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On Univalent Functions, Bloch Functions and VMOA

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1. Introduction

1.1. Let C be a (closed) Jordan curve in $\mathbb{C}$ and let $C(w_1, w_2)$ denote the smaller arc of C between the points w_1 and w_2 on C. We call C *asymptotically conformal* if

$$\max_{w \in C(w_1, w_2)} \frac{|w_2 - w| + |w - w_1|}{|w_2 - w_1|} \to 1 \quad \text{as} \quad |w_2 - w_1| \to 0 . \tag{1.1}$$

The curve C is called *quasi-conformal* if this quantity is bounded; this holds if and only if C is the image of a circle under a quasi-conformal mapping of $\mathbb{C}$ ([11, p. 105] or [18, Theorem 9.14]).

Let $\mathscr{B}$ denote the space of functions f analytic in the unit disk $\mathbb{D}$ for which $(1-|z|^2)|f'(z)|$ is bounded ("Bloch functions" [17]) and let $\mathscr{B}_0$ denote the subspace of functions with

$$(1-|z|^2)|f'(z)| \to 0 \qquad (|z| \to 1-0) . \tag{1.2}$$

Theorem 1. *Let g be analytic and univalent in $\mathbb{D}$ and let $C = \partial g(\mathbb{D})$ be a Jordan curve. Then the following three conditions are equivalent:*

i) $\log g' \in \mathscr{B}_0$;

ii) $\begin{cases} \dfrac{g(z)-g(\zeta)}{(z-\zeta)g'(\zeta)} \to 1 \qquad (|\zeta| \to 1-0) \\ \textit{uniformly in } z \in \bar{\mathbb{D}}, |z-\zeta| \leqq \lambda(1-|\zeta|) \textit{ for each } \lambda; \end{cases}$

iii) *C is asymptotically conformal.*

Problems related, in particular, to ii) have be extensively studied, mostly as a local problem. We mention only the papers of Visser [26] and Ostrowski [15] and the survey arcticles of Gattegno and Ostrowski [7, 8]; see also [16, Satz 3.15]. We apply the method of variable domains to prove iii)⇒i). This method was used in this context by Lelong-Ferrand [6]; see also [22].

We discuss the existence of a tangent and related problems in Section 3. A characterization of asymptotically conformal curves in terms of quasiconformal mappings will be given in [2].

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1.2. We introduce now a geometric condition which is stronger than (1.1). Let C be a rectifiable Jordan curve and let $l(w_1, w_2)$ denote the length of the shorter arc of C between w_1 and w_2. We call C *asymptotically smooth* if

$$\frac{l(w_1, w_2)}{|w_2 - w_1|} \to 1 \quad \text{as} \quad |w_2 - w_1| \to 0, \quad w_1, w_2 \in C, \tag{1.3}$$

and we call C *quasi-smooth* if this quantity is bounded on C. Quasi-smooth curves were first studied by Lavrent'ev [10]; for further applications of this condition see for instance [24, 25].

Let BMOA ("bounded mean oscillation" [9]) denote the space of functions $f \in H^2$ for which

$$\|f_\zeta\|_2^2 = \frac{1}{2\pi} \int_{\partial \mathbb{D}} |f(z) - f(\zeta)|^2 \frac{1 - |\zeta|^2}{|z - \zeta|^2} |dz| \tag{1.4}$$

is bounded in $\zeta \in \mathbb{D}$ where we have set

$$f_\zeta(s) = f\left(\frac{s + \zeta}{1 + \bar{\zeta} s}\right) - f(\zeta) \quad (s \in \mathbb{D}, \ \zeta \in \mathbb{D}). \tag{1.5}$$

Let VMOA ("vanishing mean oscillation" [20]) denote the subspace of functions with

$$\|f_\zeta\|_2 \to 0 \quad \text{as} \quad |\zeta| \to 1 - 0. \tag{1.6}$$

Fefferman and Stein [5] proved that BMOA is the dual space of H^1, and Neri [14] proved that H^1 is the dual space of VMOA. These two results are equivalent by Taylor's [21] theory of conjugate norms for spaces of analytic functions because VMOA is the closure of the polynomials in the BMOA-norm [20].

