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Collet, Eckmann and Hölder

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Abstract. We prove that Collet-Eckmann condition for rational functions, which requires exponential expansion only along the critical orbits, yields the Hölder regularity of Fatou components. This implies geometric regularity of Julia sets with non-hyperbolic and critically-recurrent dynamics. In particular, polynomial Collet-Eckmann Julia sets are locally connected if connected, and their Hausdorff dimension is strictly less than 2. The same is true for rational Collet-Eckmann Julia sets with at least one non-empty fully invariant Fatou component.

1. Introduction

We are interested in the dynamical characterization of geometric regularity of the Fatou components and the persistence of hyperbolic subsets in Julia sets. This direction of studies was originated by L. Carleson, P. Jones, and J.-C. Yoccoz in their work "Julia and John" on the dynamical classification of Fatou components which are John domains. They proved that the property of Fatou components being John domains is equivalent to the Misiurewicz condition (semi-hyperbolicity) for a polynomial. Hölder property is more general: every John domain is Hölder but not conversely (see [22], 5.2).

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Definition 1. *A simply connected domain Ω is called a Hölder domain (with exponent $\alpha \in (0, 1]$) if the Riemann mapping $\varphi : \mathbb{D} \to \Omega$ can be extended to a Hölder continuous (with exponent α) mapping on the closed unit disk.*

It is a well known theorem of G.H. Hardy and J.E. Littlewood (see [7], p. 74) that this definition is equivalent to

$$|\varphi'(z)| \leq C\left(\frac{1}{1-|z|}\right)^{1-\alpha} ,$$

where $z \in \mathbb{D}$ and $C > 0$ is a constant. Generally, Hölder domains can be defined in terms of quasihyperbolic distance (see Definition 8 in the Sect. 5 and [26]).

We prove that the Collet-Eckmann condition which requires exponential expansion only along the critical orbits (see Definition 2 below) yields the Hölder regularity of Fatou components. The immediate consequence of our main result is:

Corollary 1. *Suppose that the boundary of some Fatou component of a C-E rational function is connected, then it is locally connected. In particular, connected Julia set of a C-E polynomial (or any C-E rational function with at least one non-empty fully invariant Fatou component) is locally connected.*

Julia sets of rational C-E maps need not be locally connected. A well-known example of a hyperbolic Julia set which is not locally connected (and is topologically a Cantor set of circles) can be found in Sect. 11.8 in the book [3]. In the Corollaries 1 and 2 we use that if a rational Julia set has a fully invariant Fatou component (for polynomials the basin of attraction to infinity is fully invariant), then the whole Julia set coincides with its boundary. Consult the books [3, 4] for this and other basic facts from holomorphic dynamics.

The recent work of P. Jones and N. Makarov [13] gives an upper estimate of the Hausdorff (and even Minkowski) dimension of the boundary of any simply connected Hölder domain in terms of the Hölder exponent α only:

$$\mathrm{HD}(\Omega) \leq 2 - c\alpha ,$$

where c is an absolute constant. In papers of W. Smith and D.A. Stegenga [26], and P. Koskela and S. Rohde [14], the similar result is proved for any Hölder domain. See also [18] for the discussion of various related classes of domains and their metric properties.

Therefore, we arrive at the following

Corollary 2. *The boundary of Fatou component of a C-E rational function has Hausdorff dimension less than 2. In particular, the Hausdorff dimension of the Julia set for a C-E polynomial (or any C-E rational function with at least one fully invariant Fatou component) is less than 2.*

The first results on the dynamics of C-E rational maps were obtained by F. Przytycki in [23] and [24]. In these papers he studies ergodic properties of conformal measures, the existence of invariant measures in the class of conformal measures with minimal exponent and the relation between various dimensions of the Julia sets. His main result states that for C-E rational maps the Hausdorff, Minkowski, and Hyperbolic dimensions of the Julia sets coincide. Another important consequence of his approach is the existence of a polynomial with a recurrent critical point whose Julia set has the Hausdorff dimension strictly less than 2. In fact, F. Przytycki proves that $\mathrm{HD}(J) < 2$ holds for the real C-E maps which satisfy Tsuji condition (see [24]). F. Przytycki introduced in [23] a new technique of estimating distortion for holomorphic Collet-Eckmann dynamics. His method plays an important role in our estimates and hence is presented fully in Section 2.

It would be very interesting to characterize rational functions with Julia sets of Hausdorff dimension less than 2. Our result shows that polynomial C-E maps fall into this category. Further progress in this direction was obtained in the papers [25, 9].

Recently, C. McMullen [16] proved that a quadratic Julia set with Siegel disk has Hausdorff dimension less than 2 provided its rotation number is of bounded type. In this situation, M. Herman and G. Świątek theory (see [11] and [28]) together with E. Ghys construction ([8]) imply that the boundary of a Siegel disk is a quasicircle and the dynamics on the Julia set is quasiconformally equivalent to the dynamics of a corresponding Blaschke product. To the best of our knowledge there are no known examples of Siegel disks with Hölder or John property other than quasidisks. We are particularly interested in the case where a critical point lies on the boundary of the Siegel disk.

The S-unimodal C-E maps on the interval were studied more intensively. T. Nowicki and F. Przytycki proved in [20] that any non-renormalizable S-unimodal C-E map is Hölder conjugate with tent map, and conjectured the Hölder regularity of Fatou components for Collet-Eckmann quadratics. For a polynomial with connected Julia set, by a proper choice of a Riemann mapping φ to the domain of attraction to infinity, we may assume that it conjugates F on J_F with dynamics $T : z \to z^d$ on the unit circle. Hence, in this context, our theorem is a direct analogue of their result. The tent maps are piecewise linear maps which serve as prototype models in the study of the dynamics of unimodal maps.

W. de Melo and S. van Strien's book [17] is an excellent reference of the activity in this area. An important feature of smooth unimodal C-E maps is that they have always probabilistic absolutely continuous invariant measure. By the works of M. Jakobson [12], M. Benedicks and L. Carleson [1, 2] non-hyperbolic maps in the quadratic family $\{ax(1-x), a \in [0,4]\}$, which satisfy C-E condition have a positive Lebesgue measure in the parameter space. In the complex quadratic case the class of C-E maps is strictly larger than that of semi-hyperbolic (non-recurrent). The natural questions arise about possible critical orbit combinatorics of C-E quadratics, and the harmonic measure of C-E parameters on the boundary of the Mandelbrot set.

Problem 1. Do C-E parameters have full (or at least positive) harmonic measure on the boundary of the Mandelbrot set (viewed from outside)?

The definition of holomorphic C-E maps is adopted from the dynamics of S-unimodal maps (see [6, 17, 23]), with a small change: we allow critical points to be attracted to (super) attractive cycles. For example, the real map $x \mapsto x^2$ is not C-E in the real sense (compare [17]) but according to Definition 2 its complex counterpart, $z \mapsto z^2$, is a rational C-E map. As usual, we will call zeroes of the derivative of F critical points, and their images – critical values.

Definition 2. *We say that a rational function F satisfies the first Collet-Eckmann condition $\mathcal{CE}$ with constants $C_1 > 0$, $\lambda_1 > 1$ if for any critical point c whose forward orbit does not contain any other critical point and belongs to or accumulates on the Julia set, the following condition holds*

$$\left|(F^n)'(Fc)\right| > C_1 \lambda_1^n \ .$$

Such rational functions we will simply call Collet-Eckmann (or C-E).

We will also study the relation between the first and the second Collet-Eckmann conditions. T. Nowicki proved in [19] that the first C-E condition implies the second for S-unimodal maps. To our best knowledge the reversed implication was unknown even in the case of real quadratic polynomials.

Definition 3. *We say that a rational function F satisfies the second Collet-Eckmann condition $\mathcal{CE}_2(z)$ for a point z with constants $C_2 > 0, \lambda_2 > 1$ if for any preimage $y \in F^{-n}z$*

$$\left|(F^n)'(y)\right| > C_2 \lambda_2^n \ .$$

Definition 4. *We call a periodic Fatou component $\mathcal{F}$ Collet-Eckmann if for any (some – by the Koebe distortion theorem the statements are equivalent) point $z \in \mathcal{F}$ away from the critical orbits*

$$\left|(F^n)'(y)\right| > C\lambda^n \ ,$$

for any preimage $y \in F^{-n}z \cap \mathcal{F}$ with constants $C > 0$, $\lambda > 1$.

By the multiplicity $\mu(c)$ of a critical point c we mean the order of c as a zero of $F(z) - F(c)$. For simplicity we assume that no critical point belongs to another critical orbit. Otherwise a "block" of critical points

$$F : c_1 \mapsto \ldots \mapsto c_2 \mapsto \ldots \ \ \ldots \mapsto c_k \ ,$$

of multiplicities $\mu_1, \mu_2, \dots, \mu_k$ enters our statements as if it is a single critical point of multiplicity $\prod \mu_j$. If in our construction we come to the point c_k, then the process continues from the point c_1 (note, that c_1 is a critical point of multiplicity $\prod \mu_j$ for the iterate F^n mapping c_1 to Fc_k).

The main result of the paper relates dynamical condition with the analytical and geometric properties of the Julia sets and Fatou components.

