}_e \cup \mathscr{W}_h \cup \mathscr{W}_n\,,$$

together with their subfamilies $\mathscr{S}^k = \mathscr{S} \cap \mathscr{C}^k$, $\mathscr{S}^{k,j} = \mathscr{S} \cap \mathscr{C}^{k,j}$ and so on. First of all, we set $\mathscr{S}^k = \mathscr{W}^k = \emptyset$ for $k < N_0$. For $k \geq N_0$ we use a double induction. Having defined $\mathscr{S}^{k'}, \mathscr{W}^{k'}$ for all $k' < k$ and $\mathscr{S}^{k,j'}, \mathscr{W}^{k,j'}$ for all $j' < j$, we pick all squares L of $\mathscr{C}^{k,j}$ which do not have any ancestor already assigned to $\mathscr{W}$ and we proceed as follows.

(EX) We assign L to $\mathscr{W}_e^{k,j}$ if

$$\mathbf{E}(T_L, \mathbf{B}_L) > C_e \mathbf{m}_0 \mathbf{d}(L)^{2\gamma_0 - 2 + 2\delta_1} \ell(L)^{2-2\delta_1}; \tag{107}$$

(HT) We assign L to $\mathscr{W}_h^{k,j}$ if we have not assigned it to $\mathscr{W}_e$ and

$$\mathbf{h}(T_L, \mathbf{B}_L) > C_h \mathbf{m}_0^{1/4} \mathbf{d}(L)^{\gamma_0/2-\beta_2} \ell(L)^{1+\beta_2}; \tag{108}$$

(NN) We assign L to $\mathscr{W}_n^{k,j}$ if we have not assigned it to $\mathscr{W}_e \cup \mathscr{W}_h$ and it intersects a square J already assigned to $\mathscr{W}$ with $\ell(J) = 2\ell(L)$.

(S) We assign L to $\mathscr{S}^{k,j}$ if none of the above occurs.

We finally set

$$\mathbf{\Gamma} := ([-1,1]^2 \times \mathbb{R}^2) \cap \mathfrak{B} \setminus \bigcup_{L \in \mathscr{W}} L = \{0\} \cup \bigcap_{k \geq N_0} \bigcup_{L \in \mathscr{S}^k} L. \tag{109}$$

The next proposition is than a natural outcome of the way the refining process has been designed.

Proposition 17.5 (Whitney decomposition, cf. [24, Proposition 3.5])**.** *Let T, γ_0, β_2 and δ_1 be as in the Assumptions 11.3, 12.2 and 17.1. If $M_0 \geq C$, $N_0 \geq C(M_0)$, $C_e, C_h \geq C(M_0, N_0)$ (for suitably large constants) and ε_2 is sufficiently small then:*

(i) $\ell(L) \leq 2^{-N_0+1}|z_L|$ $\forall L \in \mathscr{S} \cup \mathscr{W}$;

(ii) $\mathscr{W}^k = \emptyset$ *for all* $k \leq N_0 + 6$;

(iii) $\mathbf{\Gamma}$ *is a closed set and* $\mathrm{sep}(\mathbf{\Gamma}, L) := \inf\{|x - x'| : x \in \mathbf{\Gamma}, x' \in L\} \geq 2\ell(L)$ $\forall L \in \mathscr{W}$.

Moreover, the following estimates hold with $C = C(M_0, N_0, C_e, C_h)$:

$$\mathbf{E}(T_J, \mathbf{B}_J) \leq C_e \mathbf{m}_0 \mathbf{d}(J)^{2\gamma_0-2+2\delta_1} \ell(J)^{2-2\delta_1} \qquad \forall J \in \mathscr{S}, \tag{110}$$

$$\mathbf{h}(T_J, \mathbf{B}_J) \leq C_h \mathbf{m}_0^{1/4} \mathbf{d}(J)^{\gamma_0/2-\beta_2} \ell(J)^{1+\beta_2} \qquad \forall J \in \mathscr{S}, \tag{111}$$

$$\mathbf{E}(T_H, \mathbf{B}_H) \leq C\, \mathbf{m}_0 \mathbf{d}(H)^{2\gamma_0-2+2\delta_1} \ell(H)^{2-2\delta_1} \qquad \forall H \in \mathscr{W}, \tag{112}$$

$$\mathbf{h}(T_H, \mathbf{B}_H) \leq C\, \mathbf{m}_0^{1/4} \mathbf{d}(H)^{\gamma_0/2-\beta_2} \ell(H)^{1+\beta_2} \qquad \forall H \in \mathscr{W}. \tag{113}$$

17.3. **Approximating functions and construction algorithm.** As already explained in the previous paragraphs a fundamental point is that in (a suitable portion of) each $\mathbf{B}_L$ the current T_L can be approximated efficiently with a graph of a Lipschitz multiple-valued map. The average of the sheets of this approximating map will then be used as a local model for the center manifold.

Definition 17.6 (π-approximations)**.** Let $L \in \mathscr{S} \cup \mathscr{W}$ and π be a 2-dimensional plane. If $T_L \llcorner \mathbf{C}_{32r_L}(p_L, \pi)$ fulfills the assumptions of the Lipschitz approximation Theorem (namely Proposition 16.3) in the cylinder $\mathbf{C}_{32r_L}(p_L, \pi)$, then the resulting map $f : B_{8r_L}(p_L, \pi) \to \mathcal{A}_Q(\pi^\perp)$ given by Proposition 16.3 (cf. [26, Theorem 1.5]) is a *π-approximation of T_L in $\mathbf{C}_{8r_L}(p_L, \pi)$*.

In fact, it is rather important to notice that the π-approximation above exists when π is chosen to be the "best plane" π_L:

Lemma 17.7 (Cf. [24, Lemma 3.7])**.** *Let the assumptions of Proposition 17.5 hold and assume $C_e \geq C^\star$ and $C_h \geq C^\star C_e$ for a suitably large $C^\star(M_0, N_0)$. For each $L \in \mathscr{W} \cup \mathscr{S}$ we choose a plane π_L which optimizes the excess and the height in $\mathbf{B}_L$. For any choice* *of the other parameters, if ε_2 is sufficiently small, then $T_L \llcorner \mathbf{C}_{32 r_L}(p_L, \pi_L)$ satisfies the assumptions of the Lipschitz approximation Theorem for any $L \in \mathscr{W} \cup \mathscr{S}$.*

As in [21], we wish to find a suitable smoothing of the average of the π-approximation $\boldsymbol{\eta} \circ f$. However the smoothing procedure is more complicated in the case (b) of Theorem 4.4: rather than smoothing by convolution, we need to solve a suitable elliptic system of partial differential equations. This approach can in fact be used in cases (a) and (c) as well. In several instances regarding case (a) and (c) we will have to manipulate maps defined on some affine space $q + \pi$ and taking value on $\pi^\perp$, where $q \in \Sigma$ and $\pi \subset T_q\Sigma$. In such cases it is convenient to introduce the following conventions: the maps will be regarded as maps defined on π (requiring a simple translation by q), the space $\pi^\perp$ will be decomposed into $\varkappa := \pi^\perp \cap T_q\Sigma$ and its orthogonal complement $T_q\Sigma^\perp$ and we will regard Ψ_q as a map defined on $\pi \times \varkappa$ and taking values in $T_q\Sigma^\perp$. Similarly, elements of $\pi^\perp$ will be decomposed as $(\xi, \eta) \in \varkappa \times T_q\Sigma^\perp$.

