Conformal Fractals

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Preface

The goal of this book is to introduce readers to various types of fractal sets that rise naturally in dynamical and probabilistic settings that involve conformal maps or conformal invariance. Such sets usually have some sort of approximate self-similarity, but are not "self-similar" sets in the usual strict interpretation. Instead, small pieces of the set can be blown up to unit size by maps that are not similarities, but that are conformal or have bounded distortion is some quantified sense. Although several well known examples date back well over a century, there has been explosive growth in the study of such sets in recent years.

Among the topics that we include in the "conformal fractal" category are harmonic measure, analytic capacity, limit sets of Kleinian groups, Julia sets, diffusion limited aggregation, percolation clusters and Schramm-Loewner evolutions (SLE). In each case, we seek to give some basic definitions and results, illustrated by examples, but avoid technicalities as far as possible. Each of these topics could be (indeed, many have been) the sole subject of a lenghty book; our goal is to provide an introduction to these topics, describe some of the recent progress on them, and point the reader to more specialized treatments of their favorite topics.

We have made an effort to keep the book self-contained. It is not a continuation of our earlier book "Fractals in Probabilty and Analysis" and we do not assume familarity with that text, although we will use it as a reference (among others) to avoid repeating material covered there. The only prerequisite for reading this book is measure theory and probability at the level acquired in a first graduate course.

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Conformal maps and conformal invariants

This is a book about fractals that all have some sort of invariance under conformal maps. A fundamental tool for understanding such sets are conformal invariants, i.e., numerical values that can be associated to a certain geometric configurations and that remain unchanged (or at least change in predictable ways) under the application of conformal or holomorphic maps. There are three conformal invariants that will be particularly important through the book: extremal length, harmonic measure and hyperbolic distance. Of these, extremal length is the most important because it can be defined in many situations and estimated by direct geometric arguments. The other two are defined on the disk and then transferred to other domains by a conformal map. In this chapter, we introduce extremal length, hyperboli distance and harmonic measure, and derive a famous estimate for the latter, due to Arne Beurling, using the former. As a reward for our efforts we will deduce a growth bound, due to Harry Kesten, for diffusion limited aggregation (DLA), one of the most appealing, and most challenging, conformal fractals.

1.1 Extremal length

Our first conformal invariant is extremal length. Consider a positive function ρ on a domain Ω. We think of ρ as analogous to $|f'|$ where f is a conformal map on Ω. Just as the image area of a set E can be computed by integrating $\int_E |f'|^2 dxdy$, we can use ρ to define areas by $\int_E \rho^2 dxdy$. Similarly, just as we can define $\ell(f(\gamma)) = \int_\gamma |f'(z)| ds$, we can define the ρ-length of a curve γ by $\int_\gamma \rho ds$. For this to make sense, we need γ to be locally rectifiable (so the arclength measure ds is defined) and it is convenient to assume that ρ is Borel (so that its restriction to any curve γ is also Borel and hence measurable for length measure on γ).

Suppose Γ is a family of locally rectifiable paths in a planar domain Ω and ρ is a non-negative Borel function on Ω. We say ρ is **admissible** for Γ if

$$\ell(\Gamma) = \ell_\rho(\Gamma) = \inf_{\gamma \in \Gamma} \int_\gamma \rho ds \geq 1.$$

In this case we write $\rho \in \mathscr{A}(\Gamma)$. We define the **modulus** of the path family Γ as

$$\mathrm{Mod}(\Gamma) = \inf_\rho \int_M \rho^2 dxdy,$$

where the infimum is over all admissible ρ for Γ. The **extremal length** of Γ is defined as

$$\lambda(\Gamma) = 1/M(\Gamma).$$

Note that if the path family Γ is contained in a domain Ω, then we need only consider metrics ρ are zero outside Ω. Otherwise, we can define a new (smaller) metric by setting $\rho = 0$ outside Ω; the new metric is still admissible, and a smaller integral than before. Therefore $M(\Gamma)$ can be computed as the infimum over metrics which are only nonzero inside Ω.

Modulus and extremal length satisfy several useful properties that we list as a series of lemmas.

Lemma 1.1.1 (Conformal invariance) *If Γ is a family of curves in a domain Ω and f is a one-to-one holomorphic mapping from Ω to Ω' then $M(\Gamma) = M(f(\Gamma))$.*

Proof This is just the change of variables formulas

$$\int_\gamma \rho \circ f |f'| ds = \int_{f(\gamma)} \rho ds,$$

$$\int_\Omega (\rho \circ f)^2 |f'|^2 dxdy = \int_{f(\Omega)} \rho dxdy.$$

These imply that if $\rho \in \mathscr{A}(f(\Gamma))$ then $|f'| \cdot \rho \circ f \in \mathscr{A}(f(\Gamma))$, and thus by taking the infimum over such metrics we get $M(f(\Gamma)) \leq M(\Gamma)$ Note that there might be admissible metrics for $f(\Gamma)$ that are not of this form, possibly giving a strictly small modulus. However, by switching the roles of Ω and Ω' and replacing f by f^{-1} we see equality does indeed hold. $\square$

Lemma 1.1.2 (Monotonicity) *If Γ_0 and Γ_1 are path families such that every $\gamma \in \Gamma_0$ contains some curve in Γ_1 then $M(\Gamma_0) \leq M(\Gamma_1)$ and $\lambda(\Gamma_0) \geq \lambda(\Gamma_1)$.*

Proof The proof is immediate since $\mathscr{A}(\Gamma_0) \supset \mathscr{A}(\Gamma_1)$. $\square$

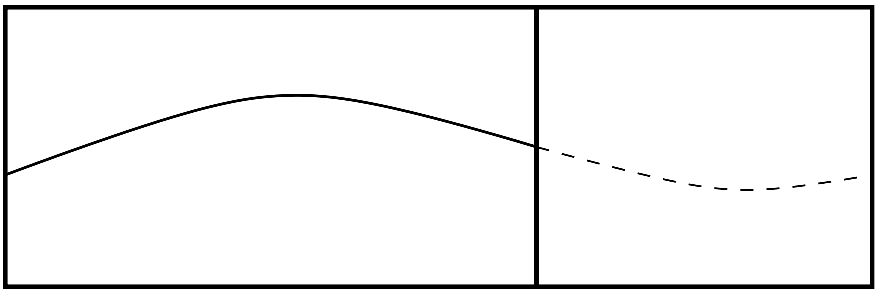


Figure 1.1.1 The Monotone rule: each curve of the first family contains a curve of the second family.

Lemma 1.1.3 (Grötsch Principle) *If Γ_0 and Γ_1 are families of curves in disjoint domains then $M(\Gamma_0 \cup \Gamma_1) = M(\Gamma_0) + M(\Gamma_1)$.*

Proof Suppose ρ_0 and ρ_1 are admissible for Γ_0 and Γ_1. Take $\rho = \rho_0$ and $\rho = \rho_1$ in their respective domains. Then it is easy to check that ρ is admissible for $\Gamma_0 \cup \Gamma_1$ and, since the domains are disjoint, $\int \rho^2 = \int \rho_1^2 + \int \rho_2^2$. Thus $M(\Gamma_0 \cup \Gamma_1) \leq M(\Gamma_0) + M(\Gamma_1)$. By restricting an admissible metric ρ to each domain, a similar argument proves the other direction. □

The Grötsch principle and the monotonicity combine to give

Corollary 1.1.4 (Parallel Rule) *Suppose Γ_0 and Γ_1 are path families in disjoint domains $\Omega_0, \Omega_1 \subset \Omega$ that connect disjoint sets E, F in $\partial\Omega$. If Γ is the path family connecting E and F in Ω, then*

$$M(\Gamma) \geq M(\Gamma_0) + M(\Gamma_1).$$

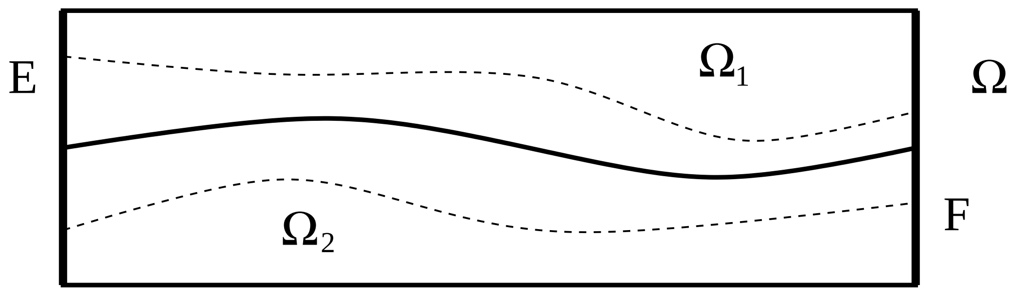


Figure 1.1.2 The Parallel Rule: curves connecting two boundary sets in the whole domain and in two disjoint subdomains.

Lemma 1.1.5 (Series Rule) *If Γ_0 and Γ_1 are families of curves in disjoint domains and every curve of $\mathscr{F}$ contains both a curve from both Γ_0 and Γ_1, then $\lambda(\Gamma) \geq \lambda(\Gamma_0) + \lambda(\Gamma_1)$.*

Proof If $\rho_j \in \mathscr{A}(\Gamma_j)$ for $j = 0,1$, then $\rho_t = (1-t)\rho_0 + t\rho_1$ is admissible for Γ. Since the domains are disjoint we may assume $\rho_0\rho_1 = 0$. Integrating ρ^2 then shows

$$M(\Gamma) \leq (1-t)^2 M(\Gamma_0) + t^2 M(\Gamma_1),$$

for each t. To find the optimal t set $a = M(\Gamma_1)$, $b = M(\Gamma_0)$, differentiate the right hand side above, and set it equal to zero

$$2at - 2b(1-t) = 0.$$

Solving gives $t = b/(a+b)$ and plugging this in above gives

$$M(\mathscr{F}) \leq t^2 a + (1-t^2)b = \frac{b^2 a a^2 b}{(a+b)^2}$$
$$= \frac{ab(a+b)}{(a+b)^2} = \frac{ab}{a+b} = \frac{1}{\frac{1}{a} + \frac{1}{b}}$$

or

$$\frac{1}{M(\Gamma)} \geq \frac{1}{M(\Gamma_0)} + \frac{1}{M(\Gamma_1)},$$

which, by definition, is the same as

$$\lambda(\Gamma) \geq \lambda(\Gamma_0) + \lambda(\Gamma_1). \quad \square$$

Next we actually compute the modulus of some path families. The fundamental example is to compute the modulus of the path family connecting opposite sides of a $a \times b$ rectangle; this serves as the model of almost all modulus estimates. So suppose $R = [0,b] \times [0,a]$ is a b wide and a high rectangle and Γ consists of all rectifiable curves in R with one endpoint on each of the sides of length a.

Lemma 1.1.6 $\mathrm{Mod}(\Gamma) = a/b$.

Proof Then each such curve has length at least b, so if we let ρ be the constant $1/b$ function on R we have

$$\int_\gamma \rho\, ds \geq 1,$$

for all $\gamma \in \Gamma$. Thus this metric is admissible and so

$$\mathrm{Mod}(\Gamma) \leq \iint_T \rho^2 dxdy = \frac{1}{b^2} ab = \frac{a}{b}.$$

To prove a lower bound, we use the well known Cauchy-Schwarz inequality:

$$(\int fgdx)^2 \le (\int f^2dx)(\int g^2dx).$$

To apply this, suppose ρ is an admissible metric on R for γ. Every horizontal segment in R connecting the two sides of length a is in Γ, so since γ is admissible,

$$\int_0^b \rho(x,y)dx \ge 1,$$

and so by Cauchy-Schwarz

$$1 \le \int_0^b (1\cdot\rho(x,y))dx \le \int_0^b 1^2dx \cdot \int_0^b \rho^2(x,y)dx.$$

Now integrate with respect to y to get

$$a = \int_0^a 1dy \le b\int_0^a\int_0^b \rho^2(x,y)dxdy,$$

or

$$\frac{a}{b} \le \iint_R \rho^2 dxdy,$$

which implies $\mathrm{Mod}(\Gamma) \ge \frac{b}{a}$. Thus $\mathrm{Mod}(\Gamma) = \frac{b}{a}$. □

Another useful computation is the modulus of the family of path connecting the inner and out boundaries of the annulus $A = \{z : r < |z| < R\}$.

Lemma 1.1.7 *If $A = \{z : r < |z| < R\}$ then the modulus of the path family connecting the two boundary components is $2\pi/\log\frac{R}{r}$. More generally, if Γ is the family of paths connecting $r\mathbb{T}$ to a set $E \subset R\mathbb{T}$, then $M(\Gamma) \ge |E|/\log\frac{R}{r}$.*

Proof By conformal invariance, we can rescale and assume $r = 1$. Suppose ρ is admissible for Γ. Then for each $z \in E \subset \mathbb{T}$,

$$1 \le (\int_1^R \rho ds)^2 \le (\int_1^R \frac{ds}{s})(\int_1^R \rho^2 sds) = \log R\int_1^R \rho^2 sds$$

and hence we get

$$\int_0^{2\pi}\int_1^R \rho^2 sdsd\theta \ge \int_E\int_1^R \rho^2 sdsd\theta \ge |E|\int_1^R \rho^2 sds \ge \frac{|E|}{\log R}.$$

When $E = \mathbb{T}$ we prove the other direction by taking $\rho = (s\log R)^{-1}$. This is an admissible metric and

$$\mathrm{Mod}(\Gamma) \le \int_0^{2\pi}\int_1^R \rho^2 sdsd\theta = \frac{2\pi}{(\log R)^2}\int_1^R \frac{1}{s}ds = \frac{2\pi}{\log R}. \quad \square$$

Given a Jordan domain Ω and two disjoint closed sets $E, F \subset \partial\Omega$, the **extremal distance** between E and F (in Ω) is the extremal length of the path family in Ω connecting E to F (paths in Ω that have one endpoint in E and one endpoint in F). The series rule is a sort of "reverse triangle inequality" for extremal distance. See Figure 1.1.3.

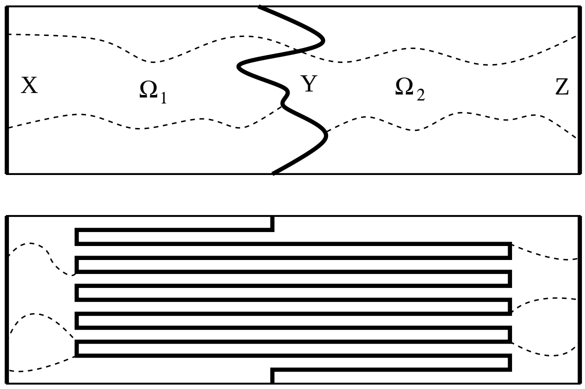


Figure 1.1.3 The series rule says that the extremal distance from X to Z in the rectangle is greater than the sum the extremal distance from X to Y in Ω_1 plus the extremal distance from Y to Z in Ω_2. The bottom figure show a more extreme case where the extremal distance between opposite sides of the rectangle is much larger than either of the other two terms.

Extremal distance can be particularly useful when both E and F are connected. In this case, their complement in $\partial\Omega$ also consists of two arcs, and the extremal distance between these is the reciprocal of the extremal distance between E and F. This holds because of conformal invariance, the fact that it is true for rectangles and an applications of the Riemann mapping theorem (we can always map Ω to a rectangle, so that E and F go to opposite sides (See Exercise 1.1).

Obtaining an upper bound for the modulus of a path family usually involves choosing a metric; every metric gives an upper bound. Giving a lower bound usually involves a Cauchy-Schwarz type argument, which can be harder to do in general cases. However, in the special case of extremal distance between arcs $E, F \subset \partial\Omega$, a lower bound for the modulus can also be computed by giving a upper bound for the reciprocal separating family. Thus estimates of both types can be given by producing metrics (for different families) and this is often the easiest thing to do.

If γ is a path in the plane let $\bar{\gamma}$ be its reflection across the real line and let

$$\gamma_u = \gamma \cap \mathbb{H}_u, \quad \gamma_\ell = \gamma \cap \mathbb{H}_l, \quad \gamma_+ = \gamma_u \cup \overline{\gamma_\ell},$$

where $\mathbb{H}_u = \{x+iy : y > 0\}$, $\mathbb{H}_l = \{x+iy : y < 0\}$ denote the upper and lower half-planes. For a path family Γ, define $\overline{\Gamma} = \{\bar{\gamma} : \gamma \in \Gamma\}$ and $\Gamma_+ = \{\gamma_+ : \gamma \in \Gamma\}$.

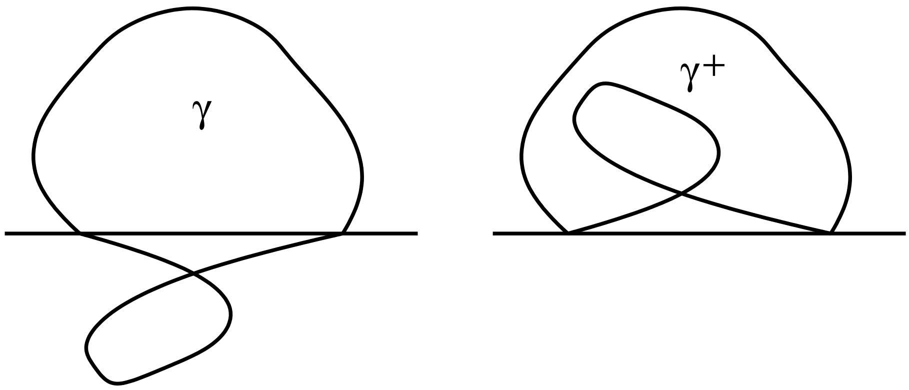


Figure 1.1.4 The curves γ and γ_+

Lemma 1.1.8 (Symmetry Rule) *If* $\Gamma = \overline{\Gamma}$ *then* $M(\Gamma) = 2M(\Gamma_+)$.

Proof We start by proving $M(\Gamma) \leq 2M(\Gamma_+)$. Given a metric ρ admissible for γ_+, define $\sigma(z) = \max(\rho(z), \rho(\bar{z}))$. Then for any $\gamma \in \Gamma$,

$$\begin{aligned} \int_\gamma \sigma ds &= \int_{\gamma_u} \sigma(z) ds + \int_{\gamma_\ell} \sigma(z) ds \\ &\geq \int_{\gamma_u} \rho(z) ds + \int_{\gamma_\ell} \rho(\bar{z}) ds \\ &= \int_{\gamma_u} \rho(z) ds + \int_{\overline{\gamma_\ell}} \rho(z) ds \\ &\geq \int_{\gamma_+} \rho ds \\ &\geq \inf_{\gamma \in \Gamma} \int_\gamma \rho ds. \end{aligned}$$

Thus if ρ admissible for Γ_+, then σ is admissible for Γ. Since $\max(a,b)^2 \leq a^2 + b^2$, integrating gives

$$M(\Gamma) \leq \int \sigma^2 dxdy \leq \int \rho^2(z) dxdy + \int \rho^2(\bar{z}) dxdy \leq 2 \int \rho^2(z) dxdy.$$

Taking the infimum over admissible ρ's for Γ_+ makes the right hand side equal to $2M(\Gamma_+)$, proving $\text{Mod}(\Gamma) \le 2\text{Mod}(\Gamma_+)$.

For the other direction, given ρ define $\sigma(z) = \rho(z) + \rho(\bar{z})$ for $z \in \mathbb{H}_u$ and $\sigma = 0$ if $z \in \mathbb{H}_l$. Then

$$\begin{aligned}
\int_{\gamma_+} \sigma ds &= \int_{\gamma_+} \rho(z) + \rho(\bar{z}) ds \\
&= \int_{\gamma_u} \rho(z) ds + \int_{\gamma_u} \rho(\bar{z}) ds + \int_{\gamma_e ll} \rho(z) + \int_{\gamma_\ell} \rho(\bar{z}) ds \\
&= \int_{\gamma} \rho(z) ds + \int_{\overline{\gamma}} \rho(\bar{z}) ds \\
&= 2 \inf_{\rho} \int_{\gamma} \rho ds.
\end{aligned}$$

Thus if ρ is admissible for Γ, $\frac{1}{2}\sigma$ is admissible for Γ_+. Since $(a+b)^2 \le 2(a^2 + b^2)$, we get

$$\begin{aligned}
M(\Gamma_+) &\le \int (\frac{1}{2}\sigma)^2 dxdy \\
&= \frac{1}{4} \int_{\mathbb{H}_u} (\rho(z) + \rho(\bar{z}))^2 dxdy \\
&\le \frac{1}{2} \int_{\mathbb{H}_u} \rho^2(z) dxdy + \int_{\mathbb{H}_u} \rho^2(\bar{z}) dxdy \\
&= \frac{1}{2} \int \rho^2 dxdy.
\end{aligned}$$

Taking the infimum over all admissible ρ's for Γ gives $\frac{1}{2}M(\Gamma)$ on the right hand side, proving the lemma. □

Lemma 1.1.9 *Let $\mathbb{D}^* = \{z : |z| > 1\}$ and $\Omega_0 = \mathbb{D}^* \setminus [R, \infty)$ for some $R > 1$. Let $\Omega = \mathbb{D}^* \setminus K$, where K is a closed, unbounded, connected set in $\mathbb{D}^*$ which contains the point $\{R\}$. Let Γ_0, Γ denote the path families in these domains with separate the two boundary components. Then $M(\Gamma_0) \le M(\Gamma)$.*

Proof We use the symmetry principle we just proved. The family Γ_0 is clearly symmetric (i.e., $\Gamma = \overline{\Gamma}$, so $M(\Gamma_0^+) = \frac{1}{2}M(\Gamma_0)$. The family Γ may not be symmetric, but we can replace it by a larger family that is. Let Γ_R be the collection of rectifiable curves in $\mathbb{D}^* \setminus \{R\}$ which have zero winding number around $\{R\}$, but non-zero winding number around 0. Clearly $\Gamma \subset \Gamma_R$ and Γ_R is symmetric so $M(\Gamma) \ge M(\Gamma_R) = 2M(\Gamma_R^+)$. Thus all we have to do is show $M(\Gamma_R^+) = M(\Gamma_0^+)$. We will actually show $\Gamma_R^+ = \Gamma_0^+$. Since $\Gamma_0 \subset \Gamma_R$ is obvious, we need only show $\Gamma_R^+ \subset \Gamma_0^+$.

