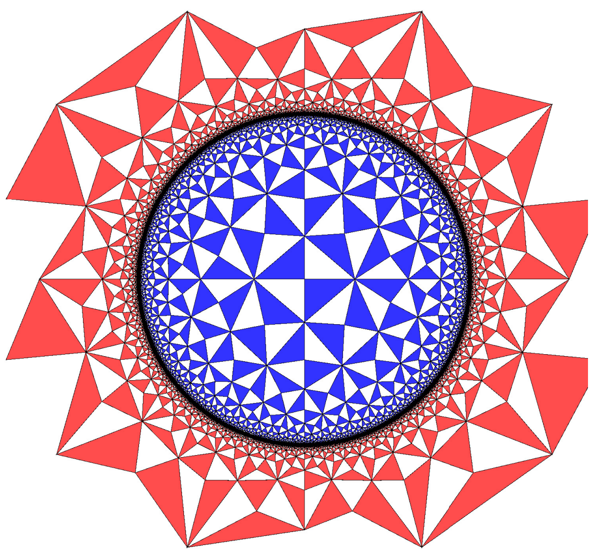
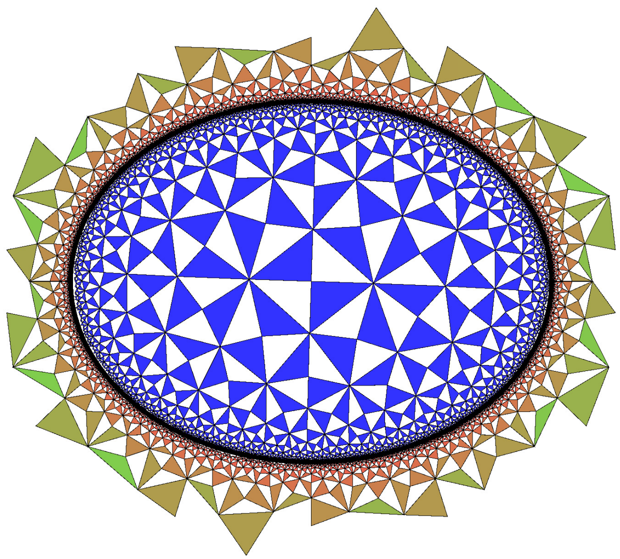
MAT 627, Spring 2025, Stony Brook University

Topics in Complex Analysis: Quasiconformal Mappings

Christopher Bishop

- Review of conformal mappings
- Extremal length and conformal modulus, log capacity, harmonic measure
- Definitions of quasiconformal mappings; geometric and analytic
- Geometric definition and basic properties
- Removable sets
- Analytic definition and measurable Riemann mapping theorem
- Conformal welding
- Further topics

Holomorphic and conformal mappings

A conformal map between planar domains is a C^1, orientation preserving diffeomorphism which preserves angles. Write $f(x, y) = (u(x, y), v(x, y))$. We can compute it derivative matrix

$$Df = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}.$$

Since f preserves orientation and angles, the linear map represented by this matrix must be an orientation preserving Euclidean similarity.

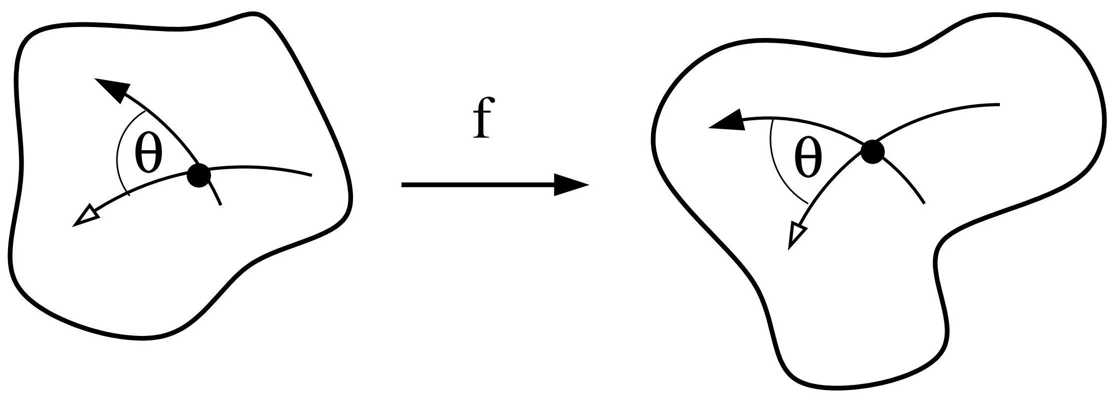
Thus it is a composition of a dilation and rotation and must have the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

which implies

$$u_x = v_y, \quad u_y = -v_x.$$

These are known as the Cauchy-Riemann equations. Thus f is conformal if it is C^1 diffeomorphism which satisfies the Cauchy-Riemann equations.



The simplest examples are the Euclidean similarities, and indeed, these are the only examples if we want maps $\mathbb{R}^2 \to \mathbb{R}^2$.

However, if we consider subdomains of $\mathbb{R}^2$, then there are many more examples. The celebrated Riemann mapping theorem says that any two simply connected planar domains (other that the whole plane) can be mapped to each other by a conformal map.

After the linear maps, the next simplest holomorphic maps are quadratic polynomials. If we take

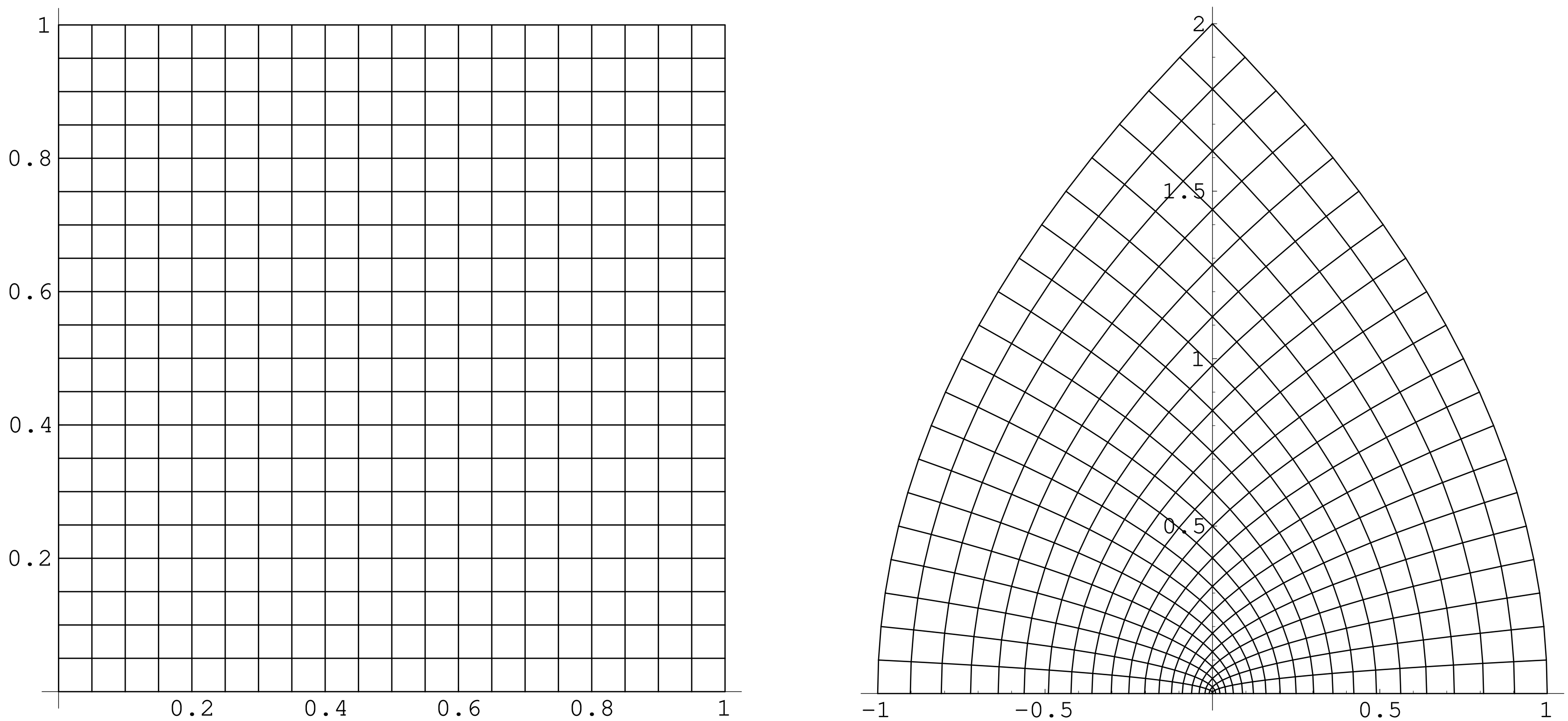
$$f(x,y) = (u(x,y), v(x,y)) = (x^2 - y^2, 2xy),$$

then we can easily check that

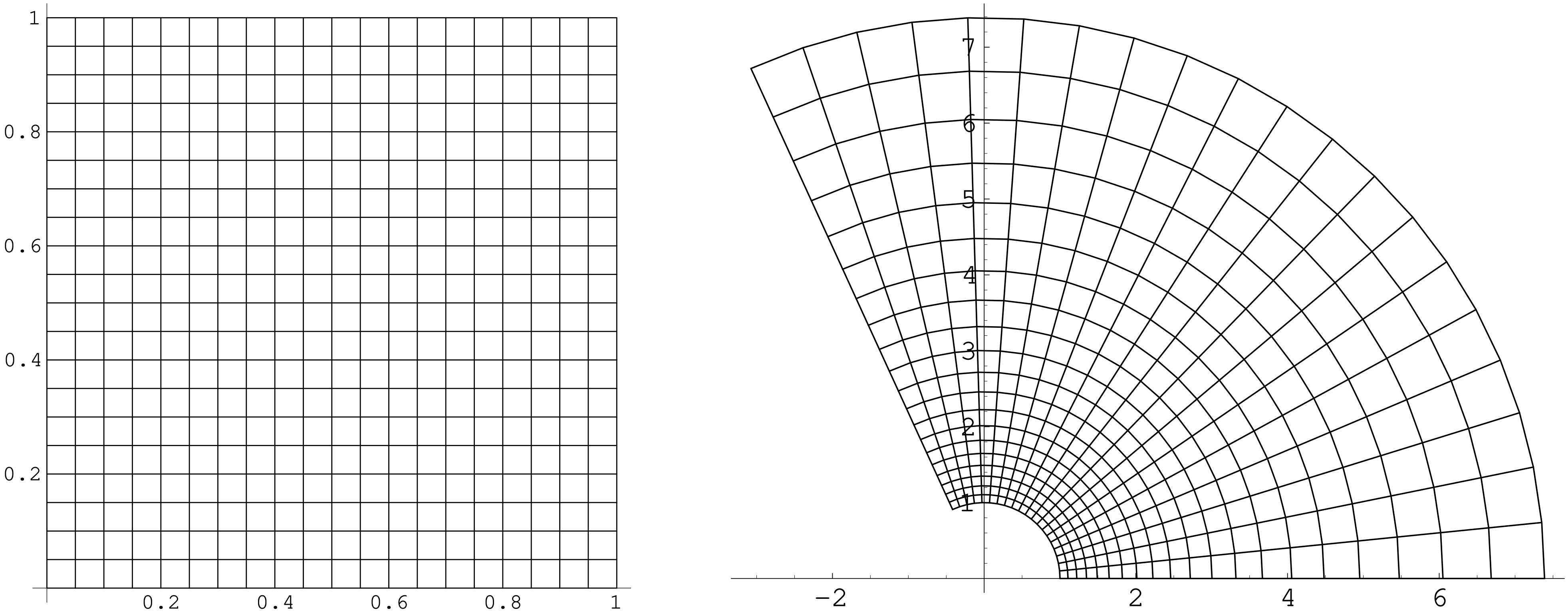
$$Df(x,y) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix},$$

so the Cauchy-Riemann equations are satisfied.

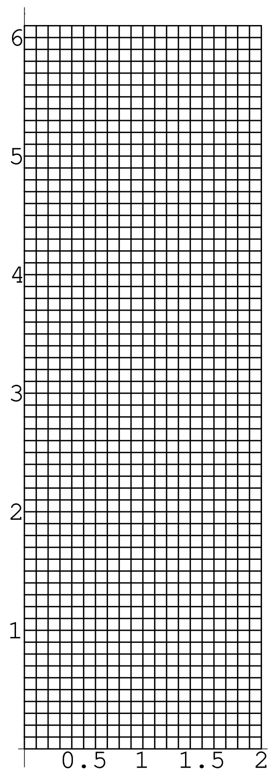
The map is not conformal on the plane since $f(-x,-y) = f(x,y)$ is 2-to-1 for $(x,y) \neq (0,0)$ and Df vanishes at the origin. However, it is a conformal map if we restrict it to a domain (an open, connected set) where it is 1-to-1, such as the open square $[0,1]^2$. The map sends this square conformally to a region in the upper half-plane.

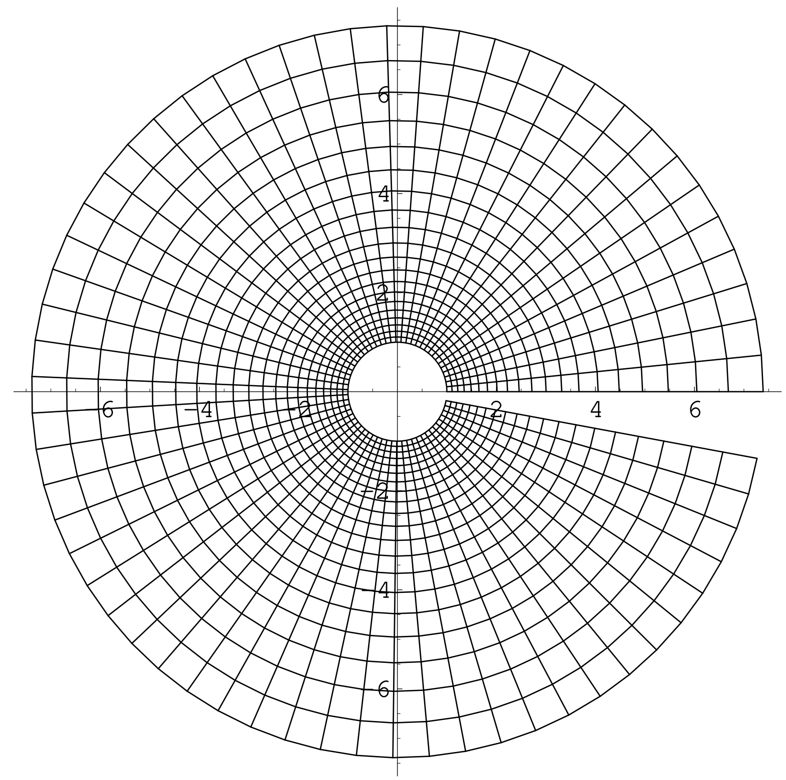


This illustrates the map $z \to z^2$ or $(x, y) \to (x^2 - y^2, 2xy)$. The top left shows a grid in the square $[0, 1]^2$. The top right shows the image under squaring map.



The same square grid of $[0,2]^2$ and its image under e^z.





This illustrates the exponential map $e^z = e^r(\cos\theta + i\sin\theta)$.. We take the image of $[0,2] \times [0,6]$. The line at height 2π will be mapped into the positive real axis. The top edge of the grid is just below this, so the image stops just before it reaches the axis.

Cauchy's Integral Formula *Suppose* γ *is a cycle contained in a region* Ω *and suppose*

$$\int_\gamma \frac{d\zeta}{\zeta - a} = 0$$

for all $a \notin \Omega$. *If* f *is analytic on* Ω *and* $z \in \mathbb{C} \setminus \gamma$ *then*

$$\frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta = f(z) \cdot \frac{1}{2\pi i} \int_\gamma \frac{1}{\zeta - z} d\zeta.$$

Cauchy's formula: Suppose Ω is bounded by a piecewise smooth curve γ and f is holomorphic on a neighborhood of $\overline{\Omega}$. Then

$$f(w) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-w} dz.$$

Pompeiu formula: Suppose Ω is bounded by a piecewise smooth curve and f is smooth on $\overline{\Omega}$.

$$f(w) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{z-w} dz - \frac{1}{\pi} \iint_\Omega \frac{f_{\overline{z}}}{z-w} dx dy.$$

Möbius transformations

A linear fractional transformation (or Möbius transformation) is a map of the form $z \to (az+b)/(cz+d)$. This is a 1-1, onto, holomorphic map of the Riemann sphere $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$ to itself.

The non-identity Möbius transformations are divided into three classes.

(1) Parabolic transformations have a single fixed point on $\mathbb{S}^2$ and are conjugate to the translation map $z \to z + 1$.

(2) Elliptic maps have two fixed points and are conjugate to the rotation $z \to e^{it}z$ for some $t \in \mathbb{R}$.

(3) The loxodromic transformations also have two fixed points and are conjugate to $z \to \lambda z$ for some $|\lambda| < 1$. If, in addition, λ is real, then the map is called hyperbolic.

Given two sets of three distinct points $\{z_1, z_2, z_3\}$ and $\{w_1, w_2, w_3\}$ there is a unique Möbius transformation that sends $w_k \to z_k$ for $k = 1, 2, 3$. This map is given by the formula

$$\tau(z) = \frac{w_1 - \zeta w_3}{1 - \zeta},$$

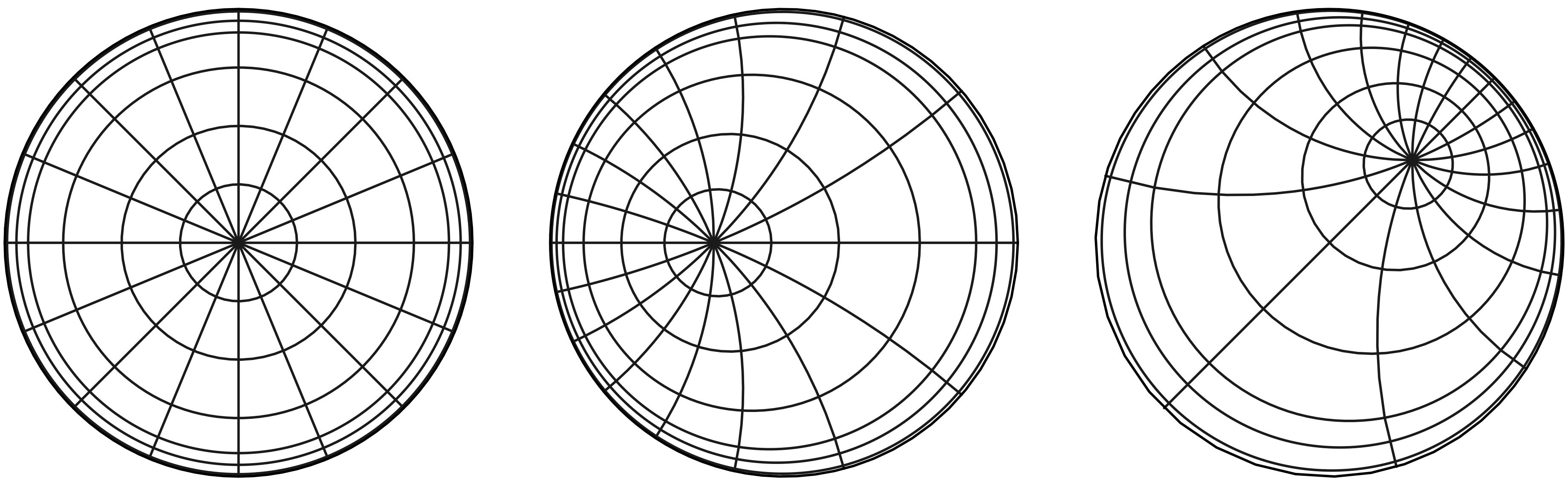
where

$$\zeta = \frac{(w_2 - w_1)}{(w_2 - w_3)} \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}.$$

A Möbius transformation sends the unit disk 1-1, onto itself iff it is if the form

$$z \to \lambda \frac{z-a}{1-\bar{a}z},$$

for some $a \in \mathbb{D}$ and $|\lambda| = 1$. In this case, any loxodromic transformation must actually be hyperbolic.



A polar grid in the disk and some images under Möbius transformations that preserve the unit disk.

Given four distinct points a, b, c, d in the plane we define their cross ratio as

$$\operatorname{cr}(a, b, c, d) = \frac{(d-a)(b-c)}{(c-d)(a-b)}.$$

Note that $\operatorname{cr}(a, b, c, z)$ is the unique Möbius transformation which sends a to 0, b to 1 and c to ∞.

This makes it clear that cross ratios are invariant under Möbius transformations; that $\operatorname{cr}(a, b, c, d)$ is real valued iff the four points lie on a circle; and is negative iff in addition the points are labeled in counterclockwise order on the circle.

Möbius transformations form a group under composition. If we identity the transformation $(az + b)/(cz + d)$ with the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then composition of maps is the same as matrix multiplication.

For any non-zero λ, the translations $(\lambda az + \lambda b)/(\lambda cz + \lambda d)$ are all the same, but correspond to different matrices.

We can choose one to represent the transformation, say the one with determinate $ad - bc = 1$, and this identifies the group of transformations the the group $\mathrm{SL}(2, \mathbb{C})$ of two by two matrices of determinate 1.

If $ad = bc$, then

$$\frac{az+b}{cz+d} = \frac{adz+bd}{cdz+d^2} = \frac{bcz+bd}{cdz+d^2} = \frac{b}{d}\frac{cz+d}{cz+d} = \frac{b}{d},$$

is constant and not a Möbius transformation.

The mapping
$$z \to \frac{az+b}{cz+d},$$
can be written as a composition of the maps
$$z \to cz+d, \quad z \to \frac{1}{z}, \quad z \to \frac{a}{c} + \frac{bc-ad}{c} z,$$
which equivalent to claiming
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} (bc-ad) & a \\ 0 & c \end{pmatrix}.$$
Either claim follows by a direct computation.

The linear maps have the property that circles map to circles an lines map to lines. The inversion also has this property, although it may interchange the two types of sets.

Lemma 1.1. *Möbius transformations map circles to circles, assuming the convention that lines are considered as circles through infinity.*

It is enough to check this for $1/z$. The equation

$$x^2 + y^2 + \alpha x + \beta y + \gamma = 0 \tag{1.1}$$

defines a circle in the plane, depending on the choice of α, β, γ. If we set $z = x + iy \neq 0$ and $\frac{1}{z} = u + iv$, then

$$u = \mathrm{Re}(\frac{x - iy}{x^2 + y^2}) = \frac{x}{x^2 + y^2},$$
$$v = \mathrm{Im}(\frac{x - iy}{x^2 + y^2}) = \frac{-y}{x^2 + y^2},$$
$$x = \frac{u}{u^2 + v^2}, \qquad y = \frac{-v}{u^2 + v^2},$$

So (1.1) becomes

$$\frac{u^2}{(u^2+v^2)^2}+\frac{v^2}{(u^2+v^2)^2}+\frac{\alpha u}{(u^2+v^2)^2}+\frac{-\beta v}{(u^2+v^2)^2}+\gamma=0.$$

After simplifying this becomes

$$\frac{1}{(u^2+v^2)^2}+\frac{\alpha u}{u^2+v^2}+\frac{-\beta v}{u^2+v^2}+\gamma=0,$$

$$1+\alpha u-\beta v+\gamma(u^2+v^2)=0,$$

which is the equation of a circle or line (depending on whether $\gamma \neq 0$ or $\gamma = 0$).

Thus $z \to \frac{1}{z}$ sends a circle missing the origin to a circle, and sends a circle though 0 to a line (which is the same as a circle passing through ∞).

The reflection through a circle $|z-c|=r$ is defined by $\arg(w^*-c)=\arg(w-c)$ and $|w-c|\cdot|w^*-c|=r^2$. Möbius transformation preserve reflections, i.e., if τ is a linear fractional transformation that send circle (or line) C_1 to circle (or line) C_2 then pairs of symmetric points for C_1 are mapped by τ to symmetric points for C_2.

Lemma 1.2. *Every Möbius transformation can be written as a even number of compositions of circle and line reflections.*

The proof is left to the reader.

In higher dimensions, reflections through planes and spheres still makes sense. In this case, Möbius transformations are defined as the group generated by any even number of compositions of such maps (even so that the result is orientation preserving).

The hyperbolic metric

The hyperbolic metric on $\mathbb{D}$ is given by

$d\rho_{\mathbb{D}} = 2|dz|/(1 - |z|^2)$. This means that

the hyperbolic length of a rectifiable curve γ in $\mathbb{D}$ is defined as

$$\ell_\rho(\gamma) = \int_\gamma \frac{2|dz|}{1 - |z|^2},$$

and the hyperbolic distance between two points $z, w \in \mathbb{D}$ is the infimum of the lengths of paths connecting them (we shall see shortly that there is an explicit formula for this distance in terms of

Corresponding metric on upper half-plane is ds/t.

This metric has constant curvature -4. Some sources use $d\rho_{\mathbb{D}} = |dz|/(1-|z|^2)$, which has curvature -1.

On the disk it is convenient to define the pseudo-hyperbolic metric

$$\rho(z, w) = |\frac{z - w}{1 - \bar{w}z}|.$$

The hyperbolic metric between two points can then be expressed as

$$\psi(w, z) = \log \frac{1 + \rho(w, z)}{1 - \rho(w, z)}.$$

On the upper half-plane the corresponding function is

$$\rho(z,w) = |\frac{z-w}{w-\bar{z}}|,$$

and ψ is given as before. A hyperbolic ball in the disk is also a Euclidean ball, but the hyperbolic and Euclidean centers are different (unless they are both the origin).

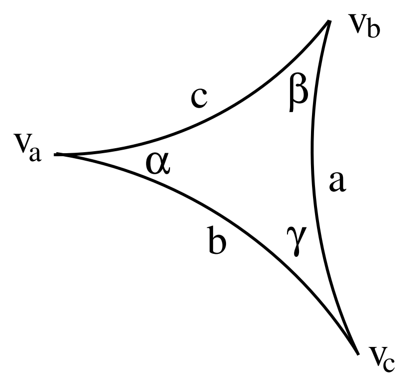
The orientation preserving isometries of the hyperbolic disk are exactly the Möbius transformations that map the disk to itself. All of these have the form

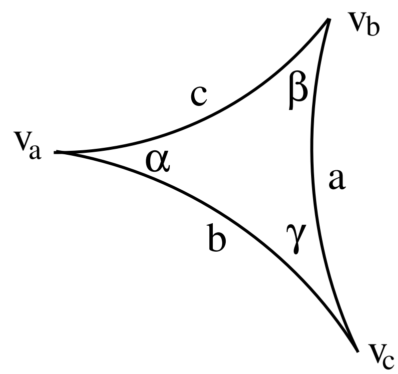
$$e^{i\theta}\frac{z-a}{1-\bar{a}z},$$

where θ is real and $a \in \mathbb{D}$.

Recall the sine and cosine rules for hyperbolic geometry (e.g., see page 148 of Beardon's book "The geometry of discrete groups".

Let T denote a hyperbolic triangle with angles α, β, γ and opposite side lengths denoted by a, b, c.





Then we have the Sine Rule,

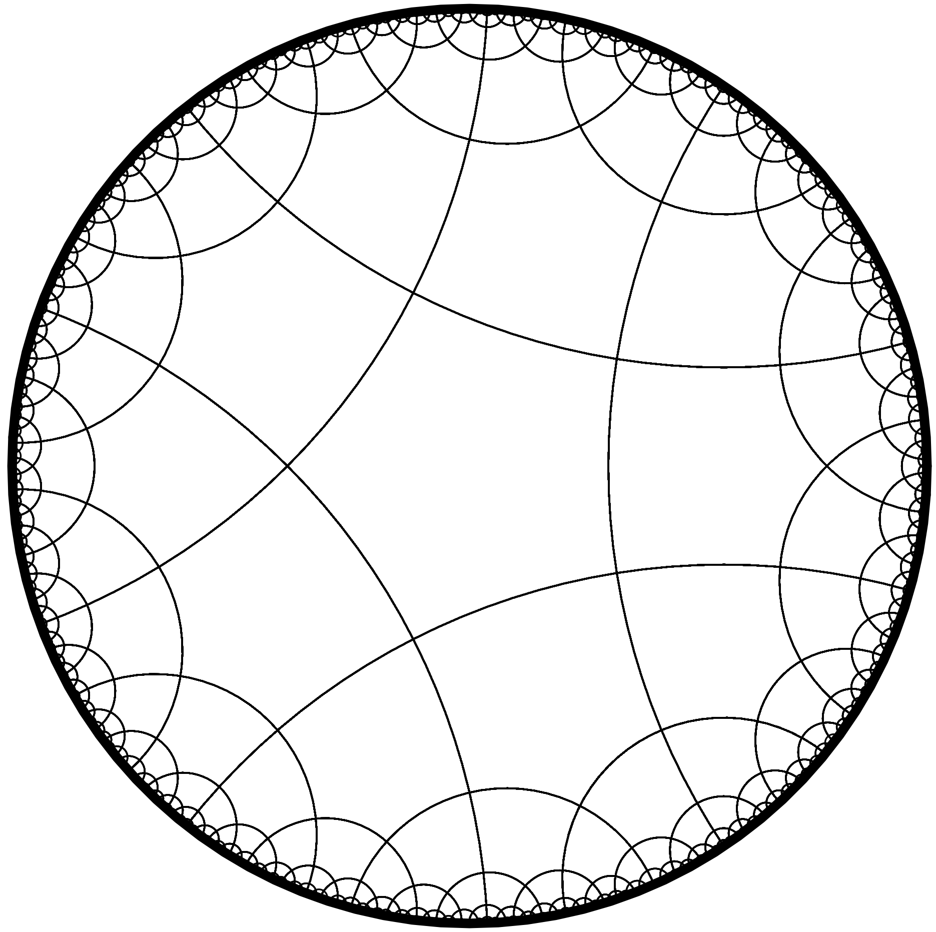
$$\frac{\sinh a}{\sin \alpha} = \frac{\sinh b}{\sin \beta} = \frac{\sinh c}{\sin \gamma} \tag{1.2}$$

the First Cosine Rule,

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \gamma \tag{1.3}$$

and the Second Cosine Rule

$$\cosh c = \frac{\cos \alpha \cos \beta + \cos \gamma}{\sin \alpha \sin \beta} \tag{1.4}$$



Normal families

A collection, or family, $\mathcal{F}$ of continuous functions on a region $\Omega \subset \mathbb{C}$ is said to be **normal on $\boldsymbol{\Omega}$** provided every sequence $\{f_n\} \subset \mathcal{F}$ contains a subsequence which converges uniformly on compact subsets of Ω.

• The family $\mathcal{F}_1 = \{f_c(z) = z + c : |c| < 1\}$ is normal in $\mathbb{C}$ but not countable.

• The family $\mathcal{F}_2 = \{z^n : n = 0, 1, \ldots\}$ is normal in $\mathbb{D}$ but the only limit function, the zero function, is not in $\mathcal{F}_2$.

• The sequence z^n converges uniformly on each compact subset of $\mathbb{D}$, but does not converge uniformly on $\mathbb{D}$.

• The family $\mathcal{F}_3 = \{g_n\}$, where $g_n \equiv 1$ if n is even and $g_n \equiv 0$ if n is odd, is normal but the sequence $\{g_n\}$ does not converge.

Definition: A family of functions $\mathcal{F}$ defined on a set $E \subset \mathbb{C}$ is

(1) **equicontinuous at $\boldsymbol{w \in E}$** if for each $\epsilon > 0$ there exist a $\delta > 0$ so that if $z \in E$ and $|z - w| < \delta$, then $|f(z) - f(w)| < \epsilon$ for all $f \in \mathcal{F}$.

(2) **equicontinuous on $\boldsymbol{E}$** if it is equicontinuous at each $w \in E$.

(3) **uniformly equicontinuous on $\boldsymbol{E}$** if for each $\epsilon > 0$ there exists a $\delta > 0$ so that if $z, w \in E$ with $|z - w| < \delta$ then $|f(z) - f(w)| < \epsilon$ for all $f \in \mathcal{F}$.

The Arzela-Ascoli Theorem: *A family $\mathcal{F}$ of continuous functions is normal on a region $\Omega \subset \mathbb{C}$ if and only if*

(1) $\mathcal{F}$ is equicontinuous on Ω, and

(2) there is a $z_0 \in \Omega$ so that the collection $\{f(z_0) : f \in \mathcal{F}\}$ is a bounded subset of $\mathbb{C}$.

This result is usually proven in MAT 532 (Chap 4 of Folland's book).

Definition: A family $\mathcal{F}$ of continuous functions is said to be **locally bounded** on Ω if for each $w \in \Omega$ there is a $\delta > 0$ and $M < \infty$ so that if $|z - w| < \delta$ then $|f(z)| \leq M$ for all $f \in \mathcal{F}$.

Theorem: *The following are equivalent for a family $\mathcal{F}$ of analytic functions on a region Ω.*

(1) $\mathcal{F}$ is normal on Ω.

(2) $\mathcal{F}$ is locally bounded on Ω.

(3) $\mathcal{F}' = \{f' : f \in \mathcal{F}\}$ is locally bounded on Ω and there is a $z_0 \in \Omega$ so that $\{f(z_0) : f \in \mathcal{F}\}$ is a bounded subset of $\mathbb{C}$.

Montel's Theorem: *A family $\mathcal{F}$ of meromorphic functions on a region Ω that omits three distinct fixed values $a, b, c \in \mathbb{C}^*$ is normal in the chordal metric.*

Picard's Great Theorem *If f is meromorphic in $\Omega = \{z : 0 < |z - z_0| < \delta\}$, and if f omits three (distinct) values in $\mathbb{C}^*$, then f extends to be meromorphic in $\Omega \cup \{z_0\}$.*

• An equivalent formulation of Picard's great theorem is that an analytic function omits at most one complex number in every neighborhood of an essential singularity.

• $f(z) = e^{1/z}$ does omit the values 0 and ∞ in every neighborhood of the essential singularity 0, so that Picard's theorem is the strongest possible statement.

• The weaker statement that a non-constant entire function can omit at most one complex number is usually called **Picard's little theorem**.

See Emile Picard

Normal families can be used to prove results like:

Koebe: *There is a* $K > 0$ *so that if* f *is analytic and one-to-one on* $\mathbb{D}$ *with* $f(0) = 0$ *and* $f'(0) = 1$, *then* $f(\mathbb{D}) \supset \{z : |z| < K\}$.

A sharper version is known and called the Koebe 1/4-theorem.

Theorem (Koebe 1/4-theorem): Assume $f(z) = z + a_2 z^2 + \dots$ is univalent on $\mathbb{D}$. Then $|a_2| \leq 2$ and

$$\operatorname{dist}(0, \partial f(\mathbb{D})) \geq \frac{1}{4}$$

.

Theorem (Koebe's estimate): Suppose f is a conformal map from $\mathbb{D}$ to a simply connected region Ω. Then for all $z \in \mathbb{D}$,

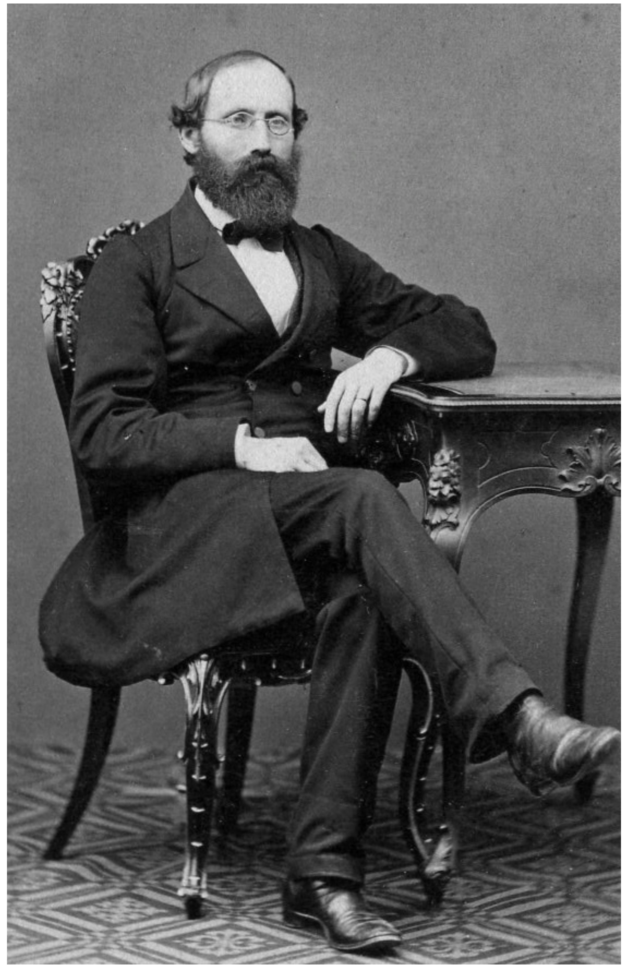
$$\frac{1}{4}|f'(z)|(1 - |z|^2) \leq \operatorname{dist}(f(z), \partial\Omega) \leq |f'(z)|(1 - \mathbf{z}|^2)$$

The Riemann mapping theorem

The Riemann Mapping Theorem *Suppose* $\Omega \subset \mathbb{C}$ *is simply-connected and* $\Omega \neq \mathbb{C}$. *Then there exists a one-to-one analytic map* f *of* Ω *onto* $\mathbb{D} = \{z : |z| < 1\}$. *If* $z_0 \in \Omega$ *then there is a unique such map with* $f(z_0) = 0$ *and* $f'(z_0) > 0$.

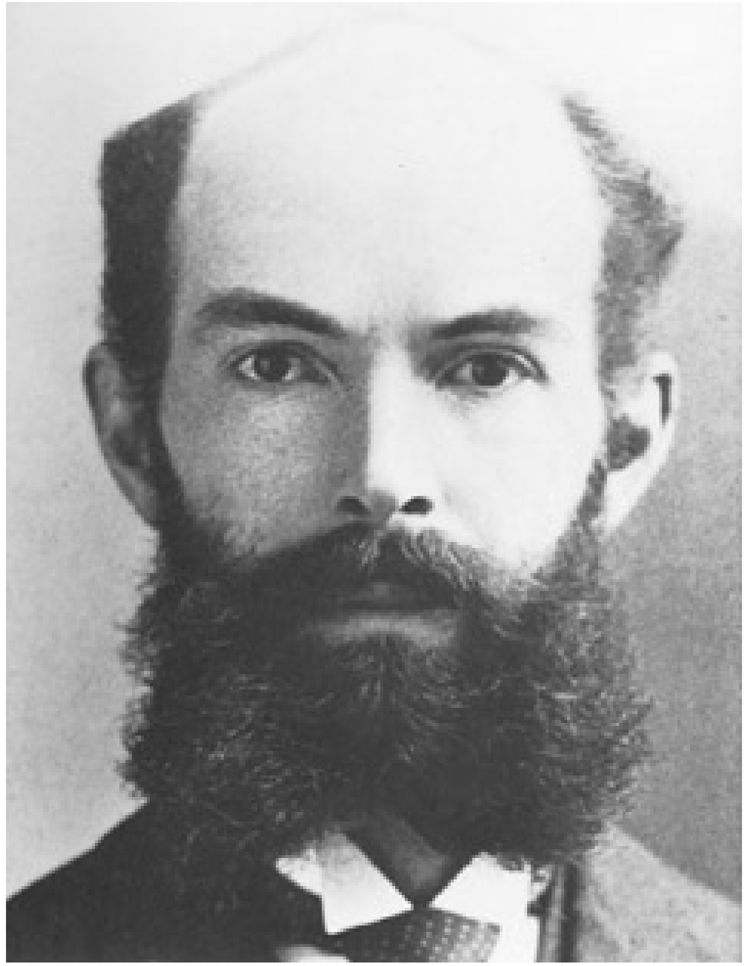
Idea of proof:

- Show there is a conformal map of Ω into $\mathbb{D}$ so that $f(z_0) = 0$ and $f'(z_0) > 0$.
- Among all such maps, choose one maximizing $f'(z_0)$. (uses normality)
- Prove this map is 1-1 and onto $\mathbb{D}$.



Georg Friedrich Bernhard Riemann

Stated RMT in 1851



William Fogg Osgood

First proof of RMT, Trans. AMS, vol. 1, 1900

The proof of Osgood represented, in my opinion, the "coming of age" of mathematics in America. Until then, numerous American mathematicians had gone to Europe for their doctorates, or for other advanced study, as indeed did Osgood. But the mathematical productivity in this country in quality lagged behind that of Europe, and no American before 1900 had reached the heights that Osgood then reached.

J.L. Walsh, "History of the Riemann mapping theorem", Amer. Math. Monthly, 1973.

Schwarz-Christoffel Formula: *Suppose* Ω *is a bounded simply-connected region whose positively oriented boundary* $\partial\Omega$ *is a polygon with vertices* $v_1, ..., v_n$. *Suppose the tangent direction on* $\partial\Omega$ *increases by* $\pi\alpha_j$ *at* v_j, $-1 < \alpha_j < 1$. *Then there exists* $x_1 < x_2 < \cdots < x_n$ *and constants* c_1, c_2 *so that*

$$f(z) = c_1 \int_{\gamma_z} \prod_{j=1}^{n} (\zeta - x_j)^{-\alpha_j} d\zeta + c_2$$

is a conformal map of $\mathbb{H}$ *onto* Ω, *where the integral is along any curve* γ_z *in* $\mathbb{H}$ *from* i *to* z.

Elwin Bruno Christtoffel Hermann Amandus Schwarz

$$f(z) = c_1 \int_{\gamma_z} \prod_{j=1}^{n} (\zeta - x_j)^{-\alpha_j} d\zeta + c_2$$

The exponents $\{\alpha_j\}$ are known from the target polygon, but the $\{x_j\}$ are not.

• The points are the preimages of the vertices under the conformal map.

• Finding these points numerically is challenging: there are several heuristics that work in practice, but are not proven to work, e.g., SC-Toolbox program by T. Driscoll.

• A provably correct algorithm is given in the paper Conformal mapping in linear time and explained in the recorded lecture Fast conformal mapping via computational and hyperbolic geometry.

A **Jordan region** is simply-connected region in $\mathbb{C}^*$ whose boundary is a Jordan curve.

Carathéodory-Tohorst Theorem: *If* φ *is a conformal map of* $\mathbb{D}$ *onto a Jordan region* Ω*, then* φ *extends to be a homeomorphism of* $\overline{\mathbb{D}}$ *onto* $\overline{\Omega}$*. In particular* $\varphi(e^{it})$ *is a parameterization of* $\partial\Omega$*.*

Although usually called “Carathéodory’s theorem, the result actually appears in the 1917 Bonn thesis of Marie Torhorst, a student of Carathéodory.

For a discussion of the history, see On prime ends and local connectivity by Lasse Rempe. Torhorst did not become an academic mathematician, but eventually became Minister of Education for the state of Thüringen in communist East Germany following WWII.

A compact set K is called "locally connected" if whenever U is a relatively open subset of K and $z \in U \subset K$, there is a relatively open subset of K that is connected and such that $z \in V \subset U$.

This is equivalent to K being a continuous image of $[0, 1]$.

Carathéodory's extends to say that a conformal map $f : \mathbb{D} \to \Omega$ has a continuous extension to the boundary iff $\partial\Omega$ is locally connected.

We will prove the theorem later in the course. Proof uses "length-area" method which is closely connected to extremal length and quasiconformal maps.

The uniformization theorem

Suppose W is a Riemann surface and $p \in W$.

The Green's function on W with pole at p_0 is a positive function $G(z, p_0)$ that is harmonic on $W \setminus \{p\}$, has a logarithmic pole at p_0 and tends to zero at ∞.

For example, $\log \frac{1}{|z|}$ is the Green's function for $\mathbb{D}$ with pole at 0.

Some Riemann surfaces have a Green's function; some do not.

Very important distinction. Many different characterizations of two cases.

A Riemann surface has a Green's function iff several other conditions hold.

(1) Brownian motion is recurrent.

(2) Geodesic flow on the unit tangent bundle of W is ergodic.

(3) Poincare series of covering group Γ diverges.

(4) Γ has the Mostow rigidity property (conjugating circle homeomorphisms are Möbius or singular).

(5) Γ has the Bowen's property (corresponding limit sets are either a circle or have dimension > 1).

(6) Almost every geodesic ray is recurrent. Equivalently, the set of escaping geodesic rays from a point $p \in W$ has zero (visual) measure.

The Uniformization, Case 1: *If W is a simply-connected Riemann surface then the following are equivalent:*

$$g_W(p, p_0) \text{ exists for some } p_0 \in W$$

$$g_W(p, p_0) \text{ exists for all } p_0 \in W,$$

$$\text{There is a one-to-one analytic map } \varphi \text{ of } W \text{ onto } \mathbb{D}.$$

Moreover if g_W exists, then

$$g_W(p_1, p_0) = g_W(p_0, p_1),$$

and $g_W(p, p_0) = -\log|\varphi(p)|$, where $\varphi(p_0) = 0$.

Paul Koebe

Proved uniformization theorem in 1907.

The dipole Green's function has two logarithmic poles with opposite signs, e.g.,

$$\log \left| \frac{z-a}{z-b} \right|$$

on the plane. This has two opposite poles and tends to 0 at infinity.

The next lemma says that a dipole Green's function always exists.

For surfaces with Green's function this is easy: take $G(z,p) - G(z,q)$ for $p \neq q$.

The Uniformization Theorem, Case 2 *Suppose* W *is a simply-connected Riemann surface for which Green's function does not exist.*
If W *is compact, then there is a one-to-one analytic map of* W *onto* $\mathbb{C}^*$*.*
If W *is not compact, there is a one-to-one analytic map of* W *onto* $\mathbb{C}$*.*

The Uniformization Theorem: *Suppose W is a simply-connected Riemann surface.*

(1) If Green's function exists for W, then there is a one-to-one analytic map of W onto $\mathbb{D}$.

(2) If W is compact, then there is a one-to-one analytic map of W onto $\mathbb{C}^*$.

(3) If W is not compact and if Green's function does not exist for W, then there is a one-to-one analytic map of W onto $\mathbb{C}$.

Theorem: *If $U = \mathbb{C}^*$, $\mathbb{C}$, or $\mathbb{D}$ and if $\mathbb{G}$ is a properly discontinuous group of LFTs of U onto U, then $U/\mathbb{G}$ is a Riemann surface. A function f is analytic, meromorphic, harmonic, or subharmonic on $U/\mathbb{G}$ if and only if there is a function h defined on U which is (respectively) analytic, meromorphic, harmonic, or subharmonic on U satisfying $h \circ \tau = h$ for all $\tau \in \mathbb{G}$ and $h = f \circ \pi$ where $\pi : U \to U/\mathbb{G}$ is the quotient map. Every Riemann surface is conformally equivalent to $U/\mathbb{G}$ for some such U and $\mathbb{G}$.*

Properly discontinuous: every point has a neighborhood U so $U \cap g(U) \neq \emptyset$ implies $g = \mathrm{Id}$.

The only Riemann surface covered by the $\mathbb{C}^*$ is $\mathbb{C}^*$ (Proposition 16.2).

The only surfaces covered by $\mathbb{C}$ are $\mathbb{C}$, $\mathbb{C} \setminus \{0\}$, and tori (Proposition 16.3).

Any other Riemann surface is covered by the disk $\mathbb{D}$.

This semester I hope to cover the following topics:

- Review of complex analysis
- Extremal length and conformal modulus,
- Logarithmic capacity, harmonic measure
- Geometric definition of quasiconformal mappings, compactness
- Applications of compactness: quasisymmetry, extension, removability, weldings
- Analytic definition and the measurable Riemann mapping theorem
- Astala's theorems on area and dimension distortion
- Quasiconformal maps on metric spaces
- Conformal dimension
- David maps

Extremal Length

Consider a positive function ρ on a domain Ω. We think of ρ as analogous to $|f'|$ where f is a conformal map on Ω.

Just as the image area of a set E can be computed by integrating $\int_E |f'|^2 dxdy$, we can use ρ to define areas by $\int_E \rho^2 dxdy$.

Similarly, we can define $\ell(f(\gamma)) = \int_\gamma |f'(z)|ds$, we can define the ρ-length of a curve γ by $\int_\gamma \rho ds$.

We need γ to be locally rectifiable (so the arclength measure ds is defined) and it is convenient to assume that ρ is Borel (so that its restriction to any curve γ is also Borel and hence measurable for length measure on γ).

Suppose Γ is a family of locally rectifiable paths in a planar domain Ω and ρ is a non-negative Borel function on Ω.

We say ρ is **admissible** for Γ if

$$\ell(\Gamma) = \ell_\rho(\Gamma) = \inf_{\gamma \in \Gamma} \int_\gamma \rho ds \geq 1.$$

In this case we write $\rho \in \mathcal{A}(\Gamma)$.

We define the **modulus** of the path family Γ as

$$\mathrm{Mod}(\Gamma) = \inf_{\rho} \int_M \rho^2 dxdy,$$

where the infimum is over all admissible ρ for Γ.

The **extremal length** of Γ is defined as $\lambda(\Gamma) = 1/M(\Gamma)$.

Note that if the path family Γ is contained in a domain Ω, then we need only consider metrics ρ are zero outside Ω.

Otherwise, we can define a new (smaller) metric by setting $\rho = 0$ outside Ω; the new metric is still admissible, and a smaller integral than before.

Therefore $M(\Gamma)$ can be computed as the infimum over metrics which are only nonzero inside Ω.

Modulus and extremal length satisfy several useful properties that we list as a series of lemmas.

Lemma 2.1 (Conformal invariance)**.** *If* Γ *is a family of curves in a domain* Ω *and* f *is a one-to-one holomorphic mapping from* Ω *to* Ω' *then* $M(\Gamma) = M(f(\Gamma))$.

Lemma 2.1 (Conformal invariance)**.** *If Γ is a family of curves in a domain Ω and f is a one-to-one holomorphic mapping from Ω to Ω' then $M(\Gamma) = M(f(\Gamma))$.*

Proof. This is just the change of variables formulas

$$\int_\gamma \rho \circ f|f'|ds = \int_{f(\gamma)} \rho ds,$$

$$\int_\Omega (\rho \circ f)^2 |f'|^2 dxdy = \int_{f(\Omega)} \rho dxdy.$$

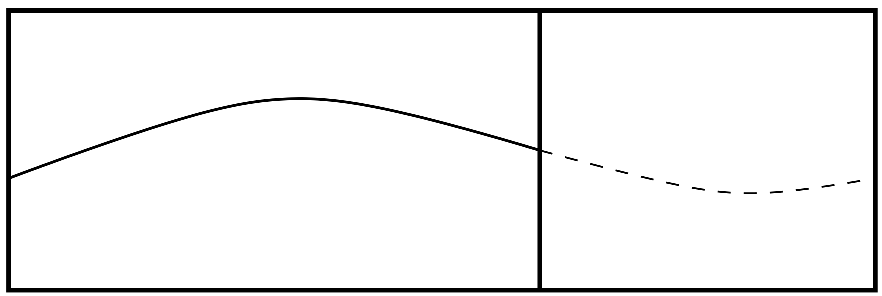
These imply that if $\rho \in \mathcal{A}(f(\Gamma))$ then $|f'| \cdot \rho \circ f \in \mathcal{A}(f(\Gamma))$, and thus by taking the infimum over such metrics we get $M(f(\Gamma)) \leq M(\Gamma)$

There might be admissible metrics for $f(\Gamma)$ that are not of this form, possibly giving a strictly smaller modulus. However, by switching the roles of Ω and Ω' and replacing f by f^{-1} we see equality does indeed hold. $\square$

Lemma 2.2 (Monotonicity)**.** *If* Γ_0 *and* Γ_1 *are path families such that every* $\gamma \in \Gamma_0$ *contains some curve in* Γ_1 *then* $M(\Gamma_0) \leq M(\Gamma_1)$ *and* $\lambda(\Gamma_0) \geq \lambda(\Gamma_1)$*.*

Lemma 2.2 (Monotonicity)**.** *If* Γ_0 *and* Γ_1 *are path families such that every* $\gamma \in \Gamma_0$ *contains some curve in* Γ_1 *then* $M(\Gamma_0) \leq M(\Gamma_1)$ *and* $\lambda(\Gamma_0) \geq \lambda(\Gamma_1)$.

Proof. The proof is immediate since $\mathcal{A}(\Gamma_0) \supset \mathcal{A}(\Gamma_1)$. $\square$



Lemma 2.3 (Grötsch Principle)**.** *If* Γ_0 *and* Γ_1 *are families of curves in disjoint domains then* $M(\Gamma_0 \cup \Gamma_1) = M(\Gamma_0) + M(\Gamma_1)$*.*

Lemma 2.3 (Grötsch Principle)**.** *If* Γ_0 *and* Γ_1 *are families of curves in disjoint domains then* $M(\Gamma_0 \cup \Gamma_1) = M(\Gamma_0) + M(\Gamma_1)$*.*

Proof. Suppose ρ_0 and ρ_1 are admissible for Γ_0 and Γ_1. Take $\rho = \rho_0$ and $\rho = \rho_1$ in their respective domains.

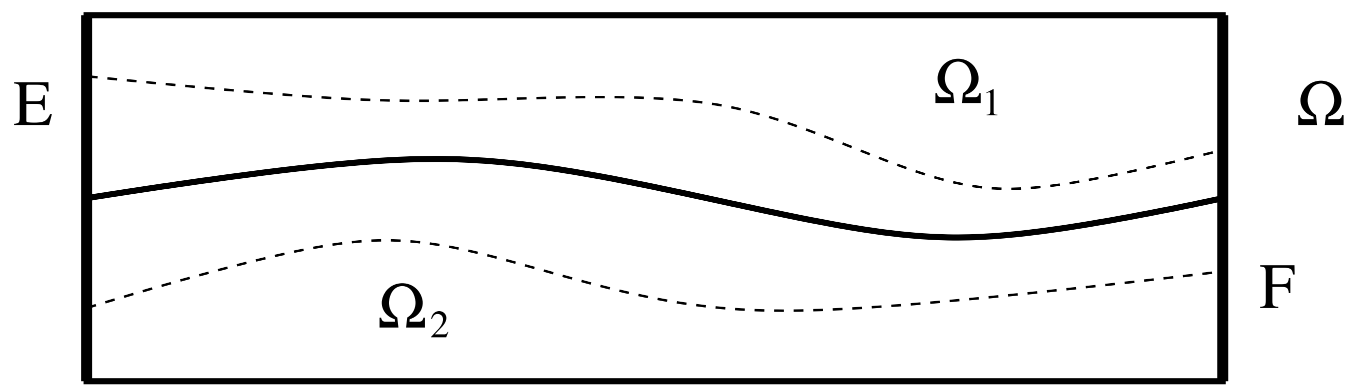
Then it is easy to check that ρ is admissible for $\Gamma_0 \cup \Gamma_1$ and, since the domains are disjoint, $\int \rho^2 = \int \rho_1^2 + \int \rho_2^2$.

Thus $M(\Gamma_0 \cup \Gamma_1) \leq M(\Gamma_0) + M(\Gamma_1)$. By restricting an admissible metric ρ for $\Gamma_0 \cup \Gamma_1$ to each domain, a similar argument proves the other direction. $\square$

Corollary 2.4 (Parallel Rule)**.** *Suppose* Γ_0 *and* Γ_1 *are path families in disjoint domains* $\Omega_0, \Omega_1 \subset \Omega$ *that connect disjoint sets* E, F *in* $\partial\Omega$*. If* Γ *is the path family connecting* E *and* F *in* Ω*, then*

$$M(\Gamma) \geq M(\Gamma_0) + M(\Gamma_1).$$

Proof. Combine the Grötsch principle and the monotonicity principle. ∎



Lemma 2.5 (Series Rule)**.** *If* Γ_0 *and* Γ_1 *are families of curves in disjoint domains and every curve of* $\mathcal{F}$ *contains both a curve from both* Γ_0 *and* Γ_1*, then* $\lambda(\Gamma) \geq \lambda(\Gamma_0) + \lambda(\Gamma_1)$*.*

Proof. If $\rho_j \in \mathcal{A}(\Gamma_j)$ for $j = 0, 1$, then $\rho_t = (1-t)\rho_0 + t\rho_1$ is admissible for Γ.

Since the domains are disjoint we may assume $\rho_0\rho_1 = 0$.

Integrating ρ^2 then shows

$$M(\Gamma) \leq (1-t)^2 M(\Gamma_0) + t^2 M(\Gamma_1),$$

for each t.

To find the optimal t set $a = M(\Gamma_1)$, $b = M(\Gamma_0)$, differentiate the right hand side above, and set it equal to zero

$$2at - 2b(1-t) = 0.$$

Solving gives $t = b/(a+b)$ and plugging this in above gives

$$M(\mathcal{F}) \ \leq \ t^2 a + (1-t^2)b = \frac{b^2 a a^2 b}{(a+b)^2} = \frac{ab(a+b)}{(a+b)^2} = \frac{ab}{a+b} = \frac{1}{\frac{1}{a}+\frac{1}{b}}$$

or

$$\frac{1}{M(\Gamma)} \geq \frac{1}{M(\Gamma_0)} + \frac{1}{M(\Gamma_1)},$$

which, by definition, is the same as

$$\lambda(\Gamma) \geq \lambda(\Gamma_0) + \lambda(\Gamma_1). \quad \square$$

The fundamental example is to compute the modulus of the path family connecting opposite sides of a $a \times b$ rectangle; this serves as the model of almost all modulus estimates.

So suppose $R = [0, b] \times [0, a]$ is a b wide and a high rectangle and Γ consists of all rectifiable curves in R with one endpoint on each of the sides of length a.

Lemma 2.6. $\mathrm{Mod}(\Gamma) = a/b.$

Proof. Each curve in Γ has length at least b, so if we let ρ be the constant $1/b$ function on R we have

$$\int_\gamma \rho ds \geq 1,$$

for all $\gamma \in \Gamma$. Thus this metric is admissible and so

$$\mathrm{Mod}(\Gamma) \leq \iint_T \rho^2 dxdy = \frac{1}{b^2}ab = \frac{a}{b}.$$

To prove a lower bound, we use the well known Cauchy-Schwarz inequality:

$$(\int fg\,dx)^2 \leq (\int f^2dx)(\int g^2dx).$$

To apply this, suppose ρ is an admissible metric on R for γ. Every horizontal segment in R connecting the two sides of length a is in Γ, so since γ is admissible,

$$\int_0^b \rho(x,y)dx \geq 1,$$

and so by Cauchy-Schwarz

$$1 \leq \int_0^b (1\cdot\rho(x,y))dx \leq \int_0^b 1^2dx \cdot \int_0^b \rho^2(x,y)dx.$$

Now integrate with respect to y to get

$$a = \int_0^a 1dy \leq b \int_0^a \int_0^b \rho^2(x,y)dxdy,$$

or

$$\frac{a}{b} \leq \iint_R \rho^2 dxdy,$$

which implies $\mathrm{Mod}(\Gamma) \geq \frac{b}{a}$. Thus $\mathrm{Mod}(\Gamma) = \frac{b}{a}$. $\square$

Lemma 2.7. *If $A = \{z : r < |z| < R\}$ then the modulus of the path family connecting the two boundary components is $2\pi / \log \frac{R}{r}$.*

More generally, if Γ is the family of paths connecting $r\mathbb{T} = \{|z| = r\}$ to a set $E \subset R\mathbb{T} = \{|z| = R\}$, then $M(\Gamma) \geq |E| / \log \frac{R}{r}$.

Proof. By conformal invariance, we can rescale and assume $r = 1$. Suppose ρ is admissible for Γ. Then for each $z \in E \subset \mathbb{T}$,

$$1 \leq (\int_1^R \rho ds)^2 \leq (\int_1^R \frac{ds}{s})(\int_1^R \rho^2 s ds) = \log R \int_1^R \rho^2 s ds$$

and hence we get

$$\int_0^{2\pi} \int_1^R \rho^2 s ds d\theta \geq \int_E \int_1^R \rho^2 s ds d\theta \geq |E| \int_1^R \rho^2 s ds \geq \frac{|E|}{\log R}.$$

When $E = \mathbb{T}$ we prove the other direction by taking $\rho = (s \log R)^{-1}$. This is an admissible metric and

$$\operatorname{Mod}(\Gamma) \leq \int_0^{2\pi} \int_1^R \rho^2 s ds d\theta = \frac{2\pi}{(\log R)^2} \int_1^R \frac{1}{s} ds = \frac{2\pi}{\log R}. \quad \square$$

Given a Jordan domain Ω and two disjoint closed sets $E, F \subset \partial\Omega$, the **extremal distance** between E and F (in Ω) is the extremal length of the path family in Ω connecting E to F (paths in Ω that have one endpoint in E and one endpoint in F).

The series rule is a sort of "reverse triangle inequality" for extremal distance.

The series rule says that the extremal distance from X to Z in the rectangle is greater than the sum the extremal distance from X to Y in Ω_1 plus the extremal distance from Y to Z in Ω_2.

Extremal distance can be particularly useful when both E and F are connected.

If so, their complement in $\partial\Omega$ also consists of two arcs, and the extremal distance between these is the reciprocal of the extremal distance between E and F.

This holds because of conformal invariance, and the fact that it is true for rectangles.

(We can conformally map Ω to some rectangle, so that E and F go to opposite sides; this follows from the Schwarz-Christoffel formula.)

Obtaining an upper bound for the modulus of a path family usually involves choosing a metric; every metric gives an upper bound.

Giving a lower bound usually involves a Cauchy-Schwarz type argument, which can be harder to do in general cases. However, in the special case of extremal distance between arcs $E, F \subset \partial\Omega$, a lower bound for the modulus can also be computed by giving a upper bound for the reciprocal separating family.

Thus estimates of both types can be given by producing metrics (for different families) and this is often the easiest thing to do.

Lemma 2.8 (Points are removable)**.** *Suppose Q is a quadrilateral with opposite sides E, F and that Γ is the path family in Q connecting E and F. If $z \in \Omega$, let $\Gamma_0 \subset \Gamma$ be the paths that do not contain z. Then* $\mathrm{mod}\,(\Gamma_0) = \mathrm{mod}\,(\Gamma)$.

This will be useful later, when we want to prove that quasiconformal map of a punctured disk is actually quasiconformal on the whole disk. The point can be replaced by larger sets.

Proof. Since $\Gamma_0 \subset \Gamma$ we have $\mod(\Gamma_0) \leq \mod(\Gamma)$ by monotonicity.

To prove the other direction we claim that any metric that is admissible for Γ_0 is also admissible for Γ.

Suppose ρ is not admissible for Γ. Then there is a $\gamma \in \Gamma$ so that $\int_\gamma \rho ds < 1 - \epsilon$.

Choose a small $r > 0$ so $D(z, r) \subset \Omega$ and note that by Cauchy-Schwarz

$$(\int_0^r [\int_0^{2\pi} \rho t d\theta] dt)^2 \leq \pi r^2 \int_{D(z,r)} \rho^2 dx dy = o(r^2).$$

Here we have used the fact that since ρ^2 is integrable on Q, we have $\int_{D(z,r)} \rho^2 dx dy \to 0$ as $r \searrow 0$ (see Folland's book).

Hence

$$\int_0^r [\int_{C_t} \rho ds]dt = \int_0^r \ell_\rho(C_t)dt = o(r),$$

where C_t is the circle of radius t around z.

