MAT 627, Spring 2025, Stony Brook University

Topics in Complex Analysis: Quasiconformal Mappings

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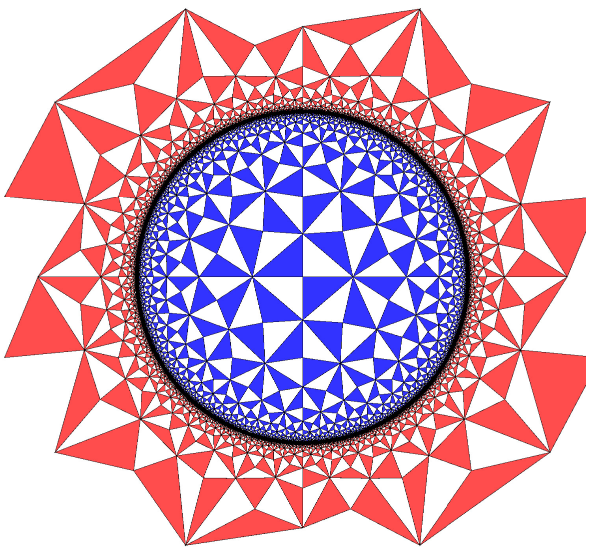
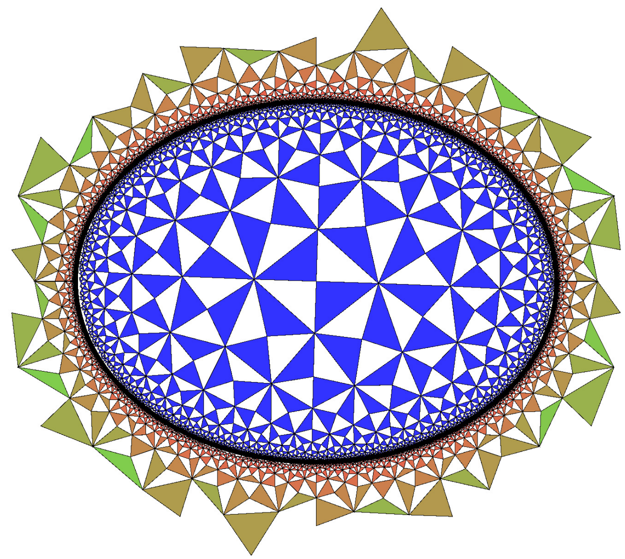




Holomorphic motions

Definition: Suppose $A \subset \mathbb{C}^\infty$. A **holomorphic motion** of A is a map $\Phi : \mathbb{D} \times A \to \mathbb{C}^\infty$ such that

(1) For each $a \in A$, the map $\lambda \to \Phi(\lambda, a)$ is holomorphic on $\mathbb{D}$.

(2) For any fixed $\lambda \in \mathbb{D}$, the map $a \to \Phi(\lambda, a) = \Phi_\lambda(a)$ is 1-to-1,

(3) The mapping Φ_0 is the identity on A.

Note that no assumption of continuity or measurability in a is made.

Astala-Martin paper on holomorphic motions.

Definition: Let $\eta : [0, \infty) \to [0, \infty)$ be an increasing homeomorphism and $A \subset \mathbb{C}$. A mapping $f : A \to \mathbb{C}$ is called η-quasisymmetric if the each triple $x, y, z \in A$,

$$\frac{|f(x_- f(y)|}{|f(x) - f(z)|} \leq \eta \left(\frac{|x - y|}{|x - z|} \right).$$

We say f is quasisymmetric if it is η-quasisymmetric for some η.

If f is defined on an open set, we also assume if preserves orientation.

It is immediate that f is continuous and injective and not hard to show f is a homeomorphism onto its image.

Easy to show that the inverse of a quasisymmetric map is quasisymmetric.

One can prove that a map $\mathbb{C} \to \mathbb{C}$ is quasisymmetric iff it is quasiconformal. Also true for broad class of metric spaces (with appropriate definition of quasi-conformal).

The λ-lemma of Mañé, Sad and Sullivan:

Theorem 9.1. *If $\Phi : \mathbb{D} \times A \to \mathbb{C}^\infty$ is a holomorphic motion, then has an extension to $\overline{\Phi} : \mathbb{D} \times \overline{A} \to \mathbb{C}^\infty$ so that*

(1) $\overline{\Phi}$ is a holomorphic motion of $\overline{A}$.

