WHAT IS...
a Quasiconformal Mapping?

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Quasiconformal mappings are generalizations of conformal mappings. They can be considered not only on Riemann surfaces, but also on Riemannian manifolds in all dimensions, and even on arbitrary metric spaces. Quasiconformal mappings occur naturally in various mathematical and often a priori unrelated contexts.

The importance of quasiconformal mappings in complex analysis was realized by Ahlfors and Teichmüller in the 1930s. Ahlfors used quasiconformal mappings in his geometric approach to Nevanlinna's value distribution theory. He also coined the term "quasiconformal" in his 1935 work on *Überlagerungsflächen* that earned him one of the first two Fields medals. Teichmüller used quasiconformal mappings to measure a distance between two conformally inequivalent compact Riemann surfaces, starting what is now called Teichmüller theory.

There are three main definitions for quasiconformal mappings in Euclidean spaces: metric, geometric, and analytic. We begin with the *metric definition*, which is the easiest to state and which makes sense in arbitrary metric spaces. It describes the property that "infinitesimal balls are transformed to infinitesimal ellipsoids of bounded eccentricity".

Let $f : X \to Y$ be a homeomorphism between two metric spaces. For $x \in X$ and $r > 0$ let

$$L_f(x,r) = \sup\{|f(x) - f(y)| : |x-y| \le r\}$$

and

$$l_f(x,r) = \inf\{|f(x) - f(y)| : |x-y| \ge r\}\,.$$

(Here and later we use the Polish notation $|a-b|$ for the distance in any metric space.) The ratio $H_f(x,r) = L_f(x,r)/l_f(x,r)$ measures the eccentricity of the image of the ball $B(x,r)$ under f. We say that f is *H-quasiconformal*, $H \ge 1$, if

$$\limsup_{r\to 0} H_f(x,r) \le H \tag{1}$$

for every $x \in X$.

Homeomorphisms that are 1-quasiconformal between domains in $\mathbb{R}^2 = \mathbb{C}$ are precisely the (complex analytic) conformal or anticonformal mappings, by a theorem of Menshov from 1937. Homeomorphisms that are 1-quasiconformal between domains in $\mathbb{R}^n$, $n \ge 3$, are precisely the Möbius transformations, or compositions of inversions on spheres in the one-point compactification $\mathbb{R}^n \cup \{\infty\}$, by the generalized Liouville theorem proved by Gehring and Reshetnyak in the 1960s. On the other hand, every diffeomorphism $f : \mathbb{R} \to \mathbb{R}$ is 1-quasiconformal according to the metric definition, as is every homeomorphism between discrete spaces. Surely not all such mappings deserve to be called quasiconformal. We will later remedy this situation.

Many early definitions for quasiconformality used some conformally invariant quantity and declared quasiconformal mappings to be those homeomorphisms that changed that quantity by a bounded amount. Here is one such *geometric definition*. Let $f : D \to D'$ be a homeomorphism between two domains in $\mathbb{R}^n$, $n \ge 2$. Then f is said to be *K-quasiconformal*, $K \ge 1$, if

$$K^{-1}\operatorname{mod}(f(\Gamma)) \le \operatorname{mod}(\Gamma) \le K\operatorname{mod}(f(\Gamma)) \tag{2}$$

for every curve family Γ in D. The *conformal modulus* $\operatorname{mod}(\Gamma)$ of a family Γ of curves in $\mathbb{R}^n$ is the infimum of the numbers $\int_{\mathbb{R}^n} \rho^n\,dx$ over all nonnegative Borel functions $\rho : \mathbb{R}^n \to [0,\infty]$ such that $\int_\gamma \rho\,ds \ge 1$ for every $\gamma \in \Gamma$. The definition of modulus is admittedly not easy to digest at one glance, but once mastered it is a powerful tool in geometric function theory. The geometric definition (2) is a global requirement that quickly yields many strong properties of quasiconformal mappings; for example, the inverse of a quasiconformal mapping is automatically quasiconformal, which is not at all obvious from the metric definition.

Also in the 1930s, an *analytic definition* for quasiconformal mappings was considered by Lavrentiev in connection with elliptic systems of partial differential equations. According to this definition, a homeomorphism $f : D \to D'$ between domains in $\mathbb{R}^n$, $n \ge 2$, is said to be *K-quasiconformal* if the first distributional partial derivatives of f are locally in the Lebesgue space L^n and if the formal differential matrix $Df = (\partial_i f_j)$ satisfies

$$\sup_{h\in\mathbb{R}^n, |h|\le 1} |Df(x)(h)|^n \le K \det Df(x) \tag{3}$$

for almost every $x \in D$. The use of distributional derivatives is essential in this context; important compactness properties of quasiconformal mappings are lost if smooth mappings only are considered.

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It is a deep fact that a homeomorphism $f\colon D \to D'$ between domains in $\mathbb{R}^n$, $n \geq 2$, is quasiconformal according to each of the three definitions—metric, geometric, and analytic—if it is quasiconformal according to any one of them. The parameters H and K depend only on each other and on n. The equivalence of these three definitions is a result of work by Gehring, Väisälä, and many others, in the 1950s and early 1960s.