Thus we can find arbitrarily small circles centered at z whose ρ-length is less than ϵ. Then for the path γ chosen above, replace it by a path that follows γ from E to the first time it hits C_t, then follows an arc of C_t, and then follows γ from the last time it hits C_t to to F.

This path is in Γ_0 but its ρ-length is at most the ρ-length of γ plus the ρ-length of C_t, and this sum is less than 1. Thus ρ is also not admissible for Γ_0. This proves the claim and the lemma. $\square$

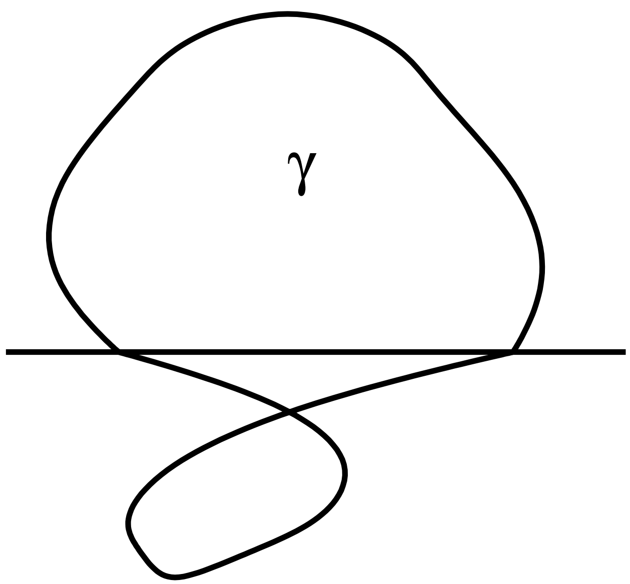
Extremal length, symmetry and Koebe's $1/4$-theorem

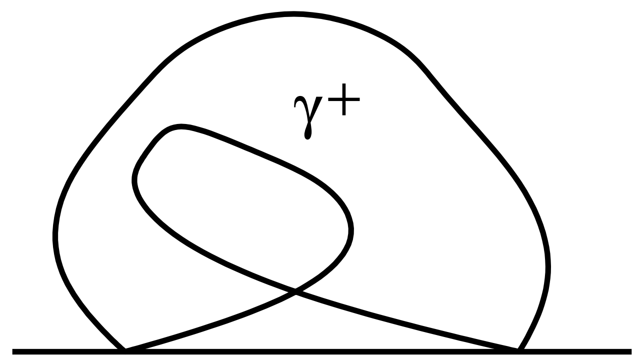
If γ is a path in the plane let $\bar{\gamma}$ be its reflection across the real line and let

$$\gamma_u = \gamma \cap \mathbb{H}, \quad \gamma_\ell = \gamma \cap \mathbb{H}_l, \quad \gamma_+ = \gamma_u \cup \overline{\gamma_\ell},$$

where $\mathbb{H} = \{x + iy : y > 0\}$, $\mathbb{H}_l = \{x + iy : y < 0\}$ denote the upper and lower half-planes.

For a path family Γ, define $\overline{\Gamma} = \{\bar{\gamma} : \gamma \in \Gamma\}$ and $\Gamma_+ = \{\gamma_+ : \gamma \in \Gamma\}$.





Lemma 2.9 (Symmetry Rule)**.** *If* $\Gamma = \overline{\Gamma}$ *then* $M(\Gamma) = 2M(\Gamma_+)$*.*

Lemma 2.9 (Symmetry Rule). *If* $\Gamma = \overline{\Gamma}$ *then* $M(\Gamma) = 2M(\Gamma_+)$.

Proof. We start by proving $M(\Gamma) \leq 2M(\Gamma_+)$.

Given a metric ρ admissible for γ_+, define $\sigma(z) = \max(\rho(z), \rho(\bar{z}))$.

Then for any $\gamma \in \Gamma$,

$$\begin{aligned}
\int_\gamma \sigma ds &= \int_{\gamma_u} \sigma(z)ds + \int_{\gamma_\ell} \sigma(z)ds \\
&\geq \int_{\gamma_u} \rho(z)ds + \int_{\gamma_\ell} \rho(\bar{z})ds \\
&= \int_{\gamma_u} \rho(z)ds + \int_{\overline{\gamma_\ell}} \rho(z)ds \geq \int_{\gamma_+} \rho ds \geq \inf_{\gamma\in\Gamma} \int_\gamma \rho ds.
\end{aligned}$$

Thus if ρ admissible for Γ_+, then σ is admissible for Γ.

Since $\max(a, b)^2 \leq a^2 + b^2$, integrating gives

$$M(\Gamma) \leq \int \sigma^2 dxdy \leq \int \rho^2(z) dxdy + \int \rho^2(\bar{z}) dxdy \leq 2 \int \rho^2(z) dxdy.$$

Taking the infimum over admissible ρ's for Γ_+ makes the right hand side equal to $2M(\Gamma_+)$, proving $\mathrm{Mod}(\Gamma) \leq 2\mathrm{Mod}(\Gamma_+)$.

For the other direction, given ρ define $\sigma(z) = \rho(z) + \rho(\bar{z})$ for $z \in \mathbb{H}$ and $\sigma = 0$ if $z \in \mathbb{H}_l$. Then

$$\begin{aligned}
\int_{\gamma_+} \sigma ds &= \int_{\gamma_+} \rho(z) + \rho(\bar{z}) ds \\
&= \int_{\gamma_u} \rho(z) ds + \int_{\gamma_u} \rho(\bar{z}) ds + \int_{\gamma_\ell} \rho(z) + \int_{\gamma_\ell} \rho(\bar{z}) ds \\
&= \int_{\gamma} \rho(z) ds + \int_{\overline{\gamma}} \rho(\bar{z}) ds \\
&= 2 \inf_{\rho} \int_{\gamma} \rho ds.
\end{aligned}$$

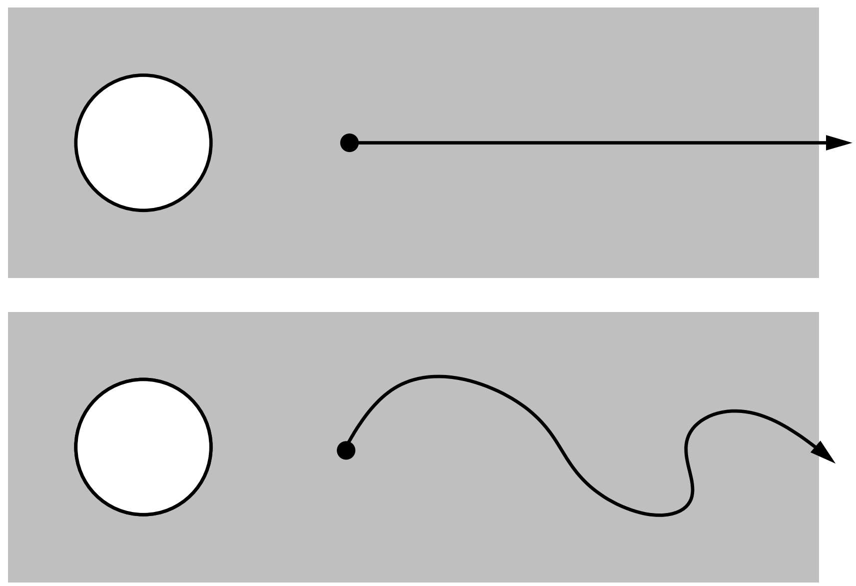
Thus if ρ is admissible for Γ, $\frac{1}{2}\sigma$ is admissible for Γ_+.

Since $(a+b)^2 \leq 2(a^2+b^2)$, we get

$$\begin{aligned} M(\Gamma_+) &\leq \int (\frac{1}{2}\sigma)^2 dxdy \\ &= \frac{1}{4}\int_{\mathbb{H}} (\rho(z)+\rho(\bar{z}))^2 dxdy \\ &\leq \frac{1}{2}\int_{\mathbb{H}} \rho^2(z)dxdy + \int_{\mathbb{H}} \rho^2(\bar{z})dxdy \\ &= \frac{1}{2}\int \rho^2 dxdy. \end{aligned}$$

Taking the infimum over all admissible ρ's for Γ gives $\frac{1}{2}M(\Gamma)$ on the right hand side, proving the lemma. $\square$

Lemma 2.10. *Let* $\mathbb{D}^* = \{z : |z| > 1\}$ *and* $\Omega_0 = \mathbb{D}^* \setminus [R, \infty)$ *for some* $R > 1$. *Let* $\Omega = \mathbb{D}^* \setminus K$, *where* K *is a closed, unbounded, connected set in* $\mathbb{D}^*$ *which contains the point* $\{R\}$. *Let* Γ_0, Γ *denote the path families in* Ω, Ω_0 *respectively that separate the two boundary components. Then* $M(\Gamma_0) \leq M(\Gamma)$.



Proof. We use the symmetry principle we just proved. The family Γ_0 is clearly symmetric (i.e., $\Gamma = \overline{\Gamma}$, so $M(\Gamma_0^+) = \frac{1}{2}M(\Gamma_0)$.

The family Γ may not be symmetric, but we can replace it by a larger family that is. Let Γ_R be the collection of rectifiable curves in $\mathbb{D}^* \setminus \{R\}$ which have zero winding number around $\{R\}$, but non-zero winding number around 0.

Clearly $\Gamma \subset \Gamma_R$ and Γ_R is symmetric so $M(\Gamma) \geq M(\Gamma_R) = 2M(\Gamma_R^+)$. Thus all we have to do is show $M(\Gamma_R^+) = M(\Gamma_0^+)$. We will actually show $\Gamma_R^+ = \Gamma_0^+$.

Since $\Gamma_0 \subset \Gamma_R$ is obvious, we need only show $\Gamma_R^+ \subset \Gamma_0^+$.

Suppose $\gamma \in \Gamma_R$. Since γ has non-zero winding around 0 it must cross both the negative and positive real axes.

If it never crossed $(0, R)$ then the winding around 0 and R would be the same, which false, so γ must cross$(0, R)$ as well.

Choose points $z_- \in \gamma \cap (-\infty, 0)$ and $z_+ \in \gamma \cap (0, R)$. These points divide γ into two subarcs γ_1 and γ_2.

Then $\gamma_+ = (\gamma_1)_+ \cup (\gamma_2)_+$. But if we reflect $(\gamma_2)_+$ into the lower half-plane and join it to $(\gamma_1)_+$ it forms a closed curve γ_0 that is in Γ_0 and $(\gamma_0)_+ = \gamma_+$. Thus $\gamma_+ \in (\Gamma_0)_+$, as desired. $\square$

Next we prove the Koebe $\frac{1}{4}$-theorem for conformal maps.

The standard proof of Koebe's $\frac{1}{4}$-theorem uses Green's theorem to estimate the power series coefficients of conformal map (proving the Bieberbach conjecture for the second coefficient).

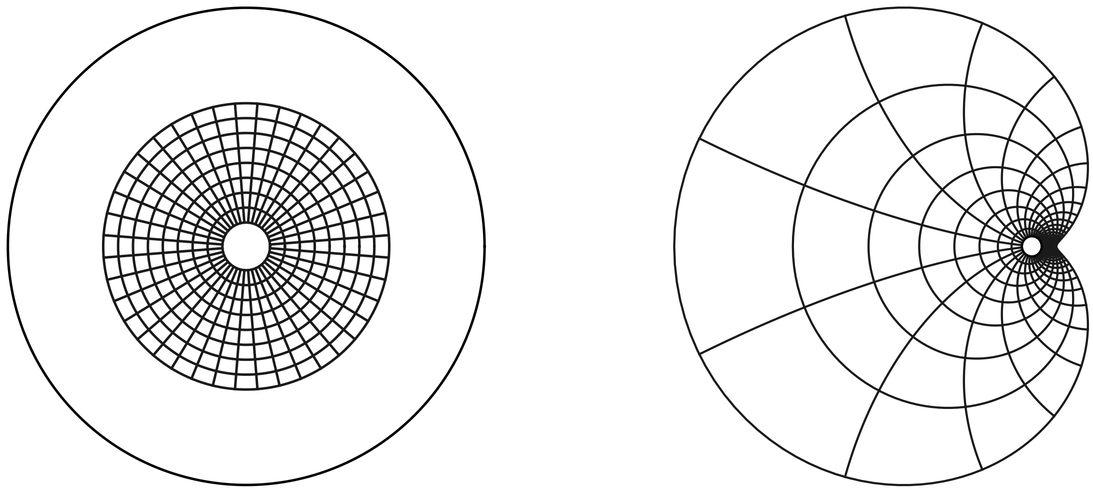
However here we will present a proof, due to Mateljevic that uses the symmetry property of extremal length.

Let $\Omega_{\epsilon,R} = \{z : |z| > \epsilon\} \setminus [R, \infty)$. Note that $\Omega_{1,R}$ is the domain considered in the previous lemma.

We can estimate the moduli of these domains using the Koebe map

$$k(z) = \frac{z}{(1+z)^2} = z - 2z^2 + 3z^3 - 4z^4 + 5z^5 - \dots,$$

This conformal maps $\{|z| < 1\}$ to $\mathbb{R}^2 \setminus [\frac{1}{4}, \infty)$ with $k(0) = 0$, $k'(0) = 1$.



Plot of the Koebe function

Then $k^{-1}(\frac{1}{4R}z)$ maps $\Omega_{\epsilon,R}$ conformally to an annular domain in the disk whose outer boundary is the unit circle and whose inner boundary is trapped between the circle of radius $\frac{\epsilon}{4R}(1 \pm O(\frac{\epsilon}{R}))$.

Thus the modulus of $\Omega_{\epsilon,R}$ is

$$2\pi \log \frac{4R}{\epsilon} + O(\frac{\epsilon}{R}). \tag{2.5}$$

Theorem 2.11 (The Koebe 1/4-Theorem)**.** *Suppose f is holomorphic, 1-1 on $\mathbb{D}$ and $f(0) = 0$, $f'(0) = 1$. Then $D(0, \frac{1}{4}) \subset f(\mathbb{D})$.*

Theorem 2.11 (The Koebe 1/4-Theorem)**.** *Suppose f is holomorphic, 1-1 on $\mathbb{D}$ and $f(0) = 0$, $f'(0) = 1$. Then $D(0, \frac{1}{4}) \subset f(\mathbb{D})$.*

Proof. Recall that the modulus of a doubly connected domain is the modulus of the path family that separates the two boundary components (and is equal to the extremal distance between the boundary components).

Let $R = \mathrm{dist}(0, \partial f(\mathbb{D}))$. Let $A_{\epsilon,r} = \{z : \epsilon < |z| < r\}$ and note that by conformal invariance

$$2\pi \log \frac{1}{\epsilon} = M(A_{\epsilon,1}) = M(f(A_{\epsilon,1})).$$

Let $\delta = \min_{|z|=\epsilon} |f(z)|$. Since $f'(0) = 1$, we have $\delta = \epsilon + O(\epsilon^2)$.

Note that $f(A_{\epsilon,1}) \subset f(\mathbb{D}) \setminus D(0,\delta)$, so

$$M(f(A_{\epsilon,1})) \leq M(f(\mathbb{D}) \setminus D(0,\delta)).$$

By Lemma 2.10 and Equation (2.5),

$$M(f(\mathbb{D}) \setminus D(0,\delta)) \leq M(\Omega_{\delta,R}) = 2\pi \log \frac{4R}{\delta} + O(\frac{\delta}{R}).$$

Putting these together gives

$$2\pi \log \frac{4R}{\delta} + O(\frac{\delta}{R}) \geq 2\pi \log \frac{1}{\epsilon},$$

or

$$\log 4R - \log(\epsilon + O(\epsilon^2)) + O(\frac{\epsilon}{R}) \geq -\log \epsilon,$$

and hence

$$\log 4R \geq -O(\frac{\epsilon}{R}) + \log(1 + O(\epsilon)).$$

Taking $\epsilon \to 0$ shows $\log(4R) \geq 0$, or $R \geq \frac{1}{4}$. $\square$

Paul Koebe

Koebe was a picturesque character whose honesty and frankness forbade him to disguise his greatness as a mathematician; in order to escape embarrassing admiration he travelled incognito, and he often said that in his birthplace Luckenwalde the street boys called after him "There goes the famous function theorist!"

– Hans Freundenthal, quoted in St Andrews biographies

This semester I hope to cover the following topics:

- Review of complex analysis
- Extremal length and conformal modulus,
- Logarithmic capacity, harmonic measure
- Geometric definition of quasiconformal mappings, compactness
- Consequences: quasisymmetry, extension, removability, weldings
- Analytic definition and the measurable Riemann mapping theorem
- Astala's theorems on area and dimension distortion
- Quasiconformal maps on metric spaces
- Conformal dimension

Logarithmic Capacity

Measures and capacities both measure the size of sets. Measures are countably additive; capacities need not be.

Many capacities are associated to a function that blows up at the origin, such as $\log 1/|z|$ or $|z|^{-\alpha}$.

Many natural problems in analysis have answers given in terms of capacities. For example, a Brownian motion in the plane hits a set E with positive probability iff the set has positive log capacity.

Hausdorff content: given as set E, let $\{D_j\} = \{D(x_j, r_j)\}$, be a covering of E by disks and define the α-Hausdorff content

$$\mathcal{H}^\alpha_\infty(E) = \inf \sum_j r_j^\alpha,$$

where the infimum is over all coverings of E.

Power r^α may be replaced by any increasing function $\phi(r)$.

Hausdorff measure: content is not a measure, but can be made into a measure by requiring covering disks to be small. We define

$$\mathcal{H}^\alpha(E) = \lim_{\delta \searrow 0} \inf \sum_j r_j^\alpha,$$

where the infimum is over all coverings with $\sup r_j \leq \delta$.

When $\alpha = 1$ this gives (a multiple of) Lebesgue measure on $\mathbb{R}$.

When $\alpha = 2$ this gives (a multiple of) Lebesgue measure on $\mathbb{R}^2$.

Hausdorff dimension: $\dim(E) = \inf\{\alpha : \mathcal{H}^\alpha(E) = 0\}$.

Standard Cantor set has dimension $\log 2/\log 3$.

Von Koch Snowflake has dimension $\log 4/\log 3$.

There are other dimensions: Minkowski, packing, Assouad,...

Lemma 3.1 (Frostman's Lemma)**.** *Let* φ *be a gauge function. Let* $K \subset \mathbb{R}^d$ *be a compact set with positive Hausdorff content,* $\mathcal{H}^{\varphi}_{\infty}(K) > 0$*. Then there is a positive Borel measure* μ *on* K *satisfying*

$$\mu(B) \leq C_d \varphi(|B|), \tag{3.6}$$

for all balls B *and*

$$\mu(K) \geq \mathcal{H}^{\varphi}_{\infty}(K).$$

Here C_d *is a positive constant depending only on* d*.*

For proof, see Chapter 3 of text by Bishop and Peres.

Suppose $\mu \geq 0$ is a finite Borel measure on $\mathbb{C}$. Define its potential function as

$$U_\mu(z) = \int \log \frac{2}{|z-w|} d\mu(w), z \in \mathbb{C}.$$

and its energy integral by

$$I(\mu) = \iint \log \frac{2}{|z-w|} d\mu(z) d\mu(w) = \int U_\mu(z) d\mu(z).$$

We put the "2" in the numerator so that the integrand is non-negative when $z, w \in \mathbb{T}$, however, this is a non-standard usage.

Measures energy needed to assemble particles with repelling force log.

Suppose E is Borel and μ is a positive measure that has its closed support inside E. We say μ is admissible for E if $U_\mu \leq 1$ on E and we define the **logarithmic capacity** of E as

$$\mathrm{cap}(E) = \sup\{\|\mu\| : \mu \text{ is admissible for } E\}$$

and we write $\mu \in \mathcal{A}(E)$.

Alternatively, the capacity of E is the infimum of $\sup U_\mu$ over all probability measures supported on E. We define the **outer capacity** (or exterior capacity) as

$$\mathrm{cap}^*(E) = \inf\{\mathrm{cap}(V) : E \subset V, V \,\mathrm{open}\}.$$

We say that a set E is **capacitable** if $\mathrm{cap}(E) = \mathrm{cap}^*(E)$.

We wil prove later that all compact sets are capacitable.

The logarithmic kernel can be replaced by other functions, e.g., $|z-w|^{-\alpha}$, and there is a different capacity associated to each one.

To be precise, we should denote logarithmic capacity as $\mathrm{cap}_{\log}$ or logcap, but to simplify notation we simply use "cap" and will often refer to logarithmic capacity as just "capacity". Since we do not use any other capacities in these notes, this abuse should not cause confusion.

WARNING: The logarithmic capacity that we have defined is **NOT** the same as is used in other texts such as Garnett and Marshall's book but it is related to what they call the Robin's constant of E, denoted $\gamma(E)$.

The exact relationship is $\gamma(E) = \frac{1}{\text{cap}(E)} - \log 2$. Garnett and Marshall define the logarithmic capacity of E as $\exp(-\gamma(E))$.

The reason for doing this is that the logarithmic kernel $\log \frac{1}{|z-w|}$ takes both positive and negative values in the plane, so the potential functions for general measures and the Robin's constant for general sets need not be non-negative.

Exponentiating takes care of this. Since we are only interested in computing the capacity of subsets of the circle, taking the extra "2" in the logarithm gave us a non-negative kernel on the unit circle, and we defined a corresponding capacity in the usual way.

Sets of zero logarithmic capacity must be very small, indeed the following computations will show that they must have dimension zero.

Corollary 3.2. *If E has positive Hausdorff dimension, then it has positive logarithmic capacity.*

Proof. By Frostman's Lemma, if E has positive dimension then there is a measure μ supported on E such that $\mu(D(x,r)) \leq Cr^{\alpha}$ for all x and some $C < \infty$ and $\alpha > 0$.

We claim μ has bounded potential. Break the integral over the plane into dyadic annuli $A_n = \{2^{-n-1} < |z| \le 2^{-n}\}$.

$$\begin{aligned}
U_\mu(z) &= \int_{\mathbb{R}^2} \log \frac{d\mu(w)}{|z-w|} \\
&= \sum_n \int_{A_n} \log \frac{d\mu(w)}{|z-w|} \\
&\le \sum_n 2^{-n\alpha} \log 2^{-(n+1)} \\
&= \log 2 \sum_n 2^{-n\alpha}(n+1) \\
&= C_\alpha.
\end{aligned}$$

Since U_μ is bounded above by C_α, the log capacity of E is bounded below by $\|\mu\|/C_\alpha = \mathcal{H}^\alpha_\infty(E)/C_\alpha > 0$. $\square$

Lemma 3.3. *U_μ is lower semi-continuous, i.e.,*

$$\liminf_{z \to z_0} U_\mu(z) \geq U_\mu(z_0).$$

Proof. Apply Fatou's lemma to the integral

$$U_\mu(z) = \int \log \frac{2}{|z-w|} d\mu(w),$$

$\square$

Recall that $\mu_n \to \mu$ weak-* if $\int f d\mu_n \to \int f d\mu$ for every continuous function f of compact support.

Lemma 3.4. *If $\{\mu_n\}$ are positive measures and $\mu_n \to \mu$ weak*, then* $\liminf_n U_{\mu_n}(z) \geq U_\mu(z)$.

Proof. If we replace $\varphi = \log \frac{2}{|z-w|}$ by the continuous kernel $\varphi_r = \max(r, \varphi)$ in the definition of U to get U^r, then weak convergence implies

$$\lim_n U^r_{\mu_n}(z) \to U^r_\mu(z).$$

So for any $\epsilon > 0$ we can choose N so that $n > N$ implies

$$U^r_{\mu_n}(z) \geq U^r_\mu(z) - \epsilon.$$

As $r \to \infty$, we have $U^r_{\mu_n} \nearrow U_{\mu_n}$, by the monotone convergence theorem (since the truncated kernels get larger). So for r large enough and $n > N$ we have

$$U_{\mu_n}(z) \geq U^r_{\mu_n}(z) \geq U_{\mu_n}(z) - \epsilon \geq U_\mu(z) - 2\epsilon.$$

Taking ϵ to zero proves the result. $\square$

Corollary 3.5. *If* $\mu_n \to \mu$ *weak-*$*$*, then* $\liminf_n I(\mu_n) \geq I(\mu)$.

Corollary 3.6. *A probability measure minimizing the energy integral exists.*

Proof of Lemma 3.5. The proof similar is to the previous lemma, except that we have to know that if $\{\mu_n\}$ converges weak-$*$, then so does the product measure $\mu_n \times \mu_n$.

However, weak convergence of $\{\mu_n\}$ implies convergence of integrals of the form

$$\iint f(x)g(y)d\mu_n(x)d\mu_n(y).$$

The Stone-Weierstrass theorem implies that the finite sums of such product functions are dense in all continuous function on the product space.

Since weak-$*$ convergent sequences are bounded, the product measures $\mu_n \times \mu_n$ also have uniformly bounded masses, and hence convergence on a dense set of continuous functions of compact support implies convergence on all continuous functions of compact support. $\square$

Lemma 3.7. *Compact sets are capacitable.*

Proof. Since $\text{cap}(E) \leq \text{cap}^*(E)$ is obvious, we only have to prove the converse.

Set $U_n = \{z : \text{dist}(z, E) < 1/n\}$ and choose a measure μ_n supported in U_n with $\|\mu_n\| \geq \text{cap}(U_n) - 1/n$. Let μ be a weak accumulation point of $\{\mu_n\}$ and note

$$U_\mu(z) = \int \log \frac{2}{|z-w|} d\mu(w) \leq \int \log \frac{2}{|z-w|} d\mu_n(w) \leq 1$$

so μ is admissible in the definition of $\text{cap}(E)$. Thus

$$\text{cap}(E) \geq \limsup \|\mu_n\| = \lim \text{cap}(U_n) = \lim \text{cap}(U_n) = \text{cap}^*(E) \quad \square.$$

Borel sets and even analytic sets are also capacitable.

Let X be a Polish topological space (compatible complete, separable metric).

If Y is Polish, then a subset $E \subset Y$ is called **analytic** if there exists a Polish space X and a continuous map $f\colon X \to Y$ such that $E = f(X)$.

Analytic sets are also called Suslin sets in honor of Mikhail Yakovlevich Suslin. The analytic subsets of Y are often denoted by $A(Y)$ or $\Sigma_1^1(Y)$.

In any uncountable Polish space there exist analytic sets which are not Borel sets. For example see "Conformal removability is hard" by C. Bishop, or textbook by Bruckner, Bruckner and Thomson.

Lemma 3.8. *If X is Polish, then every Borel set $E \subset X$ is analytic.*

For a proof see Appendix B of text by Bishop and Peres.

Lebesgue famously (falsely) claimed continuous images of Borel sets are Borel.

Every analytic set is Lebesgue measurable.

It is clear from the definitions that logarithmic capacity is monotone

$$E \subset F \quad \Rightarrow \quad \operatorname{cap}(E) \leq \operatorname{cap}(F). \tag{3.7}$$

and satisfies the regularity condition

$$\operatorname{cap}(E) = \sup\{\operatorname{cap}(K) : K \subset E, K \text{ compact}\}. \tag{3.8}$$

Lemma 3.9 (Sub-additive)**.** *For any sets* $\{E_n\}$,

$$\operatorname{cap}(\cup E_n) \leq \sum \operatorname{cap}(E_n). \tag{3.9}$$

Proof. We can write any $\mu = \sum \mu_n$ as a sum of mutually singular measures so that μ_n gives full mass to E_n.

Restrict each μ_n to a compact subset K_n of E_n so that $\mu_n(K_n) \geq (1-\epsilon)\mu_n(E_n)$.

These restrictions are admissible for each E_n and hence

$$\sum \operatorname{cap}(E_n) \geq \sum \mu_n(K_n) \geq (1-\epsilon)\sum \mu_n(E_n) = (1-\epsilon)\|\mu\|.$$

Taking $\epsilon \to 0$ proves the result. $\square$

Corollary 3.10. *A countable union of zero capacity sets has zero capacity.*

Corollary 3.11. *Outer capacity is also sub-additive.*

Proof. Given a sequence of sets $\{E_n\}$, choose open sets $V_n \supset E_n$ so that

$$\operatorname{cap}(V_n) \le \operatorname{cap}^*(E_n) + \epsilon 2^{-n}.$$

By the sub-additivity of capacity

$$\operatorname{cap}^*(\cup E_n) \le \operatorname{cap}(\cup V_n) \le \sum \operatorname{cap}(V_n) \le \epsilon + \sum \operatorname{cap}^*(E_n).$$

Taking $\epsilon \to 0$ proves the result. $\square$

Lemma 3.12. *If μ has bounded potential, then* $\operatorname{cap}(E) = 0 \Rightarrow \mu(E) = 0$.

Proof. If $\mu(E) > 0$ then μ restricted to E also has bounded potential function and proves that E has positive capacity. $\square$

Lemma 3.13. *If $\{f_n\}$ are smooth functions on $\mathbb{C}$ so that $|f_n(z)| \leq C/|z|)$ and $\|f_n\|_2 \to 0$, then $\iint_K |f_n|^2 dxdy \to 0$ over any compact K.*

Proof. It suffices to consider rectangles $K = [-L, L] \times [-R, R]$. Then

$$\begin{aligned}
\int_{[-L,L]} |f_n(x,y)|^2 dx &\geq 2\int_{-L}^{L} |f_n(x,y) - f_n(x,-L)|^2 dx + 2C^2/L \\
&\geq 2\int_{-L}^{L} \left(\int_{-R}^{y} \left| \frac{\partial f_n}{\partial y} \right| dy \right)^2 dx + 2C^2/L \\
&\leq 4L\|\nabla f_) n\|_2^2 + 2C^2/L.
\end{aligned}$$

A second integration gives the desired result when L is large compared to R.

$\square$

Lemma 3.14. *If μ is a finite signed measure with total mass zero, then $I(\mu) \geq 0$, with equality if and only if $\mu = 0$.*

Proof. First consider the case where $d\nu = h(z)dxdy$ is smooth, has compact support and $\iint hdxdy = 0$. Then for $|z|$ large,

$$|U_\mu(z)| = \iint \frac{h(z)}{|z-w|} dxdy = O(1/|z|),$$
$$|\nabla U_\mu(z)| = O(1/|z|^2).$$

Since $\Delta \log 1/|z|$ is a δ-mass of size -2π at the origin (in the sense of distributions), we have and therefore $\Delta U_\mu = -2\pi h$.

Green's theorem states that

$$\iint_\Omega (v\Delta u - u\Delta v)dxdy = \int_{\partial\Omega}\left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right)ds,$$

If $v = f^2$ is smooth with compact support and Ω is a large enough disk, then the left side vanishes. Taking $u = 1$, gives

$$\iint_\Omega \Delta f^2 dxdy = 0$$

Also note

$$\begin{aligned}\Delta(h^2) &= 2h_x \cdot h_x + 2h \cdot h_{xx} + 2h_y \cdot h_y + 2h \cdot h_{yy} \\ &= 2h\Delta h + 2\nabla h \cdot \nabla h \\ &= 2h\Delta h + 2|\nabla h|^2,\end{aligned}$$

So if h is smooth and has compact support

$$\iint 2h\Delta h = -\iint 2|\nabla h|^2.$$

Therefore

$$I(\mu) = \iint U_\mu h dx dy = \frac{-1}{2\pi} \iint U_\mu \Delta U_\mu dx dy = \frac{1}{2\pi} \iint |\nabla U_\mu|^2 dx dy t > 0.$$

If $I(\nu) = 0$ then $|\nabla U_\mu| = 0$ everywhere and $h = \Delta U_\mu = 0$ everywhere.

The general czase follows by a standard limiting argument, as follows.

Let φ be a smooth, postive, radial, compactly supported function of mass 1 and let $K_\epsilon(z) = \epsilon^{-2} K(z/\epsilon)$.

Define $h_\epsilon = \varphi * \mu$. Then h_ϵ is smooth and compactly supported so $I(h_\epsilon)$. We know that $I(h_\epsilon) > 0$. Thus if we set $L(z) = \log 1/|z|$,

$$\begin{aligned} I(h_\epsilon) &= \iint \frac{h_\epsilon h_\epsilon dxdy}{\log|z-w|} \\ &= \iint (\varphi_\epsilon(z) * \varphi_\epsilon(w) * L) d\mu(z) d\mu(w) \end{aligned}$$

where,

$$\varphi_\epsilon * \varphi_\epsilon(z) = \int \varphi_\epsilon(z-w)\varphi_\epsilon(w) dudv,$$

$$\varphi_\epsilon * \varphi_\epsilon * L(z) = \int (\varphi_\epsilon * \varphi_\epsilon) L(w-z) dxdy.$$

Since L is superharmonic the convolution with a positive, radial, mass 1 function is less than L, and the convolutions tend to L as ϵ tends to zero.

Therefore, by dominated convergence, $I(h_\epsilon) \to I(\mu)$, so $I(\mu) \geq 0$.

If $I(\mu) = 0$, write $U_\epsilon = U_{h_\epsilon}$. Then

$$\iint |\nabla U_\epsilon|^2 dxdy = I(h_\epsilon) \to 0.$$

We also have $U_\epsilon(z) = O(1/|z|)$ uniformly in ϵ so by Lemma 3.13

$$\lim_{\epsilon \to 0} \iint |\nabla U_\epsilon(z)|^2 dxdy = 0.$$

If f is smooth with compact support then by Green's theorem

$$\int f d\mu = \lim_{\epsilon \to 0} \int f d\mu_\epsilon = \lim_{\epsilon \to 0} \frac{-1}{2\pi} \int \Delta f U_\epsilon dxdy$$

so $\mu = 0$. $\square$

Lemma 3.15. *If E is compact and has positive capacity, then there exists an admissible μ that attains the maximum mass in the definition of capacity and $U_\mu(z) = 1$ everywhere on E, except possible a set of capacity zero.*

For a proof see Chapter III of Garnett and Marshall's book *Harmonic Measure*.

Proof. Let μ_n be a sequence of probability measures on E so that $\|\mu_n\| \to R$ where $R = \inf \|\mu\|$ over all probability measures supported on E.

This is finite since E has positive capacity.

By the Banach-Alaoglu theorem there is a weak-$*$ convergent subsequence with limit μ, and by Lemma 3.5,

$$I(\mu) \leq \liminf_n I(\mu_n) = R.$$

We claim that $U_\mu \geq R$ except possibly on a set of zero capacity. set of positive capacity on which $U_\mu < R - \epsilon$ and let σ be a non-zero, positive measure on T which potential bounded by 1. Define

$$\mu_t = (1-t)\mu + t\sigma.$$

This is a measure on E so that (using Fubini's theorem and Corollary)

$$\begin{aligned}
I(\mu_t) &\leq \int \log \frac{1}{|z-w|}((1-t)d\mu + td\sigma)((1-t)d\mu + td\sigma) \\
&\leq (1-t)^2 I(\mu) + (t-t^2)\int U_\mu d\sigma + (t-t^2)\int U_\mu d\sigma + t^2 I(\sigma) \\
&\leq I(\mu) - 2tI(\mu) + t\int U_\mu d\sigma + t\int U_\sigma d\mu + O(t^2) \\
&\leq I(\mu) + t\int U_\mu d\sigma + t\int U_\mu d\sigma + O(t^2) \\
&\leq I(\mu) - 2I(\mu) + 2t(1-\epsilon)\|\sigma\| + O(t^2) \\
&< I(\mu),
\end{aligned}$$

if $t > 0$ is small enough. This contradicts minimality of μ, proving the claim.

Next we show that $U_\mu \leq 1$ everywhere on the closed support of μ.

By previous step, $U_\mu \geq 1$ except on capacity zero (hence μ-measure zero).

If there is a point z in the support of μ such that $U_\mu(z) > 1$, then by lower semi-continuity of potentials, U_μ is $> 1 + \epsilon$ on some neighborhood of z and this neighborhood has positive μ measure (since z is in the support of μ) and thus $I(\mu) = \int U_\mu d\mu > \|\mu\|$, a contradiction.

Finally, let $\sigma = \mu/R$. Then the potential function of σ is bounded by 1 everywhere, so σ is admissible for E and hence $\|\sigma\| \leq \text{cap}(E)$.

If ν is any other admissible measure for E, then $\nu(\{z \in E : U_\sigma(z) < 1\}) = 0$ by Lemma 3.12. Hence

$$\|\nu\| = \int 1 d\nu = \int U_\sigma d\nu = \int U_\nu d\sigma \leq \int 1 d\sigma = \|\sigma\|,$$

and thus $\|\sigma\| \geq \operatorname{cap}(E)$.

Thus $\operatorname{cap}(E) = \|\sigma\| = \|\mu/R\| = 1/R$. $\square$

Pfluger's Theorem

Pfluger's theorem connects logarithmic capacity and extremal length.

Suppose $K \subset \mathbb{D}$ is a compact connected set with smooth boundary with 0 in the interior of K. Let K^* be the reflection of K across $\mathbb{T}$.

For any $E \subset \mathbb{T}$ that is a **finite union of closed intervals**, let Ω be the connected component of $\mathbb{C} \setminus (E \cup K \cup K^*)$ that has E on its boundary.

Let $h(z)$ be the harmonic function in Ω with boundary values 0 on K and K^* and boundary value 1 on E.

All boundary points are regular for the Dirichlet problem (since all boundary components are non-degenerate continua). Hence h extends continuously to the boundary with the correct boundary values.

h is symmetric with respect to $\mathbb{T}$, so its normal derivative on $\mathbb{T} \setminus E$ is 0.

Let $D(h) = \int_{\mathbb{D}\setminus K} |\nabla h|^2 dxdy$.

Let Γ_E denote the paths in $\mathbb{D} \setminus K$ that connect K to E.

Lemma 2.16. *With notation as above, $M(\Gamma_E) = D(h)$.*

Proof. Clearly $|\nabla h|$ is an admissible metric for Γ_E, so

$$M(\Gamma_E) \leq D(h) \equiv \int_{\mathbb{D}\setminus K} |\nabla h|^2 dxdy.$$

Thus we need only show the other direction.

Green's theorem states that

$$\iint_{\Omega} (u\Delta v - v\Delta u) dx dy = \int_{\partial\Omega} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} ds. \tag{2.10}$$

Using this and the fact that $h = 1$ on E, we have

$$\int_{\partial K} \frac{\partial h}{\partial n} ds = -\int_{\mathbb{T}} \frac{\partial h}{\partial n} ds = -\int_{E} \frac{\partial h}{\partial n} ds = -\int_{E} h\frac{\partial h}{\partial n} ds.$$

Continuing,

$$\begin{aligned}\int_{\partial K} \frac{\partial h}{\partial n} ds &= -\frac{1}{2}\int_{E} \frac{\partial (h^2)}{\partial n} ds \\ &= \frac{1}{2}\int_{\mathbb{T}\setminus E} \frac{\partial (h^2)}{\partial n} ds + \frac{1}{2}\int_{\partial K} \frac{\partial (h^2)}{\partial n} ds + \frac{1}{2}\int_{\mathbb{D}\setminus K} \Delta(h^2) dx dy.\end{aligned}$$

The first term is zero because h has normal derivative zero on $\mathbb{T} \setminus E$, and hence the same is true for h^2.

The second term is zero because h is zero on K and so $\frac{\partial}{\partial n} h^2 = 2h\frac{\partial h}{\partial n} = 0$.

To evaluate the third term, we use the identity

$$\begin{aligned}
\Delta(h^2) &= 2h_x \cdot h_x + 2h \cdot h_{xx} + 2h_y \cdot h_y + 2h \cdot h_{yy} \\
&= 2h\Delta h + 2\nabla h \cdot \nabla h \\
&= 2h \cdot 0 + 2|\nabla h|^2 \\
&= 2|\nabla h|^2,
\end{aligned}$$

to deduce

$$\frac{1}{2}\int_{\mathbb{D}\setminus K} \Delta(h^2)dxdy = \int_{\mathbb{D}\setminus K} |\nabla h|^2 dxdy.$$

Therefore,

$$\int_{\partial K} \frac{\partial h}{\partial n} ds = \int_{\mathbb{D}\setminus K} |\nabla h|^2 dxdy.$$

Thus the tangential derivative of h's harmonic conjugate has integral $D(h)$ around ∂K and therefore $2\pi h/D(h)$ is the real part of a holomorphic function g on $\mathbb{D} \setminus K$.

Then $f = \exp(g)$ maps $\mathbb{D} \setminus K$ into the annulus

$$A = \{z : 1 < |z| < \exp(2\pi/D(h))\}$$

with the components of E mapping to arcs of the outer circle and the components of $\mathbb{T} \setminus E$ mapping to radial slits.

The path family Γ_E maps to the path family connecting the inner and outer circles without hitting the radial slits, and our earlier computations show the modulus of this family is $D(h)$. $\square$

Theorem 2.17 (Pfluger's theorem)**.** *If $K \subset \mathbb{D}$ is a compact connected set with smooth boundary with 0 in the interior of K. Then there are constants C_1, C_2 so that following holds. For any $E \subset \mathbb{T}$ that is a finite union of closed intervals,*

$$\frac{1}{\mathrm{cap}(E)} + C_1 \leq \pi\lambda(\Gamma_E) \leq \frac{1}{\mathrm{cap}(E)} + C_2,$$

where Γ_E is the path family connecting K to E. The constants C_1, C_2 can be chosen to depend only on $0 < r < R < 1$ if $\partial K \subset \{r \leq |z| \leq R\}$.

Later we will extend this to compact sets $E \subset \mathbb{T}$.

Proof. Using Lemma 2.16, we only have to relate $D(h)$ to the logarithmic capacity of E.

Let μ be the equilibrium probability measure for E. We know in general that $U_\mu = \gamma$ where $\gamma = 1/\text{cap}(E)$ almost everywhere on E (since sets of zero capacity have zero measure) and is continuous off E, but since U_μ is harmonic in $\mathbb{D}$ and equals the Poisson integral of its boundary values, we can deduce $U_\mu = \gamma$ everywhere on E.

Let $v(z) = \frac{1}{2}(U_\mu(z) + U_\mu(1/\overline{z}))$. Then since ∂K has positive distance from 0, there are constants C_1, C_2 so that

$$v + C_1 \leq 0, \qquad v + C_2 \geq 0,$$

on ∂K. Note that $C_1 \geq -\gamma$ by the maximum principle and $C_2 \geq 0$ trivially.

Moreover, since μ is a probability measure supported on the unit circle, given $0 < r < R < 1$, U_μ is uniformly bounded on both the annulus $\{r \leq |z| \leq R\}$ and its reflection across the unit circle, since these both have bounded, but positive distance from the unit circle.

This proves that C_1, C_2 can be chosen to depend on only these numbers, as claimed in the final statement of the theorem.

The following inequalities are easy to check on K, K^* and E,

$$\frac{v(z)+C_1}{\gamma+C_1} \le h(z) \le \frac{v(z)+C_2}{\gamma+C_2}.$$

and hence hold on Ω by the maximum principle.

Since we have equality on E, we also get

$$\frac{\partial}{\partial n}\left(\frac{v(z)+C_1}{\gamma+C_1}\right) \le \frac{\partial h}{\partial n} \le \frac{\partial}{\partial n}\left(\frac{v(z)+C_2}{\gamma+C_2}\right)$$

for $z \in E$.

When we integrate over E, the middle term is $-D(h)$ (we computed this above) and by Green's theorem

$$-\int_E \frac{\partial}{\partial n}\frac{v(z)+C_1}{\gamma+C_1}ds = \frac{1}{\gamma+C_1}\int_{\mathbb{D}} \Delta(v)dxdy = \frac{\pi}{\gamma+C_1}$$

because v is harmonic except for a $\frac{1}{2}\log\frac{1}{|z|}$ pole at the origin.

A similar computation holds for the other term and hence

$$\frac{\pi}{\gamma + C_1} \le D(h) = M(\Gamma_E) \le \frac{\pi}{\gamma + C_2},$$

since $D(h) = \int_E \frac{\partial h}{\partial n} ds$. Hence

$$\gamma + C_1 \le \pi\lambda(\Gamma_E) \le \gamma + C_2.$$

This completes the proof of Pfluger's theorem for finite unions of intervals. $\square$

To extend Pfluger's theorem to all compact subsets of $\mathbb{T}$. First we need a continuity property of extremal length.

Recall that an extended real-valued function is lower semi-continuous if all sets of the form $\{f > \alpha\}$ are open.

Lemma 2.18. *Suppose $E \cap \mathbb{T}$ is compact, $K \subset \mathbb{D}$ is compact, connected and contains the origin, and Γ_E is the path family connecting K and E in $\mathbb{D} \setminus K$. Fix an admissible metric ρ for Γ_E and for each $z \in \mathbb{T}$, define $f(z) = \inf \int_\gamma \rho ds$ where the infimum is over all paths in Γ_E that connect K to z. Then f is lower semi-continuous.*

Proof. Suppose $z_0 \in \mathbb{T}$ and use Cauchy-Schwarz to get

$$\begin{aligned}
\int_{2^{-n-1}}^{2^{-n}} \left(\int_{|z-z_0|=r} \rho ds \right)^2 dr &\le \int_{2^{-n-1}}^{2^{-n}} \left(\int_{|z-z_0|=r} \rho^2 ds \right) dr \left(\int_{|z-z_0|=r} 1 ds \right) dr \\
&\le \int_{2^{-n-1}}^{2^{-n}} r \int_0^{2\pi} \rho^2 r d\theta dr \\
&\le \pi 2^{-n} \int_{2^{-n-1}<|z-z_0|<2^{-n}} \rho^2 dxdy \\
&= o(2^{-n}).
\end{aligned}$$

Thus there are circular cross-cuts $\{\gamma_n\} \subset \{z : 2^{-n-1} < |z - z_0| < 2^{-n}\}$ of $\mathbb{D}$ centered at z_0 and with ρ-length ϵ_n tending to 0. By taking a subsequence we may assume $\sum \epsilon_n < \infty$.

Now choose $z_n \to z_0$, with z_n separated from 0 by γ_n, and so that

$$f(z_n) \to \alpha \equiv \liminf_{z \to z_0} f(z).$$

We claim there is a path from K to z_0 whose ρ-length is $\leq \alpha + \epsilon$.

Let c_n be the infimum of ρ-lengths of paths connecting γ_n and γ_{n+1}.

By considering a path connecting K to z_n, we see that $\sum_1^n c_k \leq f(z_n)$, for all n and hence $\sum_1^\infty c_n \leq \alpha$.

Next choose $\epsilon > 0$ and n so that we can connect K to z_n (and hence to γ_n) by a path of ρ-length less than $\alpha + \epsilon$.

We can then connect γ_n to z_0 by a infinite concatenation of arcs of γ_k, $k > n$ and paths connecting γ_k to γ_{k+1} that have total length $\sum_n^\infty (\epsilon_n + c_n) = o(1)$.

Thus K is connected to z_0 by a path of ρ-length as close to α as we wish. $\square$

Corollary 2.18. *Suppose $E \subset \mathbb{T}$ is compact and $\epsilon > 0$. Then there is a finite collection of closed intervals F so that $E \subset F$ and*

$$\lambda(\Gamma_E) \leq \lambda(\Gamma_F) + \epsilon,$$

where the path families are defined as above.

Proof. Choose an admissible ρ so that $\int \rho^2 dxdy \leq M(\Gamma_E) + \epsilon$. Set

$$r = (\frac{M(\Gamma_E) + \epsilon}{M(\Gamma_E) + 2\epsilon})^{1/2} < 1.$$

By Lemma 2.18, $V = \{z \in \mathbb{T} : f(z) > r\}$ is open, and therefore we can choose a set F of the desired form inside V. Then ρ/r is admissible for Γ_F, so

$$M(\Gamma_F) \leq \int (\frac{\rho}{r})^2 dxdy = \frac{M(\Gamma_E) + 2\epsilon}{M(\Gamma_E) + \epsilon} \int \rho^2 dxdy \leq M(\Gamma_E) + 2\epsilon.$$

Thus an inequality in the opposite direction holds for extremal length. $\square$

Corollary 2.19. *Pfluger's theorem holds for all compact sets in* $\mathbb{T}$*.*

Proof. Suppose E is compact. Using Corollary 2.18 and Lemma 3.7 we can choose nested sets $E_n \searrow E$ that are finite unions of closed intervals and satisfy

$$\lambda(\mathcal{F}_{E_n}) \to \lambda(\mathcal{F}_E),$$

and

$$\mathrm{cap}(E_n) \to \mathrm{cap}(E).$$

Thus the inequalities in Pfluger's theorem extend to E. $\square$

Gehring, Hayman and Carathéodory

The boundary of a simply connected domain need not be a Jordan curve, nor even locally connected, and such examples arise naturally in complex dynamics as the Fatou components of various polynomials and entire functions.

If the boundary is locally connected, then the conformal map from the disk extends continuously to the boundary.

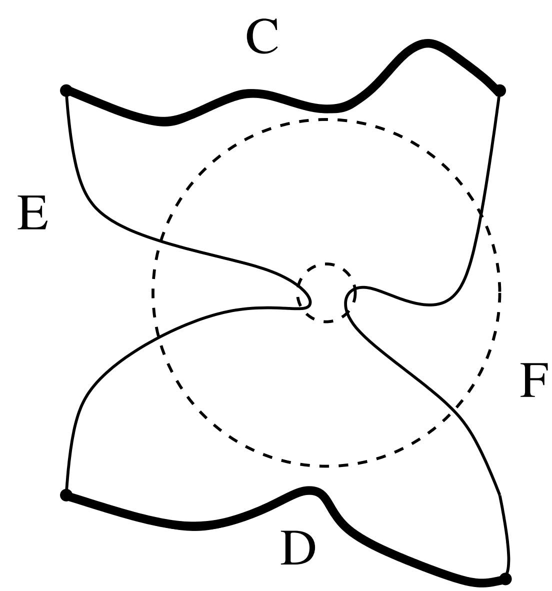
Even for general simply connected domains, the boundary values exist in some sense at most points. We will make this precise.

Lemma 2.20. *Suppose Q is a quadrilateral with opposite pairs of sides E, F and C, D. Assume*

(1) E and F can be connected in Q by a curve σ of diameter $\leq \epsilon$,

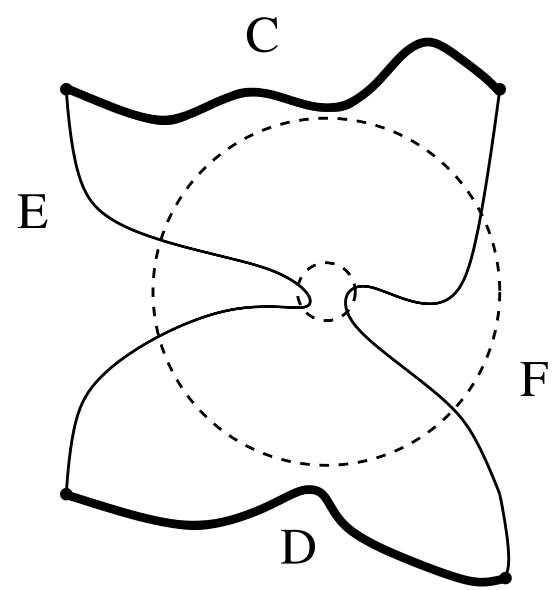
(2) any curve connecting C and D in Q has diameter at least 1.

Then the modulus of the path family connecting E and F in Q is larger than $M(\epsilon)$ where $M(\epsilon) \to \infty$ as $\epsilon \to 0$.



Proof. Define a metric on Q by $\rho(z) = \frac{1}{2}|z-a|^{-1}/\log(1/2\epsilon)$ for $\epsilon < |z-a| < 1/2$. Any curve γ connecting C and D must cross σ and since γ has diameter ≥ 1 it must leave the annulus where ρ is non-zero.

This shows that the modulus of the path family in Q separating E and F is small, hence the modulus of the family connecting them is large. $\square$



The following fundamental fact says that hyperbolic geodesics are almost the same as Euclidean geodesics.

Theorem 2.21 (Gehring-Hayman inequality)**.** *There is an absolute constant* $C < \infty$ *to that the following holds. Suppose* $\Omega \subset \mathbb{C}$ *is hyperbolic and simply connected. Given two points in* Ω*, let* γ *be the hyperbolic geodesic connecting these two points and let* σ *be any other curve in* Ω *connecting them. Then* $\mathrm{len}(\gamma) \leq \mathrm{C} \cdot \mathrm{len}(\sigma)$.

Proof. Let $f : \mathbb{D} \to \Omega$ be conformal, normalized so that γ is the image of $I = [0, r] \subset \mathbb{D}$ for some $0 < r < 1$. Without loss of generality we may assume $r = r_N 1 - 2^{-N}$ for some N. Let

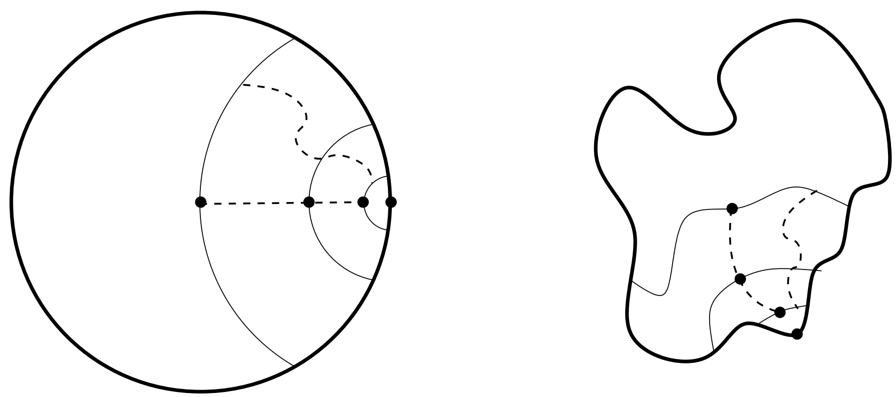
$$
\begin{aligned}
Q_n &= \{z \in \mathbb{D} : 2^{-n-1} < |z-1| < 2^{-n}\}, \\
\gamma_n &= \{z \in \mathbb{D} : |z-1| = 2^{-n}\}, \\
z_n &= \gamma_n \cap [0, 1).
\end{aligned}
$$

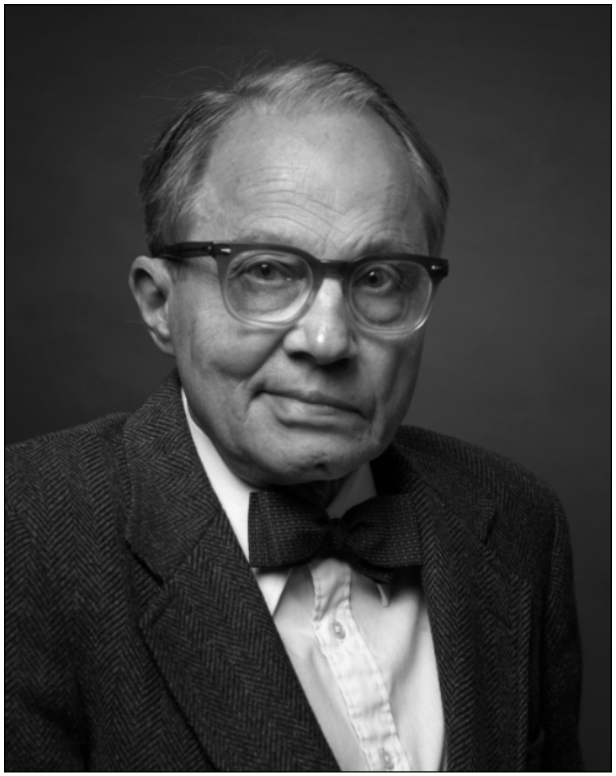
Let $Q'_n \subset Q_n$ be the sub-quadrilateral of points with $|\arg(1-z)| < \pi/6$. Each Q'_n has bounded hyperbolic diameter and, by Koebe's theorem, its image is bounded by four arcs of diameter $\simeq d_n$ and opposite sides are $\simeq d_n$ apart.

In particular, this means that any curve in $f(Q_n)$ separating $f(\gamma_n)$ and $f(\gamma_{n+1})$ must cross $f(Q'_n)$ and hence has diameter $\gtrsim d_n$. Since Q_n has bounded modulus, so does $f(Q_n)$ and so Lemma 2.20 says that the shortest curve in $f(Q_n)$ connecting γ_n and γ_{n+1} has length $\ell_n \simeq d_n$.

Thus any curve γ in Q connecting γ_n and γ_{n+1} has length at least ℓ_n, and so

$$\ell(\gamma) = O(\sum d_n) = O(\sum \ell_n) \leq O(\ell(\sigma)). \quad \square$$





Fred Gehring

Walter Hayman

If $f:\mathbb{D}\to\Omega$ is conformal define

$$a(r)=\mathrm{area}(\Omega\setminus f(r\cdot\mathbb{D})).$$

If Ω has finite area (e.g., if it is bounded), then clearly $a(r)\searrow 0$ as $r\nearrow 1$.

Lemma 2.22. *There is a $C<\infty$ so that the following holds. Suppose $f:\mathbb{D}\to\Omega$ is conformal and $\frac{1}{2}\leq r<1$. Let $E(\delta,r)=\{x\in\mathbb{T}:|f(sx)-f(rx)|\geq\delta$ for some $r<s<1\}$. Then the extremal length of the path family $\mathcal{P}$ connecting $D(0,r)$ to E is bounded below by $\delta^2/Ca(r)$.*

Proof. Let $z = f(sx)$ and suppose $w \in f(D(0,r))$. By the Gehring-Hayman estimate, the length of any curve from w to z is at least $1/C$ times the length of the hyperbolic geodesic γ between them.

But this geodesic has a segment γ_0 that lies within a uniformly bounded distance of the geodesic γ_1 from $f(rx)$ to z. By the Koebe theorem γ_0 and γ_1 have comparable Euclidean lengths, and clearly the length of γ_1 is at least δ.

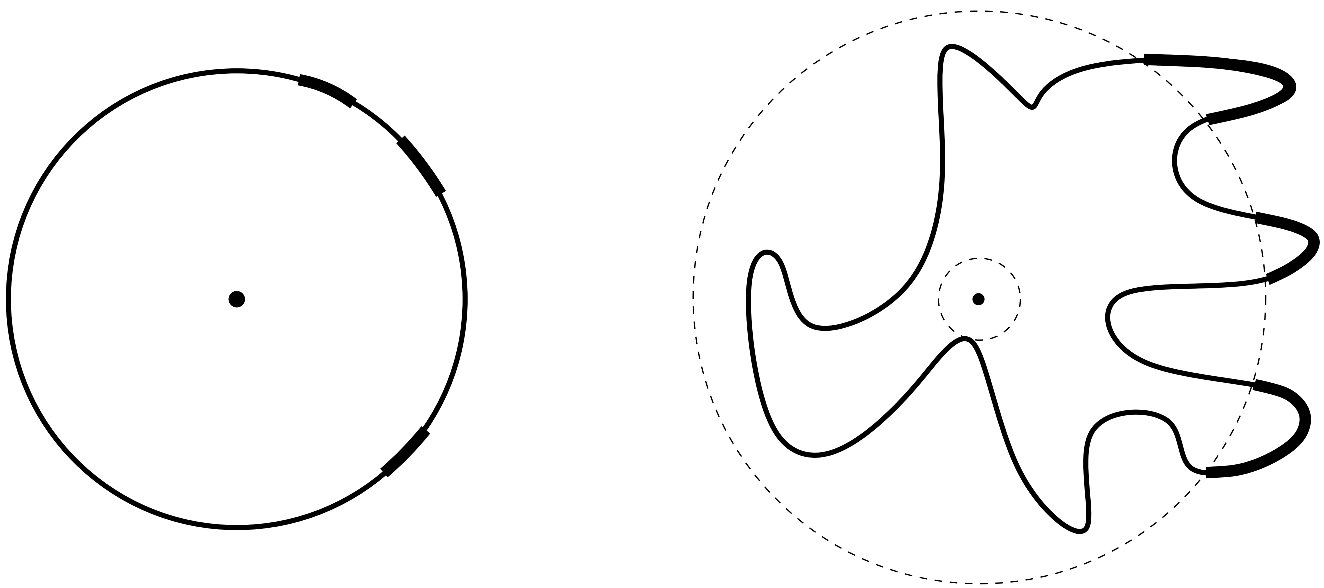
Thus the length of any path from $f(D(0,r))$ to $f(sx)$ is at least δ/C. Now let $\rho = C/\delta$ in $\Omega \setminus f(D(0,r))$ and 0 elsewhere. Then ρ is admissible for $f(\mathcal{P})$ and $\iint \rho^2 dxdy$ is bounded by $C^2 a(r)/\delta^2$.

Thus $\lambda(\mathcal{P}) \geq \frac{\delta^2}{C^2 a(r)}$. $\square$

Lemma 2.23. *Suppose $f : \mathbb{D} \to \Omega$ is conformal, and for $R \geq 1$,*

$$E_R = \{x \in \mathbb{T} : |f(x) - f(0)| \geq R \operatorname{dist}(f(0), \partial\Omega)\}.$$

Then E_R has capacity $O(1/\log R)$ if R is large enough.



Proof. Assume $f(0) = 0$ and $\mathrm{dist}(0, \partial\Omega) = 1$ and let $\rho(z) = |z|^{-1}/\log R$ for $z \in \Omega \cap \{1 < |z| < R\}$. Then ρ is admissible for the path family Γ connecting $D(0, 1/2)$ to $\partial\Omega \setminus D(0, R)$ and $\iint \rho^2 dxdy \le 2\pi/\log R$.

By definition $M(\Gamma) \le 2\pi/\log R$ and $\lambda(\Gamma) \ge (\log R)/2\pi$. By the Koebe theorem $f^{-1}(D(0, 1/2))$ is contained in a compact subset of $\mathbb{D}$, independent of Ω.

By Pfluger's theorem (Theorem 2.17),

$$\mathrm{cap}(E_r) \le \frac{2}{-2C_2 + \log R}.$$

$\square$

Corollary 2.24. *If $f : \mathbb{D} \to \Omega$ is conformal, then f has radial limits except on a set of zero capacity (and hence has finite radial limits a.e. on $\mathbb{T}$).*

Proof. Let $E_{r,\delta} \subset \mathbb{T}$ be the set of $x \in \mathbb{T}$ so that $\operatorname{diam}(f(rx, x)) > \delta$, and let $E_\delta = \bigcap_{0<r<1} E_{r,\delta}$.

If f does not have a radial limit at $x \in \mathbb{T}$, then $x \in E_\delta$ for some $\delta > 0$, and this has zero capacity by Lemma 2.22.

Taking the union over a sequence of δ's tending to zero proves the result. The set where f has a radial limit ∞ has zero capacity by Lemma 2.23, so we deduce f has finite radial limits except on zero capacity. $\square$

Combining the last two results proves

Corollary 2.25. *Given $\epsilon > 0$ there is a $C < \infty$ so that the following holds. If $f : \mathbb{D} \to \Omega$ is conformal, $z \in \mathbb{D}$ and $I \subset \mathbb{T}$ is an arc that satisfies $|I| \geq \epsilon(1 - |z|)$ and $\mathrm{dist}(z, I) \leq \frac{1}{\epsilon}(1 - |z|)$, then I contains a point w where f has a radial limit and $|f(w) - f(z)| \leq C\mathrm{dist}(f(z), \partial\Omega)$.*

Theorem 2.26 (Carathéodory)**.** *Suppose that $f : \mathbb{D} \to \Omega$ is conformal, and that $\partial\Omega$ is compact and locally path connected (for every $\epsilon > 0$ there is a $\delta > 0$ so that any two points of $\partial\Omega$ that are within distance δ of each other can be connected by a path in $\partial\Omega$ of diameter at most ϵ). Then f extends continuously to the boundary of $\mathbb{D}$.*

Lasse Rempe has pointed out this is actually due to Carathéodory's student Marie Torhorst. See Rempe's article On prime ends and local connectivity.