Theorem 1. *Rational C-E maps can have neither Siegel disks, Herman rings, nor parabolic or Cremer points. Fatou components of a rational C-E map of the Riemann sphere are Hölder domains. Additionally, we have the following relations between analytical and dynamical properties*:

(i) *The first Collet-Eckmann condition implies the second* $\mathcal{CE}_2(c)$ *for the critical points c of the maximal multiplicity* $\mu_{\max}$ (*calculated as above*), *whose backward orbits do not contain any other critical points.*

(ii) *The first Collet-Eckmann condition implies the second for all points z away from critical orbits, namely*

$$\left|(F^n)'(F^{-n}(z))\right| \gtrsim \Delta^{1-\frac{1}{\mu_{\max}}} \lambda^n \quad ,$$

where Δ *is the distance of z to the forward orbit of all critical points.*

(iii) *Attracting or superattracting Fatou component* $\mathcal{F}$ *is Hölder if and only if it is Collet-Eckmann.*

By the implication (ii) of Theorem 1, Collet-Eckmann rational maps can have neither parabolic nor elliptic Fatou components. Hence, for Collet-Eckmann rational maps Hölder property of Fatou components and the second Collet-Eckmann condition for points outside of the Julia sets are equivalent.

The distances and derivatives are calculated in the spherical metric. If the Fatou set is non-empty, one can work with the Euclidean metric on the plane, changing coordinates by a Möbius transformation so that infinity belongs to a periodic Fatou component.

In the "complex unimodal case" Theorem 1 can be restated in the following way.

Theorem 2. *For a unimodal polynomial* $F(z) = z^\ell + a$ *with connected Julia set* J_F *satisfying R-expansion property* (*see Definition* 10 *in Appendix*) *the following two conditions are equivalent:*

(i) $\mathcal{CE}$,

(ii) $\mathcal{CE}_2(0)$,

and imply the following equivalent conditions:

(iii) $\mathcal{CE}_2(z)$ *for some* (*any*) $z \in A_\infty$,

(iv) *Domain of attraction to infinity* A_∞ *is Hölder.*

The R-expansion property is needed only for the implication $(ii) \Rightarrow (i)$. We do not know whether it is necessary and what class of rational maps satisfies it. According to the recent work of S. van Strien and G. Levin [15] non-renormalizable real unimodal polynomials have R-expansion property. We prove this property for C-E dynamics in Appendix. F. Przytycki proved in [23] that rational maps satisfying the so called summability condition (weaker than Collet-Eckmann) enjoy this property also.

Finally, we are aware that that C-E complex unimodal maps cannot be infinitely renormalizable. This problem as well as many others related to measure theoretic properties, holomorphic removability and topology of Julia sets satisfying the summability condition is discussed in our recent paper [9].

Organization of the paper. We introduce distortion techniques based on the idea of shrinking neighborhoods (see [24]) in Sect. 2. In Sect. 3 we study the local behaviour of dynamics, which we later use to formulate a global induction procedure in Sect. 4. There we also prove the implication $\mathscr{CE} \Rightarrow \mathscr{CE}_2(c)$ for critical points of maximal multiplicity and the second Collet-Eckmann condition for points away from the critical orbits. The relation between $\mathscr{CE}_2(z)$, $z \notin J$ and the Hölder continuity of Fatou components is explained in Section 5.

Validity of the implication $\mathscr{CE}_2(c) \Rightarrow \mathscr{CE}$ does not belong to the main line of our current work. However, possible applications to the study of S-unimodal dynamics on the interval motivate our short discussion of the problem in Appendix.

Notation. Critical points of F are denoted by c, c', c_j, etc., and their multiplicities by $\mu(c)$, etc..

The relations $a \lesssim b, a \gtrsim b$, where a and b are real positive numbers, mean appropriately that there exists an absolute universal constant K so that $a \leq Kb$ and $b \leq Ka$. By the definition, $a \asymp b$ iff both $a \lesssim b$ and $b \lesssim a$.

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2. Distortion versus expansion

2.1. Shrinking neighborhoods

The method of shrinking neighborhoods was introduced by Przytycki in [23]. It enables to control distortion in small vicinities of expanding orbits.

We fix a decreasing sequence of positive numbers $\{\delta_n\}$ with $\prod_n (1 - \delta_n) > \frac{1}{2}$. Set $\Delta_n := \prod_{k \leq n} (1 - \delta_k)$. Let B_r be a ball of radius r with center z and $\{F^{-n}z\}$ be a sequence of preimages of z (here and below by a sequence of preimages of z we will mean a sequence of points $\{z_n = F^{-n}z\}$ such that $Fz_n = z_{n-1}$ and $z_0 = z$). We define U_n and U'_n respectively as the connected components of $F^{-n}B_{r\Delta_n}$ and $F^{-n}B_{r\Delta_{n+1}}$ which contain $F^{-n}z$. Clearly,

$$FU_{n+1} = U'_n \subset U_n \ .$$

If U_k, for $1 \le k \le n$, do not contain critical points then the distortion (i.e. how map differs from its linear approximation) of the map $F^n : U'_n \to B_{r\Delta_{n+1}}$ is bounded by a power of $\frac{1}{\delta_{n+1}}$ multiplied by an absolute constant, as formulated in the following lemma.

Lemma 1. *Assume that*

1) *The shrinking neighborhoods* U_k for $B_r(z)$, $1 \le k \le n-1$ *evade critical points,*

2) $c \in \partial U_n$.

For $y := F^{-n}z$, *let* r' *be the maximal radius such that* $B_{r'}(y) \subset F^{-n}(B_{r/2}(z))$. *Then*

$$\mathrm{dist}(Fy, Fc) \le \frac{2}{\delta_n} \left| \left(F^{n-1}\right)'(Fc) \right|^{-1} r \ .$$

Moreover, if y *is so close to* c, *that* $|F'(y)| \ge \frac{1}{M}\ \mathrm{dist}(y,c)^{\mu(c)-1}$ *with* $M > 0$, *then*

$$\left|(F^n)'(y)\right|^{\mu_{\max}} \ge \frac{\delta_n}{8^{\mu_{\max}} M} \left| \left(F^{n-1}\right)'(Fc) \right| \mathrm{dist}(y,c)^{\mu(c)-1} \left(\frac{r}{r'}\right)^{\mu_{\max}-1} \ .$$

Proof. By the Koebe distortion theorem (see Theorem 1.3 in [22]) applied to $F^{-(n-1)} : \ B_{r\Delta_{n-1}}(z) \to U_{n-1}$ we obtain that

$$\left| \left(F^{-(n-1)}\right)'(F^n c) \right| \ge \frac{\delta_n}{(2-\delta_n)^3} \left| \left(F^{-(n-1)}\right)'(z) \right| \ ,$$

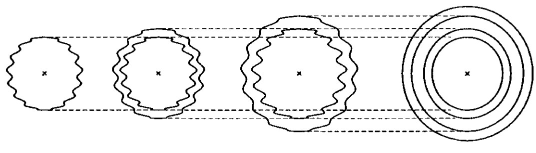

Fig. 1. Shrinking neighborhoods

and therefore

$$\left|(F^n)'(y)\right| = \left|F'(y)\right| \left|(F^{n-1})'(Fy)\right| \ge \frac{\delta_n}{8M} \left| \left(F^{n-1}\right)'(Fc) \right| \mathrm{dist}(y,c)^{\mu(c)-1} \ .$$

By the Koebe $\frac{1}{4}$-lemma (see Corollary 1.4 in [22]), the image of the map $F^{-n} : B_{r/2}(z) \to U_n$ contains a ball of radius $\frac{1}{8} r \left|(F^n)'(y)\right|^{-1}$ and the center y. Hence,

$$\left|(F^n)'(y)\right| \ge \frac{r}{8r'} \ .$$

Combining the above estimate raised to the power $(\mu_{\max} - 1)$, with the previous one, we obtain the second desired inequality.

Similarly, another application of the same Koebe distortion theorem (in invariant form, which can be obtained composing with a Möbius transformation) yields

$$\begin{aligned}\operatorname{dist}(Fy, Fc) &\le \frac{(1-\delta_n)(2-\delta_n)}{\delta_n}\Delta_{n-1} r \left|\left(F^{n-1}\right)'(Fc)\right|^{-1} \\ &< \frac{2r}{\delta_n}\left|\left(F^{n-1}\right)'(Fc)\right|^{-1} ,\end{aligned}$$

and hence the first inequality. □

3. Local analysis

In this section we will assume that $\mathscr{C}\mathscr{E}$ condition is satisfied (with constants $C_1 > 0$ and $\lambda_1 > 1$). Our main concern is local analysis. We assume that we have pieces of the backward orbit of a point. Each piece will be of a specified type. Estimates will be carried out independently for every piece. Hyperbolic structures of the backward orbits of some dynamically important points will be obtained as a result of a "global" induction (see the next section).

Scale. The scale around the critical points is given in terms of fixed numbers $R' \ll R \ll 1$. We will refer to objects which stay away from critical points and are comparable with R' as objects of the *large scale*. The proper choice of R's is one of the most important elements in the local analysis of expansion.