Definition 17.8 (Smoothing)**.** Let L and π_L be as in Lemma 17.7 and denote by f_L the corresponding π_L-approximation. In the cases (a)&(c) of Theorem 4.4 we let $\bar f(x) := \sum_i [\![\mathbf{p}_{T_{p_L}\Sigma}(f_i)]\!]$ be the projection of f_L on the tangent $T_{p_L}\Sigma$, whereas in the other case ((b), i.e. when the current is semicalibrated) we set $\bar f = f$. We let $\bar h_L$ be a solution (provided it exists) of

$$\begin{cases} \mathscr{L}_L \bar h_L = \mathscr{F}_L \\ \\ \bar h_L\big|_{\partial B_{8r_L}(p_L,\pi_L)} = \boldsymbol{\eta} \circ \bar f_L \,, \end{cases} \tag{114}$$

where $\mathscr{L}_L$ is a suitable second order linear elliptic operator with constant coefficients and $\mathscr{F}_L$ a suitable affine map: the precise expressions for $\mathscr{L}_L$ and $\mathscr{F}_L$ depend on a careful Taylor expansion of the first variations formulae and are given in Proposition 18.3 below. We then set $h_L(x) := (\bar h_L(x), \Psi_{p_L}(x, \bar h_L(x))$ in case (a) and (c) and $h_L(x) = \bar h_L(x)$ in case (b). The map h_L is the *tilted interpolating function* relative to L.

In what follows we will deal with graphs of multivalued functions f in several system of coordinates. These objects can be naturally seen as currents $\mathbf{G}_f$ (see [20]) and in this respect we will use extensively the notation and results of [20] (therefore $\mathrm{Gr}(f)$ will denote the "set-theoretic" graph). We are now ready to introduce the "bricks" with which we will construct the branched center manifolds.

Lemma 17.9 (Basic building blocks, cf. [24, Lemma 3.9])**.** *Let the assumptions of Proposition 17.5 hold and assume $C_e \geq C^\star$ and $C_h \geq C^\star C_e$ (where $C^\star$ is the constant of Lemma 17.7). For any choice of the other parameters, if ε_2 is sufficiently small the following holds. For any $L \in \mathscr{W} \cup \mathscr{S}$, there is a unique solution $\bar h_L$ of* (114) *and there is a smooth $g_L : B_{4r_L}(z_L, \pi_0) \to \pi_0^\perp$ such that $\mathbf{G}_{g_L} = \mathbf{G}_{h_L} \llcorner \mathbf{C}_{4r_L}(p_L, \pi_0)$, where h_L is the tilted interpolating function of Definition 17.8. Using the charts introduced in Definition 10.8, the map g_L will be considered as defined on the ball $B_{4r_L}(z_L, w_L) \subset \mathfrak{B}$.*

The center manifold is defined by gluing together the maps g_L.

Definition 17.10 (Interpolating functions)**.** The map g_L in Lemma 17.7 will be called the *L-interpolating function.* Fix next a $\vartheta \in C_c^\infty\big([-\frac{17}{16}, \frac{17}{16}]^m, [0,1]\big)$ which is nonnegative and is identically 1 on $[-1,1]^m$. For each k let $\mathscr{P}^k := \mathscr{S}^k \cup \bigcup_{i=N_0}^k \mathscr{W}^i$ and for $L \in \mathscr{P}^k$ define $\vartheta_L((z,w)) := \vartheta(\frac{z-z_L}{\ell(L)})$. Set

$$\hat{\varphi}_j := \frac{\sum_{L\in\mathscr{P}^j} \vartheta_L g_L}{\sum_{L\in\mathscr{P}^j} \vartheta_L} \qquad \text{on } \{(z,w) \in \mathfrak{B} : z \in [-1,1]^2 \setminus \{0\}\} \tag{115}$$

and extend the map to 0 defining $\hat{\varphi}_j(0) = 0$. In case (b) of Theorem 4.4 we set $\varphi_j := \hat{\varphi}_j$. In cases (a) and (c) we let $\bar{\varphi}_j(z,w)$ be the first $\bar{n}$ components of $\hat{\varphi}_j(z,w)$ and define $\varphi_j(z,w) = \big(\bar{\varphi}_j(z,w), \Psi(z, \bar{\varphi}_j(z,w))\big)$. φ_j will be called the *glued interpolation* at step j.

We are now ready to identify the surface which we call "branched center manifold" (again notice that for $\bar{Q} = 1$ there is certainly no branching, since the surface is a classical $C^{1,\alpha}$ graph, but we keep nonetheless the same terminology). In the statement we will need to "enlarge" slightly dyadic squares: given $L \in \mathscr{C}$ let H be the dyadic square of $\mathbb{R}^2 = \mathbb{C}$ so that L is a connected component of $\mathfrak{B} \cap (H \times \mathbb{C})$. Given $\sqrt{2}\sigma < |z_L| = |z_H|$, we let H' be the closed euclidean square of $\mathbb{R}^2$ which has the same center as H and sides of length 2σ, parallel to the coordinate axes. The square L' concentric to L and with sidelength $2\ell(L') = 2\sigma$ is then defined to be that connected component of $\mathfrak{B} \cap (H' \times \mathbb{C})$ which contains L.

Theorem 17.11 (Cf. [24, Theorem 3.11])**.** *Under the same assumptions of Lemma 17.7, the following holds provided ε_2 is sufficiently small.*

(i) *For $\kappa := \beta_2/4$ and $C = C(M_0, N_0, C_e, C_h)$ we have (for all j)*

$$|\varphi_j(z,w)| \le C\boldsymbol{m}_0^{1/4}|z|^{1+\gamma_0/2} \qquad \textit{for all } (z,w) \tag{116}$$

$$|D^l\varphi_j(z,w)| \le C\boldsymbol{m}_0^{1/2}|z|^{1+\gamma_0-l} \qquad \textit{for } l = 1,\dots,3 \textit{ and } (z,w) \neq 0 \tag{117}$$

$$[D^3\varphi_j]_{\mathcal{A}_j,\kappa} \le C\boldsymbol{m}_0^{1/2}2^{2j}\,. \tag{118}$$

(ii) *The sequence φ_j stabilizes on every square $L \in \mathscr{W}$: more precisely, if $L \in \mathscr{W}^i$ and H is the square concentric to L with $\ell(H) = \frac{9}{8}\ell(L)$, then $\varphi_k = \varphi_j$ on H for every $j,k \ge i+2$. Moreover there is an admissible smooth branching $\boldsymbol{\varphi} : \mathfrak{B}\cap([-1,1]^2\times\mathbb{C}) \to \mathbb{R}^n$ such that $\varphi_k \to \boldsymbol{\varphi}$ uniformly on $\mathfrak{B} \cap ([-1,1]^2 \times \mathbb{C})$ and in $C^3(\mathcal{A}_j)$ for every $j \ge 0$.*

(iii) *For some constant $C = C(M_0, N_0, C_e, C_h)$ and for $a' := b + \gamma_0 > b$ we have*

$$|u(z,w) - \boldsymbol{\varphi}(z,w)| \le C\boldsymbol{m}_0^{1/2}|z|^{a'}\,. \tag{119}$$

18. The elliptic system used to regularize $\boldsymbol{\eta} \circ f_L$

The main analytical difficulty in the construction of the center manifold is estimating the $C^{3,\kappa}$ norms of the building blocks g_L's and of their differences. In particular note that, when we differentiate the formula (115), the derivatives of the partition of unity functions ϑ_L's create high coefficients which multiply differences of g_L's for nearby squares. Such differences must then be relatively small in order to compensate the size of $D^j\vartheta_L$.

In fact the estimates upon g_L's involve several subtle computations since the h_L's are generated with different systems of coordinates. However this issue is solved using the tools developed in [21]. We thus focus on the following model problem: assume we fix a square L where the Whitney refining procedure has stopped and let J be the square which contains L and has been subdivided at the previous step of the refining procedure. The maps h_J and h_L are both generated with respect to two different systems of coordinates but let us assume that they are in fact the same.