Suppose $\gamma \in \Gamma_R$. Since γ has non-zero winding around 0 it must cross both

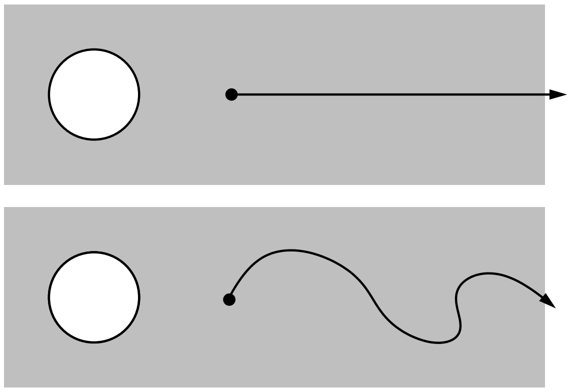


Figure 1.1.5 The topological annulus on top has smaller modulus than any other annulus formed by connecting R to ∞.

the negative and positive real axes. If it never crossed $(0,R)$ then the winding around 0 and R would be the same, which false, so γ must cross$(0,R)$ as well. Choose points $z_- \in \gamma \cap (-\infty,0)$ and $z_+ \in \gamma \cap (0,R)$. These points divide γ into two subarcs γ_1 and γ_2. Then $\gamma_+ = (\gamma_1)_+ \cup (\gamma_2)_+$. But if we reflect $(\gamma_2)_+$ into the lower half-plane and join it to $(\gamma_1)_+$ it forms a closed curve γ_0 that is in Γ_0 and $(\gamma_0)_+ = \gamma_+$. Thus $\gamma_+ \in (\Gamma_0)_+$, as desired. □

Let $\Omega_{\varepsilon,R} = \{z : |z| > \varepsilon\} \setminus [R,\infty)$. Note that $\Omega_{1,R}$ is the domain considered in the previous lemma (e.g., see the top of Figure 1.1.5). We can estimate the moduli of these domains using the Koebe map

$$k(z) = \frac{z}{(1+z)^2} = z - 2z^2 + 3z^3 - 4z^4 + 5z^5 - \dots,$$

which conformal maps the unit disk to $\mathbb{R}^2 \setminus [\frac{1}{4},\infty)$ and satisfies $k(0) = 0$, $k'(0) = 1$. Then $k^{-1}(\frac{1}{4R}z)$ maps $\Omega_{\varepsilon,R}$ conformally to an annular domain in the disk whose outer boundary is the unit circle and whose inner boundary is trapped between the circle of radius $\frac{\varepsilon}{4R}(1 \pm O(\frac{\varepsilon}{R}))$. Thus the modulus of $\Omega_{\varepsilon,R}$ is

$$2\pi \log \frac{4R}{\varepsilon} + O(\frac{\varepsilon}{R}). \tag{1.1.1}$$

Next we prove the Koebe $\frac{1}{4}$-theorem for conformal maps. The standard proof of Koebe's $\frac{1}{4}$-theorem uses Green's theorem to estimate the power series coefficients of conformal map (proving the Bieberbach conjecture for the second coefficient). However here we will present a proof, due to Mateljevic [13], that uses the symmetry property of extremal length.

Theorem 1.1.10 (The Koebe $\frac{1}{4}$ Theorem) *Suppose f is holomorphic, 1-1 on $\mathbb{D}$ and $f(0) = 0$, $f'(0) = 1$. Then $D(0, \frac{1}{4}) \subset f(\mathbb{D})$.*

Proof Recall that the modulus of a doubly connected domain is the modulus of the path family that separates the two boundary components (and is equal to the extremal distance between the boundary components). Let $R = \text{dist}(0, \partial f(\mathbb{D}))$. Let $A_{\varepsilon,r} = \{z : \varepsilon < |z| < r\}$ and note that by conformal invariance

$$2\pi \log \frac{1}{\varepsilon} = M(A_{\varepsilon,1}) = M(f(A_{\varepsilon,1})).$$

Let $\delta = \min_{|z|=\varepsilon} |f(z)|$. Since $f'(0) = 1$, we have $\delta = \varepsilon + O(\varepsilon^2)$. Note that $f(A_{\varepsilon,1}) \subset f(\mathbb{D}) \setminus D(0, \delta)$, so

$$M(f(A_{\varepsilon,1})) \leq M(f(\mathbb{D}) \setminus D(0, \delta)).$$

By Lemma 1.1.9 and Equation (1.1.1),

$$M(f(\mathbb{D}) \setminus D(0, \delta)) \leq M(\Omega_{\delta,R}) = 2\pi \log \frac{4R}{\delta} + O(\frac{\delta}{R}).$$

Putting these together gives

$$2\pi \log \frac{4R}{\delta} + O(\frac{\delta}{R}) \geq 2\pi \log \frac{1}{\varepsilon},$$

or

$$\log 4R - \log(\varepsilon + O(\varepsilon^2)) + O(\frac{\varepsilon}{R}) \geq -\log \varepsilon,$$

and hence

$$\log 4R \geq -O(\frac{\varepsilon}{R}) + \log(1 + O(\varepsilon)).$$

Taking $\varepsilon \to 0$ shows $\log 4R \geq 0$, or $R \geq \frac{1}{4}$. □

1.2 Logarithmic capacity

Logarithmic capacity associates a non-negative number to each Borel subset of the unit circle. Applying a Möbius transformation can change this value, so it is not a conformal invariant, but it will act as an intermediate between extremal and harmonic measure (a conformal invariant that will be defined later).

Suppose μ is a positive, finite Borel measure on $\mathbb{C}$ and define its potential function as

$$U_\mu(z) = \int \log \frac{2}{|z - w|} d\mu(w), z \in \mathbb{C}.$$

and its energy integral by

$$I(\mu) = \iint \log \frac{2}{|z-w|} d\mu(z)d\mu(w) = \int U_\mu(z)d\mu(z).$$

We put the "2" in the numerator so that the integrand is non-negative when $z, w \in \mathbb{T}$, however, this is a non-standard usage.

Lemma 1.2.1 *U_μ is lower semi-continuous, i.e.,*

$$\liminf_{z \to z_0} U_\mu(z) \geq U_\mu(z_0).$$

Proof Fatou's lemma. □

Recall that $\mu_n \to \mu$ weak-* if $\int f d\mu_n \to \int f d\mu$ for every continuous function f of compact support.

Lemma 1.2.2 *If $\{\mu_n\}$ are positive measures and $\mu_n \to \mu$ weak*, then* $\liminf_n U_{\mu_n}(z) \geq U_\mu(z)$.

Proof If we replace $\varphi = \log \frac{2}{|z-w|}$ by the continuous kernel $\varphi_r = \max(r, \varphi)$ in the definition of U to get U^r, then weak convergence implies

$$\lim_n U^r_{\mu_n}(z) \nearrow U^r_\mu(z).$$

Moreover, the convergence is increasing since the measures positive. So for any $\varepsilon > 0$ we can choose N so that $n > N$ implies

$$U^r_{\mu_n}(z) \geq U^r_\mu(z) - \varepsilon.$$

As $r \to \infty$ $U^r \to U$ (by the monotone convergence theorem), so for r large enough and $n > N$ we have

$$U_{\mu_n}(z) \geq U^r_{\mu_n}(z) \geq U_\mu(z) - 2\varepsilon.$$

which proves the result. □

Lemma 1.2.3 *If $\mu_n \to \mu$ weak*, then* $\liminf_n I(\mu_n) \geq I(\mu)$.

Proof The proof is almost the same as for the previous lemma, except that we have to know that if $\{\mu_n\}$ converges weak*, then so does the product measure $\mu_n \times \mu_n$. However, weak convergence of $\{\mu_n\}$ implies convergence of integrals of the form

$$\iint f(x)g(y)d\mu_n(x)d\mu_n(y).$$

and Stone-Weierstrass theorem implies that the finite sums of such product functions are dense in all continuous function on the product space. Since weak-* convergent sequences are bounded, the product measures $\mu_n \times \mu_n$ also have uniformly bounded masses, and hence convergence on a dense set of continuous functions of compact support implies convergence on all continuous functions of compact support. This, together with the fact that weak* convergent sequences are bounded ([**?**]), implies that $\mu_n \times \mu_n$ converges weak*. $\square$

Suppose E is Borel and μ is a positive measure that has its closed support inside E. We say μ is admissible for E if $U_\mu \leq 1$ on E and we define the **logarithmic capacity** of E as

$$\mathrm{cap}(E) = \sup\{\|\mu\| : \mu \text{ is admissible for } E\}$$

and we write $\mu \in \mathscr{A}(E)$. We define the **outer capacity** (or exterior capacity) as

$$\mathrm{cap}^*(E) = \inf\{\mathrm{cap}(V) : E \subset V, V \text{open}\}.$$

We say that a set E is **capacitable** if $\mathrm{cap}(E) = \mathrm{cap}^*(E)$.

The logarithmic kernel can be replaced by other functions, e.g., $|z-w|^{-\alpha}$, and there is a different capacity associated to each one. To be precise, we should denote logarithmic capacity as $\mathrm{cap}_{\log}$ or logcap, but to simplify notation we simply use "cap" and will often refer to logarithmic capacity as just "capacity". Since we do not use any other capacities in these notes, this abuse should not cause confusion.

WARNING: The logarithmic capacity that we have defined is **NOT** the same as is used in other texts such as Garnett and Marshall's book [7], but is related to what they call the Robin's constant of E, denoted $\gamma(E)$. The exact relationship is $\gamma(E) = \frac{1}{\mathrm{cap}(E)} - \log 2$. Garnett and Marshall [7] define the logarithmic capacity of E as $\exp(-\gamma(E))$. The reason for doing this is that the logarithmic kernel $\log \frac{1}{|z-w|}$ takes both positive and negative values in the plane, so the potential functions for general measures and the Robin's constant for general sets need not be non-negative. Exponentiating takes care of this. Since we are only interested in computing the capacity of subsets of the circle, taking the extra "2" in the logarithm gave us a non-negative kernel on the unit circle, and we defined a corresponding capacity in the usual way. Since the kernel is the logarithm, we feel justified in calling the corresponding capacity the logarithmic capacity, despite the divergence with usual usage.

POSSIBLE ALTERNATES : Robin's capacity, conformal capacity, circular capacity.

Lemma 1.2.4 *Compact sets are capacitable.*

Proof Since $\mathrm{cap}(E) \leq \mathrm{cap}^*(E)$ is obvious, we only have to prove the opposite direction. Set $U_n = \{z : \mathrm{dist}(z, E) < 1/n\}$ and choose a measure μ_n supported in U_n with $\|\mu_n\| \geq \text{cap}(U_n) - 1/n$. Let μ be a weak accumulation point of $\{\mu_n\}$ and note

$$U_\mu(z) = \int \log \frac{2}{|z-w|} d\mu(w) \leq \int \log \frac{2}{|z-w|} d\mu_n(w) \leq 1$$

so μ is admissible in the definition of $\text{cap}(E)$. Thus

$$\text{cap}(E) \geq \limsup \|\mu_n\| = \lim \text{cap}(U_n) = \lim \text{cap}(U_n) = \text{cap}^*(E).$$

□

It is also true that all Borel sets are capacitable. Indeed, this holds for all analytic sets (i.e., continuous images of complete separable topological spaces). See Appendix B of [2].

It is clear from the definitions that logarithmic capacity is monotone

$$E \subset F \quad \Rightarrow \quad \text{cap}(E) \leq \text{cap}(F). \tag{1.2.1}$$

and satisfies the regularity condition

$$\text{cap}(E) = \sup\{\text{cap}(K) : K \subset E, K \text{compact}\}. \tag{1.2.2}$$

Lemma 1.2.5 (Sub-additive) *For any sets* $\{E_n\}$,

$$\text{cap}(\cup E_n) \leq \sum \text{cap}(E_n). \tag{1.2.3}$$

Proof We can write any $\mu = \sum \mu_n$ as a sum of mutually singular measures so that μ_n gives full mass to E_n. We can then restrict each μ_n to a compact subset K_n of E_n so that $\mu_n(K_n) \geq (1-\varepsilon)\mu(E_n)$. These restrictions are admissible for each E_n and hence

$$\sum \text{cap}(E_n) \geq \sum \mu_n(K_n) \geq (1-\varepsilon) \sum \mu_n(E_n) = (1-\varepsilon)\|\mu\|.$$

Taking $\varepsilon \to 0$ proves the result. □

Corollary 1.2.6 *A countable union of zero capacity sets has zero capacity.*

Corollary 1.2.7 *Outer capacity is also sub-additive.*

Proof Given sets $\{E_n\}$ choose open sets $V_n \supset E_n$ so that $\text{cap}(V_n) \leq \text{cap}^*(E_n) + \varepsilon 2^{-n}$. By the sub-additivity of capacity

$$\text{cap}^*(\cup E_n) \leq \text{cap}(\cup V_n) \leq \sum \text{cap}(V_n) \leq \varepsilon + \sum \text{cap}^*(E_n).$$

Taking $\varepsilon \to$ proves the result. □

Although capacity informally "measures" the size of a set, it is not additive, and hence not a measure. See Exercise 1.4.

Lemma 1.2.8 *If E is compact, there exists an admissible μ that attains the maximum mass in the definition of capacity and $U_\mu(z) = 1$ everywhere on E, except possible a set of capacity zero.*

Proof Let μ_n be a sequence of measures on E so that $\|\mu_n\| \to \text{cap}(E)$ and $U_n = U_{\mu_n}$ is bounded above by 1 on E (such a sequence exists by the definition of logarithmic capacity). By Lemma 1.2.2, U_μ is also bounded above by 1. Also, by a standard property of weak* convergence $\|\mu\| \le \liminf_n \|\mu_n\| = \text{cap}(E)$ ([**?**]), and by Lemma 1.2.3,

$$I(\mu) \le \liminf_n I(\mu_n) \le \liminf_n \|\mu_n\| = \text{cap}(E),$$

so we must have $I(\mu) = \text{cap}(E)$.

First we claim that $U_\mu \ge 1$ except possibly on a set of zero capacity. Otherwise let $T \subset E$ be a set of positive capacity on which $U_\mu < 1 - \varepsilon$ and let σ be a non-zero, positive measure on T which potential bounded by 1. Define

$$\mu_t = (1-t)\mu + t\sigma.$$

This is a measure on E so that

$$\begin{aligned}
I(\mu_t) &\le \int \log \frac{1}{|z-w|}((1-t)d\mu + td\sigma)((1-t)d\mu + td\sigma) \\
&\le (1-t)^2 I(\mu) + 2t \int U_\mu d\sigma + t^2 I(\sigma) \\
&\le I(\mu) - 2tI(\mu) + 2t \int U_\mu d\sigma + O(t^2) \\
&\le I(\mu) - 2tI(\mu) + 2t(1-\varepsilon)\|\sigma\| + O(t^2) \\
&< I(\mu),
\end{aligned}$$

if $t > 0$ is small enough. This contradicts minimality of μ.

Next we show that $U_\mu \le 1$ everywhere on the closed support of μ. By the previous step we know $U_\mu \ge 1$ except on capacity zero, hence except on a set of μ-measure zero. If there is a point z in the support of μ such that $U_\mu(z) > 1$, then by lower semi-continuity of potentials, U_μ is $> 1 + \varepsilon$ on some neighborhood of z and this neighborhood has positive μ measure (since z is in the support of μ) and thus $I(\mu) = \int U_\mu d\mu > \|\mu\|$, a contradiction. □

The following makes a connection between logarithmic capacity and extremal length. Eventually, this will become a connection between extremal length and harmonic measure.

If $K \subset \mathbb{D}$ is a compact connected set with smooth boundary with 0 in the interior of K. Let K^* be the reflection of K across $\mathbb{T}$. For any $E \subset \mathbb{T}$ that is a finite union of closed intervals, let Ω be the connected component of $\mathbb{C} \setminus (E \cup$

$K \cup K^*$) that has E on its boundary. Let $h(z)$ be the harmonic function in Ω with boundary values 0 on K and K^* and boundary value 1 on E. By the usual theory of the Dirichlet problem (e.g. [**?**]), all boundary points are regular (since all boundary components are non-degenerate continua) and hence h extends continuously to the boundary with the correct boundary values. Moreover, h is symmetric with respect to $\mathbb{T}$, and this implies its normal derivative on $\mathbb{T} \setminus E$ is 0. Let $D(h) = \int_{\mathbb{D}\setminus K} |\nabla h|^2 dxdy.$

Lemma 1.2.9 *With notation as above,* $M(\Gamma_E) = D(h)$.

Proof Clearly $|\nabla h|$ is an admissible metric for Γ_E, so

$$M(\Gamma_E) \leq D(h) \equiv \int_{\mathbb{D}\setminus K} |\nabla h|^2 dxdy.$$

Thus we need only show the other direction.

Green's theorem states that

$$\iint_\Omega u\Delta v - v\Delta u dxdy = \int_{\partial\Omega} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} ds. \tag{1.2.4}$$

Using this and the fact that $h = 1$ on E, we have

$$\int_{\partial K} \frac{\partial h}{\partial n} ds = -\int_{\mathbb{T}} \frac{\partial h}{\partial n} ds = -\int_E \frac{\partial h}{\partial n} ds = -\int_E h\frac{\partial h}{\partial n} ds.$$

and

$$\begin{aligned}\int_{\partial K} \frac{\partial h}{\partial n} ds &= -\frac{1}{2}\int_E \frac{\partial(h^2)}{\partial n} ds \\ &= \frac{1}{2}\int_{\mathbb{T}\setminus E} \frac{\partial(h^2)}{\partial n} ds + \frac{1}{2}\int_{\partial K} \frac{\partial(h^2)}{\partial n} ds + \frac{1}{2}\int_{\mathbb{D}\setminus K} \Delta(h^2) dxdy.\end{aligned}$$

The first term is zero because h has normal derivative zero on $\mathbb{T} \setminus E$, and hence the same is true for h^2. The second term is zero because h is zero on K and so $\frac{\partial(h^2)}{\partial n} h^2 = 2h\frac{\partial h}{\partial n} = 0$. To evaluate the third term, we use the identity

$$\begin{aligned}\Delta(h^2) &= 2h_x \cdot h_x + 2h \cdot h_{xx} + 2h_y \cdot h_y + 2h \cdot h_{yy} \\ &= 2h\Delta h + 2\nabla h \cdot \nabla h \\ &= 2h \cdot 0 + 2|\nabla h|^2 \\ &= 2|\nabla h|^2,\end{aligned}$$

to deduce

$$\frac{1}{2}\int_{\mathbb{D}\setminus K} \Delta(h^2) dxdy = \int_{\mathbb{D}\setminus K} \Delta(h^2) dxdy.$$

Therefore,

$$\int_{\partial K} \frac{\partial h}{\partial n} ds = \int_{\mathbb{D}\setminus K} \Delta(h^2) dxdy.$$

Thus the tangential derivative of h's harmonic conjugate has integral $D(h)$ around ∂K and therefore $2\pi h/D(h)$ is the real part of a holomorphic function g on $\mathbb{D}\setminus K$. Then $f = \exp(g)$ maps $\mathbb{D}\setminus K$ into the annulus

$$A = \{z : 1 < |z| < \exp(2\pi/D(h))\}$$

with the components of E mapping to arcs of the outer circle and the components of $\mathbb{T}\setminus E$ mapping to radial slits. The path family Γ_E maps to the path family connecting the inner and outer circles without hitting the radial slits, and our earlier computations show the modulus of this family is $D(h)$. □

Theorem 1.2.10 (Pfluger's theorem) *If $K \subset \mathbb{D}$ is a compact connected set with smooth boundary with 0 in the interior of K. Then there are constants C_1, C_2 so that following holds. For any $E \subset \mathbb{T}$ that is a finite union of closed intervals,*

$$\frac{1}{\mathrm{cap}(E)} + C_1 \leq \pi\lambda(\Gamma_E)) \leq \frac{1}{\mathrm{cap}(E)} + C_2,$$

where Γ_E is the path family connecting K to E. The constants C_1, C_2 can be chosen to depend only on $0 < r < R < 1$ if $\partial K \subset \{r \leq |z| \leq R\}$.