Proof. Suppose $\eta > 0$ is small. Since $\partial\Omega$ is compact $\Omega \setminus f(\{|z| < 1 - \frac{1}{n}\})$ has finite area that tends to zero as $n \nearrow \infty$. Thus if n is sufficiently large, this region contains no disk of radius η.

Choose $\{z_j\}$ to be n equally spaced points on the unit circle and using Lemma 2.25 choose interlaced points $\{w_j\}$ so that f has a radial limit $f(w_j)$ at w_j and this limit satisfies $|f(w_j) - f(rw_j)| \leq C\eta$ where $r = 1 - 1/n$. Then

$$\begin{aligned} |f(w_j) - f(w_{j+1})| \ \leq \ & |f(w_j) - f(rw_j)| \\ & \qquad +|f(rw_j) - f(rw_{j+1})| \\ & \qquad\qquad +|f(rw_{j+1}) - f(w_{j+1})| \\ \leq \ & C\eta. \end{aligned}$$

The center term is bounded by Koebe's theorem and the others by definition.

Fix $\epsilon > 0$ and choose $\delta > 0$ as in the definition of locally connected.

Thus if η is so small that $C\eta < \delta$, then the shorter arc of $\partial\Omega$ with endpoints $f(w_j)$ and $f(w_{j+1})$ can be connected in $\partial\Omega$ by a curve of diameter at most ϵ.

Thus the image under f of the Carleson square with base I_j (the arc between w_j and w_{j+1}) has diameter at most $C\eta + \epsilon$. This implies f has a continuous extension to the boundary. $\square$

Uniform convergence on compact subsets of $\mathbb{D}$ does not imply uniform convergence on the boundary.

However, it is true that the conformal boundary values will converge if the image domains have some parameterizations that converge.

In other words, if a sequence of simply connected domains have boundaries with continuous parameterizations that converge uniformly to the continuous parameterization of the limiting domain, then we also get uniform convergence for the conformal parameterizations of the boundaries.

This is analogous to Carathéodory's theorem: if a domain boundary has any continuous parameterization, then the conformal parameterization is also continuous.

Lemma 2.27. *Suppose* $\{f_n\}$ *are conformal maps of* $\mathbb{D} \to \Omega_n$ *that converge uniformly on compact subsets of* $\mathbb{D}$ *to a conformal map* $f : \mathbb{D} \to \Omega$*. Suppose that the boundary of each* Ω_n *is the homeomorphic image* $\partial\Omega_n = \sigma_n(\mathbb{T})$ *and that* $\{\sigma_n\}$ *converges uniformly on* $\mathbb{T}$ *to a homeomorphism* $\sigma : \mathbb{T} \to \partial\Omega$*. Then* $f_n \to f$ *uniformly on the* $\overline{\mathbb{D}}$*.*

Proof. Fix $\epsilon > 0$ and choose n so large that if we divide $\mathbb{T}$ into n equal sized intervals $\{J_j\}_1^n$, then σ maps each of them to a set I_j of diameter at most $\epsilon/2$.

Let $I_j^k = f_k(J_j)$. Because $\sigma_k \to \sigma$ uniformly, the sets I_j all have diameter at most ϵ, if k is large enough.

Next choose $\eta > 0$ so small that if $k, m > 1/\eta$ and $\sigma_m(J_j)$ and $\sigma_k(J_i)$ contain points at most distance $C\eta$ apart, then J_i and J_k are the same or adjacent to each other.

We can do this because of the uniform convergence and the fact that σ is 1-to-1. By passing to the limit the same property holds if we replace σ_m by σ.

Next choose m so large that $f(\mathbb{D}) \setminus f(\{|z| < 1 - \frac{1}{m}\})$ is contained in an η-neighborhood of $\partial\Omega$.

Choose m points $\{z_j\}$ equally spaced on the circle $|z| = 1 - \frac{1}{m}$,and let $K_j^m \subset \mathbb{T}$ be the arc centered at $z_j/|z_j|$ of length $4\pi/m$. Fix a small number $\delta > 0$ (δ will be determined below, depending only on η).

By Lemma 2.23 choose a point $w_j \in K_j^m$ so that $|w_j - z_j| \leq 2/m$ and

$$|f(w_j) - f(w_j(1 - \frac{1}{m}))| \leq C\delta.$$

Similarly, choose points $w_j^k \in K_j^m$ so that

$$|f_k(w_j^k) - f_k(z_j)| \leq 2C\delta.$$

This is possible since $f_k \to f$ uniformly on the compact set $\{|z| \leq 1 - \frac{1}{m}\}$ and thus $\partial f_k(\mathbb{D})$ is contained in a 2δ-neighborhood of $\partial\Omega$ for k large enough, since $\partial\Omega_k$ is contained in a δ-neighborhood of $\partial\Omega$ because of the uniform convergence of the parameterizations.

By taking m larger, if necessary, we can also arrange that each I_j contains at least one of the points $f(z_m/|z_m|)$.

Thus each $f(K_j^m)$ is mapped into the union of at most 2 of the I_j and hence its image has diameter at most 2ϵ.

Also, the points $f(w_p^k)$ and $f(w_{p+1}^k)$ are at most $C\delta$ apart, so belong to the same or adjacent sets I_j. Thus $f_k(K_p)$ is a union of at most 4 such adjacent sets and hence has diameter $O(\epsilon)$.

For each w_p^k there is an arc J_j so that $f_k(w_p^k) \subset \sigma_k(J_j)$. Similarly, there is an arc J_i so that $f(w_p) \in I_i = \sigma(J_i)$.

Since $f_k \to f$ uniformly on the finite set $\{z_n\}$, we have, for k sufficiently large

$$
\begin{aligned}
|f_k(w_n^k) - f(w_n)| \;\leq\; & |f_k(w_n^k) - f_k(z_n)| \\
& \qquad +|f_k(z_n) - f(z_n)| \\
& \qquad\qquad +|f(z_n) - f(w_n)| \\
\leq\; & (2C + 1 + C)\delta.
\end{aligned}
$$

This is less than η if δ is small enough. Since I_i and I_j each have diameter at most ϵ, their union has diameter $< 2\epsilon$ and the union of the intervals adjacent to these is at most 4ϵ. Similarly for I_i^k and J_j^k. Thus $f_k(K_p)$ and $f(K_p)$ are contained in $O(\epsilon)$-neighborhoods of each other.

Thus $f_k \to f$ uniformly on $\mathbb{T}$. By the maximum principle, this implies uniform convergence on the closed disk, as desired. $\square$

Corollary 2.28. *If $\{\sigma_n\}$ are homeomorphisms of the plane that converge uniformly to a homeomorphism σ, and Q is a Jordan quadrilateral then* $M(l\sigma_n(Q)) \to M(\sigma(Q))$

Proof. Let $Q_n = \sigma_n(Q)$. By taking n large enough we can choose a point z_0 that is in every Q_n and choose conformal maps $f_n : \mathbb{D} \to Q_n$ so that $f(0) = z_0$. By normal families, we can pass to a subsequence that converges uniformly on compact subsets of $\mathbb{D}$ to a conformal map $f : \mathbb{D} \to \sigma(Q)$.

By the previous result these maps converge uniformly on $\overline{\mathbb{D}}$.

If v is a vertex of Q and $v_n \to v$ are vertices of Q_n, then the uniform convergence of f_n to f implies that preimages of v_n under f_n must converge to the preimage of v under f. Since this holds for all four vertices, and modulus on $\mathbb{D}$ is a continuous function of the four vertices, this proves the corollary. $\square$

Harmonic measure

Suppose Ω is a planar Jordan domain bounded, $z \in \Omega$, and $E \subset \partial\Omega$ is Borel.

Suppose $f : \mathbb{D} \to \Omega$ is conformal and $f(0) = z$ (use Riemann mapping theorem).

By Carathéodory's theorem, f extends continuously (even homeomorphically) to the boundary, so $f^{-1}(E) \subset \mathbb{T}$ is also Borel. We define "the harmonic measure of the set E for the domain Ω, with respect to the point z" as

$$\omega(z, E, \Omega) = |E|/2\pi,$$

where $|E|$ denotes the Lebesgue 1-dimensional measure of E.

This depends on the choice of the Riemann map f, but any two maps, both sending 0 to z, will differ only by a pre-composition with a rotation.

Thus the two possible pre-images of E differ by a rotation and hence have the same Lebesgue measure. If we fix E and Ω, then $\omega(z, E, \Omega)$ is a harmonic function of z, giving rise the name “harmonic measure”.

Since we always have $0 \leq \omega(z, E, \Omega) \leq 1$, we can deduce that if E has harmonic measure with respect to one point z in Ω then it has zero harmonic measure with respect to all points.

There are several alternate definitions:

- Hitting distribution of Brownian motion.
- Normal derivative of Green's function (need smooth boundary).
- Solution of Dirichlet problem.
- Measure minimizing log-energy (for base point ∞).

If $\partial\Omega$ is merely locally connected, then Carathéodory's theorem still implies that the Riemann map f has a continuous extension to the boundary, so the same definition of harmonic measure works.

We can define harmonic measure for general simply connected domains, by taking an increasing union of domains with Jordan boundaries, but we will postpone this discussion until later, as we will postpone the discussion of harmonic measure on multiply connected domains (defined via covering maps).

For the moment, Jordan domains and locally connected sets will provide sufficiently many interesting examples.

We want estimate harmonic measure in terms of extremal length. We have already seen how to relate extremal length to logarithmic capacity, and the following relates the latter to harmonic measure:

Lemma 2.29. *For any compact $E \subset \mathbb{T}$,*

$$\operatorname{cap}(E) \geq \frac{1}{1 + \log 2 + \pi + \log \frac{1}{|E|}}.$$

If $E \subset \mathbb{T}$ has positive Lebesgue measure, then it has positive capacity. So, if $E \subset \mathbb{T}$ is an arc, then

$$\operatorname{cap}(E) \leq \frac{1}{\log 4 + \log \frac{1}{|E|}}.$$

For arcs of small measure, the two bounds are comparable.

Proof. Let μ be Lebesgue measure restricted to E and let $x \in E$. Let I be the arc centered at x and with length $|E|$. If $y \in \mathbb{T}$ and t is the arclength distance between x and y, then $\frac{2}{\pi}t \leq |x - y| \leq t$, so

$$\begin{aligned} U_\mu(x) &= \int_E \log \frac{2}{|x-y|} dy \\ &\leq \int_I \log \frac{1}{|x-y|} dy \\ &\leq 2 \int_0^{|E|/2} \log \frac{\pi dt}{2t} = |E| \log \frac{2}{|E|} + \pi |E| \end{aligned}$$

Thus the log-capacity of E is at least

$$\|\mu\| / \sup U_\mu \leq |E| / |E| \log \frac{2}{|E|} + \pi |E| = \frac{1}{\pi + \log 2 + \log 1/|E|}$$

If E is an arc, then the center x of the arc is at most distance $|E|/2$ from any other point of the arc, and so

$$U_\mu(x) \geq \log \frac{2}{|E|/2} = \log \frac{4}{|E|} = \log \frac{1}{|E|} + \log 4,$$

for any probability measure supported on E. This gives the desired estimate. $\square$

The following is the fundamental estimate for harmonic measure, from which all other estimates flow (at least, all the ones that we will use).

Theorem 2.30. *Suppose Ω is a Jordan domain, $z_0 \in \Omega$ with $\mathrm{dist}(z_0, \partial\Omega) \geq 1$ and $E \subset \partial\Omega$. Let Γ be the family of curves in Ω which connects $D(z_0, 1/2)$ to E. Then*

$$\omega(z_0, E, \Omega) \leq C \exp(-\pi\lambda(\Gamma)).$$

If $E \subset \partial\Omega$ is an arc then the two sides are comparable.

Proof. Let $f : \mathbb{D} \to \Omega$ be conformal. By Koebe's $\frac{1}{4}$-theorem (Theorem 2.11), the disk $D(z, \frac{1}{2})$ in Ω maps to a smooth region K in the unit disk that contains the origin, and ∂K is uniformly bounded away from both the origin and the unit circle.

Thus by Pfluger's theorem applied to the curve family Γ_X connecting K and the compact set $X = f^{-1}(E)$,

$$\frac{1}{\operatorname{cap}(X)} + C_1(K) \leq \pi\lambda(\Gamma_X) \leq \frac{1}{\operatorname{cap}(X)} + C_2(K),$$

for constants C_1, C_2 that are bounded independent of all our choices.

By Lemma 2.29 the right-hand side of

$$1+\log 4+\log\frac{1}{|X|}+C_1(K)\leq \pi\lambda(\Gamma_X)\leq 1+\log 2+\log\frac{1}{|X|}+C_2(K).$$

holds in general, and the left-hand side also holds if X is an interval.

Multiply by -1 and exponentiate to get

$$\frac{|X|}{2e^{1+\pi+C_2}}\leq \exp(-\pi\lambda(\Gamma_X))\leq \frac{|X|}{4e^{C_1}}$$

under the same assumptions. Now use $\omega(z,E,\Omega)=\omega(0,X,\mathbb{D})=|X|/2\pi$ to deduce the result. $\square$

Corollary 2.31 (Ahlfors distortion theorem)**.** *Suppose* Ω *is a Jordan domain,* $z_0 \in \Omega$ *with* $\operatorname{dist}(z_0, \partial\Omega) \geq 1$ *and* $x \in \partial\Omega$*. For each* $0 < t < 1$ *let* $\ell(t)$ *be the length of* $\Omega \cap \{|w - x| = t\}$*. Then there is an absolute* $C < \infty$*, so that*

$$\omega(z_0, D(x, r), \Omega) \leq C \exp\left(-\pi \int_r^1 \frac{dt}{\ell(t)}\right).$$

Proof. Let K be the disk of radius $1/2$ around z_0 and let Γ be the family of curves in Ω which connects $D(x,r) \cap \partial\Omega$ to K.

Define a metric ρ by $\rho(z) = 1/\ell(t)$ if $z \in C_t = \{z \in \Omega : |x - z| = t\}$ and $\ell(t)$ is the length of C_t.

Any curve $\gamma \in \Gamma$ has ρ-length at least

$$L = \int_r^{1/2} \frac{dt}{\ell(t)},$$

and

$$A = \iint_\Omega \rho^2 dx dy \geq \int_r^{1/2} \int_{C_r \cap \Omega} \ell(z)^{-2} r dr d\theta = \int \ell(z)^{-1} dr = L.$$

Therefore $\lambda(\Gamma) \geq A/L^2 = 1/L$, and this proves the result. $\square$

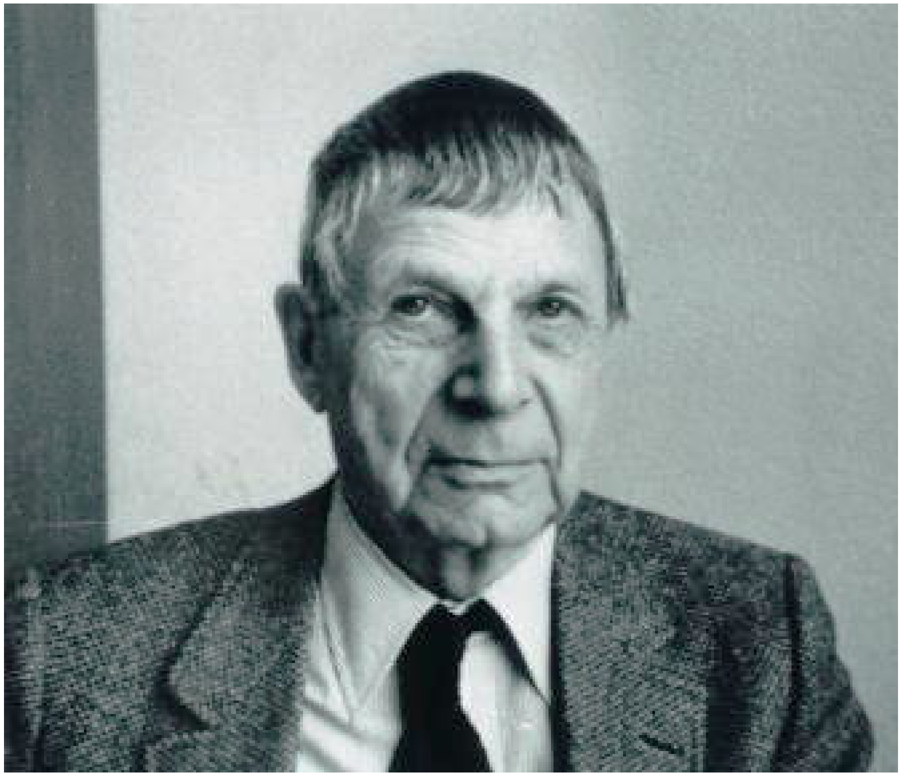
Corollary 2.32 (Beurling's estimate). *There is a $C < \infty$ so that if Ω is simply connected, $z \in \Omega$ and $d = \operatorname{dist}(z, \partial\Omega)$ then for any $0 < r < 1$ and any $x \in \partial\Omega$,*

$$\omega(z, D(x, rd), \Omega) \leq C r^{1/2}$$

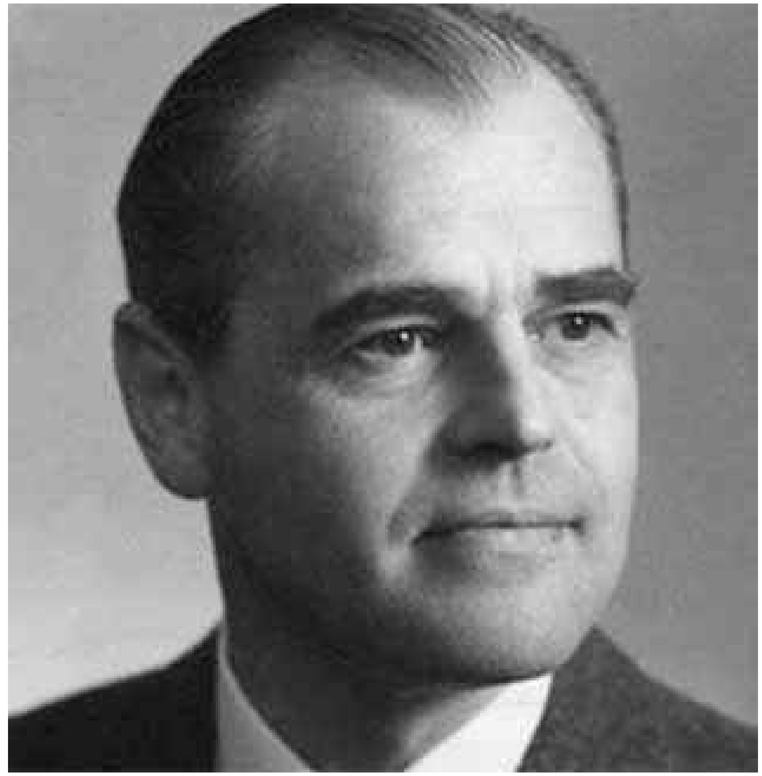
Proof. Apply Corollary 2.31 at x and use $\theta(t) \leq 2\pi t$ to get

$$\exp\left(-\pi \int_{rd}^{d} \frac{dt}{\theta(t)t}\right) \leq C \exp\left(-\frac{1}{2}\log r\right) \leq C\sqrt{r}.$$

$\square$



Lars Alhfors



Arne Beurling

Corollary 2.33. *There is an $R < \infty$ so that for any Ω is a Jordan domain and any $z \in \Omega$*

$$\omega\left(z, \partial\Omega \setminus D(z, R \cdot \operatorname{dist}(z, \partial\Omega), \Omega\right) \leq 1/2.$$

Proof. Rescale so $z = 1$ and $\operatorname{dist}(z, \partial\Omega) = 1$. Then apply $w \to 1/w$ which fixes z and maps $\partial\Omega \setminus D(z, R)$ into $D(0, 1/R - 1)$. Then Lemma 2.32 implies the result holds if $R \geq 4C^2 + 1$ (and C is as in Lemma 2.32). $\square$

Corollary 2.34. *For any Jordan domain and any* $\epsilon \in (0,1)$,

$$\omega(z, \partial\Omega \cap D(z, (1+\epsilon)\mathrm{dist}(z, \partial\Omega)), \Omega) > C\epsilon,$$

for some fixed $C > 0$.

Proof. Renormalize so $z = 0$ and 1 is a closest point of $\partial\Omega$ to z. By Corollary 2.33, the set $E = \partial\Omega \cap D(0, 1+\epsilon)$ has harmonic measure at least $1/2$ from the point $1 - \epsilon/R$ (R is as in Corollary 2.33).

Since $\omega(z, E, \Omega)$ is a positive, harmonic function on $\mathbb{D}$, Harnack's inequality says that at the origin it is larger than

$$\frac{1}{2} \cdot \frac{1 - (1 - \epsilon/R)}{1 + (1 - \epsilon/R)} \simeq \epsilon/R. \quad \square$$

This is a weak version of the Beurling projection theorem which says that the sharp lower bound is given by the slit disk $D(0, 1+\epsilon) \setminus [1, 1+\epsilon)$.

The harmonic measure of the slit in this case can be computed as an explicit function of ϵ because this domain can be mapped to the disk by sequence of elementary functions.

Theorem 2.35. *Suppose Ω is a Jordan domain and $E \subset \partial\Omega$ has zero $\frac{1}{2}$-Hausdorff measure. Then E has zero harmonic measure in Ω.*

Proof. Since dilations do not change dimension or harmonic measure, we can rescale so that Ω contains a unit disk centered at some point z. It suffices to show E has harmonic measure zero with respect to z.

The hypothesis means that for any $\epsilon > 0$, the set E can be covered by open disks $\{D(x_j, r_j)\}$ that satisfy $\sum_j r_j^{1/2} \leq \epsilon$. By Beurling's estimate, this implies

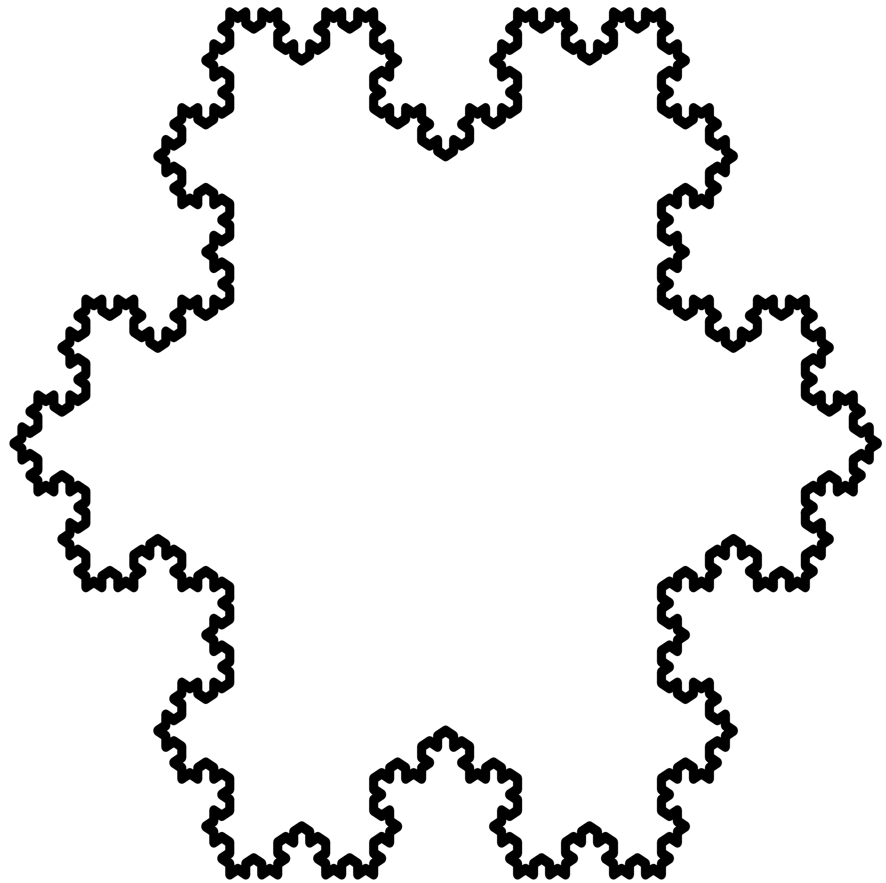
$$\omega(z, E, \Omega) \leq \sum_j \omega(z, D_j, \Omega) \leq O(\sum_j r_j^{1/2}) = O(\epsilon).$$

$\square$

This result was not improved until Lennart Carleson showed in a tour de force that the $\frac{1}{2}$ could be replaced by some $\alpha > \frac{1}{2}$ in

That result was not improved until Makarov showed it holds for all $\alpha < 1$.

Even though we have not defined harmonic measure for multiply connected domains, it is clear that no analog is possible in that case: if the boundary of Ω is a Cantor set of dimension α, then it must have full harmonic measure, even if α is small.



A famous result of Peter Jones and Tom Wolff says that harmonic measures gives full mass to a set of dimension at most 1 for any planar domain.

One might think that this holds for domain in $\mathbb{R}^n$ with bound $n-1$, but Wolff found a counterexample (Wolff snowflakes).

Currently active area of research in higher dimensions.

We recall a result from real analysis.

Theorem 2.36 (Vitali Covering Lemma)**.** *Suppose $E \subset \mathbb{R}^d$ is a measurable set and $\mathcal{B} = \{B_j\} \subset \mathbb{R}^d$ is a collection of balls so that each point of E is contained in elements of $\mathcal{B}$ of arbitrarily small diameter. Then there is a subcollection $\mathcal{C} \subset \mathcal{B}$ so that $E \setminus \cup_{B \in \mathcal{C}} B$ has zero d-measure.*

For a proof see Folland's textbook.

Corollary 2.37. *If* Ω *is Jordan domain, then harmonic measure is singular to area measure.*

Proof. By the Lebesgue density theorem, at Lebesgue almost every point z of a set E of positive area, all small enough disks satisfy

$$\text{area}(E \cap D(z,r)) \geq (1-\epsilon)\text{area}(D(z,r)), \text{ for all } .$$

In particular we must have $\theta(t) \leq \frac{\epsilon}{t}$ (angle measure of $\Omega \cap \{|w-z| = t\}$) on a set z of measure at least $r/4$ in $[r/2, r]$.

Thus by the Ahlfors distortion theorem

$$\omega(D(z, r_0 2^{-n}) \leq C \exp\left(-\pi \int_{2^{-n}r_0}^{r_0} \frac{dt}{\epsilon t}\right) \leq C 2^{-\pi n/\epsilon}.$$

This is much less than $(2^{-n}r_0)$ if n is large. Thus almost every point of $\partial\Omega$ can be covered by arbitrarily small disks so that $\omega(D(z_j, r_j)) = o(r_j^2)$.

Use Vitali's theorem to take a disjoint cover of a set of full harmonic measure, and we deduce that harmonic measure gives full mass to set of zero area. $\square$

Jean Bourgain proved this holds for general domains in higher dimensions.

Even stronger, he showed there is always a with Hausdorff dimension $\leq n - \delta_n$ that has full harmonic measure and $\delta_n > 0$ only depends on n.

Some small gaps in his proof were noticed and filled by Badger and Genschaw in Lower bounds on Bourgain's constant for harmonic measure. In $\mathbb{R}^3$, they show that harmonic measure has dimension at most

$$2.99999999999999$$

it is natural to conjecture $\delta_n = 1$ for all n, but Tom Wolff showed that for domains in $\mathbb{R}^n$ harmonic measure can have dimension either $> n-1$ or $< n-1$.

It is conjectured that the upper bound is $\delta_n = 1 - (n-2)/(n-1)$.

Jean Bourgain

Tom Wolff

This semester I hope to cover the following topics:

- Review of conformal mappings
- Extremal length and conformal modulus, log capacity, harmonic measure
- Definitions of quasiconformal mappings; geometric and analytic
- Basic properties
- Quasisymmetric maps and boundary extension
- The measurable Riemann mapping theorem
- Removable sets
- Conformal welding
- David maps
- Astala's theorems on area and dimension distortion
- Quasiconformal maps on metric spaces
- Conformal dimension

Some Linear Algebra (QC linear maps)

Conformal maps preserves angles; quasiconformal maps can distort angles, but only in a controlled way.

To make this distinction more precise we must have a way to measure angle distortion and we start with a discussion of linear maps.

Consider the linear map

$$\begin{pmatrix} x \\ y \end{pmatrix} \to M \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (ax + by, cx + dy).$$

Let M^T denote the transpose of the real matrix M, i.e., its reflection over the main diagonal. Then

$$M^T \cdot M = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix} \equiv \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

is positive and symmetric and hence has two positive eigenvalues λ_1, λ_2, assuming M in non-degenerate.

The square roots $s_1 = \sqrt{\lambda_1}$, $s_2 = \sqrt{\lambda_2}$ are the singular values of A (without loss of generality we assume $s_1 \geq s_2$). Then

$$M = U \cdot \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} \cdot V,$$

where U, V are rotations.

Thus M maps the unit circle to an ellipse whose major and minor axes have length s_1 and s_2.

Thus M preserves angles iff it maps the unit circle to a circle iff $s_1 = s_2$. Otherwise M distorts angles and we let $D = s_1/s_2$ denote the dilatation of the linear map M. This is the eccentricity of the image ellipse and is ≥ 1, with equality iff M conformal.

The inverse of a linear map with singular values $\{s_1, s_2\}$ has singular values $\{\frac{1}{s_2}, \frac{1}{s_1}\}$ and hence dilatation $D = (1/s_2)/(1/s_1) = s_1/s_2$. Thus the dilatation of a linear map and its inverse are the same.

Given two linear maps M, N with singular values $s_1 \geq s_2$ and $t_1 \geq t_2$ respectively, the singular values of the composition MN are trapped between $s_1 t_1$ and $s_2 t_2$ (this occurs for the maximum singular values since they give the operator norms of the matrices and these are multiplicative; a similar argument works for the minimum singular values and the inverse maps).

Thus the dilation is less than $(s_1 t_1)/(s_2 t_2)$ i.e., dilatations satisfy

$$D_{M \circ N} \leq D_M \cdot D_N.$$

The dilatation D can be computed in terms of a, b, c, d as follows.

The eigenvalues λ_1, λ_2 are roots of the

$$0 = \det(M^T \cdot M - \lambda I),$$

which is the same as

$$0 = (E - \lambda)(G - \lambda) - F^2 = EG - F^2 - (E + G)\lambda + \lambda^2.$$

Thus

$$\begin{aligned}
\lambda_1\lambda_2 &= EG - F^2 \\
&= (a^2 + c^2)(b^2 + d^2) - (ab + cd)^2 \\
&= a^2b^2 + a^2d^2 + c^2b^2 + d^2c^2 - (a^2b^2 + 2abcd + c^2d^2) \\
&= a^2d^2 + c^2b^2 - 2abcd \\
&= (ad - bc)^2
\end{aligned}$$

Similarly,

$$\lambda_1 + \lambda_2 = E + G = a^2 + b^2 + c^2 + d^2.$$

The values of λ_1, λ_2 can be found using the quadratic formula:

$$\begin{aligned}\{\lambda_1, \lambda_2\} &= \frac{1}{2}[E + G \pm \sqrt{(E+G)^2 - 4(EG - F^2)}] \\ &= \frac{1}{2}[E + G \pm \sqrt{(E-G)^2 + 4F^2)}].\end{aligned}$$

Thus

$$\begin{aligned}\frac{\lambda_1}{\lambda_2} &= \frac{E + G + \sqrt{(E-G)^2 + 4F^2}}{E + G - \sqrt{(E-G)^2 + 4F^2}} \\ &= \frac{(E + G + \sqrt{(E-G)^2 + 4F^2})^2}{(E+G)^2 - (E-G)^2 - 4F^2} \\ &= \frac{(E + G + \sqrt{(E-G)^2 + 4F^2})^2}{4(EG + F^2)}.\end{aligned}$$

and hence

$$D = \frac{s_1}{s_2} = \sqrt{\frac{\lambda_1}{\lambda_2}} = \frac{E + G + \sqrt{(E-G)^2 + 4F^2}}{2\sqrt{EG + F^2}}.$$

This formula can be made simpler by complexifying.

Think of the linear map M on $\mathbb{R}^2$ as a map f on $\mathbb{C}$:

$$x + iy \to ax + by + i(cx + dy) = u(x, y) + iv(x, y) = f(x + iy)$$

Then

$$M = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

and we define

$$f_z = \frac{1}{2}(f_x - if_y) = \frac{1}{2}(u_x + v_y) + \frac{i}{2}(v_x - u_y),$$
$$f_{\overline{z}} = \frac{1}{2}(f_x + if_y) = \frac{1}{2}(u_x - v_y) + \frac{i}{2}(v_x + u_y).$$

Some tedious arithmetic now shows that

$$\begin{aligned} 4|f_z|^2 &= (u_x + v_y)^2 + (v_x - u_y)^2 \\ &= u_x^2 + 2u_x v_y + v_y^2 + v_x^2 - 2v_x u_y + u_y^2 \end{aligned}$$

$$\begin{aligned} 4|f_{\overline{z}}|^2 &= (u_x - v_y)^2 + (v_x + u_y)^2 \\ &= u_x^2 - 2u_x v_y + v_y^2 + v_x^2 + 2v_x u_y + u_y^2 \end{aligned}$$

so

$$(|f_z| + |f_{\overline{z}}|)(|f_z| - |f_{\overline{z}}|) = |f_z|^2 - |f_{\overline{z}}|^2 = u_x v_y - v_x u_y = s_1 s_2 = \det(M).$$

In particular, if we assume M is orientation preserving and full rank, then $\det(M) > 0$ and we deduce $|f_z| > |f_{\overline{z}}|$.

Similarly,

$$\begin{aligned}(|f_z| + |f_{\overline{z}}|)^2 + (|f_z| - |f_{\overline{z}}|)^2 &= 2(|f_z|^2 + |f_{\overline{z}}|^2) \\ &= u_x^2 + v_x^2 + u_y^2 + v_x^2 \\ &= E + G \\ &= \lambda_1 + \lambda_2 \\ &= s_1^2 + s_2^2.\end{aligned}$$

From these equations and the facts $s_1 \geq s_2$, $|f_z| > |f_{\overline{z}}|$ we can deduce

$$s_1 = |f_z| + |f_{\overline{z}}|, \qquad s_2 = |f_z| - |f_{\overline{z}}|,$$

and hence

$$D = \frac{s_1}{s_2} = \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|}.$$

Note that $D \geq 1$ with equality iff f is a conformal linear map. It is often more convenient to deal with the complex number,

$$\mu = \frac{f_{\overline{z}}}{f_z},$$

which is called the **complex dilatation**.

Sometimes we abuse notation and just call thus the dilatation, if the meaning is clear from context.

Since $|f_{\overline{z}}| < |f_z|$, we have $|\mu| < 1$ and it is easy to verify that

$$D = \frac{1 + |\mu|}{1 - |\mu|}, \qquad |\mu| = \frac{D - 1}{D + 1},$$

so that either D or $|\mu|$ can be used to measure the degree of non-conformality.

We leave it to the reader to check that the map

$$x + iy \to (ax + by) + i(cx + dy)$$

can also be written as

$$(z, \overline{z}) \to \alpha z + \beta \overline{z},$$

where $z = x + iy$, $\overline{z} = x - iy$ and $\alpha = \alpha_1 + i\alpha_2$, $\beta = \beta_1 + i\beta_2$, satisfy

$$\alpha_1 = \frac{a+d}{2}, \quad \alpha_2 = \frac{a-d}{2}, \quad \beta_1 = \frac{c-b}{2}, \quad \beta_2 = \frac{b+c}{2},$$

In this notation $\mu = \beta/\alpha$ and

$$D = \frac{|\beta| + |\alpha|}{|\alpha| - |\beta|}.$$

As noted above, the linear map f sends the unit circle to an ellipse of eccentricity D. What point on the circle is mapped furthest from the origin?

Since

$$s_1 = |f_z| + |f_{\overline{z}}|,$$

the maximum stretching is attained when $f_z z$ and $f_{\overline{z}} \overline{z}$ have the same argument, i.e., when

$$0 < \frac{f_z z}{f_{\overline{z}} \overline{z}} = \frac{z^2}{\mu |z|^2},$$

or

$$\arg(z) = \frac{1}{2} \arg(\mu),$$

Thus $|\mu|$ encodes the eccentricity of the ellipse and $\arg(\mu)$ encodes the direction of its major axis.

If we follow f by a conformal map g, then the same infinitesimal ellipse is mapped to a circle, so we must have $\mu_{g\circ f} = \mu_f$.

If f is preceded by a conformal map g, then the ellipse that is mapped to a circle is the original one rotated by $-\arg(g_z)$, so $\mu_{f\circ g} = (|g_z|/g_z)^2 \mu_f$.

To obtain the correct formula in general we need to do a little linear algebra. Consider the composition $g \circ f$ and let $w = f(z)$ so that the usual chain rule gives

$$(g \circ f)_z = (g_w \circ f) f_z + (g_{\overline{w}} \circ f)\overline{f}_z,$$
$$(g \circ f)_{\overline{z}} = (g_w \circ f) f_{\overline{z}} + (g_{\overline{w}} \circ f)\overline{f}_{\overline{z}}.$$

or in vector notation

$$\begin{pmatrix} (g \circ f)_z \\ (g \circ f)_z \end{pmatrix} = \begin{pmatrix} f_z & \overline{f}_z \\ f_{\overline{z}} & \overline{f}_{\overline{z}} \end{pmatrix} \begin{pmatrix} (g_w \circ f) \\ (g_{\overline{w}} \circ f) \end{pmatrix}$$

The determinate of the matrix is

$$f_z\overline{f}_{\overline{z}} - \overline{f}_z f_{\overline{z}} = f_z\overline{f_z} - \overline{f_{\overline{z}}}f_{\overline{z}} = |f_z|^2 - |f_{\overline{z}}|^2 = J,$$

which is the Jacobian of f, so by Cramer's Rule,

$$\begin{aligned}(g_w \circ f) &= \frac{1}{J}[(g\circ f)_z\overline{f}_{\overline{z}} - (g\circ f)_{\overline{z}}\overline{f}_z],\\ (g_{\overline{w}} \circ f) &= \frac{1}{J}[(g\circ f)_{\overline{z}} f_z - (g\circ f)_z f_{\overline{z}}],\end{aligned}$$

so

$$\mu_g \circ f \;=\; \frac{(g\circ f)_{\overline{z}} f_z - (g\circ f)_z f_{\overline{z}}}{(g\circ f)_z\overline{f}_{\overline{z}} - (g\circ f)_{\overline{z}}\overline{f}_z} = \frac{\mu_{g\circ f} f_z - f_{\overline{z}}}{\overline{f}_{\overline{z}} - \mu_{g\circ f}\overline{f}_z} = \frac{f_z}{\overline{f_z}} \cdot \frac{\mu_{g\circ f} - \mu_f}{1 - \mu_{g\circ f}\overline{\mu_f}}.$$

Now set $h = g \circ f$ or $g = h \circ f^{-1}$ to get

$$\mu_{h \circ f^{-1}} \circ f \;=\; \frac{f_z}{\overline{f_z}} \frac{\mu_h - \mu_f}{1 - \mu_h \overline{\mu_f}}.$$

Thus if h and f have the same dilatation μ, then $g = h \circ f^{-1}$ is conformal. We will need this in the case when h is more general than an homeomorphism.

Geometric Definition of Quasiconformal Maps

A quadrilateral Q is a Jordan domain with two specified disjoint closed arcs on the boundary. (Equivalently, four distinct points and a choice of opposite edges.)

By the Riemann mapping theorem and Caratheodory's theorem, there is a conformal map from Q to a $1 \times m$ rectangle that extends continuously to the boundary with the two marked arcs mapping to the two sides of length a.

The ratio $M = M(Q) = 1/m$ is called the modulus of the four distinct marked on the boundary and is uniquely determined by Q.

The conjugate of Q is the same domain but with the complementary arcs marked. Its modulus is clearly the reciprocal of Q's modulus.

The geometric definition: A homeomorphism h, defined on a planar domain Ω, is K-quasiconformal if the

$$\frac{1}{K}M(Q) \leq M(h(Q)) \leq KM(Q),$$

for every quadrilateral $Q \subset \Omega$.

The following is a helpful sufficient condition. Many of the maps we use in practice are of this form.

The piecewise differentiable definition: h is K-quasiconformal on Ω if there are countable many analytic curves whose union is a closed set Γ of Ω such that h is continuously differentiable on each connected component of $\Omega' = \Omega \setminus \Gamma$ and $D_h \leq K$ on Ω'.

First we check that the piecewise definition implies the geometric definition.

A major goal for later is to replace piecewise differentiability with almost everywhere differentiability, but this requires some extra regularity assumptions.

Lemma 4.1. *Suppose h a homeomorphism of Ω such that there are countable many analytic curves whose union is a closed set Γ of Ω and h is continuously differentiable on each connected component of $\Omega' = \Omega \setminus \Gamma$ and $D_h \leq K$ on Ω'. Then h is K-quasiconformal.*

Proof. Using conformal maps, it suffices to consider the case when Ω and its image are both rectangles, say $\Omega = [0, a] \times [0, 1]$ and $h(\Omega) = [1, b] \times [0, 1]$.

By integrating over horizontal lines in the first rectangle, we see

$$b \leq \int_0^a (|f_z| + |f_{\overline{z}}|)dx.$$

We have used the piecewise analytic assumption here to break the integral into finitely many open segments where the fundamental theorem of calculus applies and then use the assumption that h is continuous at the endpoints to say the total integral is the sum of these sub-integrals.

Fact: if f continuous on $[a, b]$ and f' is continuous and bounded except at finitely many points, then $f(x) = \int_a^x f'(t)dt$.

Integrating in the other variable,

$$b \leq \int_0^1 \int_0^a (|f_z| + |f_{\overline{z}}|) dx dy.$$

By Cauchy-Schwarz,

$$\begin{aligned}
b^2 &\leq \left(\int_0^1 \int_0^a (|f_z| + |f_{\overline{z}}|)(|f_z| - |f_{\overline{z}}|) dx dy\right)\left(\int_0^1 \int_0^a \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|} dx dy\right) \\
&\leq \left(\int_0^1 \int_0^a (|f_z|^2 - |f_{\overline{z}}|^2) dx dy\right)\left(\int_0^1 \int_0^a \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|} dx dy\right) \\
&\leq \left(\int_0^1 \int_0^a J_f dx dy\right)\left(\int_0^1 \int_0^a D_f dx dy\right) \\
&\leq baK,
\end{aligned}$$

and so $b \leq Ka$. The other direction follows by repeating the argument for vertical lines instead of horizontal ones. $\square$

In order for the proof to work we need two things:

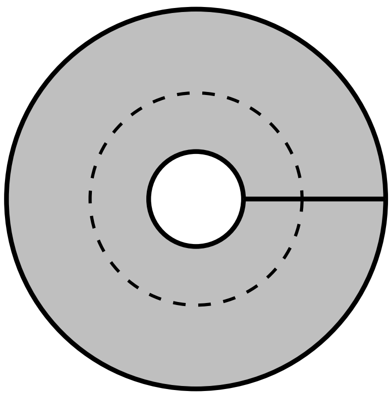
(1) the area of the range to be bounded above by integrating the Jacobian over the domain and,

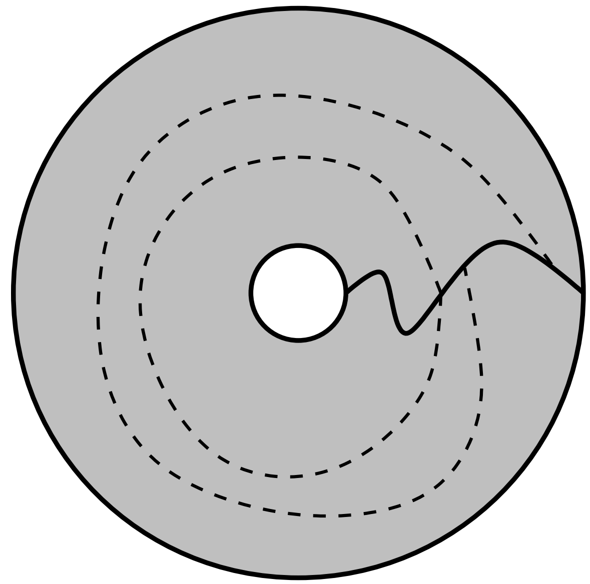
(2) each horizontal line segment S to have an image whose length is bounded above by the integral of $|f_z| + |f_{\overline{z}}|$ over S.

These certainly hold if f_z and $f_{\overline{z}}$ are piecewise continuous on a partition of the plane given by countable many analytic curves, as we have assumed, but it holds much more generally.

The geometric definition of quasiconformality actually implies that the map h has partials almost everywhere and is absolutely continuous on almost every line. This, in turn, implies the necessary estimates holds. This will be discussed later.

Corollary 4.2. *If we have a piecewise differentiable K-quasiconformal map f between annuli $A_r = \{1 < |z| < r\}$ and $A_R = \{1 < |z| < R\}$ with dilatation $\leq K$, then $\frac{1}{K} \log r \leq \log R \leq K \log r$.*





Proof. Slit A_r with $[1, r]$ to get a quadrilateral $Q \subset A_r$ and let $Q' = f(Q) \subset A_R$.

Then $M(A_R) \leq M(Q') \leq KM(Q) = M(A_r)$.

The first inequality occurs because of monotonicity of modulus (Lemma 2.2); every separating curve for the annulus connects opposite sides of Q' (but there are connecting curves that don't correspond to closed loops).

The other direction follows by considering the inverse map. $\square$

Theorem 4.3. *There is no quasiconformal map between the plane and the disk.*

Proof. Suppose $f : \mathbb{D} \to \mathbb{C}$ were a K-quasiconformal map. We may assume $f(0) = 0$.

Let $K = \{|z| \leq 1/2\}$. The modulus of the annulus $A = \mathbb{D} \setminus K$ is finite and non-zero (indeed equals $(\log 2)/2\pi$, but since $f(K)$ is compact, the topologoical annulus $\mathbb{C} \setminus f(K)$ contain the round annuli $A_R = \{\mathrm{diam}(f(K)) < |z| < R\mathrm{diam}(f(K))\}$ for any $R > 1$.

But by monotonicity

$$(\log R)/2\pi = \quad \mathrm{mod}\ (A_R) \leq \mathrm{Mod}(f(A)) \leq K\mathrm{Mod}(A) < \infty.$$

This is a contradiction for large R and shows there is no such map f. $\square$

Compactness of K-quasiconformal maps

Theorem 10.5, Arzela-Ascoli Theorem: *A family $\mathcal{F}$ of continuous functions is normal on a region $\Omega \subset \mathbb{C}$ if and only if*

(1) $\mathcal{F}$ is equicontinuous on Ω, and

(2) there is a $z_0 \in \Omega$ so that the collection $\{f(z_0) : f \in \mathcal{F}\}$ is a bounded subset of $\mathbb{C}$.

This result is usually proven in MAT 532 (Chap 4 of Folland's book).

We want to verify K-quasiconformal maps satisfy the Arzela-Ascoli theorem.

Lemma 4.4. *Suppose $\Omega \subset \mathbb{C}$ is open and simply connected and $D \subset \Omega$ is a topological closed disk. If f is K-quasiconformal on Ω and $x, y, z \in D$ with $|x - y| \leq |x - z|$. Then*

$$|f(x) - f(y)| \leq M|f(z) - f(y)|,$$

where M depends on Ω, D and K, but not on x, y or z.

Proof. After renormalizing by conformal linear maps we may assume $y = f(y) = 0$ and $z = f(z) = 1$.

Then x is in the half-plane H that lies to the left of the bisector of 0 and 1 and it suffices to show that $|f(x)|$ is bounded depending only on K, D and Ω.

Connect 1 to $\partial\Omega$ by a real segment $\sigma \subset \Omega \cap \mathbb{R}$; then $D \setminus \sigma$ is connected and there is an $\epsilon > 0$ so that and 0 can be connected to any point of $H \cap D$ by a path in D that is at least distance ϵ from σ.

Connect 0 to x by such a curve γ. Then $A = \Omega \setminus (\gamma \cup \sigma)$ is a topological annulus and $\rho = 1/\epsilon$ on $\{|z| \leq \epsilon + \mathrm{diam}(D)\}$ is admissible for the path family connecting γ and σ in Ω.

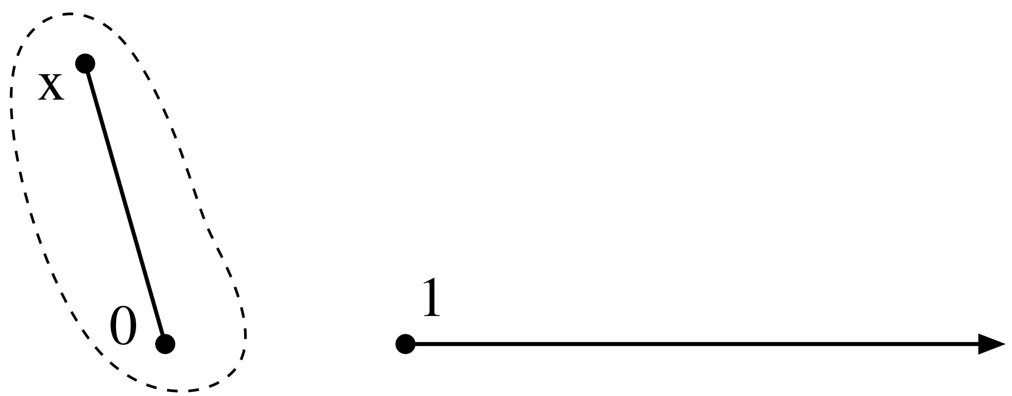
Therefore the modulus of A, which is the modulus of the family separating the two curves is greater than $\epsilon^2/(\epsilon + \mathrm{diam}(D))^2 > 0$.

Moreover, the modulus of A differs by at most a factor of K from the modulus of $B = F(A)$. However, if $|f(x)| \gg 1$, then by considering the metric $\rho(z) = 1/|z|$ on the annulus $\{z : 1 < |z| < |f(x)|\}$, we see that B has modulus tending to zero as $|f(x)| \nearrow \infty$.

Thus $|f(x)|$ is bounded in terms of K and the modulus of A, which, in turn, depends only on D and Ω. $\square$

Corollary 4.5. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is a K-quasiconformal map that fixes both 0 and 1. Then $|f(x)|$ is bounded with an estimate depending on $|x|$ and K, but not on f.*

Proof. Take $\Omega = \mathbb{C}$ and $D = \{|z| < |x| + 1\}$ in Lemma 4.4. $\square$



Lemma 4.6. *Suppose* $\Omega \subset \mathbb{C}$ *is a topological annulus of modulus* M *whose boundary consists of two Jordan curves* γ_1, γ_2 *with* γ_2 *separating* γ_1 *from* ∞. *Then* $\operatorname{diam}(\gamma_1) \leq (1-\epsilon)\operatorname{diam}(\gamma_2)$ *where* $\epsilon > 0$ *depends only on* M.

Proof. Rescale so $\mathrm{diam}(\gamma_2) = \mathrm{diam}(\Omega) = 1$ and suppose $\mathrm{diam}(\gamma_1) > 1 - \epsilon$.

Then there are points $a \in \gamma_1$ and $b \in \gamma_2$ with $|a - b| \leq \epsilon$. Let ρ be the metric on Ω defined by $\rho(z) = \frac{1}{|z-a| \log(1/2\epsilon)}$ for $\epsilon < |z - a| < 1/2$.

Then any curve $\gamma \subset \Omega$ that separates γ_1 and γ_2 satisfies $\int_\gamma \rho ds \geq 1$ and

$$\int \rho^2 dxdy \leq \frac{\pi}{4} \log^{-2} \frac{1}{2\epsilon}.$$

Thus the modulus of the path family separating the boundary components is bounded above by the right hand side, and the modulus of the reciprocal family connecting the boundary components is bounded below by $\frac{\pi}{4} \log^2 \frac{1}{2\epsilon}$.

Thus $\epsilon \geq \frac{1}{2} \exp(-\sqrt{\pi M/4})$. $\square$

A function f is **α-Hölder continuous** on a set E if there is a $C < \infty$ so that

$$|f(x) - f(y)| \leq C|x - y|^{\alpha},$$

for all $x, y \in E$.

We say f is Hölder continuous on E if this holds for some $\alpha > 0$.

We say f is locally α-Hölder on an open set Ω if each point of Ω has a neighborhood on which f is α-Hölder. This implies that f is α-Hölder on any compact set of Ω, although the multiplicative constant may depend on the set.

f is bi-Hölder if both f and f^{-1} are Hölder.

Theorem 4.7. *A K-quasiconformal map of an open set Ω is locally α-Hölder continuous for some $\alpha > 0$ that only depends on K.*

Later we will compute the actual Hölder exponent as $\alpha = 1/K$.

Proof. It is enough to show that f is Hölder on any disk D so that $3D \subset \Omega$.

Without loss of generality, assume $D = D(0, r)$, $f(0) = 0$ and $x, y \in D(0, r)$.

By Lemma 4.4, $D(0, 2r)$ is mapped into $D(0, R)$ for some $R = R(r, K)$. Surround $\{x, y\}$ by $N = \lfloor \log_2 \frac{r}{|x-y|} \rfloor$ annuli $\{A_j\}$ of modulus $\log 2$.

The image annuli $\{f(A_j)\}$ have moduli bounded away from zero, and hence $\mathrm{diam}(f(A_{j+1})) \leq (1-\epsilon)\mathrm{diam}(f(A_j))$ by Lemma 4.6. Therefore

$$\begin{aligned} |f(x) - f(y)| &\leq R(1-\epsilon)^N \leq R2^{\log_2(1-\epsilon)(1+\log_2 R - \log_2 |x-y|)} \\ &\leq C(R)|x-y|^{\log_2(1-\epsilon)}. \quad \square \end{aligned}$$

We want to show that K-quasiconformal maps have continuous boundary extensions.

This essentially follows from the fact they are Hölder continuous, but our proof of that fact is only local and may give a multiplicative constant that blows up as we approach the boundary.

We will prove that this does not happen if the boundary itself is nice enough, e.g., a circle:

Theorem 4.8. *If $\varphi : \mathbb{D} \to \mathbb{D}$ is quasiconformal and onto, then φ is α-Hölder on $\mathbb{D}$, where $\alpha > 0$ only depends on K. Thus φ extends continuously to a homeomorphism of $\mathbb{T} = \partial\mathbb{D}$ to itself.*

The proof is very similar to the Hölder estimates for quasiconformal maps in the plane, however, we will also need a trick for converting certain quadrilaterals in the disk into annuli in the plane by reflecting across the circle. The precise statement is:

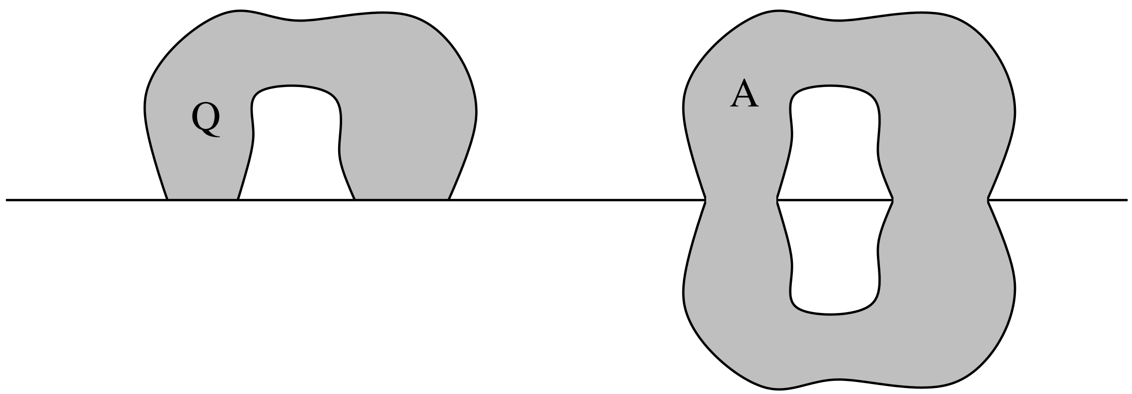
Lemma 4.9. *Suppose $Q \subset \mathbb{H}$ is a quadrilateral with a pair of opposite sides being intervals $I, J \subset \mathbb{R}$. Let A be the topological annulus formed by taking $Q \cup I \cup J \cup Q^*$ (where Q^* is the reflection of Q across $\mathbb{R}$. Then $M(A) = \frac{1}{2}M(Q)$ (here the modulus of Q refers to the modulus of the path family connecting the two sides of Q that line on the unit circle).*

Proof. Using conformal invariance, assume Q is in the upper half-plane and A is obtained by reflecting Q across the real line.

Consider the path family Γ_A in A that connects the two boundary components of A, and the path family Γ_Q in Q that separate the boundary arcs $Q \cap \mathbb{R}$. Then $(\Gamma_A)_+ = \Gamma_Q$ (notation as in Lemma 2.9), so by the Symmetry Rule

$$M(\Gamma_A) = 2M((\Gamma_A)_+) = 2M(\Gamma_Q).$$

The desired moduli are the reciprocals of these, so the result follows. $\square$



Proof of Theorem 4.8. We may assume $f(0) = 0$; the general case then follows after composing with a Möbius transformation.

We first suppose φ extends continuously to the boundary. This may seem a bit circular given the final statement of the theorem, but our plan is to prove φ is $\alpha(K)$-Hölder for assuming continuity, and then use a limiting argument to remove the continuity assumption.

More precisely, suppose $w, z \in \mathbb{D}$. We will show that

$$|\varphi(z) - \varphi(w)| \leq C|z - w|^{\alpha},$$

for constants $C < \infty$, $\alpha > 0$ that depend only on the quasiconstant K of f. This implies f is uniformly continuous and hence has a continuous extension to the boundary of $\mathbb{D}$.

Let $d = |z - w|$ and $r = \min(1 - |z|, 1 - |w|)$. There are several cases depending on the positions of the points z, w and the relative sizes of d and r.

To start, note that if $|z-w| \geq \frac{1}{10}$ we can just take $C = 20$ and $\alpha = 1$. So from here on, we assume $|z-w| < 1/10$.

Suppose $r > 1/4$, so $z, w \in \frac{3}{4}\mathbb{D}$. Surround the segment $[z, w]$ by $N \simeq \log d$ annuli with moduli $\simeq 1$. Then just as in the proof of Theorem 4.7, the image annuli have moduli $\simeq 1$ (with a constant depending on K) and hence

$$|f(z) - f(w)| \leq (1 - \epsilon(K))^N = O(|z-w|^\alpha),$$

for some $\alpha > 0$ depending only on K.

Next suppose $|z| \geq 3/4$ and $d > r$. Then separate $[z, w]$ from 0 by $N \simeq \log d$ disjoint quadrilaterals with a pair of opposite sides being arcs of $\mathbb{T}$, and all with moduli $\simeq 1$. Since $f(0) = 0$ and the image quadrilaterals have moduli $\simeq 1$, there diameters shrink geometrically, so

$$|z - w| = (1 - \epsilon(K))^N = O(d^\alpha),$$

as desired.

Finally, if $d \leq r$ we combine the two previous ideas: we start by separating $[z, w]$ from 0 by $\simeq \log d$ quadrilaterals with as above.

The smallest quadrilateral then bounds a region of diameter approximately r containing $[z, w]$ and we then construct $\simeq \log r/d$ disjoint annuli with moduli $\simeq 1$ that each separate $[z, w]$ from this smallest quadrilateral.

The same arguments as before now show

$$|z-w| = (1-\epsilon(K))^{-\log r}(1-\epsilon(K))^{\log r/d} = O(d^\alpha) = O(|z-w|^\alpha).$$

This proves the theorem assuming φ extends continuously to the boundary. Now we have to remove this extra assumption. Assume φ is any K-quasiconformal of $\mathbb{D}$ onto itself, such that $\varphi(0) = 0$. Take r close to 1 and let $\Omega_r = \varphi(\{|z| < r\})$

Then Ω_r is a Jordan domain that satisfies

$$\{|z| < 1 - \delta\} \subset \Omega_r \subset \mathbb{D},$$

with $\delta \to 0$ as $r \nearrow 1$. Let $f_r : \Omega_r \to \mathbb{D}$ be the the conformal map so that $f_r(0) = 0$ and $f_r'(0) > 0$.

By Caratheodory's theorem f_r is a homeomorphism from the closure of Ω_r to the closed unit disk, hence the K-quasiconformal map $g_r = f_r \circ \varphi$ is a homeomorphism from the closed unit disk to itself. Thus the previous argument applies to g_r, and we deduce g_r is α-Hölder.

As $r \nearrow 1$, both f_r and f_r^{-1} tend to the identity on compact subsets of $\mathbb{D}$. In particular, for $z, w \in \mathbb{D}$, we have

$$\begin{aligned} |\varphi(z) - \varphi(w)| &= \lim_{r \nearrow 1} |f_r^{-1}(g_r(z)) - f_r^{-1}(g_r(w))| \\ &= \lim_{r \nearrow 1} |g_r(z) - g_r(w)| \\ &\leq C(K)|z - w|^{\alpha}. \end{aligned}$$

By the Schwarz Lemma $g_r(z)$ and $g_r(w)$ remain in a compact subset of $\mathbb{D}$ as $r \nearrow 1$.

Thus φ is α-Hölder as well. $\square$

We have now verified that normalized K-quasiconformal maps satisfy the Arzela-Ascoli theorem, so they form a pre-compact family. To prove compactness, we need to prove:

Theorem 4.10. *If $\{f_n\}$ is a sequence of K-quasiconformal maps on Ω that converge uniformly on compact subsets to a homeomorphism f, then f is K-quasiconformal.*

This is immediate from the following result (proven earlier):

Theorem 4.11. *Suppose $\{h_n\}$ are homeomorphisms defined on a domain Ω and $Q \subset \Omega$ is a generalized quadrilateral that is compactly contained in Ω. If $\{h_n\}$ converge uniformly on compact sets to a homeomorphism h on Ω, then $M(h_n(Q)) \to M(h(Q))$*

Proof of Theorem 4.10. Any quadrilateral $Q \subset \Omega$ has compact closure in Ω so $f(Q) = \lim_n f_n(Q)$ is a quadrilateral in $f(\Omega)$ and

$$M(f(Q)) = \lim_n M(f_n(Q)) \leq K \lim_n M(Q)$$

by Lemma 2.27. The opposite inequality follows by considering the inverse maps, so we see that f is K-quasiconformal. $\square$

Lemma 4.12. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is a K-quasiconformal map that fixes both 0 and 1. Then there is a constant $0 < C < \infty$, depending only on K so that if $|z| < 1/C$, then*

$$C^{-1}|z|^K \leq |f(z)| \leq C|z|^{1/K}.$$

Proof. Since normalized K-quasiconformal maps form a compact family, there here is a constant $A = A(K)$ so that

$$f(\{|z| = 1\}) \subset \{\frac{1}{A} < |z| < A\}.$$

By rescaling we also get that for any $0 < r < \infty$

$$f(\{|z| = r\}) \subset \{\frac{|f(r)|}{A} < |z| < A|f(r)|\}.$$

Thus if $r < A^{-2}$,

$$\{A|f(r)| < |z| < \frac{1}{A}\}\} \subset f(\{r < |z| < 1\}) \subset \{|f(r)/A < |z| < A\}\}.$$

Comparing moduli in the first inclusion we get

$$\frac{1}{2\pi}\log\frac{1}{A^2|f(r)|} \le M(f(\{r<|z|<1\})) \le \frac{K}{2\pi}\log\frac{1}{r},$$

which gives $|f(r)| \ge r^K/A^2$.

