MAT 627, Spring 2025, Stony Brook University

Topics in Complex Analysis: Quasiconformal Mappings

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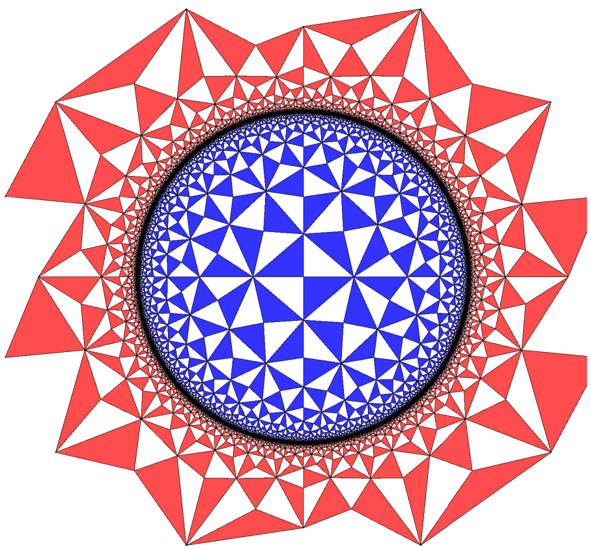

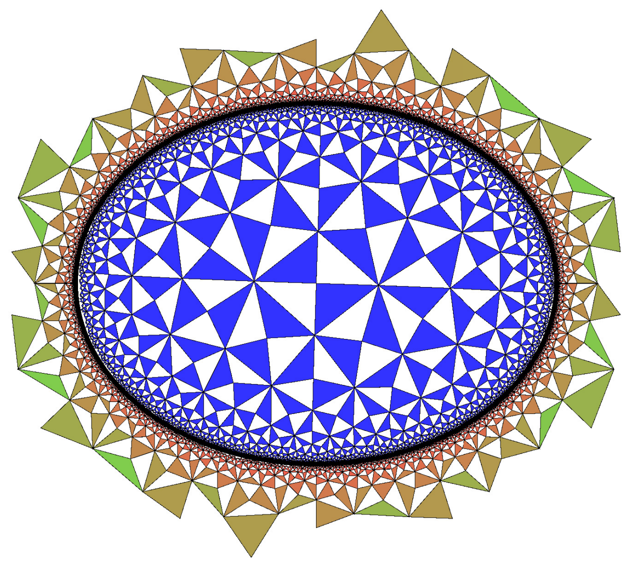

This semester I hope to cover the following topics:

- Review of complex analysis
- Extremal length and conformal modulus,
- Logarithmic capacity, harmonic measure
- Geometric definition of quasiconformal mappings, compactness
- Compactness of QC maps: quasisymmetry, extension, removability, weldings
- Analytic definition and the measurable Riemann mapping theorem
- Cauchy and Beurling transforms, analytic dependence
- Astala's theorems on area and dimension distortion
- Smirnov's $1 + k^2$ theorem
- Lehto maps
- Holomorphic motions

We have proven the measurable Riemann mapping theorem: given a dilatation μ with $\|\mu\|_\infty = k < 1$, there is a quasiconformal mapping f with dilatation μ.

Next, we would like to show $f : \mathbb{C} \to \mathbb{C}$ is unique if it normalized.

Two point normalization: $f(0) = 0$ and $f(1) = 1$.

Principle solution: if μ is compactly supported, then $f(z) = z + O(1/|z|)$ near ∞.

The latter is sometimes called the hydrodynamic normalization.

We also want to show that f depends holomorphically on μ, e.g., if z is fixed, $f_{t\mu}(z)$ is a holomorphic function of $t \in D(0, 1/k)$.

The proofs of uniqueness and holomorphic dependence both use explicit formulas involving the Cauchy transform and its "derivative", the Beurling transform.

The latter is a singular integral operator and is the 2-dimensional analog of the famous Hilbert transform on the real line.

The Cauchy Transform

Suppose $f \in L^p(\mathbb{R}^2, dxdy)$ for $p > 2$. Define

$$\mathcal{C}f(\zeta) = -\frac{1}{\pi}\iint_{\mathbb{C}} f(z)(\frac{1}{z-\zeta} - \frac{1}{z})dxdy.$$

The function $1/(z-\zeta)$ is not in L^2 locally, but is in L^q for all $q < 2$, so the integrand is locally integrable when $f \in L^p$ for some $p > 2$.