Theorem 2. *Let g be analytic and univalent in $\mathbb{D}$ and let $C = \partial g(\mathbb{D})$ be a Jordan curve. Then*

$$\log g' \in \text{VMOA} \tag{1.7}$$

if and only if C is asymptotically smooth.

A well-known theorem of Lindelöf [18, p. 295] states that $\arg g'(z)$ is continuous in $\bar{\mathbb{D}}$ if and only if C is *smooth*, that is C has a continuously turning tangent. Our Theorem 2 is a generalization because an analytic function whose imaginary part is continuous in $\bar{\mathbb{D}}$ belongs to VMOA.

Since $(1 - |\zeta|^2)|f'(\zeta)| = |f'_\zeta(0)| \leq \|f_\zeta\|_2$ by (1.5), it follows that BMOA $\subset \mathscr{B}$ and VMOA $\subset \mathscr{B}_0$. This corresponds to the fact that a quasi-smooth curve is quasiconformal and an asymptotically smooth curve is asymptotically conformal.

Corollary 1. *We consider representations of the form*

$$f = a \log g' \quad (a \in \mathbb{C}, \quad a \neq 0),$$

where g is a univalent function and $C=\partial g(\mathbb{D})$ is a Jordan curve with the property specified below. Then

a) $f\in\mathscr{B}$ $\Leftrightarrow C$ *is quasi-conformal,*
b) $f\in\mathscr{B}_0$ $\Leftrightarrow C$ *is asymptotically conformal,*
c) $f\in\mathrm{BMOA}\Leftrightarrow C$ *is quasi-smooth,*
d) $f\in\mathrm{VMOA}\Leftrightarrow C$ *is asymptotically smooth*

where $\Leftarrow$ holds for all a while $\Rightarrow$ holds for sufficiently large a.

Relation a) is well-known (for details see [19]), while c) was proved in [19]. In a recent letter, Professor Coifman and Rochberg pointed out to me that implication "$\Leftarrow$" in c) is a consequence of results of Lavrent'ev [10] on conformal mapping and results of Coifman and Fefferman [4] on the Muckenhoupt condition. Finally b) follows from a) and Theorem 1, and d) follows from c) and Theorem 2.

I want to thank Professor Warschawski and the referee for their kind advice and for pointing out several mistakes in the first version of this paper.

2. Proof of Theorem 1

i)$\Rightarrow$ii). Let $\lambda\geqq 2$, $|\zeta|=\varrho<1$ and $z\in\mathbb{D}$, $|z-\zeta|\leqq\lambda(1-|\zeta|)$. It follows from i) and (1.2) that, for $0<\varepsilon<1$, there exists $\varrho_0=\varrho_0(\varepsilon,\lambda)<1$ such that

$$\left|\log\frac{g'(z)}{g'(\zeta)}\right|=\left|\int_\zeta^z\frac{g''(t)}{g'(t)}\,dt\right|\leqq\varepsilon\log 1\Big/\left(1-\left|\frac{z-\zeta}{1-\bar\zeta z}\right|^2\right)$$

for $\varrho_0<\varrho<1$. With $s=(z-\zeta)/(1-\bar\zeta z)$, we deduce that

$$\left|\frac{g'(z)}{g'(\zeta)}-1\right|\leqq\exp\left|\log\frac{g'(z)}{g'(\zeta)}\right|-1\leqq\frac{1}{(1-|s|^2)^\varepsilon}-1\ .$$

Since $|dz/ds|=|1-\bar\zeta z|^2/(1-\varrho^2)\leqq\lambda^2(1-\varrho^2)$ we obtain by integration that, if $\varrho_0<\varrho<1$,

$$\begin{aligned}\left|\frac{g(z)-g(\zeta)}{(z-\zeta)g'(\zeta)}-1\right|&\leqq\frac{\lambda^2(1-\varrho^2)}{|z-\zeta|}\int_0^{|s|}\left(\frac{1}{(1-\sigma)^\varepsilon}-1\right)d\sigma\\&=\frac{\lambda^2(1-\varrho^2)}{|1-\bar\zeta z|}\left(\frac{1-(1-|s|)^{1-\varepsilon}}{(1-\varepsilon)|s|}-1\right)\leqq\frac{2\lambda^2\varepsilon}{1-\varepsilon}\end{aligned}$$

for $z\in\bar{\mathbb{D}}$, $|z-\zeta|\leqq\lambda(1-\varrho)$. Hence ii) holds.