The second inclusion similarly gives

$$\frac{1}{2\pi}\log\frac{A^2}{|f(r)|} \ge M(f(\{r<|z|<1\})) \ge \frac{1}{2\pi K}\log\frac{1}{r},$$

which implies $|f(r)| \le A^2 r^{1/K}$. Taking $C = A^2$ proves the lemma. $\square$

Sharpness of the exponent $1/K$ can be proven using $z \to z \cdot |z|^{(1/K)-1}$.

Corollary 4.13. *For each $K \geq 1$ there is a $C = C(K) < \infty$ so that the following holds. If $f : \mathbb{C} \to \mathbb{C}$ is K-quasiconformal and γ is a circle, then there is $w \in \mathbb{C}$ and $r > 0$ so that $f(\gamma) \subset \{z : r \leq |z - w| \leq Cr\}$.*

Proof. Without loss of generality, we can pre and post-compose so that γ is the unit circle and f fixes $0, 1$. By Lemma 4.12, $f(\gamma)$ is then contained in an annulus $\{\frac{1}{C} \leq |z| \leq C\}$, and this gives the result. $\square$

The following is then immediate.

Corollary 4.14. *If f is a K-quasiconformal mapping of the plane and D is a disk, then* $\operatorname{diam}(f(D))^2 \simeq \operatorname{area}(f(D))$, *with constants that depend only on K.*

Quasiconformality is local

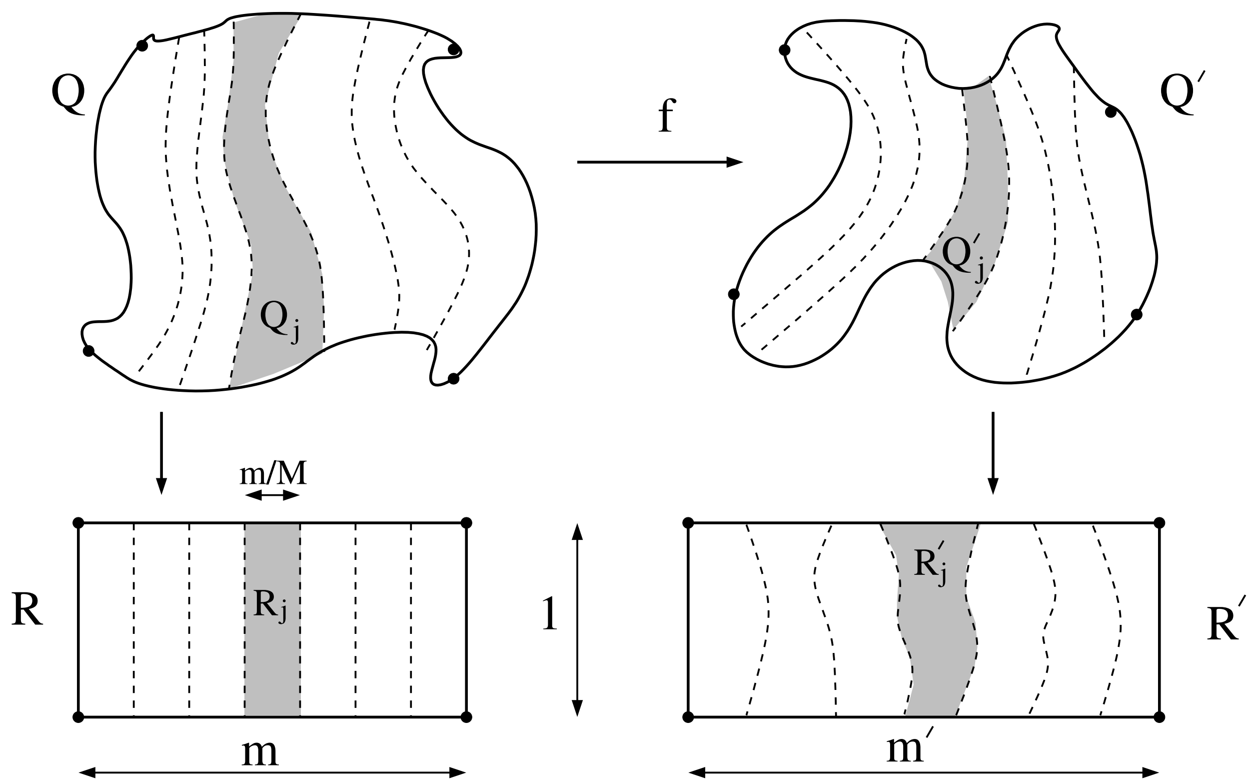
In the geometric definition of K-QC we have to consider all quadrilaterals in Ω, even those nearly as large as Ω.

The analytic definition requires only differentiability and absolute continuity, which are both local conditions.

In this section we prove that it is enough to verify the geometric definition just on all sufficiently small quadrilaterals.

Lemma 4.15. *If f is a homeomorphism of $\Omega \subset \mathbb{C}$ that is K-quasiconformal in a neighborhood of each point of Ω, then f is K-quasiconformal on Ω.*

Proof. Suppose $Q \subset \Omega$ is a quadrilateral that is conformally equivalent via a map φ to a $1 \times m$ rectangle R and $Q' = f(Q)$ is conformally equivalent a $1 \times m'$ rectangle R'. Divide R into M equal vertical strips $\{S_j\}$ of dimension $1 \times m/M$. Similarly, let $\psi : Q' \to R'$ be conformal.



We have to choose M sufficiently large that two things happen.

First choose $\delta > 0$ so that f^{-1} is K-quasiconformal on any disk of radius δ centered at any point of Q' (we can do this since Q' has compact closure in Ω).

Next, note that the closure of Q' is a union of Jordan arcs γ corresponding via $f \circ \varphi^{-1}$ to vertical line segments in R.

By the continuity of $f \circ \varphi^{-1}$ there is an $\eta > 0$ so that if $z \in R$ then $f(\varphi^{-1}(D(z,\eta)))$ has diameter $\leq \delta$.

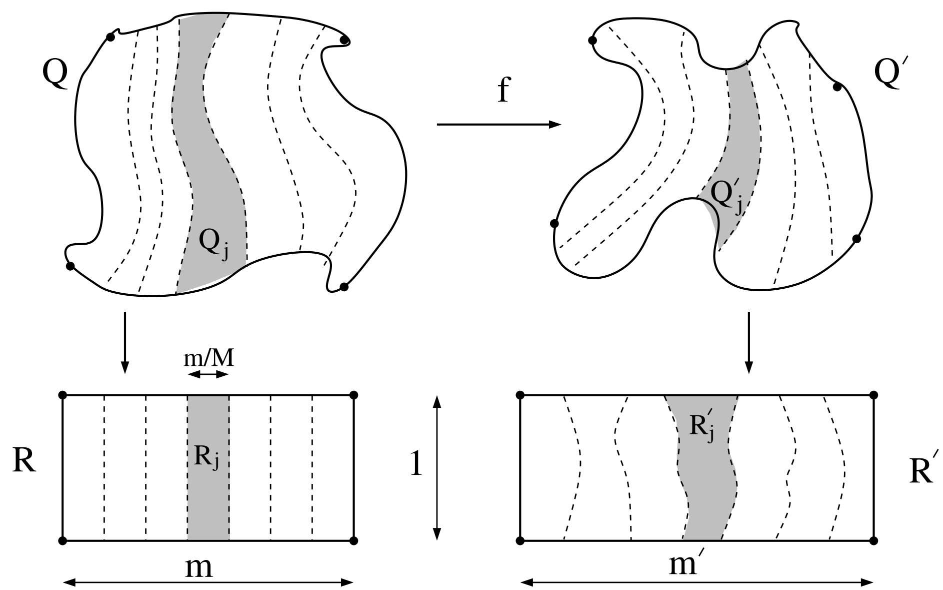
By the continuity of the inverse map, there is an $\epsilon > 0$ so that $x, y \in Q'$ and $|x - y| < \epsilon$ implies $|\varphi(f^{-1}(x)) - \varphi(f^{-1}(y))| \leq \eta$.

Thus for any $\delta > 0$ there is an $\epsilon > 0$ so that if $x, y \in \gamma \subset Q'$ are at most distance ϵ apart, then the arc of γ between then has diameter at most δ (and ϵ is independent of which γ we use).

Choose M so large that each region $Q'_j = f(\varphi^{-1}(R_j))$ contains a disk of radius at most ρ, where ρ will be chosen (later) to be very small, depending on ϵ.

Map Q'_j conformally by ϕ_j to a $1 \times m'_j$ rectangle S'_j.

Note that this rectangle is conformally equivalent to the region $R'_j = \psi(f(\varphi^{-1}(R_j))) \subset R_j$, both with the obvious choice of vertices.

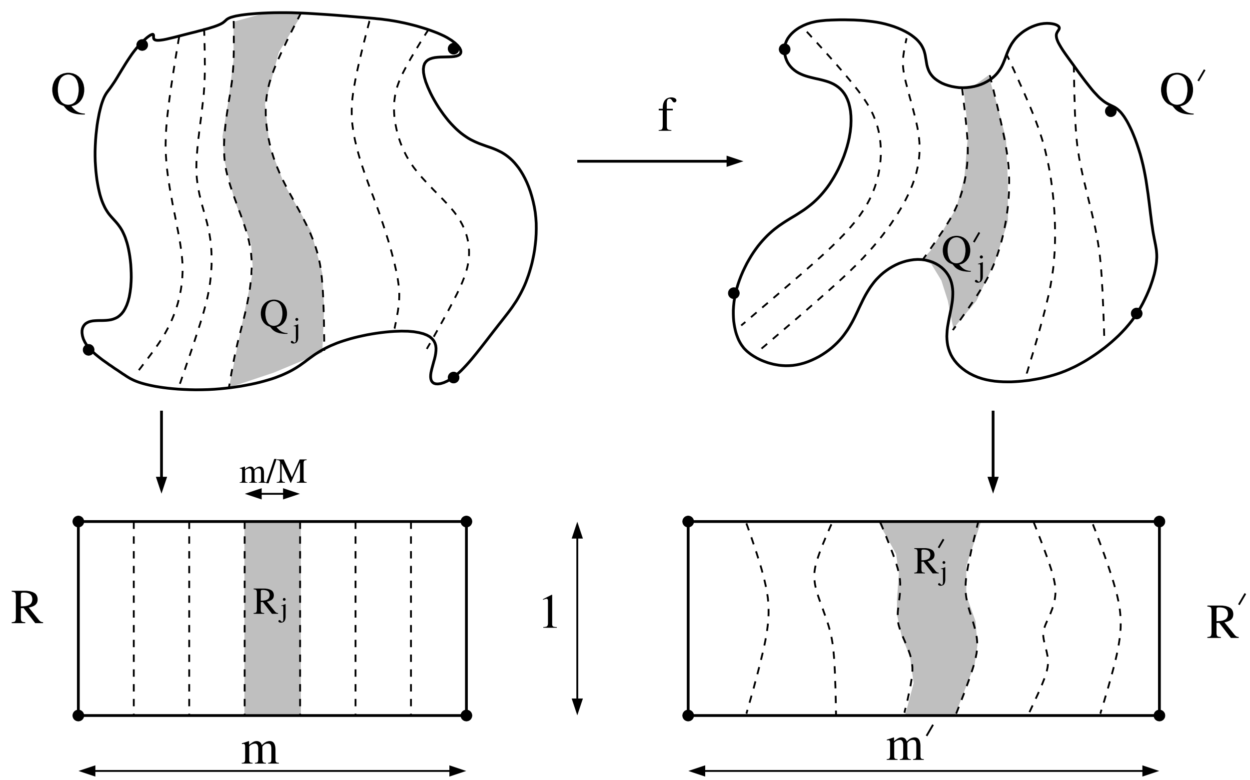


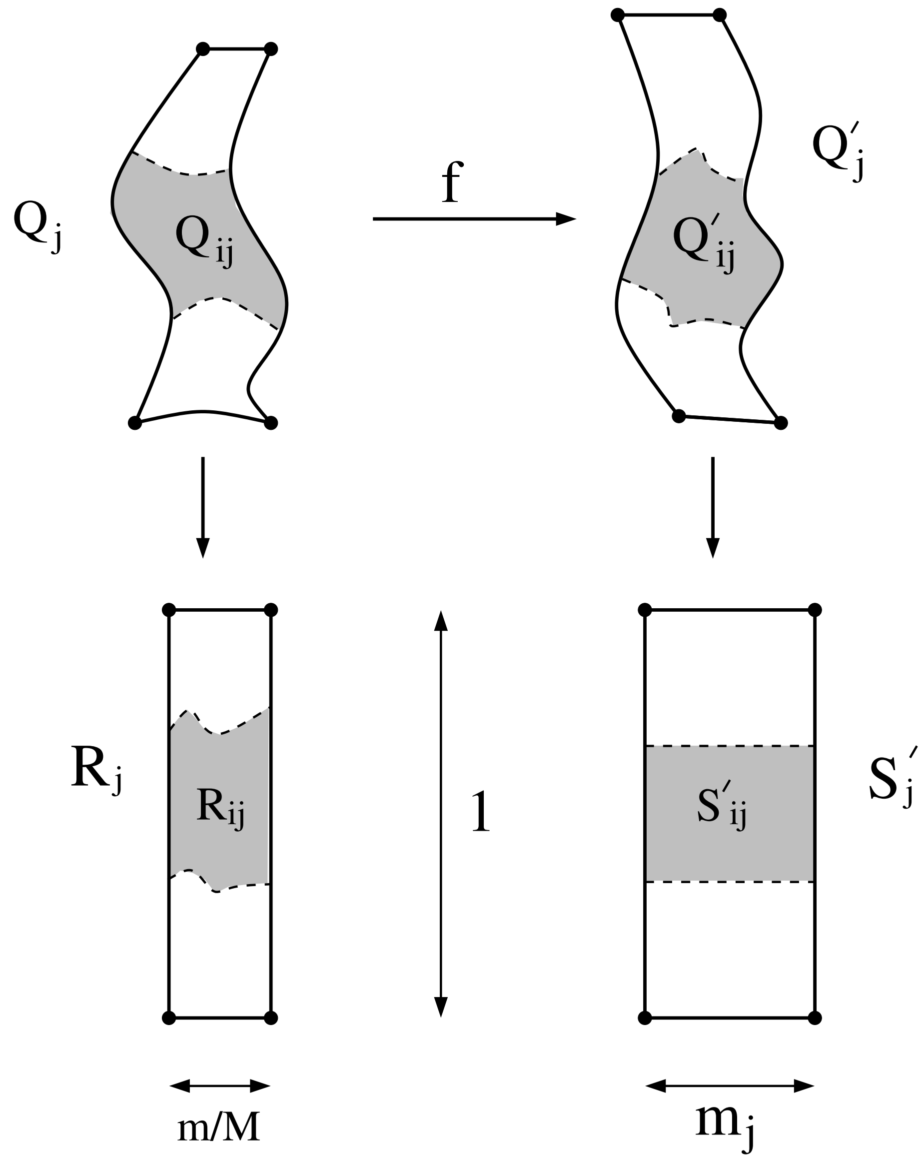
By Lemma 2.25 there is an absolute constant C so that every for every $y \in [0,1]$, there is a $t \in (0,1)$ with $|t-y| \leq Cm_j$ and so that the horizontal cross-cut of S'_j at height t maps via φ_j^{-1} to a Jordan arc of length $\leq C\rho$ in Q'_j.

Thus we can divide S'_j by horizontal cross-cuts into rectangles $\{S'_{ij}\}$ of modulus $m'_{ij} \simeq 1$ so that the preimages of these rectangles under ϕ_j are quadrilaterals with two opposite sides of length $\leq C\rho$ and which can be connected inside the quadrilateral by a curve of length $\leq C\rho$.

Taking δ as above, (so f^{-1} is K-QC on δ-balls) choose ϵ as above corresponding to $\delta/4$ and choose ρ so that $3C\rho < \min(\epsilon, \delta/4)$.

Then all four sides of the quadrilateral Q'_{ij} have diameter $\leq \delta/4$ and hence Q'_{ij} has diameter less than δ and hence lies in a disk where f^{-1} is K-quasiconformal. Let m_{ij} be the modulus of corresponding preimage quadrilateral $Q_{ij} = f^{-1}(Q'_{ij})$.





Q_j
Q_ij
f
Q′_j
Q′_ij
R_j
R_ij
1
S′_ij
S′_j
m/M
m_j

In S'_j consider the path family Γ'_j that connects the "top" and "bottom" sides of this rectangle and let m'_j denote the modulus of this path family (so $1/m'_j$ is its extremal length).

Let m_{ij} denote the modulus of the path family in the subrectangles S'_{ij} (again we take the path family connecting the top and bottom edges). These are conformally equivalent to path families connecting opposite sides of Q'_{ij} and via f^{-1} to path families in Q_{ij} whose modulus is denoted m_{ij}.

Since these quadrilaterals were chosen small enough to fit inside neighborhoods where f is K quasiconformal, we have

$$\frac{m_{ij}}{K} \le m'_{ij} \le K m_{ij}.$$

Finally, let Γ_j be the path family that connects the top and bottom of R_j and let Γ'_j be the family that connects the left and right sides of R'.

By the Series Rule

$$\frac{M}{m} = \lambda(\Gamma_j) \ge \sum_i \lambda(\Gamma_{ij}) = \sum_i \frac{1}{m_{ij}}.$$

Similarly,

$$m' = \lambda(\Gamma') \ge \sum_j \lambda(\Gamma'_j) = \sum_j m'_j.$$

We get equality in the Series Rule when a rectangle is cut by vertical lines, so

$$\frac{1}{m'_j} = \sum_i \frac{1}{m'_{ij}}.$$

Hence

$$\frac{M}{m} \geq \sum_i \frac{1}{m_{ij}} \geq \frac{1}{K} \sum_i \frac{1}{m'_{ij}} = \frac{1}{Km'_j}$$

or

$$\frac{m}{M} \leq Km'_j$$

for every j. Thus

$$m = \sum_{j=1}^{M} \frac{m}{M} \leq \sum_j Km'_j \leq Km'.$$

Applying the same result to the inverse map shows f is K-quasiconformal. $\square$

If $K = 1$, then $m = m'$ the last line of the above proof becomes

$$m' = m \leq \sum_j \frac{m}{M} \leq \sum_j m'_j \leq m'.$$

so we deduce

$$\sum_j m'_j = m',$$

whereas in general, we only have $\sum_j m'_j \leq m'$.

We claim this equality implies the curves cutting R' into the R'_j are straight segments.

We will then deduce that a 1-quasiconformal map must be conformal.

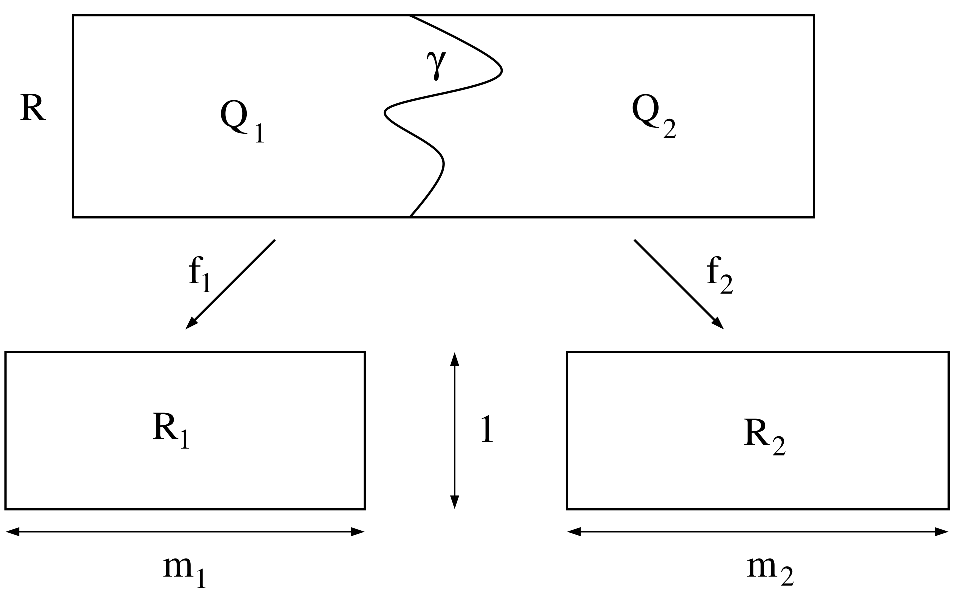
We start with:

Lemma 4.16. *Consider a* $1 \times m$ *rectangle* R *that is divided into two quadrilaterals* Q_1, Q_2 *of modulus* m_1 *and* m_2 *by a Jordan arc* γ *the connects the top and bottom edges of* R*. If* $m = m_1 + m_2$*, then the curve* γ *is a vertical line segment.*

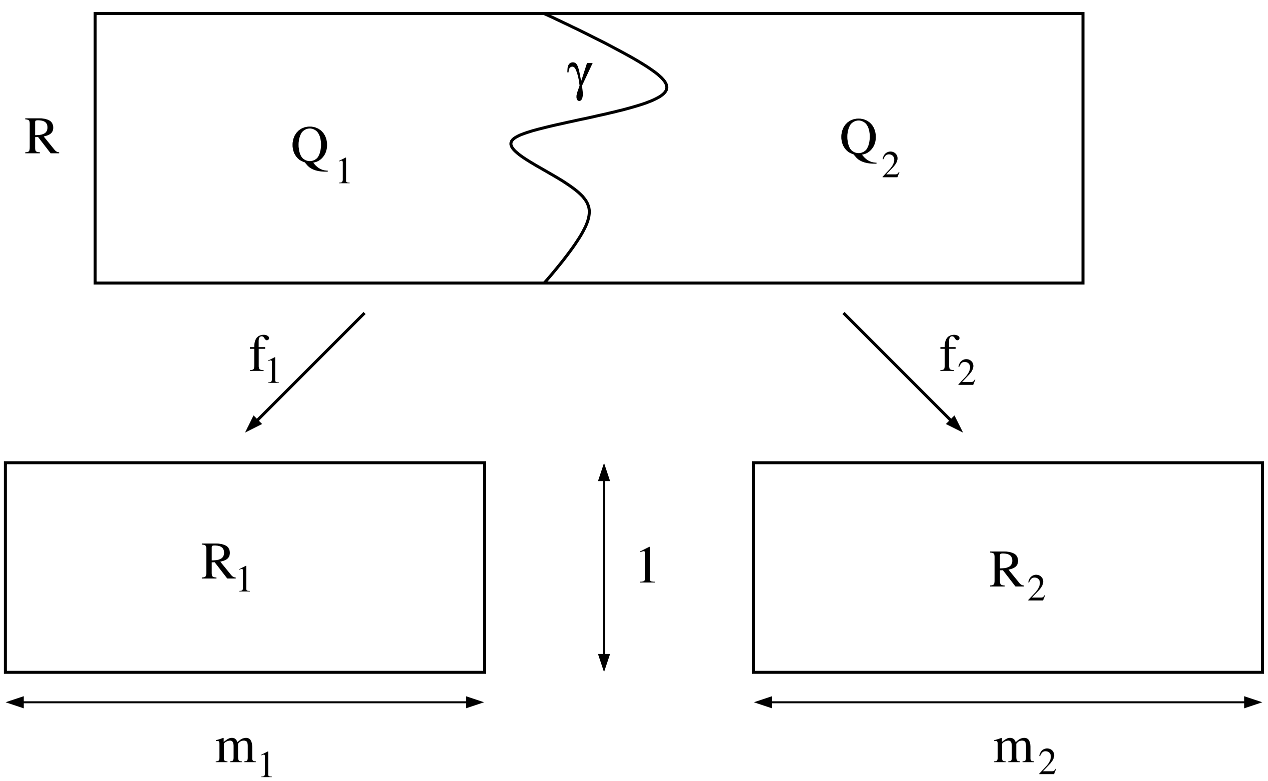
Proof. Let φ_1, φ_2 be the conformal maps of Q_1, Q_2 onto $1 \times m_1$ and $1 \times m_2$ rectangles R_1, R_2 respectively.

Set $\rho = |f_1'|$ on Q_1 and $\rho = |f_2'|$ in Q_2 and zero elsewhere. Then each horizontal line in Q is cut by γ into pieces one of which connects the left vertical edge of R to γ, and another that connect γ to the right edge of R.

The images of these connect the vertical edges of R_1 and R_2 respectively.



Thus the images have lengths at least m_1 and m_2 respectively, therefore the image of the entire horizontal segment in Q has length $\geq m_1 + m_2$.



If we integrate over all horizontal segments in Q, we see

$$\int_R (\rho - 1) dx dy \geq m_1 + m_1 - m = 0.$$

Similarly,

$$\begin{aligned}\int_R (\rho^2 - 1) dx dy &= \text{area}(f_1(Q_1) + \text{area}(f_2(Q_2)) - \text{area}(R) \\ &= (m_1 + m_2) - m \leq 0\end{aligned}$$

(we would have equality if we knew γ had zero area). Thus

$$\int_Q (\rho - 1)^2 dx dy = \int_Q (\rho^2 - 1) - 2(\rho - 1) dx dy \leq 0.$$

Since $(\rho - 1)^2 \geq 0$, this implies the integral equals zero and hence that that $\rho = 1$ almost everywhere, i.e., f_1 and f_2 are both linear and the curve γ is a vertical line segment. $\square$

Lemma 4.17. *If f is 1-quasiconformal on Ω, then it is conformal on Ω.*

Proof. If f is 1-quasiconformal in the proof of Theorem 4.15, then as noted before Lemma 4.16, we must have

$$\frac{M}{m} = \sum_i \frac{1}{m_{ij}}, \qquad \frac{1}{m'_j} = \sum_i \frac{1}{m'_{ij}}, \qquad m' = \sum_j m'_j,$$

By the previous lemma, this implies the cuts in R' forming the quadrilaterals R'_j are vertical segments, so $R'_j = S'_j$.

Thus the map $F = \psi \circ f \circ \varphi^{-1}$ sends a dense set of vertical segments in R to vertical segments in R'. Thus $F' > 0$ everywhere. Since F' is holomorphic, it must be the constant 1. Thus $f = (\psi)^{-1} \circ \mathrm{Id} \circ \varphi$ is a composition of conformal maps, hence conformal. $\square$

Lemma 4.18. *For any $\delta > 0$ and any $r > 0$, there is an $\epsilon > 0$ so that the following holds. If $f : \mathbb{C} \to \mathbb{C}$ is $(1+\epsilon)$-quasiconformal and f fixes 0 and 1, then $|z - f(z)| \le \delta$ for all $|z| < r$.*

Proof. If not, there is a sequence of $(1+\frac{1}{n})$-quasiconformal maps that all fix 0 and 1 and points $z_n \in D(0,r)$ so that $|z_n - f_n(z_n)| > \delta$.

However, there is a subsequence that converges uniformly on compact subsets of the plane to a 1-quasiconformal map that fixes 0 and 1 and that moves some point by at least δ.

However a 1-quasiconformal map is conformal on $\mathbb{C}$, hence of form $az + b$ and since it fixes both 0 and 1, it is the identity and hence doesn't move any points, a contradiction. □

Lemma 4.19. *(requires MRMT) Suppose* $E_1 \supset E_2 \supset \dots$ *are closed sets so that* $\text{area}(E_n) \to 0$. *Suppose* $K \geq 1$ *and that* $f : \mathbb{C} \to \mathbb{C}$ *is* K-*quasiconformal map with dilatation supported on* E_n, *and that* f *fixes* 0 *and* 1. *Then* f *converges to the identity uniformly on compact sets.*

Proof. By compactness, f_n converges to K-quasi conformal map that is conformal off $\cap_n E_n$, a set of zero area.

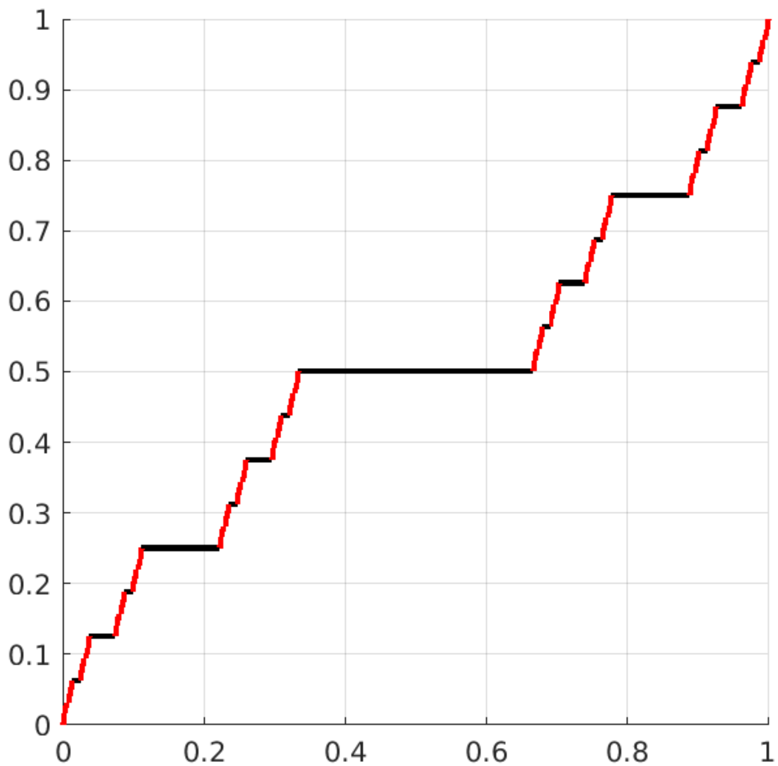
By the Measureable Riemann Mapping Theorem (to be proven later), f is conformal, hence linear and fixing $0, 1$. Hence the identity. $\square$

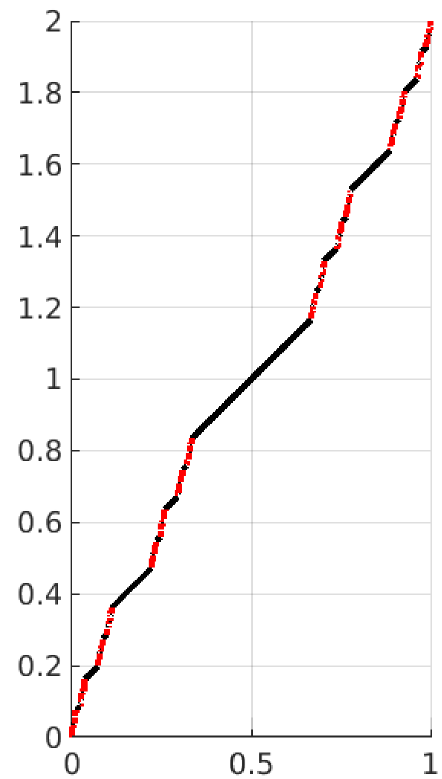
WARNING: there are homeomorphisms of the plane that are conformal except on a compact set of zero area, but are not conformal everywhere. These cannot be quasiconformal.

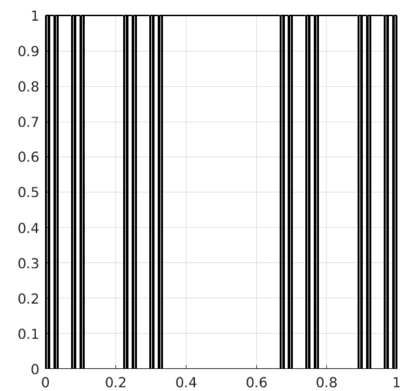
Example: suppose g is the Cantor singular function, i.e., a nonconstant, increasing function that is constant on each complementary interval of the middle thirds Cantor set $\mathcal{C}$.

Then $G(x) = g(x) + x$ is a homeomorphism of the line so that $G'(x) = 1$ except on the Cantor set.

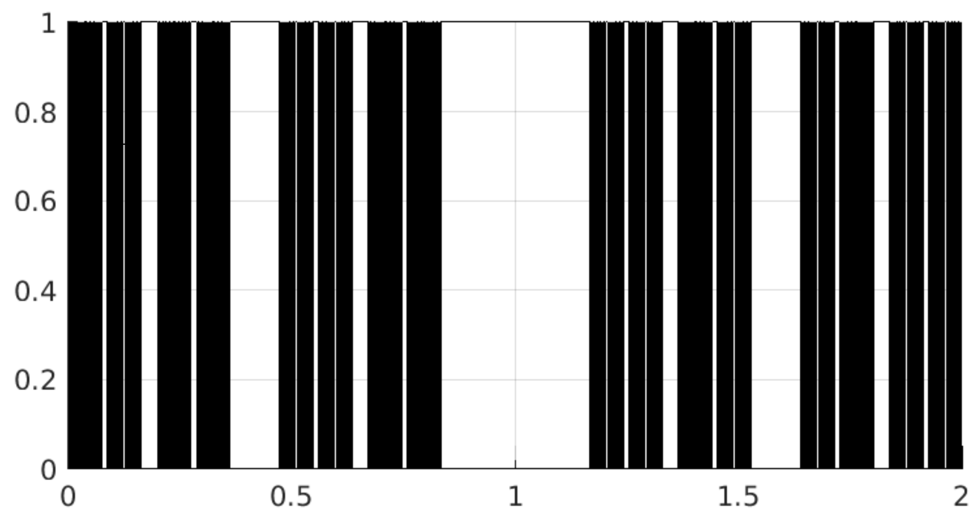
Thus $(x, y) \to (G(x), y)$ is a homeomorphism of the plane that is conformal except on the zero area set $\mathcal{C} \times \mathbb{R}$.


0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1


0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
0
0.5
1



1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.2
0.4
0.6
0.8
1



1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2

This semester I hope to cover the following topics:

- Review of complex analysis
- Extremal length and conformal modulus,
- Logarithmic capacity, harmonic measure
- Geometric definition of quasiconformal mappings, compactness
- Compactness corollaries: quasisymmetry, extension, removability, weldings
- Analytic definition and the measurable Riemann mapping theorem
- Analytic dependence on the dilatation
- Astala's theorems on area and dimension distortion
- More topics?: QC maps metric spaces, David maps, conformal dimension,...

Quasisymmetric maps

A homeomorphism $h : \mathbb{R} \to \mathbb{R}$ is called M**-quasisymmetric** if $|h(I)| \leq M|h(J)|$ whenever I and J are adjacent intervals of equal length. Equivalently,

$$\sup_{t \in \mathbb{R}, x > 0} \frac{h(x+t) - h(t)}{h(t) - h(x-t)} \leq M.$$

A homeomorphism is called **quasisymmetric** if it is M-quasisymmetric for some $M < \infty$. Later we will discuss quasisymmetric map of the unit circle to itself, but for the moment we stick to maps of $\mathbb{R}$ to $\mathbb{R}$.

The cross ratio of four points a, b, c, d on the real line

$$\frac{(a-c)(b-d)}{(b-c)(a-d)},$$

and is equal to

$$\frac{a-c}{b-c},$$

if $d = \infty$.

When f is M-quasisymmetric on $\mathbb{R}$ and a, b, c, d equal $x+t, x-t, x, \infty$, the cross ratio is -1. The cross ratio of the image points is between $-M$ and $-1/M$.

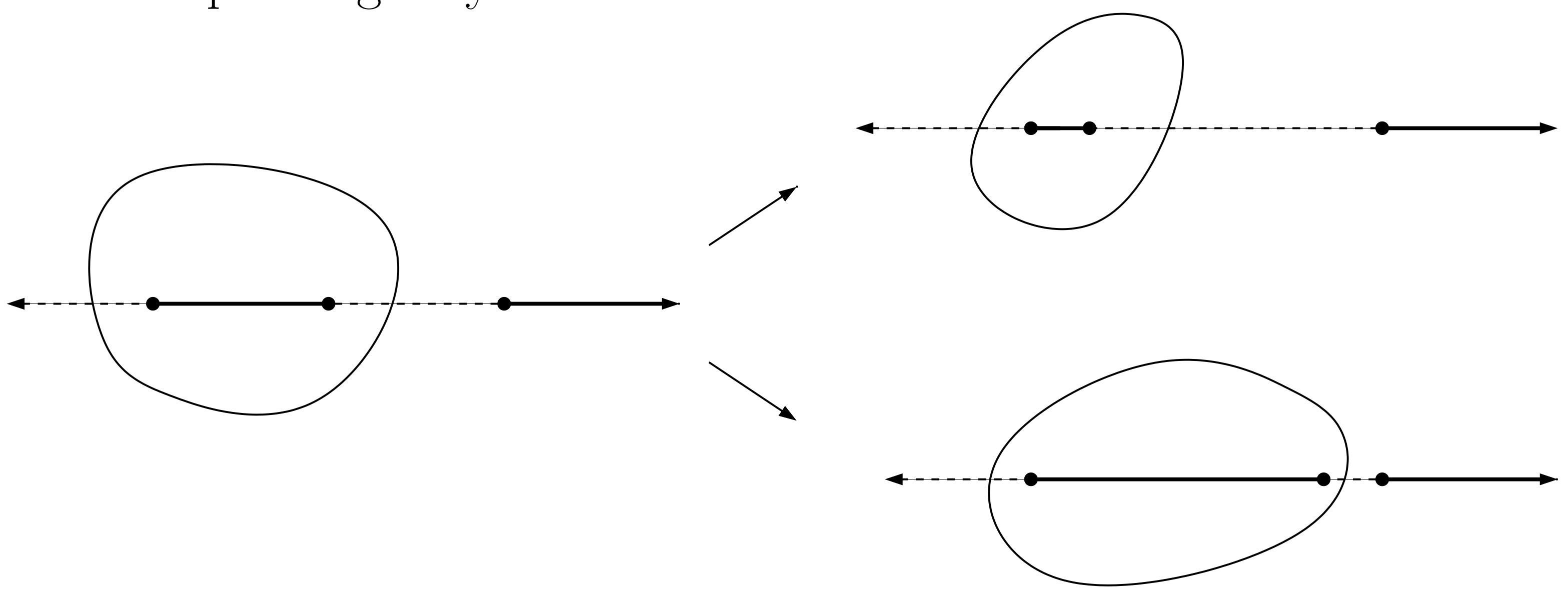
Theorem 5.1. *A homeomorphism $h : \mathbb{R} \to$ reals is quasisymmetric if and only if it extends to a quasiconformal mapping of the plane to itself.*

Proof. First we show that if f is a K-quasiconformal map of the plane that maps $\mathbb{R}$ to itself, then the restriction of f to $\mathbb{R}$ is quasisymmetric.

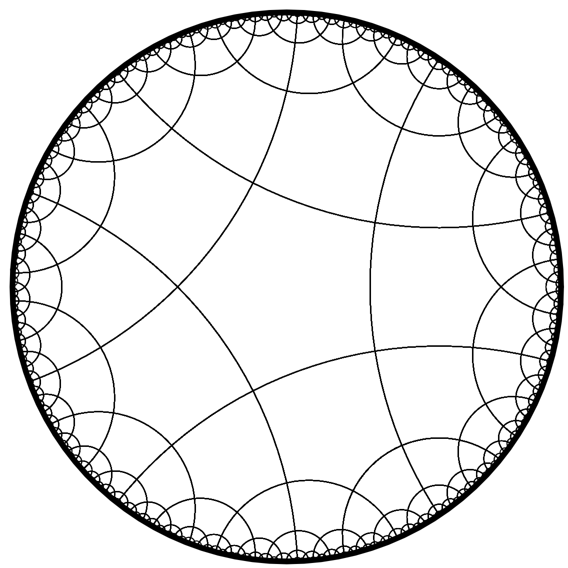
Without loss of generality we may assume $I = [0, 1/2]$ and $J = [1/2, 1]$ and that f fixes 0 and 1.

Consider the modulus of the topological annulus $A = \mathbb{C} \setminus ([0, 1] \cup [2, \infty)$. This has a fixed finite, non-zero modulus, so its image $B = f(A) = \mathbb{C}\setminus([0, x]\cup[1, \infty))$ also has modulus bounded between two positive real numbers that depend only on K.

If $x = f(1/2)$ is too close to 0 or 1, then B clear has modulus close to 0 or ∞ respectively, a contradiction. Thus x is bounded away from both 0 and 1 with an estimate depending only on K, and hence h is M-quasiconformal with a constant depending only on K.



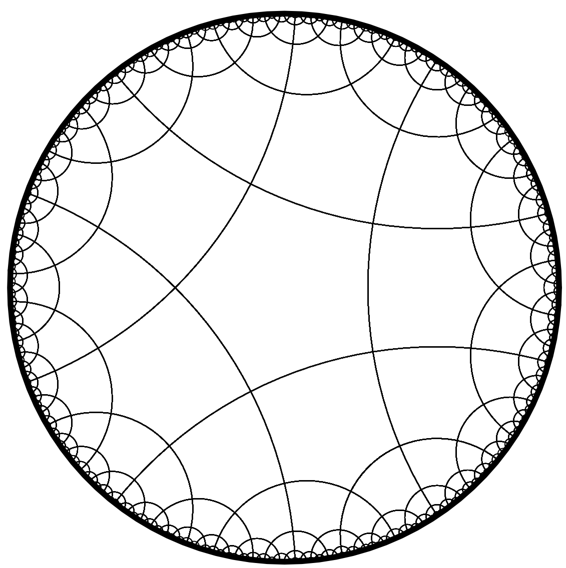
Next suppose $h : \mathbb{R} \to \mathbb{R}$ is M-quasisymmetric. We will assume h is increasing; the other case is handled by a similar argument. We will use the fact that the hyperbolic upper half-plane can be tesselated by hyperbolically identical right pentagons. The corresponding picture for the disk is shown below.



Hyperbolic space is tesselated by hyperbolically identical right pentagons. There is a corresponding picture on the upper half-plane model.

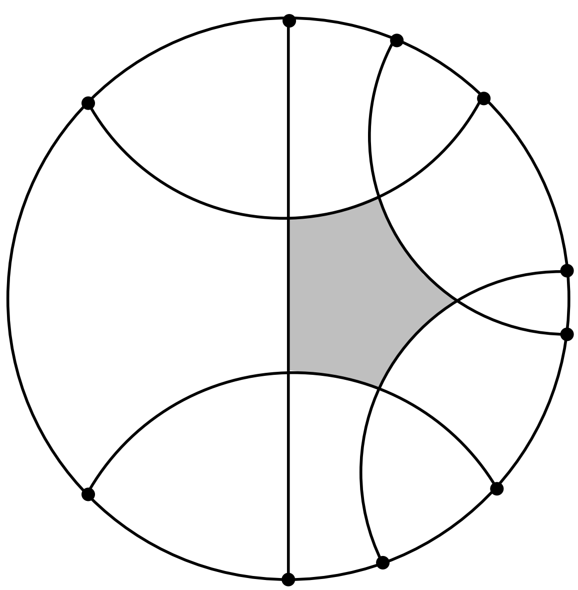
Each right pentagon in the tesselation of the upper half-plane determines five hyperbolic geodesics containing its sides, and these determine ten distinct points on the real line.

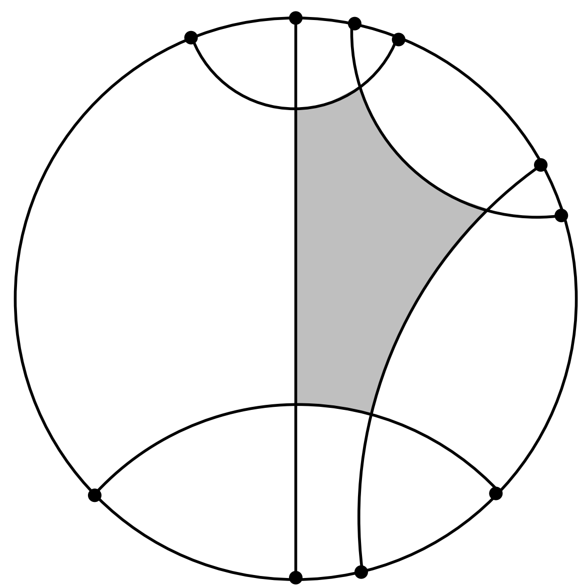
The h images of these point are also ten distinct points and the same pairs of point determine five new geodesics that define a hyperbolic pentagon (it need not be regular or right).

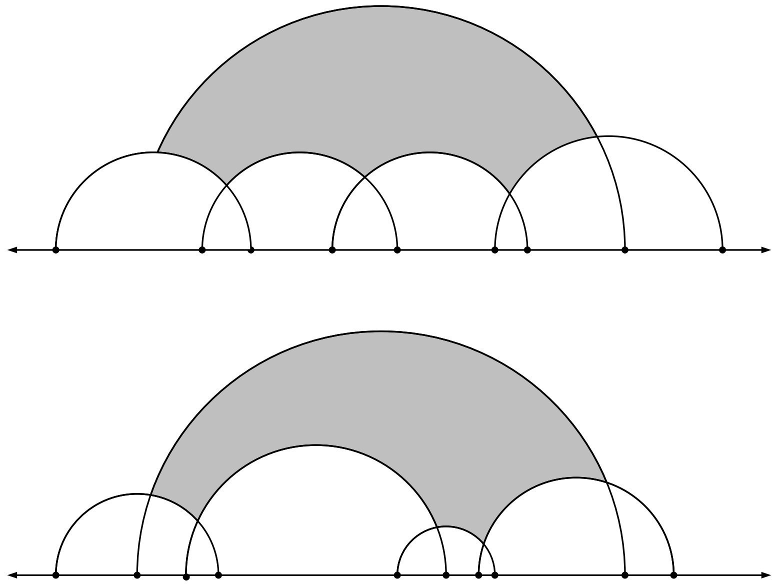


There is a diffeomorphism of the right pentagon to this new one that preserves arc-length along the edges in the sense that on each side of the pentagon length are multiplied by the ratio of the image length over the starting length.

This ensures that the diffeomorphisms defined on adjacent pentagons agree on the common sides. These diffeomorphisms come from a compact family of possibilities, thus have uniformly bounded dilatations, and hence define a quasiconformal map of the half-plane to itself that agrees with h on the boundary.



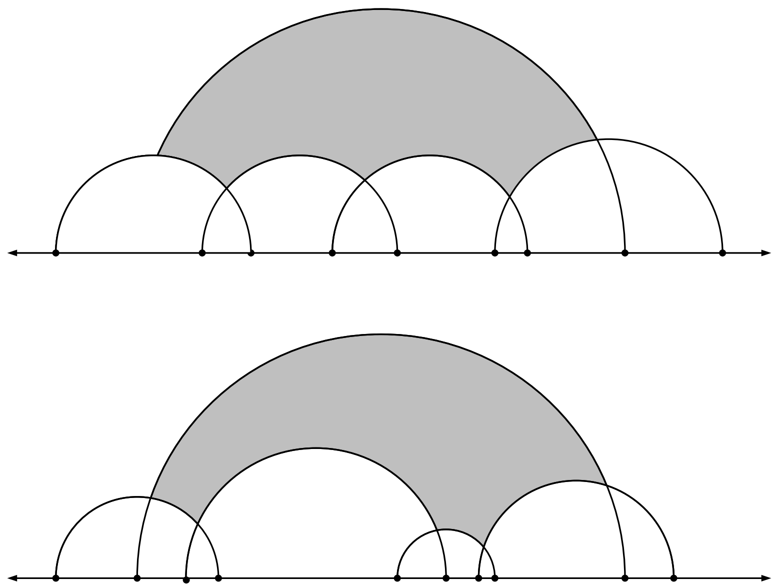




Sides of a hyperbolic right pentagon determine 5 geodesics and 10 boundary points. The images of these 10 points determine 5 geodesics, which give a hyperbolic pentagon.

We take any QC map between the pentagons that multiplies hyperbolic arclength on each edge by a constant (the ratio of the lengths of an edge and it image).

$\square$



This proof is just a more hyperbolic version of a proof due to Jerison and Kenig using a tiling of the upper half-plane by rectangles (upper halves of dyadic Carleson squares).

There are several other well known extensions. We mention two without proof.

Beurling-Ahlfors extension: Given a quasisymmetric homeomorphism f on the real line define

$$\begin{aligned} u(z) &= \int_0^1 f(x+ty)dt = \frac{1}{y}\int_x^{x+y} f(s)ds \\ v(z) &= \int_0^1 f(x-ty)dt = \frac{1}{y}\int_{x-y}^{x} f(s)ds \end{aligned}$$

and set

$$F(z) = \frac{1+i}{2}\left(u(z) + iv(z)\right).$$

Douady-Earle extension, 1986: this gives an extension E from $\mathbb{T}$ to $\mathbb{D}$ that is C^∞, biLipschitz in the hyperbolic metric (hence quasiconformal) and conformally natural, i.e.g., for any Möbius transformations ϕ and ψ, $E(\phi \circ f \circ \psi) = \phi \circ E(f) \circ \psi$. Let

$$G(z,w) = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{f(\zeta) - w}{1 - \overline{w} f(\zeta)} \frac{1 - |z|^2}{|z - \zeta|^2} |d\zeta|.$$

If $z \in \mathbb{D}$ there is a unique point w so $G(z,w) = 0$. We set $F(f)(z) = w$.

If $z = 0$, we apply Möbius transformations to f until its "average" lies at the orgin.

A different equivariant extension was given by Tukia in 1985.

It was a question of Dennis Sullivan whether there was an extension operator from quasisymmetric maps on the circle to quasiconformal maps of the disk, that was a homomorphism with respect to composition.

In 2007 Epstein and Markovic proved there is no such extension.

Quasicircles

We say that a curve γ satisfies the **3-point condition**, if there is a $M < \infty$ so that given any $x, z \in \gamma$ and y on the smaller diameter arc $\gamma(x, y) \subset \gamma$ between x, y, we have

$$|x - y| \leq M|x - z|,$$

Equivalently,

$$\operatorname{diam}(\gamma(x, z)) \leq M|x - z|.$$

This is also called the **Ahlfors M-condition** or **bounded turning**.

It is immediate from Lemma 4.5 that the image of the real line under any quasiconformal mapping of the plane is bounded turning, and below we shall prove the converse is also true.

The similar looking, but stronger, condition

$$\ell(\gamma(x, z)) \leq M|x - z|$$

where we assume γ is locally rectifiable is called the chord-arc condition. Such curves are called **chord-arc curves** or **Lavrentiev curves**, and form a special, but very important, subclass of the bounded turning curves.

It turns out that chord-arc curves are exactly the images of the real line under bi-Lipschitz maps of the plane, but we will not prove this here.

Lemma 5.2. *Suppose γ is bounded turning with constant M and $0, 1, \infty \in \gamma$.*

Suppose Ω is one of the connected components of $\mathbb{C} \setminus \gamma$ and suppose x is a point on γ between 0 and 1.

Let $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ denote the disjoint subarcs of γ from $-\infty$ to 0, from 0 to x, from x to 1 and from 1 to $+\infty$ respectively.

Let Γ be the path family joining the arc $\gamma_x \subset \gamma$ from 0 to x to the disjoint half-infinite arc $\gamma_1 \subset \gamma$ joining 1 to ∞.

Then $M(\Gamma) \to 0$ as $x \to 0$ with upper and lower bounds that depend only on $|x|$ and M

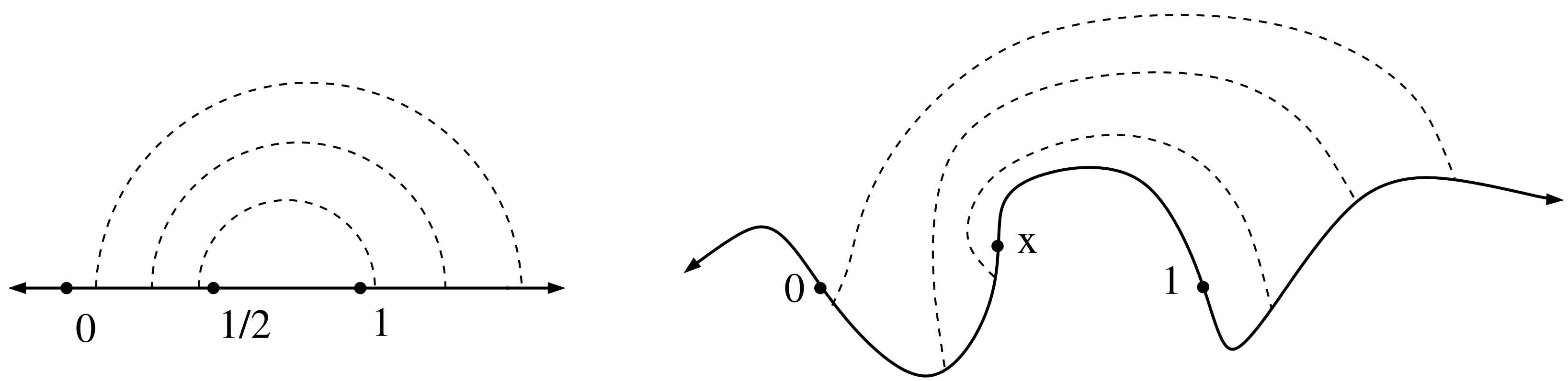
Proof. The 3-point condition implies that

$$\operatorname{dist}(\gamma_2, \gamma_4) \geq \frac{1}{M} - |x|,$$

so for $|x|$ sufficiently small every path in Γ crosses the round annulus

$$\{z : M|x| < |z| < \frac{1}{2M}\} \subset \{z : \operatorname{diam}(\gamma_2) < |z| < \operatorname{dist}(\gamma_2, \gamma_4)\}.$$

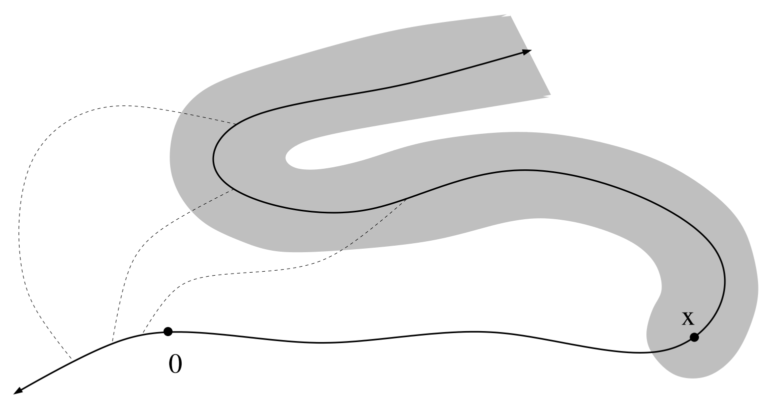
For $|x|$ small, this implies $M(\Gamma)$ is small.



To prove a lower bound on $M(\Gamma)$ it suffices to prove an upper bound on the reciprocal modulus of the path family connecting γ_1 to γ_3.

By the 3-point condition, these arcs are at least distance $|x|/M$ apart, so the metric $\rho = M/|x|$ on the disk of radius M around the origin is admissible.

The reciprocal family has modulus at most $\pi M^4/|x|^2$, so $M(\gamma) \geq |x|^2/M^4\pi$.



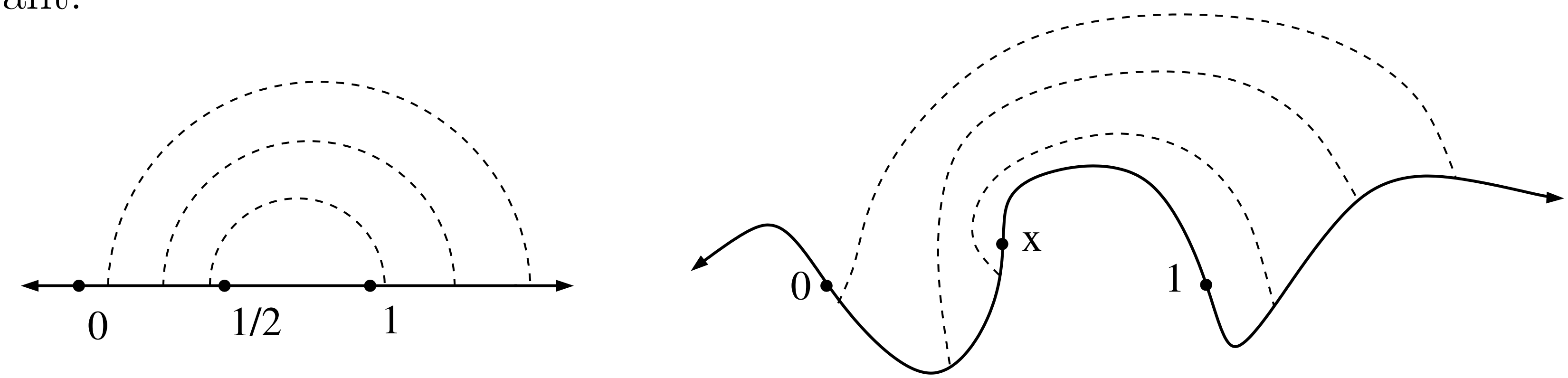
Since $\operatorname{dist}(\gamma_1, \gamma_3) \geq |x|/2M$, the metric $\rho = 1/2M$ is admissible. $\square$

Lemma 5.3. *If γ has bounded turning, and f, g are the conformal maps from the upper and lower half-planes to the two sides of γ (mapping ∞ to ∞ in both cases), then $h = g^{-1} \circ f$ is a quasisymmetric homeomorphism of the line.*

Proof. Consider two adjacent intervals of equal length on the real line.

After renormalizing by linear maps, we may assume these are $I = [0, 1/2]$ and $[[1/2, 1]$ and that h fixes both 0 and 1. By two applications of Lemma 5.2, $f(1/2)$ can't be too close to either 0 or 1, and hence $h(1/2) = g^{-1}(f(1/2))$ can't be too close to 0 or 1 either.

Thus h is quasisymmetric with a constant depending only on the 3-point constant.



The path family in the upper half-plane connecting $[0, 1/2]$ to $[1, \infty)$ has modulus 1, so its conformal image also has modulus 1. Therefore $x = f(1/2)$ can't be too close to either 0 or 1. $\square$

Lemma 5.4. *A curve γ is a quasi-line if and only if it has bounded turning.*

Proof. If γ is the quasiconformal image of a line, then it satisfies the 3-point condition by Lemma 4.5, as mentioned earlier.

On the other hand, if γ satisfies the 3-point condition, then $h = g^{-1} \circ f$ is quasisymmetric, and hence extends to a quasiconformal map H of the whole plane.

Now set $F = f$ on the lower half-plane and $F = g \circ H$ on the upper half-plane. Clearly this is quasiconformal on each half-plane and on the real line $g \circ H = g \circ g^{-1} \circ f = f$ so the two definitions agree. Thus H is quasisymmetric on the whole plane and $F(\mathbb{R}) = f(\mathbb{R}) = \gamma$. $\square$

Actually, the previous proof has a small error.

We claimed that if a homeomorphism of the plane is quasiconformal in both the upper and lower half-planes, then it is quasiconformal in the whole plane.

This is true, but not yet proven.

It is trivial from analytic definition; a little harder from the geometric definition. We will prove a much stronger result.

For the proof of Lemma 5.4 we can assume the QC map H is piecewise smooth on a hyperbolic tesselation, since we proved the QS extension theorem using an explicit construction that did this (other extensions methods even give smooth maps on whole half-plane).

Lemma 5.5. *If F is a homeomorphism of the plane that is quasiconformal on the upper and lower half-planes, and is piecewise smooth on a countable decomposition of each of these half-planes (such as given by a hyperbolic tesselation), then F is quasiconformal on the whole plane.*

We leave the proof to reader.

It follows the proof we gave that the piecewise differentiable definition of QC implies the geometric definition, except now we use that each line hits the partition boundary countably often instead of finitely often.

However, the result is true, even without assuming any smoothness.

Removable Sets

When f is continuously differentiable, it is relatively easy to check whether it is quasiconformal; we just compute the complex dilatation $\mu = f_{\overline{z}}/f_z$ and check that $|\mu| < k < 1$ everywhere.

For some applications in dynamics, functions arise that that are homeomorphisms f on $\mathbb{C}$, but which are only C^1 on an open set $\Omega = \mathbb{C} \setminus K$. If we know the dilatation is bounded on just Ω, can we still deduce that f is quasiconformal? If we can, then we say K is removable for quasiconformal mappings.

Removability depends on the "size" and "shape" of K.

We have already (implicitly) seen that K is removable if it a countable union of analytic arcs.

If K has interior, then it is easy to construct counterexamples; choose a disk $D \subset K$ and any non-quasiconformal homeomorphism of the disk to itself that is the identity on the boundary and extend it to be the identity off D.

If K has positive area, there are also counterexamples corresponding to applications of the measurable Riemann mapping theorem to a dilatation that is a non-zero constant on K and zero off K.

Even if K is quite small, there can be counterexamples. For example, given any guage function h such that $h(t) = o(t)$ as $t \searrow 0$, there is a closed Jordan curve γ and a homeomorphism of the sphere that is conformal on both components of $\mathbb{C} \setminus \gamma$ but which is not Möbius.

On the other hand, if K has finite or sigma-finite 1-measure then it is removable. These examples show that it is the "shape" rather than the "size" of K that is crucial in most cases of interest.

Recall that we proved this earlier:

Lemma 5.6. *Suppose Q is a square, $\lambda > 1$ and f is K-quasiconformal on λQ. Then*

$$\text{area}(f(Q)) \geq \epsilon \text{diam}(f(Q))^2,$$

where $\epsilon > 0$ depends only on λ and K.