The extra $1/z$ term occurs so that the difference decays like $1/|z|^2$ near infinity, and hence the difference is in L^q for all $q > 1$. Thus the integral makes sense for all $f \in L^p$, $p > 2$.

If f is compactly supported, this means that its Cauchy transform also decays like $1/|z|^2$, which will be convenient when apply the Cauchy integral formula on large circles. It also implies $\mathcal{C}(f) \in L^p$, $p > 2$, in a neighborhood of ∞, outside the support of f.

Note that $\mathcal{C}(f)(0) = 0$ since the kernel vanishes if $\zeta = 0$.

Lemma 7.1. *If $f \in L^p$, $p > 2$, then $\mathcal{C}f$ is α-Hölder continuous with $\alpha = 1 - 1/p$.*

Proof. First note that the Cauchy transform on an L^p function is bounded.

$$\begin{aligned}
|\mathcal{C}f(\zeta)| &\leq \frac{1}{\pi} \cdot \|f\|_p \cdot \|\frac{1}{z-\zeta} - \frac{1}{z}\|_q \\
&= \frac{1}{\pi} \cdot \|f\|_p \cdot \|\frac{\zeta}{(z-\zeta)z}\|_q \\
&= \frac{|\zeta|}{\pi} \cdot \|f\|_p \cdot \|\frac{1}{(z-\zeta)z}\|_q.
\end{aligned}$$

The dependence on ζ is obtained by a change of variable $z = \zeta w$

$$\iint |z(z-\zeta)|^{-q} dxdy = \iint |\zeta w(\zeta w - \zeta)|^{-q} |\zeta|^2 dudv \tag{7.18}$$
$$= |\zeta|^{2-2q} \iint |w(w-1)|^{-q} dudv.$$

Since $q = p/(p-1)$, $2 - 2q = 2/p$ and this implies

$$|\mathcal{C} f(\zeta)| \leq K_p \cdot \|f\|_p \cdot |\zeta|^{1-1/p} \tag{7.19}$$

Next,

$$
\begin{aligned}
|\mathcal{C}f(\zeta_1) - \mathcal{C}f(\zeta 2)| &= \left| \frac{1}{\pi} \iint_{\mathbb{C}} f(z) \left(\frac{1}{z-\zeta_1} - \frac{1}{z-\zeta_2} \right) dxdy \right| \\
&= \left| \frac{1}{\pi} \iint_{\mathbb{C}} f(z+\zeta_1) \left(\frac{1}{z+\zeta_2-\zeta_1} - \frac{1}{z} \right) dxdy \right| \\
&= |\mathcal{C}h(\zeta_2 - \zeta_1)|,
\end{aligned}
$$

where $h(z) = f(z+\zeta_1)$. Applying (7.19) to h, we get

$$
\begin{aligned}
|\mathcal{C}f(\zeta_1) - \mathcal{C}f(\zeta 2)| &= |\mathcal{C}h(\zeta_2 - \zeta_1) \leq K_p \|h\|_p |\zeta|^{1-1/p} \\
&= K_p \|f\|_p |\zeta|^{1-1/p}.
\end{aligned}
$$

$\square$

Lemma 7.2. *If f is smooth and has compact support, then $\mathcal{C}f$ is smooth and $(\mathcal{C}f)_{\overline{z}} = f$.*

Proof. Let $\gamma_\epsilon = \partial D(\zeta, \epsilon)$ be a small circle around ζ. The convolution of a smooth, compactly supported function is smooth and interchanging integration and differentiation gives $(\mathcal{C}f(\zeta))_{\overline{z}} = (\mathcal{C}f_{\overline{z}}(\zeta))$.

Recall

$$dzd\overline{z} = (dx + idy)(dx - idy) = idydx - idxdy = -2idxdy.$$

By Stokes theorem and using the fact that $|f| = O(1/|z|^2)$,

$$\begin{aligned}
(\mathcal{C}f(\zeta))_{\overline{z}} = (\mathcal{C}f_{\overline{z}}(\zeta)) &= -\frac{1}{\pi}\iint \frac{f_{\overline{z}}}{z-\zeta}dxdy \\
&= -\frac{1}{2\pi i}\iint \frac{f_{\overline{z}}}{z-\zeta}dzd\overline{z} \\
&= -\frac{1}{2\pi i}\iint \frac{df\,d\overline{z}}{z-\zeta} \\
&= \lim_{\epsilon\to 0}\frac{1}{2\pi i}\iint \frac{f\,dz}{z-\zeta} = f(\zeta).
\end{aligned}$$

$\square$

Corollary 7.3. *If $f \in L^p$, $p > 2$, then $(\mathcal{C}f)_{\overline{z}} = f$ in the sense of distributions.*

Proof. We must show that for any smooth ϕ with compact support,

$$\iint (\mathcal{C}f)\phi_{\overline{z}} dx dy = -\iint \phi f dx dy. \tag{7.20}$$

Take smooth functions $\{f_n\}$ of compact support converging to f. By Hölder's inequality

$$|\iint \phi(f - f_n)dxdy| \leq \|\phi\|_q \cdot \|f - f_n\|_p.$$

The first term on the product is a finite constant and the other tends to z_0, so $\iint \phi f_n \to \iint \phi f$.