We start with defining a suitable size of the shrinking neighborhoods. Fix a positive $\varepsilon \approx 0$. We choose a sequence $\delta_n := \frac{(1-q)}{2} q^n$ with $q < 1$ so that $(\lambda_1)^\varepsilon \cdot q > 1$. We require R and R' to satisfy the following conditions:

Specification of R

(i) Any two critical points are at least $100R$ apart and R is so small that $|F'(y)| \overset{M}{\asymp} \operatorname{dist}(y, c)^{\mu(c)-1}$, given that y is close to a critical point c with either $\operatorname{dist}(Fy, Fc) < R$ or $\operatorname{dist}(y, c) < 4R$.

(ii) The first return time of the critical points to $\bigcup F^{-1} B_R(Fc_i)$ is greater than a constant τ, such that

$$\lambda_1^{\tau\varepsilon} \frac{1}{8^{\mu_{\max}} M^2} C_1 \frac{(1-q)}{2} > 2^{\mu_{\max}-1} \quad .$$

(iii) R' is so small, that

$$4\left(\frac{1-q}{2}\right)^{-1} (C_1)^{-1} R' < R \quad .$$

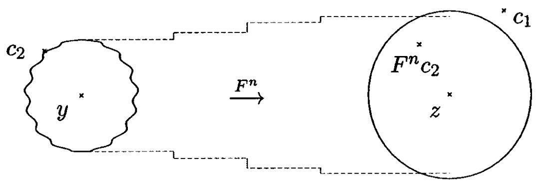

Fig. 2. Preimages of the first type

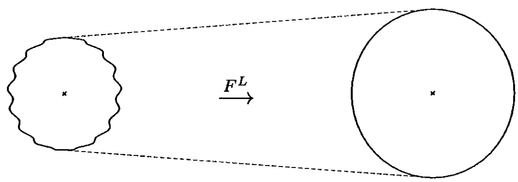

Fig. 3. Preimages of the second type

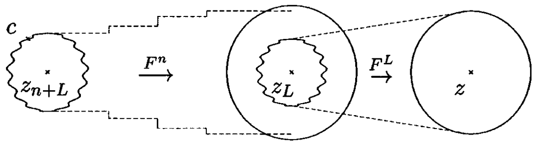

Fig. 4. Preimages of the third type

3.1. First type

Preimages of the first type form a model to study expansion along pieces of the backward orbit which "join" conformally some two critical points of F. The word "join" means here that we are able to find a ball in a vicinity of the first critical point and pull it back conformally until its boundary hits the second critical point.

Our formulation of Lemma 2 has to encompass the possibility of critical points with different multiplicities and hence it does not guarantee immediate expansion.

Definition 5. *A sequence $z, F^{-1}(z), \cdots, F^{-n}(z)$ of preimages of z is of the first type with respect to the critical points c_1 and c_2 if*

1) *Shrinking neighborhoods U_k for $B_r(z)$, $1 \leq k \leq n$, avoid critical points for some $r < 2R'$,*

2) *The critical point $c_2 \in \partial U_n$,*

3) *The critical value Fc_1 is close to z with $Fc_1 \in B_R(Fz)$.*

To simplify notation set $\mu_i := \mu(c_i)$, $d_2 := \mathrm{dist}(F^{-n}z, c_2)$, and $d_1 := \mathrm{dist}(z, c_1)$. Let r_2' be the maximal radius so that $B_{r_2'}(F^{-n}z) \subset F^{-n}(B_{r/2}(z))$. For consistency, put $r_1 := r$.

Lemma 2. *There exist* $Q_{1t} > 1$ *such that for any sequence* $F^{-1}(z), \cdots, F^{-n}(z)$ *of preimages of the first type we have that*

$$\left|(F^n)'(y)\right|^{\mu_{\max}} > 2^{\mu_{\max}}(Q_{1t})^n \frac{(d_2)^{\mu_2 - 1}}{(r_2')^{\mu_{\max}-1}} \frac{(r_1)^{\mu_{\max}-1}}{(r_1 + d_1)^{\mu_1 - 1}} \ .$$

Also

$$\operatorname{dist}(Fy, Fc_2) < R \ .$$

It is clear from the proof that by choosing R small enough we can make Q_{1t} arbitrary close to λ_1.

Proof. First note that by the first implication of Lemma 1

$$\begin{aligned}\operatorname{dist}(Fy, Fc_2) &\leq 2\ (\delta_n)^{-1}\left|\left(F^{n-1}\right)'(Fc_2)\right|^{-1} r_1 \\ &\leq 2q^{-n}\lambda_1^{-n} \cdot 2\left(\frac{1-q}{2}\right)^{-1}(C_1)^{-1} 2R' \ ,\end{aligned}$$

the last term is less than R by our choice of R', thus proving the second inequality.

To prove the first observe that condition 2 of Definition 5 implies $F^n c_2 \in \partial B_{r\Delta_n}$. Hence,

$$\operatorname{dist}(F^n c_2, c_1) \leq \operatorname{dist}(F^n c_2, z) + \operatorname{dist}(z, c_1) \leq r_1 + d_1 < 4R \ .$$

Since $\operatorname{dist}(F^n c_2, c_1)$ is small, $|F'(F^n c_2)| \overset{M}{\asymp} \operatorname{dist}(F^n c_2, c_1)^{\mu_1 - 1}$.

Therefore,

$$\begin{aligned}\left|\left(F^{n-1}\right)'(Fc_2)\right| &\geq \frac{1}{M}\left|(F^n)'(Fc_2)\right| \frac{1}{\operatorname{dist}(F^n c_2, c_1)^{\mu_1 - 1}} \\ &\geq \frac{1}{M} C_1 \lambda_1^n \frac{1}{(r_1 + d_1)^{\mu_1 - 1}} \ .\end{aligned}$$

All the hypotheses of Lemma 1 are satisfied, hence

$$\begin{aligned}\left|(F^n)'(y)\right|^{\mu_{\max}} &\geq \frac{1}{8^{\mu_{\max}} M}\delta_n\left|\left(F^{n-1}\right)'(Fc_2)\right| \frac{(d_2)^{\mu_2 - 1}}{(r_2')^{\mu_{\max}-1}}(r_1)^{\mu_{\max}-1} \\ &\geq \frac{1}{8^{\mu_{\max}} M^2}\delta_n C_1 \lambda_1^n \frac{(d_2)^{\mu_2 - 1}}{(r_2')^{\mu_{\max}-1}} \frac{(r_1)^{\mu_{\max}-1}}{(r_1 + d_1)^{\mu_1 - 1}} \ .\end{aligned}$$

The first inequality is proved with $Q_{1t} = \lambda_1^{1-2\varepsilon}$. □

3.2. Second type

The second type of preimages corresponds to a piece of the backward orbit which stays away from the critical points, i.e. there exists a neighborhood of size comparable with R' which can be pulled back along the backward orbit. The length of pieces of the second type will be always equal to L and the expansion will be deduced from the compactness and the "eventually onto" property of the Julia sets.

Definition 6. *A sequence of the preimages of z is of the second type if the ball $B_{R'}(z)$ can be pulled back by F univalently along this sequence. Additionally, we assume that* $\mathrm{dist}(z, J_F) < R'/2$.

Lemma 3. *Let $z, F^{-1}(z), \ldots, F^{-n}(z)$ be a sequence of the preimages of the second type. For every $C_{2t} > 1$ there exists $L_{2t} > 0$ so that $|(F^n)'(F^{-n}(z))| > C_{2t}$ provided $n \geq L_{2t}$.*

In order to ensure expansion in our next Lemma 4 we choose C_{2t} so that

$$\frac{1}{8^{\mu_{\max}}M}\frac{1-q}{2}\lambda_1^{-1}(R/4)^{\mu_{\max}-1}\,C_{2t} > 1 \ .$$

Remark 1. Set $Q_{2t} := (C_{2t})^{1/L_{2t}} > 1$, then for $n = L_{2t}$ the inequality above can be rewritten as $|(F^n)'(F^{-n}z)| > (Q_{2t})^n$.

Proof. Suppose that it is not so. Then, there is an infinite collection of sequences of the second type

$$z_i,\ F^{-1}z_i, \ldots, F^{-n_i}(z_i)$$

such that $n_i \to \infty$ and $|(F^{n_i})'(F^{-n_i}(z_i)| \leq C_{2t}$. Consider the preimages $F^{-n_i}(B_{R'/2}(z_i'))$, where z_i' is the closest to z_i point in J_F. Without loss of generality we can assume that $R' \ll \mathrm{diam}\, J_F$. By the Koebe $\frac{1}{4}$-lemma, any of these preimages contains a ball around $F^{-n_i}(z_i')$ of the radius larger than $\eta := R'/(8C_{2t})$. Let y be an accumulation point of the sequence $F^{-n_i}(z_i') \in J_F$. By the construction, there is an increasing subsequence $\{k_j\}$ of the sequence $\{n_j\}$ such that images of $B_{\eta/2}(y)$ under F^{k_j} are contained in $B_{R'}(z) \not\supset J_F$ and we arrived at a contradiction, since $y \in J_F$ and the Julia set has the "eventually onto" property. □

The next fact is an immediate consequence of Definition 6.