ii)$\Rightarrow$iii). Given $w_j=g(z_j)$ $(j=1,2)$ with $|z_j|=1$, we determine $\zeta\in\mathbb{D}$ such that $|z_j-\zeta|=2(1-|\zeta|)$. It follows from ii) that, uniformly in $|z-\zeta|\leqq 2(1-|\zeta|)$,

$$g(z)=g(\zeta)+(z-\zeta)g'(\zeta)(1+o(1))\quad\text{as}\quad|\zeta|\to 1-0\ .$$

If $w\in C(w_1,w_2)$ then $z=g^{-1}(w)$ lies on $\partial\mathbb{D}$ between z_1 and z_2. Hence we obtain that

$$|w-w_j|=|g'(\zeta)|(|z-z_j|+o(|z_2-z_1|))\qquad(j=1,2)\ .$$

It follows that, for $w \in C(w_1, w_2)$,

$$\frac{|w_2-w|+|w-w_1|}{|w_2-w_1|}=\frac{|z_2-z|+|z-z_1|+o(|z_2-z_1|)}{|z_2-z_1|+o(|z_2-z_1|)}\to 1$$

as $|\zeta|\to 1-0$, hence as $|w_1-w_2|\to 0$. Thus iii) holds.

iii)$\Rightarrow$i). Let iii) be satisfied. We set $\zeta=\varrho e^{i\vartheta}$ $(0\leqq\varrho<1)$ and $\varphi(s,\zeta)=e^{i\vartheta}(s+\varrho)/(1+\varrho s)$. We prove below that, for each $r<1$, there exists $R(r)<1$ with

$$(1-|\varphi(r,\zeta)|^2)\left|\frac{g''(\varphi(r,\zeta))}{g'(\varphi(r,\zeta))}\right|\leqq K_1(1-r^2) \tag{2.1}$$

for $R(r)\leqq\varrho<1$ where the constant K_1 is independent of r. This implies

$$(1-|z|^2)\left|\frac{g''(z)}{g'(z)}\right|\leqq K_1(1-r^2)\quad\text{for}\quad \frac{R+r}{1+rR}\leqq|z|<1$$

and it follows that i) holds.

Suppose that (2.1) is false. Then, for each $\lambda>0$, there exists $r<1$ and a sequence (ζ_n) such that

$$\left|\frac{\partial}{\partial r}\varphi(r,\zeta_n)\right|\left|\frac{g''(\varphi(r,\zeta_n))}{g'(\varphi(r,\zeta_n))}\right|>\lambda\qquad(n=2,3,\ldots). \tag{2.2}$$

We have $d(\zeta)\equiv\operatorname{dist}(g(\zeta),C)\to 0$ as $|\zeta|\to 1-0$. We may assume, after taking a subsequence if necessary, that $nd(\zeta_n)\to 0$ as $n\to\infty$. We define C_n as the minimal arc of C with $\{w\in C:|w-g(\zeta_n)|=nd(\zeta_n)\}\subset C_n$. Then the endpoints w_n and w_n' of C_n satisfy $|w_n-w_n'|\leqq 2nd(\zeta_n)\to 0$ as $n\to\infty$. By (1.1) we may therefore assume that

$$\max_{w\in C_n}\frac{|w_n'-w|+|w-w_n|}{|w_n'-w_n|}<1+\frac{1}{n^4}. \tag{2.3}$$

We set $\alpha_n=\arg(w_n'-w_n)$ and consider the transformations

$$t=\psi_n(w)=e^{-i\alpha_n}(w-g(\zeta_n))/d(\zeta_n)\qquad(n=1,2,\ldots). \tag{2.4}$$

