

QUASISYMMETRIC DIMENSION DISTORTION OF AHLFORS REGULAR SUBSETS OF A METRIC SPACE

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Abstract. We show that if $f: X \rightarrow Y$ is a quasisymmetric mapping between Ahlfors regular spaces, then $\dim_H f(E) \leq \dim_H E$ for “almost every” bounded Ahlfors regular set $E \subseteq X$. If additionally, X and Y are Loewner spaces then $\dim_H f(E) = \dim_H E$ for “almost every” Ahlfors regular set $E \subset X$. The precise statements of these results are given in terms of Fuglede’s modulus of measures. As a corollary of these general theorems we show that if f is a quasiconformal map of \mathbb{R}^N , $N \geq 2$, then for Lebesgue a.e. $y \in \mathbb{R}^N$ we have $\dim_H f(y + E) = \dim_H E$. A similar result holds for Carnot groups as well. For planar quasiconformal maps, our general estimates imply that if $E \subset \mathbb{R}$ is Ahlfors d -regular, $d < 1$, then some component of $f(E \times \mathbb{R})$ has dimension at most $2/(d + 1)$, and we construct examples to show this bound is sharp. In addition, we show there is a 1-dimensional set $S \subseteq \mathbb{R}$ and planar quasiconformal map f such that $f(\mathbb{R} \times S)$ contains no rectifiable sub-arcs. These results generalize work of Balogh et al. (J Math Pures Appl (2)99:125–149, 2013) and answer questions posed in Balogh et al. (J Math Pures Appl (2)99:125–149, 2013) and Capogna et al. (Mapping theory in metric spaces. <http://aimpl.org/mappingmetric>, 2016).

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1 Introduction

1.1 Dimension preservation of random translates. A quasiconformal image of a single line can be a fractal curve of Hausdorff dimension greater than 1. However, a quasiconformal mapping f of \mathbb{R}^N is ACL for $N \geq 2$, cf. [Ahl06, Väi71]. This means that f is absolutely continuous on almost all lines parallel to coordinate axes. It follows, that the image of almost every such line is a locally rectifiable curve of Hausdorff dimension 1. In particular, for a quasiconformal mapping f of \mathbb{R}^N and a line segment $L \subset \mathbb{R}^N$ we have

$$\dim_H f(y + L) = \dim_H L = 1, \quad (1.1)$$

for \mathcal{H}^N -a.e. $y \in \mathbb{R}^N$. In this paper we prove the following generalization of this result.

Theorem 1.1. *If f is a quasiconformal mapping of \mathbb{R}^N , $N \geq 2$, and $E \subset \mathbb{R}^N$ is a bounded Ahlfors regular set then*

$$\dim_H f(y + E) = \dim_H E, \quad (1.2)$$

for \mathcal{H}^N -a.e. $y \in \mathbb{R}^N$.

Here E is an Ahlfors d -regular set if Hausdorff d -measure of a ball of radius r centered at a point in E is comparable to r^d , see Section 2 for the precise definition.

Theorem 1.1 is a special case of Theorem 3.8 on Carnot groups. The latter says that for a Carnot group \mathbb{G} of Hausdorff dimension Q (with respect to its Carnot–Caratheodory metric) we have

$$\dim_H f(y \cdot E) = \dim_H E,$$

for \mathcal{H}^Q -a.e. $y \in \mathbb{G}$ and for any bounded Ahlfors regular subset $E \subset \mathbb{G}$. Theorem 3.8 is stated and proved in Section 3.7 by using Theorem 1.2 (see the next paragraph) and an estimate for modulus of measures, Lemma 3.3. Theorem 1.2, in turn, is a special case of a more general result about the modulus of the quasisymmetric image of a family of lower-regular measures with respect to an upper-regular base measure (see Theorem 4.1).

1.2 Dimension distortion of “generic” Ahlfors regular subsets. Almost as well known as the ACL property is the slightly stronger fact that a quasiconformal mapping of \mathbb{R}^N is absolutely continuous on “almost every” curve, where “almost every” is understood in the sense of conformal modulus of curve families (see Section 3 below). A fortiori, if f is a quasisymmetric mapping of \mathbb{R}^N , we have

$$\dim_H f(\gamma) = \dim_H \gamma = 1 \quad (1.3)$$

for “almost every” rectifiable curve γ in \mathbb{R}^N .

In this paper we utilize Fuglede’s modulus of families of measures [Fug57] to introduce the notion of “almost every Ahlfors d -regular subset” of a metric measure space X , see Section 3.3 below. This in turn allows us to generalize equality (1.3) and to show that dimension preservation of “generic subsets” holds under a mild assumption.

Theorem 1.2. *If $f: X \rightarrow Y$ is a quasisymmetric mapping between Ahlfors D -regular spaces, $D > 1$, which satisfies condition N^{-1} , then for every $0 < d \leq D$ we have*

$$\dim_H f(E) = \dim_H E, \quad (1.4)$$

for $\text{mod}_{D/d}$ -almost every bounded Ahlfors d -regular set $E \subset X$.

Theorem 1.2 is proven in Section 5. Recall that a homeomorphism $f: X \rightarrow Y$ between Ahlfors D -regular metric spaces satisfies Lusin's condition N if whenever $A \subseteq X$ is such that $\mathcal{H}^D(A) = 0$ then $\mathcal{H}^D(f(A)) = 0$. We say that f satisfies condition N^{-1} if f^{-1} satisfies condition N .

We say that condition N^{-1} is mild because it is known to hold, not only in the classical case where X and Y are Euclidean domains, but more generally, when X and Y are domains in D -regular, D -Loewner spaces, see e.g. [HKST01]. Recall, that Loewner spaces constitute a large class of metric spaces introduced by Heinonen and Koskela [HK98], that includes Carnot groups equipped with their Carnot–Carathéodory metrics. Indeed, conditions N and N^{-1} are always satisfied for quasiconformal mappings in this setting. See [HK98, HKST01] for the definitions of Loewner spaces, and the basic theory of QC mappings between them. Note that locally, quasiconformality and quasisymmetry are equivalent in this setting [HKST01, Theorem 9.8], hence our casual conflation of the terms when we discuss mappings in these spaces. Thus we have the following consequence of Theorem 1.2.

COROLLARY 1.3. *Let $f: X \rightarrow Y$ be a quasiconformal mapping between Ahlfors D -regular, D -Loewner spaces, $D > 1$. Then for every $0 < d \leq D$ we have*

$$\dim_H f(E) = \dim_H E, \quad (1.5)$$

for $\text{mod}_{D/d}$ -almost every bounded Ahlfors d -regular set $E \subset X$.

We do not know if the equality in Theorem 1.2 holds for quasisymmetric mappings between arbitrary Ahlfors regular spaces. However, we will prove that an inequality does hold generally.

Theorem 1.4. *If $f: X \rightarrow Y$ is a quasisymmetric mapping of Ahlfors D -regular spaces, $D > 1$, then for every $0 < d \leq D$ we have*

$$\dim_H f(E) \leq \dim_H E, \quad (1.6)$$

for $\text{mod}_{D/d}$ -almost every bounded Ahlfors d -regular set $E \subset X$.

Thus “generic non-expansion” holds true for every quasisymmetric mapping between arbitrary Ahlfors regular spaces of the same dimension. Theorem 1.4 is a special case of Corollary 4.4, which itself is an immediate corollary of Theorem 4.1; the latter is the main result of the first half of the paper and states that “non-expansion” holds under much more relaxed conditions than those in Theorem 1.4.

1.3 Exceptional “fibers” in product spaces. For a quasiconformal map of \mathbb{R}^N , the ACL condition implies that

$$\mathcal{H}^{N-1}(\{y \in \mathbb{R}^\perp : \dim f(y + \mathbb{R}) > 1\}) = 0. \tag{1.7}$$

Dimension distortion of “generic subspaces” of a Euclidean space under quasiconformal mappings has recently been studied by Balogh et al. [BMT13], along with similar explorations for Sobolev mappings $f \in W^{1,p}(\mathbb{R}^N, Y)$, $p > N$, where Y is a metric space. In particular they considered the size of the “exceptional” family of parallel n dimensional subspaces whose image experiences a prespecified jump in dimension. More precisely, it was proved in [BMT13] that if n is an integer between 1 and N and $n' \geq n$ then

$$\mathcal{H}^{\frac{n}{n'}N-n}\{y \in (\mathbb{R}^n)^\perp : \dim f(y + \mathbb{R}^n) > n'\} = 0. \tag{1.8}$$

Thus the Hausdorff dimension of the “exceptional” n -dimensional subspaces of \mathbb{R}^N whose dimension may jump over $n' > n$ is at most $\frac{n}{n'}N - n$, which is strictly less than $N - n$.

The methods in [BMT13] relied heavily on properties of Euclidean space, particularly the foliation by affine subspaces, Lipschitz retractions, the Besicovitch covering theorem, and so forth, that need not hold in greater generality. As a result, the results there assume that the source space be a Euclidean domain. The authors concluded by asking [BMT13, Problem 6.5] what could be said for more general spaces, providing the broad motivation for the present paper, which addresses this question in the setting of quasisymmetric maps between metric spaces.

The questions about dimension distortion of “generic subspaces” for more general source spaces, e.g. for the Heisenberg group, have been further investigated by Balogh et al., cf. [BMT14, BTW13, BTW16]. In fact, in [BTW13, BTW16] a general form of inequality (1.8) was obtained for quasiconformal (and Sobolev) mappings defined on metric spaces supporting Poincaré inequalities, e.g. the Heisenberg group. Our distortion estimates allow us to generalize (1.8) to even more general product spaces (thus no Poincaré inequality or even connectivity of the source space is assumed).

Theorem 1.5. *Suppose E is Ahlfors d -regular, F is doubling, and f is a quasisymmetric map on $E \times F$ with $\dim_H(f(E \times F)) \leq D'$. Then for every number $d' \leq D'$,*

$$\mathcal{H}^{\frac{d}{d'}D'-d}(\{y \in F : \dim_H f(E \times \{y\}) > d'\}) = 0. \tag{1.9}$$

Theorem 1.5 is proven in Section 6. Note that for $E = \mathbb{R}^n$ and $F = \mathbb{R}^{N-n}$, and $f: \mathbb{R}^N \rightarrow \mathbb{R}^N$ quasiconformal, we recover the result of Balogh, Monti and Tyson (1.8). However, note also that (1.9) not only generalizes (1.8) by allowing the set E to have arbitrary dimension d , but also by not requiring to have $d' \geq d$. The latter inequality of course holds necessarily in the case of (1.8), since $\dim_H(f(y + \mathbb{R}^n)) \geq n$ for every homeomorphism. In our case, on the other hand, it is possible that $\dim_H(f(E \times \{y\})) < d$ and moreover $\dim_H(f(E \times F)) \leq D' < \dim_H(E \times F)$ in which

case Theorem 1.5 estimates the dimension of y 's for which $\dim_H f(E \times \{y\})$ is greater than some $d' < \dim_H f(E \times F)$. For instance, let E be the $1/2$ dimensional “ $1/4$ -Cantor set” obtained from the unit interval $[0, 1]$ by dividing it into 4 equal intervals, removing the middle two intervals and repeating the process with the remaining intervals. Then $E \times E$ is the well known four-corner Cantor set of dimension 1, which has conformal dimension 0, see e.g. [Mac11, Theorem 1.4] or [Tys01], and therefore there is a quasisymmetric map f such that $\dim_H f(E \times E) < 1$. Theorem 1.5 then implies that the set of y 's such that $\dim_H f(E \times \{y\}) = \dim_H f(E \times E)$ is 0-dimensional.

As a consequence of Theorem 1.5, we will prove in Section 6 the following bound on the infimal dimension distortion of the fibers.

COROLLARY 1.6. *Let E, F , and f satisfy the assumptions of Theorem 1.5. Then*

$$\inf_{y \in F} \dim_H f(E \times \{y\}) \leq \frac{\dim_H f(E \times F)}{\dim_H(E \times F)} \cdot \dim_H(E). \quad (1.10)$$

Corollary 1.6 is already interesting when $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is quasiconformal, and E and F lie in the coordinate axes.

For example, consider a Borel set $S \subseteq \mathbb{R}$, with $t = \dim_H(S)$. Applying the corollary to the case $E = \mathbb{R}$ and $F = S$, inequality (1.10) becomes

$$\inf_{y \in S} \dim_H f(\mathbb{R} \times \{y\}) \leq \frac{2}{t+1}. \quad (1.11)$$

On the other hand, if we additionally assume that S is Ahlfors t -regular, then we may apply Corollary 1.6 with $E = S$ and $F = \mathbb{R}$, and obtain (interchanging the order of the factors)

$$\inf_{x \in \mathbb{R}} \dim_H f(\{x\} \times S) \leq \frac{2t}{t+1}. \quad (1.12)$$

1.4 Sharpness of dimension distortion bounds in the plane. Theorem 1.5 and Corollary 1.6 are quite sharp in the planar case. Our next result establishes the optimality of estimates (1.11) and (1.12) in one fell swoop, as consequences of a stronger result.

Theorem 1.7. *For every $0 < t < 1$, there is an Ahlfors t -regular Cantor set $S \subseteq \mathbb{R}$ such that for each $\epsilon > 0$, there is a quasiconformal mapping $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, so that for every Borel subset $A \subseteq \mathbb{R} \times S$,*

$$\dim_H(f(A)) \geq (1 - \epsilon) \frac{2\dim_H(A)}{t+1}. \quad (1.13)$$

In particular,

$$\inf_{y \in S} \dim_H f(I \times \{y\}) \geq (1 - \epsilon) \frac{2}{t+1}, \quad (1.14)$$

$$\inf_{x \in I} \dim_H f(\{x\} \times S) \geq (1 - \epsilon) \frac{2t}{t+1}, \quad (1.15)$$

for every interval $I \subset \mathbb{R}$.

This will be proven in Section 7. As a corollary to Theorem 1.7, we will also obtain the sharpness of (1.8) for $N = 2$ and $n = 1$, and answer [BMT13, Problem 6.3] in the affirmative for the planar case.

COROLLARY 1.8. *For every $d' > 1$ and $\eta > 0$ there is a quasiconformal map f of the plane such that*

$$\dim_H\{y \in \mathbb{R} : \dim_H f([a, b] \times \{y\}) > d', \forall [a, b] \subseteq \mathbb{R}\} \geq \frac{2}{d'} - 1 - \eta. \tag{1.16}$$

Sharpness of rectifiability for images of lines. It is instructive to consider the preceding results in the context of lines parallel to the x -axis. By the ACL property for quasiconformal maps, if $f([0, 1] \times \{y\})$ has infinite length for every $y \in S \subseteq [0, 1]$, then $\mathcal{H}^1(S) = 0$. It is known that this is rather sharp; Heinonen and Rhode showed there are examples where $\dim_H(S) = 1$ [HR93, Theorem 1.9]. Even so, in that result the images $f(\mathbb{R} + y)$ contain many rectifiable subarcs. The question of how many lines have purely unrectifiable images, i.e., images containing no rectifiable subarcs, has proved to be a much thornier matter. Kovalev and Onninen [KO09] proved that given any countable collection \mathcal{L} of parallel lines in \mathbb{R}^2 there is planar quasiconformal image of \mathcal{L} that contains no rectifiable subarcs, but until now, it was not known if this could be extended to uncountable families. Problem 6.4 of [BMT13] and Problem 5.3 of [CTW12] ask if this can even be improved at all, i.e., whether there is even a single uncountable family of lines with this property.

Corollary 1.8 above already answers this question with a resounding yes; if, for every $y \in S$, we require that $f(\mathbb{R} \times \{y\})$ have no rectifiable subarcs, then the corollary tells us not only that S can be uncountable, but that $\dim_H(S)$ can be arbitrarily close to 1, and furthermore, not only may the images of subarcs be unrectifiable, but they can be taken to have dimension uniformly bounded away from 1. Moreover, from Theorem 1.7 itself, we see that the dimension distortion factor may be very strongly uniform—it can be taken to apply not only to subintervals, but to arbitrary Borel subsets of the product $S \times \mathbb{R}$.