On the other hand if the support of ϕ has diameter d,

$$\begin{aligned}\left|\iint (\mathcal{C}f - \mathcal{C}f_n)\phi_{\overline{z}}dxdy\right| &\leq \|\phi_{\overline{z}}\|_1 \cdot \sup_{z\in \mathrm{supp}(\phi)} |\mathcal{C}(f-f_n)(c)| \\ &\leq \|\phi_{\overline{z}}\|_1 \cdot K_p \cdot \|f-f_n\|_p d^{1-1/p}\end{aligned}$$

and this tends to zero with n. Thus (7.20) holds. $\square$

The Beurling Transform

We will also need a few basic facts about the Beurling transform, which is usually defined as a principle value integral

$$\mathcal{T}f(\zeta) = \lim_{\epsilon \to 0} \iint_{|z-\zeta|>\epsilon} \frac{f(z)}{(z-\zeta)^2} dxdy.$$

For smooth, or even Hölder, functions of compact support this is well defined by rewriting the integral as

$$\mathcal{T}f(\zeta) = \lim_{\epsilon \to 0} \iint_{|z-\zeta|>\epsilon} \frac{f(z)-f(\zeta)}{(z-\zeta)^2} dxdy,$$

since the kernel has integral zero on any circle centered at ζ.

The Beurling transform can be extended to a bounded linear operator from $L^p(\mathbb{R}^2, dxdy)$ to itself for all $1 < p < \infty$.

We shall show below that $\mathcal{T}$ is an isometry on L^2.

The standard proof of MRMT uses that $\mathcal{T}$ is bounded for $p > 2$ with an operator norm that approaches 1 as $p \searrow 2$, but we will not need this fact; we have already proven Bojarski's theorem that $f_z \in L^p$ for a K-QC map, and this will be sufficient for our applications.

Recall

$$\int_{|z|=1} \frac{dz}{z} = 2\pi i, \qquad \int_{|z|=1} \frac{d\overline{z}}{z} = 0.$$

$$dzd\overline{z} = -2idxdy.$$

Lemma 7.4. *If f is smooth and has compact support then $\mathcal{C}f$ is smooth and $\mathcal{C}(f_z) = \mathcal{T}f - \mathcal{T}f(0)$.*

Proof. As in Lemma 7.2 we have that $\mathcal{C}f$ is smooth and $(\mathcal{C}f(\zeta))_z = (\mathcal{C}f_z(\zeta))$. Using Stokes theorem again

$$\begin{aligned}
(\mathcal{C}f_z(\zeta)) &= -\frac{1}{\pi}\iint \frac{f_z}{z-\zeta}dxdy \\
&= \frac{1}{2\pi i}\iint \frac{f_{\overline{z}}}{z-\zeta}dzd\overline{z} \\
&= \lim_{\epsilon\to 0}\left[-\frac{1}{2\pi i}\int_{|z-\zeta|=\epsilon}\frac{fd\overline{z}}{z-\zeta} + \frac{1}{2\pi i}\iint_{|z-\zeta|>\epsilon}\frac{fdzd\overline{z}}{(z-\eta)^2}\right] \\
&= \mathcal{T}f(\zeta).
\end{aligned}$$

$\square$

From the above we get

$$(\mathcal{T}f)_{\overline{z}} = \mathcal{C}(f_z)_{\overline{z}} = f_z,$$

$$(\mathcal{T}f)_z = \mathcal{C}(f_z)_z = T(f_z) = \mathcal{C}(f_{zz}) + T(f_z)(0).$$

Lemma 7.5. *The Beurling transform is an isometry on* $L^2(\mathbb{R}^2, dxdy)$.