Proof Using Lemma 1.2.9, we only have to relate $D(h)$ to the logarithmic capacity of E. Let μ be the equilibrium probability measure for E. We know in general that $U_\mu = \gamma$ where $\gamma = 1/\mathrm{cap}(E)$ almost everywhere on E (since sets of zero capacity have zero measure) and is continuous off E, but since U_μ is harmonic in $\mathbb{D}$ and equals the Poisson integral of its boundary values, we can deduce $U_\mu = \gamma$ everywhere on E. Let $v(z) = \frac{1}{2}(U_\mu(z) + U_\mu(1/\bar{z}))$. Then since ∂K has positive distance from 0, there are constants C_1, C_2 so that

$$v + C_1 \leq 0, \qquad v + C_2 \geq 0,$$

on ∂K. Note that $C_1 \geq -\gamma$ by the maximum principle and $C_2 \geq 0$ trivially. Moreover, since μ is a probability measure supported on the unit circle, given $0 < r < R < 1$, U_μ is uniformly bounded on both the annulus $\{r \leq |z| \leq R\}$ and its reflection across the unit circle, since these both have bounded, but positive distance from the unit circle. This proves that C_1, C_2 can be chosen to depend on only these numbers, as claimed in the final statement of the theorem.

The following inequalities are easy to check on K, K^* and E,

$$\frac{v(z) + C_1}{\gamma + C_1} \leq h(z) \leq \frac{v(z) + C_2}{\gamma + C_2}.$$

and hence hold on Ω by the maximum principle. Since we have equality on E, we also get

$$\frac{\partial}{\partial n}\left(\frac{v(z)+C_1}{\gamma+C_1}\right) \le \frac{\partial h}{\partial n} \le \frac{\partial}{\partial n}\left(\frac{v(z)+C_2}{\gamma+C_2}\right)$$

for $z \in E$. When we integrate over E, the middle term is $-D(h)$ (we computed this above) and by Green's theorem

$$\begin{aligned} -\int_E \frac{\partial}{\partial n}\frac{v(z)+C_1}{\gamma+C_1}ds &= \frac{1}{\gamma+C_1}\int_{\mathbb{D}} \Delta(v)dxdy \\ &= \frac{\pi}{\gamma+C_1} \end{aligned}$$

because v is harmonic except for a $\frac{1}{2}\log\frac{1}{|z|}$ pole at the origin. A similar computation holds for the other term and hence

$$\frac{\pi}{\gamma+C_1} \le D(h) = M(\Gamma_E) \le \frac{\pi}{\gamma+C_2},$$

since $D(h) = \int_E \frac{\partial h}{\partial n}ds$. Hence

$$\gamma + C_1 \le \pi\lambda(\Gamma_E) \le \gamma + C_2.$$

This completes the proof of Pfluger's theorem for finite unions of intervals. □

Next we prove Pfluger's theorem for all compact subsets of $\mathbb{T}$. First we need a continuity property of extremal length. Recall that an extended real-valued function is lower semi-continuous if all sets of the form $\{f > \alpha\}$ are open.

Lemma 1.2.11 *Suppose $E \cap \mathbb{T}$ is compact, $K \subset \mathbb{D}$ is compact, connected and contains the origin, and Γ_E is the path family connecting K and E in $\mathbb{D}\setminus K$. Fix an admissible metric ρ for Γ_E and for each $z \in \mathbb{T}$, define $f(z) = \inf \int_\gamma \rho ds$ where the infimum is over all paths in Γ_E that connect K to z. Then f is lower semi-continuous.*

Proof Suppose $z_0 \in \mathbb{T}$ and use Cauchy-Schwarz to get

$$\begin{aligned} \int_{2^{-n-1}}^{2^{-n}} \left(\int_{|z-z_0|=r} \rho ds\right)^2 dr &\le \int_{2^{-n-1}}^{2^{-n}} \left(\int_{|z-z_0|=r} \rho^2 ds\right) dr \left(\int_{|z-z_0|=r} 1 ds\right) dr \\ &\le \int_{2^{-n-1}}^{2^{-n}} r \int_0^{2\pi} \rho^2 r d\theta dr \\ &\le \pi 2^{-n} \int_{2^{-n-1}<|z-z_0|<2^{-n}} \rho^2 dxdy \\ &= o(2^{-n}). \end{aligned}$$

Therefore we can choose circular cross-cuts $\{\gamma_n\} \subset \{z : 2^{-n-1} < |z - z_0| <$

$2^{-n}\}$ of $\mathbb{D}$ centered at z_0 and with ρ-length ε_n tending to 0. By taking s subsequence we may assume $\sum \varepsilon_n < \infty$. Now choose $z_n \to z_0$ with

$$f(z_n) \to \alpha \equiv \liminf_{z \to z_0} f(z).$$

We want to show that there is a path connecting K to z_0 whose ρ-length is as close to α as we wish. Passing to a subsequence we may assume z_n is separated from K by δ_n. Let c_n be the infimum of ρ-lengths of paths connecting γ_n and γ_{n+1}. By considering a path connecting K to z_n, we see that $\sum_1^n c_k \le f(z_n)$, for all n and hence $\sum_1^\infty c_n \le \alpha$.

Next choose $\varepsilon > 0$ and choose n so that we can connect K to z_n (and hence to γ_n) by a path of ρ-length less than $\alpha + \varepsilon$. We can then connect γ_n to z_0 by a infinite concatenation of arcs of γ_k, $k > n$ and paths connecting γ_k to γ_{k+1} that have total length $\sum_n^\infty (\varepsilon_n + c_n) = o(1)$. Thus K can be connected to z_0 by a path of ρ-length as close to α as we wish. □

Corollary 1.2.12 *Suppose $E \subset \mathbb{T}$ is compact and $\varepsilon > 0$. Then there is a finite collection of closed intervals F so that $E \subset F$ and*

$$\lambda(\Gamma_E) \le \lambda(\Gamma_F) + \varepsilon,$$

where the path families are defined as above.

Proof Choose an admissible ρ so that $\int \rho^2 dxdy \le M(\Gamma_E) + \varepsilon$. Set

$$r = \left(\frac{M(\Gamma_E) + \varepsilon}{M(\Gamma_E) + 2\varepsilon}\right)^{1/2}$$

By Lemma 1.2.11 $V = \{z \in \mathbb{T} : f(z) > r\}$ is open, and therefore we can choose a set F of the desired form inside V. Then ρ/r is admissible for Γ_F, so

$$M(\Gamma_F) \le \int (\frac{\rho}{r})^2 dxdy = \frac{M(\Gamma_E) + 2\varepsilon}{M(\Gamma_E) + \varepsilon} \int \rho^2 dxdy \le M(\Gamma_E) + 2\varepsilon.$$

Thus an inequality in the opposite direction holds for extremal length. □

Corollary 1.2.13 *Pfluger's theorem holds for all compact sets in $\mathbb{T}$.*

Proof Suppose E is compact. Using Corollary 1.2.12 and Lemma 1.2.4 we can choose nested sets $E_n \searrow E$ that are finite unions of closed intervals and satisfy

$$\lambda(\mathscr{F}_{E_n}) \to \lambda(\mathscr{F}_E),$$

and

$$\mathrm{cap}(E_n) \to \mathrm{cap}(E).$$

Thus the inequalities in Pfluger's theorem extend to E. □

1.3 Hyperbolic distance

We start on the disk, and then extend to simply connected domains via the Riemann mapping theorem and to general planar domains via the uniformization theorem.

The **hyperbolic metric** on $\mathbb{D}$ is given by $d\rho(z) = |dz|/(1-|z|^2)$. This means that the hyperbolic length of a rectifiable curve γ in $\mathbb{D}$ is defined as

$$\ell_\rho(\gamma) = \int_\gamma \frac{|dz|}{1-|z|^2}, \tag{1.3.1}$$

and the hyperbolic distance between two points $z, w \in \mathbb{D}$ is the infimum of the lengths of paths connecting them (we shall see shortly that there is an explicit formula for this distance in terms of z and w). In many sources, there is a "2" in the numerator of (1.3.1), but we follow [7], where the definition is as given in (1.3.1). For most applications this makes no difference, but the reader is warned that some of our formulas may differ by a factor of 2 from the analogous formulas in some papers and books.

We define the **hyperbolic gradient** of a holomorphic function $f : \mathbb{D} \to \mathbb{D}$ as

$$D_H^H f(z) = |f'(z)| \frac{1-|z|^2}{1-|f(z)|^2}.$$

More generally, given a map f between metric spaces (X,d) and (Y,ρ) we define the gradient at a point z as

$$D_d^\rho f(z) = \limsup_{x\to z} \frac{\rho(f(z), f(x))}{d(x,z)}.$$

The use of the word "gradient" is not quite correct; a gradient is usually a vector indicating both the direction and magnitude of the greatest change in a function. We use the term in a sense more like the term "upper gradient" that occurs in metric measure theory to denote a function $\rho \geq 0$ that satisfies

$$|f(b) - f(a)| \leq \int_\gamma \rho\, ds,$$

for any curve γ connecting a and b. I hope that the slight abuse of the term will not be confusing.

In these notes, the most common metrics we will use are the usual Euclidean metric on $\mathbb{C}$, the spherical metric

$$\frac{ds}{1+|z|^2},$$

on the Riemann Sphere, S^2 and the hyperbolic metric on the disk or on some other hyperbolic planar domain. To simplify notation, we use E, S and H to denote whether we are taking a gradient with respect to Euclidean, spherical or hyperbolic metrics. For example if $f : U \to V$, the symbol $D_H^H f$ means that we are taking a gradient from the hyperbolic metric on U to the hyperbolic metric on V (assuming the domains are clear from context; otherwise we write D_U^V or $D_{\rho_U}^{\rho_v}$ if we need to be very precise.)

In this notation, the spherical derivative of a function, usually denoted

$$f^{\#}(z) = \frac{|f'(z)|}{1+|f(z)|^2},$$

is written $D_E^S f(z)$ since it is a limit of quotients where the numerator is measured in the spherical metric and the denominator is measured in the Euclidean metric. Similarly D_H^S denotes a gradient measuring expansion from a hyperbolic to the spherical metric. This particular gradient is important in the theory of normal families (e.g., see Montel's theorem in [**?**]). Another variation we will use is $D_{\mathbb{D}}^E f$. If this is bounded on the disk, then f is a Lipschitz function from the hyperbolic metric on the disk to the Euclidean metric on the plane. Such functions are called Bloch functions.

A **linear fractional transformation** is a map of the form

$$z \to \frac{a+bx}{c+dz},$$

where $a,b,c,d \in \mathbb{C}$. These exactly the 1-to-1, holomorphic maps of the Riemann sphere to itself. Such maps are also called **Möbius transformations**.

Lemma 1.3.1 *Möbius transformations of $\mathbb{D}$ to itself are isometries of the hyperbolic metric.*

Proof When f is a Möbius transformation of the disk we have

$$f(z) = \frac{z-a}{1-\bar{a}z}, \qquad f'(z) = \frac{1-|a|^2}{(1-\bar{a}z)^2}.$$

Thus

$$\begin{aligned}
D_H^H f(z) &= \frac{1-|a|^2}{(1-\bar{a}z)^2}\frac{1-|z|^2}{1-|f(z)|^2} = \frac{1-|a|^2}{(1-\bar{a}z)^2}\frac{1-|z|^2}{1-|\frac{z-a}{1-\bar{a}z}|^2} \\
&= \frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2-|z-a|^2} = \frac{(1-|a|^2)(1-|z|^2)}{(1-\bar{a}z)(1-a\bar{z})-(z-a)(\bar{z}-\bar{a})} \\
&= \frac{(1-|a|^2)(1-|z|^2)}{(1-\bar{a}z-a\bar{z}+|az|^2)-(|z|^2-a\bar{z}-z\bar{a}+|a|^2)} \\
&= \frac{(1-|a|^2)(1-|z|^2)}{(1+|az|^2-|z|^2-|a|^2)} = 1.
\end{aligned}$$

Note that

$$\ell_\rho(f(\gamma)) \leq \int_\gamma D_H^H f(z) \frac{|dz|}{1-|z|^2}.$$

Thus Möbius transformations multiply hyperbolic length by at most one. Since the inverse also has this property, we see that Möbius transformation preserve hyperbolic length. □

The segment $(-1,1)$ is clearly a geodesic for the hyperbolic metric and since isometries take geodesics to geodesics, we see that geodesics for the hyperbolic metric are circles orthogonal to the boundary.

On the disk it is convenient to define the pseudo-hyperbolic metric

$$T(z,w) = |\frac{z-w}{1-\bar{w}z}|.$$

The hyperbolic metric between two points can then be expressed as

$$\rho(w,z) = \frac{1}{2}\log\frac{1+T(w,z)}{1-T(w,z)}. \tag{1.3.2}$$

On the upper half-plane the corresponding function is

$$T(z,w) = |\frac{z-w}{w-\bar{z}}|,$$

and ρ is related as before.

Lemma 1.3.2 (Schwarz's Lemma) *If $f:\mathbb{D}\to\mathbb{D}$ is holomorphic and $f(0)=0$ then $|f'(0)|\leq 1$ with equality iff f is a rotation. Moreover, $|f(z)|\leq|z|$ for all $|z|<1$, with equality for $z\neq 0$ iff f is a rotation.*

Proof Define $g(z)=f(z)/z$ for $z\neq 0$ and $g(0)=f'(0)$. This is a holomorphic function since if $f(z)=\sum a_n z^n$ then $a_0=0$ and so $g(z)=\sum a_n z^{n-1}$ has a convergent power series expansion. Since $\max_{|z|=r}|g(z)|\leq \frac{1}{r}\max_{|z|=r}|f|\leq\frac{1}{r}$. By the maximum principle $|g|\leq\frac{1}{r}$ on $\{|z|<r\}$. Taking $r\nearrow 1$ shows $|g|\leq 1$ on $\mathbb{D}$ and equality anywhere implies g is constant. Thus $|f(z)|\leq|z|$ and $|f'(0)|=|g(0)|\leq 1$ and equality implies f is a rotation. □

In terms of the hyperbolic metric this says that

$$\rho(f(0),f(z)) = \rho(0,f(z)) \leq \mathbb{H}_r(0,z),$$

which shows the hyperbolic distance from 0 to any point is non-increasing. For an arbitrary holomorphic self-map of the disk f and any point $w\in\mathbb{D}$ we can always choose Möbius transformations τ,σ so that $\tau(0)=w$ and $\sigma(f(w))=0$, so that $\sigma\circ f\circ\tau(0)=0$. Since Möbius transformations are hyperbolic isometries, this shows

Corollary 1.3.3 *If $f : \mathbb{D} \to \mathbb{D}$ is a holomorphic then $\rho(f(w), f(z)) \le \rho(w,z)$.*

Lemma 1.3.4 *If $\{f_n\}$ are holomorphic functions on a domain Ω that converge uniformly on compact sets to f and if $z_n \to z \in \Omega$, then $f_n(z_n) \to f(z)$.*

Proof We may assume $\{z_n\}$ are contained in some disk $D \subset \Omega$ around z. Let $E = \{z_n\}_1^\infty \cup \{z\}$. This is a compact set so it has a positive distance d from $\partial\Omega$. The points within distance $d/2$ of E form a compact set F on which the functions $\{f_n\}$ are uniformly bounded on E, say by M. By the Cauchy estimate the derivatives are bounded by a constant M' on E (e.g., see [**?**]). Thus

$$|f(z) - f_n(z_n) \le |f(z) - f_n(z)| + |f_n(z) - f_n(z_n)| \le |f(z) - f_n(z)| + M'|z - z_n|,$$

and both terms on the right tend to zero by hypothesis. □

A planar domain Ω is called **hyperbolic** if $\mathbb{C} \setminus \Omega$ has at least two points.

Theorem 1.3.5 *Every hyperbolic plane domain Ω is holomorphically covered by $\mathbb{D}$ (i.e., there is a locally 1-to-1, holomorphic covering map from $\mathbb{D}$ to Ω).*

We will prove this in three steps: bounded domains, simply connected domains and finally the general case.

Uniformization for bounded domains If Ω is bounded, then by a translation and rescaling, we may assume $\Omega \subset \mathbb{D}$ and $0 \in \Omega$. We will define a sequence of domains $\{\Omega_n\}$ with $\Omega_0 = \Omega$ and covering maps $p_n : \Omega_n \to \Omega_{n-1}$ such that $p(0) = 0$. We will show that Ω_n contains hyperbolic disks centered at 0 of arbitrarily large radius and that the covering map $q_n = p_1 \circ \cdots \circ p_n : \Omega_n \to \Omega_0 = \Omega$ converges uniformly on compacta to a covering map $q : \mathbb{D} \to \Omega$.

If $\Omega_0 = \mathbb{D}$ we are done, since the identity map will work. In general assume that we have $q_n : \Omega_n \to \Omega_0$ and that there is a point $w \in \mathbb{D} \setminus \Omega_n$. Let τ and σ be Möbius transformations of the disk to itself so that $\tau(w) = 0$, choose a square root α of $\tau(0)$ and choose σ so $\sigma(\alpha) = 0$. Then $p_{n+1}(z) = \sigma(\sqrt{\tau(z)})$ and let Ω_{n+1} be the component of $U = p_{n+1}^{-1}(\Omega_n)$ that contains the origin (the set U will have one or two components; two if w is in a connected component of $\mathbb{D} \setminus \Omega_n$ that is compact in $\mathbb{D}$, and one otherwise). Since σ and τ are hyperbolic isometries and $\sqrt{z}$ expands the hyperbolic metric, we see that Ω_{n+1} contains a larger hyperbolic ball around 0 than Ω_n did.

More precisely, suppose $\text{dist}(\partial\Omega_n, 0) < r < 1$ for all n. Since $f(z) = z^2$ maps the disk to itself, it strictly contracts the hyperbolic metric; a more explicit computation shows

$$D_H^H f(z) = |2z| \frac{1 - |z|^2}{1 - |z|^4} = \frac{2|z|}{1 + |z|^2} < 1.$$

Thus $g(z) = \sqrt{z}$ is locally an expansion of the hyperbolic metric, at least on a subdomain $W \subset \mathbb{D}$ where it has a well defined branch. For $z \neq 0$,

$$D_H^H g(z) = |\frac{1}{2\sqrt{z}}| \frac{1-|z|^2}{1-|z|} \geq \frac{1+|z|}{2\sqrt{z}}. \tag{1.3.3}$$

Then (1.3.3) says that

$$D_H^H p_n(0) = D_H^H \sqrt{z}(\tau(0)) > \frac{1+r}{2\sqrt{r}} > 1,$$

since $|\tau(0)| = |w| < r$. Hence $D_H^H q_n(0)$ increases by this much at every step. But $D_H^H q_n(0) \leq 1$, which is a contradiction. Thus $d_n \to 1$.

Thus $\{q_n\}$ is a sequence of uniformly bounded holomorphic functions on the disk. By Montel's theorem, there a subsequence that converges uniformly on compact subsets of $\mathbb{D}$ to a holomorphic map $q : \mathbb{D} \to \Omega$. It is non-constant since it has non-zero gradient at the origin; moreover, by Hurwitz's theorem (see [**?**]), q' never vanishes on $\mathbb{D}$ since it is the locally uniform limit of the sequence $\{q_n'\}$, and these functions never vanish since they are all derivatives of locally univalent covering maps. Next we show that q is a covering map $\mathbb{D} \to \Omega$.

Fix $a \in \Omega$ and let $d = \text{dist}(a, \partial\Omega)$. Since Ω is bounded, this is finite. Let $D = D(a,d) \subset \Omega$. Since q_n is a covering map, every branch of q_n^{-1} is 1-to-1 holomorphic map of D into $\mathbb{D}$ and hence each q_n is a contraction from the hyperbolic metric on D to the hyperbolic metric on $\mathbb{D}$. Thus every preimage of $\frac{1}{2}D$ has uniformly bounded hyperbolic diameter.

Now fix a point $b \in q^{-1}(a)$. Since $q_n(b) \to q(b) = a$, $q_n(b) \in \frac{1}{2}D$ for n large enough, so there is branch of q_n^{-1} that contains b. Since these branches are uniformly bounded holomorphic functions, by Montel's theorem we can pass to a subsequence so that they converge to a holomorphic function g from $\frac{1}{2}D$ into $\mathbb{D}$. Moreover,

$$q(g(z)) = \lim_n q_n(q_n^{-1}(z)) = z,$$

by Lemma 1.3.4. □

This proves the existence of a covering map for bounded domains Ω. If Ω is bounded and simply connected, then we have proved the Riemann mapping theorem for Ω. For unbounded simply connected domains we use the following argument.