If we insist on uniform dimension expansion, then Corollary 1.6 shows, via estimate (1.11), that this is the best we can do—the set S in the preceding result cannot have dimension 1, in contrast to [HR93, Theorem 1.9].

Our next result shows that if we sacrifice dimension distortion, and merely ask for unrectifiability of subarcs of the images of lines, then S can indeed have dimension 1, and can in a certain sense be as large as possible, giving a different, yet equally vociferous, “yes” to [BMT13, Problem 6.4].

Theorem 1.9. *For every increasing function h on $[0, \infty)$ such that*

$$\limsup_{t \rightarrow 0} \frac{h(t)}{t} = \infty, \tag{1.17}$$

there is a compact set $S \subset [0, 1]$ and a quasiconformal map f so that

- (1) *The quasiconformal constant of f is bounded independently of h and S .*

- (2) S has infinite Hausdorff measure with respect to h (cf. Definition 2.2 below).
 (3) $f([0, 1] \times \{y\})$ contains no rectifiable subarc for any $y \in S$.

Theorem 1.9 will be proven in Section 8. In particular, taking $h(t) = t|\log t|$ in the preceding theorem produces a compact set $S \subset [0, 1]$ of Hausdorff dimension 1 and a quasiconformal map f of the plane so that $f([0, 1] \times \{y\})$ contains no rectifiable subarcs for any $y \in S$.

Note that by the ACL property, the condition (1.17) on the gauge function h is sharp.

Conformal dimension. Rewriting inequality (1.10) as follows

$$\inf_{y \in F} \frac{\dim_H f(E \times \{y\})}{\dim_H(E \times \{y\})} \leq \frac{\dim_H f(E \times F)}{\dim_H(E \times F)}, \quad (1.18)$$

we obtain the principle “*fiberwise expansion implies global expansion*”: if every fiber $E \times \{y\}$ has its dimension increased by a factor $\alpha \geq 1$, then the dimension of the whole product $E \times F$ increases by at least a factor of α as well.

This principle has an immediate implication when considering conformal dimension. Recall that the *conformal dimension* of a metric space is the infimal Hausdorff dimension of its image under any quasisymmetric map, i.e.,

$$\dim_C X = \inf_{f \in QS(X)} \dim_H f(X),$$

where $QS(X)$ is the class of all quasisymmetric maps on X . In the event that E is minimal for conformal dimension (i.e., $\dim_H E = \dim_C E$), the inequality (1.18) then implies $E \times F$ is minimal as well. When $E = \mathbb{R}$ this gives a well known result of Tyson [Tys00]. Also see the discussion after Corollary 4.4 for more general results in this vein.

REMARK 1.10. We cannot reverse (1.18) by replacing the infimum by a supremum. Consider a quasiconformal map f that maps $\{0\} \times \mathbb{R}$ to a curve of dimension $D > 1$, but is smooth elsewhere. If $F = \mathbb{R}$ and $E \subset \mathbb{R}$ contains 0 and has dimension $0 < d < D - 1$, then the right side of (1.18) is at least $D/(d + 1) > 1$, but the smoothness of f off of $\{0\} \times \mathbb{R}$ implies $\dim f(E \times \{y\})/\dim(E \times \{y\}) = 1$ for every y . In other words “global expansion does not imply fiberwise expansion”.

This paper is organized as follows. In Section 2 we review the necessary definitions and preliminary results needed in the paper. In Section 3 we define the various versions of modulus and prove Theorem 3.8 assuming Theorem 1.2. Theorem 1.1 is a particular case of Theorem 3.8. In Section 4 we state more general versions of Theorem 1.4, and some corollaries; the reader will observe that Theorem 1.4 is a special case of Corollary 4.4. In Sections 5–7 we prove the remaining theorems and corollaries given in this introduction. In Section 9 we list some open problems.

2 Measures and Mappings

2.1 Measures, dimension and Ahlfors regularity. Unless otherwise stated, the metric spaces in this paper are assumed to be separable. Given a metric space $X = (X, d_X)$ we will denote by $B(x, r)$ the closed ball in X of radius r centered at x . When $B = B(x, r)$ and $k > 0$, we denote by kB the ball $B(x, kr)$. If the metric space X is clear from the context we will often denote the metric d_X by d . The following covering lemma is well known, cf. Theorem 1.2 in [Hei01], and we will use it repeatedly in the proof of Theorem 4.1.

LEMMA 2.1 (Covering lemma). *Every family \mathcal{B} of balls of uniformly bounded diameter in a metric space X contains a countable subfamily of disjoint balls $B_i \subset \mathcal{B}$ such that*

$$\bigcup_{B \in \mathcal{B}} B \subset \bigcup_i 5B_i.$$

The space X is said to be *doubling* if there is a constant C such that every ball in X may be covered by C balls of half the radius.

Throughout the paper, the term “measure” refers to a Borel regular outer measure. A measure λ on X is *locally finite* if every $x \in X$ lies in a neighborhood $U \subset X$ with $\lambda(U) < \infty$. λ is *doubling* if there is a constant C such that for each ball $B \subseteq X$, $\lambda(2B) \leq C\lambda(B)$.

We are particularly interested in (generalized) Hausdorff measures, whose definition we now recall, cf. [Mat95], p 60.

DEFINITION 2.2. *Given a non-negative function $h: [0, \infty) \rightarrow [0, \infty)$, and a subset $E \subseteq X$, the Hausdorff h -measure of E is defined as follows. For every $\varepsilon \in (0, \infty]$, let*

$$\mathcal{H}_\varepsilon^h(E) = \inf \left\{ \sum_{i=1}^{\infty} h(r_i) : E \subset \bigcup_{i=1}^{\infty} B(x_i, r_i), r_i < \varepsilon \right\},$$

and

$$\mathcal{H}^h(E) = \lim_{\varepsilon \rightarrow 0} \mathcal{H}_\varepsilon^h(E).$$

When $h(r) = r^t$, $t \geq 0$, the resulting measure is called the *t -dimensional Hausdorff measure* and is denoted by \mathcal{H}^t .

The *Hausdorff dimension* of X is

$$\dim_H(X) = \inf \{ t : \mathcal{H}^t(X) = 0 \}.$$

A subset $E \subset X$ is *Ahlfors d -regular*, if there is a constant $C_E \geq 1$ such that for every $x \in E$ and $0 < r < \text{diam}E$ the following inequalities hold

$$\frac{1}{C_E} r^d \leq \mathcal{H}^d(E \cap B(x, r)) \leq C_E r^d. \quad (2.1)$$

We will denote by $\mathcal{A}_d(X)$ the collection of all bounded Ahlfors d -regular subsets of X , that is for every $E \in \mathcal{A}_d(X)$ the inequalities (2.1) hold for some constant $C_E \geq 1$.

We will denote the support of a measure λ on X by E_λ . We say that λ is *Ahlfors d -regular*, $d > 0$, if there are constants $C_\lambda \geq 1$ and $r_\lambda > 0$ such that

$$C_\lambda^{-1}r^d \leq \lambda(B(x, r)) \leq C_\lambda r^d, \quad (2.2)$$

for every ball $B(x, r)$ centered at $x \in E_\lambda$ with radius $r < r_\lambda$. More generally, we say λ is *upper or lower d -regular* if only the right or left inequality in (2.2) holds, respectively. We denote by $\mathcal{L}_d(X)$ the family of lower d -regular measures in X .

REMARK 2.3. If λ is d -regular, then so is the restricted Hausdorff measure $\mathcal{H}^d|_{E_\lambda}$ [Hei01, Exercise 8.11], so that if we were only interested in the two-sided regularity condition, there would be no special reason to consider d -regular measures rather than sets. On the other hand, by itself, mere upper (resp. lower) regularity of λ does not imply the same condition for $\mathcal{H}^d|_{E_\lambda}$. Whereas the full two-sided Ahlfors regularity condition is rather strong, the existence of upper and lower regular measures holds in rather great generality—the former may be obtained via the Frostman lemma (Lemma 2.5 below), and the latter exist on compact doubling metric spaces, via a theorem of Vol’berg and Konyagin [VK88]¹.

The upper regular measures given by the Frostman lemma are crucial to our applications to product spaces in Theorem 1.5 and Corollary 1.6. Also, the arc-length measure of a curve satisfies the lower regularity condition, though not necessarily the upper, so that in order to view our results as a generalization of facts about curve modulus, we must consider lower regular measures. It is for these reasons that our most general results, Theorem 4.1 and Corollary 4.2, are formulated in terms of families of measures, not sets.

We refer to [Mat95, Hei01] for proofs of the next two lemmas and for further discussion of Hausdorff measures, dimension and Ahlfors regular spaces and their properties.

LEMMA 2.4 (Mass distribution principle). *If the metric space X supports a positive upper D -regular Borel measure, then $\mathcal{H}^D(X) > 0$. In particular, $\dim_H(X) \geq D$.*

An important converse to the mass distribution principle is the following lemma, see [Mat95, Theorem 8.8].

LEMMA 2.5 (Frostman’s lemma). *If X is a doubling metric space, and $F \subseteq X$ is a Borel set such that $\mathcal{H}^s(F) > 0$, then there is an upper s -regular measure ν supported on F such that $\nu(F) > 0$.*

¹ Note that the results in [VK88] are formulated in terms of “homogeneous” measures, but on a bounded set such measures are easily seen to be lower regular.

REMARK 2.6. Frostman’s Lemma is often stated for the special case $X = \mathbb{R}^n$. However, even if (X, d_X) is only a doubling metric space, the lemma is easily obtained from the Euclidean case via Assouad’s embedding theorem [Ass83]. To see this, we first denote by X^t the metric space (X, d_X^t) for $t \in (0, 1)$, i.e. the t -snowflaked version of X . Now, if $\mathcal{H}^s(F) > 0$ for some $F \subset X$ then by Assouad’s embedding theorem [Ass83] for every $s' > s$ there is a bi-Lipschitz embedding $j: X^{s/s'} \hookrightarrow \mathbb{R}^n$ for some $n \in \mathbb{N}$. Since j is bi-Lipschitz, we have

$$\mathcal{H}^{s'}(j(F^{s/s'})) \cong \mathcal{H}^{s'}(F^{s/s'}) = \mathcal{H}^s(F) > 0,$$

where $F = (F, d_X|_F)$. Assuming Frostman’s Lemma for \mathbb{R}^n , we have that $j(F^{s/s'}) \subset \mathbb{R}^n$ supports a nontrivial, upper s' -regular measure. Since j is bi-Lipschitz, the space $F^{s/s'}$ also supports an upper s' -regular measure μ such that $\mu(F^{s/s'}) > 0$. Then, if ν is defined on F as the pullback of μ under the snowflaking, i.e. $\nu(E) = \mu(E^{s/s'})$ for every Borel set $E \subset F$, then for every ball $B_X(x, r) \subset X$ we have

$$\begin{aligned} B_X(x, r) &= \{y \in X : d_X(x, y) < r\} = \{y \in X : (d_X(x, y))^{s/s'} < r^{s/s'}\} \\ &= \{y \in X^{s/s'} : d_{X^{s/s'}}(x, y) < r^{s/s'}\} = B_{X^{s/s'}}(x, r^{s/s'}), \end{aligned}$$

and therefore

$$\nu(B_X(x, r)) = \mu(B_{X^{s/s'}}(x, r^{s/s'})) \leq C(r^{s/s'})^{s'} = Cr^s.$$

Thus F supports an upper s -regular measure ν , such that $\nu(F) = \mu(F^{s/s'}) > 0$.

2.2 Quasiconformal and quasisymmetric mappings. Given a homeomorphism $f: X \rightarrow Y$, $x \in X$ and $r > 0$ we let

$$H_f(x, r) = \frac{\sup_{d_X(x,y) \leq r} d_Y(f(x), f(y))}{\inf_{d_X(x,y) \geq r} d_Y(f(x), f(y))}.$$

The mapping f is called (metrically) *quasiconformal* if there is a constant $H < \infty$ such that

$$\limsup_{r \rightarrow 0} H_f(x, r) \leq H \tag{2.3}$$

for every $x \in X$.

Because quasiconformality is an infinitesimal property it is often hard to work with directly. For this reason one often requires a stronger, global condition from a mapping f , which we discuss next.

Let $\eta: [0, \infty) \rightarrow [0, \infty)$ be a fixed homeomorphism. A homeomorphism f between metric spaces (X, d_X) and (Y, d_Y) is called η -*quasisymmetric* if for all distinct triples $x, y, z \in X$ we have

$$\frac{d_Y(f(x), f(y))}{d_Y(f(y), f(z))} \leq \eta \left(\frac{d_X(x, y)}{d_X(y, z)} \right). \tag{2.4}$$

Quasisymmetric mappings do not distort (macroscopic) shapes too much. In particular an image of a round ball will be “roundish”, a condition which a priori holds for QC maps only on small scales depending on x .

A mapping f of a metric space (X, d_X) into (Y, d_Y) is called a *quasisymmetric embedding* if f is a quasisymmetric map of X onto $f(X) \subseteq Y$, where the metric on $f(X)$ is the restriction of the metric of Y .

It follows almost immediately from the definition that quasisymmetric maps do not distort annuli too much. More precisely, we have the following easy result, which is very similar to Lemma 3.1 in [Tys98], and which we will use in the proof of Theorem 4.1 below.

LEMMA 2.7. *If f is an η -quasisymmetric map of a separable metric space X , then for every closed ball $B' = B(y, s) \subset f(X)$, with $y = f(x)$, there is a closed ball $B = B(x, r)$ such that for each $k \geq 1$, the following inclusions hold:*

$$B' \subseteq f(B) \subseteq f(kB) \subseteq \eta(k)B'.$$

Furthermore, if f^{-1} is uniformly continuous, with modulus of continuity $\omega(t)$, then we may choose r so that $r \leq \omega(s)$.

Proof. Let $r > 0$ be the smallest number such that the first inclusion holds. Then for each $\alpha < 1$, and a point $x_\alpha \in f^{-1}(B') \setminus \alpha B$, we let $y_\alpha = f(x_\alpha) \in B' \setminus f(\alpha B)$. Therefore, whenever $y_1 = f(x_1)$, with $x_1 \in kB$, we have

$$d(y, y_1) \leq d(y, y_\alpha) \eta \left(\frac{d(x, x_1)}{d(x, x_\alpha)} \right) \leq s \eta \left(\frac{k}{\alpha} \right).$$

Passing to the limit as α goes to 1, we see that $d(y, y_1) \leq s \eta(k)$, from which the last inclusion follows. Finally, when f^{-1} has modulus of continuity ω , the choice of r implies $r \leq \omega(s)$. \square

As a consequence of Lemma 2.7 we obtain the following version of the mass distribution principle where we require an upper estimate of μ only for a limited collection of subsets.

LEMMA 2.8. *Let f be a quasisymmetric map of a metric space X . If there exist constants $C_1, C_2 \geq 1$, an integer $L \in \mathbb{N}$ and a measure μ on $f(X)$ such that*

- *there is a collection of sets $\mathcal{Q} = \{Q_i\}_{i \in \mathbb{N}}$ s.t. every ball $B = B(x, r)$ can be covered by L members of \mathcal{Q} , Q_{i_1}, \dots, Q_{i_L} , so that for $k = 1, \dots, L$ we have*

$$\text{diam} Q_{i_k} \leq C_1 r,$$

- *for every $i \geq 1$ we have $\mu(f(Q_i)) \leq C_2 \text{diam}(f(Q_i))^s$, for some $s > 0$,*

then $\dim_H f(X) \geq s$.