Proof. It is enough to check this on the dense set of smooth, compactly supported functions. Then

$$
\begin{aligned}
\iint |\mathcal{T}f|^2 dxdy &= -\frac{1}{2i}\iint |(\mathcal{C}f)_z)|^2 dz d\overline{z} \\
&= -\frac{1}{2i}\iint (\mathcal{C}f)_z\overline{(\mathcal{C}f)_z} dz d\overline{z} = -\frac{1}{2i}\iint (\mathcal{C}f)_z\overline{(\mathcal{C}f)}_{\overline{z}} dz d\overline{z} \\
&= \frac{1}{2i}\iint \mathcal{C}f\overline{(\mathcal{C}f)}_{\overline{z}z} dz d\overline{z} = \frac{1}{2i}\iint \mathcal{C}f\overline{(\mathcal{C}f)_{\overline{z}}}_{\overline{z}} dz d\overline{z} \\
&= \frac{1}{2i}\iint \mathcal{C}f\overline{f}_{\overline{z}} dz d\overline{z} = -\frac{1}{2i}\iint (\mathcal{C}f)_{\overline{z}}\overline{f} dz d\overline{z} \\
&= -\frac{1}{2i}\iint f\overline{f} dz d\overline{z} \\
&= \iint |f^2| dxdy \quad \square
\end{aligned}
$$

Uniqueness in MRMT

Lemma 7.6. *If μ is measurable, $\|\mu\|_\infty = k < 1$ and μ has compact support, then there is a unique K-quasiconformal map f (with $K = (1+k)/(1-k)$) that is absolutely continuous on almost all lines and satisfies $f_{\overline{z}} = \mu f_z$ and $f_z - 1 \in L^p(\mathbb{R}^2)$ for some $p > 1$.*

Proof. We already know uniqueness, so the L^p bound is the main point.

Suppose f is such a solution. We know $f_z \in L^p$ locally, so $f_{\overline{z}} - \mu f_z \in L^p$ on the plane. Hence $\mathcal{C}(f_{\overline{z}})$ is well defined and $(\mathcal{C} f_{\overline{z}})_{\overline{z}} = f_{\overline{z}}$ by Corollary 7.3.

Thus $(f - \mathcal{C} f_{\overline{z}})_{\overline{z}} = 0$ in the sense of distributions and hence it is analytic on the plane by Weyl's lemma.

We assumed $f_z - 1 \in L^p$, and $\mathcal{C} f_{\overline{z}} \in L^p$ for any $p > 2$ (because it is $O(|z|^{-2})$ near infinity), so the holomorphic function $F = f - \mathcal{C} f_{\overline{z}} - 1$ has $F' \in L^p$.

This is only possible if $F' = 1$ or $F(z) = z + c$.

Because we assumed $f(0) = 0$, and $\mathcal{C} f_{\overline{z}}(0) = 0$, we must have $c = 0$. Thus $f(z) = \mathcal{C}(f_{\overline{z}})(z) + z$ and $f_z = \mathcal{T}(\mu(f_z)) + 1$.

If g were another solution, then using the fact that $\mathcal{T}$ is an isometry on L^2 gives

$$\|f_z - g_z\|_2 = \|\mathcal{T}(\mu(f_z - g_z))\|_2 \leq k\|\mathcal{T}(f_z - g_z)\|_2,$$

and this is a contradiction unless $\|f_z - g_z\| - 2 = 0$.

Therefore $f_z = g_z$ almost everywhere, and hence $f_{\overline{z}} = \mu f_z = \mu g_z = g_{\overline{z}}$ almost everywhere.

Thus $f - g$ is both holomorphic and anti-holomorphic, hence constant. Since $f(0) = g(0) = 0$, they must be equal everywhere. $\square$

Alternate proof of MRMP

Consider

$$h = T(\mu h) + T\mu$$

The series

$$h = T\mu + T\mu T\mu + T\mu T\mu T\mu + \dots$$

converges in L^p if L^p norm of T is less than $1/k$, $k = \|\mu\|_\infty$.

If h is gvien by this series,set

$$f = P(\mu(h+1),$$

then $\mu(h+1) \in L^p$ and $P(\mu(h+1)$ is continuous. Moreover,

$$f_{\overline{z}} = \mu(h+1), \qquad f_z = T(\mu(h+1)] + 1 = h + 1,$$

so $f_{\overline{z}} = \mu f_z$.