Riemann mapping theorem It suffices to show any simply connected planar domain, except for the plane itself, can be conformally mapped to a bounded domain. If the domain Ω is bounded, there is nothing to do. If Ω. omits a disk

$D(x,r)$ then the map $z \to 1/(z-x)$ conformal maps Ω to a bounded domain. Otherwise, translate the domain so that 0 is on the boundary and consider a continuous branch of $\sqrt{z}$. The image is a 1-1, holomorphic image of Ω, but does not contain both a point and its negative. Since the image contains some open ball, it also omits an open ball and hence can be mapped to a bounded domain by the previous case. □

The final step is to deduce the uniformization theorem for all hyperbolic plane domains (we have only proved it for bounded domains so far). It suffices to show that any hyperbolic plane domain has a covering map from some bounded domain W, for then we can compose the covering maps $\mathbb{D} \to W$ and $W \to \Omega$. We can reduce to the following special case:

Theorem 1.3.6 *There is a holomorphic covering map from* $\mathbb{D}$ *to* $\mathbb{C}^{**} = \mathbb{C} \setminus \{0,1\}$

Proof Let

$$\Omega = \{z = x + iy : y > 0, 0 < x < 1, |z - \frac{1}{2}| > \frac{1}{2}\} \subset \mathbb{H}_u.$$

This is simply connected and hence can be conformally mapped to $\mathbb{H}_u$ with $0, 1, \infty$ each fixed. We can then use Schwarz reflection to extend the map across the sides of Ω. Every such reflection of Ω stays in $\mathbb{H}_u$ maps to either the lower or upper half-planes. Continuing this forever gives a covering map from a simply connected subdomain U of $\mathbb{H}_u$ to W. Since U is simply connected and not the whole plane (it is a subset of $\mathbb{H}_u$) it is conformally equivalent to $\mathbb{D}$ and hence a covering $q : \mathbb{D} \to W$ exists. (Actually $U = \mathbb{H}_u$, but we do not need this stronger result. See Exercise 1.8.) □

Uniformization of general planar domains Let $q : \mathbb{D} \to \mathbb{C}^{**} = \mathbb{C} \setminus \{0,1\}$. be a covering map of the twice punctured plane. If $\{a,b\} \in \mathbb{C} \setminus \Omega$ then $h(z) = bq(z) + a$ is a covering map from $U = h^{-1}(\Omega) \subset \mathbb{D}$ to Ω. Any connected component of U shows that Ω has a covering from a bounded plane domain, finishing the proof. □

We can now define a hyperbolic metric ρ on any hyperbolic domain using the covering map $p : \mathbb{D} \to \Omega$. The function ρ should be defined so that p is locally an isometry, i.e.,

$$\begin{aligned} 1 &= D_{\mathbb{D}}^{\Omega} p(w) \\ &= D_{\mathbb{D}}^{E} \mathrm{Id}(w) \cdot D_{E}^{E} p(w) \cdot D_{E}^{\rho_\Omega} \mathrm{Id}(p(w)) \\ &= \frac{1}{\rho_{\mathbb{D}}(w)} \cdot |p'(w)| \cdot \rho_\Omega(z) \end{aligned}$$

and so we take

$$\rho_\Omega(z) = \frac{|p'(w)|}{1-|w|^2} = |p'(w)|\rho_{\mathbb{D}}(w)$$

where $p(w) = z$. Different choices of p and w give the same value for $\rho_\Omega(z)$ since they differ by an isometry of $\mathbb{D}$. Thus every hyperbolic planar domain has a hyperbolic metric.

We want to give some useful estimates for ρ_Ω in terms of more geometric quantities, such as the quasi-hyperbolic metric, defined as

$$\widetilde{\rho}_\Omega(z)ds = \frac{ds}{\operatorname{dist}(z,\partial\Omega)}.$$

For simply connected domains, ρ and $\widetilde{\rho}$ are boundedly equivalent; for more general domains this can fail, but some useful estimates are still available.

The first observation is that if $f : U \to V$ is conformal and $\rho_U(z)ds$ and $\rho_V(z)ds$ are the densities of the hyperbolic metrics on U and V then

$$\rho_V(f(z)) = \rho_U(z)/|f'(z)|.$$

Applying this to the map $\tau(z) = (z+1)/(z-1)$ that maps the right half-plane $\mathbb{H}_r = \{x+iy : x > 0\}$ to the unit disk $\mathbb{D}$, we see that the hyperbolic density for the half-plane is

$$\rho_{\mathbb{H}_r}(z) = |\tau'(z)|\rho_{\mathbb{D}}(\tau(z)) = \frac{2}{|z-1|^2}\frac{1}{1-|\tau(z)|^2} = \frac{1}{2x} = \frac{1}{2\operatorname{dist}(z,\partial\mathbb{H}_r)}.$$

Thus the hyperbolic density on a half-plane is approximately the same as the quasi-hyperbolic metric. Using Koebe's theorem (Lemma 1.1.10) we can deduce that that this is true for any simply connected domain.

Lemma 1.3.7 *For simply connected domains, the hyperbolic and quasi-hyperbolic metrics are bi-Lipschitz equivalent, i.e.,*

$$d\rho_\Omega \le d\widetilde{\rho}_\Omega \le 4d\rho_\Omega. \tag{1.3.4}$$

Proof Using Koebe's theorem,

$$\rho_\Omega(f(z)) = \frac{\rho_{\mathbb{D}}(z)}{|f'(z)|} \le \rho_{\mathbb{D}}(z)\frac{1-|z|^2}{\operatorname{dist}(f(z),\partial\Omega} = \frac{1}{\operatorname{dist}(f(z),\partial\Omega} = \widetilde{\rho}(f(z)),$$

which is one half of the result. The other half is similar:

$$\rho_\Omega(f(z)) = \frac{\rho_{\mathbb{D}}(z)}{|f'(z)|} \ge \frac{1}{4}\rho_{\mathbb{D}}(z)\frac{1-|z|^2}{\operatorname{dist}(f(z),\partial\Omega)} = \frac{1}{4}\widetilde{\rho}(f(z)).$$

□

Corollary 1.3.8 *If $f : \Omega \to \Omega'$ is conformal, then*

$$\frac{\mathrm{dist}(f(z), \partial\Omega')}{4\,\mathrm{dist}(z, \partial\Omega)} \le |f'(z)| \le \frac{4\,\mathrm{dist}(f(z), \partial\Omega')}{\mathrm{dist}(z, \partial\Omega)}.$$

Proof Write $f = g \circ h^{-1}$ where $g : \mathbb{D} \to \Omega'$ and $h : \mathbb{D} \to \Omega$ and use the chain rule and Koebe's theorem. □

The following is immediate from Schwarz's lemma.

Corollary 1.3.9 *If $U \subset V$ are both hyperbolic, then $\rho_U \ge \rho_V$.*

Proof If $\Pi_U : \mathbb{D} \to U$ and $\Pi_V : \mathbb{D} \to V$ are the covering maps then the inclusion map $U \to V$ can be lifted to conformal map $\mathbb{D} \to \Pi_V^{-1}(U) \subset \mathbb{D}$. Applying Schwarz's lemma to this map (and using the fact that the projections are local isometries) gives the result. □

Lemma 1.3.10 *If $f : \mathbb{D} \to \Omega$ is conformal with $f'(0) = 1$, then $|f''(0)| \le 200$.*

Proof We can assume $f(0) = 0$. Then $\partial\Omega \cap \overline{\mathbb{D}} \ne \emptyset$, otherwise $|f'(0)| > 1$, so for $z \in \mathbb{D} \cap \Omega$, $\mathrm{dist}(z, \partial\Omega) \le 1 + |z|$. Thus on $\Omega \cap \mathbb{D}$,

$$\rho_\Omega(z) \ge \frac{1}{4}\widetilde{\rho}_\Omega(z) \ge \frac{1}{4(1+r)} \ge \frac{1}{8}.$$

Therefore $|f(z)| \le 1$ on the ball of hyperbolic radius $1/8$ around the origin, which is the same as the Euclidean ball of radius $\frac{1}{2}\log\frac{9}{7} > .1$. By the Cauchy estimate $|f''(0)| \le 200$. □

In fact, the correct bound is not 200, but 4; we have only given a quick proof of a weaker result. See Exercise **??** for how to derive the sharp estimate.

Corollary 1.3.11 *If $f : \mathbb{D} \to \Omega$ is conformal then $\varphi(z) = \log|f'(z)|$ is Lipschitz from the hyperbolic metric to the Euclidean metric, with bound that is independent of f.*

Proof We want to bound $D_H^E\varphi$ uniformly on the disk, but by pre-composing Möbius transformations, it suffices to bound $|\varphi'(0)|$ uniformly in f. By the Cauchy estimate for derivatives, it suffices to show $|\varphi(z) - \varphi(0)|$ is uniformly bounded on a uniform neighborhood of the origin, or equivalently, that $|f'(z)/f'(0)|$ is uniformly bounded on such a neighborhood. Let $d = \mathrm{dist}(f(z), \partial\Omega)$. Then every point in the Euclidean ball $D = D(f(z), d/2)$ is at most distance $3d/2$ from $\partial\Omega$, so integrating over paths from $f(z)$ to ∂D, we see that every point in ∂D is at least $\widetilde{\rho}$-distance $1/3$ from $f(z)$. By Lemma 1.3.7, every boundary point is at least hyperbolic distance $1/12$ from $f(z)$. Thus $U = f^{-1}(D)$ contains a hyperbolic disk of radius $1/12$ around the origin and on this disk (applying Corollary 1.3.8 twice),

$$|f'(z)| \leq 4\frac{\mathrm{dist}(f(z),\partial\Omega)}{1-|z|} \leq 8\,\mathrm{dist}(f(0),\partial\Omega) \leq 32|f'(0)|,$$

as desired. □

Again, this is not sharp; for a proof of the optimal bound, see Exericse 1.23.

The Lipschitz holomorphic functions from the disk with its hyperbolic metric to the plane with its Euclidean metric is called the Bloch class and is a Banach space with the norm

$$\|\varphi\|_{\mathscr{B}} = |\varphi(0)| + \sup_{|z|<1} |\varphi'(z)|(1-|z|^2).$$

In a later chapter, we shall see that Lemma 1.3.11 leads to an intimate connection between conformal maps and martingales that allows various results from probability theory about the latter to be directly to the former, e.g., Makarov's law of the iterated logarithm.

1.4 Boundary continuity

The boundary of a simply connected domain need not be a Jordan curve, nor even locally connected, and such examples arise naturally in complex dynamics as the Fatou components of various polynomials and entire functions. However, this makes little difference to the study of harmonic measure. In this section we show that, from the point view of harmonic measure, it is always enough to consider regions with locally connected boundaries.

Lemma 1.4.1 *Suppose Q is a quadrilateral with opposite pairs of sides E, F and C, D. Assume*

1. *E and F can be connected in Q by a curve σ of diameter $\leq \varepsilon$,*
2. *any curve connecting C and D in Q has diameter at least* 1.

Then the modulus of the path family connecting E and F in Q is larger than $M(\varepsilon)$ where $M(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

Proof Define a metric on Q by $\rho(z) = \frac{1}{2}|z-a|^{-1}/\log(1/2\varepsilon)$ for $\varepsilon < |z-a| < 1/2$. Any curve γ connecting C and D must cross σ and since γ has diameter ≥ 1 it must leave the annulus where ρ is non-zero. This shows that the modulus of the path family in Q separating E and F is small, hence the modulus of the family connecting them is large. □

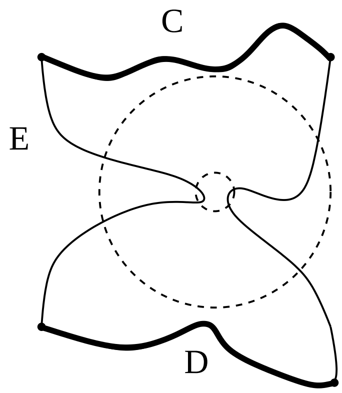


Figure 1.4.1 Proof of Lemma 1.4.1.

The following fundamental fact says that hyperbolic geodesics are almost the same as Euclidean geodesics.

Theorem 1.4.2 (Gehring-Hayman inequality) *There is an absolute constant $C < \infty$ to that the following holds. Suppose $\Omega \subset \mathbb{C}$ is hyperbolic and simply connected. Given two points in Ω, let γ be the hyperbolic geodesic connecting these two points and let σ be any other curve in Ω connecting them. Then $\ell(\gamma) \leq C\ell(\sigma)$.*

Proof Let $f : \mathbb{D} \to \Omega$ be conformal, normalized so that γ is the image of $I = [0,r] \subset \mathbb{D}$ for some $0 < r < 1$. Without loss of generality we may assume $r = r_N 1 - 2^{-N}$ for some N. Let

$$Q_n = \{z \in \mathbb{D} : 2^{-n-1} < |z-1| < 2^{-n}\},$$

and let

$$\gamma_n = \{z \in \mathbb{D} : |z-1| = 2^{-n}\},$$

$$z_n = \gamma_n \cap [0,1).$$

Let $Q_n' \subset Q_n$ be the sub-quadrilateral of points with $|\arg(1-z)| < \pi/6$. Each of these has bounded hyperbolic diameter and hence by Koebe's theorem its image is bounded by four arcs of diameter $\simeq d_n$ and opposite sides are $\simeq d_n$ apart. In particular, this means that any curve in $f(Q_n)$ separating $f(\gamma_n)$ and $f(\gamma_{n+1})$ must cross $f(Q_n')$ and hence has diameter $\gtrsim d_n$. Since Q_n has bounded modulus, so does $f(Q_n)$ and so Lemma 1.4.1 says that the shortest curve in $f(Q_n)$ connecting γ_n and γ_{n+1} has length $\ell_n \simeq d_n$. Thus any curve γ in

Q connecting γ_n and γ_{n+1} has length at least ℓ_n, and so

$$\ell(\gamma) = O(\sum d_n) = O(\sum \ell_n) \leq O(\ell(\sigma)). \quad \square$$

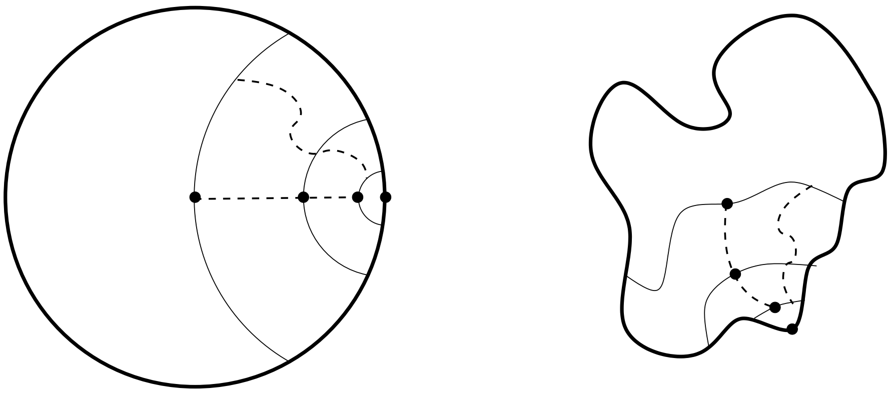


Figure 1.4.2 Proof of the Gehring-Hayman inequality.

If $f : \mathbb{D} \to \Omega$ is conformal define

$$a(r) = \text{area}(\Omega \setminus f(r \cdot \mathbb{D}).$$

If Ω has finite area (e.g., if it is bounded), then clearly $a(r) \searrow 0$ as $r \nearrow 1$.

Lemma 1.4.3 *There is a $C < \infty$ so that the following holds. Suppose $f : \mathbb{D} \to \Omega$ and $\frac{1}{2} \leq r < 1$. Let $E(\delta, r) = \{x \in \mathbb{T} : |f(sx) - f(rx)| \geq \delta$ for some $r < s < 1\}$. Then the extremal length of the path family $\mathscr{P}$ connecting $D(0,r)$ to E is bounded below by $\delta^2/Ca(r)$.*

Proof Let $z = f(sx)$ and suppose $w \in f(D(0,r))$. By the Gehring-Hayman estimate, the length of any curve from w to z is at least $1/C$ times the length of the hyperbolic geodesic γ between them. But this geodesic has a segment γ_0 that lies within a uniformly bounded distance of the geodesic γ_1 from $f(rx)$ to z. By the Koebe distortion theorem γ_0 and γ_1 have comparable Euclidean lengths, and clearly the length of γ_1 is at least δ. Thus the length of any path from $f(D(0,r))$ to $f(sx)$ is at least δ/C. Now let $\rho = C/\delta$ in $\Omega \setminus f(D(0,r))$ and 0 elsewhere. Then ρ is admissible for $f(\mathscr{P})$ and $\iint \rho^2 dxdy$ is bounded by $C^2 a(r)/\delta^2$. Thus $\lambda(\mathscr{P}) \geq \frac{\delta^2}{C^2 a(r)}$. $\square$

Lemma 1.4.4 *Suppose $f : \mathbb{D} \to \Omega$ is conformal, and for $R \geq 1$,*

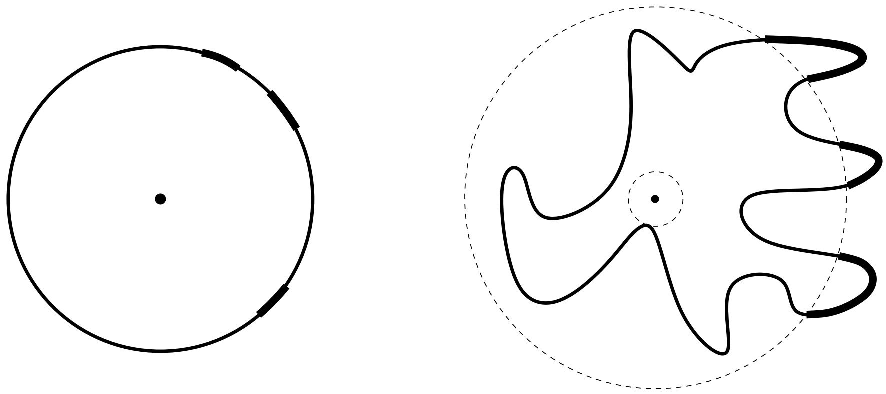
$$E_R = \{x \in \mathbb{T} : |f(x) - f(0)| \geq R \, \text{dist}(f(0), \partial\Omega)\}.$$

Then E_R has capacity $O(1/\log R)$ if R is large enough.

Proof Assume $f(0) = 0$ and $\text{dist}(0, \partial\Omega) = 1$ and let $\rho(z) = |z|^{-1}/\log R$ for $z \in \Omega \cap \{1 < |z| < R\}$. Then ρ is admissible for the path family Γ connecting $D(0, 1/2)$ to $\partial\Omega \setminus D(0, R)$ and $\iint \rho^2 dxdy \leq 2\pi/\log R$. By definition $M(\Gamma) \leq 2\pi/\log R$ and $\lambda(\Gamma) \geq (\log R)/2\pi$. By the Koebe distortion theorem $f^{-1}(D(0, 1/2))$ is contained in a compact subset of $\mathbb{D}$, independent of Ω. By Pfluger's theorem (Theorem 1.2.10),

$$\cap(E_r) \leq \frac{2}{-2C_2 + \log R},$$

which proves the result. □



Corollary 1.4.5 *If $f : \mathbb{D} \to \Omega$ is conformal, then f has radial limits except on a set of zero capacity (and hence has finite radial limits a.e. on $\mathbb{T}$).*

Proof Let $E_{r,\delta} \subset \mathbb{T}$ be the set of $x \in \mathbb{T}$ so that $\text{diam}(f(rx, x)) > \delta$, and let $E_\delta = \cap_{0<r<1} E_{r,\delta}$. If f does not have a radial limit at $x \in \mathbb{T}$, then $x \in E_\delta$ for some $\delta > 0$, and this has zero capacity by Lemma 1.4.3. Taking the union over a sequence of δ's tending to zero proves the result. The set where f has a radial limit ∞ has zero capacity by Lemma 1.4.4, so we deduce f has finite radial limits except on zero capacity. □

Combining the last two results proves

Corollary 1.4.6 *Given $\varepsilon > 0$ there is a $C < \infty$ so that the following holds. If $f : \mathbb{D} \to \Omega$ is conformal, $z \in \mathbb{D}$ and $I \subset \mathbb{T}$ is an arc that satisfies $|I| \geq \varepsilon(1 - |z|)$ and* $\text{dist}(z, I) \leq \frac{1}{\varepsilon}(1 - |z|)$*, then I contains a point w where f has a radial limit and* $|f(w) - f(z)| \leq C\,\text{dist}(f(z), \partial\Omega)$.

We can now prove:

Theorem 1.4.7 (Carathéodory) *Suppose that $f : \mathbb{D} \to \Omega$ is conformal, and that $\partial\Omega$ is compact and locally path connected (for every $\varepsilon > 0$ there is a $\delta > 0$ so that any two points of $\partial\Omega$ that are within distance δ of each other can be connected by a path in $\partial\Omega$ of diameter at most ε). Then f extends continuously to the boundary of $\mathbb{D}$.*

Proof Suppose $\eta > 0$ is small. Since $\partial\Omega$ is compact $\Omega \setminus f(\{|z| < 1 - \frac{1}{n}\})$ has finite area that tends to zero as $n \nearrow \infty$. Thus if n is sufficiently large, this region contains no disk of radius η.