Fact 3.1. *There exists a positive constant $K_{3.1}$ such that for every sequence of the preimages $z, F^{-1}(z), \ldots, F^{-n}(z)$ of the second type*

$$|(F^n)'(F^{-n}(z))| > K_{3.1} \ .$$

Proof. Denote the set of critical points by Crit. By the Koebe $\frac{1}{4}$-lemma,

$$1 \gtrsim \operatorname{dist}(F^{-n}z, \mathrm{Crit}) > \frac{1}{4} R' \left|(F^{-n})'(z)\right| \ . \qquad \square$$

3.3. Third type

The third type of preimages corresponds to pieces of the backward orbit which connect the large scale to the critical points. The third type preimages are always endowed with the hyperbolic structure.

Definition 7. *Let $z_0 = z, z_1 = F^{-1}(z), \ldots, z_{n+L} = F^{-n-L}(z)$ be a sequence of the preimages of z of the second type. The number $L = L_{2t}$ is defined in Lemma* 3. *The sequence $z_1, \ldots, z_{n+L}$ is of the third type if the following conditions are satisfied:*

1) *Shrinking neighborhoods U_k for $B_r(z_L)$, $1 \le k \le n$, avoid critical points for some $r < 2R'$,*

2) *Some critical point $c \in \partial U_n$.*

For simplicity denote $d := \operatorname{dist}(z_{n+L}, c)$, $\mu := \mu(c)$ and $L := L_{2t}$. Let r' be the maximal radius so that $B_{r'}(F^{-(n+L)}z) \subset F^{-n}(B_{r/2}(F^{-L}z))$.

Lemma 4. *There exists a constant $Q_{3t} > 1$ such that for every sequence of the preimages of the third type the following estimate holds*:

$$\left|\left(F^{n+L}\right)'(z_{n+L})\right| > (Q_{3t})^{n+L} \frac{d^{\mu-1}}{(r')^{\mu_{\max}-1}} \ .$$

Also

$$\operatorname{dist}(Fz_{n+L}, Fc) < R \ .$$

Proof. To prove the second inequality we proceed as in the proof of the second inequality of Lemma 2. Indeed, by Lemma 1

$$\begin{aligned} \operatorname{dist}(Fz_{n+L}, Fc) &\le 2(\delta_n)^{-1} \left|\left(F^{n-1}\right)'(Fc)\right|^{-1} r \\ &\le q^{-n} \lambda_1^{-n} \cdot 2\left(\frac{1-q}{2}\right)^{-1} C_1^{-1} 2R' < R \ . \end{aligned}$$

It remains to prove the first inequality. By the Koebe $\frac{1}{4}$-lemma, $F^{-L}B_R(z)$ contains a ball of radius $\frac{1}{4}R|(F^L)'(z_L)|^{-1}$ and the center z_L. By the definition of the second type, the ball does not contain $F^n(c)$. Clearly, $F^n(c) \in B_r(z_L)$. Hence,

$$r > \frac{1}{4} R \left|(F^L)'(z_L)\right|^{-1} \ .$$

We apply Lemma 1 to $y = z_{n+L} = F^{-n}(z_L)$ and then substitute the value of $\delta_n = \frac{1-q}{2} q^n$, $q\lambda_1^{\epsilon} > 1$, into the resulting inequality:

$$\begin{aligned}
\left|(F^n)'(y)\right|^{\mu_{\max}} &\geq \frac{1}{8^{\mu_{\max}} M} \delta_n \left|\left(F^{n-1}\right)'(Fc)\right| \frac{d^{\mu-1}}{(r')^{\mu_{\max}-1}} r^{\mu_{\max}-1} \\
&\geq \frac{1}{8^{\mu_{\max}} M} \frac{1-q}{2} q^n C_1 \lambda_1^{(n-1)} \cdot \frac{d^{\mu-1}}{(r')^{\mu_{\max}-1}} \left(\frac{1}{4} R \left|(F^L)'(z_L)\right|^{-1}\right)^{\mu_{\max}-1} \\
&\geq \frac{1}{8^{\mu_{\max}} M} \lambda_1^{n(1-\varepsilon)} \frac{d^{\mu-1}}{(r')^{\mu_{\max}-1}} \cdot \frac{1-q}{2} \lambda_1^{-1} (R/4)^{\mu_{\max}-1} \left|(F^L)'(z_L)\right|^{\mu_{\max}-1} .
\end{aligned}$$

By Lemma 3, $|(F^L)'(z_L)| > C_{2t}$. Therefore,

$$\begin{aligned}
\left|\left(F^{n+L}\right)'(z_{n+L})\right|^{\mu_{\max}} &= \left|(F^n)'(z_{n+L})\right|^{\mu_{\max}} \left|\left(F^L\right)'(z_L)\right|^{\mu_{\max}} \\
&\geq \lambda_1^{n(1-\varepsilon)} \frac{d^{\mu-1}}{(r')^{\mu_{\max}-1}} \frac{1}{8^{\mu_{\max}} M} \cdot \frac{1-q}{2} \lambda_1^{-1} (R/4)^{\mu_{\max}-1} \left|\left(F^L\right)'(z_L)\right| \\
&\geq \lambda_1^{n(1-\varepsilon)} \frac{d^{\mu-1}}{(r')^{\mu_{\max}-1}} \frac{1}{8^{\mu_{\max}} M} \cdot \frac{1-q}{2} \lambda_1^{-1} (R/4)^{\mu_{\max}-1} C_{2t} \\
&\geq (Q_{3t})^{n+L} \frac{d^{\mu-1}}{(r')^{\mu_{\max}-1}} .
\end{aligned}$$

The constant

$$Q_{3t} := \min\left(\lambda_1^{1-\varepsilon}, \left(\frac{1}{8^{\mu_{\max}} M} \frac{1-q}{2} \lambda_1^{-1} (R/4)^{\mu_{\max}-1} C_{2t}\right)^{1/L}\right) ,$$

is larger than 1 by the choice of C_{2t} in Lemma 3. □

4. Global induction

4.1. Preimages of critical points

Proposition 1. *The first Collet-Eckmann condition implies the second for the critical points of the maximal multiplicity.*

Proof. Suppose that c is a critical point of maximal multiplicity $\mu_{\max}$. Fix N and consider a sequence of the preimages $F^{-1}c, \ldots, F^{-N}c$. We will define by induction a sequence $\{n_j\}$, such that $n_0 = 0$, $n_{m-1} > N - L$, $n_m = N$. For the sake of simplicity, set $z_j := F^{-n_j}(c)$ and $d_j := \mathrm{dist}(z_j, c_j)$, where c_j is the closest to z_j critical point. Here are the conditions imposed on n_j:

I) For every $1 \leq j < m$, the sequence $F^{-n_{j-1}}c, \ldots, F^{-n_j}c$ is of the first, the second of length L, or the third type. The sequence $F^{-n_{m-1}}c, \ldots, F^{-n_m}c$ is of the second type.

II) Either the shrinking neighborhoods U_l for $B_{2R'}(z_j)$ and $l \leq N - n_j$ omit critical points (case IIa)), or some critical point c_j is close to z_j with $F(c_j) \in B_R(Fz_j)$ (case IIb)).

Basic inductive procedure. As a base for the induction we take $z_0 = c$, $d_0 = 0$. Suppose we have already constructed z_j.

Case IIa. If $n_j > N - L$ we put

$$m := j + 1,\ n_m := N,\ z_m := F^{-N}(0) = F^{-n_{j+1}}(z_j) \ ,$$

and the construction terminates. Suppose $n_j < N - L$. Set $y := z_j$, $y_{-L} := F^{-L}z_j$. Observe that y_{-L} is the second type preimage of y. We enlarge the ball $B_r(y_{-L})$ continuously increasing the radius r from 0 until one of the following conditions occurs:

1) for some k the shrinking neighborhood U_k for $B_r(y_{-L})$ hits some critical point c', $c' \in \partial U_k$,

2) radius r reaches the value of $2R'$.

In the case 1) we put $n_{j+1} := n_j + k + L$. The condition I) is satisfied: z_{j+1} is the third type preimage of z_j. The condition IIb) is satisfied by Lemma 4 with $c_{j+1} = c'$.

In the case 2) set $n_{j+1} := n_j + L$. Then $z_{j+1} \in J_F$ is the second type preimage of z_j of the length L. Clearly, the shrinking neighborhoods for $B_{2R'}(z_{j+1})$ satisfy the condition IIa).

Case IIb. Suppose that we have IIb), but not IIa). Set $r = 0$. The shrinking neighborhoods U_l for $B_r(z_j)$, $l \leq N - n_j$, do not contain critical points. We increase r continuously until some domain U_k hits some critical point c', $c' \in \partial U_k$. This must occur for some $r < 2R'$, since IIa) is not satisfied for z_j.

Let $n_{j+1} := n_j + k$. Then the condition I) is satisfied: z_{j+1} is the first type preimage of z_j. Lemma 2 implies the condition IIb).

Growth of the derivative. In the inductive procedure we decompose the backward orbit of the point c into pieces of the three types of preimages naturally encoded by a sequence of $1, 2, 3$. Not all combinations of $1, 2, 3$ are admissible. The restriction is that after type 2 we cannot construct the type 1. For example we could have a sequence of the form

$$\ldots\ldots 1111133222211111313221111 \ ,$$

here F acts from left to right and our inductive procedure has started from the right end. The expansion over the chains of preimages of type 2 and type 3 is guaranteed by Lemma 3 and Lemma 4 ($r' \leq d < 1$). A sequence of the first type preimages might not yield exponential expansion because of the different multiplicities of the critical points and distortion constants involved. To overcome this difficulty we will study expansion along sequences of the form $1\dots13$.