A **Whitney decomposition** of an open set Ω consists of a collection of dyadic squares $\{Q_j\}$ contained in Ω so that

(1) the interiors are disjoint,

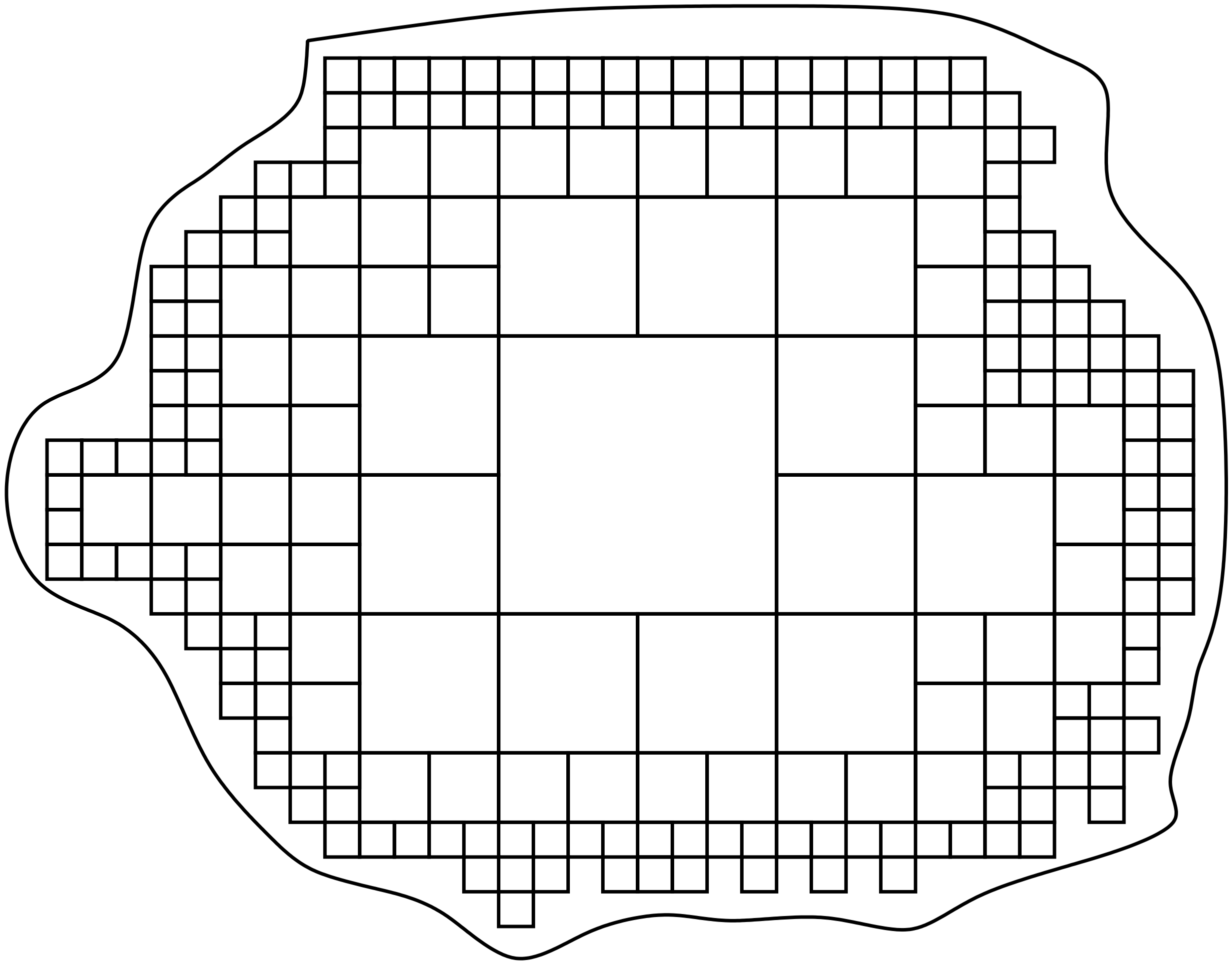
(2) the union of the closures is all of Ω,

(3) for each Q_j, $\operatorname{diam}(Q_j) \simeq \operatorname{dist}(Q_j, \partial\Omega)$.

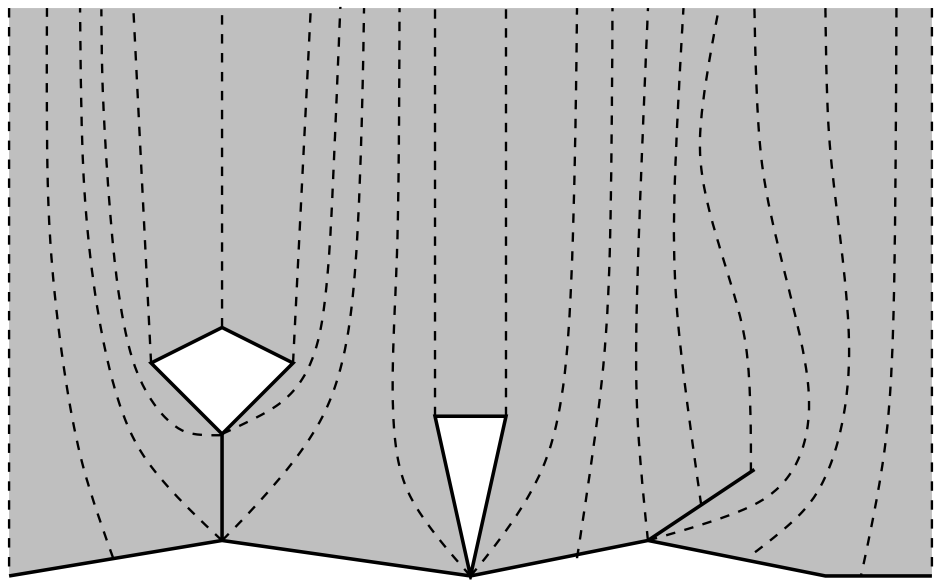
The existence of such a collection is easy to verify be taking the set of dyadic squares Q so that

$$\operatorname{diam}(Q) \leq \frac{1}{4}\operatorname{dist}(Q, \partial\Omega),$$

and that are maximal with respect to this property (i.e., the parent square fails this condition).



Suppose K is compact, $\delta > 0$ and for each $x \in K$ let γ_x be a Jordan arc in $\Omega = \mathbb{C} \setminus K$ that connects x to $\Omega_\delta = \{z \in \Omega : \text{dist}(z, K) \geq \delta\}$. For a single x, γ_x may consist of several arcs that connect x to Ω_δ.

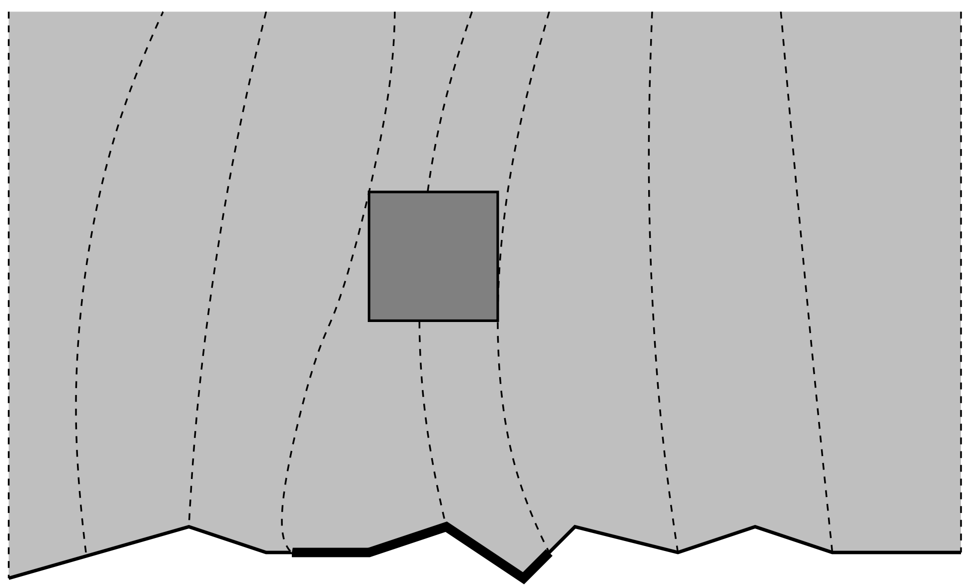


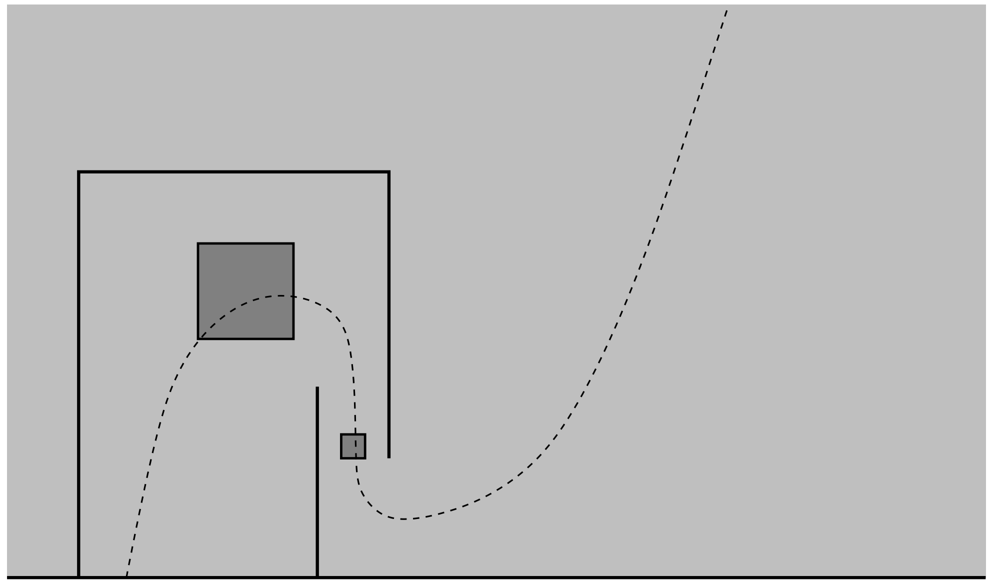
Each boundary point is connected to a point distance δ from $\partial\Omega$. Some points may be connected by more than one curve.

For each Whitney square $Q \subset \Omega$, let $S(Q) = \{x \in K : \gamma_x \cap Q \neq \emptyset\}$.

This is called the "shadow" of Q on K; the name comes from the special case when K is connected and does not separate the plane and γ_x is a hyperbolic geodesic connecting x to ∞.

If we think of ∞ as the "sun" and the geodesics as light rays, then $S(Q)$ is the part of K that blocked from ∞ by Q, i.e., it is Q's shadow.





The paths connecting a Whitney square to its shadow can sometimes hit larger Whitney squares after hitting smaller ones.

However the size of the hit squares tends to zero as the path approaches the boundary. Hence there is a“largest” square hit.

Let $C(Q)$ be the union of all Whitney squares hit by the arc γ connecting Q to some point of its shadow; this is the "filled shadow" and corresponds to a Carleson square in the unit disk.

We will assume three things about the Whitney squares and their shadows:

(S1) $S(Q)$ is closed.

(S2) $\mathrm{diam}(S(Q)) \to 0$ as $\mathrm{diam}(Q) \to 0$,

(S3) $\lim_{n\to\infty} \sum_{Q:\ell(Q)\le 2^{-n}} \mathrm{diam}(S(Q))^2 = 0$, where the sum is over all Whitney squares for Ω of side length 2^{-n}.

Theorem 5.7 (Jones-Smirnov)**.** *Suppose Ω has a Whitney decomposition so that the corresponding shadow sets satisfy conditions (S1)-(S3) above. Then $K = \partial\Omega$ is removable for quasiconformal maps.*

The Jones-Smirnov paper

If the map f is conformal off $\partial\Omega$ (i.e., $K = 1$), then we will show that the extension is conformal everywhere.

If the map f is K-quasiconformal off $\partial\Omega$ then we only prove that it is C-quasiconformal for some $C < \infty$.

However, it then follows from the analytic definition that f is actually K-quasiconformal on the whole plane.

The weaker version is sufficient for many applications.

Proof of Theorem 5.7. Recall that Ω is the complement of K. Suppose F is a homeomorphism of the plane that is quasiconformal on Ω.

Suppose that W is any bounded quadrilateral in the plane, say of modulus m and that $W' = F(W)$ has modulus m'. We want to show that $m' \leq Cm$ where $C < \infty$ depends only on K as in the statement of the theorem.

We will do this by mimicking the proof of Theorem 4.1, that showed that any piecewise differentiable map with bounded dilatation was quasiconformal (in the geometric sense).

Let $\varphi : W \to R = [0, m] \times [0, 1]$ and $\psi : W' \to [0, m'] \times [0, 1]$ be conformal maps of these quadrilaterals to rectangles R, R' (vertices mapping to vertices).

Define $X = \varphi(\partial\Omega \cap W) \subset R$.

The proof is somewhat involved because we are going to to consider three different Whitney decompositions. Let

- $\{W_j\}$ denote a Whitney decomposition for W,
- $\{Q_j\}$ a Whitney decomposition for Ω, and
- $\{U_j\}$ a Whitney decomposition for $U = R \setminus X$.

Fix some $\epsilon > 0$.

Fix a Whitney cube W_j for W.

We assume the decomposition is chosen so that $2W_j \subset W$.

Suppose $\delta > 0$ is so small (depending on our choice of W_j) that the following two conditions all hold.

(1) If Q_k is a Whitney square for Ω with diameter less than δ and the shadow $S(Q_k)$ hits W_j, then $S(Q_k) \subset 2W_j$ and the entire Whitney chain connecting any point $x \in S(Q_k)$ to Q_k is contained in $2W_j$.

This is possible by condition (S2) on shadow sets (small squares have small shadows).

Note that two points $x, y \in S(Q_k)$ can be connected by a chain of adjacent Whitney squares for Ω, all in the "shadow" of Q_k.

(2) Let $\mathcal{S}(W_j)$ denote the collections of all Whitney squares Q_k for Ω so that $\mathrm{diam}(Q_k) \le \delta$ and $S(Q_k)) \cap W_j \neq \emptyset$. Then

$$\sum_{Q_k \in \mathcal{S}(W_j)} \mathrm{diam}(S(Q_k))^2 \le \epsilon \cdot \mathrm{area}(W_j).$$

This holds for small enough δ, because by condition (S3) on shadows, this sum over all Whitney squares for Ω is finite, so removing all the squares bigger than δ gives a sum that tends to 0 as δ tends to zero.

The sum is less than $\epsilon \cdot \mathrm{area}(W_j)$ if δ is small enough (depending on W_j).

Let $\mathcal{S} = \cup_{W_j} \mathcal{S}(W_j)$.

This is the collection of all shadow sets of all Whitney squares Q_k for Ω so that (1) $\mathrm{diam}(Q_k) < \delta$ and (2) $S(Q_k)$ is contained in $\mathcal{S}(W_j)$ for some Whitney square W_j of W.

Note that each point $x \in \partial\Omega \cap W_j$ is associated to a Whitney chain that contains a square with diameter comparable to δ. There are only finitely many such squares, so their shadows form a finite collection that covers $\partial\Omega \cap W_j$.

Suppose $L = [a + iy, b + iy]$ is a horizontal segment, compactly contained in the interior of R at height y.

Let $g : R \to R'$ be the composition $\psi \circ f \circ \varphi^{-1}$. We wish to show that

$$\int_0^1 |g(b+iy) - g(a+iy)|dy \leq C\sqrt{mm'}, \tag{5.11}$$

where C depends only on K.

If we can do this, then by letting $a \to 0$ and $b \to m$ we get

$$m' \leq \lim_{a\to 0, b\to m} |g(b+iy) - g(a+iy)|,$$

and hence

$$m' \leq \lim_{a\to 0, b\to m} \int_0^1 |g(b+iy) - g(a+iy)| dy \leq C\sqrt{mm'},$$

which gives the desired inequality $m' = O(m)$.

The reversed inequality, $m = O(m')$, can be deduced from the same argument applied to the other pair of opposite sides of Q, since the corresponding path families have the reciprocal moduli. Thus it suffices to prove (5.11).

Since L is compactly contained in the interior of R and X is relatively closed in the interior of R, $L \cap X$ is compact. Thus $\varphi^{-1}(L \cap X)$ is a compact set of W, hence covered by finitely many Whitney squares for W and hence is covered by finitely many shadows sets in $\mathcal{S}$.

Let $\mathcal{X}$ be the image of the elements of $\mathcal{S}$ under φ. Then $L \cap X$ is covered by finitely many elements of $\mathcal{X}$, say $X_1, \ldots X_n$.

For $k = 1, \ldots, n$, let $Y_k = [a_k, b_k]$ be the smallest closed interval in L that contains $X_k \cap L$ (this is the convex hull of $X_k \cap L$, i.e., the interval with the same leftmost and rightmost point as $X_k \cap L$).

Then $Y_1, \ldots, Y_n$ also cover $L \cap X$ and we can extract a subcover with the property that $Y_j \cap Y_k \neq \emptyset$ implies $|j - k| \leq 1$.

Since the points a_k, b_k are both in the same set X_k, we can deduce that the preimage points $\varphi^{-1}(a_k), \varphi^{-1}(b_k)$ are both in the same element of $\mathcal{S}$.

Thus they are both in the shadow set of some Whitney square for Ω and are associated to a two sided chain of distinct Whitney squares $\{Q_m\}_{-\infty}^{\infty}$ of Whitney squares for Ω.

If two chains arising in this way, say from Y_k and Y_m with $m > k$, have a Whitney square in common, then we can combine the chains to form a chain connecting a_k to b_m consisting of distinct squares. We can replace $Y_k, Y_{k+1}, \dots Y_m$ by the single interval $Z = [a_k, b_m]$ which covers the same part of $L \cap X$..

Doing this for all intersections, we obtain a finite collection of closed intervals Z_k in L which covers the same set as the union of the Y_k's.

Furthermore, the two endpoints of each Z_k correspond to a two-sided Whitney chain in Ω and that different intervals use different Whitney squares (no overlapping chains).

Moreover, if Z_k has endpoints c_k, d_k and the corresponding chain is $\{Q_n\}$, then

$$|g(c_k) - g(d_k)| \leq \sum_n \mathrm{diam}(\psi(f(Q_n))).$$

The set $V = L \setminus \cup_k Z_k$ consists of finitely many open intervals in $U = R \setminus X$ with their endpoints in X.

We break V into countable many sub-intervals by intersecting it with the Whitney squares for $U = R \setminus X$ (without loss of generality, we can assume the endpoints of L occur on the boundary of a Whitney square for U).

On each Whitney square U_k for U we define the constant function

$$Dg = \frac{\operatorname{diam}(g(U_k))}{\operatorname{diam}(U_k)}.$$

Then if $L_j = L \cap U_j$,

$$\int_{L_j} Dg dx = \operatorname{diam}(g(U_j))/\sqrt{2}.$$

Thus if Z_L is the union of all the $Z_k \cap L$, we get

$$\int_{L \setminus Z_L} Dg dx \simeq \sum_j \operatorname{diam}(g(U_j)),$$

where the sum is over Whitney squares for U that hit L.

Thus

$$|g(b+iy) - g(a+iy)| \lesssim \int_{L \cap U} Dg dx + \sum_n \operatorname{diam}(\psi(f(Q_n))).$$

The sum is over Whitney squares Q_j for Ω that have diameter $\leq \delta$.

Now integrate in y to get

$$\int_0^1 |g(b+iy) - g(a+iy)|dy \lesssim \iint_U Dg dx + \sum_n \operatorname{diam}(\psi(f(Q_n)))\mu_n,$$

where μ_n is the Lebesgue measure in $[0,1]$ of the set of lines L_y that use the Whitney square Q_n in at least one of the two-sided chains associated to a interval $Z \subset L_y$.

The Lebesgue measure of this set is no more than its diameter, which is no more than the diameter of $X_n = \varphi(S(Q_n))$. Thus

$$\int_0^1 |g(b+iy) - g(a+iy)|dy \lesssim \iint_U Dg dx dy + \sum_n \mathrm{diam}(\psi(f(Q_n)))\mathrm{diam}(X_n),$$

We want this to be $= O(\sqrt{m \cdot m'})$.

We now estimate the second term using the Cauchy-Schwarz inequality.

$$
\begin{aligned}
&\sum_n \operatorname{diam}(\psi(f(Q_n)))\operatorname{diam}(X_n) \\
&\quad \le \left(\sum_n \operatorname{diam}(\psi(f(Q_n)))^2\right)^{1/2} \left(\sum_n \operatorname{diam}(X_n)^2\right)^{1/2} \\
&\quad \le A \left(\sum_n \operatorname{area}(\psi(f(Q_n)))\right)^{1/2} \times \\
&\qquad\qquad \left(\sum_{W_k} \left[\frac{\operatorname{diam}(\varphi(W_k))}{\operatorname{diam}(W_k)}\right]^2 \sum_{Q_n \in \mathcal{S}(W_k)} \operatorname{diam}(S(Q_n))^2\right)^{1/2} .
\end{aligned}
$$

We have used Koebe's theorem to estimate the size of the images.

Now use Lemma 5.6,

$$
\begin{aligned}
&\leq A\left(\sum_{n}\operatorname{area}(\psi(f(Q_n)))\right)^{1/2}\left(\sum_{W_k}\left[\frac{\operatorname{diam}(\varphi(W_k))}{\operatorname{diam}(W_k)}\right]^2\cdot\epsilon\cdot\operatorname{area}(W_k))\right)^{1/2}\\
&\leq A\left[\operatorname{area}(R')^{1/2}\cdot\epsilon\cdot\operatorname{area}(R)\right]^{1/2}\\
&\leq A\sqrt{\epsilon\cdot m\cdot m'}.
\end{aligned}
$$

where A just depends on the distortion estimate for conformal maps and ϵ is as small as we wish (this was Condition 2 in our choice of δ).

The other term is also bounded by Cauchy-Schwarz

$$\begin{aligned}
\iint_U Dg dx &= \sum_k \iint_{U_k} Dg dx dy \\
&\leq \left(\sum_k \iint_{U_k} (Dg)^2 dx dy \right)^{1/2} \left(\sum_k \iint_{U_k} dx dy \right)^{1/2} \\
&\leq \left(\sum_k (\mathrm{diam}(g(U_k))^2 \right)^{1/2} \left(\sum_k \mathrm{area}(U_k) \right)^{1/2} \\
&\leq C \left(\sum_k (\mathrm{area}(g(U_k)) \right)^{1/2} \mathrm{area}(R)^{1/2} \\
&\leq C \mathrm{area}(R')^{1/2} \cdot \mathrm{area}(R)^{1/2} \leq C\sqrt{m'm}.
\end{aligned}$$

Thus

$$\int_0^1 |g(b+iy) - g(a+iy)| dy \lesssim \sqrt{m'm} + O(\epsilon),$$

Taking $\epsilon \to$ gives the desired inequality. $\square$

Corollary 5.8. *If K satisfies the conditions of Theorem 5.7, then K is removable for conformal homeomorphisms, i.e., any homeomorphism of the plane that is conformal off K is conformal everywhere.*

Proof. Theorem 5.7 implies that f is quasiconformal on the plane, so the point is to show that we can take the quasiconformal constant to be 1.

If we redo the proof assuming f is conformal off $\partial\Omega$, then the piecewise constant function Dg can be replaced by the usual derivative $|g'|$.

This leads to the inequality $m' \leq \sqrt{m'm}$, or $m' \leq m$.

The reverse inequality follows by considering the reciprocal path family in each quadrilateral. Together, these imply f is 1-quasiconformal, and hence conformal.

$\square$

Corollary 5.9. *If f, g are quasiconformal maps of the upper and lower half-planes that agree on the real line, then they define a quasiconformal map on the whole plane.*

Proof. This is immediate since a line clearly satisfies the Jones-Smirnov criteria: just consider $\mathbb{R}$ as the boundary of the upper half-plane and for $x \in \mathbb{R}$, let γ_x be a vertical line ray.

Then the shadow of any square is its vertical projection, and the square of the shadows length is comparable to the area of the square.

Thus any compact segment of $\mathbb{R}$ is removable, and since quasiconformality is a local property (Theorem 4.15), the whole line is removable. $\square$

Corollary 5.10. *If f is a quasiconformal map of the upper half-plane to itself, mapping the real line to itself, then the extension of f to the whole plane by $f(\bar{z}) = \overline{f(z)}$ is quasiconformal in the whole plane.*

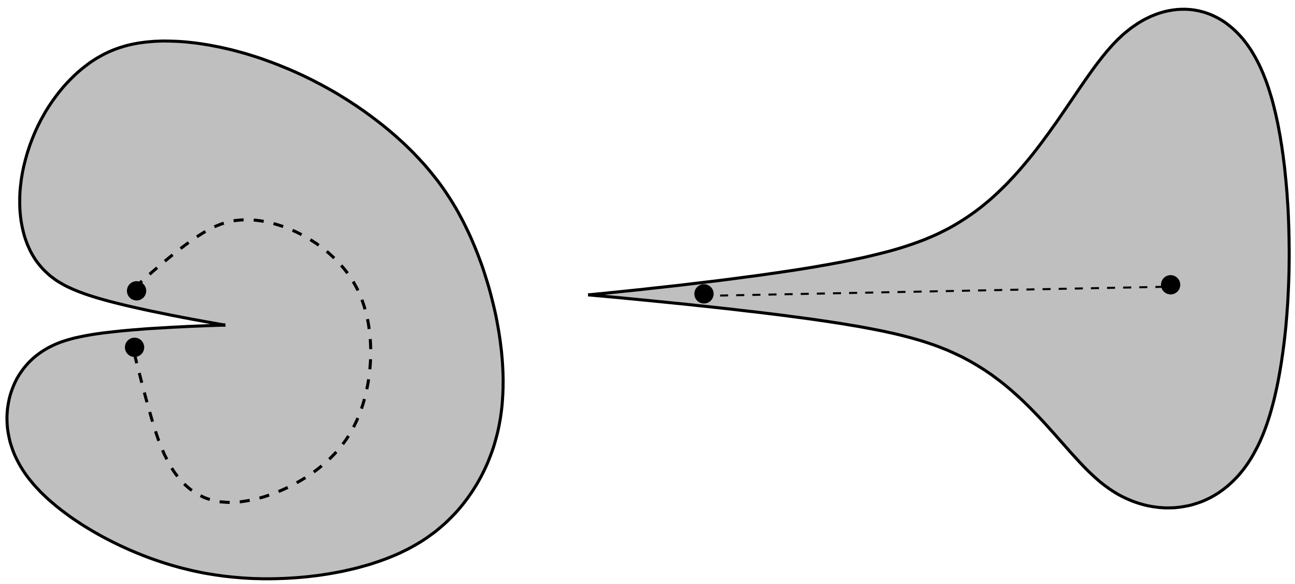
Proof. Immediate from the previous result since composing a quasiconformal map with reflections gives another quasiconformal map. $\square$

Corollary 5.11. *Quasicircles are removable.*

Proof. If $\Gamma = g(\mathbb{R})$ is a quasiconformal image of the reals and f is a homeomorphism that is quasiconformal on each side of Γ, then $h = f \circ g$ is a homeomorphism that is quasiconformal on each side of $\mathbb{R}$, then quasiconformal on the whole plane.

Thus $f = h \circ g^{-1}$ is a composition of quasiconformal maps and hence is QC. $\square$

An open, connected set Ω in $\mathbb{R}^2$ is called a **John domain** if any two points $a, b \in \Omega$ can be connected by a path γ in Ω with the property that $\operatorname{dist}(z, \partial\Omega) \gtrsim \min(|z-a|, |z-b|)$.



The domain on the left is a John domain, but the one on the left is not; inward pointing cusps are OK, but outward pointing cusps are not.

Lemma 5.12. *The Riemann map φ from the unit disk to a bounded John domain satisfies*

$$\operatorname{diam}(\varphi(I(Q))) \leq C \operatorname{diam}(\varphi(Q)),$$
$$\operatorname{dist}(\varphi(Q), \varphi(I(Q))) \leq C \operatorname{diam}(\varphi(Q)),$$

for some constant $C < \infty$ and any Whitney square Q and is shadow $I(Q)$.

Proof. The second inequality follows directly from Lemma 2.23 by considering the path family of radial lines connecting Q to I.

To prove the first inequality, consider the Whitney-Carleson boxes Q_1 and Q_2 that are adjacent to Q and of the same size. By Lemma 2.23 each is connected to its shadow by a radial segment whose image under f has length comparable to $\operatorname{diam}(f(Q))$.

Thus there is a geodesic crosscut γ of the disk that passes through Q and whose image has length comparable to $\operatorname{diam}(f(Q))$. Now suppose x is in the shadow of Q.

Any curve connecting 0 to x crosses γ, so any curve Γ connecting $f(0)$ and $f(x)$ crosses $f(\gamma)$ and hence contains a point $z \in f(\gamma) \cap \Gamma$ that is at most distance $O(\mathrm{diam}(f(Q)))$ from $\partial\Omega$. Thus by the definition of John domain, either

$$\mathrm{dist}(f(0), z) = O(\mathrm{diam}(f(Q))) \quad \text{or} \quad \mathrm{dist}(f(x), z) = O(\mathrm{diam}(f(Q))).$$

In a bounded domain, the first can only happen for finitely many Q's; for the remainder, the second must hold and hence $f(I(Q))$ is contained in a $O(\mathrm{diam}(f(Q)))$ neighborhood of $f(Q)$. $\square$

Corollary 5.13. *Boundaries of simply connected John domains are removable.*

Proof. Let Ω be a simply connected John domain and suppose $f : \mathbb{D} \to \Omega$ is conformal.

Each Whitney square Q' for Ω is covered by a uniformly bounded number images $f(Q)$ of Whitney squares for $\mathbb{D}$ and its shadows is contained in the union of corresponding shadows.

This and Lemma 5.12 imply $\mathrm{diam}(S(Q')) = O(\mathrm{diam}(Q')$.

The three conditions (1)-(3) in Theorem 5.7 follow easily.. $\square$

A simply connected plane domain Ω is called a *Hölder domain* if the Riemann map $\mathbb{D} \to \Omega$ is Hölder.

Lemma 5.14. *Boundaries of Hölder domains are removable.*

Sketch of proof. Fix a base point in Ω. The Hölder condition implies that $\{Q_j^k\}$ lists the Whitney squares of Ω approximately hyperbolic distance k from the base point then $\mathrm{diam}(S(Q_j^k)) \leq Ce^{-ak}$.

We also need an estimate of Jones and Makarov that for Hölder domains,

$$\sum_k \mathrm{diam}(S(Q_j^k))^{2-\epsilon} < M < \infty$$

for some $\epsilon > 0$ and $M < \infty$ independent of k. depending on the Hölder constant. Then

$$\sum_k \sum_j \mathrm{diam}(S(Q_j^k))^2 \leq \sum_k \sum_j \mathrm{diam}(S(Q_k))^{2-\epsilon} \mathrm{diam}(S(Q_j^k))^{\epsilon} \leq \sum_k C^{\epsilon} e^{-\epsilon a k} M < \infty$$

Corollary 5.15. *Julia sets of Collet-Eckmann polynomials are removable.*

The Jones-Smirnov result (Theorem 5.7) places restrictions on the set E, but none on the mapping (besides being a homeomorphism). An earlier result of Rickman makes an assumption on the mapping, but none on the set K:

Lemma 5.16 (Rickman's lemma)**.** *Suppose Ω is a planar domain and $K \subset \Omega$ is compact. Suppose f is homeomorphism of Ω that is quasiconformal on $\Omega \setminus K$ and F is quasiconformal on all of Ω. If $f = F$ on K, then f is quasiconformal on all of Ω.*

Proof. Isolated points of K are clearly removable and there are only countable many such points, so we may assume that K has only accumulation points.

The idea proof is the same as the proof of Theorem 5.7: we consider a quadrilateral W and its image $W' = f(W)$ and conformally map each to rectangles of modulus m and m' respectively.

Let $G = \psi \circ F \circ \varphi^{-1}$ and $g = \psi \circ f \circ \varphi^{-1}$.

Our assumption implies $g = G$ on X.

As before, we want to prove the estimate (5.11):

$$\int_0^1 |g(b+iy) - g(a+iy)|dy \leq C\sqrt{mm'},$$

However, this time we cover X by dyadic squares that are so small that G is quasiconformal on $6Q \subset R$ for each square Q used, and the image $G(Q)$ lies in R'.

The union of these squares plays the role of the set Z in the earlier proof.

Given a compact horizontal line segment L in R, we let $\{Y_k\}\{[c_k, d_k]\}$ enumerate the convex hulls of sets of the form $L \cap Q$ for Q in our cover of X.

Then defining Dg exactly as before on $R \setminus X$, and using $g = G$ on X, we get

$$\begin{aligned}
|g(b+iy) - g(a+iy)| &\le \int_{L\cap U} Dg dx + \sum_k |g(c_k) - g(d_k)| \\
&\le \int_{L\cap U} Dg dx + \sum_k |G(c_k) - G(d_k)| \\
&\le \int_{L\cap U} Dg dx + \sum_{Q:Q\cap L\neq\emptyset} \operatorname{diam}(G(Q)).
\end{aligned}$$

Integrating over y then gives

$$\int_0^1 |g(b+iy) - g(a+iy)| dy \ \le \ \int_U Dg dx + \sum_Q \operatorname{diam}(G(Q))\ell(Q).$$

The first term is bounded exactly as before and the second is bounded by

$$\begin{aligned}
\sum_Q \operatorname{diam}(G(Q))\ell(Q) \ &\le \ [\sum_Q \operatorname{diam}(G(Q))^2]^{1/2} \cdot [\sum_Q \ell(Q)^2]^{1/2} \\
&\le \ C[\sum_Q \operatorname{area}(G(Q))]^{1/2} \cdot [\sum_Q \operatorname{area}(Q)]^{1/2} \\
&\le \ C[\operatorname{area}(R')]^{1/2} \cdot [\operatorname{area}(R)]^{1/2} \\
&\le \ C\sqrt{m'm}.
\end{aligned}$$

The rest of the proof is them completed just as before. $\square$

BiLipschitz Reflections

Lemma 5.17. *A quasisymmetric map $f : \mathbb{R} \to \mathbb{R}$ can be extended to a quasiconformal map of the upper half-plane that is also biLipschitz for the hyperbolic metric.*

Proof. Go back and check the proof of the extension theorem. $\square$

Lemma 5.18. *If f is a hyperbolic biLipschitz map of the upper half-plane to itself, then f is quasiconformal.*

Proof. Easy to check that length and area change by at most a bounded factor, so modulus of any quadrilateral changes by a bounded factor (just transfer ρ without change). $\square$

Theorem 5.19. *An unbounded Jordan curve* Γ*is a quasiline iff it has a biLipschitz reflection, i.e., there is a bi-Lipschitz map of the plane that fixes* Γ *pointwise and swaps the two complements.*

Quasiline implies biLipschitz reflection. Let f and g be the conformal maps from the upper and lower half-planes to the two sides of Γ, each fixing ∞.

Since Γ is a quasiline, $h = g^{-1} \circ f$ is quasisymmetric and has a quasiconformal extension H to the lower half-plane that is biLipschitz for the hyperbolic metric.

Let $r(z) = \overline{z}$ be reflection across the real line and define $R(z) = g \circ H \circ r \circ f^{-1}$. this is a quasiconformal map from one side of Γ to the other and it fixes Γ pointwise.

If H is defined by our hyperbolic pentagon map, then each pentagon is associated to several subintervals of $\mathbb{R}$ that all have comparable harmonic measures for any point in the pentagon.

Thus the R maps the region $f^{-1}(P)$ to $g(r(P))$ and these regions have comparable diameters since the associated subintervals on Γ are the same.

Since R is a hyperbolic biLipschitz map between two domains of bounded hyperbolic diameter and comparable Euclidean size, it is a Euclidean biLipschitz map on these regions.

From this it is easy to check R is Euclidean Lipschitz everywhere. Since $R = R^{-1}$, it is automatically biLipschitz. $\square$

biLipschitz reflection implies quasiline. As above, let f and g denote conformal maps of the upper and lower half-plane to the two sides of Γ that fix ∞.

Suppose R is a biLipschitz reflection across Γ. Then $r \circ g^{-1} \circ R \circ f$ is a hyperbolic biLipschitz map of the upper half-plane to itself that extends the welding map $h = g^{-1} \circ f$.

Hyperbolic Lipschitz implies quasiconformal, so h must be quasisymmetric, which in turn implies Γ is a quasiline. $\square$

Remark: A set E is K-biLipschitz homogeneous if for any $x, y \in E$ there is a K-biLipschitz map $f : E \to E$ so that $f(x) = y$.

It is known that a biLipschitz homogeneous closed curve must be a quasicircle.

Question: is a biLipschitz homogeneous continuum a closed curve?

There are homogeneous continua for (non-biLipschitz) homeomorphisms that are not curves (e.g., the pseudo-arc and the circle of pseudo-arcs). These examples are not locally connected. Does requiring biLipschitz maps eliminate these?

A complete classification of homogeneous plane continua by L.C. Hoehn and L.G. Oversteegen, 2016.

Remark: A hyperbolic quasi-isometry $f : \mathbb{D} \to \mathbb{D}$ is a map so that

$$\frac{1}{A}\rho(z,w) - B \leq \rho(f(z), f(w)) \leq A\rho(z,w) + B.$$

Informally, these are biLipschitz at large scales.

Every quasiconformal map $f : \mathbb{D} \to \mathbb{D}$ is a hyperbolic isometry. See 2004 Annals paper by Epstein, Marden and Markovic.

Conversely, every hyperbolic quasi-isometry has boundary values that are quasisymmetric. Thus there is a quasiconformal map with the same boundary values.

Conformal Welding

Suppose Γ is a closed Jordan curve and f, g are conformal maps from $\mathbb{D}$ and $\mathbb{D}^* = \{|z| > 1\}$ to the inside and outside complementary domains of Γ.

By Carathéodorty's theorem, both these maps extend to be homeomorphisms of $\mathbb{T} \to \Gamma$, so $h = g^{-1} \circ f$ is a homeomorphism of the unit circle to itself (for brevity, we call this a circle homeomorphism).

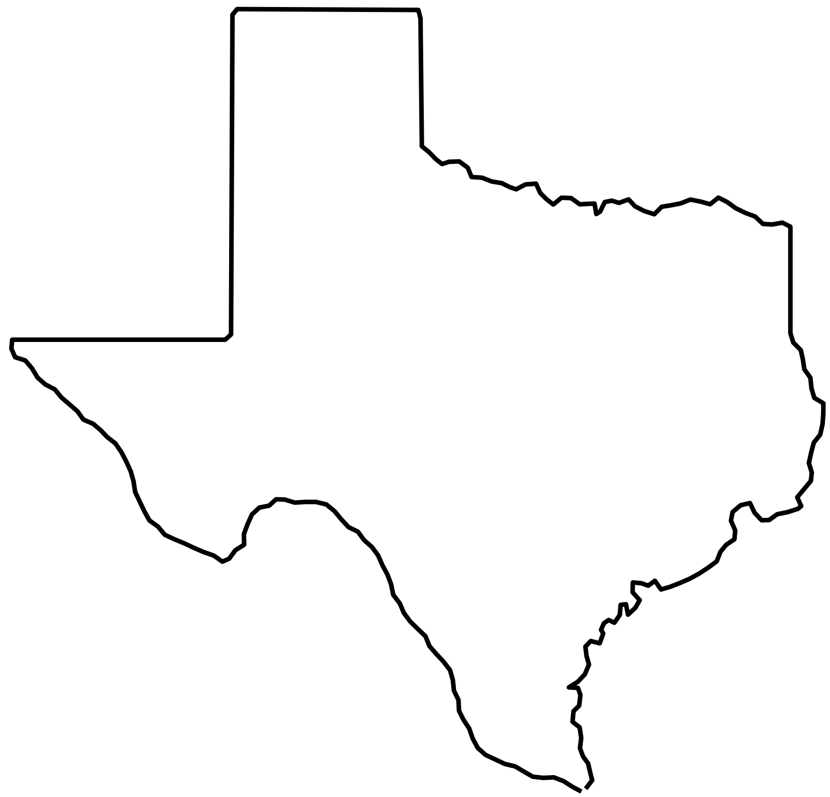
Such a circle homeomorphism is called a **conformal welding** or **welding**. Sometimes called a conformal sewing or gluing.

There is an analogous definition for unbounded Γ and homeomorphisms of $\mathbb{R}$.

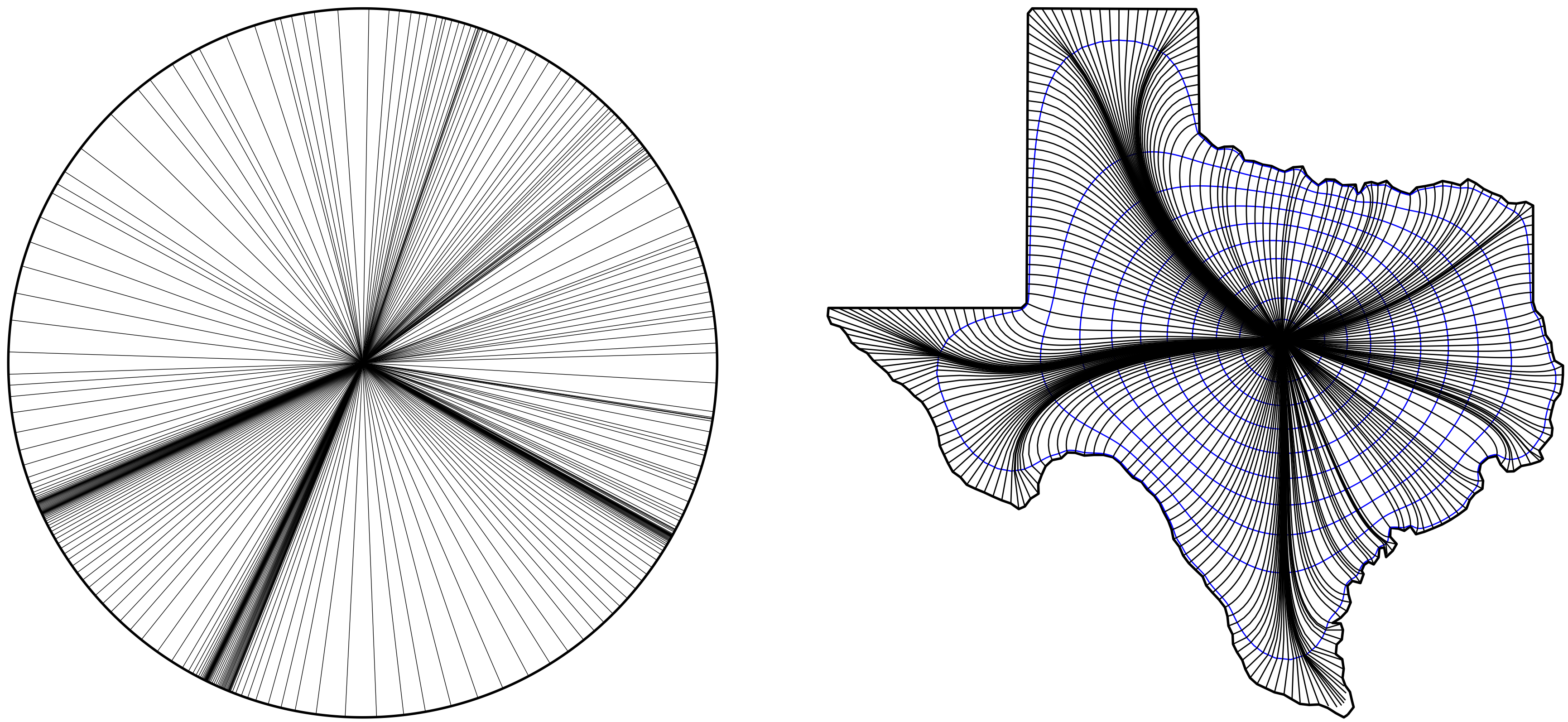
Not every homeomorphism is a welding.

Oikawa showed that if $h(x) = -|x|^\alpha$ for $x \leq 0$ and $h(x) = x^\beta$ for $x > 0$, and $\alpha \neq \beta$, then h is not a conformal welding.

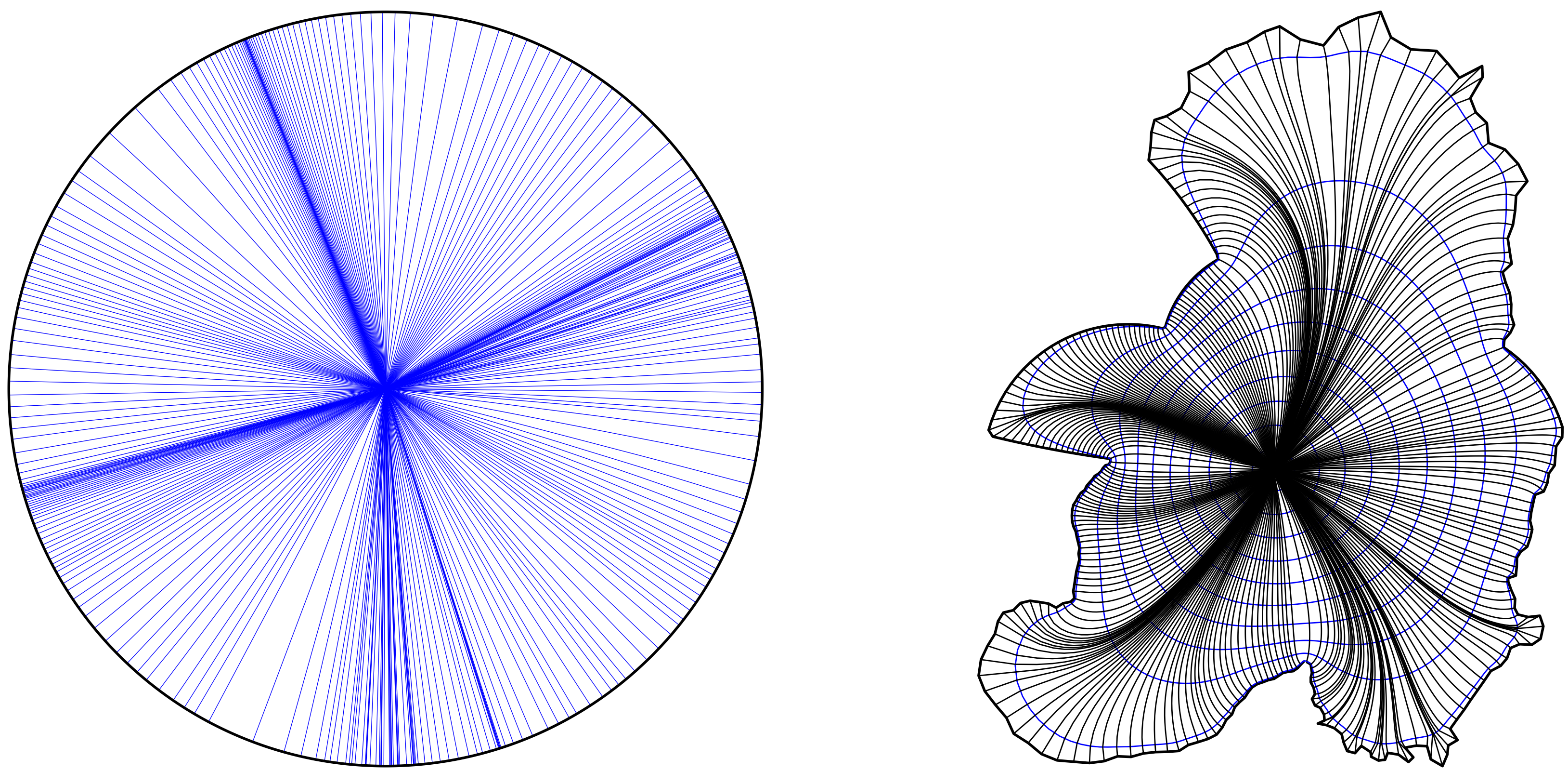
Let Γ be the union of the graph of $\sin(1/x)$ and the segment $[i, -i]$. This set divides the plane into two simply connected regions, so there are an associated conformal maps f, g that define a circle homeomorphism h. One can prove that h is not a conformal welding.

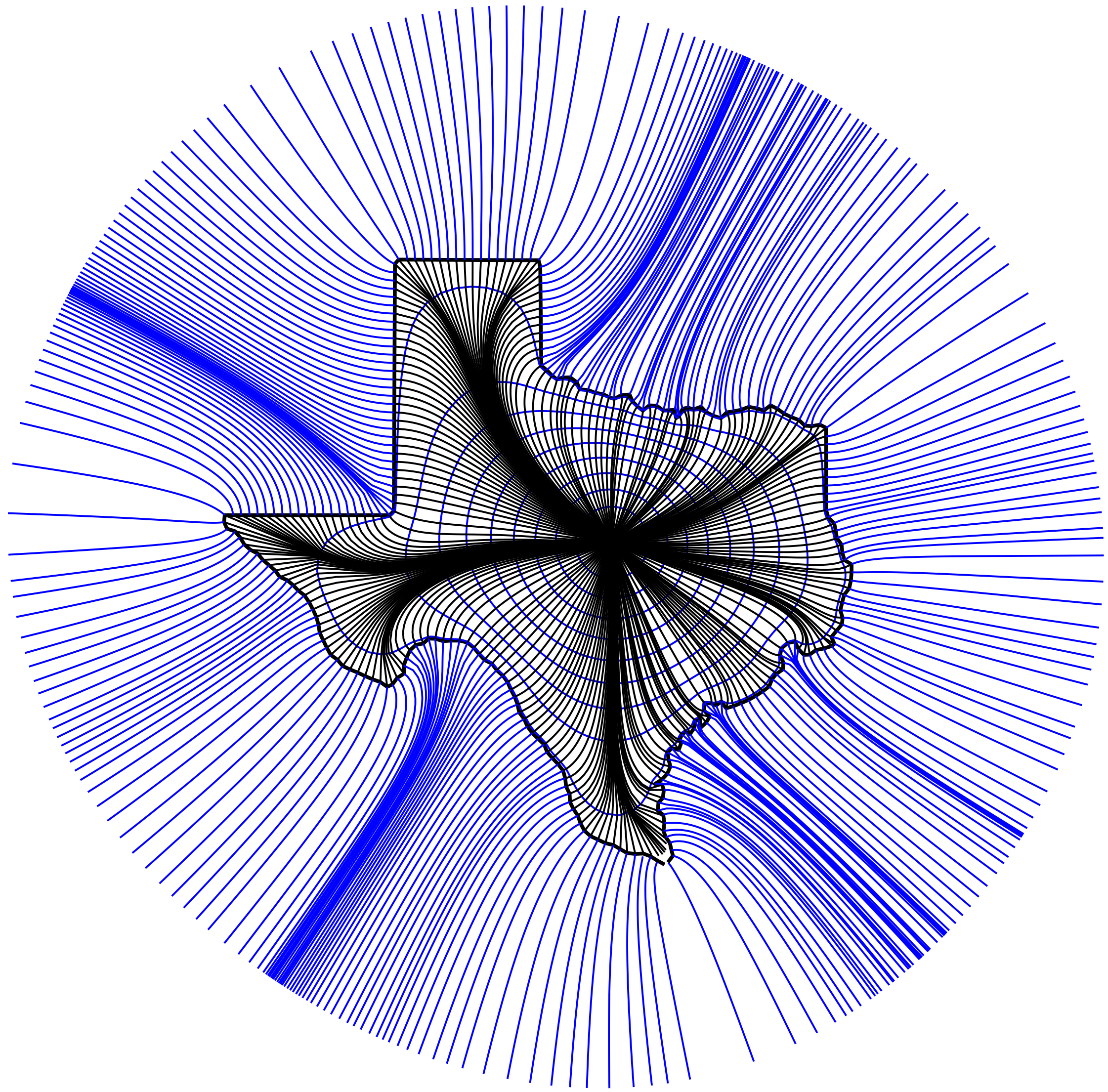


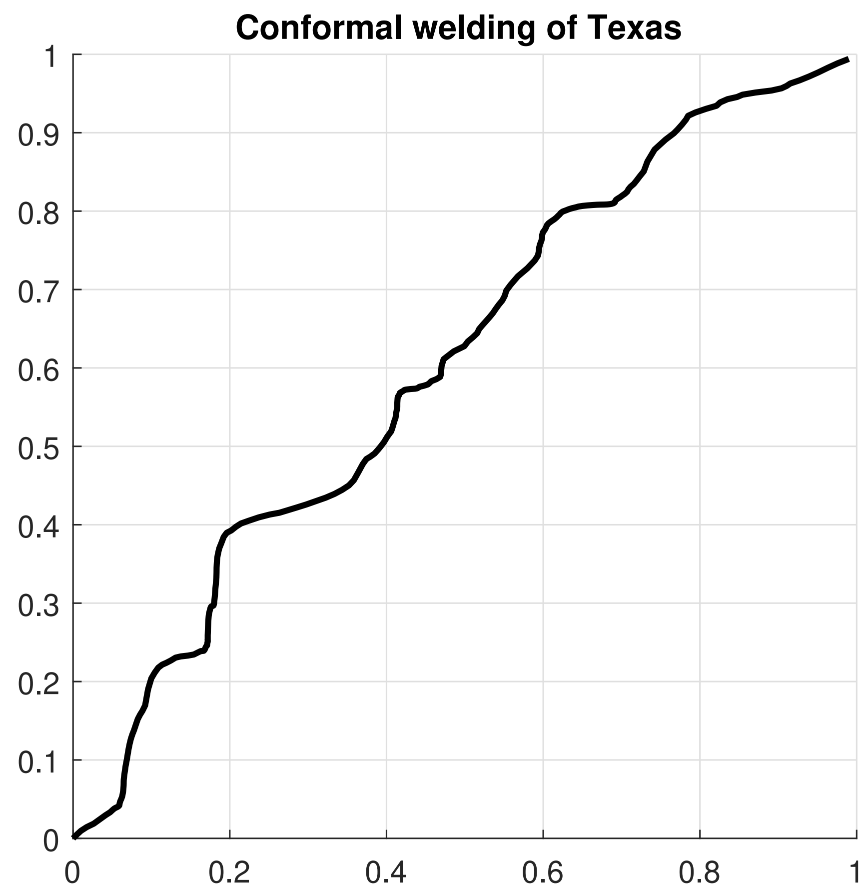
A polygon (looks like Texas)



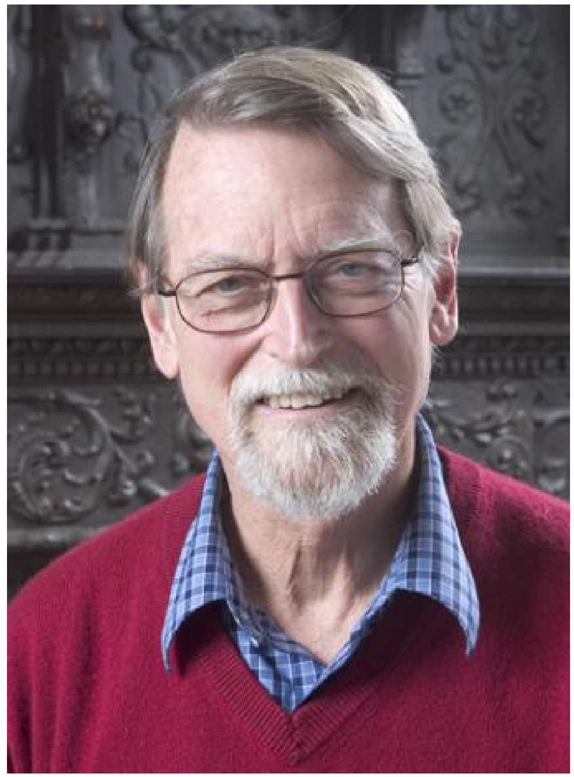
Texas reflected through a circle.







Conformal welding of Texas
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.2
0.4
0.6
0.8
1



David Mumford

2D-shape analysis using conformal mappings

by E. Sharon and D. Mumford

Theorem 5.20 (Fundamental Theorem of Conformal Welding). *A circle homeomorphism is quasisymmetric if and only if it is the conformal welding of a quasicircle.*

Of course, there are many weldings that are not quasisymmetric, e.g., the welding of any non-quasicircle.

Given any circle homeomorphism h and any $\epsilon > 0$ there is a welding map ϕ so that $h = \phi$ except on a set of Lebesgue measure ϵ. See My 2007 Annals paper.

The *Annals* paper also proves that "wild" homeomorphisms are weldings.

We say a homeomorphism h is log-singular if there is set $E \subset \mathbb{T}$ of zero logarithmic capacity so that $h(\mathbb{T} \setminus E)$ also has zero capacity.

Theorem: If h is log-singular then it is a conformal welding.

The resulting curve is very far from unique: any closed curve can be approximated by a curve with welding h.

Using this, Alex Rodriguez recently proved that any circle homeomorphism is a composition of two conformal weldings. See his paper on arXiv.

The fundamental theorem is is due to Pfluger, and has several proofs, e.g, using the measurable Riemann mapping theorem.

Assuming MRMT (for smooth μ), we can argue as follows.

Suppose $h : \mathbb{R} \to \mathbb{R}$ is quasisymmetric and let H be a QC extension to $\mathbb{R}^2$.

Choose a QC map G so that $\mu_G = \mu_H$ in upper half plane amd $\mu_G = 0$ in lower half-plane. (Then $G \circ H^{-1}$ is conformal in $\mathbb{H}$ and G is conformal in $\mathbb{H}_l$).

We claim $\Gamma = G(\mathbb{R})$ has welding h. Note $f = G$ is conformal from lower half-plane to one side of Γ. Next, $g = G \circ H^{-1}$ is conformal from upper half-plane to other side of Γ. Finally $g^{-1} \circ f = H \circ G^{-1} \circ G = H = h$ on $\mathbb{R}$.

We will give a proof that is very geometric and only uses the following facts:

- K-quasiconformal maps are compact.
- Quasisymmetric maps on $\mathbb{T}$ extend to be quasiconformal on the disk.
- Circles are removable for quasisymmetric maps.
- The uniformization theorem (for finitely connected planar domains).
- Koebe's circle domain theorem.

The first four we have discussed before.

Koebe's' circle domain theorem states that every finitely connected planar domain can be conformally mapped to a domain bounded only by circles or points.

This we will accept on faith.

One proof of this theorem is given in 2005 thesis of Karyn Lundberg.

We define a circle chain $\mathcal{C}$ to be a finite union of closed disks $\{D_k\}_1^n$ in $\mathbb{R}^2$ which have pairwise disjoint interiors and such that D_k is tangent to D_{k+1} for $k = 1, \ldots, n-1$, D_n is tangent to D_1 and there are no other tangencies. We also assume the disks are numbered in counterclockwise order.

The complement, $X = S^2 \setminus \cup_k D_k$, of a circle chain consists of two disjoint Jordan domains. We denote the bounded component by Ω and the unbounded component by Ω^*.

Let $f : \mathbb{D} \to \Omega$ and $g : \mathbb{D}^* \to \Omega^*$ be Riemann maps.

We call (f, g) a normalized circle chain pair if $f(0) = 0$, $g(\infty) = \infty$ and $\mathrm{dist}(0, \partial\Omega) = 1$.

Clearly, given a circle chain, we can always obtain a normalized pair by composing with a Möbius transformation.

Lemma 5.21. *Suppose $h : \mathbb{T} \to \mathbb{T}$ is an orientation preserving homeomorphism and suppose $\{x_k\}_1^n \subset \mathbb{T}$ is a finite collection of distinct points listed in counterclockwise order. Let $I_k = (x_k, x_{k+1}), k = 1, \dots, n$ (modulo n). Then there is a normalized circle chain pair so that for each k,*

$$f(I_k) = \partial D_k \cap \partial \Omega,$$
$$g(h(I_k)) = \partial D_k \cap \partial \Omega^*.$$

We will say that any circle chain that satisfies this conclusion corresponds to h.

Another way of stating the lemma is that given any finite positive sequences $\{a_k\}$ and $\{b_k\}$ such that $\sum_{k=1}^n a_k = \sum_{k=1}^n b_n = 1$ we can find a circle chain so that the harmonic measure of each disk in the chain satisfies

$$\omega(D_k, 0, \Omega) = a_k, \quad k = 1, \dots n,$$
$$\omega(D_k, \infty, \Omega^*) = b_k, \quad k = 1, \dots n.$$

It is a fact that this circle chain is unique up to Möbius transformations, but we will not need this here.

Proof of Lemma 5.21. We apply the Koebe circle domain theorem to a domain $\Omega = \Omega_\epsilon$ constructed as follows.

Given n points $\{x_k\}$ on the unit circle $\mathbb{T}$, let $y_k = 2h(x_k) \in 2\mathbb{T} = \{z : |z| = 2\}$. Let γ_n be disjoint smooth Jordan arcs which connect x_k to y_k in the annulus $A = \{z : 1 \leq |z| \leq 2\}$.

Let $\{I_k\} \subset \mathbb{T}$ be the arcs bounded by the points $\{x_k\}$ and let $\{J_k\}$ be the corresponding arcs on $2\mathbb{T}$. Thus J_k has harmonic measure $|h(I_k)|$ with respect to ∞. Let $\delta = \inf_k |h(I_k)|$ be the smallest of these harmonic measures.

Our domain Ω_ϵ is the union of $\mathbb{D}$, $2\mathbb{D}^* = \{z : |z| > 2\}$ and an ϵ-neighborhood of each γ_n, where ϵ is assumed to be so small that these neighborhoods are pairwise disjoint and $\partial\Omega$ has n components.

Let $f_\epsilon : \Omega_\epsilon \to \Omega_\epsilon^*$ be the map given by Koebe's theorem. Using a Möbius transformation we may assume $f(0) = 0$, $f(\infty) = \infty$ and $\text{dist}(0, \partial\Omega_\epsilon) = 1$.

We claim that the n circles in the complement of Ω_ϵ^*, are all contained in some disk $D(0, R)$ with R independent of ϵ (but R may depend on h and n).

To see this, suppose the union of closed disks satisfies $\cup_k D_k \subset \{1 \leq |z| \leq R\}$ and that it hits both boundary components. Let Ω_1 be the connected component of $f_\epsilon(\Omega_\epsilon \cap D(0, 3/2))$ containing 0.

Then for ϵ small enough, each interval I_k has harmonic measure $\geq 1/2n$ in Ω_1 and hence has capacity in Ω_1 which is bounded away from zero depending only on n.

Thus by Lemma 2.23, every disk must hit $\{|z| \leq M_1\}$, for some M_1 depending only on n.

Similarly for Ω_2 (the connected component of $f_\epsilon(\Omega \cap \{|z| > 3/2\})$ containing ∞), i.e., there is a M_2 depending only on δ such that every disk must hit $\{|z| \geq R/M_2\}$.

If R is so large that $R/M_2 > 3M_1$, then every disk in our chain hits both $\{|z| \leq M_1\}$ and $\{|z| \geq 3M_1\}$. Therefore For large n this contradicts the following fact:

Lemma 5.22. *At most 6 disjoint disks can hit both* $\{|z| = 1\}$ *and* $\{|z| = 3\}$.

Proof. Each such disk has a subdisk of radius 1 contained in the annulus $\{1 \leq |z| \leq 3\}$. Each of these intersects the circle $\{|z| = 2\}$ in an arc of angle measure $2\arctan(1/2) \approx .9273 > \pi/3$, and hence there can be at most 6 such disks. $\square$

Since we now know that the n disks all reamin inside a fixed annulus $\{1 \leq |z| < R\}$, every disk remains bounded.

Since each disk has a fixed harmonic measure from 0, its radius remains uniformly bounded away from zero.

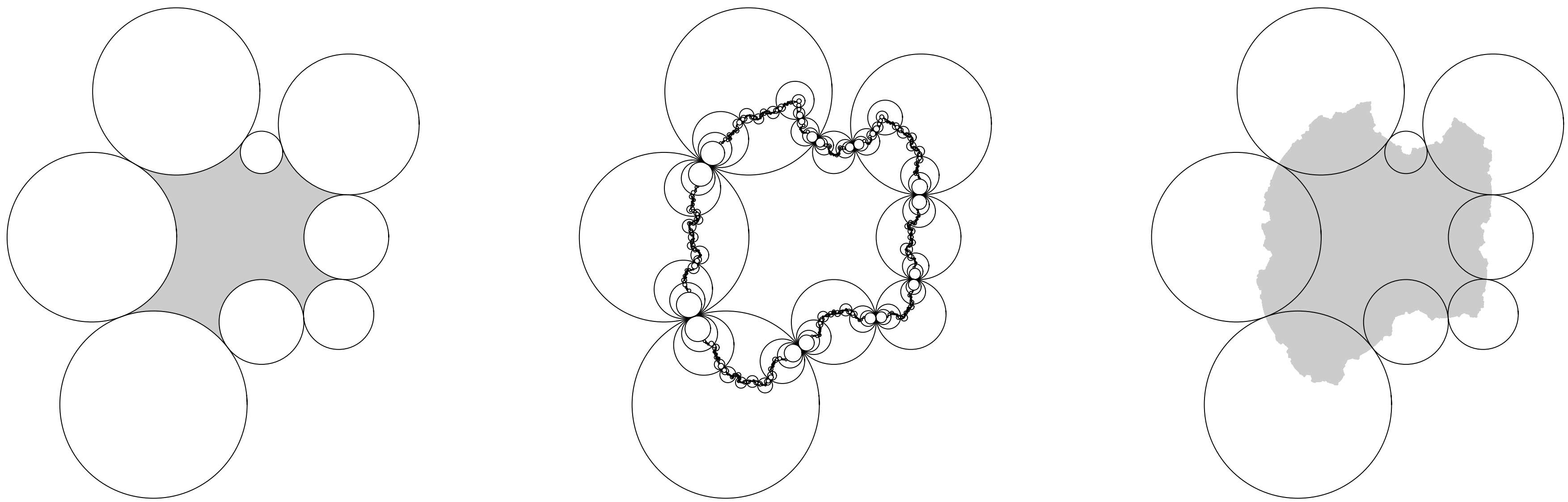
Thus we can pass to the limit as $\epsilon \to 0$, and get a circle chain of n non-degenerate tangent circles, that each have the correct harmonic measure.