Note first that the Lipschitz approximations f_L and f_J coincide on a very large set, since they both coincide with the current T except for an "error set" of rather small measure. Of course the same property holds for the averages $\boldsymbol{\eta}\circ f_L$ and $\boldsymbol{\eta}\circ f_J$. We thus can imagine that the regularization of $\boldsymbol{\eta}\circ f_L$ (namely h_L) and that of $\boldsymbol{\eta}\circ f_J$ (namely h_J) differ very little. In fact since such regularizations are reached solving a suitable elliptic system, we can hope to find efficient bounds even for the differences of the derivatives of f_J and h_L. These bounds could repeated at all larger scales. More precisely, if we consider the chain of ancestors of L, $L = J_0 \subset J = J_1 \subset J_2 \subset \ldots \subset J_N$, we then could estimate

$$\sum_{\ell=0}^{N} \|h_{J_\ell} - h_{J_{\ell-1}}\|_{C^j}\, .$$

Since the square J_N of the starting cubical decomposition of $\mathfrak{B}\setminus\{0\}$ has sidelength comparable to its distance to the origin, namely to $\mathbf{d}(J_N)$, it is then clear that $\|h_{J_N}\|_{C^j}$ depends only upon $\mathbf{d}(L) = \mathbf{d}(J_N)$. We thus gain a resulting estimate upon $\|h_L\|_{C^j}$.

Assuming that this scheme works, it is important that the regularization h_L does not differ significantly from $\boldsymbol{\eta}\circ f_L$, otherwise it cannot give a good building block for the center manifold. Recall indeed that the estimate (83) requires the graph of the approximating map $\mathcal{N}$ to coincide largely with the current T. Whereas the estimate (82) forces the average of $\mathcal{N}$ to be rather small: these two estimates would be clearly incompatible if h_L, which at the given region of interest coincides essentially with the center manifold $\mathcal{M}$, were to differ too much from $\boldsymbol{\eta}\circ f_L$. Since h_L is deduced by solving a suitable elliptic system, we conclude then that $\boldsymbol{\eta}\circ f_L$ should be an approximate solution of such system.

18.1. **The generalized maps f_{HL}.** Since, as already mentioned, the relevant systems of coordinates change from square to square, the approach of [24] (which in fact follows that of [21]) is to consider more general functions f_{HL}, $\boldsymbol{\eta}\circ f_{HL}$ and h_{HL}. The latter functions depend upon two squares, which are assumed to be appropriately related (in particular L is either an ancestor or a neighborhood of H): the square H "decides" the system of coordinates, whereas the square L decides the relevant region where we approximate the current T. More precisely

Definition 18.1. After applying the Lipschitz Approximation Theorem, namely [26, Theorem 1.5], to $T_L \llcorner \mathbf{C}_{32r_L}(p_L, \pi_H)$ in the cylinder $\mathbf{C}_{32r_L}(p_L,\pi_H)$ we denote by f_{HL} the corresponding π_H-approximation. However, rather then defining f_{HL} on the disk $B_{8r_L}(p_L,\pi_H)$, by applying a translation we assume that the domain of f_{HL} is the disk $B_{8r_L}(p_{HL},\pi_H)$

where $p_{HL} = p_H + \mathbf{p}_{\pi_H}(p_L - p_H)$. Note in particular that $\mathbf{C}_r(p_{HL}, \pi_H)$ equals $\mathbf{C}_r(p_L, \pi_H)$, whereas $B_{8r_L}(p_{HL}, \pi_H) \subset p_H + \pi_H$ and $p_H \in \mathbf{B}_{8r_L}(p_{HL}, \pi_H)$.

Observe that $f_{LL} = f_L$.

18.2. **First variations.** The next proposition is the core in the construction of the center manifold and it is the main reason behind the C^{3,γ_0} estimate for the glued interpolation. It is also the place where our proof in [24] differs most from that of [21].

Definition 18.2. Let L be either a square adjacent to H or an ancestor of H. In the cases (a) and (c) of Theorem 4.4 we denote by $\varkappa_H$ the orthogonal complement in $T_{p_H}\Sigma$ of π_H and we denote by $\bar{f}_{HL}$ the map $\mathbf{p}_{\varkappa_H} \circ f_{HL}$.

The elliptic system to be solved in order to gain our regularization h_L has the following form. Given a vector valued map $v : p_H + \pi_H \supset \Omega \to \varkappa_H$ and after introducing an orthonormal system of coordinates x^1, x^2 on π_H and $y^1, \ldots, y^{\bar{n}}$ on $\varkappa_H$, the system is given by the $\bar{n}$ equations

$$\Delta v^k + \underbrace{(\mathbf{L}_1)^k_{ij}\partial_j v^i + (\mathbf{L}_2)^k_i v^i}_{=:\mathscr{E}^k(v)} = \underbrace{(\mathbf{L}_3)^k_i (x - x_H)^i + (\mathbf{L}_4)^k}_{=:\mathscr{F}^k}, \tag{120}$$

where we follow Einstein's summation convention and the tensors $\mathbf{L}_i$ have constant coefficients. After introducing the operator $\mathscr{L}(v) = \Delta v + \mathscr{E}(v)$ we summarize the corresponding elliptic system (120) as

$$\mathscr{L}(v) = \mathscr{F}\,. \tag{121}$$

We then have a corresponding weak formulation for $W^{1,2}$ solutions of (121), namely v is a weak solution in a domain D if the integral

$$\mathscr{I}(v, \zeta) := \int (Dv : D\zeta + (\mathscr{F} - \mathscr{E}(v)) \cdot \zeta) \tag{122}$$

vanishes for smooth test functions ζ with compact support in D.

The following Proposition is then the core estimate concerning $\boldsymbol{\eta} \circ f_{HL}$: in particular the estimate (123) ensures that $\boldsymbol{\eta} \circ f_{HL}$ is close, in a suitable weak sense, to be a solution of (121). This property is crucial in ensuring that there is a solution h_{HL} which is sufficiently close to $\boldsymbol{\eta} \circ f_{HL}$.

Proposition 18.3 (Cf. [24, Proposition 6.4])**.** *Let H and L be as in Definition 18.2 (including the possibility that $H = L$) and consider f_{HL}, $\bar{f}_{HL}$ and $\varkappa_H$ as in Definition 18.1 and Definition 18.2. Then, there exist tensors with constant coefficients $\mathbf{L}_1, \ldots, \mathbf{L}_4$ and a constant $C = C(M_0, N_0, C_e, C_h)$, with the following properties:*

(i) *The tensors depend upon H and Σ (in the cases (a) and (c) of Theorem 4.4) or ω (in case (b)) and $|\mathbf{L}_1| + |\mathbf{L}_2| + |\mathbf{L}_3| + |\mathbf{L}_4| \leq C\boldsymbol{m}_0^{1/2}$.*

(ii) *If $\mathscr{I}_H$, $\mathscr{L}_H$ and $\mathscr{F}_H$ are defined through* (120), (121) *and* (122), *then*

$$\mathscr{I}_H(\boldsymbol{\eta} \circ \bar{f}_{HL}, \zeta) \leq C\boldsymbol{m}_0\, \mathbf{d}(L)^{2(1+\beta_0)\gamma_0 - 2 - \beta_2}\, r_L^{4+\beta_2}\, \|D\zeta\|_0 \tag{123}$$

for all $\zeta \in C^\infty_c(B_{8r_L}(p_{HL}, \pi_H), \varkappa_H)$.

19. The approximation on the normal bundle of the center manifold

To carry on our program for proving Theorem 13.3 we now need to approximate again our area minimizing current on the normal bundle of the branched center manifold. In order to start our considerations we introduce the "Whitney regions" which correspond roughly to those portions of the center manifold lying "above" one of the squares in the Whitney decomposition of $\mathfrak{B} \setminus \{0\}$ introduced in the previous sections to construct the center manifold.

Definition 19.1 (Center manifold, Whitney regions)**.** The manifold $\mathcal{M} := \mathrm{Gr}(\boldsymbol{\varphi})$, where $\boldsymbol{\varphi}$ is as in Theorem 17.11, is called *a branched center manifold for T relative to* $\mathbf{G}_u$. It is convenient to introduce the map $\mathbf{\Phi} : \mathfrak{B} \cap ([-1,1]^2 \times \mathbb{C}) \to \mathbb{R}^{2+n}$ given by $\mathbf{\Phi}(z,w) = (z, \boldsymbol{\varphi}(z,w))$. If we neglect the origin, $\mathbf{\Phi}$ is then a classical (C^3) parametrization of $\mathcal{M}$. $\mathbf{\Phi}(\mathbf{\Gamma})$ will be called the contact set. Moreover, to each $L \in \mathscr{W}$ we associate a *Whitney region* $\mathcal{L}$ on $\mathcal{M}$ as follows:

(WR) $\mathcal{L} := \mathbf{\Phi}(H \cap ([-1,1]^2 \times \mathbb{C}))$, where H is the square concentric to L with $\ell(H) = \frac{17}{16}\ell(L)$.