Suppose that a given sequence $1\dots13$ has the length k and the consecutive pieces of the preimages have the lengths k_i, $i=1\dots j$. Denote the multiplicities of the corresponding critical points by μ_i. Set $Q:=\min(Q_{1t},Q_{2t},Q_{3t})$.

By Lemma 2 and Lemma 4, we have that

$$\begin{aligned}\left|(F^k)'(y)\right|^{\mu_{\max}} &> \prod_{i=2}^{j} 2^{\mu_{\max}-1}(Q_{1t})^{k_i}\frac{d_{i+1}^{\mu_{i+1}-1}}{(r'_{i+1})^{\mu_{\max}-1}}\frac{r_i^{\mu_{\max}-1}}{(r_i+d_i)^{\mu_i-1}}\cdot(Q_{3t})^{k_1}\frac{d_2^{\mu_2-1}}{(r'_2)^{\mu_{\max}-1}}\\ &> Q^k\frac{d_{j+1}^{\mu_{j+1}-1}}{(r'_{j+1})^{\mu_{\max}-1}}\prod_{i=2}^{j}2^{\mu_{\max}-1}\frac{d_i^{\mu_i-1}}{(r'_i)^{\mu_{\max}-1}}\frac{r_i^{\mu_{\max}-1}}{(r_i+d_i)^{\mu_i-1}}\\ &> Q^k\ .\end{aligned}$$

Since we have $r'_i<\min(r_i,d_i)$ and $\mu_i\le\mu_{\max}$, any term in the expansion of $(r'_i)^{\mu_{\max}-1}(r_i+d_i)^{\mu_i-1}$ is dominated by $d_i^{\mu_i-1}r_i^{\mu_{\max}-1}$ and hence,

$$2^{\mu_{\max}-1}\frac{d_i^{\mu_i-1}}{(r'_i)^{\mu_{\max}-1}}\frac{r_i^{\mu_{\max}-1}}{(r_i+d_i)^{\mu_i-1}}>1\ .$$

Clearly also $r'_{j+1}<1$ and

$$\frac{d_{j+1}^{\mu_{j+1}-1}}{(r'_{j+1})^{\mu_{\max}-1}}>1\ .$$

A block of 1's which is not preceded by 3 might happen only at the beginning of the sequence. Assume that the block has length k and every corresponding piece of preimages has length k_i, $k=k_1+\cdots+k_j$. In this case $d_1=0$. By Lemma 2 we have that

$$\begin{aligned}\left|(F^k)'(y)\right|^{\mu_{\max}} &> \prod_{i=1}^{j} 2^{\mu_{\max}-1}(Q_{1t})^{k_i}\frac{d_{i+1}^{\mu_{i+1}-1}}{(r'_{i+1})^{\mu_{\max}-1}}\frac{r_i^{\mu_{\max}-1}}{(r_i+d_i)^{\mu_i-1}}\\ &> Q^k\frac{d_{j+1}^{\mu_{j+1}-1}}{(r'_{j+1})^{\mu_{\max}-1}}\cdot\prod_{i=2}^{j}2^{\mu_{\max}-1}\frac{d_i^{\mu_i-1}}{(r'_i)^{\mu_{\max}-1}}\frac{r_i^{\mu_{\max}-1}}{(r_i+d_i)^{\mu_i-1}}\cdot\frac{r_1^{\mu_{\max}-1}}{(r_1+d_1)^{\mu_1-1}}\\ &> Q^k\frac{r_1^{\mu_{\max}-1}}{(r_1+d_1)^{\mu_1-1}}>Q^k\ ,\end{aligned}$$

since $\mu_{\max}=\mu_1$ and $d_1=0$. Combining the above estimates with these of Lemma 3 and Lemma 4 for blocks of 2's and 3's we obtain that

$$\left|(F^N)'(y)\right| = \prod_{j=1}^{m}\left|(F^{n_j-n_{j-1}})'(z_j)\right|$$
$$> K_{3.1}\prod_{j=1}^{m-1} Q^{n_j-n_{j-1}} \geq K_{3.1}Q^{n_{m-1}} \geq \text{const}\, Q^N \ . \qquad \square$$

4.2. Hyperbolicity away from the critical orbits

Proposition 2. *The first Collet-Eckmann condition implies the second for every point z which is away from the forward orbits of the critical points*:

$$\left|(F^n)'(z)\right| > C_2(\lambda_2)^n \ ,$$

with $C_2 \asymp (\Delta)^{1-\frac{1}{\mu_{\max}}}$ *where* Δ *is the spherical distance from z to the orbits of the critical points.*

The reasoning for the preimages of a point which stays away from the forward orbits of the critical points is very much the same as for the preimages of the critical point with maximal multiplicity. Indeed, once started, the basic inductive procedure can be carried out for any point. The expansion along the sequences of the first and the third type was formulated in the abstract setting and does not depend on whether a point belongs to the Julia set or not.

For preimages of the second type, we need to control their distance from the Julia set. The assumptions of Lemma 3 will be satisfied for points from the $R'/2$-neighborhood $V_{R'/2}$ of the Julia set, and by the following Lemma for the preimages of points from the ϵ-neighborhood:

Lemma 5. *There exists* $\epsilon > 0$ *such that the backward orbit of* $z \in V_\epsilon := \cup_{z\in J_F} B_\epsilon(z)$ *stays in* $V_{R'/2}$ *and it does not intersect critical orbits not belonging to the Julia set.*

Proof. The proof is a combination of the Sullivan's classification of Fatou components (see [27]) with the compactness argument. We work in conformal coordinates on the Riemann sphere so that ∞ is contained in a Fatou component. Since F is a C-E map, periodic Fatou components of F cannot be parabolic. Thus they are either Siegel disks, Herman rings or sinks. Suppose that there is a sequence z_i, $\text{dist}(z_i, J_F) \to 0$ and negative integers k_i such that $F^{k_i}(z_i) \notin V_{R'/2}$. Without loss of generality we may assume that all z_i belong to the same periodic Fatou component, since only finitely many Fatou components contain a disk of the diameter larger than $R'/2$. This situation cannot occur for z_i in a Siegel disk or a Herman ring due to the existence of linear coordinates. In sinks all points are attracted to a stable periodic point, and the sequence z_i cannot exist by the compactness argument. $\square$

Let z be a point in the ϵ-neighborhood of the Julia set. Denote by Δ the distance from z to the critical orbits. We fix N and a sequence of the pre-images $F^{-1}(z), \ldots, F^{-N}(z)$. Similarly, as before, we will define by induction a sequence $\{n_j\}$ such that $n_0 = 0$, $n_{m-1} > N - L$, $n_m = N$, and

I′) For every $1 < j < m$, the sequence $F^{-n_{j-1}}z, \ldots F^{-n_j}z$ is of the first, the second of length L, or the third type. The sequence $F^{-n_{m-1}}z, \ldots F^{-n_m}z$ is of the second type. Additionally,

$$\left|(F^{n_1})'(F^{-n_1}(z))\right| > \mathrm{const}\,\Delta^{1-\frac{1}{\mu_{\max}}} Q^{n_1} \left(\frac{(d_1)^{\mu_1-1}}{(r_1)^{\mu_{\max}-1}}\right)^{1/\mu_{\max}} ,$$

II″) For $j > 0$ either the shrinking neighborhoods U_l for $B_{2R'}(z_j)$ and $l \leq N - n_j$ omit critical points (case IIa)), or some critical point c_j is close to z_j with $F(c_j) \in B_R(Fz_j)$ (case IIb)).

Inductive procedure. We will construct a sequence $z_j := F^{-n_j}(z)$ using the basic inductive procedure from Sect. 4.1. The only difference will be in the first step.

Base of induction

If the shrinking neighborhoods for $B_{2R'}(z_0)$ do not contain critical points, the condition IIa) is satisfied. We start from $j = 0$ and continue the inductive procedure as in the Section 4.1. By lemma 5, $\mathrm{dist}(z_j, J) < R'/2$, and hence sequences of the second type will yield exponential expansion.

Otherwise we take $r := \Delta$. By the definition of Δ, the shrinking neighborhoods for $B_\Delta(z)$ omit the critical points. We increase r continuously until certain shrinking neighborhood U_k hits some critical point c, i.e. $c \in \partial U_k$. It must happen for some $\Delta < r_0 < 2R'$. Set $n_1 := k$. The condition IIb) for z_1 is satisfied by the reasoning of Lemma 4.

By Lemma 1

$$\begin{aligned}\left|(F^{n_1})'(z_1)\right|^{\mu_{\max}} &\geq \frac{1}{8^{\mu_{\max}}M}\delta_n(d_1)^{\mu_1-1}\left|(F^{n_1-1})'(Fc)\right|\left(\frac{r_0}{r'_1}\right)^{\mu_{\max}-1}\\ &\geq \mathrm{const}\,\Delta^{\mu_{\max}-1}Q^{n_1}\frac{(d_1)^{\mu_1-1}}{(r'_1)^{\mu_{\max}-1}} ,\end{aligned}$$

where $d_1 := \mathrm{dist}(z_1, c)$, $\mu_1 := \mu(c)$, and $\mathrm{const} := \frac{1}{8^{\mu_{\max}}M}\frac{1-q}{2}$.