The definition of w_n and w_n' shows that

$$\psi_n(w_n)=a_n+ib_n,\ \psi_n(w_n')=-a_n+ib_n(b_n>0),\ a_n^2+b_n^2=n^2 \tag{2.5}$$

and furthermore together with (2.3) that

$$\psi_n(C\backslash C_n)\subset\{|t|\geqq n\},\ \psi(C_n)\subset\{|\operatorname{Im}t-b_n|<2a_nn^{-2}\}. \tag{2.6}$$

Since $\operatorname{dist}(0,\psi_n(C))=1$ we see that $b_n-2a_nn^{-2}\leqq 1\leqq b_n+2a_nn^{-2}$. Hence we easily obtain from (2.5) that, for $n\geqq 2$,

$$\frac{n}{2}\leqq a_n\leqq n,\qquad 1-\frac{2}{n}\leqq b_n\leqq 1+\frac{2}{n}. \tag{2.7}$$

We apply now the Carathéodory kernel theorem to the univalent functions

$$g_n(s)=\psi_n(g(\varphi(s,\zeta_n)))=\frac{g(\varphi(s,\zeta_n))-g(\zeta_n)}{e^{i\alpha_n}d(\zeta_n)}. \tag{2.8}$$

It follows from (2.6), (2.7), and (2.5) that $\{t\in\mathbb{C}: \operatorname{Im} t<1\}$ is the kernel of the domain sequence $g_n(\mathbb{D})$ with respect to $g_n(0)=0$. Taking again a subsequence we may assume that $\beta=\lim \arg g_n'(0)$ exists. We conclude that, with $b=ie^{i\beta}$,

$$g_n(s)\to -\frac{2ibs}{1-bs} \quad \text{as} \quad n\to\infty \text{ locally uniformly in } \mathbb{D}. \tag{2.9}$$

Since C is a quasi-conformal curve, the normalized function $g_n(s)/g_n'(0)$ has, in the terminology of [16, Definition 3.2], uniformly well-accessible boundary behaviour (see [16, Lemma 3.3] or [16, Satz 3.8]). It follows from [16, Folgerung 3.6] that $g_n(s)/g_n'(0)$ is uniformly bounded in $0\leqq s<1$. Thus (2.9) shows that

$$\sup_{0\leqq s<1}\left|\frac{s}{1-bs}\right|\leqq K_2 \tag{2.10}$$

for some constant K_2. Hence we obtain from (2.8) and (2.9) that, as $n\to\infty$,

$$\left|\frac{\partial}{\partial r}\varphi(r,\zeta_n)\frac{g''(\varphi(r,\zeta_n))}{g'(\varphi(r,\zeta_n))}\right|=\left|\frac{2\varrho_n}{1+\varrho_n r}+\frac{g_n''(r)}{g_n'(r)}\right|\to\left|\frac{2}{1+r}+\frac{2b}{1-br}\right|.$$

This quantity is less than $4+2K_2$ by (2.10), which contradicts (2.2) for large λ.

3. Tangents of Asymptotically Conformal Curves

Let g be analytic and univalent in $\mathbb{D}$ and let $g(\mathbb{D})$ be bounded by the Jordan curve C. The next result is closely related to a result of Warschawski [23, Satz II] and also to Ostrowski's "I. Faltensatz" [15].

Corollary 2. *If C is asymptotically conformal then, as $\varrho\to 1-0$,*

$$\frac{g(z)-g(e^{i\vartheta})}{z-e^{i\vartheta}}\sim\frac{g(e^{i\vartheta})-g(\varrho e^{i\vartheta})}{e^{i\vartheta}(1-\varrho)}\sim g'(\varrho e^{i\vartheta}) \tag{3.1}$$

uniformly in $\vartheta\in[0,2\pi]$ and, for each $\lambda>1$, uniformly in

$$z\in\bar{\mathbb{D}},\ \lambda^{-1}(1-\varrho)\leqq|z-e^{i\vartheta}|\leqq\lambda(1-\varrho). \tag{3.2}$$