For any Borel set $\mathcal{V} \subset \mathcal{M}$ we will denote by $|\mathcal{V}|$ its $\mathcal{H}^2$-measure and will write $\int_{\mathcal{V}} f$ for the integral of f with respect to $\mathcal{H}^2$. $\mathcal{B}_r(q)$ denotes the geodesic balls in $\mathcal{M}$.

We next define the open set

(V) $\mathbf{V} := \{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^n : x \in [-1,1]^2 \text{ and } |\boldsymbol{\varphi}(x,w) - y| < c_s |x|^b/2\}$.

$\mathbf{V}$ is clearly an horned neighborhood of the graph of $\boldsymbol{\varphi}$. By (66), Assumption 11.3 and Theorem 17.11 it is clear that the following corollary holds

Corollary 19.2. *Under the hypotheses of Theorem 17.11, there is $r > 0$ such that*

(i) *For every $x \in \mathbb{R}^2$ with $0 < |x| = 2\rho < 2r$, the set $\mathbf{C}_\rho(x) \cap \mathbf{V}$ consists of $\bar{Q}$ distinct connected components and* $\mathrm{spt}(T) \cap \mathbf{C}_{3r} \subset \mathbf{V}$.

(ii) *There is a well-defined nearest point projection $\mathbf{p} : \mathbf{V} \cap \mathbf{C}_{4r} \to \mathrm{Gr}(\boldsymbol{\varphi})$, which is a $C^{2,\kappa}$ map.*

(iii) *For every $L \in \mathscr{W}$ with $\mathbf{d}(L) \leq 2r$ and every $q \in L$ we have*

$$\mathrm{spt}(\langle T, \mathbf{p}, \mathbf{\Phi}(q)\rangle) \subset \left\{ y \in \mathbb{R}^{2+n} \, : |\mathbf{\Phi}(q) - y| \leq C \boldsymbol{m}_0^{1/4} \mathbf{d}(L)^{\gamma_0/2 - \beta_2} \ell(L)^{1+\beta_2} \right\}.$$

(iv) $\langle T, \mathbf{p}, p\rangle = Q\,[\![p]\!]$ *for every $p \in \mathbf{\Phi}(\mathbf{\Gamma}) \cap \mathbf{C}_{2r} \setminus \{0\}$.*

The main idea is now to use the graphs of the π_L approximations of Definition 17.6 and reparametrize them as maps on the normal bundle of $\mathcal{M}$ over a suitable region. These maps will be the model for the approximation of the current T over the corresponding Whitney region $\mathcal{L}$. This part is not substantially different from the similar one in [21] and similar estimates lead to the following approximation result

Definition 19.3 ($\mathcal{M}$-normal approximation)**.** Let r be as in Corollary 19.2 and define

(U) $\mathbf{U} := \mathbf{p}^{-1}(\mathbf{C}_{2r} \cap \mathfrak{B}_Q)$.

An $\mathcal{M}$*-normal approximation* of T is given by a pair $(\mathcal{K}, F)$ such that

(A1) $F : \mathbf{C}_{2r} \cap \mathcal{M} \to \mathcal{A}_Q(\mathbf{U})$ is Lipschitz and takes the form $F(x) = \sum_i [\![x + N_i(x)]\!]$, with $N_i(x) \perp T_x\mathcal{M}$ and $x + N_i(x) \in \Sigma$ for every x and i.

(A2) $\mathcal{K} \subset \mathcal{M}$ is closed, contains $\mathbf{\Phi}\big(\mathbf{\Gamma} \cap \mathbf{C}_{2r}\big)$ and $\mathbf{T}_F \llcorner \mathbf{p}^{-1}(\mathcal{K}) = T \llcorner \mathbf{p}^{-1}(\mathcal{K})$.

The map $N = \sum_i [\![N_i]\!] : \mathcal{M} \cap \mathbf{C}_{2r} \to \mathcal{A}_Q(\mathbb{R}^{2+n})$ is *the normal part* of F.

In the definition above it is not required that the map F approximates efficiently the current outside the set $\mathbf{\Phi}(\mathbf{\Gamma})$. However, all the maps constructed will approximate T with a high degree of accuracy in each Whitney region: such estimates are detailed in the next theorem. In order to simplify the notation, we will use $\|N|_{\mathcal{V}}\|_{C^0}$ (or $\|N|_{\mathcal{V}}\|_0$) to denote the number $\sup_{x\in\mathcal{V}} \mathcal{G}(N(x), Q[\![0]\!]) = \sup_{x\in\mathcal{V}} |N(x)|$.

Theorem 19.4 (Local estimates for the $\mathcal{M}$-normal approximation, cf. [24, Theorem 4.3])**.** *Let r be as in Corollary 19.2 and $\mathbf{U}$ as in Definition 19.3. Then there is an $\mathcal{M}$-normal approximation $(\mathcal{K}, F)$ such that the following estimates hold on every Whitney region $\mathcal{L}$ associated to $L \in \mathscr{W}$ with $\mathbf{d}(L) \leq r$:*

$$\mathrm{Lip}(N|_{\mathcal{L}}) \leq C\boldsymbol{m}_0^{\beta_0}\mathbf{d}(L)^{\beta_0\gamma_0}\,\ell(L)^{\beta_0\gamma_0} \quad \textit{and} \quad \|N|_{\mathcal{L}}\|_{C^0} \leq C\boldsymbol{m}_0^{1/4}\mathbf{d}(L)^{\gamma_0/2-\beta_2}\ell(L)^{1+\beta_2}, \tag{124}$$

$$|\mathcal{L}\setminus\mathcal{K}| + \|\mathbf{T}_F - T\|(\mathbf{p}^{-1}(\mathcal{L})) \leq C\boldsymbol{m}_0^{1+\beta_0}\,\mathbf{d}(L)^{(1+\beta_0)(2\gamma_0-2+2\delta_1)}\,\ell(L)^{2+(1+\beta_0)(2-2\delta_1)}, \tag{125}$$

$$\int_{\mathcal{L}} |DN|^2 \leq C\boldsymbol{m}_0\,\mathbf{d}(L)^{2\gamma_0-2+2\delta_1}\,\ell(L)^{4-2\delta_1}\,. \tag{126}$$

Moreover, for every Borel $\mathcal{V} \subset \mathcal{L}$, we have

$$\int_{\mathcal{V}} |\boldsymbol{\eta}\circ N| \leq C\boldsymbol{m}_0\mathbf{d}(L)^{2(1+\beta_0)\gamma_0-2-\beta_2}\,\ell(L)^{5+\beta_2/4} + C\boldsymbol{m}_0^{1/2+\beta_0}\,\mathbf{d}(L)^{2\beta_0\gamma_0+\gamma_0-1-\beta_2}\,\ell(L)^{1+\beta_2}\int_{\mathcal{V}}\mathcal{G}\big(N, Q[\![\boldsymbol{\eta}\circ N]\!]\big)\,. \tag{127}$$

The constant $C = C(M_0, N_0, C_e, C_h)$ does not depend on ε_2.