Proof. Let $B' = B(f(x), R)$. We want to show that $\mu(B') \lesssim R^s$. Let B be the ball containing $f^{-1}(B')$ given by Lemma 2.7. Since $Q_{i_k} \subset (1 + C_1)B$, from Lemma 2.7 we have for $k = 1, \dots, L$

$$f(Q_{i_k}) \subset f((1 + C_1)B) \subset \eta(1 + C_1)B'.$$

Therefore

$$\begin{aligned} \mu(B') &\leq \mu(f(B)) \leq \mu\left(\bigcup_{k=1}^L f(Q_{i_k})\right) \leq \sum_{k=1}^L \mu(f(Q_{i_k})) \\ &\leq C_2 \sum_{k=1}^L (\text{diam} f(Q_{i_k}))^s \leq C_2 \sum_{k=1}^L (\eta(1 + C_1) \cdot 2R)^s = CR^s, \end{aligned}$$

where $C = C_2 L [2\eta(1 + C_1)]^s$. Applying the mass distribution principle completes the proof. \square

Even though quasismmetry is a stronger condition than quasiconformality, it is often the case that the two notions coincide. For instance if a homeomorphism $f : \mathbb{R}^N \rightarrow \mathbb{R}^N$, $N \geq 2$ is QC then it is also QS. This was proved by Gehring for $N = 2$ in [Geh60] and by Väisälä for $N \geq 3$ [Väi61]. More recently Heinonen and Koskela extended this equivalence to a large class of metric spaces [HK98, HK95].

3 Modulus

The main tool used in this paper is the modulus of a family of curves, sets, or measures. In this section we review the definitions and basic properties for each of these concepts. In the case when the underlying measure space is a locally compact topological group G , and λ is any measure on G , we estimate from below the modulus of the family of translates of λ by elements $g \in K$, where K is any subset of G . This estimate is vital for the proof of Theorem 3.8.

3.1 Modulus of curve families. Given a metric measure space (X, μ) , a family of curves Γ in X and a real number $p \geq 1$ the p -modulus of Γ is defined as

$$\text{mod}_p \Gamma = \inf_{\rho} \int_X \rho^p d\mu,$$

where the infimum is taken over all Γ -admissible nonnegative Borel functions ρ . Here a function $\rho : X \rightarrow [0, \infty)$ is Γ -admissible if $\int_{\gamma} \rho ds \geq 1$ for every locally rectifiable curve $\gamma \in \Gamma$, where ds denotes the arlength element. We say that a property holds for p -almost every curve in X if it fails only for a curve family Γ such that $\text{mod}_p \Gamma = 0$. Notice that by definition, almost every curve is locally rectifiable. We refer to [GM05, Hei01, HK98] for further details on modulus of curve families including the definitions of rectifiability and arlength in general metric spaces.

Despite superficial differences, proofs of the aforementioned equivalence between the definitions of metric quasiconformality and quasismmetry, no matter the level

of generality, tend to follow the same broad outline: the metric definition is used, with the help of various covering arguments, to establish quasi-invariance of the conformal modulus of path families, and this invariance, along with modulus estimates for certain families, facilitates geometric arguments that yield the global distortion estimate (2.4).

As a result, one obtains the following equivalent definition of quasiconformality, the so-called “geometric” definition.

Given $D > 1$ and $K \geq 1$, a homeomorphism $f: X \rightarrow Y$ between Ahlfors D -regular spaces is called (*geometrically*) K -*quasiconformal* if for every family of curves Γ in X the following inequalities hold

$$K^{-1} \operatorname{mod}_D f(\Gamma) \leq \operatorname{mod}_D \Gamma \leq K \operatorname{mod}_D f(\Gamma), \quad (3.1)$$

where $f(\Gamma)$ denotes the image of the family Γ under f , see [Ahl54, Väi71].

For proofs of the equivalence of geometric quasiconformality to the metric definition, and to quasisymmetry, in various levels of generality, we refer the reader again to [Geh60, HK98, HK95, Väi61].

3.2 Modulus of families of measures. The notion of modulus can be extended far beyond the context of curve families. The modulus of a family of measures, with respect to an underlying measure, was defined and studied by Fuglede [Fug57] and Ziemer [Zie69]. In [Hak10] the second author used Fuglede’s modulus to study conformal dimension of various spaces. More recently, Badger studied extremal metrics and Beurling criterion for families of measures [Bad13].

Let (X, μ) be a metric measure space and $p \geq 1$. Let \mathcal{L} be a collection of measures on X . A Borel function $\rho: X \rightarrow [0, \infty]$ is said to be admissible for \mathcal{L} if

$$\int_X \rho d\lambda \geq 1$$

for every $\lambda \in \mathcal{L}$. The p -modulus of \mathcal{L} is

$$\operatorname{mod}_p(\mathcal{L}, \mu) = \inf_{\rho \text{ adm}} \int_X \rho^p d\mu,$$

where the infimum is taken over all \mathcal{L} -admissible functions ρ . Often we simply write $\operatorname{mod}_p(\mathcal{L})$, when μ is clear from context.

Next, we summarize some of the properties of modulus that will be useful for us.

LEMMA 3.1. *For every $p \geq 1$ the following properties hold.*

- (1) (MONOTONICITY) $\operatorname{mod}_p \mathcal{L} \leq \operatorname{mod}_p \mathcal{L}'$, if $\mathcal{L} \subset \mathcal{L}'$.
- (2) (SUBADDITIVITY) $\operatorname{mod}_p \mathcal{L} \leq \sum_i \operatorname{mod}_p \mathcal{L}_i$, if $\mathcal{L} = \bigcup_{i=1}^{\infty} \mathcal{L}_i$.
- (3) (ZIEMER’S LEMMA) If $1 < p < \infty$, $\mathcal{L}_1 \subset \mathcal{L}_2 \subset \dots$ are families of measures and $\mathcal{L} = \bigcup_{i=1}^{\infty} \mathcal{L}_i$ then $\operatorname{mod}_p \mathcal{L} = \lim_{i \rightarrow \infty} \operatorname{mod}_p \mathcal{L}_i$.

See [Fug57] for (1) and (2). Property (3) is due to Ziemer for families of continua in \mathbb{R}^N , see Lemma 2.3 in [Zie69]. The proof in the case of general measure families is the same as in [Zie69]. It is important to emphasize here that Ziemer’s lemma holds only under the assumption $p > 1$.

We say a property holds for mod_p -almost every $\lambda \in \mathcal{L}$ if it fails only for a family $\mathcal{L}_0 \subset \mathcal{L}$ such that $\text{mod}_p(\mathcal{L}_0) = 0$.

3.3 Modulus of families of Ahlfors regular sets. The notion of modulus defined above is quite general, but also a bit technical and abstract, and presumes we have at hand not merely a single underlying measure, but a family of other measures as well. In the greatest generality, this complication is unavoidable—we cannot speak of the modulus of a family of sets without some measures on hand to formulate the admissibility condition. Our motivation for working in such generality was discussed earlier in Remark 2.3.

Ahlfors regularity, on the other hand, is fundamentally a metric notion, in the sense that the existence of any Ahlfors d -regular measure on a set E is equivalent to d -regularity of the Hausdorff measure $\mathcal{H}^d|_E$, and the latter property is determined entirely by the metric. With this in mind, given any family $\mathcal{E} \subseteq \mathcal{A}_d(X)$ of Ahlfors d -regular sets, we define the p -modulus of \mathcal{E} to be

$$\text{mod}_p(\mathcal{E}, \mu) = \text{mod}_p(\{\mathcal{H}^d|_E\}_{E \in \mathcal{E}}, \mu).$$

As before, when μ is clear from context, we omit it. In particular, we do this when X is D -regular and $\mu = \mathcal{H}^D$. The dimension d will always be clear from context, so we do not include it in the notation either, and in any case, a given set can be Ahlfors d -regular for at most one value of d .

Finally, the most important modulus in (quasi)-conformal geometry, when considering d -dimensional subsets of D -dimensional spaces, is the conformal modulus $\text{mod}_{D/d}$. Thus when $\mathcal{E} \subseteq \mathcal{A}_d(X)$, and X is Ahlfors D -regular, we unambiguously define $\text{mod}(\mathcal{E}) := \text{mod}_{D/d}(\mathcal{E})$, and simply say “ $\text{mod}_{D/d}$ almost every $E \in \mathcal{E}$ ” to refer to a property that holds for all $E \in \mathcal{E} \setminus \mathcal{E}_0$, for some subfamily \mathcal{E}_0 with $\text{mod}(\mathcal{E}_0) = 0$.

3.4 Modulus and products. Let (E, λ) and (F, ν) be two metric measure spaces with $0 < \lambda(E) < \infty$. Denote $X = E \times F$ and $\mu = \lambda \times \nu$. Let $\mathcal{F} = \{E \times \{y\} : y \in F\}$, and for each $y \in F$, let λ_y be the pushforward of λ by the map $x \mapsto (x, y)$, so that for each Borel set $A \subseteq X$, $\lambda_y(A) = \lambda(\{x \in E : (x, y) \in A\})$.

LEMMA 3.2. *With the notation as above, let $\mathcal{L}_F = \{\lambda_y : y \in F\}$. Then for every $p \geq 1$ we have*

$$\text{mod}_p \mathcal{L}_F = \frac{\mu(X)}{\lambda(E)^p} = \frac{\nu(F)}{\lambda(E)^{p-1}}. \tag{3.2}$$

Proof. This proof is the same as in the classical case of curve families. We give it here for completeness. First note that since the function $\rho(x, y) \equiv \lambda(E)^{-1}$ is admissible

for \mathcal{L}_F , we have $\text{mod}_p \mathcal{L}_F \leq \frac{\mu(X)}{\lambda(E)^p}$. To obtain the lower bound, note that for every \mathcal{L}_F -admissible ρ we have $\int_E \rho(x, y) d\lambda \geq 1, \forall y \in Y$, and therefore by Hölder's inequality we obtain that for every $y \in Y$ the following holds

$$1 \leq \lambda(E)^{p-1} \int_E \rho^p(x, y) d\lambda.$$

Integrating both sides of this inequality with respect to ν we obtain

$$\nu(F) \leq \lambda(E)^{p-1} \int_X \rho^p(x, y) d\mu,$$

and therefore

$$\frac{\mu(X)}{\lambda(E)^p} = \frac{\nu(F)\lambda(E)}{\lambda(E)^p} \leq \int_X \rho^p(x, y) d\mu.$$

Hence $\text{mod}_p \mathcal{L}_F \geq \frac{\mu(X)}{\lambda(E)^p}$. □

3.5 Modulus and group translations. In this subsection we consider another example of a family of measures - a family of translates of a given measure λ by elements of a set $K \subset G$, where G is a topological group, which in particular could be \mathbb{R}^n . We will first show that the modulus of the family of translates of λ depends on the measure of K , and then will consider an example of translates of the Cantor set in the real line \mathbb{R} .

Suppose $G = (G, \nu)$ is a locally compact topological group, with right invariant Haar measure ν . Let λ be another measure on G . For each $y \in G$, denote by $y_*\lambda$ the pushforward of λ by left multiplication by y .

If $K \subseteq G$ is measurable, let $K_*\lambda = \{y_*\lambda: y \in K\}$.

LEMMA 3.3. For every $p \geq 1$ we have

$$\text{mod}_p(K_*\lambda, \nu) \geq \frac{\nu(K)}{\lambda(G)^p}. \quad (3.3)$$

Proof of Lemma 3.3. Let $\rho: G \rightarrow [0, \infty]$ be admissible for the family $K_*\lambda$, and fix $y \in K$. Since translation by y maps G to itself, we have $y_*\lambda(G) = \lambda(G)$. Thus by Hölder's inequality,

$$\int_G \rho(yx)^p d\lambda(x) = \int_G \rho^p dy_*\lambda \geq \frac{(\int_G \rho dy_*\lambda)^p}{\lambda(G)^{p-1}} \geq \frac{1}{\lambda(G)^{p-1}}.$$

Thus we obtain

$$\begin{aligned} \lambda(G) \int_G \rho^p d\nu &= \int_G \int_G \rho(y)^p d\nu(y) d\lambda(x) = \int_G \int_G \rho(yx)^p d\nu(y) d\lambda(x) \quad (\text{right invariance of } \nu) \\ &= \int_G \int_G \rho(yx)^p d\lambda(x) d\nu(y) \quad (\text{Fubini's theorem}) \\ &\geq \int_K \int_G \rho(yx)^p d\lambda(x) d\nu(y) \\ &\geq \int_K \left(\frac{1}{\lambda(G)^{p-1}} \right) d\nu = \frac{\nu(K)}{\lambda(G)^{p-1}}. \end{aligned}$$

Dividing each side by $\lambda(G)$ and infimizing over all admissible functions ρ , we obtain the desired inequality (3.3). \square

Using the terminology of Section 3.3 we have the following consequence of Lemma 3.3.

COROLLARY 3.4. *Let $n \geq 1$, $0 < d < n$. If E is a nonempty, bounded Ahlfors d -regular subset of \mathbb{R}^n and $K \subseteq \mathbb{R}^n$ is a Lebesgue measurable set “of translates” of E , then the family of translates $\{y + E : y \in K\}$ has positive p -modulus, for any $p \geq 1$, whenever K has positive n -dimensional Lebesgue measure. More precisely, if $\mathcal{H}^n(K) > 0$ then*

$$\text{mod}_p(\{\mathcal{H}^d \llcorner_{y+E} : y \in K\}, \mathcal{H}^n) > 0,$$

for every $p \geq 1$.

Proof. Let $(G, \nu) = (\mathbb{R}^n, \mathcal{H}^n)$ and $\lambda = \mathcal{H}^d \llcorner_E$. Then $0 < \lambda(G) = \mathcal{H}^d(E) < \infty$ since E is a nonempty, bounded Ahlfors regular set. Moreover $y_*\lambda = \mathcal{H}^d \llcorner_{y+E}$. The result follows immediately from inequality (3.3). \square

REMARK 3.5. Note that the converse of Corollary 3.4 is not true; it is possible to have a set $K \subset \mathbb{R}^2$ of zero Lebesgue measure such that the family $\{y + E : y \in K\}$ has positive modulus. Indeed, if $E = [0, 1]$, $\lambda = \mathcal{H}^1 \llcorner_{[0,1]}$ and K is the vertical segment of length one, connecting the origin to the point $(0, 1) \in \mathbb{R}^2$, then the family of translates $K_*\lambda$ coincides with the product family $\mathcal{L}_{[0,1]}$ as in Lemma 3.2, i.e. with the family of restrictions of the 1-dimensional Lebesgue measure to the horizontal segments $[0, 1] \times \{y\}$, with $y \in [0, 1]$. But $\text{mod}_p(\mathcal{L}_{[0,1]}, \mathcal{H}^2) = 1$ for every $p \geq 1$, by Lemma 3.2, even though $\mathcal{H}^2(K) = 0$.

3.6 Modulus and Minkowski sum. From the previous remark it follows that the positivity of modulus of a family of translates $K_*\lambda$ of a measure λ is not characterized by the measure $\nu(K)$ of the set of translates. However, the example in that remark may lead the reader to think that one may be able to characterize the positivity of modulus (at least in \mathbb{R}^n) in terms of the measure of the Minkowski sum $K + E$, i.e. the union of the supports of $y_*\lambda$ as y runs through K . We will show that this is also not true. For this we will consider the middle thirds Cantor set $C \subset [0, 1]$ and the Bernoulli probability measure λ supported on C , and will show that there are two sets of translates $K_1, K_2 \subseteq [0, 1]$ and $p \geq 1$ such that

$$\text{mod}_p(K_{1*}\lambda, \mathcal{H}^1) > 0 \quad \text{and} \quad \text{mod}_p(K_{2*}\lambda, \mathcal{H}^1) = 0$$

even though

$$K_1 + C = K_2 + C.$$

Let $K_1 = [0, 1]$ then clearly $K_1 + C = [0, 2]$. Moreover, by Lemma 3.3 for every $p \geq 1$ we have

$$\text{mod}_p(K_{1*}\lambda, \mathcal{H}^1) \geq \frac{\mathcal{H}^1([0, 1])}{\lambda(C)^{p-1}} = 1 > 0.$$

Let $K_2 = C$. It is well known that $C + C = [0, 2]$, see e.g. [Ran40]. Next we show that $\text{mod}_p(K_{2*}\lambda, \mathcal{H}^1) = 0$ for some values of p .