Proof. It follows from Theorem 1ii) that, with $\zeta=\varrho e^{i\vartheta}$,

$$\frac{g(z)-g(\zeta)}{z-\zeta}=(1+o(1))\frac{g(e^{i\vartheta})-g(\zeta)}{e^{i\vartheta}-\zeta} \quad (\varrho\to 1-0)$$

uniformly in ϑ and in $z\in\bar{\mathbb{D}}$, $|z-e^{i\vartheta}|\leqq\lambda(1-\varrho)$. We conclude that

$$\frac{g(z)-g(e^{i\vartheta})}{z-e^{i\vartheta}}=\frac{g(e^{i\vartheta})-g(\zeta)}{e^{i\vartheta}-\zeta}\left(1+\frac{z-\zeta}{z-e^{i\vartheta}}o(1)\right).$$

Hence the first relation (3.1) holds uniformly in z satisfying (3.2). The second relation (3.1) follows at once from Theorem 1ii).

Corollary 3. *Let C be asymptotically conformal and let $\zeta \in \partial \mathbb{D}$. If the limit*

$$\lim_{z \to \zeta} \arg \frac{g(z) - g(\zeta)}{z - \zeta} \quad or \quad \lim_{z \to \zeta} \frac{g(z) - g(\zeta)}{z - \zeta} \tag{3.3}$$

exists with z restricted to some arc $\subset \bar{\mathbb{D}}$ ending at ζ, then it also exists as an unrestricted limit.

Proof. A first application of (3.1) shows that the radial limit exists, and a second application of (3.1) then shows that the unrestricted limit exists.

Let T be the set of points on the Jordan curve C where C has a tangent. A theorem of Lindelöf [18, p. 302] states that, for $|\zeta| = 1$,

$$\lim_{z \to \zeta, z \in \bar{\mathbb{D}}} \arg \frac{g(z) - g(\zeta)}{z - \zeta} \text{ exists} \Leftrightarrow g(\zeta) \in T. \tag{3.4}$$

There are quasi-conformal curves with $T = \emptyset$ [18, p. 304]. We call a set *uncountably dense* on C if every arc of C contains uncountably many points of that set.

Corollary 4. *If C is asymptotically conformal then T is uncountably dense on C. There exist asymptotically conformal curves such that T has zero linear Hausdorff measure.*

Proof. a) The function $f = \log g'$ belongs to $\mathscr{B}_0$ by Theorem 1. By the method used to prove [13, Theorem 3] together with [1, Theorems 4.1, 4.2] (or [18, Theorem 10.7] and [18, Theorem 9.5]), it can be shown that $\operatorname{Im} f = \arg g'$ has a radial limit on an uncountably dense subset of $\partial \mathbb{D}$. Hence the first assertion follows from Corollary 2 and from (3.4).

b) We sketch another, geometric, proof of the first assertion. Let $G = g(\mathbb{D})$ and let $\varphi(w)$ $(w \in G)$ be a point on C of minimal distance from w. Then the disk of radius $|\varphi(w) - w|$ around w lies in G and touches C at $\varphi(w)$. It can be deduced from (1.1) that $\varphi(w) \in T$. If C is not a circle we can find, given $z \in G$, a small line segment S with endpoint z that does not lie on the normal to C at $\varphi(z)$. Then φ is one-to-one on S. It follows that T is uncountably dense on C because $|\varphi(w) - w| \to 0$ as $w \to C$.

c) We construct now examples to prove the second assertion. Let $b_k \to 0 (k \to \infty)$ but $\sum |b_k|^2 = \infty$. The Hadamard gap series

$$f(z) = \sum_{k=1}^{\infty} b_k z^{2^k} \qquad (z \in \mathbb{D})$$

belongs to $\mathscr{B}_0$ because $b_k \to 0$ [17]. According to Corollary 1b) there is a univalent function g with $f = a \log g'$ such that C is asymptotically conformal.