20. Splitting before tilting and proof of Theorem 13.3

We have now achieved our branched center manifold $\mathcal{M}$ and the map $\mathcal{N}$, which is simply a conformal reparametrization of the map N in Theorem 19.4. It remains to show that $\mathcal{N}$ satisfies all the estimates in Theorem 13.3. The strategy is to use the local estimates in Theorem 19.4 and sum them over the different Whitney regions. There are however two fundamental issues that we need to address:

(B) The estimates in Theorem 19.4 are in terms of the two parameters $\mathbf{d}(L)$ and $\ell(L)$, whereas we need to control them appropriately with integral quantities involving either $|\mathcal{N}|$ or $|D\mathcal{N}|$. If we ignore for the moment the squares of type (NN), we could take advantage of the stopping conditions (HT) and (EX) to get appropriate bounds in terms of the height and the excess of the current in the balls $\mathbf{B}_L$. In turn we hope to use the approximation theorem again to say that the latter quantities can be compared to the L^2 norm of $\mathcal{N}$ over $\mathcal{L}$ and the Dirichlet energy of $\mathcal{N}$ over $\mathcal{L}$. The squares of type (NN) will then be treated observing that they must be close to some square of type (EX).

(S) The estimates in Theorem 13.3 are in terms of the quantities $\mathbf{D}(r)$ and $\mathbf{F}(r)$, which are integrals over geodetic balls $B_r(0) \subset \mathfrak{B}$. If a Whitney regions $\mathcal{L}$ falls partly in $B_r(0)$ and partly in the complement of $B_r(0)$, we then run into troubles when we try to use the idea in (B). A crucial point is that we can control the L^2 norm of $\mathcal{N}$ and the Dirichlet norm of $\mathcal{N}$ over the regions $\mathcal{L}$ with corresponding integral quantities in any nearby smaller regions with comparable size. For squares of type (HT) this is a consequence of a refined "uniform separation estimate", first shown in [21] for area minimizing currents, which must be extended also to our almost minimizing settings. For squares of type (EX) the arguments are more subtle and they are based on what we call splitting before tilting phenomenon, which is exploited in [21] as well and carefully explained in [17]. In a nutshell, the reason is that the current T is close to the graph of a Dir-minimizer which has a nontrivial separation among its sheets. Such maps have the property that their Dirichlet energy on the domain of definition is comparable to the Dirichlet energy in any smaller region of comparable size. The information that the Dir-minimizing approximation is nontrivial comes from the usual De Giorgi-type decay of the excess which would result from being close to a Q copy of a classical harmonic function: such decay is incompatible with the stopping condition (EX)!

We collect here the corresponding precise statements. The first statement gives the needed uniform "separation" of sheets which occur in regions where the (HT) condition stops the Whitney refining procedure.

Proposition 20.1 (Separation, cf. [24, Proposition 4.4])**.** *There is a dimensional constant $C^\sharp > 0$ with the following property. Assume the hypotheses of Theorem 19.4, and in addition $C_h^4 \geq C^\sharp C_e$. If ε_2 is sufficiently small, then the following conclusions hold for every $L \in \mathscr{W}_h$ with $\mathbf{d}(L) \leq r$:*

(S1) *$\Theta(T_L, p) \leq Q - 1$ for every $p \in \mathbf{B}_{16 r_L}(p_L)$.*
(S2) *$L \cap H = \emptyset$ for every $H \in \mathscr{W}_n$ with $\ell(H) \leq \frac{1}{2}\ell(L)$.*
(S3) *$\mathcal{G}\big(N(x), Q\,[\![\boldsymbol{\eta} \circ N(x)]\!]\big) \geq \frac{1}{4} C_h \boldsymbol{m}_0^{1/4}\, \mathbf{d}(L)^{\gamma_0/2 - \beta_2} \ell(L)^{1+\beta_2}$ $\forall x \in \mathbf{\Phi}(B_{4\ell(L)}(z_L, w_L))$.*

A simple corollary of the previous proposition is the fact that no (NN) square is adjacent to an (HT) square.

Corollary 20.2 (Domains of influence)**.** *For any $H \in \mathscr{W}_n$ there is a chain $L = L_0, \ldots, L_n = H$ such that*

(a) *$L_0 \in \mathscr{W}_e$ and $L_k \in \mathscr{W}_n$ for all $k > 0$;*
(b) *$L_k \cap L_{k-1} \neq \emptyset$ and $\ell(L_k) = \frac{\ell(L_{k-1})}{2}$ for all $k > 0$.*

In particular $H \subset B_{3\sqrt{2}\ell(L)}(z_L, w_L)$.

We use this last corollary to partition $\mathscr{W}_n$ in "areas of influence" of squares of type (EX).

Definition 20.3 (Domains of influence)**.** We first fix an ordering of the squares in $\mathscr{W}_e$ as $\{J_i\}_{i \in \mathbb{N}}$ so that their sidelengths do not increase. Then $H \in \mathscr{W}_n$ belongs to $\mathscr{W}_n(J_0)$ (the domain of influence of J_0) if there is a chain as in Corollary 20.2 with $L_0 = J_0$. Inductively,

$\mathscr{W}_n(J_r)$ is the set of squares $H \in \mathscr{W}_n \setminus \cup_{i<r}\mathscr{W}_n(J_i)$ for which there is a chain as in Corollary 20.2 with $L_0 = J_r$.

We finally give a precise statement for the splitting before tilting phenomenon in (EX) squares explained in our discussion above.

Proposition 20.4 (Splitting, cf. [24, Proposition 4.7])**.** *There are positive constants C_1,$C_2(M_0)$, $\bar r(M_0, N_0, C_e)$ such that, if $M_0 \geq C_1$, $C_e \geq C_2(M_0)$, if the hypotheses of Theorem 19.4 hold and ε_2 is chosen sufficiently small, then the following holds. If $L \in \mathscr{W}_e$ with $\mathbf{d}(L) \leq \bar r$, $q \in \mathfrak{B}$ with* $\operatorname{dist}(L, q) \leq 4\sqrt{2}\,\ell(L)$ *and $\Omega := \mathbf{\Phi}(B_{\ell(L)/8}(q))$, then:*

$$C_e \boldsymbol{m}_0\, \mathbf{d}(L)^{2\gamma_0-2+2\delta_1}\ell(L)^{4-2\delta_1} \leq \ell(L)^2\mathbf{E}(T_l, \mathbf{B}_L) \leq C\int_\Omega |DN|^2\,, \tag{128}$$

$$\int_{\mathcal{L}} |DN|^2 \leq C\ell(L)^2\mathbf{E}(T, \mathbf{B}_L) \leq C\ell(L)^{-2}\int_\Omega |N|^2\,, \tag{129}$$

where $C = C(M_0, N_0, C_e, C_h)$.

Having all these estimates at disposal, Theorem 13.3 follows from a careful covering argument, which is essentially borrowed from [22].

21. Proof of Theorem 14.1, part I: almost Dir-minimality of $\mathscr{N}$ and Poincaré inequality

The proof of Theorem 14.1, given in [25], is split in six steps, which we will list briefly in the next paragraph.

(i) First of all we deduce an almost minimizing property for the map $\mathscr{N}$ in terms of its Dirichlet energy, using the Taylor expansion of the area and the fact that the graph of $\mathscr{N}$ coincides with the support of the current T on a rather large set.

(ii) We then exploit the almost minimizing property and compare the Dirichlet energy of $\mathscr{N}$ with that of a suitable extension of its boundary value on any given ball, which is essentially an adapted version of the centered harmonic extension described in Section 8.2.

(iii) In the thid step, we use the comparison above and a first variation argument to derive a suitable Poincaré-type inequality for $\mathscr{N}$.

(iv) We then compute again the first variations of the Dirichlet energy of $\mathscr{N}$ and use the Poincaré inequality to bound efficiently several error terms.

(v) Using the latter bounds, we prove an almost monotonicity property for the frequency function and show the existence and boundedness of its limit, which is indeed the number I_0 of Theorem 14.1. The almost minimality of $\mathscr{N}$ and the energy comparison of Step (ii) will then allow us to conclude an exponential rate of decay to this limit.

(vi) From the decay of the previous step we capture the asymptotic behavior of $\mathscr{N}$ and show the existence of the map g of Theorem 14.1.

The overall strategy follows the ideas and some of the computations in [10]. However several adjustments are needed to carry on the proof in the cases (b) and (c) of Theorem

4.4. In particular, when we show the decay of the frequency function towards its limit, we need to introduce a suitable modification of the usual frequency function to handle case (b) of Theorem 4.4.