Consequtive circles must touch in the limit, since the extremal distance beween them is zero. Non-conequtive circles do not touch because their extremal distance is positive.

This proves Lemma 5.21. $\square$

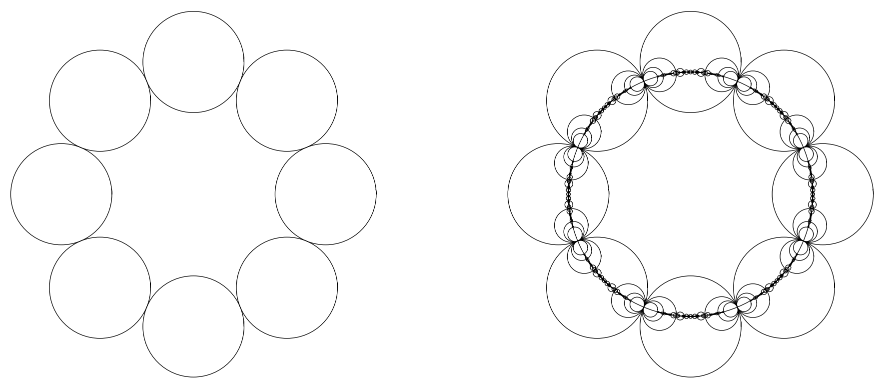
Proof of the Fundamental Theorem. Given a homeomorphism h and n equidistributed points $\{x_k\}_1^n \subset \mathbb{T}$, let $y_k = h(x_k)$ for $k = 1, \ldots n$ and consider the corresponding circle chain $\mathcal{C}_n$ as given by Lemma 5.21.

As before, let Ω_n, Ω_n^* denote the bounded and unbounded complementary domains. By reflecting through each circle we obtain a new chain with $n(n-1)$ circles. Continuing in this way we obtain, in the limit, a Jordan curve Γ_n, with complementary components D_n (bounded) and D_n^* (unbounded).



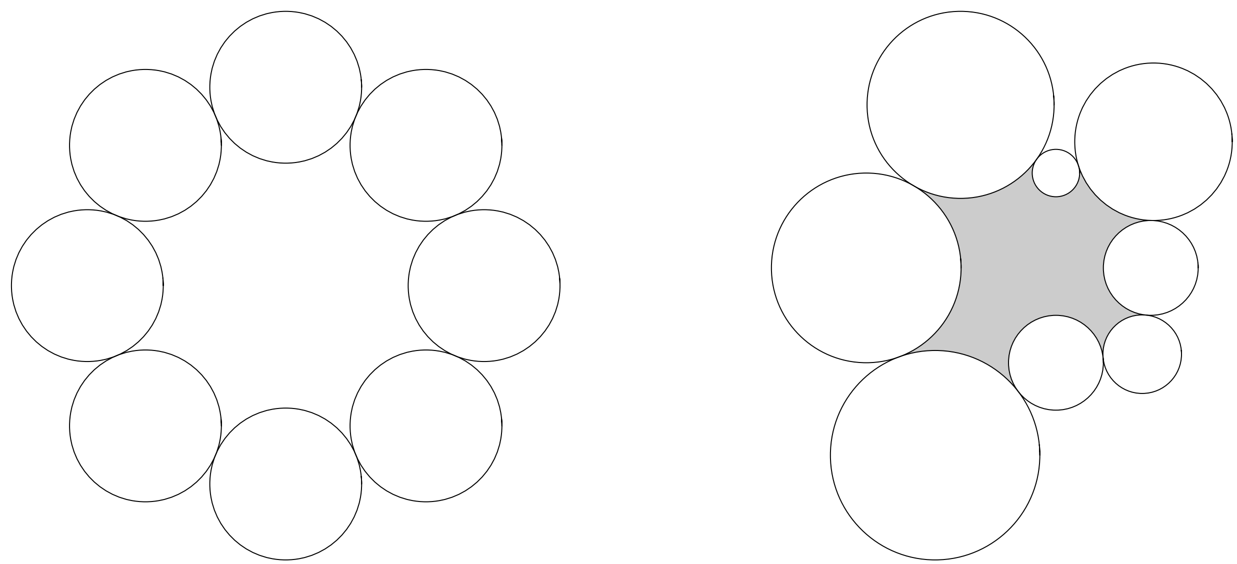
This shows the original chain and the domain Ω_n on the left, three iterations of the reflections in the center and the corresponding domain D_n on the right.

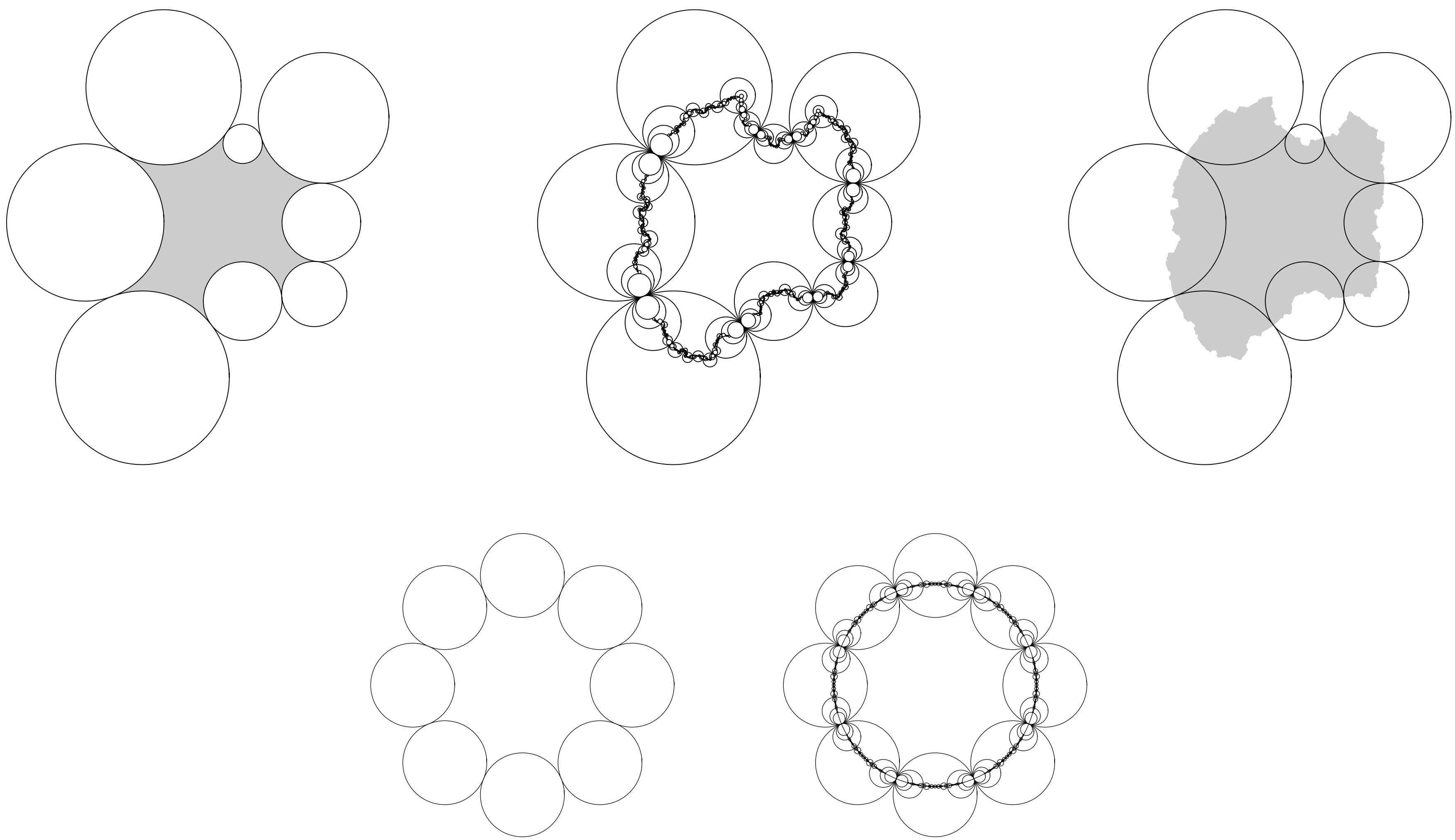
Similarly, given a circle chain $\mathcal{D}_n$ of n circles of equal size, with tangent points along the unit circle, we can reflect through the circles, getting a nested sequence of circle chains which limit on the unit circle, as shown below.



If h quasisymmetric, we know it is the boundary extension of some K-quasiconformal selfmap of the disk.

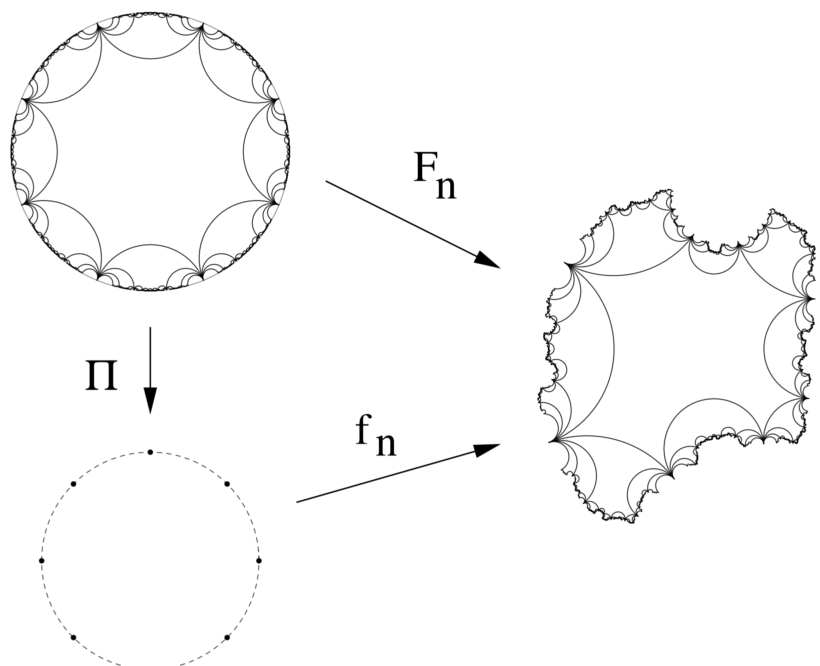
We claim there is a K-quasiconformal map of the plane sending the circles this figure to the circles in the previous figure. We will prove this by constructing the map separately inside and outside the unit circle.





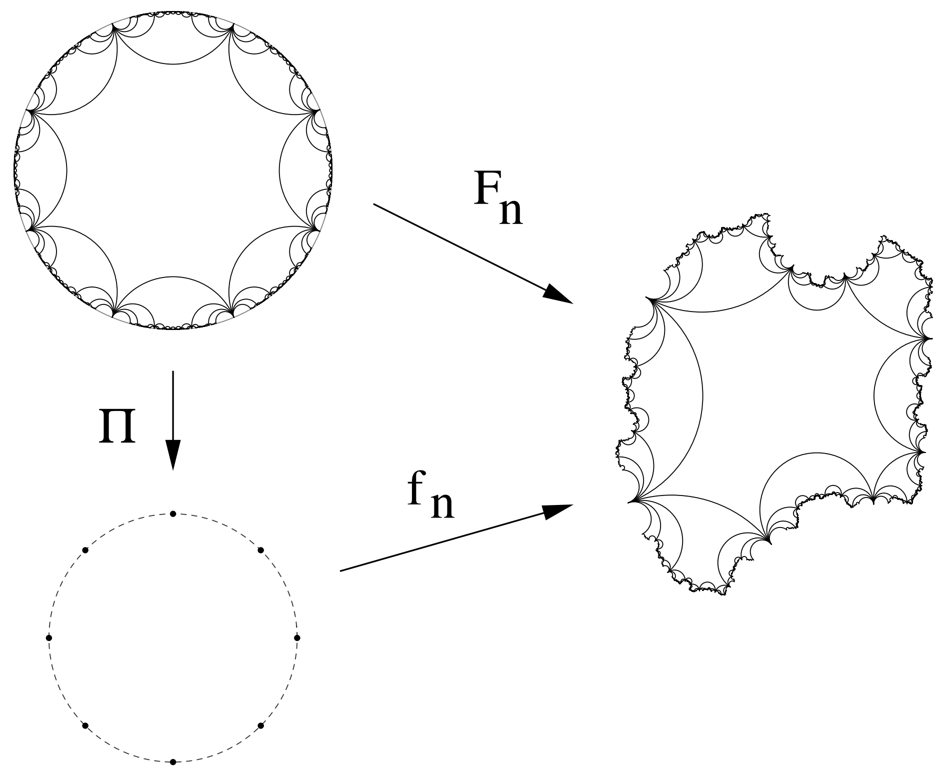
Let $W_n = S^2 \setminus \{x_1, \ldots, x_n\}$. We may assume $n \geq 3$, so there is a universal covering map $\Pi : \mathbb{D} \to W_n$.

Let U_n be the component of $\Pi^{-1}(\mathbb{D})$ containing the origin, and note that by symmetry U_n may be chosen to be bounded by hyperbolic geodesics with end-points at the x_k's (the arcs $\mathbb{T} \setminus \cup\{x_k\}$ are hyperbolic geodesics in W_n; this is even clearer if we map $\mathbb{T}$ to $\mathbb{R}$ by a Möbius transformation).



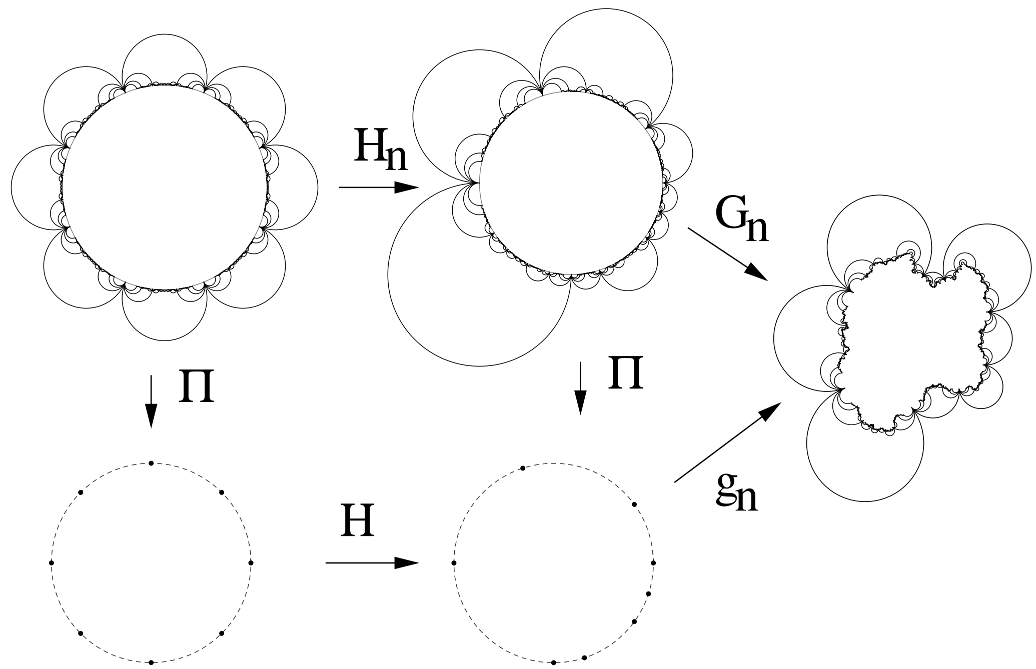
Reflecting these arcs across $\mathbb{T}$ gives the circle chain $\mathcal{D}_n$ in the figure below.

The conformal map $f_n \circ \Pi : U_n \to \Omega_n$ can be extended by repeated Schwarz reflection to a conformal map $F_n : \mathbb{D} \to D_n$.



Similarly, Koebe's theorem gives a conformal map $g_n : \mathbb{D}^* \to \Omega_n^*$. Let $W_n^* = S^2 \setminus \{y_1, \dots, y_n\}$ and consider $\Pi : \mathbb{D}^* \to W_n^*$ as the universal cover of W_n^*.

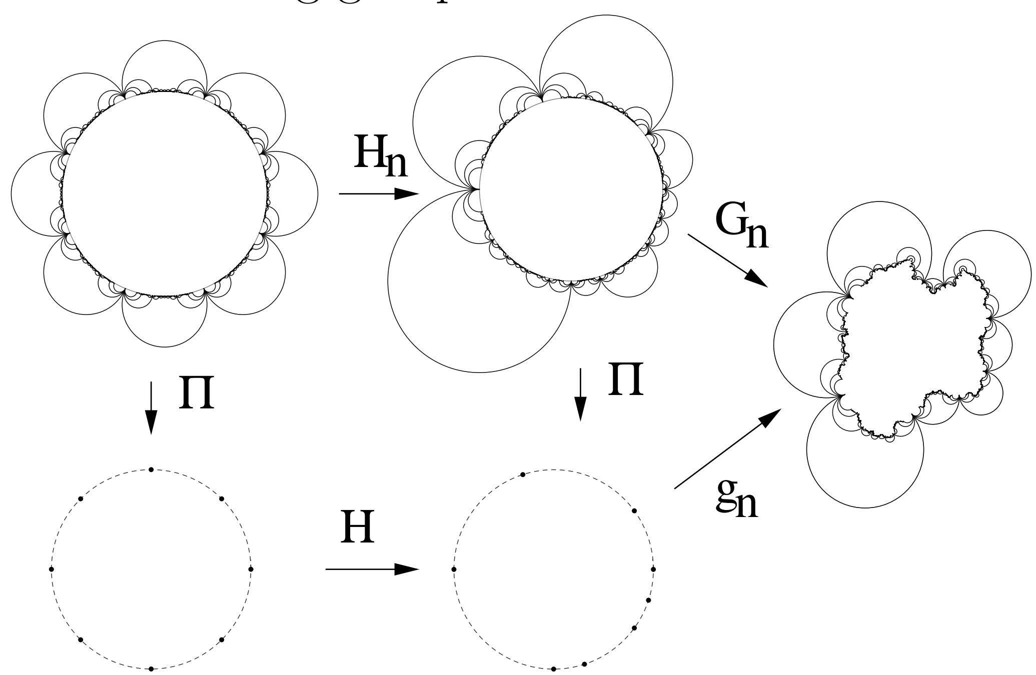
As above, we can lift g_n to map of $\Pi^{-1}(\mathbb{D}^*) \to \Omega_n^*$ and use Schwarz reflection to extend it to a map G_n from $\mathbb{D}^* \to D_n^*$. See below.



By assumption h is the boundary extension of a K-QC map of the disk to itself. By reflection we can extend this is a K-QC map H of S^2 to itself.

Then H maps W_n to W_n^* and lifts to a K-QC map of the universal covers.

We can represent these by $\mathbb{D}^*$ so we get a K-quasiconformal map $H_n : \mathbb{D}^* \to \mathbb{D}^*$ which conjugates the covering groups.



Thus $G_n \circ H_n$ is a K-quasiconformal map of $\mathbb{D}^*$ to D^* whose boundary values agree with F_n on $\mathbb{T}$, and hence these maps together define a K-quasiconformal map of S^2, since circles are removable for QC mappings.

This map takes $\mathbb{T}$ to Γ_n and the circle chain $\mathcal{D}_n$ to the chain $\mathcal{C}_n$.

Taking $n \to \infty$, using the uniform continuity of K-quasiconformal mappings and passing to a subsequence if necessary, we see that our circle chains converge uniformly to a K-quasicircle and that h is the corresponding conformal welding, as desired. $\square$

This semester I hope to cover the following topics:

- Review of conformal mappings
- Extremal length and conformal modulus, log capacity, harmonic measure
- Geometric definition quasiconformal mappings
- Basic properties
- Quasisymmetric maps and boundary extension
- Removable sets
- Conformal welding
- Analytic definition of quasiconformal mappings
- The measurable Riemann mapping theorem
- Further topics

Statement of the MRMP

Our goal in this section is to prove:

Theorem 6.1. *[Measurable Riemann Mapping Theorem] Given any measurable function μ on the plane with $\|\mu\|_\infty = k < 1$, there is a $K = (k+1)/(k-1)$ quasiconformal map f with dilatation $\mu_f = \mu$ Lebesgue almost everywhere on $\mathbb{C}$.*

The idea of the proof is fairly simple.

Given a measurable μ find a sequence of "nice" functions $\{\mu_n\}$ with $\mu_n \to \mu$ pointwise and $\sup_{\mathbb{C}} |\mu_n(z)| \leq k = \|\mu\|_\infty < 1$.

For nice dilatations, there are corresponding K-QC map f_n with dilatation μ_n, and we may assume these maps are normalized to fix 0 and 1.

By compactness of K-QC maps there is a subsequence that converges uniformly on compact subsets of the plane to a K-QC map f.

Finally, we have to prove f is differentiable almost everywhere, and its dilatation μ_f equals μ almost everywhere.

The last step is the hard one, and requires two deep theorems.

Proposition 6.2. *A K-quasiconformal map f defined on a planar domain Ω is differentiable almost everywhere on Ω. The dilatation $\mu_f = f_{\bar{z}}/f_z$ is well defined and less than $k < 1$ almost everywhere.*

Proposition 6.3. *Suppose $\{f_n\}$, f are all K-quasiconformal maps on the plane with dilatations $\{\mu_n\}$, μ_f respectively, that $f_n \to f$ uniformly on compact sets and that $\mu_n \to \mu$ pointwise almost everywhere. Then $\mu_f = \mu$ almost everywhere.*

On the other hand, finding the "nice" dilatations is relatively easy.

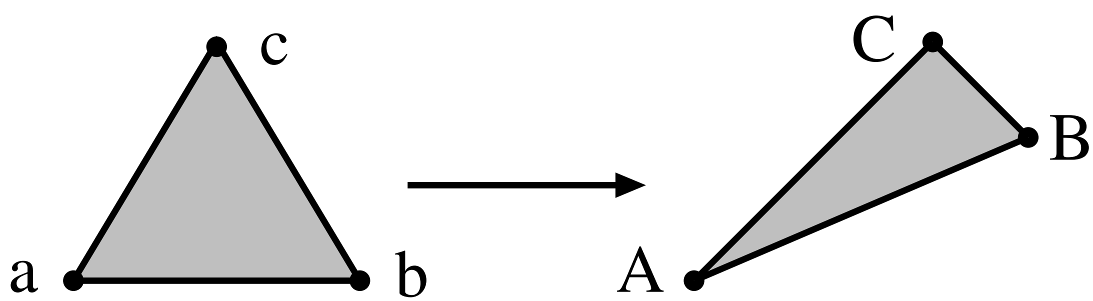
For now, “nice” dilatation will mean piecewise constant on a triangulations.

Later, we will also show that we can take smooth dilatations of compact support, and prove existence using integral operators..

We say that a linear map f is **K-quasiconformal** if $D_f \leq K$. The linear map need not be defined on the whole plane.

Given two triangles T_1, T_2 with vertices a, b, c and A, B, C, there is a unique affine map $T_1 \to T_2$ taking $a \to A$, $b \to B$ and $c \to C$.

The map is orientation preserving if both triangles were labeled in the same orientation.



There is an obvious affine map between these triangles and we can easily compute its quasiconformal constant of this map as follows.

First use a conformal linear map to send each triangle to one of the form $\{0, 1, a\}$ and $\{0, 1, b\}$. The affine map is then of the form $f(z) \to \alpha z + \beta \bar{z}$ where $\alpha + \beta = 1$ and $\beta = (b - a)/(a - \bar{a})$ and from this we see that

$$K_f = \frac{1 + |\mu_f|}{1 - |\mu_f|},$$

where

$$\mu_f = \frac{f_{\bar{z}}}{f_z} = \frac{\beta}{\alpha} = \frac{b - a}{b - \bar{a}},$$

If the triangle T' is degenerate, or has the opposite orientation as T, we simply give ∞ as our QC bound K.

Triangulate the plane using a triangular grid with elements of size δ_n.

Given a measurable μ on the plane, define μ_n to be the average of μ on each triangle of the grid.

Clearly $\|\mu_n\|_\infty \leq \|\mu\|_\infty$ and $\mu_n \to \mu$ (by the Lebesgue differentiation theorem).

For each triangle T in the grid let T' be the triangle so that affine map between them has constant dilatation $\mu_n|_T$.

Then attach these triangles T' in the same pattern as the T's.

We get a simply connected, non-compact Riemann surface R_n and a QC map $g_n : \mathbb{C} \to R_n$.

By the uniformization theorem, R_n is conformally equivalent either the plane or the disk.

Since this surface is QC equivalent to the plane, it must be the plane, i.e., there is a conformal map $f_n \to R_n \to \mathbb{C}$. (Can't be the disk by Lemma 4.3.)

This gives a quasiconformal map $\phi_n = f_n \circ g_n : \mathbb{C} \to \mathbb{C}$ with dilatation μ. By composing with a conformal linear map, we can assume 0 and 1 are fixed by f_n.

Since the dilatations μ_n have absolute value bounded above by $\|\mu\|_\infty < 1$, there is a subsequence that converges uniformly on compact sets to a quasiconformal map f.

As noted above, we now have to show the hard part: f has a well defined dilatation and this is equal to μ.

The main technical difficulty involves Pompeiu's formula:

$$f(w) = \frac{1}{2\pi i}\int_{\partial\Omega}\frac{f(z)}{z-w}dz - \frac{1}{\pi}\iint_{\Omega}\frac{f_{\overline{z}}}{z-w}dxdy. \tag{6.12}$$

However, it is not even clear whether this formula makes sense for a quasiconformal map; since f is continuous, the first integral is well defined, but it is not clear whether the second integral is well defined in general; we need to verify that $f_{\overline{z}}$ is defined.

We expect (but have not yet proved) that

$$\begin{aligned}
\text{area}(f(\Omega)) &= \int_\Omega J_f dxdy \\
&= \int_\Omega |f_z|^2 - |f_{\overline{z}}|^2 dxdy \\
&= \int_\Omega |f_z|^2(1 - |\mu_f|^2) dxdy,
\end{aligned}$$

which would imply f_z and $f_{\overline{z}}$ are in L^2 locally.

However, $|z - w|^{-1}$ is not in L^2, so we can't be sure that the area integral in the Pompeiu formula is convergent.

But $|z-w|^{-1}$ it in L^q locally for every $q<2$, so the integral will be bounded if we can show $f_{\overline{z}} \in L^p$ locally for some $p>2$.

This is a fundamental result of Bojarski in $\mathbb{C}$ and of Gehring in dimensions ≥ 2 and we will prove it later in this chapter, using the 2-dimensional version of Gehring's proof.

Most of the work consist of showing that for a K-quasiconformal map f, $f_z \in L^p$ for some $p>2$ that depends only on K.

Some facts from Real Analysis I

Next we recall some facts from real analysis.

Theorem 6.4 (Wiener's Covering Lemma)**.** *Let $\mathcal{B} = \{B_j\}$ be a finite collection of balls in $\mathbb{R}^d$. Then there is a finite, disjoint subcollection $\mathcal{C} \subset \mathcal{B}$ so that*

$$\cup_{B\in\mathcal{B}} B \subset \cup_{B\in\mathcal{C}} 3B.$$

In particular, the Lebesgue measure of the set covered by the subcollection is at least 3^{-d} times the measure covered by the full collection.

Theorem 6.5 (Vitali Covering Lemma)**.** *Suppose $E \subset \mathbb{R}^d$ is a measurable set and $\mathcal{B} = \{B_j\} \subset \mathbb{R}^d$ is a collection of balls so that each point of E is contained in elements of $\mathcal{B}$ of arbitrarily small diameter. Then there is a subcollection $\mathcal{C} \subset \mathcal{B}$ so that $E \setminus \cup_{B\in\mathcal{C}} B$ has zero d-measure.*

Theorem 6.6 (Lebesgue Dominated Convergence theorem)**.** *Suppose* $g \in L^2(\mu)$ *and* $\{f_n\}$ *satisfy* $|f_n| \leq g$ *and* $\lim f_n = f$ *pointwise. Then* $\lim \int f_n d\mu = \int f d\mu$.

Theorem 6.7 (Egorov's Theorem)**.** *Suppose* μ *is a finite positive measure and* $\{f_n\}$ *is a sequence of measurable functions that converge to* f *pointwise almost everywhere on a set* E *with respect to* μ*. Then for every* $\epsilon > 0$ *there is a subset* $F \subset E$ *so that* $\mu(E \setminus F) < \epsilon$ *and* $f_n \to f$ *uniformly on* F*.*

Lemma 6.8 (The Calderon-Zygmund lemma). *) Suppose Q is a square, $u \in L^1(Q, dxdy)$ and suppose*

$$\alpha > \frac{1}{\text{area}(Q)} \int_Q |u| dxdy.$$

Then there is a countable collection of pairwise disjoint open dyadic subsquares of Q so that

$$\alpha \leq \frac{1}{\text{area}(Q_j)} \int_{Q_j} |u| dxdy < 4\alpha, \tag{6.13}$$

$$|u| \leq \alpha \text{ almost everywhere on } Q \setminus \cup_j Q_j, \tag{6.14}$$

$$\sum \text{area}(Q_j) \leq \frac{1}{\alpha} \int_Q |u| dxdy \tag{6.15}$$

Proof. We say a subsquare of Q has property P is the first conclusion above holds and we define a collection of subsquares by iteratively dividing squares that do not have property P into four, equal sized disjoint subsquares, and stopping when property P is achieved.

If the average of u over a square is less than α then average over each of the four subsquares is $< 4\alpha$, so every stopped square has property P.

Any point not in a stopped square is a limit of squares where the average of u is $< \alpha$, so by the Lebesgue differentiation theorem $u \leq \alpha$ at almost every such point. Finally,

$$\int_Q |u| dx dy \geq \sum_j \alpha \text{area}(Q_j),$$

which proves the third property. $\square$

For a locally integrable function f, the **Hardy-Littlewood maximal function** of f is defined as

$$\mathcal{H}Lf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r))} |f(y)dy.$$

.

Here the supremum is over balls centered at x, but it is easy to see that we get some of comparable size we take all ball containing x.

Theorem 6.9 (Hardy-Littlewood maximal theorem)**.** *$\mathcal{H}L$ maps L^1 into weak-L^1, i.e., there is a constant d so that for all $\alpha > 0$*

$$|\{x : \mathcal{H}Lf(x) > \alpha\}| \leq \frac{C}{\alpha} \int |f(x)|dx.$$

Also, $\mathcal{H}L$ is a bounded operator on L^p for $1 < p \leq \infty$, i.e., there is a constant C_p so that $\|\mathcal{H}Lf\|_p \leq C_p \|f\|_p$.

Lemma 6.10. *If $\phi \geq 0$ is a compactly supported, radial, decreasing function with $\|\phi\|_1 = 1$ and f is locally integrable, then $|f * \phi(x)| \leq \mathcal{H}Lf(x)$.*

Theorem 6.11 (Marcinkiewicz interpolation theorem)**.** *Suppose* (X, μ) *and* (Y, ν) *are measure spaces, and suppose* $p_0, q_0, p_1, q_1 \in [1, \infty]$, *such that* $p_0 \leq q_0$, $p_1 \leq q_1$ *and*

$$\frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1}, \frac{1}{q} = \frac{1-t}{q_0} + \frac{t}{q_1}$$

for some $0 < t < 1$. *If* T *is a sub-linear map from* $L^{p_0}(\mu) + L^{p_1}(\mu)$ *to the space of measurable functions on* Y *that is weak-type* (p_0, q_0)

In particular, a sublinear operator that maps $L^1(\mu$ *boundedly into weak-*L^1 *and is bounded on* L^∞ *is also bounded from* L^p *to* L^p *for all* $1 < p < \infty$.

For the proof see Theorem 6.28 Folland's book.

Absolute Continuity on Lines

The main type of K-quasiconformal maps used in this text are piecewise C^1 functions that satisfy

$$|f_{\overline{z}}| \leq k|f_z|, \tag{6.16}$$

where $k - (K - 1)/(K + 1)$.

We discussed earlier that this equation holding almost everywhere is not enough to guarantee a map is quasiconformal.

For example, suppose $g : [0, 1] \to [0, 1]$ is the usual Cantor singular function.e., a continuous function that increases from 0 to 1 on $[0, 1]$ and is constant on each complementary component $\{I_j\}$ of the Cantor middle-$\frac{1}{3}$ set E. Then the map

$f(x, y) = (x + g(x), y)$, is a homeomorphism of $[0, 1] \times [0, 1]$ to $[0, 2] \times [0, 1]$ that is a translation (hence conformal) on each rectangle $I_j \times [0, 1]$, where I_j is a complementary interval of the Cantor set. Thus $f_{\overline{z}} = 0$ almost everywhere, but there are several way to check that f is not quasiconformal, It is not conformal because does not preserve the modulus of $[0,1]^2$.

If I is a covering interval of the Cantor set of length 2^{-n} whose image under g has length 3^{-n}, then the modulus of $I \times [0,1]$ is changed by a factor of $(3/2)^n$.

A map $f : \mathbb{R} \to \mathbb{C}$ is absolutely continuous if for every compact interval $I \subset \mathbb{R}$ and $\epsilon > 0$ there is a $\delta > 0$ so that $E \subset I$ and $|E| < \delta$ imply $|f(E)| < \epsilon$.

It is a theorem of real analysis that a function is absolutely continuous if it is differentiable almost everywhere, its derivative is locally in L^1, and the fundamental theorem of calculus holds: $f(b) - f(a) = \int_a^b f'(x)dx$.

Theorem 6.12. *If f is quasiconformal, then f is absolutely continuous on almost every line in any given direction.*

Proof. After applying a Euclidean similarity, we may consider horizontal lines in $Q = [0,1]^2$. Define

$$A(y) = \text{area}(f([0,1] \times [0,y])).$$

Then $A(0) = 0$, $A(1) = \text{area}(f(Q)) < \infty$ and A is increasing.

A is continuous except on a countable set and has a finite derivative a.e..

Fix a value of y where both this things happen, and we will show that f is absolutely continuous on the horizontal line $L_y = [0,1] \times \{y\}$.

The main idea is that if this failed, then modulus estimates relating length to area will force $A'(y) = \infty$.

Consider the long, narrow rectangle $R = [0,1] \times [y, y + \frac{1}{n}]$ and divide it into $m << n$ disjoint $\frac{1}{m} \times \frac{1}{n}$ sub-rectangles $\{R_j\}$.

Let $R'_j = f(R_j)$ and let the "left", "right", and "bottom" edges of R'_j be the images under f of corresponding edges of R_j.

Let b_j any value strictly less than the length of $f(L_y \cap \partial R_j)$, i.e., the length of the bottom edge of R'_j. This length might be finite or infinite, but b_j is finite.

Fix $\epsilon > 0$.

By taking n large enough, we can insure that any curve in $f(R_j)$ than joins the images of the vertical sides of R_j has length $\geq b_j$.

This follows because as $n \to \infty$, any curve in $f(R_j)$ joining the opposite "vertical" sides limits on the bottom edge and hence the liminf of the lengths of such curves is at least the length of the bottom edge of R'_j.

Since R_j is $(1/n) \times (1/m)$, by quasiconformality we deduce

$$M(R'_j) \geq M(R_j)/K = \frac{m}{Kn}.$$

Using the metric $\rho = 1/b_j$ on R'_j, shows

$$M(R'_j) \leq \frac{\text{area}(R'_j)}{b_j^2}.$$

or

$$b_j^2 \leq \frac{\text{area}(R'_j)}{M(R'_j)} \leq \frac{\text{area}(R'_j)}{m/Kn}.$$

Using these inequalities and Cauchy-Schwarz,

$$
\begin{aligned}
(\sum_{j=1}^{m} b_j)^2 &\leq (\sum_{j=1}^{m} b_j^2 m)(\sum_{j=1}^{m} \frac{1}{m}) = m \sum_{j=1}^{m} b_j^2 \\
&\leq m \sum_{j=1}^{m} \frac{\text{area}(R'_j)}{M(R'_j)} \\
&\leq m \sum_{j=1}^{m} \frac{\text{area}(R'_j)}{m/Kn} \\
&\leq K \sum_{j=1}^{m} \frac{\text{area}(R'_j)}{1/n} \\
&\leq K \cdot \frac{A(y+\frac{1}{n}) - A(y)}{1/n} \to K \cdot A'(y).
\end{aligned}
$$

If we can take $\sum b_j$ arbitrarily large, then $A'(y) = \infty$. So if $A'(y) < \infty$, then $f(L_y)$ has finite length.

Given a compact set $E \subset L_y$, suppose E is hit by N of the rectangles R_j and that m has been chosen so large that $N/m \leq 2m_1(E)$.

Now repeat the argument above, but only summing over the j's so that the bottom edges of R_j hit E.

$$
\begin{aligned}
(\sum_j b_j)^2 &\leq (\sum_j b_j^2 m)(\sum_j \frac{1}{m}) \\
&\leq N \sum_j \frac{\text{area}(R'_j)}{M(R'_j)} \\
&\leq N \sum_j \frac{\text{area}(R'_j)}{m/Kn} \\
&\leq \frac{N}{m} \sum_{j=1}^{m} \text{area}(R'_j) Kn \\
&\leq K \cdot m_1(E) \cdot \frac{A(y+\frac{1}{n}) - A(y)}{1/n} \to K \cdot m_1(E) \cdot A'(y).
\end{aligned}
$$

Thus $m_1(E)$ small, implies $\sum b_j$ is small, and hence $f(E)$ has small 1-dimensional measure. Hence f is absolutely continuous on L_y, as desired. $\square$

Basic theorems of real analysis say that if f is absolutely continuous on a line L , then its partial derivative along that lines exists almost everywhere and

$$f((b) - f(a) = \int_a^b f_n ds,$$

where $a, b \in L$ and f_n is the partial in the direction from a to b.

Since we have shown that quasiconformal maps are absolutely continuous on almost every horizontal and almost every vertical line, we see that the partial f_x, f_y exist almost everywhere and hence $f_z, f_{\overline{z}}$ are defined almost everywhere,

So is $\mu_f = f_{\overline{z}}/f_z$ almost everywhere that f_z in non-zero.

Next we want to say that at a point w where these $f_z, f_{\overline{z}}$ exist, we have

$$f(z) = f(w) + f_z(w)(z - w) + f_{\overline{z}}(w)(\overline{z} - \overline{w}) + o(|z - w|),$$

i.e., f is differentiable at w.

However, as explained in most calculus texts, the existence of partial derivatives at at a point does not imply a function is differentiable there (consider $f(x, y) = xy/(x^2 + y^2)$ at the origin).

Theorem 6.13. *[Gehring-Lehto] If f is a homeomorphism of $\Omega \subset \mathbb{C}$ and has partials almost everywhere, then it is differentiable almost everywhere.*

Proof. By Egorov's theorem the limits

$$f_x(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h},$$
$$f_y(z) = \lim_{h \to 0} \frac{f(z+ih) - f(z)}{h},$$

are uniform and converge to a continuous functions on a compact set $E \subset \Omega$ so that $\mathrm{area}(\Omega \setminus E)$ is as small as we wish.

Almost every point of E is a point of density for the intersection of E with both the vertical and horizontal lines through z_0, so if suffices to prove differentiability at such points.

For simplicity we assume 0 is such a point. The proof follows the usual case in calculus where we assume the partials are continuous, except that here we have to replace continuous on a neighborhood of 0 with continuous on a set E that is measure dense around 0.

Because of the continuity and uniform convergence on E, for any $\epsilon > 0$ there is a $\delta > 0$ so that

$$|f_x(0) - f_x(z)|, \quad |f_y(0) - f_y(z)| < \epsilon,$$

if $z \in E \cap D(0, \delta)$-neighborhood of 0 and

$$|f_x(z) - \frac{f(z+h) - f(z)}{h}|, \quad |f_y(z) - \frac{f(z+ih) - f(z)}{h}| < \epsilon,$$

if $z \in E \cap D(0, \delta)$ and $h \in [-\delta, \delta]$.

Note that for $z = x + iy$,

$$\begin{aligned} f(z) - f(0) - xf_x(0) - yf_y(0) &= [f(z) - f(x) - yf_y(x)] \\ &\qquad +[f(x) - f(0) - xf_x(0)] \\ &\qquad\qquad +[yf_y(x) - yf_y(0)] \\ &= I + II + III. \end{aligned}$$

If $|z| < \delta$ and $x \in E$, then by the inequalities above,

$$\begin{aligned} I &< \epsilon|y|, \\ II &< \epsilon|x|, \\ III &< \epsilon y. \end{aligned}$$

thus the term on the far left is bounded by $3\epsilon|z|$, which proves differentiability if $x \in E$. A similar proof works if $iy \in E$.

Fix $\epsilon > 0$ and choose δ so small that if $0 < x < \delta$, then $E \cap (\frac{x}{1+\epsilon}, x) \neq \emptyset$ (this must be possible since $E \cap \mathbb{R}$ has density 1 at 0) and $E \cap (\frac{iy}{1+\epsilon}, y) \neq \emptyset$.

Thus if $0 < |x|, |y| \leq \delta/(1+\epsilon)$ can find points

$$x_1 \in E \cap (\frac{x}{1+\epsilon}, x) \quad x_2 \in (x, (1+\epsilon)x)$$

$$y_1 \in E \cap (\frac{y}{1+\epsilon}, y) \quad y_2 \in (y, (1+\epsilon)y)$$

so that $z = x + iy$ is inside the rectangle $R = (x_1, x_2) \times (y_1, y_2)$.

Since f is a homeomorphism (all we need is that it is continuous and open), $|f|$ takes its maximum on the boundary, so

$$\begin{aligned}
\sup_{z=x+iy\in R} & |f(z)-f(0)-xf_x(0)-yf_y(0)| \\
&\leq \sup_{w=u+iv\in\partial R} |f(w)-f(0)-xf_x(0)-yf_y(0)| \\
&\leq \sup_{w=u+iv\in\partial R} |f(w)-f(0)-uf_x(0)-vf_y(0)| \\
&\qquad +|(x-u)f_x(0)|+|(y-v)f_y(0)| \\
&\leq 3\epsilon|w|+\sup_{w=u+iv\in\partial R} |x-u||f_x(0)|+|y-v||f_y(0)| \\
&\leq 3\epsilon(1+\epsilon)|z| = o(|z|).
\end{aligned}$$

$\square$

Corollary 6.14. *A K-quasiconformal map f defined on a planar domain Ω is differentiable almost everywhere on Ω.*

Proof. This is immediate from Theorems 6.12 and 6.13. $\square$

Lemma 6.15. *If f is K-quasiconformal then for every square Q,*

$$\int_Q J_f dxdy \leq \mathrm{area}(f(Q)) \leq \pi \mathrm{diam}(f(Q))^2.$$

Proof. We only use the quasiconformal hypothesis to deduce f is differentiable almost everywhere; the result holds for all such maps.

The second inequality area $\leq \pi\mathrm{diam}^2$ is obvious.

At any point x where f is differentiable we can choose a small square Q_x containing x such that

$$\mathrm{area}(f(Q_x)) \geq (1-\epsilon) J_f(x) \mathrm{area}(Q_x),$$

and by the Lebesgue differentiation theorem, for almost every x we have

$$\int_Q J_f dxdy \leq (1+\epsilon) J_f(x) \mathrm{area}(Q),$$

for all small enough squares Q centered at x.

Combining these two estimates and using the Vitali covering theorem to extract a collection of disjoint squares $\{Q_j\}$ with centers x_j and with these properties that cover almost every point of Q, we get

$$\begin{aligned}
\int_Q J_f dxdy &\leq \sum_j \int_{Q_j} J_f dxdy \\
&\leq (1+\epsilon) J_f(x_j) \text{area}(Q_j) \\
&\leq \frac{1+\epsilon}{1-\epsilon} \text{area}(f(Q_j)) \\
&\leq \frac{1+\epsilon}{1-\epsilon} \text{area}(f(Q)).
\end{aligned}$$

Taking $\epsilon \searrow 0$, gives $\text{area}(f(E)) \geq \int_E J_f dxdy$. $\square$

Since $J_f = |f_z|^2 - |f_{\overline{z}}|^2$, we have

$$J_f = |f_z|^2 - |\mu|^2|f_z|^2 = (1 - \|\mu|^2)|f_z|^2 \geq (1 - k^2)|f_z|^2$$

so

$$|f_z|^2 \leq \frac{J_f}{(1 - k^2)}.$$

Corollary 6.16. *If f is K-quasiconformal then for every square Q,*

$$\int_Q |f_z|^2 dxdy \leq \frac{\pi}{1 - k^2} \mathrm{diam}(f(Q))^2.$$

Lemma 6.17. *If f is K-quasiconformal, then*

$$\frac{(\int_Q |f_z| dxdy)^2}{\text{area}(Q)} \gtrsim \text{diam}(f(Q))^2.$$

with a uniform constant for every square Q.

Proof. The path family connecting opposite sides of a square Q has modulus 1, so the image of this family in $f(Q)$ has modulus between K and $1/K$.

If the shortest path in $f(Q)$ connecting the opposite sides of $f(Q)$ was $M \cdot \text{diam}(f(Q))$ than taking the constant metric $\rho = 1/M\text{diam}(f(Q))$ implies the modulus of this path family is $\leq \pi/M^2$, a contradiction if M is large.

This implies the shortest path in $f(Q)$ connecting the opposite sides of $f(Q)$ has length $\simeq \mathrm{diam}(f(Q))$

Thus so the integral of $|f_z| + |f_{\overline{z}}|$ along any horizontal segment crossing Q is at least $C\mathrm{diam}(f(Q))$ for some fixed $C > 0$ (depending only on K).

Since $|f_z| \leq |f_z| + |f_{\overline{z}}| \leq (1+k)|f_z|$, the same is true for the integral of $|f_z|$.

Integrating over all horizontal segments crossing Q gives

$$\int_Q |f_z| dxdy \gtrsim \operatorname{diam}(Q)\operatorname{diam}(f(Q)).$$

Hence

$$\frac{(\int_Q |f_z| dxdy)^2}{\operatorname{area}(Q)} \gtrsim \frac{[\operatorname{diam}(Q)\operatorname{diam}(f(Q))]^2}{\operatorname{area}(Q)} \gtrsim \operatorname{diam}(f(Q))^2.$$

$\square$

Lemma 6.18. *If f is K-quasiconformal, then*

$$\int_Q |f_z|^2 dxdy \leq C \frac{(\int_Q |f_z| dxdy)^2}{\text{area}(Q)}$$

Note, this goes in the opposite direction of the usual Cauchy-Schwarz estimate.

Proof. Note that for K-quasiconformal maps, $|\mu_f| \leq k = (K-1)/(K+1)$ and

$$|f_z|^2(1-k^2) \leq |f_z|^2(1-|\mu|^2) \leq |f_z|^2 - |f_{\overline{z}}|^2 = J_f \leq |f_z|^2,$$

so that J_f and $|f_z|^2$ are the same up to a bounded factor.

Thus combining the two previous results,

$$\int_Q |f_z|^2 dxdy \lesssim \operatorname{diam}(f(Q))^2 \lesssim \frac{(\int_Q |f_z| dxdy)^2}{\operatorname{area}(Q)}$$

or

$$\int_Q |f_z|^2 dxdy \leq C \cdot \frac{(\int_Q |f_z| dxdy)^2}{\operatorname{area}(Q)}$$

for some constant C that depends only on the quasiconformal constant of f (and not on the choice of the square Q). $\square$

Hölder's inequality implies

$$\int_Q |f_z| dxdy \leq \left(\int_Q |f_z|^2 dxdy\right)\left(\int_Q 1 dxdy\right) = \text{area}(Q) \cdot \left(\int_Q |f_z|^2 dxdy\right)$$

The inequality in the lemma goes in the opposite direction, and is called a reverse Hölder inequality.

Such inequalities are fundamental to certain parts of PDE.

We shall see later that it has profound implications for the behavior of f_z.

Gehring's inequality and Bojarski's theorem

Hölder's inequality says that

$$\int fg d\mu \leq (\int f^p d\mu)^{1/p} (\int g^q d\mu)^{1/q},$$

where $1 \leq p, q \leq \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$.

Applying this to a non-negative function v and the constant 1 on a square Q, and using $p - 1 = p/q$, we get

$$\int_Q v dx dy \, eq \left(\int_Q v^p dx dy \right)^{1/p} (\text{area}(Q))^{1/q}$$

$$\left(\frac{1}{\text{area}(Q)} \int_Q v dx dy \right)^p \leq \frac{1}{\text{area}(Q)} \int_Q v^p dx dy.$$

with equality if and only if v is a.e. constant.

Thus the "reverse Hölder inequality "

$$\frac{1}{\text{area}(Q)} \int_Q v^p dxdy \le \left(\frac{K}{\text{area}(Q)} \int_Q v dxdy \right)^p,$$

can only hold if $K \ge 1$.

If it holds for single Q, this does not say much, except that $v \in L^p \cap L^1$.

However, if it holds for all Q's, we can deduce that $v \in L^{p+\epsilon}$ for some $\epsilon > 0$.

This remarkable "self-improvement" estimate is due to Gehring, although the proof we give follows the presentation in Garnett's book *Bounded Analytic Functions* (Theorem VI.6.9).

Recall the distribution function of a measurable function f on a measure space (X, μ) is

$$d_f(t) = \mu(\{x : |f(x)| > t\}),$$

and the L^p norm of f can be computed as

$$\int |f|^p d\mu = p \int_0^\infty t^{p-1} d_f(t) dt.$$

We start with a technical lemma.

Lemma 6.19. *Suppose that* $p > 1$, $v \geq 0$, $E_\lambda = \{z : v(z) > \lambda\}$, *and that*

$$\int_{E_\lambda} v^p dxdy \leq A\lambda^{p-1} \int_{E_\lambda} vdxdy,$$

for all $\lambda \geq 1$. *Then there is* $r > p$ *and* $C < \infty$ *so that*

$$\left(\int_Q v^r dxdy\right)^{1/r} \leq C \cdot \left(\int_Q v^p dxdy\right)^{1/p}.$$

Proof. This is basically just arithmetic with distribution functions. Note that it suffices to assume area$(Q) = 1$ and $\int_Q v^p dxdy = 1$. Then if $v > 1$,

$$v^{r-p} - 1 = \int_1^v (r-p)\lambda^{r-p-1} d\lambda$$
$$v^{r-p} = 1 + (r-p)\int_1^v (\lambda^{r-p-1} d\lambda$$

so

$$\begin{aligned}
\int_{E_1} v^r dxdy &= \int_{E_1} v^p v^{r-p} dxdy \\
&= \int_{E_1} v^p(1 + (r-p)\int_1^v \lambda^{r-p-1} d\lambda) dxdy \\
&= \int_{E_1} v^p + (r-p)\int_1^\infty \lambda^{r-p-1} \int_{E_\lambda} v^p dxdyd\lambda
\end{aligned}$$

By our assumption,

$$\begin{aligned}\int_{E_1} v^r dxdy &\leq \int_{E_1} v^p + A(r-p)\int_1^\infty \lambda^{r-2}\int_{E_\lambda} vdxdyd\lambda \\ &\leq \int_{E_1} v^p + A(r-p)\int_{E_1} v(\int_0^v \lambda^{r-2}d\lambda)dxdy \\ &\leq \int_{E_1} v^p + A\frac{r-p}{r-1}\int_{E_1} v^r dxdy \\ &\leq \int_{E_1} v^p + \frac{1}{2}\int_{E_1} v^r dxdy\end{aligned}$$

where the last inequality holds if r is close enough to p (depending on A and p).

Subtracting the last term of the last step from the first step gives

$$\int_{E_1} v^r dxdy \leq 2 \int_{E_1} v^p dxdy.$$

Off E_1 we have $v \leq 1$ so $v^r \leq v^p$ and hence

$$\int_Q v^r dxdy \leq 3 \int_Q v^p dxdy.$$

Because of our normalizations, this proves the lemma. $\square$

Theorem 6.20. *Let $p > 1$. If $v(x) \geq 0$ and $v \in L^p(Q, dxdy)$, and if the "reverse Hölder inequality"*

$$(\frac{1}{\text{area}(Q)} \int_Q v^p dxdy) \leq (K \frac{1}{\text{area}(Q)} \int_Q v dxdy)^p,$$

holds for all subsquares of a square Q_0, then there is an $r > p$ so that

$$(\frac{1}{\text{area}(Q_0)} \int_{Q_0} v^r dxdy)^{1/r} \leq \frac{C(K, p, r)}{\text{area}(Q_0)} \int_{Q_0} v dxdy.$$

Proof. We need only verify the hypothesis of Lemma 6.19.

Fix λ and set $\beta = 2K\lambda$.

We split the integral

$$\int_{E_\lambda} v^p dxdy = \int_{E_\lambda \setminus E_\beta} v^p dxdy + \int_{E_\beta} v^p dxdy$$

into two pieces. The second piece is trivial to bound by the correct estimate because

$$\int_{E_\lambda \setminus E_\beta} v^p dxdy \leq \beta^{p-1} \int_{E_\lambda \setminus E_\beta} v dxdy \leq (2K\lambda)^{p-1} \int_{E_\lambda} v dxdy.$$

To bound the other piece of the integral, we use the Calderon-Zygmund lemma (Lemma 6.8) to find a sequence of disjoint squares $\{Q_j\}$ so that

$$\beta^p \leq \frac{1}{\text{area}(Q_j)} \int_{Q_j} v^p dxdy < 4\beta^p,$$

and $v \leq \beta$ almost everywhere off $\cup Q_j$.

Thus $E_\beta \setminus \cup Q_j$ has measure zero and

$$\int_{E_\beta} v^p dxdy \leq \sum_j \int_{Q_j} v^p dxdy \leq 4\beta^p \sum \text{area}(Q_j).$$

We now make use of the reverse Hölder hypothesis to write

$$\beta^p \leq \frac{1}{\text{area}(Q_j)} \int_{Q_j} v^p dxdy \leq \left(\frac{K}{\text{area}(Q_j)} \int_{Q_j} vdx \right)^p,$$

$$\begin{aligned}
\text{area}(Q_j) &\leq \frac{K}{\beta} \int_{Q_j} vdxdy \\
&\leq \frac{K}{\beta} \left(\int_{Q_j \cap E_\lambda} vdxdy + \lambda \cdot \text{area}(Q_j) \right) \\
&\leq \frac{K}{\beta} \int_{Q_j \cap E_\lambda} vdxdy + \frac{1}{2} \text{area}(Q_j)
\end{aligned}$$

since $\beta = 2K\lambda$. Solving for area(Q_j) gives

$$\text{area}(Q_j) \leq \frac{2K}{\beta} \int_{Q_j \cap E_\lambda} vdxdy \leq \frac{1}{\lambda} \int_{Q_j \cap E_\lambda} vdxdy.$$

Thus by the defining property of the Q_j's,

$$\begin{aligned}
\int_{E_\beta} v^p dxdy &\leq \sum_j \int_{Q_j} v^p dxdy \\
&\leq 4\beta^p \sum_j \operatorname{area}(Q_j) \\
&\leq 4\beta^p \lambda^{-1} \sum_j \int_{Q_j \cap E_\lambda} vdx \\
&\leq 2^{p+2} K^p \lambda^{p-1} \int_{E_\lambda} vdx.
\end{aligned}$$

Thus the hypothesis of Lemma 6.19 holds with $A = (2K)^{p-1} + 2^{p+2}K^p$, and we deduce that $v \in L^r(Q, dxdy)$ for some $r > p$. $\square$

Theorem 6.21 (Bojarski's Theorem)**.** *If* $1 \leq K < \infty$*, there is a* $p > 2$ *and* $A, B < \infty$ *so that the following holds. If* $f : \mathbb{C} \to \mathbb{C}$ *is* K*-quasiconformal, and* $Q \subset \mathbb{C}$ *is a square, then*

$$(\frac{1}{\text{area}(Q)} \iint_Q |f_z|^p dxdy)^{1/p} \leq A(\frac{1}{\text{area}(Q)} \int_Q |f_z|^2 dxdy)^{1/2} \leq B\frac{\text{diam}(f(Q))}{\text{diam}(Q)}$$

Proof. To apply Gehring's inequality to the partial derivatives of quasiconformal maps, we have to show that these partial satisfy a reverse Hölder inequality. What we want is

$$\int_Q |f_z|^2 dxdy \leq \frac{C}{\text{area}(Q)}(\int_Q |f_z| dxdy)^2,$$

with a uniform C for all squares in the plane. This was Lemma 6.18. $\square$

Lemma 6.22. *If f fixes $0, 1, \infty$, then*

$$\int_Q |L_f(x) - 1|^2 dxdy \leq \epsilon \cdot \text{area}(Q),$$

where $L_f = |f_z| + |f_{\overline{z}}|$ and $\epsilon \to 0$ as $\|\mu_f\|_\infty \to 0$.

Proof. Fix a square Q with sides parallel to the axes, let $\ell(Q)$ denote its side length and let S_1, S_2 denote the two vertical sides of S.

By Cauchy-Schwarz

$$0 \leq (\frac{1}{\text{area}(Q)} \int_Q |L_f - 1| dxdy)^2 \leq \frac{1}{\text{area}(Q)} \int_Q |L_f - 1|^2 dxdy.$$

Now expand and rearrange

$$
\begin{aligned}
&= \frac{1}{\text{area}(Q)} \int_Q (L_f^2 - 2L_f + 1)dxdy \\
&= \frac{1}{\text{area}(Q)} \int_Q (L_f^2 - 1 - 2L_f + 2)dxdy \\
&= \frac{1}{\text{area}(Q)} \int_Q (L_f^2 - 1)dxdy - \frac{2}{\text{area}(Q)} \int_Q (L_f - 1)dxdy
\end{aligned}
$$

Now use $(Lf)^2 = (|f_z| + |f_{\overline{z}}|)^2 \leq K(|f_z| - |f_{\overline{z}}|)(|f_z| + |f_{\overline{z}}|) = KJ_f$ to get

$$
\begin{aligned}
&\leq \frac{1}{\text{area}(Q)} \int_Q (KJ_f - 1) - \frac{2}{\text{area}(Q)} \int_Q (L_f - 1) dxdy \\
&\leq \frac{1}{\text{area}(Q)} \int_Q (KJ_f + J_f - J_f - 1) - \frac{2}{\text{area}(Q)} \int_Q (L_f - 1) dxdy \\
&= \frac{1}{\text{area}(Q)} \int_Q (K-1) J_f dxdy \\
&\qquad + \frac{1}{\text{area}(Q)} \int_Q (J_f - 1) dxdy \\
&\qquad\qquad - \frac{2}{\text{area}(Q)} \int_Q (L_f - 1) dxdy
\end{aligned}
$$

We claim each terms tends to zero with $\|\mu\|_\infty$. Since the quantity we are bounding is non-negative, we have to find upper bounds for the first two terms tending to 0, and a lower bound for the last integral tending to zero.

First,

$$
\begin{aligned}
\frac{1}{\text{area}(Q)}\int_Q (K-1)J_f\,dxdy &= O(\|\mu\|_\infty)\frac{1}{\text{area}(Q)}\int J_f\,dxdy \\
&= O(\|\mu\|_\infty).
\end{aligned}
$$

Next, since f tends to the identity on Q,

$$
\begin{aligned}
\frac{1}{\text{area}(Q)}\int_Q (J_f-1)\,dxdy &= \frac{1}{\text{area}(Q)}\int_Q J_f\,dxdy - \frac{1}{\text{area}(Q)}\int_Q 1\,dxdy \\
&= \text{area}(f(Q)) - \text{area}(Q) \to 0.
\end{aligned}
$$

Finally, the integral of $|f_z| + |f_{\overline{z}}|$ over a horizontal segment in Q gives an upper bound for the length of the image curve, and this must be at least the distance between the two vertical sides of $f(Q)$. Thus

$$\begin{aligned}
\frac{2}{\mathrm{area}(Q)} \int_Q (L_f - 1)dxdy &= 2\left(\frac{1}{\mathrm{area}(Q)} \int_Q L_f dxdy - 1)\right) \\
&\geq 2\left(\frac{\mathrm{dist}(S_1, S_2)\ell(Q)}{\mathrm{area}(Q)} - 1\right) \\
&\geq 2\left(\frac{\mathrm{dist}(S_1, S_2)}{\ell(Q)} - 1\right)
\end{aligned}$$

where S_1, S_2 are the vertical sides of $f(Q)$. This tends to zero since f tends uniformly to the identity on Q.

Because of the negative sign in front of the third term in our sum of integrals, this proves the result. $\square$

Corollary 6.23. *If f fixes $0, 1, \infty$, then there is a $p > 2$, so that*

$$\int_Q |L_f(x) - 1|^p dxdy \to 0,$$

where $L_f = |f_z| + |f_{\overline{z}}|$ as $\|\mu_f\|_\infty \to 0$.

Proof. We know there is a $t = 2 + 2\epsilon > 2$ so that $L^f \in L^t(Q)$ with a bound depending only on t and Q. Taking $s = (q+2)/2 = 2+\epsilon$, then $2 < s < t$ and we can use Hölder's inequality with exponents $p = 4/s < 2$ and $q = (4+4\epsilon)/s > 2$ to write

$$\|L_f - 1\|_s \leq \|L_f\|_2^{s/4} \cdot \|L_f - 1\|_t^{s/(4+4\epsilon)}.$$

The L^2 norm on the right tends to zero by Lemma 6.22 and the L^t is uniformly bounded by Bojarski's theorem, if t is chosen close enough to 2. Thus the product tends to zero. ∎

This will be important later when we want to show the map $\mu \to f_\mu$ is continuous from the unit ball of L^∞ to Hölder continuous functions.

Corollary 6.24 (Pompeiu formula)**.** *If Ω has a piecewise C^1 boundary and f is quasiconformal on Ω, then*

$$f(w) = \frac{1}{2\pi i}\int_{\partial\Omega}\frac{f(z)}{z-w}dz - \frac{1}{\pi}\iint_{\Omega}\frac{f_{\overline{z}}}{z-w}dxdy. \tag{6.17}$$

Proof. Smooth and take a limit using the L^p boundedness of the the Hardy-Littlewood maximal theorem and the Lebesgue dominated convergence theorem. □

Corollary 6.25. *If f is quasiconformal, then f maps sets of zero area to zero area and*

$$\mathrm{area}(f(E)) = \int_E J_f dxdy.$$

Proof. Since $\nu(E) = \mathrm{area}(f(E))$ and $\mu(E) = \int_E J_f dxdy$ are both non-negative Borel measures, it suffices to show that they are equal for some convenient basis of sets, say squares with sides parallel to the coordinate axes. Let Q be such a square.