Choose $\{z_j\}$ to be n equally spaced points on the unit circle and using Lemma **??** choose interlaced points $\{w_j\}$ so that f has a radial limit $f(w_j)$ at w_j and this limit satisfies $|f(w_j) - f(rw_j)| \leq C\eta$ where $r = 1 - 1/n$. Then

$$\begin{aligned} |f(w_j) - f(w_{j+1})| \leq\ & |f(w_j) - f(rw_j)| \\ & + |f(rw_j) - f(rw_{j+1})| \\ & + |f(rw_{j+1}) - f(w_{j+1})| \\ \leq\ & C\delta, \end{aligned}$$

where the center term is bounded by Koebe's theorem and the other two by definition.

Fix $\varepsilon > 0$ and choose $\delta > 0$ as in the definition of locally connected. Thus if η is so small that $C\eta < \delta$, then the shorter arc of $\partial\Omega$ with endpoints $f(w_j)$ and $f(w_{j+1})$ can be connected in $\partial\Omega$ by a curve of diameter at most ε. Thus the image under f of the Carleson square with base I_j (the arc between w_j and w_{j+1}) has diameter at most $C\eta + \varepsilon$. This implies f has a continuous extension to the boundary. □

It is an inconvenient fact is that conformal maps do not have to extend continuously to the boundary. We noted above however, that radial do exist almost everywhere. Another convenient substitute for full continuity says that every conformal map is continuous on a subdomain of $\mathbb{D}$ whose boundary hits "most of" $\partial\mathbb{D}$. The precise statement requires a new definition.

Given a compact set $E \subset \mathbb{T}$ we will now define the associated "sawtooth" region W_E Suppose $\{I_n\}$ are the connected components of $\mathbb{T} \setminus E$ and for each n let $\gamma_n(\theta)$ be the circular arc in $\mathbb{D}$ with the same endpoints as I_n and which makes angle θ with I_n (so $\gamma_n(0) = I_n$ and $\gamma_n(\pi/2)$ is the hyperbolic geodesic with the same endpoints as I_n). Let $C_n(\theta)$ be the region bounded by I_n and $\gamma_n(\theta)$, and let $W_E(\theta) = \mathbb{D} \setminus \cup_n C_n(\theta)$. Let $W_E = W_E(\pi/8)$ (and let $W_E^* \subset \overline{\mathbb{D}}^c$ be its reflection across $\mathbb{T}$).

If $f : \mathbb{D} \to \Omega$ and $0 < r < 1$, then define

$$d_f(r) = \sup\{|f(z) - f(w)| : |z| = |w| = r \text{ and } |z - w| \leq 1 - r\}. \quad (1.4.1)$$

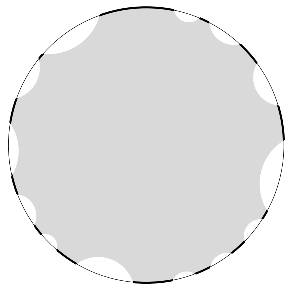


Figure 1.4.3 The sawtooth domain W_E

If $\partial\Omega$ is bounded in the plane, then it is easy to see this goes to zero as $r \nearrow 1$, since otherwise any neighborhood of $\partial\Omega$ would contain infinitely many disjoint disks of a fixed, positive size by Koebe's theorem (Theorem 1.1.10).

Lemma 1.4.8 *Suppose $f : \mathbb{D} \to \Omega \subset S^2$ is conformal. Then for any $\varepsilon > 0$ there is a compact set $X \subset \mathbb{T}$ with* $\operatorname{cap}(\mathbb{T} \setminus X) < \varepsilon$ *such that f is continuous on $\overline{W_X}$.*

Proof By applying a square root and a Möbius transformation, we may assume that $\partial\Omega$ is bounded in the plane. Given $r < 1$ let

$$E(\delta, r) = \{x \in \mathbb{T} : |f(sx) - f(tx)| > \varepsilon \text{ for some } r < s < t < 1\}$$

and note that by Pfluger's theorem (Theorem 1.2.10) and Lemma 1.4.3

$$\operatorname{cap}(E(\delta, r)) \leq \exp(-\pi\varepsilon^2/Ca(r)),$$

where $a(r) = \operatorname{area}(f(\mathbb{D}) \setminus f(r \cdot \mathbb{D}))$, as before. Moreover, this set is open since f is continuous at the points sx and tx. Fix $\varepsilon > 0$, take $\varepsilon_n = 2^{-n}$, and choose r_n so close to 1 that $\operatorname{cap}(E_n) \equiv \operatorname{cap}(E(\varepsilon_n, r_n)) \leq \varepsilon 2^{-n}$. If we define $X = \mathbb{T} \setminus \cup_{n>1} E_n$, then X is closed and $\mathbb{T} \setminus X$ has capacity $\leq \varepsilon$ by subadditivity.

To show f is continuous at every $x \in \overline{W_X}$, we want to show that $|x - y|$ small implies $|f(x) - f(y)|$ is small. We only have to consider points $x \in \partial W_X \cap \mathbb{T}$. First suppose $y \in \partial W_X \cap \mathbb{T}$. Choose the maximal n so that $s = |x - y| \leq 1 - r_n$. Then $x, y \notin E_n$, so

$$|f(x) - f(y)| \leq |f(x) - f(sx)| + |f(sx) - f(sy)| + |f(sy) - f(y)|.$$

The first and last terms on the right are $\leq \varepsilon_{n-1}$ by the definition of X. The middle term is at most $d_f(1-s)$ (defined in (1.4.1), which tends to 0 as $s \to 0$. Thus $|f(x)-f(y)|$ is small if $|x-y|$ is.

Now suppose $x \in \partial W_X \cap \mathbb{T}$, $y \in \partial W_X \setminus \mathbb{T}$. From the definition of W_X it is easy to see there is a point $w \in \partial W_X \cap \mathbb{T}$ such that $|w-y| \leq 2(1-|y|) \leq 2|x-y|$. For the point w we know by the argument above that $|f(x)-f(w)|$ is small. On the other hand,

$$|f(y)-f(w)| \leq |f(y)-f(|y|w)| + |f(|y|w)-f(w)|.$$

The first term is bounded by $Cd_f(|y|)$ and the second is small since $w \notin E_n$. Thus $|f(x)-f(y)|$ is small depending only on $|x-y|$. Hence f is continuous on $\overline{W_X}$. □

1.5 Harmonic measure

Suppose Ω is a planar domain bounded by a Jordan curve, $z \in \Omega$ and $E \subset \partial\Omega$ is Borel. Suppose $f : \mathbb{D}\Omega$ is conformal and $f(0) = z$ (by the Riemann mapping theorem there is always such a map). By Carathéodory's theorem, f extends continuously (even homeomorphically) to the boundary, so $f^{-1}(E) \subset \mathbb{T}$ is also Borel. We define "the harmonic measure of the set E for the domain Ω, with respect to the point z" as

$$\omega(z, E, \Omega) = |E|/2\pi,$$

where $|E|$ denotes the Lebesgue 1-dimensional measure of E. This depends on the choice of the Riemann map f, but any two maps, both sending 0 to z, will differ only by a pre-composition with a rotation. Thus the two possible pre-images of E differ by a rotation and hence have the same Lebesgue measure. If we fix E and Ω, then $\omega(z, E, \Omega)$ is a harmonic function of z (Exercise 1.12), giving rise the name "harmonic measure". Since we always have $0 \leq \omega(z, E, \Omega) \leq 1$, we can deduce that if E has harmonic measure with respect to one point z in Ω then it has zero harmonic measure with respect to all points (Exercise 1.13). If $\partial\Omega$ is merely locally connected, then Carathéodory's theorem still implies that the Riemann map f has a continuous extension to the boundary, so the same definition of harmonic measure works.

Theorem 1.4.8 allows us to define harmonic measure on a general simply connected proper subdomain of $\mathbb{C}$ by

$$\omega(z, E, \Omega) = \sup_n \omega(z, E \cap \partial\Omega_n, \Omega_n),$$

where $f : \mathbb{D} \to \Omega$ is conformal with $f(0) = z$, $\Omega_n = f(W_{F_n})$ and $\{F_n\}$ are nested, increasing compact sets with measure tending to $|\mathbb{T}|$ chosen using Lemma 1.4.8 so that f is continuous on each $\overline{W_{F_n}}$. It is easy to verify that this definition does not depend on any of the choice involved.

In general, we can not assume that Ω_n in the previous paragraph is a Jordan domain. For example, if $\Omega = \mathbb{D} \setminus [0,1)$ is a slit disk, then any approximating domains will have to hit both sides of the slit in nearly full harmonic measure, and thus $\partial\Omega$ will contain self-intersections. However, if we are willing to give up approximation of the whole boundary, and only approximate sets of positive measure, then we can do this with Jordan subdomains. This will be discussed in Section **??**, after we have proven the Moore triod theorem and the F. and M. Riesz Theorem.

We want estimate harmonic measure in terms of extremal length. We have already seen how to relate extremal length to logarithmic capacity, and the following relates the latter to harmonic measure:

Lemma 1.5.1 *For any compact $E \subset \mathbb{T}$,*

$$\operatorname{cap}(E) \geq \frac{1}{1 + \log 2 + \pi + \log \frac{1}{|E|}}.$$

If $E \subset \mathbb{T}$ has positive Lebesgue measure, then it has positive capacity. In particular, if $E \subset \mathbb{T}$ is an arc, then

$$\operatorname{cap}(E) \leq \frac{1}{\log 4 + \log \frac{1}{|E|}}.$$

For arcs of small measure, the two bounds are comparable.

Proof If μ is Lebesgue measure restricted to E, then clearly the corresponding potential function is less than potential function of an arc I of the same measure evaluated at the center x of that arc. Since $\frac{2}{\pi}t \leq |x-y| \leq t$ if the arclength between $x, y \in \mathbb{T}$ is t, this value is at most

$$\int_I \log \frac{2}{|x-y|} dy \leq 2 \int_0^{|E|/2} \log \frac{\pi}{t} dt = |E| \log \frac{2}{|E|} + (1+\pi)|E|$$

If we normalize the measure to have mass one, then we get

$$U_\mu \leq \log \frac{2}{|E|} + 1 + \pi = \log \frac{1}{|E|} + 1 + \log 2 + \pi.$$

If E is an arc, then the center x of the arc is at most distance $|E|/2$ from any other point of the arc, and so

$$U_\mu(x) \geq \log \frac{2}{|E|/2} = \log \frac{4}{|E|} = \log \frac{1}{|E|} + \log 4,$$

for any probability measure supported on E. This gives the desired estimate. □

The following is the fundamental estimate for harmonic measure, from which all other estimates flow (at least, all the ones that we will use).

Theorem 1.5.2 *Suppose Ω is a Jordan domain, $z_0 \in \Omega$ with* $\mathrm{dist}(z_0, \partial\Omega) \geq 1$ *and $E \subset \partial\Omega$. Let Γ be the family of curves in Ω which connects $D(z_0, 1/2)$ to E. Then*

$$\omega(z_0, E, \Omega) \leq C \exp(-\pi\lambda(\Gamma)).$$

If $E \subset \partial\Omega$ is an arc then the two sides are comparable.

Proof Let $f : \mathbb{D} \to \Omega$ be conformal. By Koebe's $\frac{1}{4}$-theorem (Theorem 1.1.10), the disk $D(z, \frac{1}{2})$ in Ω maps to a smooth region K in the unit disk that contains the origin, and ∂K is uniformly bounded away from both the origin and the unit circle. Thus by Pfluger's theorem applied to the curve family Γ_X connecting K and the compact set $X = f^{-1}(E)$,

$$\frac{1}{\mathrm{cap}(X)} + C_1(K) \leq \pi\lambda(\Gamma_X) \leq \frac{1}{\mathrm{cap}(X)} + C_2(K),$$

for constants C_1, C_2 that are bounded independent of all our choices.

By Lemma 1.5.1 the right-hand side of

$$1 + \log 4 + \log \frac{1}{|X|} + C_1(K) \leq \pi\lambda(\Gamma_X) \leq 1 + \log 2 + \log \frac{1}{|X|} + C_2(K).$$

holds in general, and the left-hand side also holds if X is an interval. Multiply by -1 and exponentiate to get

$$\frac{|X|}{2e^{1+\pi+C_2}} \leq \exp(-\pi\lambda(\Gamma_X)) \leq \frac{|X|}{4e^{C_1}}$$

under the same assumptions. Now use $\omega(z, E, \Omega) = \omega(0, X, \mathbb{D}) = |X|/2\pi$ to deduce the result. □

One of the most famous and most useful applications of this result is

Corollary 1.5.3 (Ahlfors distortion theorem) *Suppose Ω is a Jordan domain, $z_0 \in \Omega$ with* $\mathrm{dist}(z_0, \partial\Omega) \geq 1$ *and $x \in \partial\Omega$. For each $0 < t < 1$ let $\ell(t)$ be the length of $\Omega \cap \{|w - x| = t\}$. Then there is an absolute $C < \infty$, so that*

$$\omega(z_0, D(x, r), \Omega) \leq C \exp(-\pi \int_r^1 \frac{dt}{\ell(t)}).$$

Proof Let K be the disk of radius $1/2$ around z_0 and let Γ be the family of curves in Ω which connects $D(x,r) \cap \partial\Omega$ to K. Define a metric ρ by $\rho(z) = 1/\ell(t)$ if $z \in C_t = \{z \in \Omega : |x-z| = t\}$ and $\ell(t)$ is the length of C_t. Any curve $\gamma \in \Gamma$ has ρ-length at least

$$L = \int_r^{1/2} \frac{dt}{\ell(t)},$$

and

$$A = \iint_\Omega \rho^2 dxdy \geq \int_r^{1/2} \int_{C_r \cap \Omega} \ell(z)^{-2} r dr d\theta = \int \ell(z)^{-1} dr = L.$$

Therefore

$$\lambda(\Gamma) \geq A/L^2 = 1/L,$$

and this proves the result. □

Corollary 1.5.4 (Beurling's estimate) *There is a $C < \infty$ so that if Ω is simply connected, $z \in \Omega$ and $d = \text{dist}(z, \partial\Omega)$ then for any $0 < r < 1$ and any $x \in \partial\Omega$,*

$$\omega(z, D(x, rd), \Omega) \leq Cr^{1/2}$$

Proof Apply Corollary 1.5.3 at x and use $\theta(t) \leq 2\pi t$ to get

$$\exp(-\pi \int_{rd}^{d} \frac{dt}{\theta(t)t}) \leq C \exp(-\frac{1}{2} \log r) \leq C\sqrt{r}.$$

□

Corollary 1.5.5 *There is an $R < \infty$ so that for any Ω is a Jordan domain and any $z \in \Omega$*

$$\omega(z, \partial\Omega \setminus D(z, R\,\text{dist}(z, \partial\Omega), \Omega) \leq 1/2.$$

Proof Rescale so $z = 1$ and $\text{dist}(z, \partial\Omega) = 1$. Then apply $w \to 1/w$ which fixes z and maps $\partial\Omega \setminus D(z, R)$ into $D(0, 1/R - 1)$. Then Lemma 1.5.4 implies the result if $R \geq 4C^2 + 1$ (C is as in Lemma 1.5.4). □

Corollary 1.5.6 *For any Jordan domain and any $\varepsilon > 0$,*

$$\omega(z, \partial\Omega \cap D(z, (1+\varepsilon)\,\text{dist}(z, \partial\Omega)), \Omega) > C\varepsilon,$$

for some fixed $C > 0$.

Proof Renormalize so $z = 0$ and 1 is a closest point of $\partial\Omega$ to z. By Corollary 1.5.5, the set $E = \partial\Omega \cap D(0, 1+\varepsilon)$ has harmonic measure at least $1/2$ from the point $1 - \varepsilon/R$. Since $\omega(z, E, \Omega)$ is a positive, harmonic function on $\mathbb{D}$, Harnack's inequality says it is larger than $C\varepsilon/R$ at the origin. □

This is a weak version of the Beurling projection theorem which says that the sharp lower bound is given by the slit disk $D(0,1+\varepsilon)\setminus[1,1+\varepsilon)$. The harmonic measure of the slit in this case can be computed as an explicit function of ε because this domain can be mapped to the disk by sequence of elementary functions.

Theorem 1.5.7 *Suppose Ω is a Jordan domain and $E \subset \partial\Omega$ has zero $\frac{1}{2}$-Hausdorff measure. Then E has zero harmonic measure in Ω.*

Proof Since dilations do not change dimension or harmonic measure, we can rescale so that Ω contains a unit disk centered at some point z. By Exercise 1.13, it suffices to show E has harmonic measure zero with respect to z.

By definition, the hypothesis means that for any $\varepsilon > 0$, the set E can be covered by open disks $\{D(x_j,r_j)\}$ that satisfy $\sum_j r_j^{1/2} \leq \varepsilon$. By Beurling's estimate, this implies

$$\omega(z,E,\Omega) \leq \sum_j \omega(z,D_j,\Omega) \leq O(\sum_j r_j^{1/2}) = O(\varepsilon).$$

□

This result was not improved until Lennart Carleson [4] showed in a tour de force that the $\frac{1}{2}$ could be replaced by some $\alpha > \frac{1}{2}$ in [4]. That result was not improved until Makarov showed it holds for all $\alpha < 1$ [12]. We will prove Makarov's theorem in Chapter **??**. Even though we have not defined harmonic measure for multiply connected domains, it is clear that no analog is possible in that case: if the boundary of Ω is a Cantor set of dimension α, then it must have full harmonic measure, even if α is small.

Corollary 1.5.8 *If Ω is Jordan domain, then harmonic measure is singular to area measure.*

Proof By the Lebesgue density theorem, at Lebesgue almost every point z of a set E of positive area, all small enough disks satisfy

$$\text{area}(E\cap D(z,r)) \geq (1-\varepsilon)\,\text{area}(D(z,r)),$$

for all $r < r_0$. In particular we must have $\ell(t) \leq \frac{\varepsilon}{t}$ on a set of measure at least $r/4$ in $[r/2,r]$. Thus by the Ahlfors distortion theorem

$$\omega(D(z,r_0 2^{-n}) \leq C\exp(-\pi\int_{2^{-n}r_0}^{r_0}\frac{dt}{\varepsilon t}) \leq C2^{-\pi n/\varepsilon}.$$

This is much less than $(2^{-n}r_0)$ if n is large. Thus almost every point of $\partial\Omega$ can be covered by arbitrarily small disks so that $\omega(D(z_j,r_j)) = o(r_j^2)$. Use Vitali's covering theorem to take a disjoint cover of a set of full harmonic measure, and we deduce that harmonic measure gives full mass to set of zero area. □

Corollary 1.5.9 *There is an* $\varepsilon > 0$ *so that harmonic measure on a planar Jordan domain always gives full measure to a set of Hausdorff dimension at most* $2-\varepsilon$.

Proof Fix a large integer b and consider the b-adic squares in the plane. Take one such square Q that intersects $\partial\Omega$ and consider its b^2 children squares. We claim that if b is large enough, then at least one of them has harmonic measure that is less than $(2b^2)^{-1}$ times the harmonic measure of Q. If there is a subsquare that misses $\partial\Omega$, then its harmonic measure is zero, and the claim is true. Therefore we may assume every subsquare hits $\partial\Omega$. Suppose Q has side length 1 and define a finite sequence of squares S_k, concentric with Q and with side lengths $\frac{1}{b}, \frac{3}{b}, \frac{6}{b}, \dots, 1$. If $z \in \partial S_k$, then $\mathrm{dist}(z, \partial\Omega) \leq \sqrt{2}/b$ and $\mathrm{dist}(z, S_{k-1}) > 3/b$, so by Corollary **??** ,

$$\max_{z\in\partial S_k} \omega(z, \partial\Omega \cap S_{k-1}, \Omega \setminus S_{k-1}) < 1-\delta,$$

for some uniform $\delta > 0$ (independent of k and b). By the maximum principle and induction,

$$\omega(S_1) \leq (1-\delta)^{b/3},$$

and this is less than $1/(2b^2)$ if b is large enough. This prove the claim, that ω deviates from the uniform distribution on the sub-squares by a fixed amount.

The rest is standard. The deviation from uniformity implies that the entropy

$$h(\mu) = -\sum_{k=1}^{b^2} \omega(Q_j) \log_b \omega(Q_j),$$

is strictly less than 2, the maximum that occurs when every square has equal measure (Exercise **??**). The strong law of large numbers and Billingsley's lemma now imply that ω has dimension strictly less than 2, with a bound that depends on b, but not on Ω. □

Jean Bourgain [3] proved this holds for general domains in higher dimensions, with a δ that depends only on the dimension. We shall see later that the bound $\dim(\omega) \leq 1$ holds in the plane.

1.6 Diffusion Limited Aggregation

Start with a unit disk centered at the origin. Imagine another unit disk, whose center moves as a Brownian motion starting near infinity unit the it hits the first disk and the stops. Now send in another random disk until it hits one of the first two. Continue in this way until n disks have accumulated to form a connected set as illustrated in Figure 1.6.1.