21.1. **The almost minimizing property.** Recall the maps $N := \mathcal{N} \circ \mathbf{\Psi}^{-1}$ and $F := \sum_{i=1}^{Q} [\![p + N_i(p)]\!]$. In order to state the almost minimizing property of $\mathcal{N}$ we introduce an appropriate notion of competitor.

Definition 21.1. A Lipschitz map $\mathcal{L}\colon B_r \to \mathcal{A}_Q(\mathbb{R}^{n+2})$ is called a competitor for $\mathcal{N}$ in the ball B_r if

(a) $\mathcal{L}|_{\partial B_r} = \mathcal{N}|_{\partial B_r}$;
(b) $\mathrm{spt}(\mathcal{G}(z,w)) \subset \Sigma$ for all $(z,w) \in B_r$, where $\mathcal{G}(z,w) := \sum_{j=1}^{Q} [\![\mathbf{\Psi}(z,w) + \mathcal{L}_j(z,w)]\!]$.

We are now ready to state the almost minimizing property for $\mathcal{N}$. We use the notation $\mathbf{p}_{T_p\Sigma}$ for the orthogonal projection on the tangent space to Σ at p. We recall that, given our choice of coordinates, $\mathbf{p}_{T_0\Sigma}$ is the projection on $\mathbb{R}^{2+\bar{n}} \times \{0\}$. Since this projection will be used several times, we will denote it by $\mathbf{p}_0$. By the $C^{3,\varepsilon}$ regularity of Σ, there exists a map $\Psi_0 \in C^{3,\varepsilon}(\mathbb{R}^{2+\bar{n}}, \mathbb{R}^l)$ such that

$$\Psi_0(0) = 0\,,\ D\Psi_0(0) = 0 \quad \text{and} \quad \mathrm{Gr}(\Psi_0) = \Sigma\,.$$

Next, for each function $\mathcal{L}$ satisfying Condition (b) in Definition 21.1 we consider the map $\bar{\mathcal{L}} := \mathbf{p}_0 \circ \mathcal{L}$, which is a multivalued $\bar{\mathcal{L}} : \mathfrak{B} \to \mathcal{A}_Q(\mathbb{R}^{2+\bar{n}})$. We observe that it is possible to determine $\mathcal{L}$ from $\bar{\mathcal{L}}$. In particular, fix coordinates $(\xi,\eta) \in \mathbb{R}^{2+\bar{n}} \times \mathbb{R}^l$ and let $\mathcal{L} = \sum [\![\mathcal{L}_i]\!]$, $\bar{\mathcal{L}} = \sum [\![\bar{\mathcal{L}}_i]\!]$, where $\bar{\mathcal{L}}_i = \mathbf{p}_0 \circ \mathcal{L}_i$. Then the formula relating $\mathcal{L}_i$ and $\bar{\mathcal{L}}_i$ is

$$\mathcal{L}_i(z,w) = \big(\bar{\mathcal{L}}_i(z,w), \Psi_0\big(\mathbf{p}_0(\mathbf{\Psi}(z,w)) + \bar{\mathcal{L}}_i(z,w)\big) - \Psi_0(\mathbf{p}_0(\mathbf{\Psi}(z,w)))\big)\,. \tag{130}$$

Proposition 21.2 (Energy comparison, cf. [25, Proposition 3.2])**.** *There exists a constant $C_{21.2} > 0$ such that the following holds. If $r \in (0,1)$ and $\mathcal{L}\colon B_r \to \mathcal{A}_Q(\mathbb{R}^{2+n})$ is a Lipschitz competitor for $\mathcal{N}$ with $\|\mathcal{L}\|_\infty \leq r$ and $\mathrm{Lip}(\mathcal{L}) \leq C_{21.2}^{-1}$, then*

$$\begin{aligned}\int_{B_r} |D\mathcal{N}|^2 \leq (1 + C_{21.2}\, r) \int_{B_r} |D\bar{\mathcal{L}}|^2 + C_{21.2}\, \mathrm{Err}_1(\mathcal{N}, B_r)&\\ + C_{21.2}\, \mathrm{Err}_2(\mathcal{L}, B_r) + C_{21.2}\, r^2 \mathbf{D}'(r)\,,&\end{aligned} \tag{131}$$

where $\bar{\mathcal{L}} := \mathbf{p}_0 \circ \mathcal{L}$ and the the error terms $\mathrm{Err}_1(\mathcal{N}, B_r)$, $\mathrm{Err}_2(\mathcal{L}, B_r)$ are given by the following expressions:

$$\mathrm{Err}_1(\mathcal{N}, B_r) = \mathbf{\Lambda}^{\gamma_0}(r)\, \mathbf{D}(r) + \mathbf{F}(r) + \mathbf{H}(r) + \boldsymbol{m}_0^{1/2}\, r^{1+\gamma_0} \int_{\partial B_r} |\boldsymbol{\eta} \circ \mathcal{N}| \tag{132}$$

and

$$\mathrm{Err}_2(\mathcal{L}, B_r) = \boldsymbol{m}_0^{1/2} \int_{B_r} |z|^{\gamma_0 - 1} |\boldsymbol{\eta} \circ \mathcal{L}|\ . \tag{133}$$

21.2. **Centered harmonic competitor.** The most natural choice for the competitor $\mathcal{L}$ is the centered harmonic extension of the boundary value $\mathcal{N}|_{\partial B_r}$. Following the ideas of [10] we estimate carefully the energy of such competitor. The computations follow then closely those of Section 9.5: combined with the Proposition 21.2 we then conclude the following estimates.

Proposition 21.3 (Cf. [25, Proposition 4.1])**.** *There are constants $C > 0$, $\sigma > 0$ such that, for every $r \in (0,1)$ there exists a competitor $\mathcal{L}\colon B_r \to \mathcal{A}_Q(\mathbb{R}^{2+n})$ for $\mathcal{N}$ with the following additional properties:*

(i) $\mathrm{Lip}(\mathcal{L}) \leq C_{21.2}^{-1}$, $\|\mathcal{L}\|_0 \leq Cr$.

(ii) *The following estimates hold:*

$$\int_{B_r} |D\bar{\mathcal{L}}|^2 \leq C\, r \int_{\partial B_r} |D\bar{\mathcal{N}}|^2 \leq Cr\mathbf{D}'(r)\,, \tag{134}$$

$$\int_{B_r} |z|^{\gamma_0 - 1} |\boldsymbol{\eta} \circ \mathcal{L}| \leq C\, r^{\gamma_0} \int_{\partial B_r} |\boldsymbol{\eta} \circ \mathcal{N}| + C\,\mathbf{H}(r)\,. \tag{135}$$

(iii) *For every $a > 0$ there exists $b_0 > 0$ such that, for all $b \in (0, b_0)$, the following estimate holds:*

$$(2\,a + b) \int_{B_r} |D\bar{\mathcal{L}}|^2 \leq r \int_{\partial B_r} |D_\tau \mathcal{N}|^2 + \frac{a\,(a+b)}{r} \int_{\partial B_r} |\mathcal{N}|^2 + Cr^{1+\sigma}\mathbf{D}'(r)\,. \tag{136}$$

Using this competitor in Proposition 21.2, we then infer the following corollary.