We have already proved the "$\geq$" direction in Lemma 6.15.

To prove the other direction, we use the fact that $J_f \in L^p(Q, dxdy)$ for some $p > 1$. Define a smoothed version f_n of f by convolving f with a smooth, non-negative bump function φ_n of total mass 1 and support in $D(0, \frac{1}{n})$.

Since f is continuous on $\mathbb{C}$, $f_n \to f$ uniformly on Q. Since convolution is linear, the partials of f_n are the partials of f convolved with φ_n and therefore the supremum over n of these partials is bounded by the Hardy-Littlewood maximal function of f_z, i.e.,

$$\sup_n |(f_n)_z(x)| \leq \mathcal{H}L(f_z)(x),$$

and similarly for $f_{\overline{z}}$.

Because the Hardy-Littlewood maximal operator is bounded on L^p for $1 < p < \infty$, and $f_z, f_{\overline{z}} \in L^p$ for some $p > 1$, we see that $\{(f_n)_z)\}$, $\{(f_n)_{\overline{z}})\}$ are dominated by an L^p function and hence by an L^2 function on Q (since $L^p \subset L^2$ on bounded sets).

Thus the sequence of Jacobians $\{J_{f_n}\}$ is dominated by an L^1 function on Q, so by the Lebesgue dominated convergence theorem,

$$\int_Q J_{f_n} dxdy \to \int_Q J_f dxdy.$$

Moreover, since f_n is smooth

$$\int_Q J_{f_n} dxdy \geq \text{area}(f_n(Q)).$$

(equality may not hold since we don't know f_n is 1-to-1, and the integral computes area with multiplicity).

Since $f_n \to f$ uniformly, $f_n(Q)$ eventually contains any compact subset of $f(Q)$ and hence

$$\limsup_n \text{area}(f_n(Q)) \geq \text{area}(f(Q)).$$

Thus $\text{area}(f(Q)) \leq \int_Q J_f dxdy$, as desired. $\square$

Corollary 6.26. *If f is quasiconformal, then $|f_z| > 0$ almost everywhere.*

Proof. The inverse of f is also quasiconformal and so f^{-1} maps zero area sets to zero area. Thus f can't map sets of positive measures to zero measure. Thus J_f can't vanish on a set of positive measure. Neither can f_z since $J_f/(1-k^2) \leq |f_z|^2$. $\square$

Later we will need the following result, which can be proven using parts of the preceding argument.

Corollary 6.27. *Suppose f is K-quasiconformal. Then f can be approximated, uniformly on compact sets, by smooth K-quasiconformal maps $\{f_n\}$ whose dilatations $\{\mu_n\}$ converge pointwise to the dilatation μ of f, and such that for any measurable set E, $|f_n(E)| \to |f(E)|$.*

Weak convergence of dilatations

Lemma 6.28. *Suppose* $\{g_n\} \in L^p(R, dxdy)$ *for some* $p > 2$ *and*

$$\lim_n \iint_R \frac{g_n(z)}{z-w} dxdy = 0$$

for all $w \in R$. *Then* $\lim_n \iint_R g_n dxdy = 0$.

Proof. Fix rectangles $R'' \subset R' \subset R$, each compactly contained in the interior of the next.

Using the Cauchy integral formula for the constant function 1 on the curve $\partial R'$ we see that we can uniformly approximate the constant function 1 on R'' by a finite sum $s(z) = \sum \frac{a_k}{z-w_k}$ with $w_k \in \partial R'$ and $\sum |a_k|$ is uniformly bounded.

Then

$$\begin{aligned}
\iint_R g_n(z)dxdy &= \iint_R g_n(z)s(z)dxdy + \iint_R g_n(z)(1-s(z))dxdy \\
&= o(1) + \iint_{R''} g_n(z)(1-s(z))dxdy \\
&\qquad + \iint_{R\setminus R''} g_n(z)(1-s(z))dxdy.
\end{aligned}$$

For a fixed n, the integral

$$\iint_{R''} g_n(z)(1-s(z))dxdy$$

can be made as close to zero as we wish by taking s close to 1 on R''.

The other integral

$$\iint_{R\setminus R''} g_n(z)(1 - s(z))dxdy$$

be made small by taking area$(R \setminus R'') \to 0$

This implies the L^p norm of g_n on $R \setminus R''$ tends to zero whereas the L^q norm of s remains uniformly bounded (it is a combination of L^q functions with bounded norm $O(\sum |a_j|)$).

So by Hölder's inequality, the integral of the product tends to zero.

Thus is $\iint_R g_n dxdy$ as small as we wish for n large, proving the lemma. $\square$

Lemma 6.29. *If $\{g_n\}$ are K-quasiconformal maps that converge uniformly on compact sets to a quasiconformal map g, then for any rectangle R,*

$$\iint_R [(g_n)_z - g_z] dxdy \to 0,$$

$$\iint_R [(g_n)_{\overline{z}} - g_{\overline{z}}] dxdy \to 0.$$

and $(g_n)_z \to g_z$ and $(g_n)_{\overline{z}} \to g_{\overline{z}}$ weakly.

Proof. First consider the $\overline{z}$-derivative. Let $h_n = (g_n)_{\overline{z}} - g_{\overline{z}}$.

By the Pompeiu formula

$$g(w) = \frac{1}{2\pi i} \int_{\partial R} \frac{g(z)}{z-w} dz - \frac{1}{\pi} \iint_R \frac{g_{\overline{z}}}{z-w} dxdy.$$

Since $g_n \to g$ uniformly on R, we know that for $z \notin \partial R$ the first two terms with g_n converge to the corresponding terms for g. Thus the third term also must converge, i.e.,

$$\lim_{n\to\infty} \iint_R \frac{h_n(z)}{z-w} dxdy = 0$$

for any $w \in R$. Then $\iint_R h_n dxdy \to 0$, follows from Lemma 6.28.

To prove weak conference, take any continuous f of compact support and uniformly approximate it to within ϵ by a function $\tilde{f}$ that is constant on finite union of rectangles. Then

$$\iint fh_n dxdy = \iint (f - \tilde{f})h_n dxdy + \iint \tilde{f}h_n dxdy.$$

The first integral is bounded by $\epsilon \iint |h_n| dxdy$, which is small since $\|h_n\|_1 \leq C\|h_n\|_p$ is uniformly bounded on a large ball containing the support of both f and $\tilde{f}$.

The second integral tends to zero since is a finite linear combination of integrals of h_n over rectangles.

The result for z-derivatives follows from the same proof applied to the complex conjugates of g and $\{g_n\}$, using the fact that $(\bar{f})_{\overline{z}} = \overline{f_z}$. $\square$

Completing the proof of the MRMT

Theorem 6.30. *Suppose $\{f_n\}$, f are all K-quasiconformal maps on the plane with dilatations $\{\mu_n\}$, μ_f respectively, that $f_n \to f$ uniformly on compact sets and that $\mu_n \to \mu$ pointwise almost everywhere. Then $\mu_f = \mu$ almost everywhere.*

Proof. We restrict attention to some domain Ω with compact closure. We know that $f_{\bar{z}} = \mu_f f_z$ almost everywhere, and we know that f_z is non-zero almost everywhere, so it suffices to show that for almost every w,

$$f_{\bar{z}}(w) - \mu(w) f_z(w) = 0.$$

By the Lebesgue dominated convergence theorem, it suffices to show that the integral of $f_{\bar{z}}(w) - \mu(w) f_z(w)$ over any rectangle R is zero.

We re-write this function as

$$
\begin{aligned}
f_{\bar{z}}(w) - \mu(w) f_z(w) &= [f_{\bar{z}}(w) - (f_n)_{\bar{z}}(w)] \\
&\quad +[(f_n)_{\bar{z}}(w) - \mu_n(w) \cdot (f_n)_z(w)] \\
&\quad +[\mu_n(w) \cdot (f_n)_z(w) - \mu(w) \cdot (f_n)_z(w)] \\
&\quad +[\mu(w) \cdot (f_n)_z(w) - \mu(w) f_z(w)] \\
&= I + II + III + IV.
\end{aligned}
$$

Term II equals zero almost everywhere, so we need only show that the integrals of the other three terms over any rectangle R tend to zero as n tends to ∞.

Term I: The integral of $f_{\bar{z}}(w) - \mu(w) f_z(w)$ over R tends to zero by Lemma 6.29.

Term III: By Cauchy-Schwarz, the integral of the third term is bounded by

$$\left(\iint_R (\mu - \mu_n)^2 dxdy\right)^{1/2} (\iint_R |(f_n)_z|^2 dxdy)^{1/2},$$

The first integrand tends to zero pointwise and is bounded above by 2 almost everywhere, so this integral tends to zero by the Lebesgue dominated convergence theorem.

On the other hand

$$\left(\iint_R |(f_n)_z|^2 dxdy\right)^{1/2} \simeq \mathrm{diam}(f_n(R)),$$

by Lemma 6.16, and since $\{f_n\}$ converges uniformly on compact sets, this remains bounded.

Thus the integral of III is bounded by the product of a uniformly bounded term and a term tending to zero. Hence it also tends to zero.

Term IV: The same lemma as in Case I, but applied to $f_z = (\bar{f})_{\bar{z}}$.

Since $(\bar{f})_{\bar{z}} = \overline{(f_z)}$, we have

$$\iint_R (f_z - (f_n)_z) dxdy \to 0.$$

Now approximate μ in the $L^q(R, dxdy)$ norm by a function ν that is constant on a finite collection of disjoint squares (such functions are dense in L^q). Then

$$\begin{aligned}
\lim_n \iint_R \mu \cdot (f_z - (f_n)_z) dxdy &= \lim_n \iint_R (\mu - \nu)(f_z - (f_n)_z) dxdy \\
&\leq \lim_n \|\mu - \nu\|_q \|(f_z - (f_n)_z)\|_p.
\end{aligned}$$

The first term is as small as we wish and the second is uniformly bounded, so the product is as small as we wish. Thus the limit must be zero, as desired. $\square$

This completes the proof of the measurable Riemann mapping theorem in the general case.

Theorem 6.31. *If two quasiconformal maps have the same dilatation and both fix* $0, 1, \infty$*, then they are the same map.*

Proof. Suppose f and g are two such maps.

The map $h = f \circ g^{-1}$ is quasiconformal and has dilatation 0, hence $h_{\overline{z}} = 0$. Since h is abosulutely continuous on almost all lines it is holmorphic in the sense of distributions, so by Weyl's lemma, it a a classical holomorphic funtion. (See Thm 9.26, the Elliptic Regularity Theorem in Folland's book).

Since h is a holomorphic homeomorphism of the plane, it is linear, and since it fixes 0 and 1, it must be the identity map. Thus $f = g$.

□

Theorem 6.32. *Every K-quasiconformal map can be written as a finite composition of n quasiconformal mappings with dilatation $K^{1/n}$.*

Proof. We may assume $f = f^\mu$ is defined on the whole plane. For each z let $\{\mu_k(z)\}_{k=1}^n$ be n equally spaced (in the hyperbolic metric) on the segement between 0 and $\mu(z)$ and set $f_f = f^{\mu_k}$. Then

$$\mu_{f_{k+1}\circ f_k^{-1}} = \left(\frac{\mu_{k+1} - \mu_k}{1 - \mu_{k+1}\overline{\mu_k}} \frac{(f_k)_z}{\overline{((f_k)_{\overline{z}}}}\right) \circ f_k^{-1}.$$

Setting $g_k = f_{k+1} \circ f_k^{-1}$, we have $f = g_n \circ \cdots \circ g_1$ and each g_k is $K^{1/n}$-quasiconformal (this is clearer in the right half-plane model of hyperbolic space, were $a,^2, a^3, \cdots \in \mathbb{R}_+$ are equally spaced with respect to the hyperbolic metric.)

$\square$

This semester I hope to cover the following topics:

- Review of complex analysis
- Extremal length and conformal modulus,
- Logarithmic capacity, harmonic measure
- Geometric definition of quasiconformal mappings, compactness
- Compactness of QC maps: quasisymmetry, extension, removability, weldings
- Analytic definition and the measurable Riemann mapping theorem
- Cauchy and Beurling transforms, analytic dependence
- Astala's theorems on area and dimension distortion
- Smirnov's $1+k^2$ theorem
- Lehto maps
- Holomorphic motions

We have proven the measurable Riemann mapping theorem: given a dilatation μ with $\|\mu\|_\infty = k < 1$, there is a quasiconformal mapping f with dilatation μ.

Next, we would like to show $f : \mathbb{C} \to \mathbb{C}$ is unique if it normalized.

Two point normalization: $f(0) = 0$ and $f(1) = 1$.

Principle solution: if μ is compactly supported, then $f(z) = z + O(1/|z|)$ near ∞.

The latter is sometimes called the hydrodynamic normalization.

We also want to show that f depends holomorphically on μ, e.g., if z is fixed, $f_{t\mu}(z)$ is a holomorphic function of $t \in D(0, 1/k)$.

The proofs of uniqueness and holomorphic dependence both use explicit formulas involving the Cauchy transform and its “derivative”, the Beurling transform.

The latter is a singular integral operator and is the 2-dimensional analog of the famous Hilbert transform on the real line.

The Cauchy Transform

Suppose $f \in L^p(\mathbb{R}^2, dxdy)$ for $p > 2$. Define

$$\mathcal{C}f(\zeta) = -\frac{1}{\pi}\iint_{\mathbb{C}} f(z)(\frac{1}{z-\zeta} - \frac{1}{z})dxdy.$$

The function $1/(z-\zeta)$ is not in L^2 locally, but is in L^q for all $q < 2$, so the integrand is locally integrable when $f \in L^p$ for some $p > 2$.

The extra $1/z$ term occurs so that the difference decays like $1/|z|^2$ near infinity, and hence the difference is in L^q for all $q > 1$. Thus the integral makes sense for all $f \in L^p$, $p > 2$.

If f is compactly supported, this means that its Cauchy transform also decays like $1/|z|^2$, which will be convenient when apply the Cauchy integral formula on large circles. It also implies $\mathcal{C}(f) \in L^p$, $p > 2$, in a neighborhood of ∞, outside the support of f.

Note that $\mathcal{C}(f)(0) = 0$ since the kernel vanishes if $\zeta = 0$.

Lemma 7.1. *If $f \in L^p$, $p > 2$, then $\mathcal{C}f$ is α-Hölder continuous with $\alpha = 1 - 1/p$.*

Proof. First note that the Cauchy transform on an L^p function is bounded.

$$
\begin{aligned}
|\mathcal{C}f(\zeta)| &\leq \frac{1}{\pi} \cdot \|f\|_p \cdot \|\frac{1}{z-\zeta} - \frac{1}{z}\|_q \\
&= \frac{1}{\pi} \cdot \|f\|_p \cdot \|\frac{\zeta}{(z-\zeta)z}\|_q \\
&= \frac{|\zeta|}{\pi} \cdot \|f\|_p \cdot \|\frac{1}{(z-\zeta)z}\|_q.
\end{aligned}
$$

The dependence on ζ is obtained by a change of variable $z = \zeta w$

$$
\begin{aligned}
(7.18) \qquad \iint |z(z-\zeta)|^{-q} dxdy &= \iint |\zeta w(\zeta w - \zeta)|^{-q} |\zeta|^2 dudv \\
&= |\zeta|^{2-2q} \iint |w(w-1)|^{-q} dudv.
\end{aligned}
$$

Since $q = p/(p-1)$, $2 - 2q = 2/p$ and this implies

$$
(7.19) \qquad |\mathcal{C}f(\zeta)| \leq K_p \cdot \|f\|_p \cdot |\zeta|^{1-1/p}
$$

Next,

$$\begin{aligned}
|\mathcal{C}f(\zeta_1) - \mathcal{C}f(\zeta 2)| &= \left|\frac{1}{\pi}\iint_{\mathbb{C}} f(z)\left(\frac{1}{z-\zeta_1} - \frac{1}{z-\zeta_2}\right) dxdy\right| \\
&= \left|\frac{1}{\pi}\iint_{\mathbb{C}} f(z+\zeta_1)\left(\frac{1}{z+\zeta_2-\zeta_1} - \frac{1}{z}\right) dxdy\right| \\
&= |\mathcal{C}h(\zeta_2-\zeta_1)|,
\end{aligned}$$

where $h(z) = f(z+\zeta_1)$. Applying (7.19) to h, we get

$$\begin{aligned}
|\mathcal{C}f(\zeta_1) - \mathcal{C}f(\zeta 2)| &= |\mathcal{C}h(\zeta_2-\zeta_1) \leq K_p\|h\|_p|\zeta|^{1-1/p} \\
&= K_p\|f\|_p|\zeta|^{1-1/p}.
\end{aligned}$$

$\square$

Lemma 7.2. *If f is smooth and has compact support, then $\mathcal{C}f$ is smooth and $(\mathcal{C}f)_{\overline{z}} = f$.*

Proof. Let $\gamma_\epsilon = \partial D(\zeta, \epsilon)$ be a small circle around ζ. The convolution of a smooth, compactly supported function is smooth and interchanging integration and differentiation gives $(\mathcal{C}f(\zeta))_{\overline{z}} = (\mathcal{C}f_{\overline{z}}(\zeta))$.

Recall

$$dzd\overline{z} = (dx + idy)(dx - idy) = idydx - idxdy = -2idxdy.$$

By Stokes theorem and using the fact that $|f| = O(1/|z|^2)$,

$$\begin{aligned}
(\mathcal{C}f(\zeta))_{\overline{z}} = (\mathcal{C}f_{\overline{z}}(\zeta)) &= -\frac{1}{\pi}\iint \frac{f_{\overline{z}}}{z-\zeta}dxdy \\
&= -\frac{1}{2\pi i}\iint \frac{f_{\overline{z}}}{z-\zeta}dzd\overline{z} \\
&= -\frac{1}{2\pi i}\iint \frac{df\,d\overline{z}}{z-\zeta} \\
&= \lim_{\epsilon\to 0}\frac{1}{2\pi i}\iint \frac{f\,dz}{z-\zeta} = f(\zeta).
\end{aligned}$$

$\square$

Corollary 7.3. *If $f \in L^p$, $p > 2$, then $(\mathcal{C}f)_{\overline{z}} = f$ in the sense of distributions.*

Proof. We must show that for any smooth ϕ with compact support,

$$\iint (\mathcal{C}f)\phi_{\overline{z}} dxdy = -\iint \phi f dxdy. \tag{7.20}$$

Take smooth functions $\{f_n\}$ of compact support converging to f. By Hölder's inequality

$$|\iint \phi(f - f_n)dxdy| \leq \|\phi\|_q \cdot \|f - f_n\|_p.$$

The first term on the product is a finite constant and the other tends to z_0, so $\iint \phi f_n \to \iint \phi f$.

On the other hand if the support of ϕ has diameter d,

$$\begin{aligned}\left|\iint(\mathcal{C}f-\mathcal{C}f_n)\phi_{\overline{z}}dxdy\right| &\leq \|\phi_{\overline{z}}\|_1\cdot\sup_{z\in\operatorname{supp}(\phi)}|\mathcal{C}(f-f_n)(c)| \\ &\leq \|\phi_{\overline{z}}\|_1\cdot K_p\cdot\|f-f_n\|_p d^{1-1/p}\end{aligned}$$

and this tends to zero with n. Thus (7.20) holds. $\square$

The Beurling Transform

We will also need a few basic facts about the Beurling transform, which is usually defined as a principle value integral

$$\mathcal{T}f(\zeta) = \lim_{\epsilon \to 0} \iint_{|z-\zeta|>\epsilon} \frac{f(z)}{(z-\zeta)^2} dxdy.$$

For smooth, or even Hölder, functions of compact support this is well defined by rewriting the integral as

$$\mathcal{T}f(\zeta) = \lim_{\epsilon \to 0} \iint_{|z-\zeta|>\epsilon} \frac{f(z) - f(\zeta)}{(z-\zeta)^2} dxdy,$$

since the kernel has integral zero on any circle centered at ζ.

The Beurling transform can be extended to a bounded linear operator from $L^p(\mathbb{R}^2, dxdy)$ to itself for all $1 < p < \infty$.

We shall show below that $\mathcal{T}$ is an isometry on L^2.

The standard proof of MRMT uses that $\mathcal{T}$ is bounded for $p > 2$ with an operator norm that approaches 1 as $p \searrow 2$, but we will not need this fact; we have already proven Bojarski's theorem that $f_z \in L^p$ for a K-QC map, and this will be sufficient for our applications.

Recall

$$\int_{|z|=1} \frac{dz}{z} = 2\pi i, \qquad \int_{|z|=1} \frac{d\overline{z}}{z} = 0.$$

$$dzd\overline{z} = -2idxdy.$$

Lemma 7.4. *If f is smooth and has compact support then $\mathcal{C}f$ is smooth and $\mathcal{C}(f_z) = \mathcal{T}f - \mathcal{T}f(0)$.*

Proof. As in Lemma 7.2 we have that $\mathcal{C}f$ is smooth and $(\mathcal{C}f(\zeta))_z = (\mathcal{C}f_z(\zeta))$. Using Stokes theorem again

$$
\begin{aligned}
(\mathcal{C}f_z(\zeta)) &= -\frac{1}{\pi}\iint \frac{f_z}{z-\zeta}dxdy \\
&= \frac{1}{2\pi i}\iint \frac{f_{\overline{z}}}{z-\zeta}dzd\overline{z} \\
&= \lim_{\epsilon\to 0}\left[-\frac{1}{2\pi i}\int_{|z-\zeta|=\epsilon}\frac{fd\overline{z}}{z-\zeta} + \frac{1}{2\pi i}\iint_{|z-\zeta|>\epsilon}\frac{fdzd\overline{z}}{(z-\eta)^2}\right] \\
&= \mathcal{T}f(\zeta).
\end{aligned}
$$

$\square$

From the above we get

$$(\mathcal{T}f)_{\overline{z}} = \mathcal{C}(f_z)_{\overline{z}} = f_z,$$

$$(\mathcal{T}f)_z = \mathcal{C}(f_z)_z = T(f_z) = \mathcal{C}(f_{zz}) + T(f_z)(0).$$

Lemma 7.5. *The Beurling transform is an isometry on* $L^2(\mathbb{R}^2, dxdy)$.

Proof. It is enough to check this on the dense set of smooth, compactly supported functions. Then

$$\begin{aligned}
\iint |\mathcal{T}f|^2 dxdy &= -\frac{1}{2i}\iint |(\mathcal{C}f)_z)|^2 dzd\overline{z} \\
&= -\frac{1}{2i}\iint (\mathcal{C}f)_z\overline{(\mathcal{C}f)_z}dzd\overline{z} = -\frac{1}{2i}\iint (\mathcal{C}f)_z\overline{(\mathcal{C}f)}_{\overline{z}}dzd\overline{z} \\
&= \frac{1}{2i}\iint \mathcal{C}f\overline{(\mathcal{C}f)}_{\overline{z}z}dzd\overline{z} = \frac{1}{2i}\iint \mathcal{C}f\overline{(\mathcal{C}f)_{\overline{z}}}_{\overline{z}}dzd\overline{z} \\
&= \frac{1}{2i}\iint \mathcal{C}f\overline{f}_{\overline{z}}dzd\overline{z} = -\frac{1}{2i}\iint (\mathcal{C}f)_{\overline{z}}\overline{f}dzd\overline{z} \\
&= -\frac{1}{2i}\iint f\overline{f}dzd\overline{z} \\
&= \iint |f^2|dxdy \quad \square
\end{aligned}$$

Uniqueness in MRMT

Lemma 7.6. *If μ is measurable, $\|\mu\|_\infty = k < 1$ and μ has compact support, then there is a unique K-quasiconformal map f (with $K = (1+k)/(1-k)$) that is absolutely continuous on almost all lines and satisfies $f_{\overline{z}} = \mu f_z$ and $f_z - 1 \in L^p(\mathbb{R}^2)$ for some $p > 1$.*

Proof. We already know uniqueness, so the L^p bound is the main point.

Suppose f is such a solution. We know $f_z \in L^p$ locally, so $f_{\overline{z}} - \mu f_z \in L^p$ on the plane. Hence $\mathcal{C}(f_{\overline{z}})$ is well defined and $(\mathcal{C} f_{\overline{z}})_{\overline{z}} = f_{\overline{z}}$ by Corollary 7.3.

Thus $(f - \mathcal{C} f_{\overline{z}})_{\overline{z}} = 0$ in the sense of distributions and hence it is analytic on the plane by Weyl's lemma.

We assumed $f_z - 1 \in L^p$, and $\mathcal{C} f_{\overline{z}} \in L^p$ for any $p > 2$ (because it is $O(|z|^{-2})$ near infinity), so the holomorphic function $F = f - \mathcal{C} f_{\overline{z}} - 1$ has $F' \in L^p$.

This is only possible if $F' = 1$ or $F(z) = z + c$.

Because we assumed $f(0) = 0$, and $\mathcal{C} f_{\overline{z}}(0) = 0$, we must have $c = 0$. Thus $f(z) = \mathcal{C}(f_{\overline{z}})(z) + z$ and $f_z = \mathcal{T}(\mu(f_z)) + 1$.

If g were another solution, then using the fact that $\mathcal{T}$ is an isometry on L^2 gives

$$\|f_z - g_z\|_2 = \|\mathcal{T}(\mu(f_z - g_z))\|_2 \leq k\|\mathcal{T}(f_z - g_z)\|_2,$$

and this is a contradiction unless $\|f_z - g_z\| - 2 = 0$.

Therefore $f_z = g_z$ almost everywhere, and hence $f_{\overline{z}} = \mu f_z = \mu g_z = g_{\overline{z}}$ almost everywhere.

Thus $f - g$ is both holomorphic and anti-holomorphic, hence constant. Since $f(0) = g(0) = 0$, they must be equal everywhere. $\square$

Alternate proof of MRMP

Consider

$$h = T(\mu h) + T\mu$$

The series

$$h = T\mu + T\mu T\mu + T\mu T\mu T\mu + \dots$$

converges in L^p if L^p norm of T is less than $1/k$, $k = \|\mu\|_\infty$.

If h is gvien by this series,set

$$f = P(\mu(h+1),$$

then $\mu(h+1) \in L^p$ and $P(\mu(h+1)$ is continuous. Moreover,

$$f_{\overline{z}} = \mu(h+1), \qquad f_z = T(\mu(h+1)] + 1 = h+1,$$

so $f_{\overline{z}} = \mu f_z$.

Analytic dependence

Lemma 7.7. *Suppose $\mu_t = \mu(z,t)$ is a path of dilatations that is differentiable at $t = 0$. Then the corresponding normalized QC maps are also differentiable at $t = 0$.*

More precisely, suppose $\mu(z,t) = r\nu(z) + t\epsilon(z,t)$ where $\nu, \epsilon \in L^\infty$ and $\|\epsilon(\cdot,t)\|_\infty \to 0$ for $t \searrow 0$. Let $f^\mu = f(z,t)$ be the quasiconformal map with dilatation $\mu(z,t)$ and normalized to have fixed points $0, 1, \infty$. Then

$$\dot{f}(\zeta) = \frac{1}{\pi}\int_{\mathbb{C}} \nu(z) R(z,\zeta)\,dxdy$$

where

$$R(z,\zeta) = \frac{1}{z-\zeta} - \frac{\zeta}{z-1} + \frac{\zeta-1}{z} = \frac{\zeta(\zeta-1)}{z(z-1)(z-\zeta)}.$$

Proof. We follow the proof in Ahlfors's book.

For $|\zeta| < 1$ the Pompeiu formula (Lemma 6.24) says

$$f(\zeta) = \frac{1}{2\pi i}\int_{|z|=1} \frac{f(z)dz}{z-\zeta} - \frac{1}{\pi}\iint_{|z|<1} \frac{f_{\overline{z}}(z)}{z-\zeta}dxdy. \tag{7.21}$$

We want to manipulate the line integral to get an integral formula for f in terms of $f_{\overline{z}}$ over the whole plane.

Since $|\zeta| < |z| = 1$, we can write

$$\begin{aligned}
\frac{1}{z-\zeta} &= \frac{1}{z} \cdot \frac{1}{1-\zeta/z} \\
&= \frac{1}{z} \cdot \sum_{n=0}^{\infty} (\zeta/z)^n \\
&= \frac{1}{z} \cdot \left[1 + \frac{\zeta}{z} + \frac{\zeta^2}{z^2} \sum_{n=0}^{\infty} (\zeta/z)^n \right] = \frac{1}{z} + \frac{\zeta}{z^2} + \frac{\zeta^2}{z^2} \frac{1}{z-\zeta}.
\end{aligned}$$

Using this, rewrite the line integral in (7.23) as

$$\frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)dz}{z-\zeta} = A + B\zeta + \frac{\zeta^2}{2\pi i} \int_{|z|=1} \frac{f(z)dz}{z^2(z-\zeta)}. \tag{7.22}$$

Apply the substitution $z = 1/w$, $dz = -dw/w^2$ in the last integral to obtain

$$\zeta^2 2\pi i \int_{|z|=1} \frac{f(z)dz}{z^2(z-\zeta)} = -\zeta^2 2\pi i \int_{|w|=1} \frac{f(1/w)dw}{(w^2)(1/w)^2(1/w-\zeta)} \tag{7.23}$$

$$= -\zeta^2 2\pi i \int_{|w|=1} \frac{f(1/w)wdw}{1-w\zeta}.$$

Let $g(z) = 1/f(1/z)$. Then g is quasiconformal and $g(0) = 0$.

It is easy to check that $(1/g)_{\overline{z}} = g_{\overline{z}}/g^2$ and that if h is holomorphic, then $(gh)_{\overline{z}} = g_{\overline{z}}h$

Now appy $g(0) = 0$ and the Pompieu formula again,

$$\begin{aligned}
\frac{-\zeta^2}{2\pi i}\int_{|w|=1}\frac{f(1/w)wdw}{1-w\zeta} &= \frac{-\zeta^2}{2\pi i}\int_{|w|=1}\frac{g(w)^{-1}wdw}{1-w\zeta} \\
&= \frac{-\zeta^2}{2\pi i}\int_{|w|<1}\frac{g_{\overline{z}}(w)wdw}{g^2(w)(1-w\zeta)}.
\end{aligned}$$

The integrals converge because quasiconformal maps are biHölder and hence $|g(w)| > c\sqrt{|w|}$ if $\|\mu\|_\infty$ is small enough. (Then $|w|/|g(w)|$ is bounded.)

We know that f is given by some formula of the form:

$$
\begin{aligned}
f(\zeta) &= A + B\zeta - \frac{1}{\pi}\int_{|z|<1} \frac{f_{\overline{z}}(z)}{z-\zeta} dxdy \\
&\qquad - \frac{1}{\pi}\iint_{|z|<1} \frac{g_{\overline{z}}(z)}{g(z)^2}\left(\frac{\zeta^2 z}{1 - z\zeta}\right) dxdy.
\end{aligned}
$$

We guess (or solve for) the correct values of A, B:

$$
\begin{aligned}
f(\zeta) &= \zeta - \frac{1}{\pi}\int_{|z|<1} f_{\overline{z}}(z)\left(\frac{1}{z-\zeta} - \frac{\zeta}{z-1} + \frac{\zeta - 1}{z}\right) dxdy \\
&\qquad - \frac{1}{\pi}\iint_{|z|<1} \frac{g_{\overline{z}}(z)}{g(z)^2}\left(\frac{\zeta^2 z}{1 - z\zeta} - \frac{\zeta z}{1-z}\right) dxdy
\end{aligned}
$$

and can check this is correct by verifying $f(0) = 0$ and $f(1) = 1$.

In the first integral set $f_{\overline{z}} = \mu f_z = \mu(f_z - 1) + \mu$ and use a corresponding expression for $g_{\overline{z}}$ with $\mu_g(z) = (z/\overline{z})^2\mu(1/z)$.

Because $\|f_z - 1\|_p \to 0$ as $\|\mu\|_\infty \to 0$ by Corollary 6.23, and $\mu/t \to \nu$,

$$\begin{aligned}
\dot{f}(\zeta) &= \lim_{t\to 0}\frac{f(\zeta)-\zeta}{t} \\
&= \frac{1}{\pi}\int_{|z|<1}\nu(z)\left(\frac{1}{z-\zeta} - \frac{\zeta}{z-1} + \frac{\zeta-1}{z}\right)dxdy \\
&\qquad -\frac{1}{\pi}\iint_{|z|<1}\nu(1/z)\left(\frac{\zeta^2 z}{1-z\zeta} - \frac{\zeta z}{1-z}\right)dxdy.
\end{aligned}$$

If $1/z$ is taken as the integration variable in the second integral, it transforms to the same integrand as in the first, so

$$\dot{f}(\zeta) \;=\; \frac{1}{\pi}\int_{\mathbb{C}} \nu(z)R(z,\zeta)dxdy$$

where

$$R(z,\zeta) = \frac{1}{z-\zeta} - \frac{\zeta}{z-1} + \frac{\zeta-1}{z} = \frac{\zeta(\zeta-1)}{z(z-1)(z-\zeta)}.$$

$\square$

Theorem 7.8. *If $\mu(z,t)$ is a holomorphic function of t, let $f^{\mu}(z,t)$ be the quasiconformal map with dilatation $\mu(z,t)$, normalized to fix 0, 1 and ∞, then for each fixed z, $f^{\mu}(z,t)$ is a holomorphic function of t.*

Corollary 7.9. *Suppose $\|\mu\|_{\infty} = k < 1$ is the dilatation of f. Let $\mu(z,t) = (t/k)\mu(z)$. Then for each z, $f^t(z)$ is a holomorphic function of $t \in \mathbb{D}$ so that $f^k = f$.*

Proof of Theorem 7.8. It suffices to show that $f^t(z)$ is differentiable at each t. We have already done this for $t = 0$.

For arbitrary t_0, since $\mu(z,t)$ is differentiable in t, we may assume

$$\mu(z,t) = \mu(z,t_0) + \nu(z,t_0)(t-t_0) + o(t-t_0),$$

and consider

$$f^{\mu(t)} = f^{\lambda} \circ f^{\mu(t_0)},$$

where (using the composition law for dilatations)

$$\lambda = \lambda(t) = \left(\frac{\mu(t) - \mu(t_0)}{1 - \mu(t)\overline{\mu(t_0)}} \right) \circ (f^{\mu_0})^{-1}.$$

Then

$$\dot{\lambda}(t) = \left(\frac{\nu(t_0)}{1 - |\mu_0|^2} \cdot \frac{f_z^{\mu_0}}{\overline{f_{\bar{z}}^{\mu_0}}} \right) \circ (f^{\mu_0})^{-1},$$

and

$$\begin{aligned}
\frac{\partial}{\partial t} f(z,t) &= \dot{f} \circ f^{\mu_0} \\
&= -\frac{1}{\pi} \iint \left(\frac{\nu(t_0)}{1 - |\mu_0|^2} \cdot \frac{f_z^{\mu_0}}{\overline{f_{\bar{z}}^{\mu_0}}} \right) \circ (f^{\mu_0})^{-1} R(z, f^{\mu_0}(\zeta)) dx dy \\
&= -\frac{1}{\pi} \iint \nu(t_0, z)(f_z^{\mu_0})^2 R(f^{\mu_0}(z), f^{\mu_0}(\zeta)) dx dy.
\end{aligned}$$

This is the general formula for the derivative. $\square$

Area Distortion

Theorem 8.1 (Astala's Theorem)**.** *Suppose f is K-quasiconformal of $\mathbb{D}$ to itself with $f(0) = 0$. Then for any measurable $E \subset \mathbb{D}$, we have*

$$|f(E)| \leq C(K) \cdot |E|^{1/K}.$$

Astala's original proof uses idea from dynamics. Here we will follow a shorter proof due to Alexandre Eremenko and David Hamilton.

Kari Astala

Alex Eremenko

David Hamilton

We prove Astala's theorem by first assuming $\mu = 0$ on E and then assuming $\mu = 0$ off E, and then combining these cases.

First a technical lemma.

Lemma 8.2. *Let* $\{p_j\}_1^n > 0$ *and* $\{q_j\}_1^n > 0$ *be probability distributions on* $1, \ldots, n$. *Then*

$$\sum_{j=1}^{n} p_j \log p_j \geq \sum_{j=1}^{n} p_j \log q_j.$$

Recall Jensen's inequality: if μ is a probability measure and ϕ is convex, then

$$\phi(\int \mu) \geq \int \phi d\mu.$$

Proof. Note that $\phi(x) = x \log x$ is convex by computing ϕ''. By Jensen's inequality,

$$\begin{aligned}
\sum_{j=1}^{n} p_j \log p_j - \sum_{j=1}^{n} p_j \log q_j &= \sum_{j=1}^{n} q_j (p_j/q_j) \phi(p_j/q_j) \\
&= \sum_{j=1}^{n} q_j \phi(p_j/q_j) \\
&\geq \phi\left(\sum_{j=1}^{n} q_j \cdot p_j/q_j\right) \\
&= \phi(1) = 0 \quad \square
\end{aligned}$$

Side remark:

The Kullback-Leibler divergence between probability measures is

$$\sum p_j \log(p_j/q_j).$$

Very useful in statistics. Roughly measures the "surprise" in seeing data distribution $\{q_j\}$ when actual distribution is $\{p_j\}$.

Is always ≥ 0 and equals zero iff two distributions are the same.

Wikipedia article on KL divergence

Discrete version:

Lemma 8.3. *Let $\{a_j\}_1^n$ be positive functions on the unit disk such that $\log a_j$ are harmonic and for $|\lambda| < 1$,*

$$\sum_{j=1}^{n} a_j(\lambda) \leq 1. \tag{8.24}$$

Then for $|\lambda| < 1$,

$$\sum_{j=1}^{n} a_j(\lambda) \leq \left(\sum_{j=1}^{n} a_j(0)\right)^{(1-|\lambda|)/(1+|\lambda|)}. \tag{8.25}$$

Note we have equality at $\lambda = 0$ and RHS $\to 1$ as $|\lambda| \nearrow 1$.

If a is holomorphic and never zero, then $|a|$ is positive and $\log |a|$ is harmonic.

Proof. For $\lambda, z \in \mathbb{D}$, define the probability distributions

$$p_j = \frac{a_j(\lambda)}{\sum a_j(\lambda)}, \qquad q_j = \frac{a_j(z)}{\sum a_j(z)}.$$

and for fixed λ, define

$$\begin{aligned} H(z) &= -\sum_{j=1}^{n} p_j \log a_j(z) + \sum_{j=1}^{n} p_j \log p_j \\ &= -\sum_{j=1}^{n} p_j \log q_j(z) - \log \sum_{k} a_k(z) + \sum_{j=1}^{n} p_j \log p_j \end{aligned}$$

By our assumption on a_j, H is harmonic in z (λ fixed).

By Lemma 8.2

$$H(z) + \log \sum_{j=1}^{n} a_j(z) = \left(-\sum_{j=1}^{n} p_j \log a_j(z) + \sum_{j=1}^{n} p_j \log p_j \right) + \log \sum_{j=1}^{n} a_j(z)$$
$$= -\sum_{j=1}^{n} p_j \log q_j(z) + \sum_{j=1}^{n} p_j \log p_j \geq 0.$$

Hence

$$H(z) \geq -\log \sum a_j(z) \geq 0,$$

since $\sum a_j(z) \leq 1$. Thus by Harnack's inequality

$$H(z) \geq \frac{1 - |z|}{1 + |z|} H(0).$$

Setting $z = \lambda$, we have

$$
\begin{aligned}
H(\lambda) &= -\sum_{j=1}^{n} p_j \log a_j(\lambda) + \sum_{j=1}^{n} p_j \log p_j \\
&= -\sum_{j=1}^{n} p_j \log p_j - \sum_{j} \log a_j(\lambda) + \sum_{j=1}^{n} p_j \log p_j \\
&= -\sum_{j} \log a_j(\lambda)
\end{aligned}
$$

So, using Harnack's inequality

$$
\begin{aligned}
-\sum_{j} \log a_j(\lambda) = H(\lambda) &\geq \frac{1-|\lambda|}{1+|\lambda|}\; H(0) \\
&= \frac{1-|\lambda|}{1+|\lambda|} \left(-\sum_{j=1}^{n} p_j \log a_j(0) + \sum_{j=1}^{n} p_j \log p_j \right)
\end{aligned}
$$

Now apply Lemma 8.2 (in third line) to get

$$
\begin{aligned}
-\sum_j \log a_j(\lambda) \;&\geq\; \frac{1-|\lambda|}{1+|\lambda|}\left(-\sum_{j=1}^n p_j \log a_j(0) + \sum_{j=1}^n p_j \log p_j\right) \\
&=\; \frac{1-|\lambda|}{1+|\lambda|}\left(-\sum_{j=1}^n p_j \log \frac{a_j(0)}{\sum_k a_k(0)} - \log \sum_k a_k(0) + \sum_{j=1}^n p_j \log p_j\right) \\
&\geq\; \frac{1-|\lambda|}{1+|\lambda|}\left(-\log \sum_k a_k(0)\right)
\end{aligned}
$$

Thus

$$- \log \sum_j a_j(\lambda) \geq \frac{1 - |\lambda|}{1 + |\lambda|} \Big(- \log \sum a_j(0) \Big) \, .$$

Switching signs and exponentiating gives the desired inequality,

$$\sum_j a_j(\lambda) \leq \left(\log \sum_j a_j(0) \right)^{(1-|\lambda|)/(1+|\lambda|)} \, . \quad \square$$

Continuous version of Lemma 8.3:

Corollary 8.4. *Suppose* $a(z,\lambda) > 0$ *is defined on* $E \times \mathbb{D}$ *and assume* $\log a(z,\lambda)$ *is harmonic in* λ. *Also suppose that* $z = x + iy$ *and for all* $|\lambda| < 1$,

$$\int_E a(z,\lambda)dxdy \leq 1. \tag{8.26}$$

Then for $|\lambda| < 1$,

$$\int_E a(z,\lambda)dxdy \leq \left(\int_E a_j(z,0)dxdy\right)^{(1-|\lambda|)/(1+|\lambda|)}. \tag{8.27}$$

Proof. Write the integral as a limit of Riemann sums, apply Lemma 8.2 and take the limit. $\square$

Lemma 8.5 (Area theorem)**.** *Suppose f is conformal on $\mathbb{D}^* = \{|z| > 1\}$ and $f(z) = z + o(1)$ near infinity. Then $|\mathbb{C} \setminus f(\mathbb{D}^*)| \leq \pi$.*

Proof. For $r > 1$, $f(\{|z| = r\})$ is a smooth Jordan curve γ. Let $A(r)$ be the area of the region Ω enclosed by this curve. By Green's theorem

$$\begin{aligned}
A(r) &= \int_\gamma x dy = -\int_\gamma y dx = \frac{i}{2}\int_\gamma w d\overline{w} \\
&= \frac{i}{2}\int_0^{2\pi} f(z)\overline{f'(z)}\,\overline{izdt} \\
&= \frac{1}{2}\int_0^{2\pi} (z + a_1/z + \dots)\overline{(z - a_1/z - \dots)}dt \\
&= \frac{1}{2}\int_0^{2\pi} (1 - |a_1|^2 - 2|a_2|^2 - \dots)dt \\
&\leq \pi \quad \square
\end{aligned}$$

The following is Theorem 13.1.1 of the Astala-Iwaniec-MArtin

Lemma 8.6. *Suppose μ is measurable, compactly supported and $\|\mu\|_\infty = k < 1$. For $\lambda \in \mathbb{D}$, let $f(\lambda, z)$ be the principle solution of $f_{\overline{z}} = \mu f_z$. If μ vanishes in a neighborhood U of a point z, then the derivative $\partial_z f(\lambda, z)$ of the analytic function $z \to f(\lambda, z)$ depends holomorphically on λ.*

Theorem 8.7. *Suppose f is K-quasiconformal on the plane and is conformal outside $\mathbb{D}$, and assume $f(z) = z + o(1)$ near infinity. If the dilatation μ of f is zero on $E \subset \mathbb{D}$, then $|f(E)| \leq \pi^{1-1/K}|E|^{1/K}$.*

Proof. We may assume E is open.

To deduce the general case we choose nested open sets $\{E_n\}$ containing E so that $\text{area}(E_n \setminus E) \to 0$ and let μ_n be the restriction of μ to E_n^c and let f_n be the corresponding maps. Then since $J_f = |f_z|^2$ when $\mu = 0$,

$$\begin{aligned}
\left| \int_E J_f dxdy - \int_{E_n} J_{f_n} dxdy \right| &\leq \left| \int_{\mathbb{D}} |f_z|^2 - |(f_n)_z|^2 dxdy \right| \\
&\leq \left| \int_{\mathbb{D}} |f_z| - |(f_n)_z| \cdot |f_z| + |(f_n)_z| dxdy \right| \\
&\leq \|f_z - (f_n)_z\|_2 \cdot \|f_z + (f_n)_z\|_2 \to 0
\end{aligned}$$

since the first term tends to zero and the second is bounded.

Thus $\text{area}(f_n(E_n)) \to \text{area}(f(E))$ and so the estimate for open sets implies the same estimate for measurable sets.

For $|\lambda| < 1$ define a K_λ-quasiconformal map f_λ with dilatation

$$\mu_\lambda(z) = \lambda \cdot \frac{K+1}{K-1} \cdot \mu(z) = \frac{\lambda}{k} \cdot \mu(z),$$

and normalized so that $f(z) = z + o(1)$ near infinity.

Note that $f_k = f$ and dilatation of f_λ is $K_\lambda = (1 + |\lambda|)/(1 - |\lambda|)$.

The Jacobian of f_λ is

$$J_\lambda(z) = |\partial_z f_\lambda(z)|^2(1 - |\mu_\lambda(z)|^2).$$

Define

$$a(z, \lambda) = \frac{1}{\pi} J_\lambda(z)$$

Since f is conformal on E, i.e., $\mu = 0$ on E, we have $\mu_\lambda = 0$ on E too, so

$$a(z, \lambda) = \frac{1}{\pi} J_\lambda(z) = \frac{1}{\pi} |\partial_z f_\lambda(z)|^2.$$

Since $f_\lambda(z)$ is a non-vanishing holomorphic function of λ, so is $\partial_z f_\lambda(z)$.

Fact: if g is holomorphic and never zero, then $\log |g|^2 = 2 \log |g|$ is harmonic.

Hence $\log a(z, \lambda)$ is harmonic in λ.

By the area theorem for conformal maps, $\mathrm{area}(f_\lambda(\mathbb{D})) \leq \pi$, so

$$\int_{\mathbb{D}} a(z,\lambda)dxdy = \frac{1}{\pi}\int_{\mathbb{D}} J_\lambda(z)dxdy \leq 1.$$

Thus $a(z,\lambda)$ satisfies Corollary 8.4, and hence

$$\frac{1}{\pi}|f(E)| = \frac{1}{\pi}\int_E J_\lambda(z)dxdy \leq \left(\frac{|E|}{\pi}\right)^{(1-|\lambda|)/(1+|\lambda|)}.$$

Setting $\lambda = (K-1)/(K+1)$ gives $\mu_\lambda = \mu$ and thus $|f(E)| \leq \pi^{1-1/K}|E|^{1/K}$. $\square$

Theorem 8.8. *Suppose that f is K-quasiconformal on the plane, that $E \subset \mathbb{D}$, and that the dilatation μ of f is zero on $\mathbb{C} \setminus E$, Assume $f(z) = z + o(1)$ near infinity. Then $|f(E)| \leq K|E|$.*

Proof. If suffices to prove this for compact E, since for general sets, the area is just the supremum of the areas of all compact subsets.

It suffices to prove this when f is smooth, since we can find smooth approximations to f whose dilatations are supported in a neighborhood U of E whose area is a close to E as we wish.

Set $\omega = f_{\overline{z}}$. If $\mathcal{T}$ denotes the Beurling transform, then $f_z = 1 + \mathcal{T}\omega$ and

$$\omega = \mu(1 + \mathcal{T}\mu + \mathcal{T}\mu\mathcal{T}\mu + \dots)$$

$$\mathcal{T}\omega = \mathcal{T}\mu + \mathcal{T}\mu\mathcal{T}\mu + \mathcal{T}\mu\mathcal{T}\mu\mathcal{T}\mu + \dots.$$

Then

$$
\begin{aligned}
|f(E)| &= \int_E J_f \, dxdy = \int_E |f_z|^2 - |f_{\overline{z}}|^2 dxdy \\
&= \int_E |1 + \mathcal{T}\omega|^2 - |\omega|^2 dxdy \\
&= \int_E (1 + \mathcal{T}\omega)\overline{(1 + \mathcal{T}\omega)} - |\omega|^2 dxdy \\
&= \int_E (1 + 2\mathrm{Re}(\mathcal{T}\omega) + |\mathcal{T}\omega|^2 - |\omega|^2) dxdy.
\end{aligned}
$$

Since $\mathcal{T}$ is an isometry on L^2, and ω is supported on E,

$$\int_E |\mathcal{T}\omega|^2 dxdy \leq \int_{\mathbb{C}} |\mathcal{T}\omega|^2 dxdy = \int_{\mathbb{C}} |\omega|^2 dxdy = \int_E |\omega|^2 dxdy.$$

Thus

$$|f(E)| \ \leq \ |E| + 2\int_E \mathrm{Re}(\mathcal{T}w) dxdy.$$

Let $(\mathcal{T}\mu)^1 = \mathcal{T}\mu$ and inductively define the kth iterate $(\mathcal{T}\mu)^k = \mathcal{T}(\mu(\mathcal{T}\mu)^{k-1})$ for $k = 2, \ldots$.

Observe that by Cauchy-Schwarz and since $\mathcal{T}$ is an isometry on L^2 , the kth iterate satisfies

$$
\begin{aligned}
\int_E |(\mathcal{T}\mu)^k| dxdy &\leq \left(\int_E 1 dxdy\right)^{1/2} \left(\int_{\mathbb{C}} |(\mathcal{T}\mu)^k|^2 dxdy\right)^{1/2} \\
&= |E|^{1/2} \left(\int_E |\mu(\mathcal{T}\mu)^{k-1}|^2 dxdy\right)^{1/2} \\
&= \|\mu\|_\infty |E|^{1/2} \left(\int_E |(\mathcal{T}\mu)^{k-1}|^2 dxdy\right)^{1/2}.
\end{aligned}
$$

Applying induction we deduce

$$\int_E |(\mathcal{T}\mu)^k| dxdy \;=\; \|\mu\|_\infty^k |E|^{1/2} \left(\int_E 1 dxdy \right)^{1/2} = \|\mu\|_\infty^k |E|.$$

Since $\|\mu\|_\infty = k = (K-1)/(K+1)$, we therefore have

$$\begin{aligned}
|f(E)| &\le |E| + 2|E|(\|\mu\|_\infty + \|\mu\|_\infty^2 + \dots) \\
&= |E| + 2|E|(-1 + 1 + \|\mu\|_\infty + \|\mu\|_\infty^2 + \dots) \\
&= |E| \left(-1 + \frac{2}{1-k} \right) \\
&= K|E|. \quad \square
\end{aligned}$$

The following result is Astala's theorem with a slightly different normalization.

Corollary 8.9. *Suppose f is K-quasiconformal on the plane and is conformal outside $\mathbb{D}$, and assume $f(z) = z + o(1)$ near infinity. If $E \subset \mathbb{D}$, then $|f(E)| \leq K\pi^{1-1/K}|E|^{1/K}$.*

Proof. Write $f = h \circ g$ where g is conformal on E and h is conformal off $g(E)$. Then

$$|f(E)| = |h(g(E))| \leq K|g(E)| \leq K\pi^{1-1/K}|E|^{1/K}.$$

$\square$

Proof of Astala's theorem, Theorem 8.1. Astala's theorem is for self-maps of the disk, whereas what we have done following Eremenko and Hamilton is for maps that are conformal outside the unit disk.

A K-quasiconformal self-map of the disk f can be written as a composition of two K-quasiconformal maps $f = h \circ g$ where g is conformal on $\{|z| > 2\}$, with $g(z) = z + o(1)$ near infinity, and h is conformal from $\Omega = g(2\mathbb{D})$ to $\Omega' = f(2\mathbb{D})$. Both these Jordan domains have diameter $\simeq 1$.

Then $|f(E)| = |h(g(E))|$ and we know $|g(E)| \leq C(K)|E|^{1/K}$, so it is enough to know that h multiplies the area of $g(E)$ by at most a factor depending only on K.

Note that $g(E)$ is separated from $\partial\Omega$ by the topological annulus $A = g(2\mathbb{D} \setminus \overline{\mathbb{D}})$, that has modulus bounded above in terms of K.

Hence the distance between the boundaries of A is bounded below, depending only on K. By Koebe's theorem, the absolute value of the derivative of h on E is bounded above by a constant depending only on K.

This gives the desired estimate. $\square$

Define $p(K) = \sup\{p : J_f \in L^p_{\mathrm{loc}}(\Omega)\}$ where the supremum is over all K-quasiconformal maps f on Ω.

We have seen previously that $p(K) > 1$; Bojarski's Theorem, Theorem 6.21.

Theorem 8.10. *For any planar domain* Ω, $p(K) = K/(K-1)$

Proof. First we prove $p(K) \leq K/(K-1)$.

We claim that Setting $f(z) = z|z|^{(1/K)-1}$ shows that $p(K) \leq K/(K-1)$.

To prove this, note that the partials are $O\left(|z|^{(1/K)-1}\right)$, so

$$J_f^p = O\left(|z|^{2p(1-K)/K}\right)$$

In order to be locally integrable, we need

$$2p(1-K)/K > -2,$$

which is equivalent to $p < K/(K-1)$).

Next we prove $p(K) \geq K/(K-1)$.

First consider a K-quasiconformal map $f : \mathbb{D} \to \mathbb{D}$ and for $s \geq 0$, set

$$E_s = \{x \in \mathbb{D} : J_f(x) \geq s\}.$$

By Astala's area theorem

$$s|E_s| \leq \int_{E_s} J_f dx dy = |f(E_s)| \leq C(K)|E_s|^{1/K}$$

or, solving for $|E_s|$,

$$|E_s| \leq \left(\frac{C(K)}{s}\right)^{K/(K-1)}.$$

For such a map

$$\int_{\mathbb{D}} J_f^p dxdy \leq \pi + p \int_1^{\infty} s^{p-1}|E_s|ds = \pi + M(K) \int_0^{\infty} s^{p-1} s^{-K/(K-1)} ds.$$

This is finite if

$$(p-1) - K/(K-1) < -1$$

or, equivalently,

$$p < K/(K-1).$$

This completes the proof for $f : \mathbb{D} \to \mathbb{D}$.

For a general K-quasiconformal map on a domain Ω, choose a compact disk D with $2D \subset \Omega$. Let ψ and ϕ be conformal maps of $2D$ and $f(2D)$ respectively to the unit disk. Then the previous argument applies to $g = \phi \circ f \circ \psi^{-1}$.

But by Koebe's theorem the derivative of ϕ and ψ are both comparable to constants on D and $f(D)$ and thus J_f^p is integrable on D if and only if J_g is.

This proves the result in general. $\square$

Theorem 8.11. *Suppose $f : \Omega \to \Omega'$ is K-quasiconformal and $E \subset \Omega$ is compact. Then*

$$\dim(f(E)) \leq \frac{2K \dim(E)}{2 + (K-1)\dim(E)}.$$

The estimate in the theorem can be re-written as

$$\frac{1}{K}\left(\frac{1}{\dim(E)} - \frac{1}{2}\right) \leq \frac{1}{\dim(f(E))} - \frac{1}{2} \leq K\left(\frac{1}{\dim(E)} - \frac{1}{2}\right).$$

Astala gives examples showing equality is possible for some Cantor sets.

For $K = (1+k)/(1-k)$ and a line segment E, the estimate says

$$\dim(f(E)) \leq \frac{2K}{K+1} = 1+k.$$

Astala conjectured, and Smirnov later proved, that

$$\dim(f(E)) \leq 1+k^2 < 1+k,$$

but Ivrii has show this is not sharp either (at least for small k).

We will prove Smirnov's bound later.

Lemma 8.12. *Suppose* $0 < t < 1$, $f : \mathbb{D} \to \mathbb{D}$ *is* K*-quasiconformal with* $f(0) = 0$, *and* $\{B_j\}$ *are pairwise disjoint balls in* $\mathbb{D}$*. If*

$$\frac{tK}{1 + t(K-1)} < p \leq 1,$$

then

$$\sum_j |f(B_j)|^p \leq C(K, t, p) \left(\sum_j |B_j|^t \right)^{1/(1+t(K-1)} .$$

Here $|B_j|$ denotes the area of B_j.

Proof. If $1 < p_0 < K/(K-1)$, then the conjugate exponent $q_0 = p_0/(p_0-1)$ satisfies $K < q_0 < \infty$ or $1 < q_0/K < \infty$.

Since

$$\frac{p(1+t(K-1))}{tK} > 1,$$

we can choose p_0 and q_0 so that

$$1 < \frac{q_0}{K} < p \cdot \frac{1+t(K-1)}{tK}.$$

Now apply Hölder's inequality to the integral with exponents p_0, q_0 to deduce

$$\begin{aligned}\sum_j |f(B_j)|^p &= \sum_j \left(\int_{B_j} J_f dxdy \right)^p \\ &= \sum_j \left[\left(\int_{B_j} J_f^{p_0} dxdy \right)^{p/p_0} |B_j|^{p/q_0} \right].\end{aligned}$$

Next apply Hölder's inequality to the sum with conjugate exponents p_0/p and $p_0/(p_0-p)$ to get

$$
\begin{aligned}
\sum_j |f(B_j)|^p &\leq \sum_j \left[\left(\int_{B_j} J_f^{p_0} dxdy \right)^{p/p_0} |B_j|^{p/q_0} \right] \\
&= \sum_j \left(\sum \int_{B_j} J_f^{p_0} dxdy \right)^{p/p_0} \left(\sum_j |B_j|^{(p/q_0)p_0/(p_0-p)} \right)^{(p_0-p)/p_0} \\
&\leq \sum_j \left(\int_{\mathbb{D}} J_f^{p_0} dxdy \right)^{p/p_0} \left(\sum_j |B_j|^{(p/q_0)p_0/(p_0-p)} \right)^{(p_0-p)/p_0} .
\end{aligned}
$$

Since $p_0 < P(K) = K/(K-1)$, the first term is finite.

Some arithmetic shows that if $p_0 = K/(K-1)$ and $p = tK/(1+t(K-1))$, then

$$\frac{p_0}{p_0 - p} = 1 + t(K-1).$$

The left side is decreasing in p_0 (for $p_0 > 1$) and increasing in p (for $0 < p < 1$), so if

$$1 < p_0 < \frac{K}{K-1}$$

and

$$1 > p > \frac{tK}{1+t(K-1)},$$

then

$$\frac{p_0}{p_0 - p} > 1 + t(K-1).$$

We proceed, assuming p and p_0 satisfy these bounds.

Not that if $\alpha < \beta$ and $0 < x < 1$ then then $x^\beta < x^\alpha$. Similarly, if $0 < x < C$ then $x^\beta \le C(\alpha, \beta) x^\alpha$. Thus

$$\sum_j |f(B_j)|^p \le \sum_j \left(\int_{\mathbb{D}} J_f^{p_0} dx dy \right)^{p/p_0} \left(\sum_j |B_j|^{(p/q_0)p_0/(p_0-p)} \right)^{(p_0-p)/p_0}$$

$$\le C(K, t, p) \left(\sum_j |B_j|^{(p/q_0)(1+t(K-1)} \right)^{(p_0-p)/p_0} .$$

We choose q_0 so that $p/q_0 > t/(1 + t(K-1)$, so this becomes

$$\sum_j |f(B_j)|^p \le C(K, t, p) \left(\sum_j |B_j|^t \right)^{1/(1+t(K-1))} . \quad \Box$$

Proof of Astala's dimension estimate. Suppose $f : \Omega \to \Omega'$ is K-quasiconformal and that $E \subset \Omega$ is compact with Hausdorff dimension strictly less than 2.

Choose $\dim(E)/2 < t < 1$ and cover E by squares with disjoint interiors. Each square contains an inscribed ball of comparable size, giving a collect $\{B_j\}$ of pairwise disjoint balls whose doubles cover E.

We know $\operatorname{diam}(B_j)^2 \simeq |B_j|$. Thus if

$$\delta > \frac{2tK}{1+t(K-1)}$$

by Lemma 8.12 we have

$$\sum_j \operatorname{diam}(f(B_j))^\delta = C\left(\sum_j \operatorname{diam}(B_j)^{2t}\right)^{1/(1+t(K-1)}.$$

For any $t > \dim(E)/2$, the sum on the right can be made as small as we wish, by an appropriate choice of covering squares. Thus $\dim(f(E)) \leq \delta$ for any $\delta > 2tK/(1+t(K-1))$ and thus any

$$\begin{aligned}\delta &> \frac{\dim(E)K}{1+\dim(E)(K-1)/2}\\ &= \frac{2\dim(E)K}{2+\dim(E)(K-1)} \quad \square\end{aligned}$$

Lemma 8.13. *If $E \subset \mathbb{D}$ is closed and has zero Hausdorff* 1*-measure, then any bounded holomorphic map f on $\Omega = \mathbb{D} \setminus E$ extends to be holomorphic on $\mathbb{D}$.*

Proof. We can choose R arbitrarily close to 1 so that e circle $C_R = \{|z| = R\}$ does not hit E, since otherwise E hits all large enough circles and hence has positive length.

Cover the part of E inside C_R by balls whose total boundary length is less than ϵ. For any z inside C_R but outside the balls, we use the Cauchy integral formula to write $f(z)$ as the Cauchy integral over C_R and a contour γ of length at most ϵ contained in the union of the boundaries of the ball.

Since f is bounded, the contribution of γ tends to zero with ϵ and hence f agrees with its Cauchy integral over C_R, which defines a holomorphic function on the entire interior of C_R. Taking $R \nearrow 1$, shows f extends to be holomorphic on all of $\mathbb{D}$.

$\square$

Corollary 8.14. *A planar compact set E with* $\dim(E) < 2/(K+1)$ *is removable for bounded K-quasiregular maps.*

Astala constructs sets of any dimension $> 2/(K+1)$ that are not removable.

Proof. It suffices to consider maps defined on a disk.

Any K-quasiregular map f can be factored as $f = \phi \circ g$ where ϕ is holomorphic on $\mathbb{D}$ and $g : \mathbb{D} \to \mathbb{D}$ is K-quasiconformal.

If $\dim(E) < 2/(K+1)$, then

$$\dim(g(E)) < \frac{2K\frac{2}{K+1}}{2 + (K-1)\frac{2}{K+1}} = \frac{4K}{2K+2+2K-2} = 1.$$

Thus $g(E)$ is removable for ϕ, i.e., ϕ extends to be holomorphic on the whole plane and hence f extends to be quasiregular on the plane. $\square$

Conformal dimension

The conformal dimension of a set E is the infimum of the Hausdorff dimensions of $f(E)$ over all quasiconformal maps in the plane.

Introduced by Pansu in 1989.

One can replace quasiconformal maps by quasisymmetric maps into metric spaces. This may change the value. Sometimes use "C-dim" for metric space images, and "QC-dim" for images in $\mathbb{R}^n$.

One could also consider other types of dimension.

Conformal Dimension: theory and application by John Mackay and Jeremy Tyson, 2010 Sometimes use "C-dim" for metric space images, and "QC-dim" for images in $\mathbb{R}^n$.