LEMMA 3.6. *Let $C \subset I := [0, 1]$ be the middle-thirds Cantor set and λ be the Bernoulli probability measure on C . Then for $1 \leq p < \log_2 3$ we have*

$$\text{mod}_p(C_*\lambda, \mathcal{H}^1) = 0. \quad (3.4)$$

Proof. To find $\text{mod}_p(C_*\lambda, \mathcal{H}^1)$ we let

$$\rho_i = 2^i \chi_{[1-\frac{1}{3^i}, 1+\frac{1}{3^i}]}$$

for every $i \geq 1$. We will show ρ_i is admissible for $C_*\lambda$ for every $i \geq 1$. To see that, fix a point $y \in C$ and consider the measure $y_*\lambda$ supported on $y+C$. Note, that $1 \in y+C$, since $1-y \in C$ if $y \in C$. Next for $i \geq 1$ let J_i denote the i th generation interval of length 3^{-i} used in the standard construction of the Cantor set, which contains the point $1-y$. Then, by definition of λ we have that $\lambda(J_i) = 2^{-i}$. Moreover, since $y + J_i$ contains the point $1 \in \mathbb{R}$ we also have

$$y + J_i \subset \left[1 - \frac{1}{3^i}, 1 + \frac{1}{3^i}\right].$$

Thus,

$$\begin{aligned} \int \rho_i d(y_*\lambda) &= 2^i (y_*\lambda) \left(\left[1 - \frac{1}{3^i}, 1 + \frac{1}{3^i}\right] \right) \geq 2^i (y_*\lambda)(y + J_i) \\ &\geq 2^i \lambda(J_i) = 2^i 2^{-i} = 1, \end{aligned}$$

where we used the fact that $\lambda(J_i) = 2^{-i}$ on every i 'th generation interval J_i of the Cantor set C . Thus ρ_i is admissible for $C_*\lambda$ for every $i \geq 1$, and we estimate the modulus of $C_*\lambda$ as follows

$$\text{mod}_p(C_*\lambda, \mathcal{H}^1) \leq \int_{C+C} \rho_i^p d\mathcal{H}^1 \leq \int_0^2 \left(2^i \chi_{[1-\frac{1}{3^i}, 1+\frac{1}{3^i}]} \right)^p d\mathcal{H}^1 = 2 \left(\frac{2^p}{3} \right)^i \xrightarrow{i \rightarrow \infty} 0,$$

if $1 \leq p < \frac{\log 3}{\log 2}$. Thus, $\text{mod}_p(C_*\lambda, \mathcal{H}^1) = 0$ for $p \in [1, \log_2 3)$. \square

REMARK 3.7. Lemma 3.3 can be generalized to the case where λ is a measure on the semigroup \mathcal{F}_ν of ν -preserving transformations on a measure space (Y, ν) . In the above proof, instead of taking $x, y \in G$, one takes $\phi \in \mathcal{F}_\nu$, $y \in Y$, and integrates accordingly, replacing yx with $\phi(y)$. We leave the details to the interested reader.

3.7 Carnot groups and left translates. Lemma 3.3 allows us to generalize Theorem 1.1 from Euclidean space to Carnot groups, as was discussed in Section 1.1. For the proof we also assume Theorem 1.2, which will be proven in Section 5. We refer the reader to [HK00, Chapter 11] for definitions and background on Carnot groups in the context of metric space analysis.

Theorem 3.8. *Let $\mathbb{G} = (\mathbb{G}, \cdot)$ be a Carnot group of homogeneous dimension $Q > 1$, equipped with its left invariant Carnot–Carathéodory metric. Suppose $E \subset \mathbb{G}$ is a bounded q -Ahlfors regular set, $0 < q \leq Q$, and $f : \mathbb{G} \rightarrow \mathbb{G}$ is a quasiconformal mapping. Then*

$$\dim_H f(y \cdot E) = \dim_H E, \quad (3.5)$$

for \mathcal{H}^Q -a.e. $y \in \mathbb{G}$.

Proof of Theorem 3.8. We first prove the theorem in the case when E is a bounded set. Since the metric is left-translation invariant, the sets $y \cdot E$ are isometric to E , and hence Ahlfors q -regular. Moreover, the Hausdorff measure \mathcal{H}^Q is positive and locally finite, and left invariant. Since Carnot groups are unimodular (i.e., left and right Haar measures coincide), \mathcal{H}^Q is right invariant as well (though see Remark 3.9 below).

Let $K \subseteq \mathbb{G}$ be the set of points $y \in \mathbb{G}$ for which Equation (3.5) fails. Since left translations are isometries, the measures $y_* \mathcal{H}^q \llcorner_E = \mathcal{H}^q \llcorner_{y \cdot E}$ are all q -regular as well. By Theorem 1.2, we have $\text{mod}_{Q/q}(\{y \cdot E : y \in K\}, \mathcal{H}^Q) = 0$. Applying Lemma 3.3 with $G = \mathbb{G}$, $\nu = \mathcal{H}^Q$, $\lambda = \mathcal{H}^q \llcorner_E$, and $p = Q/q$, we have that $\mathcal{H}^Q(K) = 0$ as desired. \square

REMARK 3.9. In the preceding proof, we did not really need to use the fact that \mathbb{G} is unimodular. In any locally compact topological group, left and right Haar measures are comparable on compact subsets, so that right Haar measure ν is locally Ahlfors Q -regular. Theorem 1.2 easily generalizes to allow replacement of \mathcal{H}^Q with the locally Q -regular measure ν , so that

$$\text{mod}_{Q/q}(\{y \cdot E : y \in K\}, \nu) = 0.$$

Lemma 3.3 implies $\nu(K) = 0$ as before, so using again the fact that ν and \mathcal{H}^Q are locally comparable, one has $\mathcal{H}^Q(K) = 0$ as well.

4 General Versions of Non-Expansion

We begin with our most general (and technical) dimension distortion theorem, from which all of our upper bounds on dimension distortion derive. We recall that $\mathcal{L}_d(X)$ denotes the family of lower d -regular measures in X , and that we denote the support of a measure $\lambda \in \mathcal{L}_d(X)$ by E_λ .

Theorem 4.1. *Let $D > d > 0$, and $D' > d' > 0$, with $\frac{D}{d} \geq \frac{D'}{d'}$. Suppose that μ is an upper D -regular measure on a separable metric space X , and that $f: X \rightarrow Y$ is a quasisymmetric embedding.*

- (1) *If $\mathcal{H}^{D'} \llcorner_{f(X)}$ is locally finite, then for $\text{mod}_{D/d}$ -almost every $\lambda \in \mathcal{L}_d(X)$, $\mathcal{H}^{d'} \llcorner_{f(E_\lambda)}$ is locally finite.*
- (2) *If $\mathcal{H}^{D'}(f(X)) = 0$, then for $\text{mod}_{D/d}$ -almost every $\lambda \in \mathcal{L}_d(X)$, $\mathcal{H}^{d'}(f(E_\lambda)) = 0$.*

By fixing the values of D and d and varying D' and d' , we obtain the following corollary.

COROLLARY 4.2. *Let $D > d > 0$, and let μ , X , and f satisfy the assumptions of Theorem 4.1. Then for $\text{mod}_{\frac{D}{d}}$ -almost every $\lambda \in \mathcal{L}_d(X)$,*

$$\frac{\dim_H f(E_\lambda)}{\dim_H f(X)} \leq \frac{d}{D}. \quad (4.1)$$

In particular, if $\text{mod}_{D/d}(\mathcal{L}_d(X)) > 0$ as well, then there is a measure $\lambda \in \mathcal{L}_d(X)$ satisfying (4.1).

Theorem 4.1 and Corollary 4.2 will be proven in Section 5. Readers interested in analysis on metric spaces, particularly in terms of Newton–Sobolev theory, may wish to keep in mind the special case of curve modulus. In this setting, integration with respect to arc-length along a rectifiable curve $\gamma: [0, l] \rightarrow Y$ (parametrized by arc-length) is the same as integration with respect to the push-forward of Lebesgue measure, $\gamma_*(\mathcal{H}_1)$, which is easily seen to be a lower 1-regular measure. Since almost every curve is locally rectifiable, we may apply Corollary 4.2 to curve families.

COROLLARY 4.3. *Suppose that μ is an upper D -regular measure, $D > 1$, on a separable metric space X , and that $f: X \rightarrow Y$ is a quasisymmetric embedding. Then for mod_D -almost every curve γ in X ,*

$$\dim_H f(\gamma) \leq \frac{\dim_H f(X)}{D}. \quad (4.2)$$

Proof. For each locally rectifiable curve γ , the corresponding arclength measure λ is lower-regular and $\gamma = E_\lambda$. Moreover, in this case $d = 1$, so inequality (4.1) of Corollary 4.2 implies (4.2). \square

If X is Ahlfors D -regular, we may apply Corollary 4.2 to the family $\mathcal{A}_d(X)$ of bounded Ahlfors d -regular subsets of X , by letting $\mu = \mathcal{H}^D$ and observing that $\{\mathcal{H}^d \llcorner_E\}_{E \in \mathcal{A}_d(X)} \subseteq \mathcal{L}_d(X)$. Note that in this case $\dim_H E = d$ and $\dim_H X = D$, and recall from the previous section that in this context, the notion of “almost every d -regular set” is well-defined.

COROLLARY 4.4. *Let $D > d > 0$, let X be Ahlfors D -regular, and let $f: X \rightarrow Y$ be a quasisymmetric mapping. Then for $\text{mod}_{D/d}$ -almost every $S \in \mathcal{A}_d(X)$,*

$$\frac{\dim_H f(S)}{\dim_H S} \leq \frac{\dim_H f(X)}{\dim_H X}. \quad (4.3)$$

In particular, if Y is also D -dimensional, then $\text{mod}_{D/d}$ -almost every $S \in \mathcal{A}_d(X)$ satisfies

$$\dim_H f(S) \leq \dim_H S. \tag{4.4}$$

Proof. For each S , let λ be d -dimensional Hausdorff measure restricted to S , so $S = S_\lambda$. Then Corollary 4.2 immediately implies (4.3). \square

REMARK 4.5. Inequality (4.3) may be thought of as a generalization of the fiberwise expansion estimate (1.18) for products. Indeed, if both (E, λ) and (F, ν) are Ahlfors regular spaces then $(X, \mu) = (E \times F, \lambda \times \nu)$ is also Ahlfors regular and we may apply Corollary 4.4 with $S = E \times \{y\}$. Inequality (4.3) then will imply that for $\text{mod}_{D/d}$ -almost every $E \times \{y\}$, or more precisely for $\text{mod}_{D/d}$ -almost every λ_y like in Section 3.4, the following holds

$$\frac{\dim_H f(E \times \{y\})}{\dim_H E \times \{y\}} \leq \frac{\dim_H f(E \times F)}{\dim_H E \times F},$$

where as usual D is the dimension of $X = E \times F$. Moreover, by Lemma 3.2, if $F' \subset F$ then the family $\{\lambda_y\}_{y \in F'}$ has positive modulus if and only if $\nu(F') > 0$. Therefore we obtain the following strengthening of (1.18):

$$\text{ess sup}_{y \in F'} \frac{\dim_H f(E \times \{y\})}{\dim_H E \times \{y\}} \leq \frac{\dim_H f(E \times F)}{\dim_H E \times F}, \tag{4.5}$$

where ess sup is taken with respect to the measure ν on F . Thus, we obtain the following generalized principle of “*fiberwise expansion implies global expansion*”: if there is a set $F' \subset F$ such that $\nu(F') > 0$ and the fibers $E \times \{y\}, y \in F'$ have their dimensions increased by f by a factor $\alpha \geq 1$, then the dimension of the whole product $E \times F$ increases by at least a factor of α as well.

REMARK 4.6. As mentioned before, an important part of Theorem 4.1 is the relaxation of the regularity conditions on the underlying space X as well as the measure μ . Most significantly, *we do not assume that μ is a doubling measure*, i.e. $\mu(B(x, 2r)) \leq C\mu(B(x, r))$ for all $x \in X$ and $r > 0$. Instead, we assume only upper regularity of μ . This relaxation is of paramount importance to the proof of Theorem 1.5, as Frostman’s lemma only gives us upper regularity (see Remark 4.7 below for further discussion of this). As a consequence, we cannot use the well known “Bojarsky lemma”, which is usually used in similar situations when estimating the modulus from above, see e.g. [Hei01, Theorem 15.10] or [HK95, Proposition 2.9]. Instead, our argument is more in the spirit of the proof of [Wil12, Theorem 1.2], in that a supremum must be used in place of a summation when constructing admissible functions, see (5.1) below. This method, in turn, relies on the quasi-preservation of annuli guaranteed by Lemma 2.7, and so our applications to quasiconformal maps in the plane depend heavily on the equivalence between quasiconformality and quasisymmetry.

REMARK 4.7. It is important to keep in mind the distinction between a doubling metric space and a doubling measure on a metric space. It is easy to show that a metric space with nonzero doubling measure must itself be doubling. Volberg and Konyagin[VK88] proved that, conversely, compact doubling metric spaces admit doubling measures, and Luukainen and Saksman[LS98] extended this result to arbitrary complete metric spaces.

On the one hand, in order to invoke Frostman's lemma in the first place, F must be Borel in its completion, and must be doubling as a metric space. Even so, the conclusion of Frostman's lemma only gives upper regularity for μ , and so even though a doubling measure exists, we cannot assume that μ simultaneously has the doubling property and the desired regularity.

5 Proofs of Theorems 1.2, 4.1 and Corollary 4.2

Proof of Theorem 4.1. We first observe that if the conclusions of the theorem hold for d' , then they hold for every number larger than d' as well, and so we may assume with no loss of generality that $\frac{D}{d} = \frac{D'}{d'}$.

To begin, we suppose $U \subseteq X$ is bounded. Then f and f^{-1} are uniformly continuous, and we may let $\omega(t)$ be a modulus of continuity for f^{-1} .

Fix $\epsilon > 0$, and let $\delta = \omega(\epsilon)$. Since $\mathcal{H}^{D'} \llcorner_{f(X)}$ is locally finite, we may choose balls

$$B'_i = B(y_i, s_i) \subseteq f(X) \subseteq Y$$

such that $\bigcup_{i=1}^{\infty} B'_i \supseteq f(U)$, each $s_i < \epsilon$, and $\sum_{i=1}^{\infty} s_i^{D'} \leq \mathcal{H}^{D'}(f(U)) + \epsilon$. By Lemma 2.7, there is at each point $x_i = f^{-1}(y_i)$ a radius $r_i < \delta$ such that the balls $B_i = B(x_i, r_i)$ satisfy

$$B'_i \subseteq f(B_i) \subseteq f(10B_i) \subseteq \eta(10)B'_i,$$

where η is the distortion function for f .

Now, let

$$g_{\delta}(x) = \sup_{i \in \mathbb{N}} \frac{s_i^{d'}}{r_i^d} \chi_{2B_i}(x), \quad (5.1)$$

and for each $M > 0$, define the family $\mathcal{L}_{U,\epsilon}^M$ of measures $\lambda \in \mathcal{L}_d$, whose supports E_{λ} are distorted significantly, as follows

$$\mathcal{L}_{U,\epsilon}^M := \{\lambda \in \mathcal{L}_d(X) \mid r_{\lambda} > 2\delta \text{ and } \mathcal{H}_{c\eta(10)}^{d'}(f(E_{\lambda} \cap U)) > MC_{\lambda}\}, \quad (5.2)$$

where C_{λ} and r_{λ} are the constants in the definition of lower d -regularity (2.2).

Next, to estimate $\int g_{\delta} d\lambda$ from below we let $I_{\lambda} \subseteq \mathbb{N}$ be the set of indices i such that $B_i \cap E_{\lambda} \neq \emptyset$. By the basic covering lemma, there is a subset $J_{\lambda} \subseteq I_{\lambda}$ such that

$$\bigcup_{j \in J_{\lambda}} 10B_j \supseteq \bigcup_{i \in I_{\lambda}} 2B_i \supseteq E_{\lambda} \cap U,$$

and for $j_1 \neq j_2$, $2B_{j_1} \cap 2B_{j_2} = \emptyset$.