Analytic dependence

Lemma 7.7. *Suppose $\mu_t = \mu(z,t)$ is a path of dilatations that is differentiable at $t = 0$. Then the corresponding normalized QC maps are also differentiable at $t = 0$.*

More precisely, suppose $\mu(z,t) = r\nu(z) + t\epsilon(z,t)$ where $\nu, \epsilon \in L^\infty$ and $\|\epsilon(\cdot,t)\|_\infty \to 0$ for $t \searrow 0$. Let $f^\mu = f(z,t)$ be the quasiconformal map with dilatation $\mu(z,t)$ and normalized to have fixed points $0, 1, \infty$. Then

$$\dot{f}(\zeta) = \frac{1}{\pi} \int_{\mathbb{C}} \nu(z) R(z,\zeta) dxdy$$

where

$$R(z,\zeta) = \frac{1}{z-\zeta} - \frac{\zeta}{z-1} + \frac{\zeta - 1}{z} = \frac{\zeta(\zeta-1)}{z(z-1)(z-\zeta)}.$$

Proof. We follow the proof in Ahlfors's book.

For $|\zeta| < 1$ the Pompeiu formula (Lemma 6.24) says

$$f(z) = \frac{1}{2\pi i}\int_{|z|=1} \frac{f(z)dz}{z-\zeta} - \frac{1}{\pi}\iint_{|z|<1} \frac{f_{\overline{z}}(z)}{z-\zeta}dxdy. \tag{7.21}$$

We want to manipulate the line integral to get an integral formula for f in terms of $f_{\overline{z}}$ over the whole plane.

Since $|\zeta| < |z| = 1$, we can write

$$\begin{aligned}\frac{1}{z-\zeta} &= \frac{1}{z}\cdot\frac{1}{1-\zeta/z} \\ &= \frac{1}{z}\cdot\sum_{n=0}^{\infty}(\zeta/z)^n \\ &= \frac{1}{z}\cdot\left[1+\frac{\zeta}{z}+\frac{\zeta^2}{z^2}\sum_{n=0}^{\infty}(\zeta/z)^n\right] = \frac{1}{z}+\frac{\zeta}{z^2}+\frac{\zeta^2}{z^2}\frac{1}{z-\zeta}.\end{aligned}$$

Using this, rewrite the line integral in (7.23) as

$$\frac{1}{2\pi i}\int_{|z|=1}\frac{f(z)dz}{z-\zeta} = A + B\zeta + \frac{\zeta^2}{2\pi i}\int_{|z|=1}\frac{f(z)dz}{z^2(z-\zeta)}. \tag{7.22}$$

Apply the substitution $z = 1/w$, $dz = -dw/w^2$ in the last integral to obtain

$$\zeta^2 2\pi i \int_{|z|=1} \frac{f(z)dz}{z^2(z-\zeta)} = -\zeta^2 2\pi i \int_{|w|=1} \frac{f(1/w)dw}{(w^2)(1/w)^2(1/w-\zeta)} \tag{7.23}$$

$$= -\zeta^2 2\pi i \int_{|w|=1} \frac{f(1/w)w\,dw}{1-w\zeta}.$$

Let $g(z) = 1/f(1/z)$. Then g is quasiconformal and $g(0) = 0$.

It is easy to check that $(1/g)_{\overline{z}} = g_{\overline{z}}/g^2$ and that if h is holomorphic, then $(gh)_{\overline{z}} = g_{\overline{z}}h$

Now appy $g(0) = 0$ and the Pompieu formula again,

$$\begin{aligned}
\frac{-\zeta^2}{2\pi i}\int_{|w|=1}\frac{f(1/w)wdw}{1-w\zeta} &= \frac{-\zeta^2}{2\pi i}\int_{|w|=1}\frac{g(w)^{-1}wdw}{1-w\zeta} \\
&= \frac{-\zeta^2}{2\pi i}\int_{|w|<1}\frac{g_{\overline{z}}(w)wdw}{g^2(w)(1-w\zeta)}.
\end{aligned}$$

The integrals converge because quasiconformal maps are biHölder and hence $|g(w)| > c\sqrt{|w|}$ if $\|\mu\|_\infty$ is small enough. (Then $|w|/|g(w)|$ is bounded.)