There is a powerful existence theorem, proved by Morrey in 1938, that lends a special flavor to the 2-dimensional theory: given a measurable complex valued function μ in $\mathbb{R}^2$ such that $||\mu||_\infty < 1$, there exists a quasiconformal homeomorphism $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ (unique when properly normalized) such that $\overline{\partial} f = \mu\, \partial f$ in the sense of distributions, where $\overline{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$ and $\partial = \frac{1}{2}(\partial_x - i\partial_y)$ in complex notation. That is, one can measurably preassign the eccentricity and angle of the ellipses that are carried to a circle by the almost everywhere defined derivative of f. This fact has had tremendous impact on complex analysis and dynamics, Teichmüller theory, and low-dimensional topology.

The interplay of all three aspects of quasiconformality (metric, geometric, and analytic) is an important feature of the theory; one cannot rely on just one of them. Everyone who gives a graduate course on quasiconformal mappings faces the dilemma that a reasonably self-contained proof (using contemporary real and harmonic analysis) of the equivalence of all the preceding definitions would easily take up half of the semester. I would like to emphasize the fact, perhaps not widely realized, that from the technical point of view the quasiconformal mapping theory even in dimension $n = 2$ is much more part of real than complex analysis. Instead of power series, integral representations, or algebraic techniques, the theory relies on singular integrals, geometric measure theory, and Sobolev spaces.

There is another metric approach to quasiconformality that lately has found applications in a variety of contexts. A homeomorphism, or more generally an embedding, $f\colon X \to Y$ is said to be *η-quasisymmetric* if $\eta\colon [0,\infty) \to [0,\infty)$ is a homeomorphism and if $|x-y| \leq t|x-z|$ implies $|f(x) - f(y)| \leq \eta(t)|f(x) - f(z)|$ for every triple of points $x, y, z \in X$. An η-quasisymmetric mapping is obviously quasiconformal according to the metric definition, with $H = \eta(1)$, but the converse is not true in general. It is another deep fact that the infinitesimal condition (1) implies quasisymmetry for homeomorphisms $f\colon \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 2$. Although this fact is a statement about two purely metric conditions, all known proofs use delicate geometry and analysis; in particular, all known proofs give an η that depends on H and dimension n (but nothing else). It is an open problem whether a self-homeomorphism of an infinite-dimensional Hilbert space that satisfies (1) is also quasisymmetric.

Quasisymmetry is the right definition for quasiconformal mappings in dimension one, e.g., on the real line. More generally, the concept of quasisymmetry is a good analogue of quasiconformality in arbitrary metric spaces, where condition (1) is often too weak to give an interesting theory. On the other hand, quasisymmetry is a global condition and simple examples show, for instance, that conformal mappings between planar domains need not be quasisymmetric. The following *egg yolk principle* describes a precise relationship between quasisymmetry and quasiconformality for homeomorphisms between domains in $\mathbb{R}^n$, $n \geq 2$: $f\colon D \to D'$ is quasiconformal if and only if there is η such that $f|B(x, \frac{1}{2}\mathrm{dist}(x, \partial D))$ is η-quasisymmetric for every $x \in D$. The assertion is moreover quantitative in that the various parameters depend only on each other and n.

Today quasiconformal mappings are used everywhere in complex analysis of one variable. But early on, the theory found applications beyond the classical framework. Mostow's proof of his celebrated rigidity results in general rank-one symmetric spaces required a quasiconformal mapping theory in sub-Riemannian manifolds. Sullivan showed that all topological manifolds outside dimension four carry quasiconformal structures, a fact later used by him, Connes, and Teleman to develop a theory of characteristic classes on topological manifolds. In the past ten years, it has become known that a full-fledged quasiconformal mapping theory exists in rather general metric measure spaces. This theory has subsequently been applied to new rigidity studies in geometric group theory. There is also a budding theory of quasiconformal mappings in infinite-dimensional Banach spaces, based on the concept of quasisymmetry. From the time of Lavrentiev and Morrey, the connection between quasiconformal analysis and elliptic partial differential equations via the analytic definition (3) has been manifest. In harmonic analysis, an important self-improving phenomenon associated with reverse Hölder inequalities was first discovered in connection with quasiconformal mappings by Gehring in 1973. New generalizations have emerged from connections to elasticity theory.

Quasiconformal mappings are fascinating objects in mathematics. They are flexible enough to be ubiquitous, yet they harbor enough subtle analytic and geometric properties so as to be useful in a variety of contexts.

References

[1] L. V. Ahlfors, Lectures on quasiconformal mappings, (with additional chapters by C. J. Earle, I. Kra, M. Shishikura, J. H. Hubbard), *University Lecture Series* **38** (2006), Amer. Math. Soc., Providence, RI.

[2] J. Heinonen, *Lectures on Analysis on Metric Spaces*, Springer-Verlag, 2001.

[3] T. Iwaniec and G. Martin, Geometric function theory and non-linear analysis, *Oxford Mathematical Monographs*, Clarendon Press, Oxford University Press, 2001.