Since $\sum |b_k|^2 = \infty$ it follows from Zygmund's gap series theorem [26, p. 203] that $\log g'$ has a finite radial limit at almost no point of $\partial \mathbb{D}$. Hence the McMillan twist point theorem ([12, Theorem 1] or [18, Theorem 10.15]) shows that $g^{-1}(T)$ has zero measure, and it follows from another result of McMillan [12, Theorem 2] that T has zero linear Hausdorff measure.

4. Proof of Theorem 2

4.1. The following lemma is proved at once by integration by parts.

Lemma 1. *Let p, φ, ψ be real absolutely continuous functions in $[a,b]$. If $\varphi(t) \leqq \psi(t)$ and $p(t) \geqq 0$, $p'(t) \leqq 0$ for $a \leqq t \leqq b$, then*

$$\int_a^b \varphi'(t)p(t)dt \leqq \int_a^b \psi'(t)p(t)dt + (\psi(a) - \varphi(a))p(a).$$

We turn now to the proof of Theorem 2. Let $f = \log g'$ where g is univalent in $\mathbb{D}$ and $C = \partial g(\mathbb{D})$ is asymptotically smooth. Then $f \in \mathrm{BMOA} \subset H^1$ by Corollary 1. Let $\zeta = \varrho e^{i\vartheta}(0 < \varrho < 1)$. We set

$$v(t) = \mathrm{Re}[f(e^{i(\vartheta+t)}) - f(\zeta)] = \log|g'(e^{i(\vartheta+t)})/g'(\zeta)|, \tag{4.1}$$

$$p(t) = \frac{1}{2\pi} \frac{1-\varrho^2}{|e^{it} - \varrho|^2} \quad \text{(the Poisson kernel)}. \tag{4.2}$$

Since $e^x \geqq 1 + x + x^2/2$ for $x \geqq 0$ and $e^x \geqq 1 + x$ for $x \leqq 0$, we see that

$$\int_{-\pi}^{\pi} e^{v(t)}p(t)dt \geqq \int_{-\pi}^{\pi} (1 + v(t))p(t)dt + \frac{1}{2}\int_A v(t)^2 p(t)dt, \tag{4.3}$$

where $A = \{t \in [-\pi, \pi] : v(t) \geqq 0\}$. Since

$$\int_{-\pi}^{\pi} v(t)p(t)dt = 0$$

by (4.1) and the Poisson integral formula, we conclude from (4.3) by the Schwarz inequality that

$$\left(\int_{-\pi}^{\pi} |v(t)|\, p(t)dt\right)^2 = \left(2\int_A v(t)p(t)dt\right)^2 \leqq 4\int_A v(t)^2 p(t)dt$$
$$\leqq 8\left(\int_{-\pi}^{\pi} e^{v(t)}p(t)dt - 1\right). \tag{4.4}$$

We proceed now to estimate the last integral. Since C is, in particular, asymptotically conformal, we have $f \in \mathscr{B}_0$ by Theorem 1i). Hence there exists $r < 1$ such that

$$|\log g'(\zeta) - \log g'(\zeta^*)| = \left|\int_{\zeta^*}^{\zeta} f'(z)dz\right| \leqq \frac{1}{2}\log\frac{1-\varrho^*}{1-\varrho} \tag{4.5}$$

for $\zeta^* = \varrho^* e^{i\vartheta}$, $r \leqq \varrho^* \leqq \varrho$. If $\lambda > \pi/(1-r)$ we set, for $\varrho > 1 - \pi/\lambda$,

$$\alpha = \lambda(1-\varrho), \quad \zeta_t = \left(1 - \frac{(1-r)t}{\pi}\right)e^{i\vartheta} \quad (\alpha \leqq t \leqq \pi). \tag{4.6}$$

Since $r \leqq |\zeta_t| \leqq \varrho$ we obtain from (4.5) that, for $\alpha \leqq t \leqq \pi$,

$$|g'(\zeta_t)| \leqq \left(\frac{1-|\zeta_t|}{1-\varrho}\right)^{1/2} |g'(\zeta)| = \left(\frac{(1-r)t}{(1-\varrho)\pi}\right)^{1/2} |g'(\zeta)|. \tag{4.7}$$