Induction and expansion

The point z_1 satisfies I′) and IIb). Take z_1 as a base of the induction. We use the basic inductive procedure to pick points $z_2, \ldots, z_m$ and decompose the backward orbit into pieces of preimages of type 1, 2 and 3. The expansion along the blocks of 2's, 3's, and 1 … 13 is exponential. The first block of 1's yields exponential expansion up to the power of Δ. Combining all these estimates together we obtain the claim of Proposition 2 for the points z in the ϵ-neighborhood of the Julia set which stay away from the critical orbits. If there are non-empty Fatou components then the distortion argument implies the claim of Proposition 2 for all z outside the Julia set and away from the critical orbits.

5. Collet-Eckmann condition and Hölder domains

First we will consider the case of a polynomial with simply-connected domain of attraction to ∞, where we have clear and nice relation between Hölder and expansion exponents.

Let F be a polynomial of degree d with the Julia set J_F. Denote by $A_\infty := \{z : F^n z \to \infty\}$ the basin of attraction to infinity. If J_F is connected there exists a conformal map

$$\varphi : \mathbb{D} = \{|z| < 1\} \to A_\infty, \quad \varphi(0) = \infty \ .$$

Without loss of generality φ conjugates F with dynamics $T : z \mapsto z^d$ on $\mathbb{D}$:

$$F \circ \varphi = \varphi \circ T \ .$$

Lemma 6. *The following conditions are equivalent*:

(i) *Domain A_∞ is Hölder with exponent* α,

(ii) *For some constant* C_1

$$|\varphi'(\zeta)| < C_1(1 - |\zeta|)^{\alpha-1}, \quad \zeta \in \mathbb{D} \ ,$$

(iii) *For some* (*any*) *point $z \in A_\infty$ and constant $C_2 = C_2(z)$*

$$\left|(F^n)'(y)\right| > C_2 d^{n\alpha}, \quad y \in F^{-n}z, \ n \in \mathbb{Z}_+ \ .$$

Proof. Properties (i) and (ii) are equivalent for any domain, the proof is in [7], p. 74. Hence, it is sufficient to prove the equivalence of (ii) and (iii).

We differentiate the identity $F^n \circ \varphi = \varphi \circ T^n$. As a result we obtain

$$F_n' \circ \varphi \cdot \varphi' = \varphi' \circ T^n \cdot T^{n\prime} \ .$$

We apply the above equality to the preimages of $\zeta \in T^{-n}\xi$. If a point ξ is inside an annulus $A := \left\{r^d \le |\xi| \le r\right\}$ for fixed $r < 1$ then the right-hand side is approximately $\asymp d^n$.

We obtain up to a constant that

$$d^{-n}|\varphi'(\zeta)| \asymp \left|F_n'(y)\right|^{-1} \ ,$$

where $y = \varphi(\zeta)$ is a corresponding preimage of $z = \varphi(\xi)$ under F^n.

Since any point ζ close to $\partial\mathbb{D}$ after a number of iterations gets into A and $(1 - |\zeta|) \asymp d^{-n}$, we obtain that (ii) is equivalent to the uniform $\mathcal{CE}_2(z)$ for all $z \in \varphi(A)$ and hence (by the distortion argument) to (iii). $\square$

Definition 8. *We will call (possibly non-simply-connected) domain $\mathcal{F}$ Hölder if it satisfies a quasihyperbolic boundary condition:*

$$\mathrm{dist}_{\mathrm{qh}}(z, z_0) \leq \frac{1}{\varepsilon} \log \frac{1}{\mathrm{dist}(z, \partial\mathcal{F})} + C \ ,$$

for a fixed $z_0 \in \mathcal{F}$ and any $z \in \mathcal{F}$.

Above the quasihyperbolic distance $\mathrm{dist}_{\mathrm{qh}}(y, z)$ between points $y, z \in \mathcal{F}$ is defined as the infimum of

$$\int_\gamma \frac{|d\zeta|}{\mathrm{dist}(\zeta, \partial\mathcal{F})} \ ,$$

over all rectifiable curves γ joining y, z in $\mathcal{F}$.

For simply connected domains quasihyperbolic and hyperbolic metrics are comparable and this definition of Hölder domains is equivalent to one given in Introduction.

Proposition 3. *Attracting or super-attracting periodic Fatou component $\mathcal{F}$ is Hölder if and only if it is Collet-Eckmann.*

Without loss of generality we may assume that F fixes a Fatou component $\mathcal{F}$. Throughout the rest of this section we will always mean by F^{-n} a branch mapping $\mathcal{F}$ to itself.

Take a subdomain $\Omega \Subset \mathcal{F}$ with a nice boundary containing all critical points from $\mathcal{F}$ such that $F\Omega \Subset \Omega$. Define

$$\begin{aligned} \Omega_0 &:= \mathcal{F} \backslash \bar{\Omega} \ , \\ \Omega_n &:= F^{-n}\Omega_0 \ , \\ \Omega'_n &:= \Omega_n \backslash \Omega_{n+1} \ , \\ \tau_n &:= \partial\Omega_n \ . \end{aligned}$$

Any point z eventually gets to Ω under some iterate of F, meaning that for $z \in \Omega_0$ there is a unique index $n = n(z)$ such that $z \in \Omega'_n$. Also fix some $z_0 \in \Omega'_0$.

Lemma 7. *Suppose that $z \in \Omega_0$ and $n = n(z)$. Then*

$$\begin{aligned} \mathrm{dist}(z, \partial\mathcal{F}) &\asymp \left|(F^n)'(z)\right|^{-1} \ , \\ \mathrm{dist}_{\mathrm{qh}}(z, z_0) &\asymp n \ , \end{aligned}$$

up to some constant depending on $\mathscr{F}$ and our choice of Ω only.

Proof. First note that it is sufficient to prove first relation for z sufficiently close to the boundary. Let R be so small that R-neighborhood of $\partial\mathscr{F}$ is contained in Ω_0. Take a finite cover of $\partial\mathscr{F}$ by balls B_r from Technical lemma 8. Denoting by V their union, we note that for any $y \in V$ and positive integer k

$$\operatorname{dist}\left(F^{-k}y, \partial\mathscr{F}\right)\left|\left(F^k\right)'(F^{-k}y)\right| \asymp \operatorname{dist}(y, \partial\mathscr{F}) \ ,$$

by the Technical lemma 8 applied to an appropriate ball B_r. Let m be the minimal integer such that $\Omega_m \subset V$. Thus for arbitrary $z \in \Omega_m$ with $n := n(z) > m$ we have $F^{n-m}z \in \Omega'_m \subset V$ and

$$\operatorname{dist}(z, \partial\mathscr{F}) \asymp \left|\left(F^{n-M}\right)'(z)\right|^{-1}\operatorname{dist}(F^{n-m}z, \partial\mathscr{F}) \asymp \left|(F^n)'(z)\right|^{-1} \ ,$$

which proves the first relation.

To prove the second relation, first prove $\operatorname{dist}_{\mathrm{qh}}(z, z_0 t) \gtrsim n$. In fact, we can join z and z_0 with a quasihyperbolic geodesic γ. Set $\gamma_k := \gamma \cap \Omega'_k$. Then

$$\begin{aligned}\operatorname{length}_{\mathrm{qh}}(\gamma_k) &= \int_{\gamma_k} \frac{|d\zeta|}{\operatorname{dist}(\zeta, \partial\mathscr{F})} \\ &= \int_{\gamma_k} \frac{\left|dF^k(\zeta)\right|}{\left|(F^k)'(\zeta)\right|\operatorname{dist}(\zeta, \partial\mathscr{F})} \asymp \int_{F^k\gamma_k} |d\xi| \gtrsim 1 \ ,\end{aligned}$$

since $F^k\gamma_k$ must join τ_1 with τ_0. Hence $\operatorname{length}_{\mathrm{qh}}(\gamma) \geq \sum_{k=0}^{n-1}\operatorname{length}_{\mathrm{qh}}(\gamma_k) \gtrsim n$.