(2) Each $\overline{\Phi}_\lambda : \overline{A} \to \mathbb{C}^\infty$ is quasisymmetric.

(3) $\overline{\Phi}(\lambda, a)$ is jointly continuous in λ and a.

Proof. we may assume A has at least three points and that $\{0, 1, \infty\} \in A$. We normalize Φ so the motion fixes $\{0, 1, \infty\}$ by setting

$$(\lambda, a) \to \frac{\Phi(\lambda,1) - \Phi(\lambda,0)}{\Phi(\lambda,1) - \Phi(\lambda,\infty)} \cdot \frac{\Phi(\lambda,1) - \Phi(\lambda,\infty)}{\Phi(\lambda,1) - \Phi(\lambda,0)}$$

The new map is still denoted Φ.

Let ρ be the hyperbolic metric on $\mathbb{C} \setminus \{0, 1\}$.

It follows from properties of the hyperbolic metric that there is some function $\eta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ so that

$$|w| \leq \eta(\rho(w, z), |z||)$$

and that for $\eta(x, \epsilon) \to 0$ uniformly as $\epsilon \to 0$ as long as $x \in (0, M]$, for a fixed $M < \infty$.

If $a_1, a_2, a_3 \in A$ are distinct, define

$$g(\lambda) = \frac{\Phi_\lambda(a_1) - \Phi_\lambda(a_2)}{\Phi_\lambda(a_1) - \Phi_\lambda(a_3)}.$$

This is holomorphic in λ with values in $\mathbb{C} \setminus \{0, 1\}$.

The Schwarz lemma says that holomorphic maps are contractions of the hyperbolic metric on any hyperbolic domain (this follows from the disk case and the uniformization theorem).

Thus g is a contraction of the hyperbolic metric from $\mathbb{D}$ to $\mathbb{C} \setminus \{0, 1\}$. Hence

$$\rho(g(\lambda), g(0)) \leq \rho_{\mathbb{D}}(\lambda, 0) = \log \frac{1 + |\lambda|}{1 - |\lambda|}.$$

Since

$$g(0) = \frac{a_1 - a_2}{a_1 - a_3}$$

we have

$$\left| \frac{\Phi_\lambda(a_1) - \Phi_\lambda(a_2)}{\Phi_\lambda(a_1) - \Phi_\lambda(a_3)} \right| \leq \eta \left(\log \frac{1 + |\lambda|}{1 - |\lambda|}, \left| \frac{a_1 - a_2}{a_1 - a_3} \right| \right).$$

This is the definition of Φ being quasisymmetric on A, and implies Φ is uniformly continuous on A, hence extends continuously to the closure of A.

We claim the extension is injective.

If not, there are points x, y in the closure that get mapped to the same point z.

Choose a_2 so that $\Phi(a_2) \neq z$ (we can do this since Φ is injective on A and A contains at least three points).

Then as a_1 approaches x and a_3 approaches y,

$$\left|\frac{\Phi_\lambda(a_1) - \Phi_\lambda(a_2)}{\Phi_\lambda(a_1) - \Phi_\lambda(a_3)}\right|$$

would blow up, contrary to what we have proved. Thus the extension is 1-to-1.

Thus the extension is a homeomorphism of the compact set $\overline{A}$.

For $a \in \overline{A} \setminus A$, the function $\lambda to \overline{\Phi}(\lambda, a)$ is a local uniform limit of holomorphic functions, so it is also holomorphic on $\mathbb{D}$.

The function is jointly continuous because for every $0 < r < 1$, the family $\{\overline{\Phi}_\lambda : \lambda \in r\mathbb{D}\}$ is equicontinuous. Note that

$$\begin{aligned}|\overline{\Phi}(\lambda_1, a_1) - \overline{\Phi}(\lambda_2, a_2)| \leq\ & |\overline{\Phi}(\lambda_1, a_1) - \overline{\Phi}(\lambda_1, a_2)| \\ & + |\overline{\Phi}(\lambda_1, a_2) - \overline{\Phi}(\lambda_2, a_2)|\end{aligned}$$