Since the balls $2B_j$ are disjoint, we have

$$\int g_\delta d\lambda \geq \int \sup_{j \in J_\lambda} \frac{s_j^{d'}}{r_j^d} \chi_{2B_j} d\lambda = \int \sum_{j \in J_\lambda} \frac{s_j^{d'}}{r_j^d} \chi_{2B_j} d\lambda = \sum_{j \in J_\lambda} \frac{s_j^{d'}}{r_j^d} \lambda(2B_j) \geq \frac{2^d}{C_\lambda} \sum_{j \in J_\lambda} s_j^{d'},$$

where the last inequality holds because $\lambda \in \mathcal{L}_d$.

Now, since $s_j < \varepsilon$ and

$$\bigcup_{j \in J_\lambda} \eta(10)B'_j \supseteq \bigcup_{j \in J_\lambda} f(10B_j) \supseteq f(E_\lambda \cap U),$$

from the definition of $\mathcal{L}_{U,\varepsilon}^M$ we obtain

$$\sum_{j \in J_\lambda} (\eta(10)s_j)^{d'} \geq \mathcal{H}_{\varepsilon\eta(10)}^{d'}(f(E_\lambda \cap U)) \geq MC_\lambda.$$

Combining the last two estimates we obtain

$$\int g_\delta d\lambda \geq \frac{2^d}{C_\lambda} \sum_{j \in J_\lambda} s_j^{d'} \geq \frac{2^d}{C_\lambda} \cdot \frac{MC_\lambda}{\eta(10)^{d'}} = \frac{2^d M}{\eta(10)^{d'}}. \tag{5.3}$$

Therefore, if $C = \frac{\eta(10)^{d'}}{2^d M}$ then Cg_δ is admissible for $\mathcal{L}_{U,\varepsilon}^M$. Since μ is upper D -regular, we can estimate the modulus of this family as follows:

$$\begin{aligned} \text{mod}_{D/d}(\mathcal{L}_{U,\varepsilon}^M) &\leq C^{D/d} \int g_\delta^{D/d} d\mu = C^{D/d} \int \sup_{i \in \mathbb{N}} \left(\frac{s_i^{d'}}{r_i^d} \chi_{2B_i} \right)^{D/d} d\mu \\ &\leq C^{D/d} \int \sum_{i=1}^\infty \left(\frac{s_i^{d'}}{r_i^d} \chi_{2B_i} \right)^{D/d} d\mu \leq C^{D/d} \sum_{i=1}^\infty \frac{s_i^{D'}}{r_i^D} \mu(2B_i) \\ &\leq C^{D/d} 2^D C_\mu \sum_{i=1}^\infty s_i^{D'} \leq C^{D/d} 2^D C_\mu \left(\mathcal{H}_{\varepsilon\eta(10)}^{D'}(f(U)) + \varepsilon \right). \end{aligned} \tag{5.4}$$

We would like to let ε approach 0 in (5.4), so we have an estimate in terms of $\mathcal{H}^{D'}(f(U))$ rather than $\mathcal{H}_{\varepsilon\eta(10)}^{D'}(f(U))$. For that note, that if $\varepsilon_1 > \varepsilon_2$ then $\mathcal{L}_{U,\varepsilon_1}^M \subseteq \mathcal{L}_{U,\varepsilon_2}^M$. Thus, if we define

$$\mathcal{L}_U^M = \bigcup_{n=1}^\infty \mathcal{L}_{U,\frac{1}{n}}^M,$$

then \mathcal{L}_U^M consists precisely of those measures $\lambda \in \mathcal{L}_d(X)$ for which

$$\mathcal{H}^{d'}(f(E_\lambda \cap U)) > MC_\lambda.$$

By Ziemer's lemma (see Lemma 3.1) we have

$$\text{mod}_{D/d}(\mathcal{L}_U^M) \leq \lim_{n \rightarrow \infty} \text{mod}_{D/d} \left(\mathcal{L}_{U,\frac{1}{n}}^M \right),$$

combining which with (5.4) we obtain the key modulus estimate

$$\text{mod}_{D/d}(\mathcal{L}_U^M) \leq \eta(10)^{\frac{Dd'}{d}} C_\mu \frac{\mathcal{H}^{D'}(f(U))}{M^{D/d}}. \quad (5.5)$$

To prove (1), note that if we define $\mathcal{L}_U^\infty = \bigcap_{k=1}^\infty \mathcal{L}_U^k$ then \mathcal{L}_U^∞ consists of all the measures $\lambda \in \mathcal{L}_d(X)$ for which $\mathcal{H}^{d'}(f(E_\lambda \cap U)) = \infty$. Therefore, from the monotonicity of modulus and inequality (5.5) it follows that for every $k \in \mathbb{N}$ we have

$$\text{mod}_{D/d}(\mathcal{L}_U^\infty) \leq \text{mod}_{D/d}(\mathcal{L}_U^k) \leq \eta(10)^{\frac{Dd'}{d}} C_\mu \frac{\mathcal{H}^{D'}(f(U))}{k^{D/d}}.$$

In particular, if $\mathcal{H}^{D'}(f(U)) < \infty$ then

$$\text{mod}_{D/d}(\mathcal{L}_U^\infty) = \text{mod}_{D/d}\{\lambda \in \mathcal{L}_d(X) \mid \mathcal{H}^{d'}(f(E_\lambda \cap U)) = \infty\} = 0. \quad (5.6)$$

Finally, by countable subadditivity of modulus and of Hausdorff measure, along with the separability of X , we obtain (1) from (5.6).

To prove (2) let $\mathcal{L}_U^0 = \bigcup_{k=1}^\infty \mathcal{L}_U^{1/k}$. Note that \mathcal{L}_U^0 consists of those measures $\lambda \in \mathcal{L}_d(X)$ for which $\mathcal{H}^{d'}(f(E_\lambda \cap U)) > 0$. If $\mathcal{H}^{D'}(f(U)) = 0$ then by (5.5) $\text{mod}(\mathcal{L}_U^{1/k}) = 0$ for every $k \in \mathbb{N}$ and therefore by the countable subadditivity of modulus we obtain that

$$\text{mod}_{D/d}(\mathcal{L}_U^0) = \text{mod}_{D/d}\{\lambda \in \mathcal{L}_d(X) \mid \mathcal{H}^{d'}(f(E_\lambda \cap U)) > 0\} = 0. \quad (5.7)$$

Finally, again we obtain (2) from (5.7) by using the countable subadditivity of modulus, of Hausdorff measure and the separability of X . \square

REMARK 5.1. As noted before, one of the most important features of the proof of Theorem 4.1 is the construction of the admissible function g_δ by the formula (5.1), which allows us to prove the key inequality (5.5) without assuming that μ is a doubling measure. This is similar to the arguments of Williams [Wil12].

Proof of Corollary 4.2. To prove inequality (4.1), we suppose that $D' > \dim_H(f(X))$, and that $\frac{D}{d} = \frac{D'}{d'}$. Then by part (2) of Theorem 4.1, we know that for $\text{mod}_{D/d}$ -almost every $\lambda \in \mathcal{L}_E$, $\mathcal{H}^{d'}(f(E_\lambda)) = 0$, and so $\dim_H(f(E_\lambda)) \leq d'$, whereby

$$\frac{\dim_H(f(E_\lambda))}{\dim_H(f(X))} \leq \frac{d'}{\dim_H(f(X))}.$$

Since this holds for $D' = \dim_H(f(X)) + \frac{1}{n}$ for each n , we obtain by countable subadditivity that for $\text{mod}_{D/d}$ -almost every $\lambda \in \mathcal{L}_d(X)$,

$$\frac{\dim_H(f(E_\lambda))}{\dim_H(f(X))} \leq \frac{d'}{D'} = \frac{d}{D}. \quad \square$$

Proof of Theorem 1.2. In light of Theorem 4.1, we only need to show that $\dim f(E) \geq d$ for mod $_{D/d}$ -almost every $E \in \mathcal{A}_d(X)$. We shall actually show more, namely, that $\mathcal{H}^d(f(E)) > 0$ for almost every E .

When $d = D$ the theorem follows immediately from condition N^{-1} . Suppose then that $d < D$. Let $L_{f^{-1}}(y, r) = \sup_{y' \in B(y, r)} d(f^{-1}(y'), f^{-1}(y))$, and $L_{f^{-1}}(y) = \limsup_{r \rightarrow 0} \frac{L_{f^{-1}}(y, r)}{r}$. Quasisymmetry implies that $L_{f^{-1}}(y)^D \lesssim J_{f^{-1}}(y) < \infty$, at almost every $y \in Y$. Here

$$J_{f^{-1}}(y) := \limsup_{r \rightarrow 0} \frac{\mathcal{H}^D(f^{-1}(B(y, r)))}{\mathcal{H}^D(B(y, r))} = \frac{df_* \mathcal{H}^D \llcorner_X}{d\mathcal{H}^D \llcorner_Y}$$

is the volume derivative of f^{-1} .

Condition N^{-1} implies that at almost every $x \in X$, we have $L_{f^{-1}}(f(x)) < \infty$. Egorov’s theorem then gives sets A_ϵ , with $\mathcal{H}^D(X \setminus A_\epsilon) < \epsilon$, on which $\frac{L_{f^{-1}}(f(x), r)}{r}$ is uniformly bounded for all $x \in A_\epsilon$ and $r < R_\epsilon$. It follows that the restriction $f^{-1}|_{f(A_\epsilon)}$ is locally Lipschitz.

Now let $A = \bigcup_{\epsilon > 0} A_\epsilon$. Since $\mathcal{H}^D(X \setminus A) = 0$, D/d -almost every measure λ satisfies $\lambda(X \setminus A) = 0$ (this is easy to see by taking the admissible function $\rho = \infty \cdot \chi_{X \setminus A}$ for the exceptional family). It follows that D/d -almost every $E \in \mathcal{A}_d(X)$ satisfies $\mathcal{H}^d(E \cap A_\epsilon) > 0$ for some ϵ . Since $f^{-1}|_{f(A_\epsilon)}$ is locally Lipschitz, we have $\mathcal{H}^d(f(E)) \geq \mathcal{H}^d(f(E \cap A)) > 0$, and the theorem is proved. \square

6 Products and the Proof of Theorem 1.5

To apply Frostman’s lemma in the proof of Theorem 1.5, we need the next lemma, which follows quickly from a similar result in [BMT13], see Lemma 3.1 in that paper. Though the statement there is restricted to maps from Euclidean spaces, the proof uses no metric properties of the domain, employing only the fact that $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$ is equipped with the product topology. Hence it applies in our setting as well. We give the argument from [BMT13] here for the reader’s convenience.

LEMMA 6.1. *Let E and F be topological spaces, with E σ -compact, let Z be a metric space, and let $f: E \times F \rightarrow Z$ be continuous. Then for each $s \geq 0$, the set*

$$F^s = \{y \in F: \dim_H(f(E \times \{y\})) > s\}$$

is a Borel set.

Proof. Suppose first that E is compact. It suffices to show that for each $t \geq 0$ and $\epsilon > 0$ the set

$$F_{\infty, \epsilon}^t = \{y \in F: \mathcal{H}_\infty^t(f(E \times \{y\})) \geq \epsilon\}$$

is closed, since $F^s = \bigcup_{t > s} \bigcup_{\epsilon > 0} F_{\infty, \epsilon}^t$. To this end, let $y \in F \setminus F_{\infty, \epsilon}^t$. Then there is a sequence of open balls $B(z_i, r_i)$, with each $z_i \in Z$, and with $\sum_{i=1}^\infty r_i^s < \epsilon$, such that

$$f(E \times \{y\}) \subset \bigcup_{i=1}^{\infty} B(z_i, r_i),$$

or equivalently,

$$E \times \{y\} \subseteq \bigcup_{i=1}^{\infty} f^{-1}(B(z_i, r_i)).$$

By the continuity of f , $\bigcup_{i=1}^{\infty} f^{-1}(B(z_i, r_i))$ is open, so by the compactness of E , there is an open set $U \ni y$ such that

$$E \times U \subseteq \bigcup_{i=1}^{\infty} f^{-1}(B(z_i, r_i)),$$

so that $U \subseteq F \setminus F_{\infty, \epsilon}^t$. Thus $F \setminus F_{\infty, \epsilon}^t$ is open, whereby $F_{\infty, \epsilon}^t$ is closed as desired.

Finally, suppose $E = \bigcup_{i=1}^{\infty} E_i$, with each E_i compact. Then $F^s = \bigcup_{i=1}^{\infty} F_i^s$, where

$$F_i^s = \{y \in F : \dim_H(f(E_i \times \{y\})) > s\}.$$

Since these sets are Borel by the compact case of the lemma, F^s is Borel as well. \square

Proof of Theorem 1.5. The theorem is trivial if $d' = D'$. We also observe that since quasisymmetric maps are uniformly continuous on bounded sets, they extend to the completions of the spaces on which they are defined. We may therefore assume that $d' < D'$ and that E and F are complete. Note that d -regularity implies the doubling property, so that E is a complete doubling metric space. As such, E is proper (balls are compact), and a fortiori σ -compact.

Let $D = \frac{d}{d'} D'$, and $F_{d'} = \{y \in F : \dim_H f(E \times \{y\}) > d'\}$. Suppose by way of contradiction that

$$\mathcal{H}^{\frac{d}{d'} D' - d}(F_{d'}) > 0. \quad (6.1)$$

By Lemma 6.1, $F_{d'}$ is Borel set. Thus it follows from Frostman's lemma that there is a nonzero upper $(D - d)$ -regular measure ν on $F_{d'}$.

Since E (and hence \mathcal{H}^d) is d -regular, the measure $\mu = \mathcal{H}^d \times \nu$ is upper D -regular on $X = E \times F_{d'}$, and so by Lemma 3.2, we have that $\text{mod}_{D/d}(\{\mathcal{H}^d|_{E \times \{y\}} : y \in F_{d'}\}) > 0$. From this and Corollary 4.2 (which applies by the lower regularity of each measure $\mathcal{H}^d|_{E \times \{y\}}$), we then conclude that there is some $y \in F_{d'}$ such that

$$\dim_H f(E \times \{y\}) \leq \frac{d}{D} D' = d',$$

contradicting the definition of $F_{d'}$. \square

Proof of Corollary 1.6. Let $d' < \inf_{y \in F} \dim_H f(E \times \{y\})$, and $D' = \dim_H(f(E \times F))$. Using the notation of the preceding proof, we have that $F = F_{d'}$, and so by Theorem 1.5, we obtain

$$\mathcal{H}^{\frac{d}{d'} D' - d}(F) = 0,$$

so that $\dim_H(F) \leq \frac{d}{d'} D' - d$.

Since E is Ahlfors d -regular, the packing dimension of E is equal to its Hausdorff dimension (see Theorem 6.13 of [Mat95]). Therefore we also have (see e.g. Corollary 8.11 of [Mat95]) that

$$\dim_H(E \times F) = \dim_H E + \dim_H F \leq \frac{d}{d'} D',$$

whereby

$$d' \leq \frac{dD'}{\dim_H(E \times F)} = \frac{\dim_H(f(E \times F))}{\dim_H(E \times F)} \cdot \dim_H(E).$$

Since this holds for every $d' < \inf_{y \in F} \dim_H f(E \times \{y\})$, the proof is complete. \square

7 Proofs of Theorem 1.7 and Corollary 1.8

Proof of Theorem 1.7. For $0 < t < 1$, let $\alpha = 2^{-1/t}$, and let S be the Cantor set given by a standard iterative construction that starts with $I_0 = [0, 1]$ and replaces each n th generation interval I with two $(n + 1)$ st generation intervals of length $\alpha|I|$ and distance $\frac{1}{3}(1 - 2\alpha)|I|$ from each other and from the endpoints of I . It is easy to check that $\dim(S) = t$. The proof is somewhat cleaner when we restrict to the case that α^{-k} is an integer for some positive integer k (which may depend on α). Note that this holds whenever $t \in \mathbb{Q}$. The proof in the general case follows the same idea, with a few technical modifications, described in the ensuing Remark 7.4. Taking $\alpha^{-k} = N$, we have $N^t = 2^k$. By taking multiples of k , we may assume N is as large as we wish.