Let $K_1, K_2, \ldots$ denote constants depending only on f. We deduce from (1.1) that, for $0 < t \leqq \pi$,

$$\int_{\vartheta}^{\vartheta+t} |g'(e^{i\tau})| d\tau < K_1 |g(e^{i(\vartheta+t)}) - g(e^{i\vartheta})| < K_2 t |g'(\zeta_t)| ;$$

the last inequality is proved as in [19, Lemma 2b] using (1.1) and [18, Corollary 10.3]. Hence we conclude from (4.1) and (4.7) that

$$\int_0^t e^{v(\tau)} d\tau < K_3 (1-\varrho)^{-1/2} t^{3/2} \qquad (\alpha \leqq t \leqq \pi),$$

and it follows from Lemma 1 that

$$\int_{\alpha}^{\pi} e^{v(t)} p(t) dt < \frac{3}{2} K_3 \int_{\alpha}^{\pi} \frac{t^{1/2}}{(1-\varrho)^{1/2}} p(t) dt + \frac{K_3 \alpha^{3/2}}{(1-\varrho)^{1/2}} p(\alpha).$$

Let $\varepsilon > 0$ be given. Since $p(t) < \pi(1-\varrho) t^{-2}$ and since $\alpha = \lambda(1-\varrho)$, we deduce that

$$\int_{\alpha}^{\pi} e^{v(t)} p(t) dt < K_4 (1-\varrho)^{1/2} \alpha^{-1/2} = K_4 \lambda^{-1/2} < \varepsilon \tag{4.8}$$

if $\lambda = \lambda(\varepsilon)$ is chosen sufficiently large.

Furthermore it follows from (4.1) and (1.3) that

$$\int_0^t e^{v(\tau)} d\tau = \int_{\vartheta}^{\vartheta+t} \left| \frac{g'(e^{i\tau})}{g'(\zeta)} \right| d\tau < \frac{1+\varepsilon}{|g'(\zeta)|} |g(e^{i(\vartheta+t)}) - g(e^{i\vartheta})|$$

if $0 < t < t_0(\varepsilon)$. Thus we obtain from Theorem 1 ii) that

$$\int_0^t e^{v(\tau)} d\tau < (1+2\varepsilon) t + \varepsilon(1-\varrho) \qquad (0 \leqq t \leqq \alpha)$$

if $\varrho > \varrho_0(\varepsilon)$, and it follows from Lemma 1 and from (4.2) that

$$\int_0^{\alpha} e^{v(t)} p(t) dt < (1+2\varepsilon) \int_0^{\alpha} p(t) dt + \varepsilon(1-\varrho) p(0) < \tfrac{1}{2} + 3\varepsilon .$$

Adding this inequality to (4.8) we see that

$$\int_{-\pi}^{\pi} e^{v(t)} p(t) dt < 1 + 8\varepsilon \qquad (\varrho_0(\varepsilon) < \varrho < 1)$$

because the integral over $[-\pi, 0]$ can be estimated in an analogous way. Hence we obtain from (4.4) that

$$\int_{-\pi}^{\pi} |v(t)| p(t) dt < 8 \sqrt{\varepsilon} \qquad (\varrho_0(\varepsilon) < \varrho < 1). \tag{4.9}$$

Since $p(t) \geqq [2\pi(1-\varrho)]^{-1}$ for $|t| \leqq 1-\varrho$, it follows from (4.1) that

$$\frac{1}{2(1-\varrho)} \int_{-(1-\varrho)}^{1-\varrho} |\mathrm{Re}\, f(e^{i(t+\vartheta)}) - \mathrm{Re}\, f(\zeta)| dt < 8\pi \sqrt{\varepsilon} .$$

Hence $\mathrm{Re}\, f(e^{it})$ belongs to VMO, and since $f \in H^1$ we conclude that $f \in \mathrm{VMOA}$.

4.2. For the proof of the converse we need the following lemma which is related to results of John and Nirenberg [9] and of Cima and Schober [3] on BMOA.