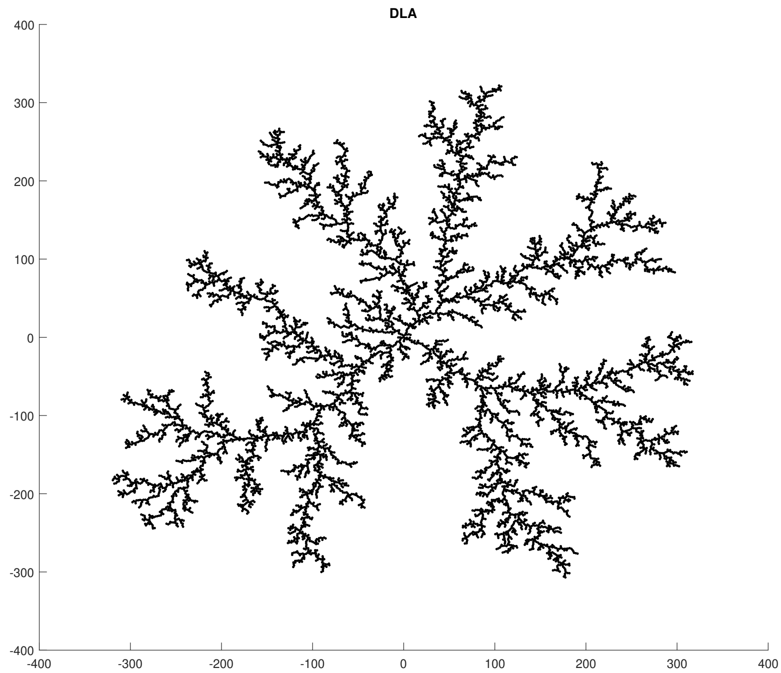


Figure 1.6.1 A diffusion limited aggregates, $n = 10,000$.

It is conjectured that these aggregates, properly rescaled, will have continuous limits that are fractals of dimension approximately 1.71 (based on large numerical simulations), but almost nothing is known rigorously. Indeed, the only rigorous result about DLA is the following upper bound due to Harry Kesten (see [8], [9], [10]), although our presentation follows the one in [11].

Theorem 1.6.1 *Almost surely, the diameter of DLA at the nth step is* $O(n^{2/3})$.

Proof The first step is to make the definition of DLA a little more precise. A moving disk will hit a set E when the center is precisely distance 1 from that set. In our case, the set is a union of n unit disks centered at a finite set of points $P_n = \{p_1, \ldots, p_n\}$. Thus the process of adding the next disk by letting

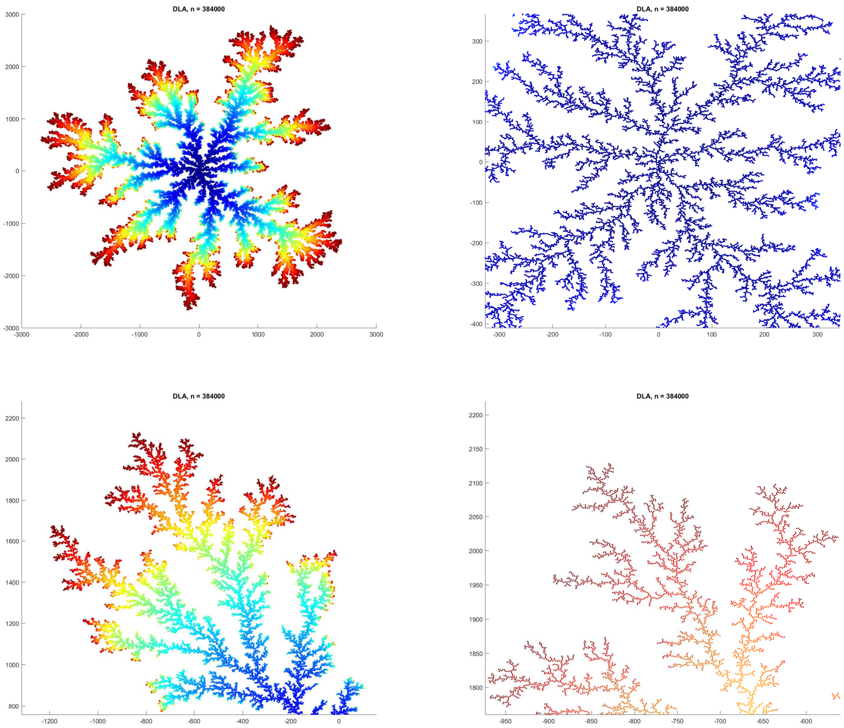


Figure 1.6.2 A diffusion limited aggregation with 384,000 disks. The disks are colored according to when they entered the cluster. Upper left is the full cluster. Upper right is an enlargement of the center. The bottom two pictures are successive enlargements of tip.

it wander by Brownian motion, is precisely the same as choosing a point p_{n+1} on the set

$$E_n = \{z : \mathrm{dist}(z, P_n) = 2\},$$

with respect to harmonic measure at ∞ for the domain Ω_n that is the unbounded complementary component of E_n.. Since E_n is, by definition, a connected set, Ω_n is simply connected and will be bounded by a finite number of circular arcs.

Actually, almost surely Ω_n will be the entire complement of E_n. Otherwise, we must have chosen a disk that made contact with two or more earlier disks. But there are only a finite number of points on E_k where this happens, and finite sets have harmonic measure zero (e.g., Beurling's theorem), so the probability of making such a choice is zero. Thus, almost surely, each disk in the cluster (except the one at the origin) hits exactly one previously chosen disk, although it may be hit by several (at most four, almost surely) later ones.

Consider

$$\mathrm{rad}(n) = \max\{|p| : p \in P_n\},$$

which measures the size of the DLA cluster in terms of a disk around the origin, and its inverse

$$\mathrm{exit}(m) = \max\{n : \mathrm{rad}(n) \le m\},$$

which measures how soon the cluster grows beyond a given radius. The theorem is stated in terms of an upper bound for $\mathrm{rad}(n)$, but is equivalent to a lower bound for $\mathrm{exit}(m)$:

$$\liminf_{m\to\infty} \frac{\mathrm{exit}(m)}{m^{3/2}} \ge \beta, \tag{1.6.1}$$

holds almost surely for some constant $\beta > 0$. More precisely, we define

$$V_m = \{\mathrm{exit}(m) \le \beta m^{3/2}\},$$

and we will prove that $\sum_m \mathbb{P}(V_m) < \infty$. The Borel-Cantelli lemma then implies that the probability that V_m occurs infinitely often is zero. Thus almost surely V_m only occurs finitely often, which gives (1.6.1).

We estimate the probability of V_m by placing these events inside larger events and estimating those. If V_m occurs, it means that the DLA cluster contains a path of at most $\beta m^{3/2}$ disks $\{D_1,\dots D_N\}$ that starts at the origin and ends with a disk that hits the circle $\{|z| = m\}$. Moreover, every D_{j+1}, $j = 1,\dots N-1$ was selected after D_j in the growth process. Otherwise suppose $Dj+1$ is the first counterexample in the path. Then D_{j+1} is the unique earlier disk hit by D_j, so D_{j-1}, which also touches D_j, must have been chosen later than D_j, making D_j a counterexample too.

Every unit disk contains a point in the lattice $\mathbb{N}\times\mathbb{N}$, so for each path of unit disks as above, we can choose a sequence of lattice points $\mathbf{z} = \{z_1,\dots z_N\}$ such $z_j \in D_j$, $j = 1,\dots z_N$ and $|z_j - |z_{j+1}| \le 4$ since the union of two touching unit disks has diameter 4. We will say that sequence of distinct lattice points $\{z_1,\dots,z_k\}$ is m-admissible if

$$|z_1| \le m/2, \quad |z_k| \ge m, \quad |z_j - z_{j+1}| \le 4.$$

Note that there are at most $m^2 80^{k-1}$ m-admissible sequences of length k; there are m^2 possible choices for z_1, and each following choice is made from a 9×9 square, omitting the center. Moreover, the length of an m-admissible sequence is at least $m/8$ since the first and last points are at least distance $m/2$ apart.

Given an m-admissible sequence $\mathbf{z}$ of length k, we define $W_m(\mathbf{z})$ to be the set

of clusters so that:
(1) the cluster contains at most $\beta m^{3/2}$ disks,
(2) the cluster contains the sequence $\mathbf{z}$, and
(3) the disk containing z_{j+1} was chosen after the disk containing z_j. By our comments above each cluster in V_m contained in the event $W_m(\mathbf{z})$ for some m-admissible sequence of length $k \leq \beta m^{3/2}$. Thus all of V_m is contained W_m, the union of $W_m(\mathbf{z})$ over all m-admissible sequences of length at most $\beta m^{3/2}$.

We claim that if $\mathbf{z}$ has length k, then

$$\mathbb{P}(W_m(\mathbf{z})) \leq (C\beta)^k. \tag{1.6.2}$$

We will finish the proof of the theorem assuming this is true, and then prove the estimate. Given (1.6.2)

$$\begin{aligned}\mathbb{P}(W_m) &\leq \sum_{\mathbf{z}} \mathbb{P}(W(\mathbf{z})) \\ &\leq \#(m-\text{admissible } \mathbf{z}) \cdot (C\beta)^k \\ &\leq m^2 80^{k-1} (C\beta)^k \\ &\leq m^2 (80C\beta)^k \\ &\leq m^2 (80C\beta)^{m/4},\end{aligned}$$

since $k \geq m/4$. Thus

$$\sum_m \mathbb{P}(V_m) \leq \sum_m \mathbb{P}(W_m) \leq \sum_m m^2 (80C\beta)^{m/4} < \infty,$$

if we choose $\beta < 1/80C$. This completes the proof of Theorem 1.6.1, except for the proof of (1.6.2).

First we explain the general idea for proving (1.6.2). Suppose we have already grown a cluster that contains the points $z_1, \dots, z_j$. How long do we have to wait before the cluster contains z_{j+1}? We must add a disk within distance 4 of the disk containing z_j. Since the cluster has diameter at least $m/2$, by Beurling's estimate (Lemma 1.5.4) the probability of choosing such a disk is less that $C/\sqrt{m}$. Therefore the expected number of disks we add before covering z_{j+1} is at least $\sqrt{m}/C$. This has to happen k times, so we expect that $k\sqrt{m}/C$ disks need to be added to the cluster before the whole sequence $\mathbf{z}$ is covered. Since $k \geq m/8$, we therefore expect to need about $m^{3/2}/C$ disks to be added. However, clusters in the event $W_m(\mathbf{z})$ only use $\beta m^{3/2}$ disks to cover $\mathbf{z}$. If β is small compared to $1/C$, this event should have small probability.

To make this idea precise, let $D_1, \dots$ be an enumeration of the disks in the cluster, in the order they are added. Suppose z_j is contained in disk $D_{k(j)}$ and let $w(j) = k(j+1) - k(j)$; this is the time we "wait" between covering z_j and

z_{j+1}. Then

$$\mathbb{P}(w(j) > t) \geq (1-p)^t,$$

where $p \leq C/\sqrt{m}$. Therefore $w(j)$ is bounded below by a geometric random variable (the same one for each j), and $\sum_j w(j)$ will be bounded below by the corresponding sum of geometric variables. We estimate this distribution using:

Lemma 1.6.2 *Suppose $X_1, \ldots X_n$ are independent geometric random variables, i.e., $\mathbb{P}(X_j = s) = p(1-p)^{s-1}$ for some $0 < p < 1/2$, and $Y = \sum_{j=1}^n X_j$. If $a \geq 2p$, then*

$$\mathbb{P}(Y \leq an/p) \leq (2e^2 a)^n.$$

Proof As usual, we define the moment generating function of the random variable Y as the expected value of $\exp(tY)$. If X is a geometric random variable, then

$$\mathbb{E}(e^{tX}) = \sum_{j=1}^{\infty} e^{tj} p(1-p)^{j-1} = pe^t \sum_{j=0}^{\infty} (e^t(1-p))^j = \frac{p}{1-e^t(1-p)}.$$

Since Y is a sum of independent copies of X,

$$\mathbb{E}(e^{tY}) = \prod_{j=1}^{\infty} \mathbb{E}(e^{tX}) = \left[\frac{p}{e}^t 1 - e^t(1-p)\right]^w.$$

By Chebyshev's inequality

$$\mathbb{P}(Y < \frac{\ln\lambda}{-t}) = \mathbb{P}(e^{-tY} > \lambda) \leq \frac{1}{\lambda}\mathbb{E}(e^{-tY}).$$

Set $\lambda = \exp(-ant/p)$ to get

$$\mathbb{P}(Y < an/p) \leq \exp(ant/p)\mathbb{E}(e^{-tY}) = \frac{\exp(ant/p)e^{-nt}p^n}{(1-e^{-t}(1-p))^n} = \frac{\exp(ant/p)p^n}{(e^t-(1-p))^n}$$

Now set $t = \ln(a(1-p)/(a-p)$ and this becomes

$$\begin{aligned}
\mathbb{P}(Y < an/p) &\le \frac{p^n \left(\frac{a(1-p)}{a-p}\right)^{an/p}}{(\frac{(a(1-p)}{a-p} - (1-p))^n} \\
&\le \frac{p^n \left(\frac{a(1-p)}{a-p}\right)^{an/p}}{(1-p)^n(\frac{(a}{a-p} - 1)^n} \\
&\le \frac{p^n \left(\frac{a(1-p)}{a-p}\right)^{an/p}}{(1-p)^n(\frac{p}{a-p})^n} \\
&\le \left(\frac{a(1-p)}{a-p}\right)^{an/p} \left(\frac{a-p}{1-p}\right)^n.
\end{aligned}$$

Using $p < 1/2$ and $a \ge 2p$, we get $a \le 2(a-p)$ and $1-p > 1/2$, so

$$\begin{aligned}
\mathbb{P}(Y < an/p) &\le \left(\frac{a(1-p)}{a-p}\right)^{an/p} (2a)^n \\
&\le \left(\frac{a}{a-p}\right)^{an/p} (2a)^n \\
&\le \left(1 + \frac{p}{a-p}\right)^{an/p} (2a)^n \\
&\le \left(1 + \frac{p}{a-p}\right)^{2(a-p)n/p} (2a)^n \\
&\le (e^2 2a)^n,
\end{aligned}$$

since $(1+\frac{1}{x})^x \le e$. □

To finish the proof of (1.6.2), apply Lemma 1.6.2 with $a = \beta k/p \ge C_1\beta m^{3/2}$

$$\begin{aligned}
\mathbb{P}(W_m) &\le \mathbb{P}(\sum_{j=1}^{k} w(j) < \beta m^{3/2}) \\
&\le \mathbb{P}(\sum_{j=1}^{k} X_j < C_1\beta k/p) \\
&\le (2e^2 C_1 \beta)^k = (C_2\beta)^k,
\end{aligned}$$

as desired. This completes the proof of (1.6.2) and hence of Theorem 1.6.1.

□

1.7 Notes

Diffusion limited aggregation was introduced by Witten and Sander in 1981. See [15], [16]. There have been numerous numerical simulations of DLA and heuristic arguments for estimating its growth and geometry, but after thirty years, Kesten's bound is the only rigorously provable thing we know about DLA.

Many variants of DLA have also been proposed and studied. See, for example, [5], [6], [14], [1],

Our discussion of DLA assumed disks were added by moving them continuously by Brownian motion until they made contact with the existing cluster. An alternative model is to use a random walk on a lattice. In this case, the DLA cluster is a connected collection of lattice sites. This is a common formulation of the problem and was the version used in Kesten's papers [10], [8], [9]. The bound and proof are essentially the same as we have given (indeed, our proof is modeled on the discrete proof given by Lawler in []), but one needs a discrete version of Beurling's harmonic measure estimate, Lemma 1.5.4. We choose to give the continuous version of DLA in order to make use of the classical version of Beurling's estimate, which we will also need for other applications in this book.

We have only considered DLA in two dimensions. It is known that in 3 dimensions, the diameter is almost surely $O(n^{1/2}(\log n)^{1/4})$ and in dimensions $d \geq 4$ it is $O(n^{2/(d+1)})$. See [10]. It seems unbelievable that there is no non-trivial lower bound for the diameter. The trivial bound in the plane is of order $n^{1/2}$, since no more than $O(n)$ disjoint unit disks can be packed into a disk of radius $\sqrt{n}$ region. However, so far as the authors know, there is no proof that

$$\lim_{n\to\infty} \frac{\mathrm{diam}(\mathrm{DLA}(n))}{\sqrt{n}} = \infty.$$

It also seems very likely that the bound $2/3$ in Kesten's theorem can be improved; the numerics indicate this and looking at the pictures quickly convinces one that we should be able to improve the square root estimate in Beurling's theorem, which is only sharp for line segments (and DLA does not look like a line segment!). Even more difficult questions include whether DLA has a continuous scaling limit, and what the dimension of such a limit might be.

Stas Smirnov has warned that graduate students and postdocs not be allowed to work on DLA. Apparently they are particularly susceptible to a debilitating condition known as "diffusion limited aggravation".

1.8 Exercises

Exercise 1.1 If Ω is a Jordan domain and $E, F \subset \partial\Omega$ are disjoint closed subarcs, then there is a conformal map of Ω to some rectangle so that E and F map to opposite sides.

Exercise 1.2 If Ω is a topological annulus bounded by two Jordan curves, show that it can be conformally mapped to a round annulus.

Exercise 1.3 Let $E \subset \mathbb{C}$ be a closed set and z a point not in E. Compute the modulus of the path family connecting E to $\{z\}$.

Exercise 1.4 Let $E_n \subset \mathbb{T}$ be defined by $\{z : \mathrm{Re}(z^n) > 0\}$. Show that $\mathrm{Cap}_{\log}(\mathrm{E_n}) \to \mathrm{Cap}_{\log}(\mathbb{T})$ as $n \to \infty$. Since $\mathbb{T} \setminus E_n$ clearly has the same capacity as E_n, this implies capacity is not additive.

Exercise 1.5 Show that the linear fractional transformations that map $\mathbb{D}$ 1-to-1, onto itself are exactly those of the form $z \to \lambda(z-a)/(1-\overline{a}z)$ where $|a| < 1$ and $|\lambda| = 1$.

Exercise 1.6 Show a hyperbolic ball in the disk is also a Euclidean ball, but the hyperbolic and Euclidean centers are different (unless they are both the origin). Compute the Euclidean center and radius of a hyperbolic ball of radius r centered at z in $\mathbb{D}$.

Exercise 1.7 Show that the only isometries of the hyperbolic disk are Möbius transformations and their reflections across $\mathbb{R}$.

Exercise 1.8 Show that the domain U constructed in the proof of Theorem 1.3.6 is equal to $\mathbb{H}_u$.

Exercise 1.9 If $\{f_n\}$ are holomorphic functions on a domain Ω that converge uniformly on compact sets to f and if $z_n \to z \in \Omega$, then $f_n(z_n) \to f(z)$.

Exercise 1.10 Suppose E is compact and supports a positive measure μ so that $\mu(D(x,r)) \le \varphi(r)$, where

$$\sum_{n=0}^{\infty} n\varphi(2^{-n}) < \infty,$$

Then E has positive capacity.

Exercise 1.11 If $E \subset \mathbb{T}$ is compact and has positive Hausdorff dimension, then it has positive capacity.

Exercise 1.12 Suppose Ω is a planar Jordan domain and $E \subset \partial\Omega$ is Borel. Prove that $\omega(z, E, \Omega)$ is a harmonic function of z.

Exercise 1.13 Suppose Ω is a planar Jordan domain and $E \subset \partial\Omega$ is Borel. Show that if $\omega(z, E, \Omega) = 0$ for some $z \in \Omega$, then it is zero on all of Ω.

Exercise 1.14 If $\{p_k\}_{k=1}^n$ are non-negative numbers and $\sum_{k=1}^n p_k = 1$, show that $h = -\sum_{k=1}^n p_k \log p_k$ is maximized uniquely when $p_k = 1/n$ for all k.

Exercise 1.15 Suppose $g(z) = \frac{1}{z} + b_0 + b_1 z + \dots$ is univalent in $\mathbb{D}$. Then $\sum_{n=0}^{\infty} n|b_n|^2 \leq 1$. In particular, $|b_1| \leq 1$. This is the area theorem.

Exercise 1.16 Use the area theorem to prove that if $\varphi(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is univalent on the unit disk with $\varphi'(0) = 1$, then $|a_2| \leq 2$. This is the case $n = 2$ of the Bieberbach conjecture (later to become deBrange's theorem [], []).

Exercise 1.17 Use the previous exercise to give a second proof of the Koebe $\frac{1}{4}$-theorem.

Exercise 1.18 If f is conformal on the disk, and $\varphi = \log f'$, then $|\varphi'(z)| \leq 6/(1-|z|^2)$ for all $z \in \mathbb{D}$.

Exercise 1.19 If φ is conformal on $\mathbb{D}$ then

$$\frac{1-|z|}{(1+|z|)^3} \leq |\varphi'(z)| \frac{1+|z|}{(1-|z|)^3}.$$

This is the distortion theorem. See e.g., Theorem I.4.5 of [7].

Exercise 1.20 If φ is conformal on $\mathbb{D}$ then

$$\frac{|z|}{(1+|z|)^2} \leq |\varphi(z)| \frac{|z|}{(1-|z|)^2}.$$

This is the growth theorem. See e.g., Theorem I.4.5 of [7].