The basic building block in the construction of f is a quasiconformal map of the unit square $Q = [0, 1]^2$,

$$\Phi: [0, 1]^2 \rightarrow [0, 1]^2. \tag{7.1}$$

Let $\{I_j\}$, $j = 1, \dots, 2^k$ be the k th generation covering intervals of S , each of which has length $\alpha^k = \frac{1}{N}$, and let $R_j = [0, 1] \times I_j$ be rectangles with their short edges on the vertical sides of Q . Note that each of these is isometric to the $1 \times \frac{1}{N}$ rectangle

$$R = \{(x, y) \in \mathbb{R}^2: 0 < x < 1, 0 < y < 1/N\}.$$

The map Φ will be conformal on each of these rectangles and will be quasiconformal on the rest of Q . Our construction also gives that Φ is the identity on the top and bottom edges of Q and it is symmetric with respect to the vertical bisector of Q , so $\Phi(1, y) = \Phi(0, y) + (1, 0)$ on the vertical sides of Q . This means that Φ can be extended to a quasiconformal map of the whole plane by simply mapping each square $Q + (n, m)$ to itself by $(x, y) \rightarrow \Phi(x - n, y - m) + (n, m)$.

The map Φ is constructed by specifying a generalized quadrilateral $\mathcal{T} \subset Q$ (\mathcal{T} for “tube”) that has two opposite sides on the vertical sides of Q and conformal modulus N (the same as R). This means there is a conformal map

$$\phi: R \mapsto \mathcal{T}$$

that maps vertices to vertices. This map is used to define a conformal map of each R_j to a translate \mathcal{T}_j of \mathcal{T} that connects the left and right sides of Q . The tubes \mathcal{T}_j will have disjoint closures that do not hit the top and bottom edges of Q and so the complement of these tubes in Q are $2^k + 1$ regions. We define a quasiconformal map from each component of $Q \setminus \cup_j R_j$ to the corresponding component of $Q \setminus \cup_j \mathcal{T}_j$ so that it extends the mapping on each R_j , is the identity on the top and bottom edges of Q and is symmetric on the vertical edges of Q . The quasiconformal constant K of this map depends on the geometry of and spacing between the \mathcal{T}_j , but is finite for the examples we will build.

The tube \mathcal{T} will be constructed so that

$$|\phi'| \geq C_1 N^{\frac{1-t}{2}}, \quad (7.2)$$

on all of R and with some constant $C_1 > 0$ that is independent of k and N . First we use this estimate to finish the proof of the theorem, and then we construct a tube for which this estimate is true.

Let $f_1 = \Phi$. The rectangle R has an obvious decomposition into N squares of side length $1/N$. Define a map $f_2: Q \rightarrow Q$ as the identity outside $\cup_j R_j$ and inside each R_j use a scaled version of Φ to map each subsquare of R_j to itself. In general, $f_n: Q \rightarrow Q$ is defined as the identity off the 2^{nk} rectangles corresponding to the intervals of generation nk covering S and is defined using a scaled copy of Φ on the N^n squares, of sidelength $\alpha^{nk} = N^{-n}$, making up each such rectangle. Then

$$g_n = f_1 \circ \cdots \circ f_n$$

is quasiconformal with constant K (at most one map in the composition is non-conformal when applied to any point). Thus, the limiting map is also K -quasiconformal, and we finally define f as follows,

$$f = \lim_{n \rightarrow \infty} g_n = \lim_{n \rightarrow \infty} f_1 \circ \cdots \circ f_n.$$

Moreover, on every generation nk square \tilde{Q} in one of the scaled copies of R , f restricts to a map of the square onto itself, followed by a succession of n conformal maps, each satisfying inequality (7.2). We therefore have, for every generation nk square \tilde{Q} in a scaled copy of R , the estimate

$$\text{diam}(f(\tilde{Q})) \geq C_1^n N^{n(\frac{1-t}{2})} \text{diam}(\tilde{Q}) = C_1^n N^{-n(\frac{1+t}{2})}. \quad (7.3)$$

Since C_1 does not depend on our choice of N , we may suppose as well that N was chosen large enough so that $C_1 \geq N^{-(\frac{\epsilon}{1-\epsilon})(\frac{1+t}{2})}$, whereby

$$\text{diam}(f(\tilde{Q})) \geq N^{-n(\frac{1+t}{2(1-\epsilon)})}.$$

Consider a Borel subset $A \subseteq \mathbb{R} \times S$, and let $\delta < \dim_H(A)$. By Frostman's lemma, there is a positive measure μ supported on A such that

$$\mu(\tilde{Q}) \leq C_\mu (\text{diam} \tilde{Q})^\delta = C_\mu N^{-n\delta}.$$

Let $\nu = f_{\#}\mu$ be the pushforward measure, and let $s = \frac{2\delta}{(t+1)}(1 - \epsilon)$. Then for every generation nk square \tilde{Q} in a scaled copy of R ,

$$\text{diam}(f(\tilde{Q}))^s \geq N^{-ns} \left(\frac{1+t}{2(1-\epsilon)} \right) = N^{-n\delta} \geq C_{\mu}^{-1} \mu(\tilde{Q}) = C_{\mu}^{-1} \nu(f(\tilde{Q})). \tag{7.4}$$

Next, note that every ball $B = B(x, r)$ in $\mathbb{R} \times S$ can be covered by a uniformly bounded number of generation nk squares of comparable diameter in the scaled copies of R . Indeed, choose the smallest $n \in \mathbb{N}$ so that $N^{-n} \geq 2r$. Then B intersects at most 2 rectangles of width N^{-k} and thus at most 2^k rectangles of width $N^{-(k+1)}$. Since each rectangle contains N squares, we can cover B by $L = 2^k \cdot 2N$ squares \tilde{Q} of side length $N^{-(k+1)}$. Since $\text{diam}(\tilde{Q}) < r$ by Lemma 2.8, $\text{dim}(f(A)) \geq \frac{2\delta}{(t+1)}(1 - \epsilon)$. Since this holds for arbitrary $\delta < \text{dim}_H(A)$, we obtain $\text{dim}(f(A)) \geq \frac{2\text{dim}_H(A)}{(t+1)}(1 - \epsilon)$.

We are now done, except we must build a tube $\mathcal{T} \subset [0, 1]^2$ that has the proper estimate on the conformal map to a rectangle.

The “Tube” construction. As before, let $N = \alpha^{-k}$. The $1 \times \frac{1}{N}$ rectangle R is divided into N disjoint squares and there are 2^k such rectangles. Since $\alpha < \frac{1}{2}$, we have $N \gg 2^k$ when k is large. Let

$$m = \left\lfloor \sqrt{\frac{N-1}{2^{k-1}}} \right\rfloor,$$

so that

$$\frac{1}{2}N \leq M = m^2 2^{k-1} + 1 \leq N,$$

if N is large enough.

We start by constructing a large copy of the tube \mathcal{T} , which we will denote by T . Consider the $(2^k m + 1) \times m$ rectangle W shown in Figure 1. The shaded (dark and light grey) unit area squares in W form a tube T that connects the two vertical

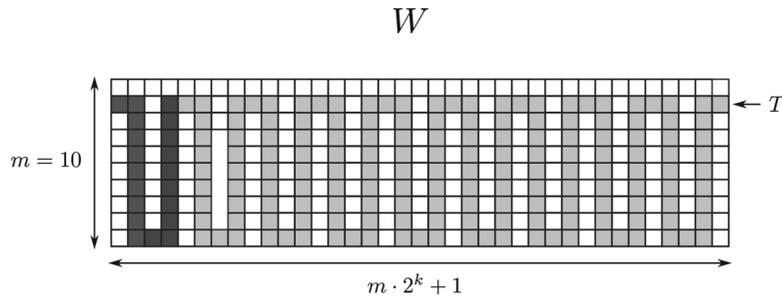


Figure 1: The first version of the “large” tube $T \subset W$ is made up of $M = m^2 2^{k-1} + 1$ shaded squares forming a connected subset of a $m \times (2^k m + 1)$ grid. Here we have taken $k = 2$ and $m = 10$. The dark grey squares make up the pattern, which when repeated periodically $m2^{k-2}$ times, forms T , except for the “last” or rightmost sub-square which is adjacent to the right edge of W

sides of W . More precisely, denoting by $Q_{p,q}$ the square $[p-1, p] \times [q-1, q]$ in the plane, the tube T is obtained by considering the union of $2m$ (dark grey) squares

$$Q_{1,m-1}; Q_{2,m-1}, \dots, Q_{2,1}; Q_{3,1}; Q_{4,1}, \dots, Q_{4,m-1},$$

repeating this pattern periodically $m \cdot 2^{k-2}$ times ($k \geq 2$) and attaching the rightmost square $Q_{m2^k+1,m-1}$. Alternatively, we may write the tube T as follows

$$\begin{aligned} T = & \bigcup_{i=0}^{m \cdot 2^{k-2} - 1} Q_{4i,m-1} \cup (Q_{4i+1,m-1} \cup \dots \cup Q_{4i+1,1}) \\ & \cup \bigcup_{i=0}^{m \cdot 2^{k-2} - 1} Q_{4i+2,1} \cup [Q_{4i+3,1} \cup \dots \cup Q_{4i+3,m-1}] \cup Q_{m2^k+1,m-1}. \end{aligned}$$

Let M be the number of disjoint subsquares in T , which is also the area of T . Note, that the number of the subsquares of T , which do not intersect the right side of W is exactly the half of the subsquares of W , which do not intersect its right side. Indeed, for each column of the grid containing only one square in T , move that square to the top row of the next column to the right. This gives a sequence of alternating “full” and “empty” columns, see Figure 1. Since there are m subsquares of W in any column, we have

$$M - 1 = \frac{\mathcal{H}^2(W) - m}{2} = \frac{m \times (2^k m + 1) - m}{2} = m^2 \cdot 2^{k-1}.$$

Therefore, $M = m^2 \cdot 2^{k-1} + 1$.

We think of T as a generalized quadrilateral with two sides (the “short sides”) on the vertical sides of W and two other sides (the “long sides”) that connect the vertical sides of W .

Up to a similarity, the region T is almost the tube we want, but it is convenient to “round the corners” as shown in Figure 2. Rounding the corners will imply that the derivative of the conformal map of a rectangle R (of the same modulus as T) to T (as well as to the final tube \mathcal{T}) is everywhere comparable to the ratio of the widths of R and T , as is demonstrated in Lemma 7.2 below. Next, we carry out this procedure in more detail.

Let γ be the “core curve” which connects the midpoints of the short sides of T . More precisely, if Q_i is not a “corner subsquare” of T then $\gamma \cap Q_i$ is a horizontal or vertical interval in Q_i connecting the midpoints of opposite sides. On the other hand, if Q_i is a “corner subsquare” then $\gamma \cap Q_i$ is the arc of the circle of radius $1/2$ connecting the midpoints of two adjacent sides of Q_i , see Figure 2.

For $0 < w \leq 1$ we denote by T^w the intersection of T and the $w/2$ -neighborhood of γ . Thus T^w is a tube around γ of “width” w . Moreover, for $w < 1$ the long sides of T^w are $C^{1+\alpha}$ curves for every $\alpha < 1$, since they are differentiable curves consisting of line segments and circular arcs. Next, we will show that w can be chosen so that

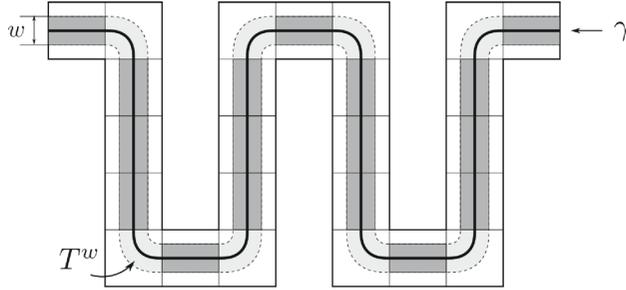


Figure 2: The tube T^w has rounded corners, so that the conformal map to the tube from the rectangle of the same modulus has a “nice derivative” (the minimum and maximum expansion have bounded ratio). Moreover, T^w has a smaller width w ; this increases the modulus of the family connecting the two long edges of the tube. By “thinning” the tube we can make its modulus exactly what we want. Since we start within a factor of 4 of the desired modulus, only a bounded amount of thinning is needed. The darker rectangles in “non-corner” squares are used to estimate the modulus from below

T^w is conformally equivalent to the $1 \times 1/N$ rectangle R . For that we will need the following lemma.

LEMMA 7.1. For $w \in (0, 1]$ let Γ^w be the path family connecting the long sides of T^w . Then we have the following estimates

$$\frac{M - 2^k m}{w} \leq \text{mod}_2 \Gamma^w \leq \frac{l(\gamma)}{w} < \frac{M}{w}, \tag{7.5}$$

where $l(\gamma)$ is the length of γ . In particular if $2^k m < M/2$ then for $w = 1$ we have

$$\frac{1}{2}M \leq M - 2^k m \leq \text{mod}_2 \Gamma^1 < M. \tag{7.6}$$

Proof. To prove the right hand side of (7.5), take the constant metric $\rho \equiv w^{-1}$ on T^w . Since every path connecting the long sides of T^w has length $\geq w$, we see that ρ is admissible for Γ . Hence,

$$\text{mod}_2 \Gamma \leq \int_T w^{-2} dx dy = \frac{\mathcal{H}^2(T^w)}{w^2} \leq \frac{l(\gamma)w}{w^2} = \frac{l(\gamma)}{w}.$$

To prove the left hand inequality of (7.5) consider all the “non-corner” squares in T . Note that there are $2^k m$ “corner” squares and therefore only $M - 2^k m$ “non-corner” squares, and we denote them by $Q_i, i = 1 \dots, M - 2^k m$. For each such “non-corner” square Q_i let Γ_i^w be the subfamily of paths in Γ^w contained in Q_i . Note, that the paths in Γ_i^w connect the opposite sides of the rectangle $Q_i \cap T^w$ of length 1 and therefore $\text{mod}_2(\Gamma_i^w) = 1/w$. Since Γ_i^w 's are disjoint families, we have

$$\text{mod}_2(\Gamma^w) \geq \text{mod}_2 \left(\bigcup_{i=1}^{M-2^k m} \Gamma_i^w \right) = \sum_{i=1}^{M-2^k m} \text{mod}_2(\Gamma_i^w) = \frac{M - 2^k m}{w}. \quad \square$$

Note, that $\text{mod}_2(\Gamma^w)$ changes continuously with w and by (7.5) can be made as large as desired by taking w small enough. Thus, there is a $w_0 > 0$ such that $\text{mod}_2(\Gamma^{w_0}) = N$. Moreover, from (7.5) it follows that for $w_1, w_2 \in (0, 1]$ we have

$$\frac{1}{2} \frac{\text{mod}(\Gamma^{w_2})}{\text{mod}(\Gamma^{w_1})} \leq \frac{w_1}{w_2} \leq 2 \frac{\text{mod}(\Gamma^{w_2})}{\text{mod}(\Gamma^{w_1})}.$$

In particular, from (7.6) we obtain

$$\frac{1}{4} \leq 2^{-1} \frac{M}{N} \leq w_0 < 1$$

and the width of the thinner tube is comparable to the width of the original tube.

Slightly abusing the notation, we let T denote the large rounded, thinned tube T^{w_0} of modulus N and finally define the tube \mathcal{T} by

$$\mathcal{T} = \sigma(T)$$

where σ is the similarity of the plane

$$\sigma(x, y) = (m2^k + 1)^{-1}(x, y).$$

Therefore \mathcal{T} is a copy of T which connects the vertical sides of Q and intersects only M subsquares of $[0, 1]^2$ of sidelength $(2^k m + 1)^{-1}$.