Lemma 2. *If* $f \in \mathrm{VMOA}$ *and* $\zeta = \varrho e^{i\vartheta}$, *then*

$$\frac{1}{2(1-\varrho)} \int_{-(1-\varrho)}^{1-\varrho} |\exp[f(e^{it+i\vartheta}) - f(\zeta)] - 1| \, dt \to 0 \qquad (|\zeta| \to 1-0). \tag{4.10}$$

Proof. Let $\|\cdot\|_2$ denote the H^2-norm and $\|\cdot\|_*$ the BMOA-norm. It was shown in [19, Lemma 1] that

$$\left\| \int_0^z \varphi'(s)\psi(s)\,ds \right\|_2 \leqq K \|\varphi(z)\|_* \, \|\psi(z)\|_2 , \tag{4.11}$$

where K is an absolute constant.

Let $0 < \varepsilon < 1/(2K)$. Since $f \in \mathrm{VMOA}$ there exists $r < 1$ [20, Theorem 1] such that $\|f(z) - f(rz)\|_* < \varepsilon$. We write

$$f = q + h, \qquad q(z) = f(z) - f(rz), \qquad h(z) = f(rz) \tag{4.12}$$

and use the notation (1.5). Since $\|q_\zeta\|_* \leqq \|q\|_* < \varepsilon$ for $|\zeta| < 1$ it follows from (4.11) that

$$\|e^{q_\zeta} - 1\|_2 \leqq K \|q_\zeta\|_* \, \|e^{q_\zeta}\|_2 < \varepsilon K (\|e^{q_\zeta} - 1\|_2 + 1).$$

Since $\varepsilon K < 1/2$ we see that $\|e^{q_\zeta} - 1\|_2 < 2\varepsilon K$ for $|\zeta| < 1$ and therefore from (4.12) that

$$\begin{aligned} \|e^{f_\zeta} - 1\|_2 &= \|(e^{q_\zeta} - 1)e^{h_\zeta} + (e^{h_\zeta} - 1)\|_2 \\ &< 2\varepsilon K \exp\left(\max_{|z|=1} |h_\zeta(z)| \right) + \|e^{h_\zeta} - 1\|_2 . \end{aligned}$$

Since h is analytic in $\bar{\mathbb{D}}$ we deduce from (1.5) that $h_\zeta(z)$ remains uniformly bounded in $|z| \leqq 1$ and that $\|h_\zeta\|_2 \to 0$ as $|\zeta| \to 1-0$. Hence

$$\frac{1}{2\pi} \int_{\partial\mathbb{D}} |e^{f(z)-f(\zeta)} - 1|^2 \frac{1-|\zeta|^2}{|z-\zeta|^2} |dz| = \|e^{f_\zeta} - 1\|_2^2 \to 0$$

as $\varrho \to 1-0$, and the assertion (4.10) of the lemma easily follows.

We can now conclude the proof of Theorem 2. Let $f = \log g' \in \mathrm{VMOA}$. For given $w_1, w_2 \in C$ we choose $\zeta = \varrho e^{i\vartheta}$ such that $g^{-1}(w_{1,2}) = e^{i(\vartheta \pm \alpha)}$ where $\alpha = 1 - \varrho$. Then it follows from Lemma 2 that

$$l(w_1, w_2) = \int_{-\alpha}^{\alpha} |g'(e^{it})| \, dt < 2\alpha(1+\varepsilon)|g'(\zeta)|$$

for $\varrho_1(\varepsilon) < |\zeta| < 1$. Since $\mathrm{VMOA} \subset \mathscr{B}_0$ we obtain from Theorem 1 ii) that

$$\begin{aligned} |w_2 - w_1| &= |e^{i(\vartheta+\alpha)} - e^{i(\vartheta-\alpha)} + o(|e^{i\alpha} - \varrho|)| \, |g'(\zeta)| \\ &> 2\alpha(1-\varepsilon)|g'(\zeta)| \end{aligned}$$

for $\varrho > \varrho_2(\varepsilon) > \varrho_1(\varepsilon)$. Hence (1.3) holds because $|w_1 - w_2| \to 0$ is equivalent to $\varrho \to 1-0$, and C is therefore asymptotically smooth.

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