We know that f is given by some formula of the form:

$$
\begin{aligned}
f(\zeta) &= A + B\zeta - \frac{1}{\pi}\int_{|z|<1} \frac{f_{\overline{z}}(z)}{z-\zeta} dxdy \\
&\quad - \frac{1}{\pi}\iint_{|z|<1} \frac{g_{\overline{z}}(z)}{g(z)^2}\left(\frac{\zeta^2 z}{1-z\zeta}\right) dxdy.
\end{aligned}
$$

We guess (or solve for) the correct values of A, B:

$$
\begin{aligned}
f(\zeta) &= \zeta - \frac{1}{\pi}\int_{|z|<1} f_{\overline{z}}(z)\left(\frac{1}{z-\zeta} - \frac{\zeta}{z-1} + \frac{\zeta-1}{z}\right) dxdy \\
&\quad - \frac{1}{\pi}\iint_{|z|<1} \frac{g_{\overline{z}}(z)}{g(z)^2}\left(\frac{\zeta^2 z}{1-z\zeta} - \frac{\zeta z}{1-z}\right) dxdy
\end{aligned}
$$

and can check this is correct by verifying $f(0) = 0$ and $f(1) = 1$.

In the first integral set $f_{\overline{z}} = \mu f_z = \mu(f_z - 1) + \mu$ and use a corresponding expression for $g_{\overline{z}}$ with $\mu_g(z) = (z/\overline{z})^2 \mu(1/z)$.

Because $\|f_z - 1\|_p \to 0$ as $\|\mu\|_\infty \to 0$ by Corollary **??**, and $\mu/t \to \nu$,

$$\begin{aligned} \dot{f}(\zeta) &= \lim_{t \to 0} \frac{f(\zeta) - \zeta}{t} \\ &= \frac{1}{\pi} \int_{|z|<1} \nu(z) \left(\frac{1}{z - \zeta)} - \frac{\zeta}{z - 1} + \frac{\zeta - 1}{z} \right) dxdy \\ &\qquad - \frac{1}{\pi} \iint_{|z|<1} \nu(1/z) \left(\frac{\zeta^2 z}{1 - z\zeta} - \frac{\zeta z}{1 - z} \right) dxdy. \end{aligned}$$

If $1/z$ is taken as the integration variable in the second integral, it transforms to the same integrand as in the first, so

$$\dot{f}(\zeta) \ = \ \frac{1}{\pi}\int_{\mathbb{C}} \nu(z)R(z,\zeta)dxdy$$

where

$$R(z,\zeta) = \frac{1}{z-\zeta} - \frac{\zeta}{z-1} + \frac{\zeta-1}{z} = \frac{\zeta(\zeta-1)}{z(z-1)(z-\zeta)}.$$

$\square$

Theorem 7.8. *If $\mu(z,t)$ is a holomorphic function of t, let $f^\mu(z,t)$ be the quasiconformal map with dilatation $\mu(z,t)$, normalized to fix 0, 1 and ∞, then for each fixed z, $f^\mu(z,t)$ is a holomorphic function of t.*

Corollary 7.9. *Suppose $\|\mu\|_\infty = k < 1$ is the dilatation of f. Let $\mu(z,t) = (t/k)\mu(z)$. Then for each z, $f^t(z)$ is a holomorphic function of $t \in \mathbb{D}$ so that $f^k = f$.*

Proof of Theorem 7.8. It suffices to show that $f^t(z)$ is differentiable at each t. We have already done this for $t = 0$.

For arbitrary t_0, since $\mu(z,t)$ is differentiable in t, we may assume

$$\mu(z,t) = \mu(z,t_0) + \nu(z,t_0)(t-t_0) + o(t-t_0),$$

and consider

$$f^{\mu(t)} = f^{\lambda} \circ f^{\mu(t_0)},$$

where (using the composition law for dilatations)

$$\lambda = \lambda(t) = \left(\frac{\mu(t) - \mu(t_0)}{1 - \mu(t)\overline{\mu(t_0)}}\right) \circ (f^{\mu_0})^{-1}.$$

Then

$$\dot{\lambda}(t) = \left(\frac{\nu(t_0)}{1-|\mu_0|^2} \cdot \frac{f_z^{\mu_0}}{\overline{f_{\bar{z}}^{\mu_0}}} \right) \circ (f^{\mu_0})^{-1},$$

and

$$\begin{aligned}
\frac{\partial}{\partial t} f(z,t) &= \dot{f} \circ f^{\mu_0} \\
&= -\frac{1}{\pi} \iint \left(\frac{\nu(t_0)}{1-|\mu_0|^2} \cdot \frac{f_z^{\mu_0}}{\overline{f_{\bar{z}}^{\mu_0}}} \right) \circ (f^{\mu_0})^{-1} R(z, f^{\mu_0}(\zeta)) dxdy \\
&= -\frac{1}{\pi} \iint \nu(t_0, z)(f_z^{\mu_0})^2 R(f^{\mu_0}(z), f^{\mu_0}(\zeta)) dxdy.
\end{aligned}$$

This is the general formula for the derivative. $\square$