Corollary 21.4. *For every $r \in (0,1)$ the following inequality holds*

$$\mathbf{D}(r) \leq C\, r\, \mathbf{D}'(r) + C\,\mathbf{H}(r) + C\,\mathbf{F}(r) + C\,\boldsymbol{m}_0^{1/2}\, r^{\gamma_0} \int_{\partial B_r} |\boldsymbol{\eta} \circ \mathcal{N}|\,. \tag{137}$$

For every $a > 0$ there exists $b_0 > 0$ such that, for all $b \in (0, b_0)$ and all $r \in]0,1[$

$$\mathbf{D}(r) \leq (1 + Cr) \left[\frac{r}{(2\,a+b)} \int_{\partial B_r} |D_\tau \mathcal{N}|^2 + \frac{a\,(a+b)}{r\,(2\,a+b)} \mathbf{H}(r) \right] + C\,\mathcal{E}_{QM}(r) + Cr^{1+\sigma}\mathbf{D}'(r)\,, \tag{138}$$

with

$$\mathcal{E}_{QM}(r) \leq \boldsymbol{\Lambda}(r)^{\eta_0}\mathbf{D}(r) + \mathbf{F}(r) + \mathbf{H}(r) + \boldsymbol{m}_0^{1/2}\, r^{\gamma_0} \int_{\partial B_r} |\boldsymbol{\eta} \circ \mathcal{N}|\,.$$

21.3. **Outer variations and the Poincaré inequality.** In the third step, we employ the first variations of the area functional on T in conjunction with the estimates of the previous section to derive the following Poincaré inequality:

Theorem 21.5 (Poincaré inequality, cf. [25, Proposition 5.1])**.** *There exists a constant $C_{21.5} > 0$ such that if r is sufficiently small, then*

$$\mathbf{H}(r) \leq C_{21.5}\, r\, \mathbf{D}(r)\,. \tag{139}$$

We record however the two main tools used to prove Theorem 21.5. They are the suitable modifications of similar computations used to derive the monotonicity of the frequency function in the case of Dir-minimizers. The first identity is the elementary formula which gives the derivative of $\mathbf{H}$. In order to state it we introduce the quantity

$$\mathbf{E}(r) := \int_{\partial B_r} \sum_{j=1}^{Q} \langle \mathcal{N}_j, D_\nu \mathcal{N}_j \rangle \,. \tag{140}$$

Lemma 21.6. $\mathbf{H}$ *is a Lipschitz function and the following identity holds for a.e.* $r \in (0,1)$

$$\mathbf{H}'(r) = \frac{\mathbf{H}(r)}{r} + 2\,\mathbf{E}(r)\,. \tag{141}$$

The second identity is a consequence of the first variations of T under specific vector fields, which we call "outer variations": such variations "stretch" the normal bundle of $\mathcal{M}$ suitably and they are defined using the map $\mathcal{N}$. They are therefore the obvious counterparts of the outer variations used to derive the monotonicity of the frequency function for Dir-minimizers, see Section 8.3.

In the case of semicalibrated currents it is convenient to modify the Dirichlet energy suitably to gain a new quantity which enjoys better estimates. Thus, from now on $\mathbf{\Omega}$ will denote $\mathbf{D}$ in the cases (a) and (c) of Theorem 4.4, whereas in the case (b) it will be given by

$$\begin{aligned}\mathbf{\Omega}(r) :=& \mathbf{D}(r) + \mathbf{L}(r)\\ :=& \mathbf{D}(r) + \int_{\mathbf{\Psi}(B_r)} \sum_{i=1}^{Q} \langle \xi_1(p)\wedge D_{\xi_2}N_i(p)\wedge N_i(p) + D_{\xi_1}N_i(p)\wedge \xi_2(p)\wedge N_i(p), d\omega(p)\rangle\, dp\,.\end{aligned}$$

Proposition 21.7 (Outer variations, cf. [25, Proposition 5.3])**.** *There exist constants* $C_{21.7} > 0$ *and* $\kappa > 0$ *such that, if* $r > 0$ *is small enough, then the inequality*

$$|\mathbf{\Omega}(r) - \mathbf{E}(r)| \le C_{21.7}\,\mathcal{E}_{OV}(r) \tag{142}$$

holds with

$$\mathcal{E}_{OV}(r) = \mathbf{\Lambda}(r)^\kappa \Big(\mathbf{D}(r) + \frac{\mathbf{H}(r)}{r} + r\mathbf{D}'(r)\Big) + \mathbf{F}(r) + r^{1+\gamma_0}\frac{d}{dr}\|T - \mathbf{T}_F\|(\mathbf{p}^{-1}(\mathbf{\Psi}(B_r)))\,. \tag{143}$$

Moreover

$$|\mathbf{L}(r)| \le C\,\boldsymbol{m}_0^{1/2}\, r^{2-\gamma_0}\mathbf{D}(r) + C\,\boldsymbol{m}_0^{1/2}\,\mathbf{F}(r). \tag{144}$$

22. Proof of Theorem 14.1, part II: inner variations and decay of the frequency function

22.1. **Inner variations.** Using the Poincaré inequality in Theorem 21.5, we can give very simple estimates of the error terms in the "inner variations" of the current T. The latter correspond to deformations of T along appropriate vector fields which are tangent to $\mathcal{M}$ and therefore they are the obvious analog of the inner variations used to derive the monotonicity of the frequency function for Dir-minimizers, see Section 8.3. In order to state our main conclusion we need to introduce yet another quantity

$$\mathbf{G}(r) := \int_{\partial B_r} |D_\nu \mathcal{N}|^2 \,. \tag{145}$$

Proposition 22.1 (Inner Variations, cf. [25, Theorem 6.1])**.** *There exist constants $C_{22.1} > 0$ and $\eta > 0$ such that, if $r > 0$ is small enough, than the following holds*

$$|\mathbf{D}'(r) - 2\,\mathbf{G}(r)| \le C\,\mathcal{E}_{IV}(r)\,, \tag{146}$$

where

$$\begin{aligned}\mathcal{E}_{IV}(r) = r^{2\eta-1}\mathbf{D}(r) + \mathbf{D}(r)^\eta\,\mathbf{D}'(r) + \frac{\boldsymbol{m}_0^{1/2}}{r^{1-\gamma_0}}\int_{\partial B_r}|\boldsymbol{\eta}\circ\mathcal{N}(z,w)|\\ + \frac{d}{dr}\|T - \mathbf{T}_F\|(\mathbf{p}^{-1}(\mathbf{\Psi}(B_r)))\,.\end{aligned} \tag{147}$$

The next lemma summarizes then all the estimates achieved so far.

Lemma 22.2. *There exist constant $C_{22.2} > 0$ and $\eta > 0$ such that for every r sufficiently small the following holds:*

$$\begin{aligned}
\mathbf{F}(r) + r\mathbf{F}'(r) \le& C_{22.2}\, r^{\gamma_0}\mathbf{D}(r) && (148)\\
|\mathbf{L}(r)| \le& C_{22.2}\, r\,\mathbf{D}(r) && (149)\\
|\mathbf{L}'(r)| \le& C_{22.2}\, \left(\mathbf{H}(r)\,\mathbf{D}'(r)\right)^{1/2} && (150)\\
\mathcal{E}_{OV} \le& C_{22.2}\,\mathbf{D}^{1+\eta}(r) + C_{22.2}\,\mathbf{F}(r) + C_{22.2} r\mathbf{D}^\eta(r)\mathbf{D}'(r) + C_{22.2}\, r\,\mathcal{E}_{BP}(r), && (151)\\
\mathcal{E}_{IV}(r) \le& C_{22.2}\, r^{2\eta-1}\mathbf{D}(r) + C_{22.2}\,\mathbf{D}(r)^\eta\,\mathbf{D}'(r) + C_{22.2}\,\mathcal{E}_{BP}(r), && (152)
\end{aligned}$$

where

$$\mathcal{E}_{BP}(r) := \frac{\boldsymbol{m}_0^{1/2}}{r^{1-\gamma_0}}\int_{\partial B_r}|\boldsymbol{\eta}\circ\mathcal{N}| + \frac{d}{dr}\|T - \mathbf{T}_F\|(\mathbf{p}^{-1}(\mathbf{\Psi}(B_r)))$$

Moreover, for every $a > 0$ there exist constants $b_0(a), C(a) > 0$ such that

$$\mathbf{D}(r) \le \frac{r\,\mathbf{D}'(r)}{2(2\,a+b)} + \frac{a(a+b)\,\mathbf{H}(r)}{r(2\,a+b)} + C(a)\,r\,\mathcal{E}_{IV}(r) \quad \forall\, b < b_0(a). \tag{153}$$