Exercise 1.21

Exercise 1.22

Exercise 1.23

Solutions (eventually move to end of book)

Solution 1.1 First map Ω to the disk by the Riemann mapping theorem. Then use a Möbius transformation to arrange for the images of E and F to be arcs centered at ± 1 and symmetric with respect to the real line. Then the Schwarz-Christoffel formula gives a map to the desired rectangle.

Solution 1.2 Use uniformization theorem to get covering by disk. Then use Riemann map to get covering by vertical strip with deck transformations being vertical translations. Then use exponential map to send strip to annulus and collapsing orbits to single points.

Solution 1.3 Take an annulus around the point that is disjoint from E, but has modulus close to zero, and use monotonicity.

Solution 1.4 The logarithmic capacity of the circle is $1/\log 2$. Compute the potential of Lebesgue measure restricted to E_n and show that it is bounded by $1/2\log 2+o(1)$ Therefore approximately twice this measure is still admissible, which means the capacity of E_n is close to the capacity of the circle, if n is large..

Solution 1.8

Solution 1.9 We may assume $\{z_n\}$ are contained in some disk $D\subset\Omega$ around z. Let $E=\{z_n\}\cup\{z\}$. This is a compact set so it has a positive distance d from $\partial\Omega$. The points within distance $d/2$ of E form a compact set F on which the functions $\{f_n\}$ are uniformly bounded on E, say by M. By the Cauchy estimate the derivatives are bounded by a constant M' on E. Thus

$$|f(z)-f_n(z_n)\leq|f(z)-f_n(z)|+|f_n(z)-f_n(z_n)|\leq|f(z)-f_n(z)|+M'|z-z_n|,$$

and both terms on the right tend to zero by hypothesis.

Solution 1.10 The condition easily implies U_μ is bounded, hence $\text{supp}(\mu)$ has positive capacity.

Solution 1.11 This follows from Frostman's theorem (Theorem **??**) since if $\dim(E)>0$ then E supports a measure that satisfies $\mu(D(x,r))=O(r^\varepsilon)$ for some $\varepsilon>0$ and $\sum_n 2^{-\varepsilon n}<\infty$.

Solution 1.12 Show that $\omega(z,E,\mathbb{D})$ must agree with the Poisson integral of the indicator function of E (the function that is 1 on E and 0 off E). This holds because the derivative of a Möbius transformation of the disk to itself has absolute value equal to the Poisson kernel when restricted to the unit circle.

Solution 1.13 By the maximum principle, a harmonic function that attains a minimum or maximum is constant.

Solution 1.15 For $0<r<1$ let $D_r=\mathbb{C}\setminus g(D(0,r))$. If $z=g(w)$ and $w=e^{i\theta}$ then $dw=iwd\theta$, so by (**??**),

$$\text{area}(D_r)=\iint_{D_r}dxdy=\frac{1}{2i}\int_{\partial D_r}\bar{z}dz=\frac{-1}{2i}\int_{\partial D(0,r)}\bar{g}(w)g'(w)dw.$$

To evaluate the right hand side note that

$$g(z)=\frac{1}{z}+b_0+b_1z+\dots,$$
$$g'(z)=\frac{1}{z^2}+0+b_1+2b_2z+\dots,$$

so that

$$\int_{|w|=r} \bar{g}(w)g'(w)dw = i\int \bar{g}(w)g'(w)wd\theta$$
$$= i\int (\frac{1}{\bar{w}} + \bar{b}_0 + \bar{b}_1\bar{w} + \dots)(-\frac{1}{w} + b_1 w + 2b_2 w + \dots)d\theta$$
$$= 2\pi i(-\frac{1}{r^2} + |b_1|^2 r^2 + 2|b_2|r^4 + \dots)$$

Thus,

$$0 \le \text{area}(D_r) = \pi(\frac{1}{r^2} - \sum_{n=1}^{\infty} n|b_n|^2 r^{2n}).$$

Taking $r \to 1$ gives the result.

Solution 1.16 Let $F(z) = z\sqrt{f(z^2)/z^2}$. Then the quantity inside the square root is even and doesn't vanish in $\mathbb{D}$, so F is odd, univalent and

$$F(z) = z + \frac{a_2}{2}z + \dots.$$

Thus

$$g(z) = \frac{1}{F(z)} = \frac{1}{z} - \frac{a_2}{2}z + \dots,$$

is univalent and satisfies Theorem **??**, so $|a_2| \le 2$.

Solution 1.17 By pre-composing with a Möbius transformation and post-composing by a linear map, we may assume $z = 0$, $f(0) = 0$ and $f'(0) = 1$. Then the right hand inequality is just Schwarz's lemma applied to f^{-1}. To prove the left hand inequality, suppose f never equals w in $\mathbb{D}$. Then

$$g(z) = \frac{wf(z)}{w - f(z)}$$
$$= w(z + a_2 z^2 + \dots)\frac{1}{w}[(1 + \frac{1}{w}(z + a_2 z^2 + \dots) + \frac{1}{w^2}(z + a_2 z^2 + \dots)^2 + \dots)]$$
$$= z + (a_2 + \frac{1}{w})z^2 + \dots,$$

is univalent with $f(0) = 0$ and $f'(0) = 1$. Applying Corollary 1.16 to f and g gives

$$\frac{1}{|w|} \le |a_2| + |a_2 + \frac{1}{w}| \le 2 + 2 = 4.$$

Thus the omitted point w lies outside $D(0, 1/4)$, as desired.

Solution 1.18 Define

$$F(z) = \frac{f(\tau(z)) - f(w)}{(1-|w|^2)f'(w)},$$

where

$$\tau(z) = \frac{z+w}{1-\bar{w}z}.$$

Then F is conformal, $F(0) = 0$ and $F'(0) = 1$, so Lemma **??** says that $|F''(0)| \le 4$. A computation shows

$$F''(0) = \frac{f''(z)}{f'(z)}(1-|z|^2) + (-2\bar{z}),$$

and $\varphi' = (\log f')' = f''/f'$, so

$$|\varphi'|(1-|z|^2) \le |F''(0)| + |2z| \le 4+2 = 6.$$

Solution 1.19 Fix a point $w \in \mathbb{D}$ and write the Koebe transform of f,

$$F(z) = \frac{f(\tau(z)) - f(w)}{(1-|w|^2)f'(w)},$$

where

$$\tau(z) = \frac{z+w}{1-\bar{w}z}.$$

This is univalent, so by Corollary 1.16, $|a_2(w)| \le 2$. Differentiation and setting $z = 0$ shows

$$F'(z) = \frac{f'(\tau(z))\tau'(z)}{(1-|w|^2)f'(w)},$$

$$F''(z) = \frac{f''(\tau(z))\tau'(z)^2 + f'(\tau(z))\tau''(z)}{(1-|w|^2)f'(w)},$$

$$\tau'(0) = 1-|w|^2, \tau''(0) = -2(1-|w|^2),$$

$$F''(0) = \frac{f''(w)}{f(w)}(1-|w|^2) - 2\bar{w}.$$

This implies that the coefficient of z^2 (as a function of w) in the power series of F is

$$a_2(w) = \frac{1}{2}((1-|w|^2)\frac{f''(w)}{f'(w)} - 2\bar{w}).$$

Using $|a_2| \le 2$ and multiplying by $w/(1-|w|^2)$, we get

$$|\frac{wf''(w)}{f'(w)} - \frac{2|w|^2}{1-|w|^2}| \le \frac{4|w|}{1-|w|^2}.$$

Thus

$$\frac{2|w|^2-4|w|}{1-|w|^2} \leq \frac{wf''(w)}{f'(w)} \leq \frac{4|w|+2|w|^2}{1-|w|^2}.$$

Now divide by $|w|$ and use partial fractions,

$$\frac{-1}{1-|w|}+\frac{-3}{1+|w|} \leq \frac{1}{|w|}\frac{wf''(w)}{f'(w)} \leq \frac{3}{1-|w|}+\frac{1}{1+|w|}$$

$$\begin{aligned}\frac{\partial}{\partial r}\log|f'(re^{i\theta})| &= \frac{\partial}{\partial r}\mathrm{Re}\log f'(z)\\ &= \mathrm{Re}\frac{z}{|z|}\frac{\partial}{\partial z}\log f'(z)\\ &= \frac{1}{|z|}\mathrm{Re}(\frac{zf''(z)}{f'(z)})\end{aligned}$$

Since $w = re^{i\theta}$ and $f'(0) = 1$, we can integrate to get

$$\log(1-r)-3\log(1+r) \leq \log|f'(re^{i\theta})| \leq -3\log(1-r)+\log(1+r).$$

Exponentiating gives the result.

2
Conformal maps and martingales

We start by showing any harmonic Bloch function on the unit disk defines a dyadic martingale on the unit circle that appoximates it to within a bounded additive factor. This allows us to immediately deduce Makarov's law of the iterated logarithm for conformal maps from the similar law for martingales, at least for domains with "nice" boundary. We then give the slighly more involved argument that handles arbitrary Jordan domians and even arbitrary simply connected domains. We will also that the conformal map of the disk to the interior of the von Koch snowflake is essentially sharp for Makarov's theorem.

2.1 Bloch martigales

For g analytic on the disk we define

$$\|g\|_{\mathscr{B}} = |g(0)| + \sup_{z\in\mathbb{D}} |g'(z)|(1-|z|^2),$$

which is called the Bloch norm of g. The collection of analytic functions with finite Bloch norm is called the Bloch space $\mathscr{B}$. Note that these are exactly the holomorphic Lipschitz functions from the hyperbolic metric on the disk to the Euclidean metric on the plane.

One example of a Bloch function is the lacunary series

$$\varphi(z) = \sum_{n=1}^{\infty} z^{2^n}.$$

To prove this is Bloch, fix a point $z \in \mathbb{D}$ and choose n so that

$$(n-1)^{-1} \geq 1 - |z| > n^{-1}.$$

Split the sum defining φ at n and use the fact that $(1-1/n)^n < e^{-1}$ to get

$$
\begin{aligned}
|\varphi'(z)| &\le \sum_{k:2^k\le n} 2^k |z|^{2^k-1} + \sum_{k:2^k>n} 2^k |z|^{2^k-1} \\
&\le \sum_{k:2^k\le n} 2^k + \sum_{j>0} n2^j |(1-\frac{1}{n})^{n-1}|^{2^j} \\
&\le 2n + n\sum_{j>0} 2^j e^{-2^j} \\
&\le Cn \\
&\le \frac{C}{1-|z|}.
\end{aligned}
$$

Of course, a similar computation works for $\sum z^{b^n}$ for any integer $b \ge 2$. The example is suggestive because $\{z^{2^n}\}$ looks roughly like a sum if independent random variables. In fact, a precise statement like this holds for all Bloch functions and will be the basis of everything that follows.

If f is univalent on the disk then f' never vanishes, so $\log f'$ is an analytic function on the disk. Lemma 1.23 stated that this function is in the Bloch with uniformly bounded constant and Exercise 1.18 showed that an upper bound for this constant is 6. We saw in the lacunary example that some Bloch functions look like sums of independent random variables. In fact, we shall show that all Bloch functions look like martingales.

Consider a Whitney decomposition of the disk, as illustrated in Figure **??**. The innermost part of the decomposition is a central disk of radius $1/4$. Outside of the central disk, the annulus $A_1 = \{\frac{1}{4} < |z| < \frac{1}{2}\}$ is divided into eight equal sectors, the annulus $A_2 = \{\frac{1}{2} < |z| < \frac{3}{4}\}$ into sixteen sectors, and so on, as shown in Figure 2.1.1. These sectors are called Whitney boxes. Each Whitney box has two radial sides and two circular arc sides concentric with the origin. The circular arc closer to the origin is called the top of the box and the arc further from the origin is called the bottom. Each bottom arc is divided into two pieces by the tops of the Whitney boxes below it ("below" means between the given box and the unit circle). We call these the left and right sides of the bottom arc (left is the one further clockwise). The sides and bottoms of Whitney boxes we will call the Whitney edges, their endpoints we call Whitney vertices. The union of these edges and vertices forms an infinite graph in $\mathbb{D}$ which we call the Whitney graph. The radial projection of a closed Whitney box B onto the unit circle, $\mathbb{T}$, is a closed arc that we denote B^* (this is sometimes called the "shadow" of B, thinking of a light source at the origin). The union of a closed Whitney box B and all the closed Whitney boxes B' so that $(B')^* \subset B^*$ is called the Carleson square with base $I = B^*$.

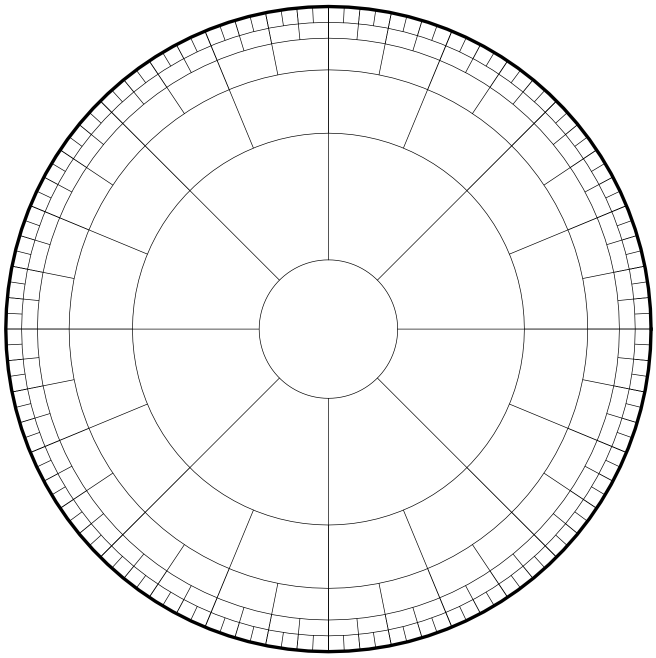


Figure 2.1.1 The Whitney decomposition of the disk.

A **dyadic martingale** on the unit cirlce is a sequence $\{f_n\}$ of functions so that each f_n is constant on the interiors of the nth level dyadic intervals and so that value of f_n on any such interval I is the sum of values of f_{n+1} on the two children of I. The dyadic martingale on the circle is called a Bloch martingale if

$$\|\{f_n\}\|_{\mathscr{B}} \equiv \sup_n \sup_I |f_n - f_{n+1}| < \infty.$$

The quantity on the left is called the Bloch norm of the martingale.

We shall use f_n to denote the martingale as a function on the circle and f_I to denote the value taken by f_n on I, if I is a nth generation dyadic interval. Thus we can write

$$f_n(x) = \sum_I f_I \mathbf{1}_I(x),$$

where the sum is over all nth generation dyadic intervals.

Lemma 2.1.1 *For any harmonic Bloch function u on the disk, there is a Bloch* *martingale* $\{f_n\}$ *on the circle so that* $\|\{f_n\}\|_{\mathscr{B}} \leq C\|u\|_{\mathscr{B}}$ *and*

$$\sup_{I\in\mathscr{D}_n} |u(z_I) - f_n(I)| \leq C\|u\|_{\mathscr{B}}.$$

Proof Suppose u is a harmonic Bloch function. Without loss of generality we may assume its Bloch norm is 1. Suppose $I \subset \mathbb{T}$. We claim that the limit

$$u_I = \lim_{r\nearrow 1} \int_I u(re^{i\theta})d\theta \tag{2.1.1}$$

exists and satisfies

$$|u_I - u(z_I)| = O(\|u\|_{\mathscr{B}}). \tag{2.1.2}$$

If so, then $\{u_I\}$ as I ranges over all dyadic subintervals of $\mathbb{T}$ defined the desired martigale.

We apply Green's theorem over the truncated Carleson box

$$Q_r = \{se^{i\theta} : r < 1-s < |I|, e^{i\theta} \in I\},$$

for $r \ll |I|$. Taking $v = \log \frac{1}{|z|}$, Green's theorem says that since both u and v are harmonic in Q_r, the boundary integral

$$\int_{Q_r} u\frac{\partial v}{\partial n} + v\frac{\partial u}{\partial n} ds = 0.$$

Thus the integral over the "bottom" of the truncated box is the negative of the integral over the other three sides. The integral over the top side is

$$\int_I u(|I|e^{i\theta})d\theta = u(z_I) + O(1)$$

since u itself varies by less than $O(1)$ over this arc. The integrand over each radial side of Q_r is bounded by

$$|u\frac{\partial v}{\partial n}| + |v\frac{\partial u}{\partial n}| \leq 0 + \log\frac{1}{t}\frac{1}{1-t} = O(1),$$

which is integrable on $[1-|I|, 1)$. Hence the limits over the radial sides as $r \nearrow 1$ exists and are $O(1)$. Thus the limit in (2.1.1) exists and satisfies (2.1.2) as desired. □

Lemma 2.1.2 *If* $\{f_n\}$ *is a real-valued Bloch martingale, then*

$$\liminf_{n\to\infty} f_n(\theta) < \infty,$$

for almost every θ.

Proof Without loss of generality assume $f_0 = 0$ and that the Bloch norm is 1. Define a stopping time τ by the first time that $f_n(\theta)$ is either less than -2 or greater than $M \gg 0$. Let X be the set where we step because we are greater than M, and let Y be rest of the cirlce. Since the martingale has Bloch norm 1, $f_tau \geq -3$ on Y, so

$$0 = \int f_\tau d\theta = \int_X f_\tau d\theta + \int_Y f_\tau d\theta \geq M|X| - 3|Y|,$$

or $|X| \leq 3/M$. Thus $\{\theta : \sup_n f_n(\theta) > M\}$ has small measure. In particular, $\{f_n(\theta)\}$ cannot converge to $+\infty$ on positive measure. □

Lemma 2.1.3 *If $\Phi : \mathbb{D} \to \Omega$ is a conformal map then*

$$\liminf_{r \nearrow 1} |\Phi'(re^{i\theta})| < \infty,$$

for almost every θ.

Proof Let $\{f_n\}$ be the dyadic martingale associated to the real-valued harmonic Bloch funtion $u = \text{Re}(\log \Phi')$. By Lemmas **??**, the martingale as finite liminf almost everywhere, and by Lemma 2.1.1, so does u. Since $|\Phi'| = \exp(u)$, the lemma follows. □

Lemma 2.1.4 *If $\{f_n\}$ is a real-valued Bloch martingale, then for almost every θ either*

$$\lim_{n\to\infty} f_n(\theta),$$

exists and is finite or both

$$\liminf_{n\to\infty} f_n(\theta) = -\infty, \limsup_{n\to\infty} f_n(\theta) < \infty$$

hold. In other words, for almost every θ, the sequence $\{f_n(\theta)\}$ either converges to a finite limit or oscilates between $-\infty$ and $+\infty$.

Proof YUVAL. I KNOW HOW TO PROVE THIS USING A MAXIMAL THEOREM FOR MARTINGALES (USUAL PROOF: TRIVAL FOP L^infty,WEAK TYPE L^1, ITERPOLATE) AND L^2 CONDITION FOR CONVERGENCE. NOY TOO HARD¡ BUT IS THERE A SHORT-CUT? □

2.2 Harmonic measure has dimension 1

The dimension of a measure μ is defined to be

$$\dim(\mu) = \inf\{\dim(A) : A \text{ has full } \mu \text{ measure}\}.$$

Suppose $\Omega \subset \mathbb{R}^d$ is open and suppose ω is harmonic measure on $\partial\Omega$. What can we say about $\dim(\omega)$? Harmonic measure depends on a choice of base point in Ω, but the different points all give mutually absolutely continuous measures, so that this question does not depend on the base point. For $d \geq 3$ there are a few results, but still many open questions. For $d = 2$, things are much better understood. One of the key results is due to Makarov who proved that if $\Omega \subset \mathbb{R}^2$ is simply connected then $\dim(\omega) = 1$.

Theorem 2.2.1 (Makarov's upper bound) *Suppose Ω is a simply connected plane domain with a locally connected boundary. Then there exists $E \subset \partial\Omega$ with full harmonic measure and σ-finite $\mathscr{H}^1$ measure.*

Proof Divide the unit circle into three disjoint sets E_1, E_2, E_3 with the properties

1. f' has a non-tangential limit at all $e^{i\theta} \in E_1$.
2. $\liminf_{z \to e^{i\theta}, z \in \Gamma(e^{i\theta})} |f'(z)| = 0$ for all $e^{i\theta} \in E_2$.
3. $\mathscr{H}^1(E_3) = 0$.

By the conformal invariance of harmonic measure, the harmonic measure for Ω is supported on $\Phi(E_1)$ and $\Phi(E_2)$.