It remains to prove that $\operatorname{dist}_{\mathrm{qh}}(z, z_0) \lesssim n$. To do so, it suffices to construct for any $y \in \Omega'_k$ a point $y' \in \tau_k$ such that $\operatorname{dist}_{\mathrm{qh}}(y, y') \lesssim 1$. In fact, consider $w := F^k y \in \Omega'_0$, we can join it with some point $w' \in \tau_0$ by a curve $\gamma \subset \Omega'_0$ of length $\lesssim 1$. Pulling γ back by a branch of F^{-k} sending w to y, we obtain for $y' := F^{-k}w' \in \tau_k$:

$$\begin{aligned}\operatorname{dist}_{\mathrm{qh}}(y, y') &\leq \int_{F^{-k}\gamma} \frac{|d\zeta|}{\operatorname{dist}(\zeta, \partial\mathscr{F})} \\ &= \int_{F^{-k}\gamma} \frac{\left|dF^k(\zeta)\right|}{\left|(F^k)'(\zeta)\right|\operatorname{dist}(\zeta, \partial\mathscr{F})} \asymp \int_{\gamma} |d\xi| \lesssim 1 \ .\end{aligned}$$

Now for $z \in \Omega'_n$ we can construct by induction a sequence: $z_k \in \tau_k$, $n \geq k \geq 1$, $z = z_{n+1}$, with $\operatorname{dist}_{\mathrm{qh}}(z_k, z_{k+1}) \lesssim 1$. Therefore,

$$\operatorname{dist}_{\mathrm{qh}}(z,z_0) \le \operatorname{dist}_{\mathrm{qh}}(z_0,z_1) + \sum_{k=1}^{n} \operatorname{dist}_{\mathrm{qh}}(z_k,z_{k+1}) \lesssim 1 + \sum_{k=1}^{n} 1 \lesssim n \ . \qquad \square$$

Technical Lemma 8. *For any domain* $\mathscr{F}$, *point* $z \in \partial\mathscr{F}$, *and positive* R, *there exists positive* $r \ll R$ *such that for any point* $y \in \mathscr{F} \cap B_r(z)$ *and conformal mapping* $\phi : \mathscr{F}' := \mathscr{F} \cap B_R(z) \to \mathbb{C}$ we have

$$|\phi'(y)|\operatorname{dist}(y,\partial\mathscr{F}) \asymp \operatorname{dist}(\phi(y),\partial\phi(\mathscr{F}'))$$
$$\asymp \operatorname{dist}\Big(\phi(y),\partial\phi(\mathscr{F}') \cap \overline{\phi(\mathscr{F}' \cap B_{R/2}(z))}\Big)$$

up to a constant depending on $\mathscr{F}$, z, *and* R.

Proof. First note that the first relation is true by the Koebe distortion theorem.

Suppose that the second is not, i.e.

$$\rho := \operatorname{dist}\Big(\phi(y),\partial\phi(\mathscr{F}') \cap \overline{\phi(\mathscr{F}' \cap B_{R/2}(z))}\Big) \ll \operatorname{dist}(\phi(y),\partial\phi(\mathscr{F}')) =: \mathrm{P} \ .$$

Consider the family Γ of curves joining $\partial B_\rho(\phi(y))$ with $\partial B_{\mathrm{P}}(\phi(y))$ in $\phi(\mathscr{F}')$, its extremal length will be large: $> \log(P/\rho)/2\pi \gg 1$.

On the contrary, the family $\phi^{-1}\Gamma$ contains a subfamily of curves joining two opposite sides of some strip, separating y and z in $B_R(z)$. Since y and z

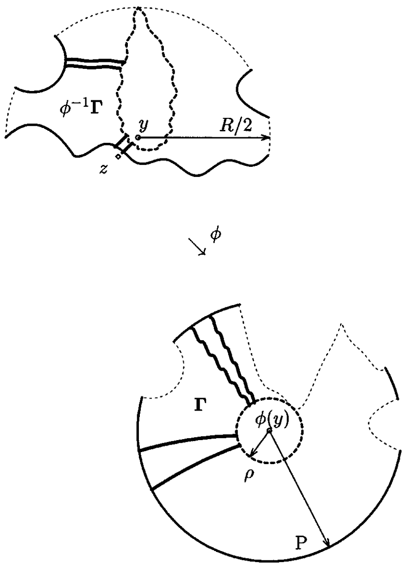

Fig. 5. Curve families Γ and $\phi^{-1}\Gamma$ from Technical Lemma

are close, the extremal length of $\phi^{-1}\Gamma$ should be small (namely $\lesssim (1+\log(R/r))^{-1} \ll 1$), and we arrive at a contradiction.

On Figure 5 the families Γ and $\phi^{-1}\Gamma$ are plotted as extra bold curves joining bold and bold dotted ($\partial B_{\mathrm{P}}(\phi(y)) \cap \phi(\mathscr{F}')$ and ∂B_ρ correspondingly, and their preimages under ϕ) lines. □

Proof (*of the Proposition*). Take z close to the boundary of $\mathscr{F}$. Then by Lemma 7 we have

$$\mathrm{dist}_{\mathrm{qh}}(z, z_0) \asymp n(z) \text{and } \mathrm{dist}(z, \partial\mathscr{F}) \asymp \left|\left(F^{n(z)}\right)'(z)\right|^{-1} ,$$

hence the quasihyperbolic boundary condition for the point z is equivalent to the inequality

$$\log\left|\left(F^{n(z)}\right)'(z)\right| \gtrsim n(z) ,$$

which by the distortion theorem is equivalent to the analogous inequality for the corresponding preimage of z_0 under $F^{-n(z)}$. Therefore the quasihyperbolic boundary condition holds for all points if and only if the second Collet-Eckmann condition for preimages of z_0 in $\mathscr{F}$ does, i.e. the Fatou component $\mathscr{F}$ is Collet-Eckmann. □

Lemma 9. *If Fatou component $\mathscr{F}$ is Hölder then so are its preimages.*

Proof. By Koebe distortion theorem, the distance to the boundary and hence the quasihyperbolic distance are changed under the conformal mapping at most by a multiplier of 4. Therefore preimage of a Hölder domain under conformal mapping is also Hölder.

It remains to consider the case when $F : \mathscr{F}' \to \mathscr{F}$ has some critical points in $\mathscr{F}'$. But then we can find a finite open cover of $\mathscr{F}'$ such that F maps conformally each element into $\mathscr{F}$ (sending $\partial\mathscr{F}'$ to $\partial\mathscr{F}$), and since Hölder property is local, Hölder regularity of $\mathscr{F}$ will imply the same for $\mathscr{F}'$. □

Now it remains to notice that Collet-Eckmann rational maps cannot have parabolic points, and the second Collet-Eckmann condition excludes Siegel disks and Herman rings since they have local coordinates. Thus all Fatou components are preimages of periodic attractive or superattractive ones and hence are Hölder domains.

To exclude the possibility of Cremer points we will prove much stronger statement that all periodic points are uniformly expanding, i.e. there is a constant $Q > 1$, such that for every periodic point $x \in J_F$, $F^m x = x$, the inequality $|(F^m)'(x)| > Q^m$ holds. In one-dimensional real dynamics the above condition is called *uniform hyperbolic structure* (on cycles).

Lemma 10. *C-E rational maps have uniform hyperbolic structure.*

Proof. First we rule out the existence of neutral periodic points $F^m x = x$ in the Julia set. We repeat the inductive procedure for a periodic point x and the inverse branch of F^{-km} which fixes x. If we choose R smaller than the distance of the cycle $\{x, \dots, F^m(x)\}$ to the critical points then the shrinking neighborhoods defined in the inductive procedure cannot hit the critical points. Hence, only blocks of the second type are admissible in the procedure and $|(F^m)'(x)| > Q_{2t}^m > 1$.

If x is a repelling periodic point, $F^m x = x$, then there exists a neighborhood U of x so that the inverse branch of F^m fixing x is bi-holomorhically equivalent to the multiplication by $1/(F^m)'(x)$. Let c be any critical point of the maximal multiplicity. Since the backward orbit of c is dense in J, we can find a positive number k and the branch F^{-k} so that $F^{-k}(c) \in U$. By the first claim of Theorem 1 and the bounded distortion of F^{-rm} on U, there exists a constant $Q > 1$ so that the following estimate holds

$$\begin{aligned} |(F^m)'(x)| &= \lim_{r\to\infty} |(F^{rm})'(F^{-rm}x)|^{1/r} \geq \liminf_{r\to\infty} |(F^{rm})'(F^{-k-rm}c)|^{1/r} \\ &= \liminf_{r\to\infty} |(F^{rm+k})'(F^{-k-rm}c)|^{1/r} \geq \liminf_{r\to\infty} Q^{k+rm/r} = Q^m \ . \qquad \square \end{aligned}$$

A natural question arises:

Problem 2. Is uniform hyperbolic structure on cycles equivalent to the $\mathscr{CE}$ condition for rational maps of the Riemann sphere?

Note also that, if the Fatou set is non-empty, Cremer points cannot exist since then the Julia set would contain non-accessible points (see [21]). However (as was pointed to us by the referee) the reasoning above is necessary, since M. Herman has constructed examples of transitive (hence with empty Fatou set) rational maps with Cremer points – see [10].

Summing it up, we arrive at the following

Corollary 3. *All Fatou components of a rational C-E map are Hölder domains. Rational C-E maps can have neither Siegel disks, Herman rings, nor parabolic or Cremer points.*

6. Appendix

In the Appendix we assume that F is a polynomial with connected Julia set which satisfies the second Collet-Eckmann condition for the preimages of the critical points. Our studies here are motivated by the results about S-unimodal maps on the real line.

Two types of preimages. We will decompose the forward orbit of a critical point into parts. The first will consists of blocks of the reversed first type which is a modification of the first type defined in the Sect. 3. The second part of the orbit will stay at a certain distance from the critical points.

Parameters and scales

We put $\delta_n := \frac{1-q}{2} q^n$ with $q = (\lambda_2)^{-\varepsilon}$ for small positive ε. Large scale will be considered with parameter $R \ll 1$, which satisfy the following conditions:

(i) Any two critical points are at least $100R$ apart and R is so small that $|F'(y)| \underset{\asymp}{M} \mathrm{dist}(y,c)^{\mu(c)-1}$ and $\mathrm{dist}(Fy, Fc) \underset{\asymp}{M} \mathrm{dist}(y,c)^{\mu(c)}$ given that $\mathrm{dist}(y,c) < R$ for a critical point c.