Theorem 8.15 (Bishop, 1999)**.** *For any set* $E \subset \mathbb{R}^n$ *with* $\dim(E) > 0$ *and any* $\epsilon > 0$ *there is a QC map* f *so* $\dim(f(E)) > n - \epsilon$.

In other words, you can always increase dimension by QC maps.

The dimension of a line segment cannot by lowered by any QC (indeed homeomorphic) image. Thus it is "minimal" for conformal dimension.

Theorem 8.16 (Tyson, 2000)**.** *If $E \subset \mathbb{R}^n$ is Ahlfors regular for dimension α, then $E \times [0,1] \subset$ is minimal.*

Tyson's paper

Ahlfors regular means that for any ball $B(x,r)$ the α-Hausdorff measure satisfies

$$\mathbb{H}_\alpha(E \cap B(x,r)) \simeq r^\alpha.$$

Self similar Cantor sets have this property.

Thus there exists minimal sets of every dimension ≥ 1.

These sets can be taken to be Cantor sets.

Theorem 8.17 (Kovalev, 2006)**.** *If* $\dim(E) < 1$ *then its conformal dimension is* 0*.*

In other words, conformal dimension never takes values in $(0, 1)$.

Kovalev's paper

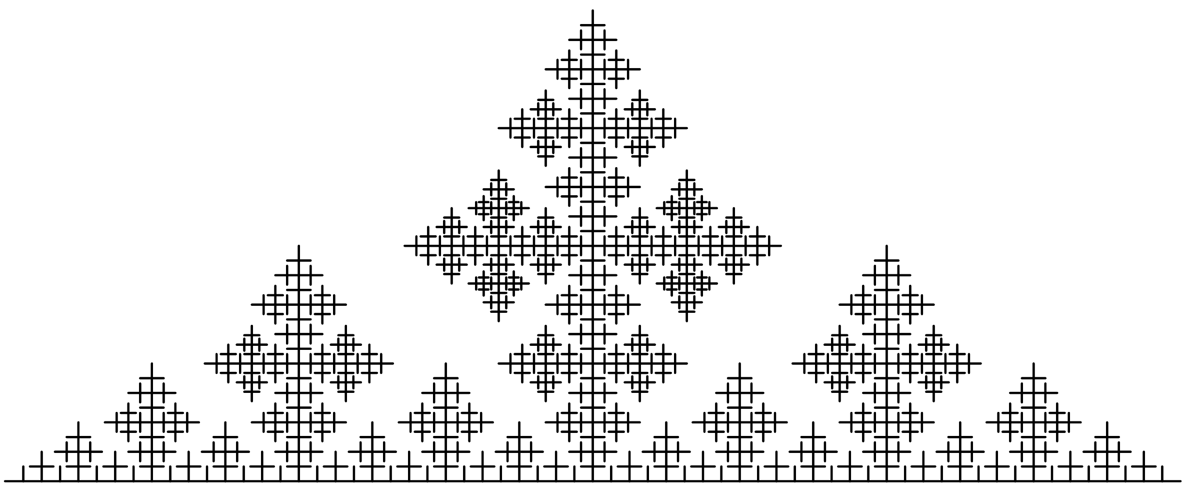
Theorem 8.18 (Hakobyan, 2006 Stony Brook thesis)**.** *There are minimal sets in* $\mathbb{R}$ *of dimension* 1 *but zero Lebegue measure.*

Hakobayan's paper

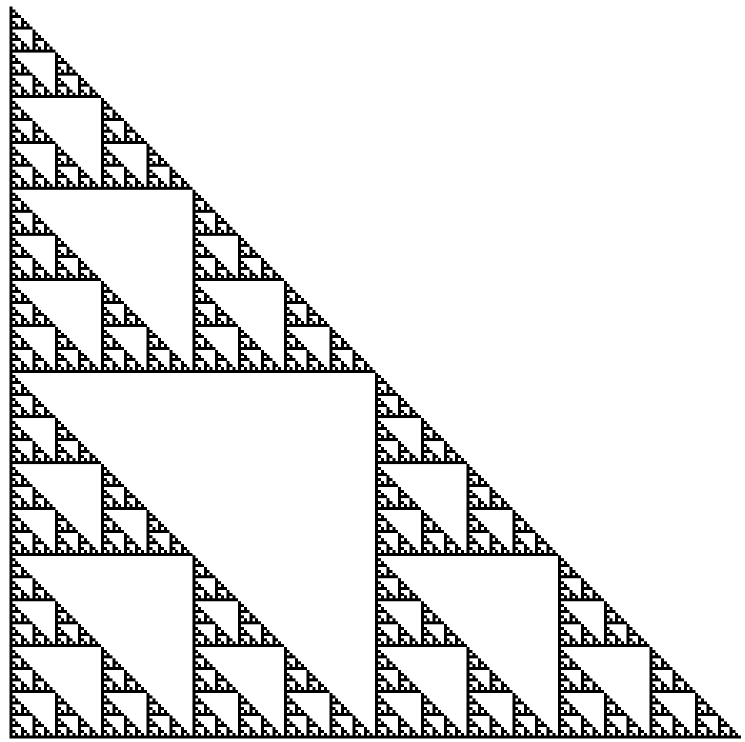
Theorem 8.19 (Bishop-Tyson, 2001)**.** *For each $1 \leq \alpha < n$ and $K < \infty$, there is a set $E \subset \mathbb{R}^n$ of dimension α and conformal dimension 0 but so that $\dim(f(E)) = \alpha$ for every K-quasiconformal map f.*

For this set, the dimension can be lowered to zero, but a big dilatation K is needed to lower the dimension even an little. paper

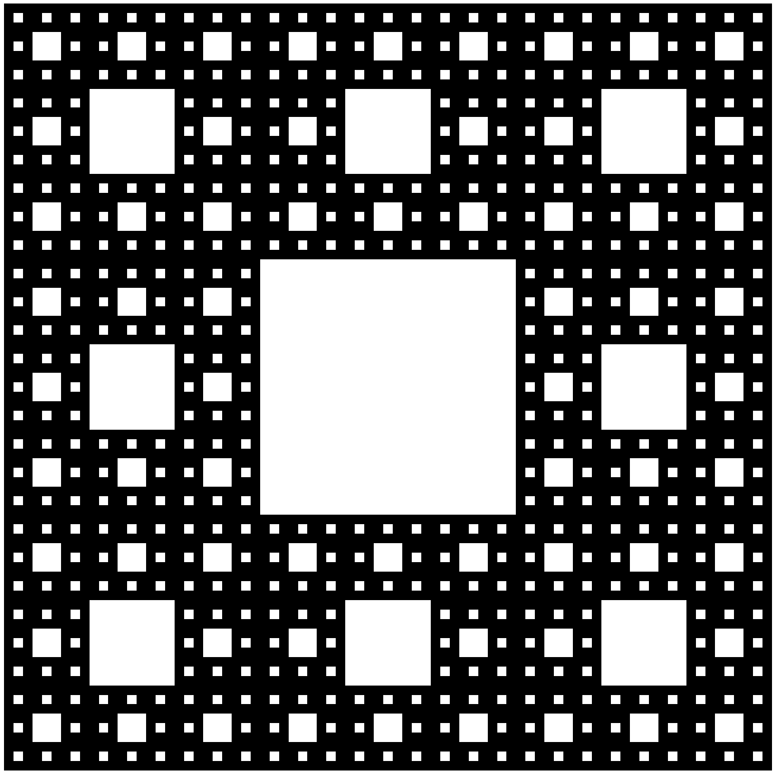
Theorem 8.20 (Bishop-Tyson, 2001)**.** *There is a compact set $E \subset \mathbb{R}^2$ with conformal dimension 1, but this dimension is not attained by any quasisymmetric map into any metric space. paper*



Theorem 8.21 (Tyson-Wu, 2003)**.** *The Sierpinski gasket has conformal dimension* 1*.*



Sierpinski gasket, $\dim = \log 3/\log 2 \approx 1.58$



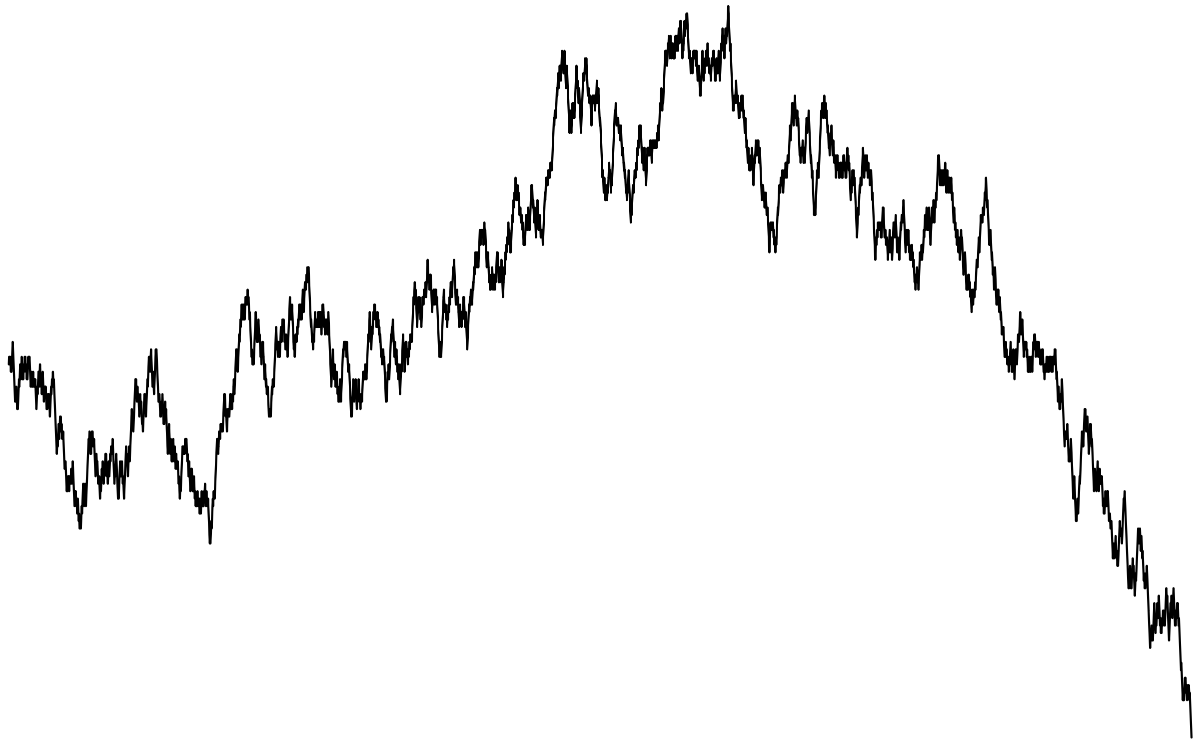
Sierpinski carpet, dim $= \log 8/\log 3 \approx 1.89$

Exact conformal dimension is unknown

Must be $\geq 1 + \log 2/\log 3 \approx 1.63$.

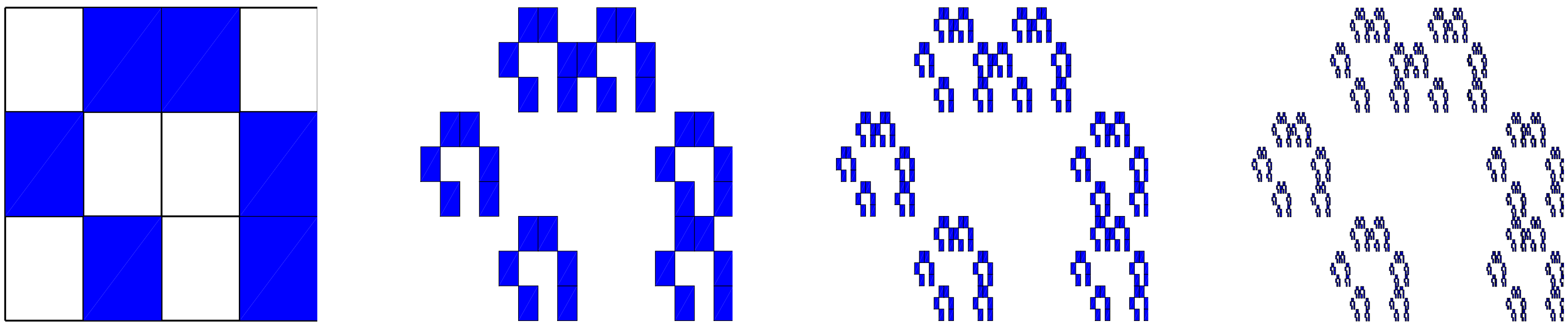
Proven $> 1 + \log 2 \log 3$ by Kwapisz in 2019

2016 Lecture slides by Lukas Geyer and Rob Malo



Graph of 1-dimensional Brownian motion, dim $= 3/2$

Is minimal, Binder, Hakobyan, Li 2024



They also prove certain Bedford-McMullen carpets are minimal

They require each row to have the same number of rectangles.

Theorem 8.22 (Bishop-Hakobyan-Williams, 2016)**.** *If $E \subset \mathbb{R}^n$ is Ahlfors α-regular and f is quasiconformal then* $\dim(f(E+y)) = \dim(E) = \alpha$ *for almost every $y \in \mathbb{R}^n$.*

Can be extended (with modifications) to quasisymmetric maps between certain metric spaces (e.g., Carnot groups).

Can estimate dimension of the set of y where $\dim(f(E+y)) > d > \dim(E)$.

paper constucts Cantor set (dim near 1) of parallel lines and a quasiconformal map that increases dimension of every subarc of every line.

Smirnov's $1 + k^2$ bound

Theorem 8.23. *Suppose f is K-quasiconformal on the plane and $K = (1+k)/(1-k)$. Then* $\dim(f(\mathbb{R})) \leq 1 + k^2$.

Oleg Ivrii has shown that a better bound is $1 + \Sigma^2 k^2 + O(k^{8/3} - \epsilon)$ where Σ^2 is a constant less than 1 (by deep work of Hedenmalm).

Σ^2 is defined as $\sup \sigma^2(S\mu) < 1$ where the supremum is over measurable functions μ so that $|\mu| \leq 1$ on $\mathbb{D}$ and is 0 elsewhere, and σ^2 is the asymptotic variance of a Bloch function

$$\sigma^2(g) = \lim_{R \searrow 1} \frac{1}{2\pi |\log(R-1)|} \int_{|z|=R} |g(z)|^2 |dz|.$$

Recall the connection between ellipse fields and dilatations.

The eccentricity of an ellipse at z is determined by $|\mu(z)|$ and its major axis is in the direction $\arg(\sqrt{\mu(z)})$.

The ambiguity in the square root makes no difference, since the major axis is given by both directions.

A map with dilatation μ maps the corresponding ellipse field to the "all circles" field, which we will denote by T in what follows.

Consider the symmetry condition

$$\mu_\phi(\overline{z}) = \overline{\mu_\phi(z)} \tag{8.28}$$

i.e., the ellipses at conjugate points are reflections of each other across the x-axis.

A QC map f with such a dilatations (and that fixes $0, 1$) preserves $\mathbb{R}$, i.e., $f(\mathbb{R}) = \mathbb{R}$.

Such maps do not increase dimension of $\mathbb{R}$ at all.

Smirnov considers the condition

$$\mu_\phi(\overline{z}) = -\overline{\mu_\phi(z)}. \tag{8.29}$$

This says the ellipse at $\overline{z}$ is a reflection across y-axis of the ellipse at z.

If A is an ellipse field, we let $\|A\|$ denote the essential supremum of the eccentricities. This is the same as the dilatation K of the crresponding QC map.

Let T denote the all circles ellipse field.

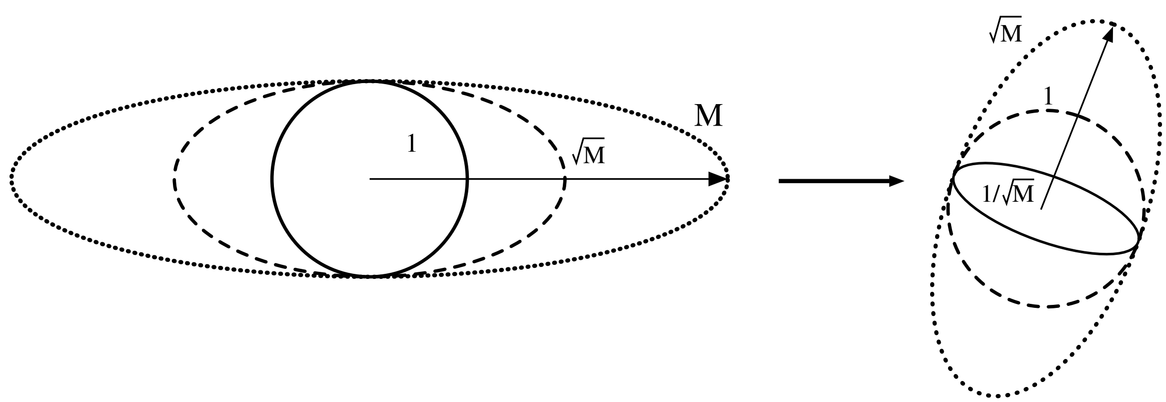
If ϕ is a QC map sending T to A, then $\|\phi(T)\| = \|A\|$.

In this case, $\phi(T)$ at $\phi(z)$ has same pointwise eccentricities as A at z, but the ellipse is rotated.

If M is an ellipse field, then $\sqrt{M}$ is the ellipse field with the same major axis (no rotation), but we take the square root of the eccentricity.

Note that if a linear map sends an ellipse of eccentricity $\sqrt{M}$ to a circle, then the "parallel" ellipse of eccentricity M is sent to an ellipse of eccentricity $\sqrt{M}$,

A circle is sent to an ellipse of eccentricity $\sqrt{M}$, but with the major axis rotated by $\pi/2$.



To prove Smirnov's theorem we need two preliminary results:

- Chararterize dialatatione of "optimal" maps.
- Stonger version of Harnack's inequality.

Theorem 8.24. *The following are equivalent:*

(i) Γ *is a* k*-quasiline.*

(ii) $\Gamma = \psi(\mathbb{R})$ *with* $\|\mu_\psi\|_\infty \le 2k/(1+k^2)$ *and* $\mu_\psi = 0$ *on* $\mathbb{H} = \{x+iy : y > 0\}$.

(iii) $\Gamma = \phi(\mathbb{R})$ *with* $\|\mu_\phi\|_\infty \le k$ *and*

$$\mu_\phi(\overline{z}) = -\overline{\mu_\phi(z)}. \tag{8.30}$$

Since (iii) $\Rightarrow$ (i) is trivial, we need only prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

We follow Smirnov's notation and proof closely, giving the proof in terms of ellipse fields.

Smirnov's paper

Proof of (i) $\Rightarrow$ (ii) .

Suppose $\|\mu\|_\infty = k$ and let $N(z)$, $\|N\| \le K = (1+k)/(1-k)$, be the ellipse field representing a k-quasiconformal map η, which maps $\mathbb{R}$ onto Γ.

Define an ellipse field A:

$$A(z) = \begin{cases} \overline{N(\bar{z})}, & z \in \mathbb{H}_l \\ N(z), & z \in \mathbb{H} \end{cases} .$$

Let α be the QC corresponding to A fixing $0, 1$. It preserves $\mathbb{R}$.

Set $\psi := \eta \circ \alpha^{-1}$. Note $\psi(\mathbb{R}) = \eta(\alpha^{-1}(\mathbb{R})) = \eta(\mathbb{R}) = \Gamma$.

For $z \in \mathbb{H}$, η and α both send the ellipse field $N(z)$ to the field of circles, hence the map $\psi = \eta \circ \alpha^{-1}$ preserves the field of circles and is conformal in the upper half-plane.

In $\mathbb{H}_l$ both η and α change eccentricities by at most K, so ψ changes eccentricities by at most K^2. Thus

$$\|\mu_\psi\| \leq (K^2 - 1)/(K^2 + 1) = 2k/(1 + k^2).$$

$\square$

Proof of (ii) $\Rightarrow$ (iii) .

Let $M(z)$ be the ellipse field corresponding to the $2k/(1+k^2)$-quasiconformal map ψ, with quasiconstant

$$K' = \frac{1+2k/(1+k^2)}{1-2k/(1+k^2)} = \left(\frac{1+k}{1-k}\right)^2 = K^2 .$$

Recall M is zero in the lower half plane (so ψ is conformal there).

Let β be a quasiconformal map corresponding to the ellipse field

$$B(z) = \begin{cases} \sqrt{M(z)}, & z \in \mathbb{H}_l \\ \sqrt{\overline{M(\bar{z})}}, & z \in \mathbb{H} \end{cases}.$$

Here $\sqrt{M}$ denotes the ellipse with the same alignment whose eccentricity is the square root of M's eccentricity; the ellipses are not rotated, only their eccentricities change.

Since $\overline{B(z)} = B(\overline{z})$, $\beta(\mathbb{R}) = \mathbb{R}$

Define $\phi := \psi \circ \beta^{-1}$. Since β preserves $\mathbb{R}$, we have $\phi(\mathbb{R}) = \psi(\mathbb{R}) = \Gamma$.

Recall, T denotes the field of circles.

By definition, ψ maps M to T and β maps B to T. Thus β^{-1} maps B to T.

Let $L(z)$ be the image of the ellipse field $M(z)$ under β. Since M is the preimage of T under ψ, L is also the preimage of T under ϕ. Thus $\|L\|$ is the maximum dilatation of ϕ in the lower half-plane.

For $z \in \mathbb{H}_l$, β maps $B = \sqrt{M}$ to T and ψ maps T to T (it is conformal in $\mathbb{H}_l$. Thus M is mapped by β to a field with the same eccentricities as $\sqrt{M}$.

Hence $\|L\| = \|\sqrt{M}\| = \sqrt{K'} = K$.

In the upper half-plane, β maps $\sqrt{M}$ to T and ψ is conformal, so $\phi = \psi \circ \beta^{-1}$ maps $\beta(T)$ to T. Since

$$\|\beta(T)\| = \|\beta^{-1}(T)\| = \|\sqrt{M}\| = K,$$

we see that ϕ also has dilatation K in the upper half-plane.

Thus ϕ is the desired K-QC map sending $\mathbb{R}$ to Γ. $\square$

Lemma 8.25. *Let h be a positive harmonic function in the unit disc $\mathbb{D}$, whose partial derivative at the origin vanishes in the direction of some $\lambda \in \mathbb{D}$: $\partial_\lambda h(0) = 0$. Then h satisfies*

$$\frac{1-|\lambda|^2}{1+|\lambda|^2} \cdot h(\lambda) \leq h(0) \leq \frac{1+|\lambda|^2}{1-|\lambda|^2} \cdot h(\lambda) \ .$$

This is a stronger version of Harnack's inequality:

$$\frac{1-|\lambda|}{1+|\lambda|} \cdot h(\lambda) \leq h(0) \leq \frac{1+|\lambda|}{1-|\lambda|} \cdot h(\lambda) \ .$$

that holds for all positive harmonic functions.

Proof. By precomposing with a rotation we map assume $h_z(0) = 0$.

By replacing h by $g(z) = (h(z) + h(\overline{z})/(2h(0))$, we may assume $h(0) = 1$ and $h(\overline{z}) = h(z)$. Thus $h_y(0) = 0$ also, and hence the gradient vanishes at 0.

Since $h(z) = g(z) = \frac{1}{2}(h(z)+h(\overline{z}))$ when $z \in \mathbb{R}$, it suffices to prove the desrired bounds for the new function (which we continue to denote by h).

If $\widetilde{h}$ is the harmonic conjugate of h vanishing at 0, then $h + i\widetilde{h}$ map the disk to the right half-plane, and the function

$$f := \tau \circ (h + i\widetilde{h}) = \frac{z-1}{z+1} \circ (h + i\widetilde{h})$$

maps the disk to itself and satisfies $f(0) = f'(0) = 0$. By the Schwarz lemma, $|f(z)/z| \leq |z|$, hence $|f(z)| \leq |z|^2$.

Thus $h(\{|z| < \lambda\}) \subset \tau^{-1}(\{|z| < \lambda^2\})$ and a direct calculation shows the latter set lies between the vertical lines

$$\{x = \tau^{-1}(-\lambda^2) = \frac{1+\lambda^2}{1-\lambda^2}\}$$

and

$$\{x = \tau^{-1}(\lambda^2) = \frac{1-\lambda^2}{1+\lambda^2}\}.$$

This proves the lemma. $\square$

Proof that quasicircles have dimension $\leq 1 + k^2$.

Suppose Γ is a k-quasiline, i.e., $\Gamma = \phi(\mathbb{R})$ where $\|\mu_\phi\|_\infty \leq k < 1$.

Assume $\mu = \mu_\phi$ satisfies Smirnov's condition (8.30): $\mu_\phi(\overline{z}) = -\overline{\mu_\phi(z)}$.

Define a holomorphic motion ϕ_λ with Beltrami coefficients $\mu_\lambda := \mu \cdot \lambda/k$ and which preserve points 0, 1, ∞. As usual, $\phi_0 = \text{id}$ and $\phi_k = \phi$.

Because of (8.30) holds for μ_k we can deduce that μ_λ satisfies (8.30) for real λ and that (8.28) $\mu_\phi(\overline{z}) = \overline{\mu_\phi(z)}$ holds for imaginary λ.

Moreover, for real values of λ one has the additional symmetry

$$\phi_\lambda(z) \;=\; \overline{\phi_{-\lambda}(\bar{z})}.$$

Fix $\rho \in (1/2, 1)$, and consider λ inside the slightly smaller disk $\rho\mathbb{D}$. Within this region, the maps ϕ_λ are uniformly quasisymmetric, so there is a constant $C = C_\rho$ such that

$$(8.31) \quad |z - x| \le |y - x| \Rightarrow |\phi_\lambda(z) - \phi_\lambda(x)| \le C_\rho \cdot |\phi_\lambda(y) - \phi_\lambda(x)|,$$

$$(8.32) \quad C_\rho \cdot |z - x| \le |y - x| \Rightarrow 2|\phi_\lambda(z) - \phi_\lambda(x)| \le |\phi_\lambda(y) - \phi_\lambda(x)|.$$

Since ϕ_λ fixes $0, 1$ (8.31) implies that $\phi_\lambda(\mathbb{D}) \subset D(0, C_\rho)$.

Lemma 8.26. *With notation as above, if $\{I_j\}$ are disjoint intervals of length $1/n$ in $[-1,1]$, then the images $\{\phi_\lambda(I_j)\}$ are contained in balls $\{B_j\}$ with bounded overlaps, i.e., any point is in at most C_ρ of the balls.*

Proof. Suppose $J_1 = [a_1, b_1]$ and $J_2 = [a_2, b_2]$ are two such intervals and let B_j, for $j = 1, 2$, be the smallest ball centered at $\phi_\lambda(a_j)$ containing $\phi_\lambda(J_j)$.

We may assume B_1 is larger than B_2. By (8.32), $\text{dist}(J_1, J_2) \geq C_\rho/n$ implies $\text{dist}(\phi_\lambda(a_2)) \geq 2 \cdot \text{radius}(B_1)$. Hence these balls are disjoint.

Thus we can divide the intevals in $[0, 1]$ into at most C_ρ collections, so that all the balls corresponding to each collecdtion are pairwise disjoint. $\square$

We continue with the proof. It suffices bound the dimension of $\phi_\lambda([0,1])$.

Cover $[0,1]$ by n intervals $I_j = [a_j, b_j]$ of length $1/n$, and let $B_j(\lambda)$ be the smallest ball centered at $\phi_\lambda(a_j)$ containing $\phi_\lambda(I_j)$.

Its actual radius is comparable to within a factor of C_ρ of its "complex radius"

$$r_j(\lambda) := \phi_\lambda(b_j) - \phi_\lambda(a_j).$$

This is a holomorphic function of λ.

Also note r_j is never zero if $a_j \neq b_j$, so $\log |r_j|$ is harmonic on $\mathbb{D}$.

$\phi([0,1])$ is covered by the images of the I_j, and by (8.31).

$$\operatorname{diam}(I_j) \leq C_\rho |r_j(\lambda)|.$$

To esimate $\dim(\phi([0,1])$ it suffices to bound sums of the form

$$\sum_j \operatorname{diam}(\phi_\lambda(I_j))^p \leq C_\rho^p \sum_j |r_j(\lambda)|^p. \tag{8.33}$$

We will estimate the logarithm of the right-hand side.

Recall Jensen's inequality: if f is concave down on $(0, \infty)$, and μ is a probability measure on $(0, \infty)$, then

$$\int f d\mu \leq f(\int d\mu)$$

Moreover, if f is strictly concave (not linear on any sub-interval) then equality ocurs iff μ is a point mass.

Since $\log x$ is concave, if $\{\nu_j\}$ is a probability vector, then Jensen's inequality applied to the measure ν with mass $\{\nu_j\}$ at $\{\log |r_j(\lambda)|^p/\nu_j\}$ gives

$$\begin{aligned}
\log \sum_j |r_j(\lambda)|^p &= \log \sum_j \nu_j \frac{|r_j(\lambda)|^p}{\nu_j} \\
&\geq \sum_j \nu_j \log \left(\frac{|r_j(\lambda)|^p}{\nu_j} \right) \\
&\geq -\sum_j \nu_j \log \nu_j + \sum_j \nu_j \log |r_j(\lambda)|^p \\
&= I_\nu - p\,\Lambda_\nu(\lambda) \ ,
\end{aligned} \tag{8.34}$$

where $I_\nu := -\sum_j \nu_j \log \nu_j$ is the "entropy" and $\Lambda_\nu(\lambda) := -\sum_j \nu_j \log |r_j(\lambda)|$ is the "Lyapunov exponent" of the probability distribution $\{\nu_j\}$.

Note $\Lambda_\nu(\lambda)$ is a harmonic function of λ, since $r_j(\lambda)$ are holomorphic.

Since log is strictly concave, equality is achieved if and only if all the mass is concentrated at one point, i.e., $|r_j(\lambda)|/\nu_j$ is independent of j.

This would imply $|r_j(\lambda)|$ is proportional to ν_j.

Thus we have the "variational principle"

$$\log \sum_j |r_j(\lambda)|^p = \sup_\nu \{I_\nu - p\Lambda_\nu(\lambda)\} \ , \tag{8.35}$$

where the supremum is over all probability distributions ν, with equality when

$$\nu_j = \frac{|r_j(z)|}{\sum_j |r_j(z)|}.$$

Fix some $\nu = \{\mu_j\}$ and define

$$H(\lambda) := 2\Lambda_\nu(\lambda) - I_\nu + 3\log C_\rho.$$

Then H is harmonic in λ (since Λ_ν is) and is an even function on the real line (because of the symmetry of our motion $r_j(\lambda) = \overline{r_j(-\lambda)}$ for $\lambda \in \mathbb{R}$).

Thus H has partial derivative zero at $z = 0$ in the postive real direction.

By (8.32) the balls $B_j(\lambda)$ cover every point at most C_ρ times.

By (8.31) their union is contained in a ball of radius C_ρ. Hence

$$\pi \sum_j |r_j(\lambda)| \leq \sum_j \operatorname{area}(B_j) \leq C_\rho \cdot \operatorname{area}(B(0, C_\rho)),$$

and thus

$$\sum_j |r_j(\lambda)|^2 \leq C_\rho^3,$$

and so by the variational principle (8.35) we have $I_\nu - 2\Lambda_\nu(\lambda) \leq \log C_\rho^3$.

Therefore $H(\lambda) = 2\Lambda_\nu(\lambda) - I_\nu + 3 \log C_\rho \geq 0$ on $\rho\mathbb{D}$.

Moreover, by (8.35) with $p = 1$ we have

$$I_\nu - \Lambda_\nu(0) \le \log \sum |r_j(0)| = \log 1 = 0.$$

Thus

$$\begin{aligned} H(0) &= 2\Lambda_\nu(0) - I_\nu + 3\log C_\rho \\ &= 2(\Lambda_\nu(0) - I_\nu) + I_\nu + 3\log C_\rho \\ &\ge I_\nu + 3\log C_\rho. \end{aligned}$$

Apply Lemma 8.25 (stronger Harnack inequality) in the disk $\rho\mathbb{D}$ to obtain

$$\begin{aligned} 2\Lambda_\nu(k) - I_\nu + 3\log C_\rho &= H(k) \\ &\geq \frac{1-k^2\rho^{-2}}{1+k^2\rho^{-2}} \cdot H(0) \\ &\geq \frac{1-k^2\rho^{-2}}{1+k^2\rho^{-2}}\{I_\nu + 3\log C_\rho\}, \end{aligned}$$

The equation (from the previous page)

$$2\Lambda_\nu(k) - I_\nu + 3\log C_\rho \;\geq\; \frac{1-k^2\rho^{-2}}{1+k^2\rho^{-2}}\{I_\nu + 3\log C_\rho\},$$

implies

$$\frac{2}{1+k^2\rho^{-2}} I_\nu - 2\Lambda_\nu(k) \leq \frac{2k^2\rho^{-2}}{1+k^2\rho^{-2}} 3\log C_\rho,$$

which can be rewritten as

$$I_\nu - (1+k^2\rho^{-2})\Lambda_\nu(k) \leq 3\frac{k^2}{\rho^2}\log C_\rho.$$

The last equation holds for all distributions ν, so by the variational principle (8.35) if we set $p = 1 + k^2/\rho^2$, and take $\rho > 1/2$, then we get

$$
\begin{aligned}
\log \sum_j |r_j(k)|^p &= \sup_\nu \{I_\nu - p\Lambda_\nu(k)\} \\
&\leq k^2\rho^{-2} 3 \log C_\rho \\
&\leq 12 \log C_\rho.
\end{aligned}
$$

Sending n to infinity, (8.33) and (8.36) imply that the p-dimensional Hausdorff measure of $\phi[0,1]$ is bounded by $C_\rho^{12+p} \leq C_\rho^{14}$, since $p \leq 2$.

Hence $\dim(\phi([0,1])) \leq p = 1 + k^2\rho^{-2}$. Let $\rho \nearrow 1$ to prove the theorem. $\square$

Holomorphic motions

Definition: Suppose $A \subset \mathbb{C}^\infty$. A **holomorphic motion** of A is a map $\Phi : \mathbb{D} \times A \to \mathbb{C}^\infty$ such that

(1) For each $a \in A$, the map $\lambda \to \Phi(\lambda, a)$ is holomorphic on $\mathbb{D}$.

(2) For any fixed $\lambda \in \mathbb{D}$, the map $a \to \Phi(\lambda, a) = \Phi_\lambda(a)$ is 1-to-1,

(3) The mapping Φ_0 is the identity on A.

Note that no assumption of continuity or measurability in a is made.

Astala-Martin paper on holomorphic motions.

Definition: Let $\eta : [0,\infty) \to [0,\infty)$ be an increasing homeomorphism and $A \subset \mathbb{C}$. A mapping $f : A \to \mathbb{C}$ is called η-quasisymmetric if the each triple $x, y, z \in A$,

$$\frac{|f(x_-f(y)|}{|f(x) - f(z)|} \leq \eta\left(\frac{|x-y|}{|x-z|}\right).$$

We say f is quasisymmetric if it is η-quasisymmetric for some η.

If f is defined on an open set, we also assume if preserves orientation.

It is immediate that f is continuous and injective and not hard to show f is a homeomorphism onto its image.

Easy to show that the inverse of a quasisymmetric map is quasisymmetric.

One can prove that a map $\mathbb{C} \to \mathbb{C}$ is quasisymmetric iff it is quasiconformal. Also true for broad class of metric spaces (with appropriate definition of quasi-conformal).

The λ-lemma of Mañé, Sad and Sullivan:

Theorem 9.1. *If* $\Phi : \mathbb{D} \times A \to \mathbb{C}^\infty$ *is a holomorphic motion, then has an extension to* $\overline{\Phi} : \mathbb{D} \times \overline{A} \to \mathbb{C}^\infty$ *so that*

(1) $\overline{\Phi}$ *is a holomorphic motion of* $\overline{A}$*.*

(2) Each $\overline{\Phi}_\lambda : \overline{A} \to \mathbb{C}^\infty$ *is quasisymmetric.*

(3) $\overline{\Phi}(\lambda, a)$ *is jointly continuous in* λ *and* a*.*

Proof. we may assume A has at least three points and that $\{0, 1, \infty\} \in A$. We normalize Φ so the motion fixes $\{0, 1, \infty\}$ by setting

$$(\lambda, a) \to \frac{\Phi(\lambda, 1) - \Phi(\lambda, 0)}{\Phi(\lambda, 1) - \Phi(\lambda, \infty)} \cdot \frac{\Phi(\lambda, 1) - \Phi(\lambda, \infty)}{\Phi(\lambda, 1) - \Phi(\lambda, 0)}$$

The new map is still denoted Φ.

Let ρ be the hyperbolic metric on $\mathbb{C} \setminus \{0, 1\}$.

It follows from properties of the hyperbolic metric that there is some function $\eta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ so that

$$|w| \leq \eta(\rho(w, z), |z||)$$

and that for $\eta(x, \epsilon) \to 0$ uniformly as $\epsilon \to 0$ as long as $x \in (0, M]$, for a fixed $M < \infty$.

If $a_1, a_2, a_3 \in A$ are distinct, define

$$g(\lambda) = \frac{\Phi_\lambda(a_1) - \Phi_\lambda(a_2)}{\Phi_\lambda(a_1) - \Phi_\lambda(a_3)}.$$

This is holomorphic in λ with values in $\mathbb{C} \setminus \{0, 1\}$.

The Schwarz lemma says that holomorphic maps are contractions of the hyperbolic metric on any hyperbolic domain (this follows from the disk case and the uniformization theorem).

Thus g is a contraction of the hyperbolic metric from $\mathbb{D}$ to $\mathbb{C} \setminus \{0, 1\}$. Hence

$$\rho(g(\lambda), g(0)) \leq \rho_{\mathbb{D}}(\lambda, 0) = \log \frac{1 + |\lambda|}{1 - |\lambda|}.$$

Since

$$g(0) = \frac{a_1 - a_2}{a_1 - a_3}$$

we have

$$\left|\frac{\Phi_\lambda(a_1) - \Phi_\lambda(a_2)}{\Phi_\lambda(a_1) - \Phi_\lambda(a_3)}\right| \leq \eta\left(\log\frac{1+|\lambda|}{1-|\lambda|}, \left|\frac{a_1 - a_2}{a_1 - a_3}\right|\right).$$

This is the definition of Φ being quasisymmetric on A, and implies Φ is uniformly continuous on A, hence extends continuously to the closure of A.

We claim the extension is injective.

If not, there are points x, y in the closure that get mapped to the same point z.

Choose a_2 so that $\Phi(a_2) \neq z$ (we can do this since Φ is injective on A and A contains at least three points).

Then as a_1 approaches x and a_3 approaches y,

$$\left|\frac{\Phi_\lambda(a_1) - \Phi_\lambda(a_2)}{\Phi_\lambda(a_1) - \Phi_\lambda(a_3)}\right|$$

would blow up, contrary to what we have proved. Thus the extension is 1-to-1.

Thus the extension is a homeomorphism of the compact set $\overline{A}$.

For $a \in \overline{A} \setminus A$, the function $\lambda to \overline{\Phi}(\lambda, a)$ is a local uniform limit of holomorphic functions, so it is also holomorphic on $\mathbb{D}$.

The function is jointly continuous because for every $0 < r < 1$, the family $\{\overline{\Phi}_\lambda : \lambda \in r\mathbb{D}\}$ is equicontinuous. Note that

$$
\begin{aligned}
|\overline{\Phi}(\lambda_1, a_1) - \overline{\Phi}(\lambda_2, a_2)| \leq\ & |\overline{\Phi}(\lambda_1, a_1) - \overline{\Phi}(\lambda_1, a_2)| \\
& + |\overline{\Phi}(\lambda_1, a_2) - \overline{\Phi}(\lambda_2, a_2)|
\end{aligned}
$$

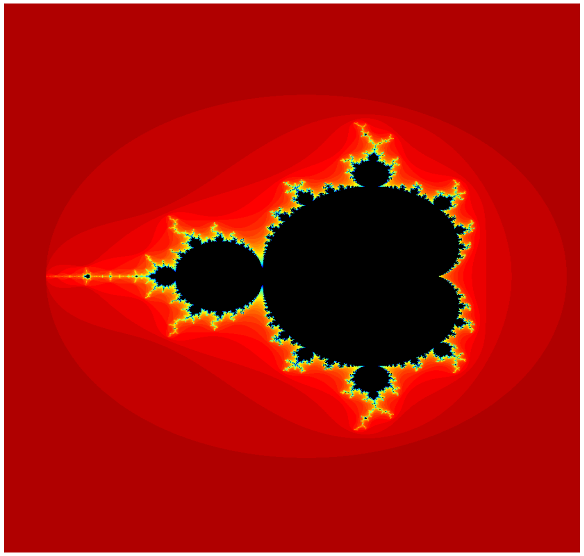
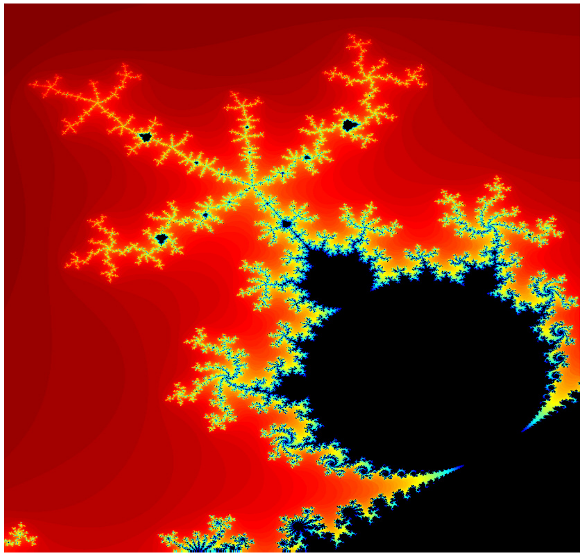
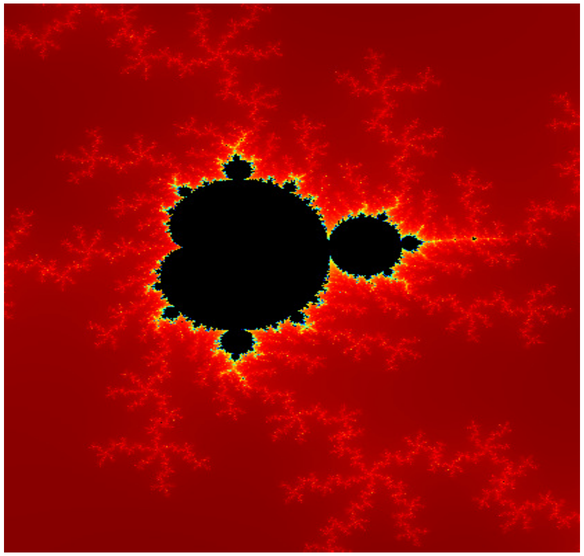
The first term is small because for a fixed λ, $\overline{\Phi}_\lambda$ is uniformly continuous with a bound depending only on an upper bound for $|\lambda| < 1$.

The second term is small because for a fixed a, $\overline{\Phi}(\lambda, a)$ is holomorphic in λ, hence continuous. $\square$

One application of the λ-lemma is to Julia sets of quadratic polynomials $z^2 + c$.

The Mandelbrot set has several hyperbolic components. Each of these are simply connected, and these maps have a single attracting periodic point.

The repelling periodic points are dense in the Julia set and move holomorphically as a function of c inside each hyperbolic component.

One can prove the repelling points do not collide. If they do, another attracting periodic point must result. This is impossible as each such point attracts a critical orbit and there is only one critical point.

By the λ-lemma says the holomorphic motion of the repelling points can be extended to the Julia set. Thus all the Julia sets in a hyperbolic component are quasisymmetrically equivalent.

Astala-Martin 2001 paper on holomorphic motions.

Complex Dynamics and Renormalization by Curt McMullen. Chapter 4 is titled "Holomorphic motions and the Mandelbrot set".

The extended λ-lemma:

Lemma 9.2. *Every holomorphic motion on a set $A \subset \mathbb{C}$ can be extended to a holomorphic motion of $\mathbb{C}$.*

Due to Slodkowski in 1991 using methods of several complex variables.

Proof in book of Astala-Iwaniec-Martin follows an argument of Chirka based on PDE; a non-linear Cauchy problem.

We will not give a proof in this class.

Degenerate Beltrami Equations

Quasiconformal maps correspond to homeomorphic solutions of

$$f_{\overline{z}} = \mu f_z \text{ where } \|\mu\|_\infty = k < 1.$$

For some applications we want to consider μ with $\|\mu\|_\infty = 1$.

This is possible if $\{z : |\mu(z)| > 1 - \epsilon\}$ is "small" when ϵ is small.

We follow Chapter 20 of *Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane* by Kari Astala, Tadeusz Iwaniec and Gaven Martin (but will only cover a small part of this chapter).

A mapping $f = u + iv$ on Ω is called a mapping of **finite distortion** if

(1) $f \in W^{1,1}_{loc}(\Omega)$.

(2) $J(\cdot, f) = u_x v_y - v_y v_x \in L^1_{loc}$

(3) There is a measurable $K(z)$ that is finite almost everywhere and so that almost everywhere in Ω

$$|Df(z)|^2 \leq K(z)J(z, f).$$

The smallest such K denoted $K(z, f)$. This is the **distortion function** of f.

Suppose μ is compactly supported and $\|\mu\|_\infty \leq 1$. A solution f to a degenerate Beltrami equation $f_{\overline{z}} = \mu f_z$ is called a **principle solution** if

(1) f is a homeomorphism of $\mathbb{C}$.

(2) For some discrete set $E \subset \mathbb{C}$, f lies in the Sobolev space $W^{1,1}_{loc}(\mathbb{C} \setminus E)$.

(3) At infinity f has the Taylor expansion

$$f(z) = z + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots.$$

Suppose $\mu = f_{\bar{z}}/f_z$ and $K(z) = (1+\mu(z))/(1-\mu(z))$. We will allow K to be unbounded, but only "slightly"

Let $A_n = \{2^{-n-1} < |z| < 2^{-n}\}$. Suppose $1 \le K_n \nearrow \infty$ and define $f(z) = z \cdot c_n \cdot |z|^{1-1/K_n}$ where $c_0 = 1$ and c_n is chosen to make f continuous.

The dilatation of f on A_n is K_n.

The image of A_n is a round annulus $\{r < |z| < R\}$ where

$$\log \frac{R}{r} = K_n \cdot \log 2,$$

or

$$\frac{r}{R} = \exp(-(\log 2)/K_n).$$

The diameters of $f(A_n)$ tends to zero iff

$$\prod_{n=0}^{\infty} \exp(-(\log 2)/K_n) = \exp\left(-\log 2 \sum_{n=0}^{\infty} 1/K_n\right) = 0$$

and stays bounded away from zero iff

$$\sum_{n=0}^{\infty} 1/K_n < \infty.$$

In this case, f is discontinuous at 0.

For example, take $K_n = n(\log n)^{1+\epsilon}$ for any $\epsilon > 0$.

For $K_n = n(\log n)^{1+\epsilon}$, the function $K(z)$ is integrable on $\mathbb{D}$:

$$\int_{\mathbb{D}} K(z)dxdy = \sum_0^\infty K_n \text{area}(A_n) = \pi \sum_0^\infty n(\log n)^{1+\epsilon} 4^{-n} < \infty.$$

Similarly, $K \in L^p(dxdy)$ for all $p < \infty$.

However K is not exponentially integrable:

$$\int_{\mathbb{D}} \exp(K(z))dxdy = \sum_0^\infty \exp(n\left[(\log n)^{1+\epsilon} - \log 4\right]) = \infty.$$

The following is Theorem 20.4.9 in Astala-Iwaniec-Martin:

Theorem 10.1. *Suppose* $\exp(K(z)) \in L^p$ *for some* $p > 0$ *and* $\mu(z) = 0$ *for* $|z| > 1$. *Then the Beltrami equation* $f_{\overline{z}} = \mu f_z$ *admits a unique principle solution* f *such that* $f \in W^{1,s}_{loc}$ *for all* $s < 2$.

Actually, the partials of f satisfy the stronger condition

$$\int_{\mathbb{D}} \frac{|f_z|^2}{\log(e + |f_z|)} dxdy < \infty.$$

The original version of this is due to Guy David. We will not prove this, but we will prove a related result due to Olli Lehto.

Theorem 10.2. *Suppose that μ is measurable, is compactly supported and $|\mu(z)| < 1$ for almost all $z \in \mathbb{C}$. Suppose*

$$K(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|}$$

is locally integrable , $K \in L^1_{loc}(\mathbb{C})$, and

$$\lim_{r \to 0} \int_r^1 \frac{1}{\int_0^{2\pi} K(z + \rho e^{i\theta}) d\theta} \frac{d\rho}{\rho} = \infty.$$

for all $z \in \mathbb{C}$. Then the Beltrami equation $f_{\overline{z}} = \mu f_z$ a.e. has a homeomorphic solution $f \in W^{1,1}$.

Idea of proof:

- Truncate K to make it bounded, and get f_n that are quasiconformal.
- Show $\{f_n\}$ is equicontinuous, so f_n converges to a continuous f.
- Show f is a homeomorphism
- Show f is a solution of Beltrami equation.

We need some preliminary results.

Lemma 10.3 (Loewner property)**.** *Suppose* $\Omega = \mathbb{C} \setminus (E \cup F)$ *is a topological annulus and suppose* E *is bounded and* F *is unbounded. Then*

$$M(\Omega) \leq \Phi\left(\frac{\operatorname{dist}(E, F)}{\operatorname{diam}(E)}\right),$$

where $\Phi : [0, \infty) \to [0, \infty)$ *is an increasing homeomorphism, independent of* Ω*.*

Here $M(\Omega)$ is the modulus of the path family separating the boundary components. For $\{r < |z| < R\}$ this is $2\pi \log(R/r)$.

Proof. This is very standard modulus estimate. Applying a linear map we can assume $-1, 1 \in E$ and $\mathrm{diam}(E) = 2$.

For $\mathrm{dist}(E, F) = r > 2$, Ω contains the round annulus $\{1 < |z| < r\}$ so $M(\Omega) \leq 2\pi r$.

For $\mathrm{dist}(E, F) = r < 2$ we can take

$$\rho(z) = \frac{1}{|z - x| \log 2/r},$$

where $x \in E$ is a point within distance r of F. This gives a bound for $M(\Omega)$ tending to zero with r. $\square$

Lemma 10.4. *Suppose f is a homeomorphism of finite distortion from $A = \{1 < |z| < r\}$ to $A' = \{1 < |z| < r'\}$. Then*

$$2\pi \int_1^r \frac{1}{[\int_0^{2\pi} K(\rho e^{i\theta}, f) d\theta]} \cdot \frac{d\rho}{\rho} < \log r'$$

Proof. Since f is finite distortion, it is in $W^{1,1}_{loc}$, and this means it is absolutely continuous on almost every circle around z.

By considering a branch of logarithm, we see that for a.e. $\rho \in [1, r]$,

$$\int_0^{2\pi} \frac{f_\theta(\rho e^{i\theta})}{f(\rho e^{i\theta})} d\theta = 2\pi i.$$

The Cauchy-Riemann equations in polar coordinates are

$$f_z = \frac{1}{2} e^{-i\theta}(f_\rho - \frac{i}{\rho} f_\theta),\ qquad f_{\overline{z}} = \frac{1}{2} e^{i\theta}(f_\rho + \frac{i}{\rho} f_\theta),$$

which implies

$$\rho^{-2} |f_\theta|^2 \le (|f_z| + |f_{\overline{z}}|)^2 \le K(z, f) J(z, f).$$

Hence

$$2\pi \leq \int_0^{2\pi} \frac{\sqrt{K(\rho e^{i\theta}, f) J(\rho e^{i\theta}, f)}}{|f(\rho e^{i\theta})|^2} d\theta.$$

By Hölder's inequality

$$4\pi^2 \leq \rho^2 \int_0^{2\pi} K(\rho e^{i\theta} d\theta \cdot \int_0^{2\pi} \frac{J(\rho e^{i\theta})}{|f(\rho e^{i\theta}|^2} d\theta$$

or

$$\frac{4\pi^2}{\rho \int_0^{2\pi} K(\rho e^{i\theta} d\theta} \leq \rho \int_0^{2\pi} \frac{J(\rho e^{i\theta})}{|f(\rho e^{i\theta}|^2} d\theta$$

Now integrate with respect to ρ

$$\int_1^r \int_0^{2\pi} \frac{J(\rho e^{i\theta})}{|f(\rho e^{i\theta}|^2} \rho d\theta d\rho = \int_A \frac{J(\rho e^{i\theta})}{|f(\rho e^{i\theta}|^2} dxdy \leq \int_{A'} \frac{1}{|z|^2} dxdy = 2\pi \log r'. \quad \square$$

Since both sides of the previous lemma are conformally invariant we deduce the following.

Corollary 10.5. *Suppose f is a homeomorphism of finite distortion from $A = \{1 < |z| < r\}$ to a topological annulus Ω. Then*
$$2\pi \int_1^r \frac{1}{\int_0^{2\pi} K(\rho e^{i\theta}, f)d\theta} \frac{d\rho}{\rho} < M(\Omega)$$

Here $M(\Omega)$ is the modulus of the path family that separates the boundary components.

Proof of Theorem 10.2. For $n \in \mathbb{N}$, define

$$\mu_n(z) = \begin{cases} \mu(z), & \text{if } |\mu(z)| \leq 1 - 1/n \\ (1 - 1/n)\mu(z)/|\mu(z)|, & \text{otherwise} \end{cases}.$$

The there is a unique principle solutions f_n that is a quasiconformal so that $(f_n)_{\overline{z}} = \mu(f_n)_z$. We want to show the family $\{f_n\}$ is equicontinuous.

For $z_0 \in \mathbb{D}$, let $A(r) = \{r < |z - z_0| < 1\}$.

By Koebe's theorem (applied at $z = \infty$), $f_n(2\mathbb{D})$ is uniformly bounded. Thus $f_n(A(r)) \subset f_n(2\mathbb{D}) \subset C\mathbb{D}$ for some fixed C (can take $C = 8$).

Set $E = \overline{f_n(\mathbb{D})}$ and $F = \mathbb{C} \setminus f_n(\mathbb{D})$. Then $\operatorname{dist}(E, F) \le C$.

By Lemma 10.3 and Corollary 10.5

$$\begin{aligned}\int_r^1 \frac{1}{\int_0^{2\pi} K(z+\rho e^{i\theta})d\theta}\frac{d\rho}{\rho} &\le \int_r^1 \frac{1}{\int_0^{2\pi} K(z+\rho e^{i\theta}, f)d\theta}\frac{d\rho}{\rho} \\ &\le \le \frac{1}{4\pi^2}\Phi\left(\frac{C}{\operatorname{diam}(f_n(D(z_0,r)))}\right).\end{aligned}$$

Our hypothesis is that the left hand side tends to ∞ and therefore $\operatorname{diam}(f_n(D(z_0, r))$ tends to zero at a rate independent of n.

Therefore $\{f_n\}$ is equicontinuous and there is a subsequence converging locally uniformly to a continuous function f on $\mathbb{C}$.

Next we have to show that f is a homeomorphism. We will do this by showing that the inverse functions $\{f_n^{-1}\}$ are also equicontinuous and thus converge to a continuous limit which must be f^{-1}. Hence f is a homeomorphism. This requires a few preliminary lemmas.

Lemma 10.6. *If f is quasiconformal and $g = f^{-1}$ then*

$$g_{\overline{z}}(z) = \frac{-f_{\overline{z}}(z)}{|f_z(z)|^2 - |f_{\overline{z}}(z)|^2} = -\frac{f_{\overline{z}}(z)}{J(z,f)},$$

$$\overline{g_z(z)} = \frac{f_z(z)}{|f_z(z)|^2 - |f_{\overline{z}}(z)|^2} = \frac{f_z(z)}{J(z,f)}.$$

Proof. Any 2D real-linear mapping can be expressed in complex notation as

$$az + b\overline{z}.$$

In particular, the derivative map of f can be written as

$$Df = f_z z + f_{\overline{z}}\overline{z}$$

where

$$f_z = \frac{1}{2}(f_x - if_y) \qquad f_{\overline{z}} = \frac{1}{2}(f_x + if_y).$$

If we compose two linear maps $az + b\overline{z}$ and $cz + d\overline{z}$ we get

$$\begin{aligned} a(cz + d\overline{z}) + b\overline{(cz + d\overline{z})} &= acz + ad\overline{z} + b\overline{cz} + b\overline{d}z \\ &= (ac + b\overline{d})z + (ad + b\overline{c})\overline{z} \end{aligned}$$

If we apply this and the chain rule to compute derivatives of a composition of maps, we get (setting $w = f(z)$):

$$\begin{aligned} (g \circ f)_z(z) &= g_w(f(z))f_z(z) + g_{\overline{w}}(f(z))\overline{f_{\overline{z}}(z)} \\ (g \circ f)_{\overline{z}}(z) &= g_w(f(z))f_{\overline{z}}(z) + g_{\overline{w}}(f(z))\overline{f_z(z)}. \end{aligned}$$

If $g = f^{-1}$ we get the equations

$$\begin{aligned} 1 &= g_w(f(z))f_z(z) + g_{\overline{w}}(f(z))\overline{f_{\overline{z}}(z)} \\ 0 &= g_w(f(z))f_{\overline{z}}(z) + g_{\overline{w}}(f(z))\overline{f_z(z)}. \end{aligned}$$

and solving these for g_z and $g_{\overline{z}}$ gives the lemma. $\square$

Lemma 10.7. *Suppose $f \in W^{1,2}(2\mathbb{D})$ is a homeomorphism. Then for all Lebesgue points $a, b \in \mathbb{D}$ we have*

$$|f(a) - f(b)|^2 \leq \frac{\pi \int_{2\mathbb{D}} |\nabla f|^2 dxdy}{\log(e + 1/|a-b|)}.$$

Proof. Suppose $a, b \in \mathbb{D}$ and let $D(t)$ be the disk radius t centered at $(a+b)/2$. If $t \geq |a-b|$ then $f(a), f(b) \in f(D(t))$ and hence

$$|f(a) - f(b)| \leq \operatorname{diam} f(D(t)) \leq \frac{1}{2}\ell(\partial f(D(t))).$$

For almost every t, f is absolutely continuous on $\partial D(t)$, so

$$|f(a) - f(b)| \leq \frac{1}{2} \int_{\partial f(D(t))} |\nabla f| ds$$

or

$$\frac{|f(a) - f(b)|}{\pi t} \leq \frac{1}{2\pi t} \int_{\partial f(D(t))} |\nabla f| ds.$$

Jensen's inequality then gives

$$\left(\frac{|f(a) - f(b)|}{\pi t} \right)^2 \leq \frac{1}{2\pi t} \int_{\partial f(D(t))} |\nabla f|^2 ds.$$

Now multiply both sides by t and integrate over $(|a - b|/2, 1]$

$$|f(a) - f(b)|^2 \log \frac{2}{|a - b|} \leq \frac{\pi}{2} \int_{2\mathbb{D}} |\nabla f|^2 dx dy.$$

$\square$

Recall Jensen's inequality: if μ is a probability measure on $[a, b]$ and ϕ is convex, then

$$\phi(\int f(t)d\mu) \leq \int \phi(f(t))d\mu.$$

To prove this let $x = \int f d\mu$ and let $L(t) = at + b$ be a linear functions so that $L(t) \leq \phi(t)$ for all t and $L(x) = \phi(x)$. Then

$$\phi(\int f(t)d\mu) = L(\int f(t)d\mu) = \int L(f(t))d\mu \leq \int \phi(f(t))d\mu \quad \square.$$

Lemma 10.8. *Suppose f is quasiconformal, μ is supported in the unit disk, and that f is the principle solution of $f_{\overline{z}} = \mu f_z$. Let $g = f^{-1}$. Then for $|a|, |b| \leq 1$, we have*

$$|g(a) - g(b)|^2 \leq \frac{C + \int_{\mathbb{D}} K(z, f) dxdy}{\log(e + 1/|a - b|)}.$$

Proof. By Lemma 10.7

$$|g(a) - g(b)|^2 \leq \frac{\pi \int_{2\mathbb{D}} |\nabla g|^2 dxdy}{\log(e + 1/|a - b|)}.$$

By Lemma 10.6

$$\int_{f(\mathbb{D})} |\nabla g|^2 \leq \int_{\mathbb{D}} |g_z|^2 + |g_{\overline{z}}|^2 = \int_{\mathbb{D}} \frac{|f_z|^2 + |f_{\overline{z}}|^2}{J(z, f)} = \int_{\mathbb{D}} K(x, f)$$

Outside $f(\mathbb{D})$, g is conformal and $\int_{2\mathbb{D}\setminus f(\mathbb{D})} |\nabla g|^2$ is bounded by the area of $f^{-1}(2\mathbb{D})$, which is uniformly bounded by Koebe's theorem. $\square$

This completes the proof that the inverses $\{f_n^{-1}\}$ are equicontinuous, and hence that f is a homeomorphism.

The final step in the proof of Theorem 10.2 is to show that the limiting function f satisfies the correct Beltrami equation.

Note that

$$|Df_n(z)| \leq \sqrt{K(z, f_n)J(z, f_n)} \leq \sqrt{K(z)J(z, f_n)},$$

so Hölder's inequality implies

$$\int |Df_n(z)|^2 \leq \int K(z) \int J(z, f_n) < \infty$$

since we assumed K is locally integrable and all the f_n send any compact set into a uniformly bounded set. Thus the derivatives of f_n converge weakly in $W^{1,1}_{loc}$ to the corresponding derivatives of f. Thus $f \in W^{1,1}_{loc}$.

Suppose ϕ is a compactly supported smooth function. Consider
$$\int \phi((f_n)_{\overline{z}} - \mu(f_n)_z) = \int \phi(\mu_n - \mu)(f_n)_z.$$
Since the derivative of f_n converge weakly to the derivatives of f, the left side tends to
$$\int \phi(f_{\overline{z}} - \mu f_z).$$

The right side tends to zero since

$$\begin{aligned}\int |\phi| \cdot |\mu_n - \mu| \cdot |Df_n| &\le \int |\phi| \cdot |\mu_n - \mu| \cdot \sqrt{K(z)J(z, f_n)} \\ &\le \left(\int_{\mathbb{D}} |\phi|^2 \cdot |\mu_n - \mu|^2 \cdot K\right)^{1/2} \left(\int_{\mathbb{D}} J(z, f_n)\right)^{1/2}\end{aligned}$$

where the Jacobian integral is uniformly bounded and the first integral tends to zero by the Lebesgue dominated convergence theorem.

Thus $\int \phi(f_{\overline{z}} - \mu f_z) = 0$ for every smooth function ϕ of compact support, which implies $f_{\overline{z}} = \mu f_z$ almost everywhere.

This completes the proof of Lehto's theorem. $\square$

A special case occurs when $K(z)$ is bounded by a radial function $\phi(|z|)$ in the unit disk and is $= 1$ outside $\mathbb{D}$.

Then the integral condition in Lehto's theorem only needs to be checked when $|z| = 1$. It is easy to check that the result holds if

$$K(z) \leq 1 + \log \frac{1}{1 - |z|}.$$

In this case, the set where $|\mu| > 1 - \epsilon$ has area decaying like $\exp(-1/\epsilon)$.