Take the unit square $Q = [0, 1]^2$ and subdivide it into $(2^k m + 1)^2$ disjoint subsquares of side length $(2^k m + 1)^{-1}$, grouped into 2^k rectangles of dimension $(2^k m + 1) \times m$ and the single $(2^k m + 1) \times 1$ strip at the bottom. See Figure 3.

Inside each rectangle, place a (shifted up) copy of the tube \mathcal{T} . These are the \mathcal{T}_j 's mentioned earlier. Finally, we show in Lemma 7.2 that the conformal map from R_j to \mathcal{T}_j (mapping vertices to vertices), or equivalently the map $\phi : R \rightarrow \mathcal{T}$ has derivative

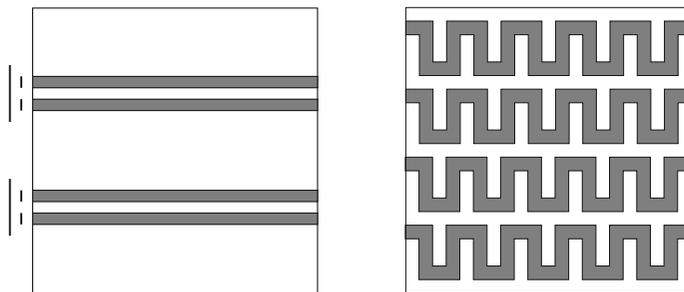


Figure 3: On the left is Q and the shaded rectangles R_j . On the right are the tubes T_j (we have omitted the rounding and thinning to make the picture simpler). Each R_j is conformally mapped to the corresponding T_j (they have the same modulus, so we can send vertices to vertices) and we quasiconformally extend these maps to map of Q onto itself that is the identity on the top and bottom edges and is symmetric on the left and right edges (the tubes are symmetric with respect to the vertical bisector of Q , so this is possible)

everywhere comparable to $\frac{\text{width}(\mathcal{T})}{\text{width}(R)}$. But, since $N^t = 2^k$ and thus $m \simeq \sqrt{\frac{N}{2^k}} = N^{\frac{1-t}{2}}$ it follows, that

$$|\phi'| \simeq \frac{\text{width}(\mathcal{T})}{\text{width}(R)} = \frac{w_0(2^k m + 1)^{-1}}{N^{-1}} \simeq \frac{N}{2^k m} \simeq \frac{N}{N^t N^{\frac{1-t}{2}}} = N^{\frac{1-t}{2}}$$

thus completing the proof of Theorem 1.7 (except for the proof of Lemma 7.2).

LEMMA 7.2. *The conformal map $\phi : R \rightarrow \mathcal{T}$ taking the vertices of R to vertices of \mathcal{T} , has a derivative that is everywhere comparable to the width of \mathcal{T} divided by the width of R , i.e.*

$$|\phi'(z)| \simeq \frac{\text{width}(\mathcal{T})}{\text{width}(R)}. \tag{7.7}$$

Proof. We will show that the absolute value of the derivative of $\phi^{-1} : \mathcal{T} \rightarrow R$ is comparable to $\frac{\text{width}(R)}{\text{width}(\mathcal{T})}$ everywhere in \mathcal{T} .

Using complex notation we consider the linear maps

$$s_{\mathcal{T}}(z) = \sigma^{-1}z = (2^k m + 1)z = \frac{w_0}{\text{width}(\mathcal{T})}z, \quad s_R(z) = \frac{1}{\text{width}(R)}z = Nz$$

and note that if we define

$$\psi = s_R \circ \phi^{-1} \circ s_{\mathcal{T}}^{-1},$$

then ψ maps the large rounded tube T onto a rectangle of width 1 and we have the following expression for the derivative of ϕ ,

$$|(\phi^{-1})'| = |(s_R^{-1} \circ \psi \circ s_{\mathcal{T}})'| = |(s_R)'|^{-1} |\psi'| |(s_{\mathcal{T}})'| = w_0 \frac{\text{width}(R)}{\text{width}(\mathcal{T})} |\psi'| \simeq \frac{\text{width}(R)}{\text{width}(\mathcal{T})} |\psi'|.$$

Thus, to obtain the estimate (7.7) it is enough to show that if ψ is the conformal mapping of T onto a rectangle of width 1, taking vertices to vertices, then we have that $|\psi'| \simeq 1$ everywhere in T .

Now, the conformal map ψ from the tube T to the rectangle of width 1 can be extended by Schwarz reflection across the “short ends” of the tube and rectangle to a map $\tilde{\psi}$ from an infinite tube \tilde{T} of width w_0 to the infinite strip $S = \{x + iy : 0 < y < 1\}$. Assume we have done this (to avoid separate arguments near the short ends). The Koebe $\frac{1}{4}$ -theorem, cf. [GM05] Corollary 4.4, then implies that a conformal map $\tilde{\psi} : \tilde{T} \rightarrow S$ has derivative

$$|\tilde{\psi}'(z)| \simeq \frac{\text{dist}(\tilde{\psi}(z), \partial S)}{\text{dist}(z, \partial \tilde{T})},$$

so it suffices to prove that the right hand side is uniformly bounded and bounded away from zero.

Let v be the imaginary part of $\tilde{\psi}$; it is a harmonic function on the infinite tube \tilde{T} with boundary values $v = 1$ on one boundary component (call it X_1) and $v = 0$ on the other (call it X_0). Then $\text{dist}(\tilde{\psi}(z), \partial S) = \min(v(z), 1 - v(z))$ and it suffices to prove the following implications:

$$\text{dist}(z, X_0) \rightarrow 0 \implies v(z) \simeq \text{dist}(z, X_0) \tag{7.8}$$

$$\text{dist}(z, X_1) \rightarrow 0 \implies 1 - v(z) \simeq \text{dist}(z, X_1). \tag{7.9}$$

We will prove (7.8) in detail; the estimate (7.9) follows from an essentially identical argument.

To prove (7.8) we will show that for any point $x \in X_0$ there exist functions ζ_x and ξ_x defined on the segment I_x which passes through x and is orthogonal to ∂T such that for all $z \in I_x$ we have

$$\zeta_x(z) \leq v(z) \leq \xi_x(z), \tag{7.10}$$

and such that as $\text{dist}(z, X_0) \simeq 0$ the following estimates hold with constants independent of $x \in X_0$,

$$\text{dist}(z, X_0) \lesssim \zeta_x(z), \tag{7.11}$$

$$\xi_x(z) \lesssim \text{dist}(z, X_0), \tag{7.12}$$

thus implying (7.8).

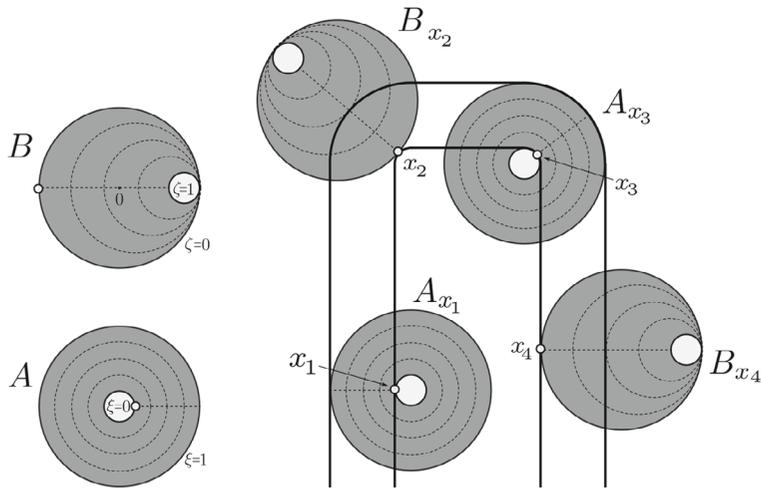


Figure 4: On the *left* are the domains of definition of ξ and ζ and their boundary values. A special boundary point is marked with a *white dot*. Explicit computations show that ξ and ζ vanish at this point and grow almost linearly on the interior segment normal to this point. The *dashed lines* show level *lines* of the two functions. The *righthand side of the figure* shows how these domains can be positioned with the special point at any boundary point of the tube; this shows that v has the correct estimate on every normal segment crossing the tube, hence on the whole tube

The functions ξ_x and ζ_x will be the “transported versions” of harmonic functions ξ and ζ which we define next, see also Figure 4.

First, let $A = \{z \in \mathbb{C} : \frac{1-w_0}{2} < |z| < \frac{1+w_0}{2}\}$, and define $\xi : A \rightarrow \mathbb{R}$ to be the harmonic function in A with boundary values

$$\xi(z) = \begin{cases} 0 & \text{for } |z| = \frac{1-w_0}{2}, \\ 1 & \text{for } |z| = \frac{1+w_0}{2}. \end{cases} \tag{7.13}$$

The function ξ can be explicitly written as $\xi(z) = (\log \frac{1+w_0}{1-w_0})^{-1} \log \frac{2|z|}{1-w_0}$. Then, as $z \in A$ approaches $\frac{1-w_0}{2}$ along the real axis, we have the estimate $\xi(z) \simeq \text{dist}(z, \frac{1-w_0}{2})$, which follows from the fact that the partial derivative of ξ at $\frac{1-w_0}{2}$ with respect the first coordinate is positive [this is clear from the construction, but could also be checked using the formula for $\xi(z)$ given above].

Now, for a fixed $x \in X_0$ there is a unique isometry r_x of the plane such that $r_x(\frac{1-w_0}{2}) = x$ (i.e. the special point of A is mapped to $x \in X_0$), and the circle $r_x(\{|z| = \frac{1-w_0}{2}\})$ is tangent to X_0 at $x \in X_0$ and is located outside of the (open) tube \tilde{T} . We denote $A_x = r_x(A)$ and define the harmonic function ξ_x on A_x by $\xi_x = \xi \circ r_x^{-1}$. Then

$$\begin{aligned} v \leq 1 &= \xi_x && \text{on } \partial A_x \cap \tilde{T} \\ v = 0 &\leq \xi_x && \text{on } \partial \tilde{T} \cap A_x, \end{aligned}$$

therefore $v \leq \xi_x$ on the whole boundary of $A_x \cap \tilde{T}$. By the maximum principle, $v \leq \xi_x$ on $A_x \cap \tilde{T}$, which give one side of (7.10).

From the behaviour of ξ near the special point $\frac{1-w_0}{2} = r_x^{-1}(x)$ of A , we have that as z approaches x along the normal segment to X_0 the following estimates hold

$$\xi_x(z) = \xi(r_x^{-1}(z)) \simeq \text{dist}(r_x^{-1}(z), r_x^{-1}(x)) = \text{dist}(z, x) = \text{dist}(z, X_0).$$

Therefore (7.11) holds as z approaches ∂X along the normal segment I_x and the constants are independent of x , since they depend only on $\partial_1 \xi(\frac{1-w_0}{2})$ (here ∂_1 denotes the partial derivative in the first coordinate).

To obtain the left hand inequality in (7.10) we define the second function ζ on the cusped region $B = \{|z| < \frac{1+w_0}{2}\} \setminus D_0$ where D_0 is the (closed) disk of radius $(1-w_0)/2$ that is tangent to the outer circle at the point $\frac{1+w_0}{2} \in \mathbb{C}$ and is contained in $\{|z| < \frac{1+w_0}{2}\}$. We set ζ to be the harmonic function on B such that $\zeta = 0$ on the boundary of the larger disk and $\zeta = 1$ on the boundary of the smaller disk. The dashed curves are circles and show the level lines of ζ . In Figure 4, the special boundary point $-\frac{1+w_0}{2} \in \partial B$ is marked as a white dot; it is opposite to the point where the two boundary circles are tangent.

The function ζ can be computed explicitly by mapping the cusp region to the infinite strip $\{0 < y < 1\}$ by a Möbius transformation and the estimate $\partial_1 \zeta(-\frac{1+w_0}{2}) > 0$ clearly holds in this case as well. Indeed, since ζ is a Möbius transformation the complex derivative ζ' (hence also partial derivatives) does not vanish in \mathbb{C} . Also

$\zeta(-\frac{1+w_0}{2}) = 0$ and $\zeta(z) > 0$ for $z \in B$, so $\partial_1 \zeta$ is not negative at $-\frac{1+w_0}{2}$. Therefore we have the $\zeta(z) \simeq \text{dist}(z, -\frac{1+w_0}{2})$ as z approaches $-\frac{1+w_0}{2}$ along the real axis.

Just like before, for $x \in X_0$ there is an isometric copy B_x of the region B (i.e. $B_x = s_x(B)$ where s_x is an isometry of the plane), such that $B_x \cap \tilde{T} \neq \emptyset$ and B_x is tangent to \tilde{T} at its special point $s(-\frac{1+w_0}{2}) = x \in X_0$. Next, defining ζ_x on B_x by $\zeta_x = \zeta \circ s_x^{-1}$, we obtain that the boundary circle where $\zeta_x = 1$ lies outside \tilde{T} on the other side of X_1 (the boundary component of \tilde{T} where $v = 1$). As above, it is easy to check that $\zeta_x \leq v$ on the intersection $B_x \cap \tilde{T}$ and in particular, this is true for z which are on the normal segment to X_0 at x , which proves completely (7.10).

Finally, the estimate $\zeta_x(z) \simeq \text{dist}(z, X_0)$ is obtained from the corresponding estimate for $\zeta(z)$ the same way as (7.11) followed for ξ_x .

Thus we proved (7.10)–(7.12) which in turn imply (7.8). The argument for (7.9) that $1 - v(z) \simeq \text{dist}(z, X_1)$ is identical to the one above, so this proves the lemma.

As mentioned before, this completes the proof of Theorem 1.7.

REMARK 7.3. The rounding of the tube T in the proof is not actually necessary. If we leave the corners of T then the derivative of the conformal map of R to T will tend to zero or ∞ at the corners, but will have uniform bounds on any subregion that is bounded away from the corners. The next generation of the construction will take place inside such a sub-region, and so the proof of the theorem would work even without rounding the corners (however, rounding is easy and gives a cleaner estimate).

REMARK 7.4. The case when t is irrational follows from the rational case considered above. Indeed, suppose $0 < t < 1$ and $\varepsilon > 0$ are like in the statement of Theorem 1.7, and t is irrational. Then we may choose $t < t' < 1$ and $\varepsilon' > 0$ with $t' \in \mathbb{Q}$ so that

$$\frac{1 - \varepsilon'}{t' + 1} \geq \frac{1 - \varepsilon}{t + 1},$$

e.g. if ε is small enough we may take $t' = t + \varepsilon(1 + t)$ and $\varepsilon' \leq \varepsilon^2$. By the case considered above there is an Ahlfors- t' regular Cantor set $S' \subset \mathbb{R}$, such that for every Borel subset A of $\mathbb{R} \times S'$ we have

$$\frac{\dim_H f(A)}{\dim_H A} \geq \frac{2(1 - \varepsilon')}{t' + 1} \geq \frac{2(1 - \varepsilon)}{t + 1}.$$

Now, by a theorem of Mattila and Saaranen [MS09] there is an Ahlfors- t regular subset $S \subset S'$, since $t < t'$. Thus, considering Borel subsets A of $\mathbb{R} \times S \subset \mathbb{R} \times S'$ completes the proof for the irrational values of t .

Proof of Corollary 1.8. Let d' and η be as in the statement of the corollary. Without loss of generality assume that $1 < d' < 2$. Let $d = \frac{2}{d'} - 1 - \eta$. Then $\frac{2}{d+1} = \frac{d'}{1 - \eta d'} > d'$, and we may therefore choose $\epsilon > 0$ so that $\frac{2}{d+1} - \epsilon > d'$.