An important corollary of the previous lemma is the following

Corollary 22.3. *There exists a constant $C_{22.3} > 0$ such that, if η is the constant of Lemma 22.2, then for every $0 \le \gamma < \eta$ and r sufficiently small, the nonnegative functions $\frac{\mathcal{E}_{IV}(r)}{r^\gamma\,\mathbf{D}(r)}$ $\frac{\mathcal{E}_{OV}(r)}{r^{1+\gamma}\mathbf{D}(r)}$ are both integrable. Moreover, if we define the functions*

$$\mathbf{\Sigma}_{IV}(r) := \int_0^r \frac{\mathcal{E}_{IV}(s)}{s^\gamma\,\mathbf{D}(s)}\,ds\,, \tag{154}$$

$$\mathbf{\Sigma}_{OV}(r) := \int_0^r \frac{\mathcal{E}_{OV}(s)}{s^\gamma\,\mathbf{D}(s)}\,ds\,, \tag{155}$$

$$\mathbf{\Sigma}(r) := \mathbf{\Sigma}_{IV}(r) + \mathbf{\Sigma}_{OV}(r)\,, \tag{156}$$

then

$$\mathbf{\Sigma}(r) \leq C_{22.3}\, r^{\eta-\gamma}\,. \tag{157}$$

22.2. **Decay of the frequency function.** We now have the tools needed to study the asymptotic behavior of the normal approximation $\mathcal{N}$. First of all we prove the approximate monotonicity of the frequency function and derive appropriate decay estimates.

For every $r \in (0,1)$ such that $\mathbf{H}(r) > 0$, we set $\bar{\mathbf{I}}(r) := \frac{r\,\mathbf{\Omega}(r)}{\mathbf{H}(r)}$ where we recall that

$$\mathbf{\Omega}(r) := \begin{cases} \mathbf{D}(r) & \text{in the cases (a) and (b) of Theorem 4.4;} \\ \mathbf{D}(r) + \mathbf{L}(r) & \text{in case (c).} \end{cases}$$

Furthermore we define $\bar{\mathbf{K}}(r) := \bar{\mathbf{I}}(r)^{-1}$ whenever $\mathbf{\Omega}(r) \neq 0$. By (149) there exists $r_0 > 0$ such that

$$\frac{1}{2}\mathbf{D}(r) \leq (1 - C\,r)\mathbf{D}(r) \leq \mathbf{\Omega}(r) \leq (1 + C\,r)\mathbf{D}(r) \leq 2\,\mathbf{D}(r) \qquad \forall r \leq r_0\,. \tag{158}$$

Having fixed r_0, $\bar{\mathbf{K}}(r)$ is well defined whenever $\mathbf{D}(r) > 0$ and hence, by the Poincaré inequality, whenever $\bar{\mathbf{I}}(r)$ is defined. Moreover, if for some $\rho \leq r_0$ $\bar{\mathbf{K}}(\rho)$ is not well defined, that is $\mathbf{\Omega}(\rho) = 0$, then obviously $\mathbf{\Omega}(r) = \mathbf{D}(r) = 0$ for every $r \leq \rho$.

We are now ready to state the first important monotonicity estimate. From now on we assume of having fixed a γ

Theorem 22.4. *There exists a constant $C_{22.4} > 0$ with the following property: if $\mathbf{D}(r) > 0$ for some $r \leq r_0$, then (setting $\gamma = 0$ in* (154) *and* (155)*) the function*

$$\bar{\mathbf{K}}(r)\,\exp(-4r - 4\mathbf{\Sigma}_{IV}(r)) - 4\,\mathbf{\Sigma}_{OV}(r) \tag{159}$$

is monotone non-increasing on any interval $[a,b]$ where $\mathbf{D}$ is nowhere 0. In particular, either there is $\bar{r} > 0$ such that $\mathbf{D}(\bar{r}) = 0$ or $\bar{\mathbf{K}}$ is well-defined on $]0, r_0[$ and the limit $K_0 := \lim_{r\to 0}\bar{\mathbf{K}}(r)$ exists.

A fundamental consequence of Theorem 22.4 is the following dichotomy.

Corollary 22.5. *There exists $\bar{r} > 0$ such that*

(A) *either $\bar{\mathbf{K}}(r)$ is well-defined for every $r \in]0, r_0[$, the limit*

$$K_0 := \lim_{r\downarrow 0}\bar{\mathbf{K}}(r) \tag{160}$$

is positive and thus there is a constant C and a radius $\bar{r}$ such that

$$C^{-1}\,r\,\mathbf{D}(r) \leq \mathbf{H}(r) \leq C\,r\,\mathbf{D}(r) \qquad \forall\, r \in]0, \bar{r}[\,; \tag{161}$$

(B) *or $T \llcorner \mathbf{p}^{-1}(\mathbf{\Psi}(B_{\bar{r}})) = Q\,[\![\mathbf{\Psi}(B_{\bar{r}})]\!]$ for some positive $\bar{r}$.*

In turn, using the above dichotomy we can show

Theorem 22.6 (Decay of the frequency function, cf. [25, Theorem 7.3])**.** *Assume that condition (i) in Theorem 14.1 fails. Then the frequency $\bar{\mathbf{I}}(r)$ is well-defined for every sufficiently small r and its limit $I_0 = \lim_{r\to 0}\bar{\mathbf{I}}(r) = K_0^{-1}$ exists and it is finite and positive.*

Moreover there exist constants $\lambda, C_{22.6}, H_0, D_0 > 0$ such that, for every r sufficiently small the following holds:

$$\left|\mathbf{I}(r) - I_0\right| + \left|\frac{\mathbf{H}(r)}{r^{2I_0+1}} - H_0\right| + \left|\frac{\mathbf{D}(r)}{r^{2I_0}} - D_0\right| \le C_{22.6}\, r^{\lambda}\,. \tag{162}$$

22.3. **Uniqueness of the tangent map and conclusion.** As already remarked for Dir-minimizers, as a consequence of the decay estimate in Theorem 22.6 we can show that suitable rescalings of the normal approximation $\mathcal{N}$ converge to a unique limiting profile. To this aim we consider for every $r \in (0,1)$ the functions $f_r : \partial B_1 \to \mathcal{A}_{Q_1}(\mathbb{R}^{2+n})$ given by

$$f_r(z,w) := \frac{\mathcal{N}(i_r(z,w))}{r^{I_0}},$$

where we recall that $i_r(z,w) = \big(rz, r^{1/\bar{Q}}w\big)$. We recall also that $T_0\mathcal{M} = \mathbb{R}^2 \times \{0\}$, and $T_0\Sigma = \mathbb{R}^2 \times \mathbb{R}^{\bar{n}} \times \{0\}$. In the following, with a slight abuse of notation, we write $\mathbb{R}^{\bar{n}}$ for the subspace $\{0\} \times \mathbb{R}^{\bar{n}} \times \{0\}$.

The final step in the proof of Theorem 14.1 is then the following proposition.

Proposition 22.7 (Uniqueness of the tangent map, cf. [25, Proposition 8.1])**.** *Assume alternative (i) in Theorem 14.1 fails and let I_0 and λ be the positive numbers of Theorem 22.6. Then $I_0 > 1$ and there exists a function $f_0 : \partial B_1 \to \mathcal{A}_Q(\mathbb{R}^{\bar{n}})$ such that*

(i) *$\boldsymbol{\eta} \circ f_0 = 0$ and $f_0 \not\equiv Q_1 [\![0]\!]$;*

(ii) *for every r sufficiently small*

$$\mathcal{G}(f_r(z,w), f_0(z,w)) \le C\, r^{\lambda/16} \quad \forall\, (z,w) \in \partial B_1\,; \tag{163}$$

(iii) *the I_0-homogeneous extension $g(z,w) := |z|^{I_0} f_0\left(\frac{z}{|z|}, \frac{w}{|w|}\right)$ is nontrivial and* Dir-*minimizing.*

In particular, by (iii) $\mathrm{Im}(g) \setminus \{0\} \subset \mathbb{R}^{2+n}$ *is a real analytic submanifold.*

Theorem 14.1 follows immediately from Proposition 22.7 and Theorem 22.6.

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