(ii) The first return time of the critical points to $\bigcup B_R(c_i)$ is greater then a certain constant τ, such that

$$2M^2 \left(\frac{1-q}{2}\right)^{-1} (\lambda_2)^{\tau(\varepsilon-1)} < 1/2 \ ,$$

$$\frac{1}{8M} \frac{1-q}{2} C_2 (\lambda_2)^{\tau\varepsilon} > 1 \ .$$

Reversed first type

Definition 9. *A sequence of preimages of z: $z, F^{-1}(z), \ldots, F^{-n}(z)$, is of the reversed first type with respect to two critical points c_1 and c_2 if*

1) *Shrinking neighborhoods U_k for $B_r(F^{-1}z)$, $1 \le k \le n-1$ avoid critical points,*

2) $\mathrm{dist}(F^{-1}z, c_1) = r/2 < R$,

3) $c_2 \in U_n$.

To simplify notation set $y := F^{-n}z$ and $d_2 := \mathrm{dist}(F^{-1}y, c_2)$. For consistency let $d_1 := r/2 = \mathrm{dist}(F^{-1}z, c_1)$.

Lemma 11. *There exists a constant $Q_1 > 1$ such that for a sequence of preimages of the first reversed type we have that*

$$\left|(F^n)'(y)\right| > (Q_1)^n \frac{(d_1)^{\mu_1 - \mu_{\max}}}{(d_2)^{\mu_2 - \mu_{\max}}} \ .$$

Also

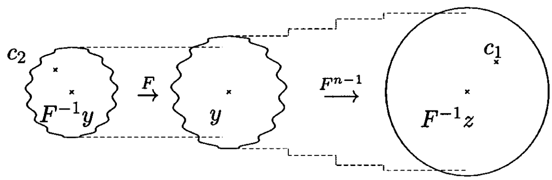

Fig. 6. Preimages of the reversed first type

$$\operatorname{dist}\left(F^{-1}y, c_2\right) < R \ .$$

Proof. Set $u := F^{-n}(c_1)$. By the Koebe distortion theorem (see Theorem 1.3 in [22]) applied to $F^{-(n-1)} : B_{r\Delta_{n-1}}(F^{-1}) \to U_{n-1}$ we obtain that

$$\begin{aligned}
(\operatorname{diam}(U_n))^{\mu_2} &\le M \operatorname{diam}(U'_{n-1}) \\
&\le M \frac{(1-\delta_n)(2-\delta_n)}{\delta_n} r \left| \left(F^{n-1}\right)'(Fu) \right|^{-1} \\
&\le 2M^2 (\delta_n)^{-1} r \operatorname{dist}(u, c_2)^{\mu_2 - 1} \left| (F^n)'(u) \right|^{-1} \\
&\le 2M^2 (\delta_n)^{-1} (\operatorname{diam}(U_n))^{\mu_2 - 1} \left(C_2 \lambda_2^n\right)^{-1} \ ,
\end{aligned}$$

hence by the choice of R (since $n \ge \tau$)

$$d_2 \le \operatorname{diam}(U_n) \le 2 M^2 (\delta_n)^{-1} (\lambda_2)^{-n} r \le r/2 = d_1 \ ,$$

which implies the second desired inequality.

To prove the first, write (using the Koebe distortion theorem again)

$$\begin{aligned}
\left|(F^n)'(y)\right| &\ge \frac{1}{M} \left| \left(F^{n-1}\right)'(y) \right| \operatorname{dist}\left(F^{-1}z, c_1\right)^{\mu_1 - 1} \\
&\ge \frac{1}{M} \frac{\delta_n}{(2-\delta_n)^3} \left| \left(F^{n-1}\right)'(Fu) \right| (d_1)^{\mu_1 - 1} \\
&\ge \frac{1}{8M} \delta_n \left| (F^n)'(u) \right| \\
&\qquad \cdot \ \operatorname{dist}(u, c_2)^{-(\mu_2 - 1)} (d_1)^{\mu_{\max} - 1} (d_1)^{\mu_1 - \mu_{\max}} \\
&\ge \left(\frac{1}{8M} \delta_n C_2 (\lambda_2)^n \right) \\
&\qquad \cdot \ \operatorname{diam}(U_n)^{-(\mu_2 - 1)} \operatorname{diam}(U_n)^{\mu_{\max} - 1} (d_1)^{\mu_1 - \mu_{\max}} \\
&\ge \left(\frac{1}{8M} \frac{1-q}{2} C_2 (\lambda_2)^{n\varepsilon} \right) (\lambda_2)^{n(1-2\varepsilon)} \\
&\qquad \cdot \ \operatorname{diam}(U_n)^{\mu_{\max} - \mu_2} (d_1)^{\mu_1 - \mu_{\max}} \\
&\ge (Q_1)^n (d_2)^{\mu_{\max} - \mu_2} (d_1)^{\mu_1 - \mu_{\max}} \ ,
\end{aligned}$$

for $Q_1 := \lambda^{1-2\varepsilon}$ by the choice of R. □

Expansion away from the critical points

Let us define an important property of Julia sets which we call R-expansion.

Definition 10. *We say that a Julia set is R-expansive if for any positive R there exist parameters $n(R)$ and $K(R) > 1$ such that every forward orbit of length* *greater than $n(R)$ which stays away from the critical points at the distance at least R has the derivative greater than $K(R)$.*

Up till now there are known only few examples of R-expansive Julia sets. Among these we have non-renormalizable quadratic polynomials and non-renormalizable real unimodal polynomials $z^l + c$ (see [15]).

It is easy to see that if the orbit $z, Fz \ldots, F^k z$ is R away from the critical points, then R-expansion implies

$$\left|(F^k)'(z)\right| > \text{const}\,(Q_2)^k \ ,$$

for $Q_2 := K(R)^{1/n(R)} > 1$.

Proposition 4. *The Julia set of a rational C-E map is R-expansive.*

Proof. We proceed as in the proof of Lemma 10. Let $x \in J$ be a point whose forward orbit stays away from the critical points at the distance at least Δ. We choose R, which is a parameter in the inductive procedure defined in Sect. 4.1, to be smaller than Δ. We repeat the inductive procedure for points $F^n(x)$, $1 \ll n$, and the inverse branches F^{-n} which map $F^n(x)$ to x. By our choice of R, only second type preimages are admissible. Thus by Lemma 3, there exists $n(R) > 0$ such that $|(F^n)'(x)| > 2$ for every $n > n(R)$. □

Proposition 5. *If the Julia set of a rational map F is R-expansive then the second Collet-Eckmann condition for the preimages of critical points implies the first for the critical points of maximal multiplicity.*

Proof. Let c be a critical point of maximal multiplicity: $\mu(c) = \mu_{\max}$. Fix N and consider a sequence of the images

$$z := F^N(F(c)),\ F^{-1}(z) = F^{N-1}(F(c)), \ldots, F^{-N}(z) = F(c),\ F^{-(N+1)}(z) = c \ .$$

Let n_0 be the smallest positive integer such that $F^{-(n_0+1)}(z)$ is in the R-neighborhood of some critical point. We will define by induction a sequence $\{n_j\}$ such that $n_m = N$. For simplicity, set $z_j := F^{-n_j}(z)$. Here are the conditions imposed on n_j:

I) The sequence $F^{-n_{j-1}}z, \ldots, F^{-n_j}z$ is of the first reversed type;

II) Some critical point $c_j \in B_R(F^{-1}z_j)$.

As a base for the induction we take $z_0 = F^{-n_0}(z)$. Suppose we have already constructed z_j. Let k be the first time shrinking neighborhoods U_k for $B_r(F^{-1}z_j)$ with $r = d_j := \text{dist}(F^{-1}z_j, c_j) < R$ hit some critical point c_{j+1}: $c_{j+1} \in U_k$, clearly $k \le N - n_j$. Set $n_{j+1} := n_j + k$, condition I) is satisfied since z_{j+1} is a reversed first type preimage of z_j. Condition II) follows from Lemma 11. The construction naturally terminates when for some j we get $n_j = N$.

Expansion

We estimate $|(F^N)'(Fc)|$ in the usual way. Put $\mu_j := \mu(c_j)$, then

$$\begin{aligned}\left|\left(F^N\right)'(Fc)\right| &= \left|(F^{n_0})'(z_0)\right| \prod_{j=1}^{m} \left|(F^{n_j - n_{j-1}})'(z_j)\right| \\ &\geq \text{const}\,(Q_2)^{n_0} \prod_{j=1}^{m} (Q_1)^{n_j - n_{j-1}} \\ &\quad \cdot\ (d_m)^{\mu_{\max} - \mu_m} / (d_0)^{\mu_{\max} - \mu_0} \\ &= \text{const}\, Q^{n_0} Q^{n_m - n_0} (d_0)^{\mu_0 - \mu_{\max}} > \text{const}\, Q^N \ ,\end{aligned}$$

for $Q := \min\{Q_1, Q_2\} > 1$. Here we used that $\mu_m = \mu(c) = \mu_{\max}$, $\mu_0 \leq \mu_{\max}$, and $d_0 < 1$. □

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