Theorem 1.7 implies that for every $0 < d < 1$ there is a d -regular Cantor set $S \subseteq \mathbb{R}$ and a quasiconformal mapping f of \mathbb{R}^2 satisfying inequality (1.14). That is, for every $y \in S$,

$$\dim_H f(I \times \{y\}) \geq \frac{2}{d+1} - \epsilon > d',$$

so that $\{y \in \mathbb{R} : \dim_H f(I \times \{y\}) > d'\} \supseteq S$. Since $\dim_H S = d$, it follows that

$$\dim_H \{y \in \mathbb{R} : \dim_H f(I \times \{y\}) > d'\} \geq \dim_H(S) = d = \frac{2}{d'} - 1 - \eta. \quad \square$$

8 Proof of Theorem 1.9

Proof of Theorem 1.9. The idea is quite simple. We start by mapping horizontal tubes to nearly horizontal tubes that oscillate. Inside these, we build thinner tubes that oscillate on a smaller scale, and continue by induction, obtaining in the limit a Cantor set of curves each of which oscillates at infinitely many scales, hence has no rectifiable subarc. Using sufficiently many sufficiently thin tubes we can keep the quasiconstant bounded while making the Hausdorff measure as large as we want.

The proof is essentially a sequence of pictures. First, choose a diffeomorphism ϕ of the unit square $Q = [0, 1]^2$ to itself of the form $\phi(x, y) = (x, \psi(x, y))$ that is the identity on the boundary, and translates the vertical segment $V = \{\frac{1}{2}\} \times [\frac{1}{4}, \frac{3}{4}]$ up by $1/8$ to the segment $\{\frac{1}{2}\} \times [\frac{3}{8}, \frac{7}{8}]$. Thus segments of the form $S_y = [0, 1] \times \{y\} \subset Q$ have curved images with the same endpoints as S_y , but deviate by at least $\frac{1}{8}$ from S_y for $y \in [\frac{1}{4}, \frac{3}{4}]$. See Figure 5.

Let $\gamma = \phi(S_y)$ for $y \in [\frac{1}{4}, \frac{3}{4}]$. Choose a large integer n and divide γ into subcurves with endpoints $z_k = (x_k, y_k)$ where $x = \frac{k}{n}$, $k = 0, 1, \dots, n$. Let γ_n be the polygonal path with these vertices. At each z_k , $k = 1, \dots, n - 1$, draw a segment perpendicular to and above γ whose length is the same as the $(k - 1)$ st segment in γ_n . The other endpoint is denoted w_k . Let $\tilde{\gamma}$ be the polygonal curve with vertices w_0, \dots, w_n . See Figure 6.

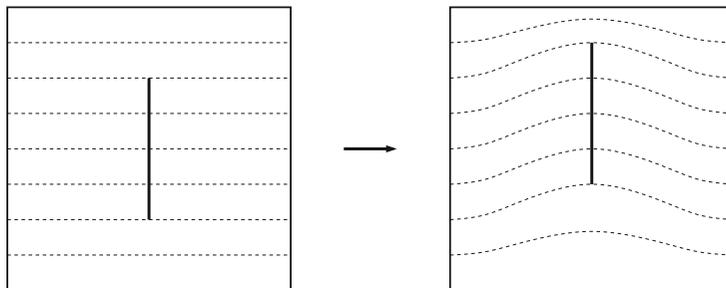


Figure 5: Choose a smooth self-map of Q that is the identity on ∂Q , but causes *horizontal* segments near the *middle* of Q to bend by a definite amount

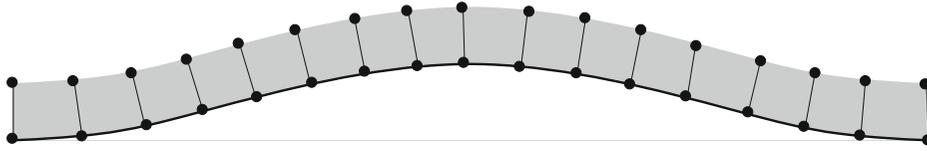


Figure 6: We inscribe a *polygonal* curve γ_n in γ and add *perpendicular* segments. We then connect the new vertices to form a path $\tilde{\gamma}_n$ that is almost *parallel* to γ_n

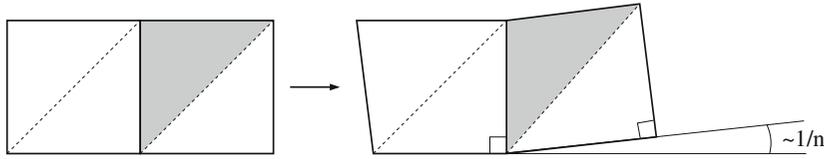


Figure 7: Two adjacent *squares* are mapped to adjacent *quadrilaterals* that are almost *squares* [with error $O(1/n)$]

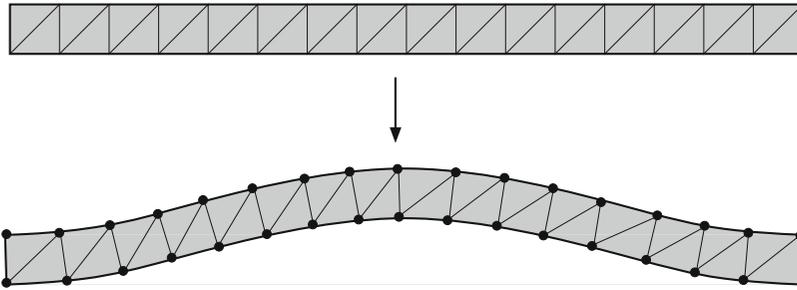


Figure 8: Piecewise linear maps define a $1 + O(\frac{1}{n})$ -QC map from a *rectangle* to a *tube*

The reason we defined $\tilde{\gamma}_n$ as we did (and did not simply translate γ_n upwards), is so that the region between the curves can be divided into quadrilaterals that are very nearly squares. We will denote this region T_n and call it a “tube”. Since γ is a smooth curve, adjacent segments of γ_n have angles that agree to within $O(1/n)$; thus adjacent perpendicular segments have angles that agree to within $O(1/n)$. Thus each of the quadrilaterals formed by $\gamma_n, \tilde{\gamma}_n$ and the segments joining their vertices have all angles within $O(1/n)$ of 90 degrees. See Figure 7.

By adding diagonals and using the unique piecewise linear map between corresponding triangles, we get a quasiconformal map from a true square to each of our quadrilaterals with dilatation bounded by $O(1/n)$. Piecing these together gives us a map from a $1 \times \frac{1}{n}$ rectangle to our tube T_n . See Figure 8.

By repeating the construction we can map several parallel straight tubes to several curved tubes as in Figure 9. We assume there are $\simeq n$ tubes, all have width $\simeq 1/n$ and are vertically separated by $\simeq 1/n$. This, combined with the fact that the edges of the tubes have bounded slope, implies that the piecewise linear map previously defined on the union of tubes can be extended to a quasiconformal map \tilde{f}_n of Q to itself with uniformly bounded constant K (independent of n).

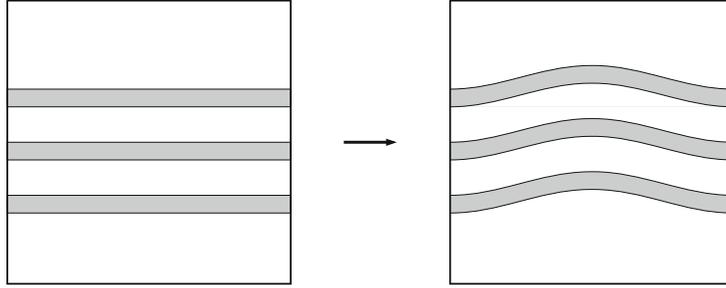


Figure 9: Repeat the construction on several *parallel* tubes

Now we must repeat the construction at smaller scales, without letting the dilatations blow up. However, this is quite simple. We construct a Cantor set as before, with nested families $\mathcal{I}_1 \supset \mathcal{I}_2 \supset \dots$ of intervals.

For any $I \in \mathcal{I}_k$, define a subcollection of $\simeq n_{k+1}$ sub-intervals in the middle half of I that have length $|I|/n_{k+1}$ and are separated from each other by at least $|I|/n_{k+1}$. Doing this for each $I \in \mathcal{I}_k$ defines \mathcal{I}_{k+1} .

Now, let $\mathcal{R}_k = \{[0, 1] \times I : I \in \mathcal{I}_k\}$ be the family of k th generation rectangles. Each of these may be divided into squares, on which we apply a scaled copy of the map $\tilde{f}_{n_{k+1}}$, to obtain a K -quasiconformal map of Q to itself, which restricts to the identity on $Q \setminus \bigcup \mathcal{R}_k$, and is $1 + O(\frac{1}{n_{k+1}})$ -QC on $\bigcup \mathcal{R}_{k+1}$. Denote this map by f_{n_k} .

Now, let $g_k = f_{n_1} \circ \dots \circ f_{n_k}$. Then g_k is $K + O(\frac{1}{n_1} + \dots + \frac{1}{n_k})$ quasiconformal, as every point in $\bigcup \mathcal{R}_j \cup \mathcal{R}_{j+1}$ is hit by some number of iterations of the identity, then by a K -QC map, and then by successive $1 + \frac{1}{n_i}$ quasiconformal maps, $i < j$.

Thus if we choose n_k to increase so quickly that $\sum_k n_k^{-1}$ converges to a small value, the dilatations will be uniformly bounded. In this case, we can take a limit and obtain a map and a Cantor set S so that every segment $[0, 1] \times \{y\}$, $y \in S$ has a definite oscillation near every point at infinitely many scales, and hence is nowhere rectifiable.

The set S comes with a covering by the nested collections of intervals \mathcal{I}_k . The Hausdorff measure of S can be bounded from below in the usual way of defining a measure on S by distributing mass from one of these intervals to all its children equally. Since $\limsup_{n \rightarrow \infty} nh(\delta/n) = \infty$ for any fixed $\delta > 0$, by choosing n_k large enough we can insure that

$$\sum h(|I_j|) \geq 2h(|I|),$$

where the sum is over the \mathcal{I}_{k+1} of $I \in \mathcal{I}_k$. A standard argument then shows that S has infinite h -measure. □

9 Remarks and Open Questions

Many natural questions in the vein of our results still remain open. The following question was suggested to us by Nages Shanmugalingam.

QUESTION 9.1. *Let $f: X \rightarrow Y$ be a quasiasymmetric mapping between Ahlfors D -regular (D -Loewner) spaces, $D > 1$. Is it true that for a fixed $d \in [0, D]$ and for $\text{mod}_{D/d}$ almost every Ahlfors d -regular set $E \subset X$, we have that $f(E)$ is Ahlfors d -regular as well?*

This question is open even in \mathbb{R}^N , except for $d = N$, which follows from a theorem of Gehring and Kelly [GK74]. The case of $d = D$ for more general metric spaces was considered by Korte et al. [KMS12].

9.1 Families of translates. The questions below are formulated in the case of the Euclidean space, but would also be interesting for Carnot groups.

How large is the collection of translates of a set E the dimension of which can simultaneously jump up by a prespecified amount? To be more precise, let $E \subset \mathbb{R}^N$ be a bounded Ahlfors d -regular set, $d \in (0, N)$, and f a quasiconformal map of \mathbb{R}^N . For $d' \in (d, N)$, we would like to estimate from above the Hausdorff dimension of the points y s.t. $\dim_H f(y + E) > d'$?

QUESTION 9.2. *Suppose $E \subset \mathbb{R}^N$ is a bounded d -regular set. Is it true that for $d' \in (d, N)$ and for every QC map f of \mathbb{R}^N we have*

$$\dim_H \{y \in \mathbb{R}^N : \dim_H f(y + E) > d'\} \leq \frac{d}{d'} N? \quad (9.1)$$

To estimate the dimension of the set appearing in (9.1) we would like to use Corollary 4.2 and a stronger version of Lemma 3.3, which we formulate in the case of \mathbb{R}^N .

Recall from Section 3.7, that if $y \in \mathbb{R}^N$ and λ is a measure on \mathbb{R}^N we denoted by $y_*\lambda$ the pushforward of λ by the translation by y , and for $K \subset \mathbb{R}^N$ we let $K_*\lambda = \{y_*\lambda : y \in K\}$.

QUESTION 9.3. *Let $p \geq 1$, $1 < D < N$ and $K \subset \mathbb{R}^N$. Suppose that $\text{mod}_p(K_*\lambda, \mu) = 0$, whenever μ is an upper D -regular measure on \mathbb{R}^N . Is $\dim_H K \leq D$?*

By Corollary 4.2, an affirmative answer to Question 9.3 would imply an affirmative answer to Question 9.2.

9.2 Sharpness for nonplanar mappings. Though Theorems 1.4 and 1.5 are quite general, our constructions in Theorems 1.7 and 1.9 seem difficult to extend beyond the planar case, as we do not have flexibility in constructing conformal mappings in higher dimensions. This leads to the following questions:

QUESTION 9.4. *Does an analogue to Theorem 1.7 remain true in higher dimensions? For example, given positive integers m, n , $N = m + n \geq 3$, and $t \in (0, n)$, is there a t -regular subset $S \subseteq \mathbb{R}^n$ such that for each $\epsilon > 0$, there is a quasiconformal map $f: \mathbb{R}^N \rightarrow \mathbb{R}^N$, such that for every Borel set $A \subseteq \mathbb{R}^m \times S$,*

$$\dim_H(f(A)) \geq (1 - \epsilon) \frac{N \dim_H(A)}{t + m}?$$

QUESTION 9.5. Suppose m, n, N are as before, and that

$$\lim_{t \rightarrow 0} \frac{h(t)}{t^n} = \infty. \quad (9.2)$$

Is there a compact set $S \subseteq [0, 1]^n$ and a quasiconformal map $f: \mathbb{R}^N \rightarrow \mathbb{R}^N$ so that

- (1) The quasiconformal constant of f is bounded independent of h and S .
- (2) S has infinite Hausdorff measure with respect to h (cf. Definition 2.2 below).
- (3) $f([0, 1]^m \times \{y\})$ contains no m -rectifiable subset for any $y \in S$.

Recall that a set is m -rectifiable if it is the union of an \mathcal{H}^n -nullset and a countable family of Lipschitz images of subsets of \mathbb{R}^m .

In the preceding question, one can consider other conditions in place of the absence of rectifiable subsets. For example, one could still ask in higher dimensions that the surfaces $f([0, 1]^m \times \{y\})$ contain no rectifiable curves.

An even stronger requirement would be that the surfaces can be parameterized by a map satisfying $|\phi(x) - \phi(y)| > \eta(|x - y|)$ for some η with $\eta(t)/t \rightarrow \infty$ as $t \rightarrow 0$. It is easy to see from the proof that our planar example in Theorem 1.7 has this latter property.

In higher dimensions, examples of such highly non-rectifiable surfaces were constructed by David and Toro [DT99] and Bishop [Bis99].

9.3 Distortion by Sobolev mappings. We lastly point out that these questions may be, and have been, explored in other classes of mappings.

As mentioned before, frequency of dimension distortion of Sobolev mappings $f \in W^{1,p}(\mathbb{R}^n, Y)$ were studied in [BMT13] for $p \geq n$. Dimension distortion in the case $1 < p < n$ has been explored by Hencl and Honzík, cf. [HH12, HH15]. Similar questions for more general source spaces, e.g. for the Heisenberg group, have been considered by Balogh et al. [BMT14, BTW13, BTW16]. Hencl and Honzík also considered dimension distortion for mappings in Triebel–Lizorkin spaces in [HH14].

As for results in the vein of Theorems 1.7 and 1.9, examples of mappings which simultaneously expand large families of subspaces or subsets have been exhibited in [BMT13, BTW13, HH12, HH14] in various contexts. In fact it was shown in these works that such mappings are “generic” in some sense. We refer the reader to the mentioned papers for more precise statements and definitions.

Even so, it is unclear to what extent our results extend to the Sobolev case. For example, we do not know the answer to the following question.

QUESTION 9.6. Suppose $N \geq 2$ and $0 < d < N$. Let $f: \mathbb{R}^N \rightarrow \mathbb{R}^N$ be a continuous Sobolev mapping, $f \in W^{1,p}(\mathbb{R}^N, \mathbb{R}^N)$, $p > N$. Is it true that for D/d -almost every Ahlfors d -regular set $E \subset \mathbb{R}^N$ we have $\dim_H f(E) \leq d$?

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