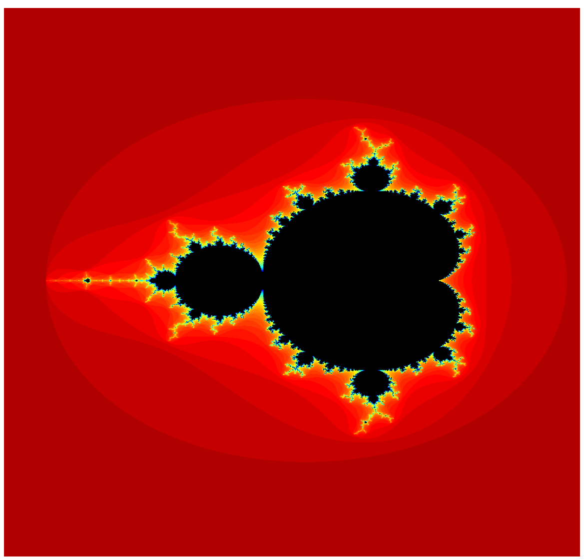
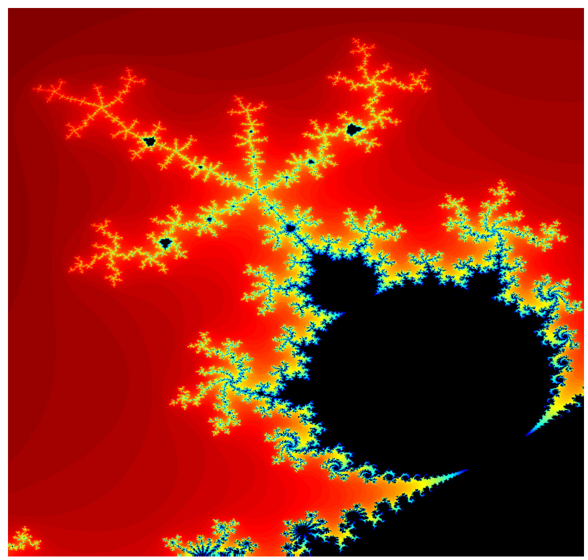
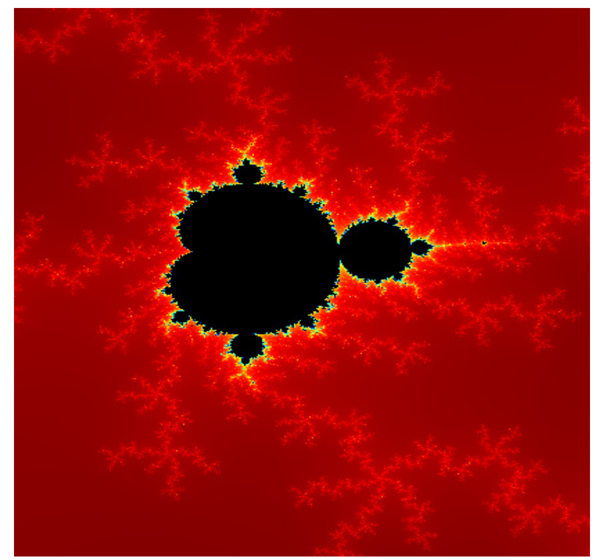
The first term is small because for a fixed λ, $\overline{\Phi}_\lambda$ is uniformly continuous with a bound depending only on an upper bound for $|\lambda| < 1$.

The second term is small because for a fixed a, $\overline{\Phi}(\lambda, a)$ is holomorphic in λ, hence continuous. $\square$

One application of the λ-lemma is to Julia sets of quadratic polynomials $z^2 + c$.

The Mandelbrot set has several hyperbolic components. Each of these are simply connected, and these maps have a single attracting periodic point.

The repelling periodic points are dense in the Julia set and move holomorphically as a function of c inside each hyperbolic component.

One can prove the repelling points do not collide. If they do, another attracting periodic point must result. This is impossible as each such point attracts a critical orbit and there is only one critical point.

By the λ-lemma says the holomorphic motion of the repelling points can be extended to the Julia set. Thus all the Julia sets in a hyperbolic component are quasisymmetrically equivalent.

Astala-Martin 2001 paper on holomorphic motions.

Complex Dynamics and Renormalization by Curt McMullen. Chapter 4 is titled "Holomorphic motions and the Mandelbrot set".

The extended λ-lemma:

Lemma 9.2. *Every holomorphic motion on a set $A \subset \mathbb{C}$ can be extended to a holomorphic motion of $\mathbb{C}$.*

Due to Slodkowski in 1991 using methods of several complex variables.

Proof in book of Astala-Iwaniec-Martin follows an argument of Chirka based on PDE; a non-linear Cauchy problem.

We will not give a proof in this class.