First we will show that there is a subset $F \subset \Phi(E_2)$ so that $\omega(F) = \omega(\Phi(E_2))$ and $\mathscr{H}^1(\Phi(F)) = 0$. Fix an integer k and for each z in the disk where $|\Phi'(z)| \leq 2^{-k}$ let I_z denote the largest dyadic arc on the unit circle with containing $z/|z|$ and length $\leq 1 - |z|$. Each point of E_2 is in infinitely many such arcs (with arbitrarily small size) so by the Vitali covering theorem, we can choose a disjoint subcollection of the arcs $\{I_j^k\}_{j=1}^{\infty}$ so that $\mathscr{H}^1(E \setminus \cup_j I_j^k) = 0$. Let $\{z_j^k\}$ be the points in the disk corresponding to the chosen arcs. Also set

$$w_j^k = \Phi(z_j^k), \qquad d_j^k = \operatorname{dist}(w_j^k, \partial\Omega),$$

$$D_j^k = \{|w - w_j^k| \leq Ck^2 d_j^k\}, \qquad G_k = \cup_j D_j^k,$$

$$F_n = \cup_{k \geq n} G_k, \qquad F = \cap_n F_n.$$

Where C is an in Beurling's estimate, Lemma 1.5.4.

Then

$$\omega(w_j^k, D_j^k \cap \partial\Omega, \Omega) \geq 1 - \frac{1}{k} > 0,$$

and so

$$\omega(z_j^k, \Phi^{-1}(G_k), \mathbb{D}) \geq 1 - \frac{1}{k},$$

which implies

$$|I_j^k \setminus G_k| \le O(1/k).$$

Thus for an interval I_j^k,

$$|(E \cap I_j^k) \setminus G_k| \le |I_j^k \setminus G_k| \le O(|I_j^k|/k).$$

Thus

$$|E \setminus F_n| \le inf_{k \ge n} |E \setminus G_k| = 0.$$

Since $\{F_k\}$ are nested decreasing, $\{E \setminus F_k\}$ is nested increasing and their measures converge to the measure of $E \setminus F$, which therefore must be zero.

Finally, we just have to show $\mathscr{H}^1(F) = 0$. By Koebe's theorem $|D_j^k| \sim k^2 |\Phi'(z_j^k)||I_j^k|$,

$$\begin{aligned}
\mathscr{H}^1(F) &\le \inf_n \sum_{k>n} \sum_j |D_j^k| \\
&\le C \inf_n \sum_{k>n} \sum_j k^2 |\Phi'(z_j^k)||I_j^k| \\
&\le C \inf_n \sum_{k>n} 2^{-k} k^2 \sum_j |I_j^k| \\
&\le C \inf_n \sum_{k>n} 2^{-k} k^2 \sum_j 2\pi \\
&= 0.
\end{aligned}$$

Now we have to deal with E_1. For each integer $n \ge 1$, let E_1^n be the subset of E_1, where $|f'|$ is radially bounded by n. The union of these sets is all of E_1. Choose a compact subset F_1^n so that $|E_1^n \setminus F_1^n| \le 1/n$. By definition $\mathrm{Re}\log f'$ is in Bloch and so is bounde by $\log n + O(1)$ on any hyperbolic neighborhood of a radial segment ending in F_1^n, hence $|f'| = O(n)$ on the region W_F. The boundary of W_F has length at most $2\pi^2$, so its image under f has length at most $O(n)$, and this includes the set $f(F_1^n)$. Since $\cup_n F_1^n$ is a full mesure subset of E_1, this completes the proof of Makarov's upper bound. □

Next we prove a reverse inequality, that harmonic measure has dimension at least one. To avoid technicalities we will make a regularity asumption on the boundary of Ω; this assumption is removed in Sections **??** and **??**. For a Bloch martingale, $|f_n(\theta)| = O(n)$ for every θ by definition. We need the following slight improvement of this.

Lemma 2.2.2 *If $\{f_n\}$ is a Bloch martingale, then for almost every θ, we have $|f_n(\theta)| = o(n)$.*

Proof Let C be the Bloch norm of $\{f_n\}$. Note that $\{f_k - f_{k+1}\}$ are orthogonal, so

$$\int f_n^2 d\theta = \sum_{k=0}^{n-1} |f_{k+1} - f_k|^2 d\theta \le C^2 n,$$

so by Chebyshev's inequality

$$\{f_n > \lambda\} \le \frac{1}{\lambda}\|f_n\|_1 \le \frac{1}{\lambda}\|f_n\|_2^{1/2} \le \frac{C\sqrt{n}}{\lambda}.$$

Taking $\lambda = \varepsilon n$ we get

$$\{f_n > \varepsilon n\} = O(\frac{1}{\varepsilon\sqrt{n}}).$$

Taking $n = m^3$, this becomes

$$\{f_{m^3} > \varepsilon m^3\} = O(\frac{1}{\varepsilon m^{3/2}}),$$

which is summable over m, so by Borel-Cantelli

$$\limsup_{m\to\infty} \frac{f_{m^3}(\theta)}{m^3} \le \varepsilon,$$

holds almost everywhere. For $m^3 < n < (m+1)^3$, we have $n - m^3 = O(m^2)$, so the bounded difference condition for Bloch martingales implies, for almost every θ,

$$f_n(\theta) \le f_{m^3}(\theta) + O(m^2) = (\varepsilon + \frac{1}{m})O(m^3) = (\varepsilon + \frac{1}{\sqrt{n}})O(n).$$

Since $\varepsilon > 0$ was arbitary, this proves the lemma. □

Corollary 2.2.3 *If $f : \mathbb{D} \to \Omega$ is a conformal map and $\varepsilon > 0$, then*

$$\liminf_{r\to 1} \frac{|f'(re^{i\theta})|}{(1-r)^{\varepsilon}} \ge 1$$

for almost every θ.

Proof This is immediate from the martingale version (Lemma 2.2.2) and Lemma 2.1.1 □

We say that a closed Jordan curve γ is a **quasidisk** if there is a $M < \infty$ so that $\mathrm{diam}(\gamma(x,y)) \le M|x-y|$, where $\gamma(x,y)$ is the subarc of γ between x and y of smaller diameter. Such curves are also called "bounded turning", or said to satisfy Ahlfors' 3-point condition. The name "quasicircle" comes from the fact that these curves are exactly the images of the unit circle under quasiconformal mappings of the plane to itself. Although we will not prove this, we will use the word “quasicircle” since this is the most common term for this class of curves. Similary, a bounded domain whose boundary is a quasicircle is called a **quasidisk**. The definition is sufficiently general to include many fractal curves, such as the von Koch snowflake.

Theorem 2.2.4 *If Ω is a quasidisk and ω is harmonic measure for Ω, then* $\dim(\omega) = 1$.

Proof We have already see $\dim(\omega) \leq 1$, so we only need to prove the other direction. Fix $\varepsilon > 0$. Suppose $X \subset \partial\Omega$ has positive harmonic measure. By Lemma 2.2.3 we can choose a compact set $E \subset [0, 2\pi]$ and $0 < s < 1$ so that $Y = f(E) \cap X$ has positive harmonic measure and

$$|f'(re^{i\theta})| \geq (1-r)^{\varepsilon}$$

for all $r > s$ and $\theta \in E$. We claim that $\dim(Y) \geq 1-\varepsilon$. Suppose $\{D_j\}$ is a cover of Y by disks. By the quasicircle assumption, we can associate to each disk an arc γ_j so that $D_j \cap \partial\Omega \subset \gamma_j$ and $\operatorname{diam}(\gamma_j) \simeq \operatorname{diam}(D_j)$. Each γ_j corresponds to an arc $I_j \subset \mathbb{T}$. By assumption I_j contains a point $e^{i\theta}$ of E, and by the Koebe $\frac{1}{4}$-theorem,

$$|f'(z_j)| \gtrsim |I_j|^{\varepsilon},$$

where $z_j = z_{I_j}$. Therefore

$$\operatorname{diam}(\gamma_j)^{1-\varepsilon} \geq (|I_j| \cdot |f'(z_j)|)^{1-\varepsilon} \geq |I_j|^{(1+\varepsilon)(1-\varepsilon)} \geq |I_j|^{(1-\varepsilon^2)} \geq |I_j|.$$

Since $\{I_j\}$ covers the set $f^{-1}(Y)$, we deduce that

$$\sum_j |I_j| \geq |f^{-1}(Y)| > 0,$$

is bounded away from zero. Hence the $(1-\varepsilon)$ Hausdorff content of Y is also bounded away from zero, so $\dim(X) \geq \dim(Y) \geq 1-\varepsilon$. Since $\varepsilon > 0$ was arbitrary, we have shown $\dim(X) \geq 1$ for any set X of positive harmonic measure. □

This result can improved in at least two ways. First, quasidisks can be replaced by general Jordan domains (or even general simply connected domains). Second the dimension estimate can be replaced by a much more precise guage estimate. Both improvements will be given later sections.

2.3 Makarov's law of the iterated logarithm

In this section we considerably strengthen Lemma 2.2.2 on the almost everywhere growth of martinagles, and derive from this a stronger lower bound, in terms of guage funtions, for the dimension of harmonic measure on "nice" Jordan domains.

Theorem 2.3.1 *Suppose $\{f_n\}$ is Bloch martingale of norm 1. Then*

$$\limsup_{n\to\infty} \frac{|f_n(\theta)|}{\sqrt{n \log\log n}} = O(1)$$

for almost every θ.

Proof YUVAL WILL FILL IN □

Theorem 2.3.2 *Suppose $\{f_n\}$ is Bloch martingale of norm 1 and there are $N < \infty$ and $\delta > 0$ so that for every dyadic interval I, there is a dyadic interval $J \subset I$ so that $|J| \geq 2^{-N}|I|$ and $|f_I - f_J| > \delta$. Then*

$$\limsup_{n\to\infty} \frac{|f_n(\theta)|}{\sqrt{n \log\log n}} \geq C(N, \delta) > 0$$

for almost every θ

Proof YUVAL WILL FILL IN □

Theorem 2.3.3 (Makarov's LIL for Bloch functions) *There is constant $C < \infty$ so that the following holds. Suppose u is a real-valued Bloch function and*

$$\psi(t) = \sqrt{\log\frac{1}{t} \log\log\frac{1}{t}}.$$

Then

$$\limsup_{r\nearrow 1} u(re^{i\theta})\psi(1-r) \leq O(\|u\|_{\mathscr{B}}),$$

for almost every θ.

Proof Immediate from martingale version and Theorem 2.1.1. □

Theorem 2.3.4 (Makarov's LIL for harmonic measure) *There is constant $C < \infty$ so that the following holds. Suppose Ω is a quasidisk and $E \subset \partial\Omega$ has zero φ-measure for the guage function*

$$\varphi_C(t) = t \exp(C\sqrt{\log\frac{1}{t} \log\log\frac{1}{t}}).$$

Then E has zero harmonic measure in Ω.

Proof The proof is essentially the same as Theorem 2.2.4, except that instead of the easy $o(n)$ upper bound for martingales, we use the more difficult Theorem **??**.

CHRIS WILL FILL IN DETAILS □

2.4 The snowflake is sharp

DEFINE VON KOCH SNOWFLAKE

COMPUTE DIMENSION ?

Theorem 2.4.1 *If $f : \mathbb{D} \to \Omega$ is the conformal map to the interior of the von Koch snowflake, then there is a $c > 0$ $E \subset \partial\Omega$ that has full harmonic measure, but zero Hausdorff φ-measure for*

$$\varphi(t) = t\exp(c\sqrt{\log\frac{1}{t}\log\log\frac{1}{t}}).$$

In other word, the snowflake shows Makarov's LIL is sharp.

Proof CHRIS WILL FILL IN

NEED TO SHOW THE MARTINGALE HAS THE NEEDED LOWER BOUND ON VARIANCES EVERYWHERE AT ALL SCALES

Given a segment I in the nth generation of the construction we want to use extremal length to show that there is an $\alpha > 0$ and segments in the $n+k$th generation whose harmonic measures are, respectively, greater than $\omega(I)3^{-(1-\alpha)k}$ and less than $\omega(I)3^{-(1+\alpha)k}$. Thus $\log|f'| = \mathrm{Re}\log f'$ differs by at least $2\alpha k$ at interior point corresponding to these arcs, and the corresponding martigales differs by $2\alpha k - O(1)$. Choosing k large enough gives the lower bound on variance needed in the lower bound version of the law of the iterated logarithm.

Comment: It is somewhat easier to show that $\mathrm{Im}\log f' = \arg(f')$ *has the desired lower bound on its variance, since here we just have to compute the change in angles for segements in generations n and $n+k$. But we would then have to transfer this to the real part, using some version of harmonic conjugation or the Hilbert transform, and this seems much more ackward than estimating* $|f'|$.

□

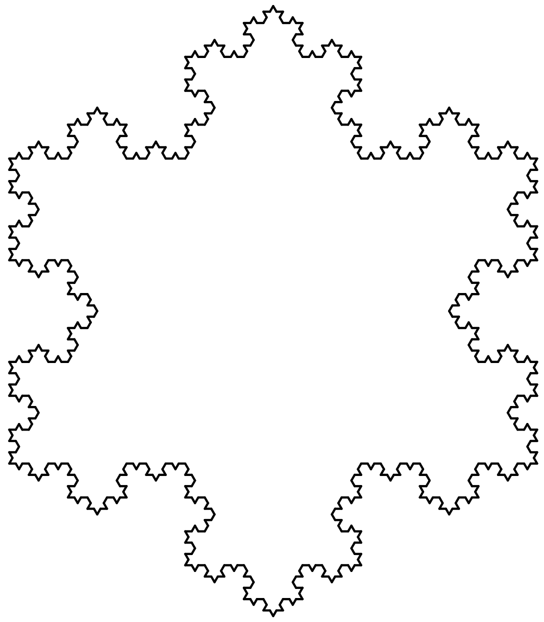


Figure 2.4.1 The von Koch snowflake

2.5 Jordan domains

Next we remove the quasidisk assumtion from Makarov's LIL (Theorem **??**), and prove the result for all Jordan domains. Later (Theorem 3.1.6) we will see that this implies it for all simply connected domains. In the previous case we assumed that for any disk D, all the components of $D \cap \partial\Omega$ were contained in a single arc of $\partial\Omega$ whose diameter was comparable to the diameter of D. In general, this is not true, so we consider the components of $2D \cap \partial\Omega$ separately. Although there may be infinitely many such components, using extremal length we can show that at most $O(-\log \operatorname{diam}(D))$ of these components account for most of the harmonic measure of $D \cap \partial\Omega$ and the extra logarithmic factor can be absorbed into the Makarov's guage function by changing the constant.

in the previous paragraph. We say σ is a **crosscut** of Ω if σ is a Jordan arc in Ω with both endpoints on $\partial\Omega$. By Corollary **??** we can choose a δ so any disk of radius δ has harmonic measure less than $1/n$. (Also see Exercise **??** of Chapter **??**.) The following lemma says that the diameter of a crosscut can be estimated in terms of the size of its preimage and and the estimates on $|f'|$.

Let

$$E_n = \{e^{i\theta} : |f'(re^{i\theta})| \geq \frac{\varphi_{-a}(1-r)}{1-r}, 1 - \frac{1}{n} \leq r < 1\},$$

Lemma 2.5.1 *Suppose σ is a crosscut on Ω contained in some disk D of radius $\leq \delta$. Let β be the subarc of $\partial\Omega$ separated from z_0 by σ and $I \subset \mathbb{T}$ be the arc corresponding arc to β. Assume $I \cap E_n \neq \emptyset$. Then*

$$\text{diam}(D_j) \geq \text{diam}(\sigma) \geq C\varphi_{-a}(|I|).$$

Proof The left hand inequality is trivial, so to prove the other, choose $e^{i\theta} \in I \cap E_n$ and let $z = (1-|I|)e^{i\theta}$. By our choice of δ, $|I| \leq 1/n$, so

$$|f'(z)| \geq \varphi_{-a}(|I|)|I|^{-1}.$$

By the Koebe 1/4 theorem (Theorem **??**),

$$d = \text{dist}(f(z), \partial\Omega) \geq C\varphi_{-a}(|I|),$$

and so by Corollary **??**

$$\omega(f(z), \sigma, \Omega \setminus \sigma) \leq C(\frac{|\sigma|}{d})^{1/2},$$

for some $C < \infty$. On the other hand,

$$\begin{aligned}\omega(f(z), \sigma, \Omega \setminus \sigma) &= \omega(z, f^{-1}(\sigma), \mathbb{D} \setminus f^{-1}(\sigma)) \\ &\geq \min\{\omega(z, I, \mathbb{D}), \omega(z, I^c, \mathbb{D})\},\end{aligned}$$

depending on which side of $f^{-1}(\sigma)$ the point z lies. By the definition of z we see that both these terms are bigger than some absolute constant and so

$$|\sigma| \geq Cd \geq C\varphi_{-a}(|I|),$$

as required. □

The following lemma says that a neighborhood on $\partial\Omega$ does not have too many preimages on the unit circle with large measure. The proof uses some simple facts about extremal length.

Lemma 2.5.2 *Suppose D is a disk of radius $r < r_0$ such that $\omega(D) \geq r$ and let $2D$ be the concentric disk with twice the radius. Then there are crosscuts $\{\sigma_j\} \subset 2D$, $j = 1, \dots, m$ with associated arcs $\{\beta_j\}$, $\{I_j\}$ such that*

$$m \leq \frac{2\pi}{\log 2} \log \frac{1}{r},$$

$$\sum_j |I_j| \geq \frac{3}{4}\omega(D).$$

Proof To normalize the situation assume that $\operatorname{dist}(z_0, \partial\Omega) = 1$ and let K be a disk of radius $1/2$ centered at z_0.

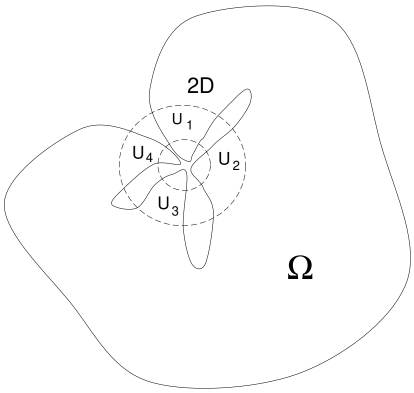


Figure 2.5.1 Regions in proof of Lemma 2.5.2.

Let Ω_0 be the component of $\Omega \setminus D$ containing $\{z_0\}$ and let $\{U_j\}$ be the components of $\Omega_0 \cap 2D$ whose boundary contains arcs on ∂D. Since $\omega(D) > 0$ this collection is nonempty.

Fix j and consider U_j. It is a Jordan domain and $\partial U_j \cap (2D \setminus D)$ is a union of arcs exactly two of which Γ_1^j, Γ_2^j connect ∂D to $\partial 2D$. Their complement in ∂U_j consists of arcs, one of which, call it δ_j, hits ∂D. Then the set $(\partial U_j \cap \partial D) \setminus \partial\Omega$ is a union of arcs $\{\gamma_k\}$ of ∂D each of which is a crosscut of Ω with associated arcs β_k of $\partial\Omega$. Let $E_j = \cup_k \gamma_k$.

Let $\mathscr{F}_j$ be the family of all arcs separating K from E_j and let $\tilde{\mathscr{F}}_j$ be the family of all arcs in U_j connecting Γ_1^j to Γ_2^j and $\mathscr{F}$ the family of all arcs in $2D \setminus D$ separating the two boundary circles. Then by the estimate Corollary 1.5.2 relating extremal length to harmonic measure,

$$M(\mathscr{F}_j) \leq \frac{1}{\pi}\log\left(\frac{C}{\omega(E_j)}\right).$$

By monotonicity (lemma 1.1.2),

$$M(\tilde{\mathscr{F}}_j) \leq M(\mathscr{F}_j).$$

By Lemma 1.1.5,

$$\sum_j \lambda(\tilde{\mathscr{F}}_j) \leq \lambda(\mathscr{F}) = \frac{2\pi}{\log 2},$$

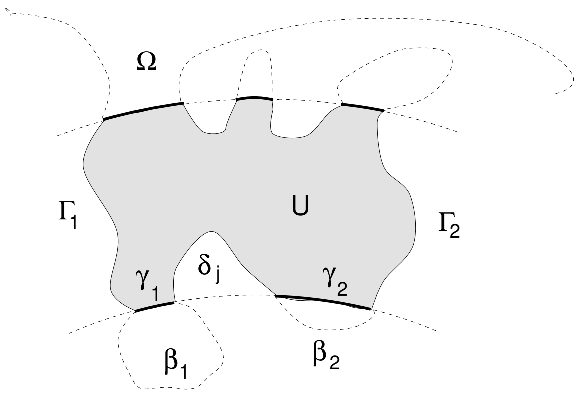


Figure 2.5.2 Detail of region U in proof of Lemma 2.5.2.

or

$$\sum_j M(\tilde{\mathscr{F}}_j)^{-1} \le M(\mathscr{F})^{-1} = \frac{2\pi}{\log 2},$$

so

$$\sum (\log \frac{C}{\omega(E_j)})^{-1} \le \frac{2}{\log 2}.$$

By Tchebyshev's inequality,

$$|\{j : \omega(E_j) \ge Cr^{\pi k}\}| \le \frac{2\pi}{\log 2} k \log \frac{1}{r}.$$

Hence

$$\begin{aligned}\sum_{\omega(E_j)\le Cr^{\pi}} \omega(E_j) &\le \sum_{k=1}^{\infty} (\frac{2\pi}{\log 2}(k+1)\log\frac{1}{r})Cr^{\pi k} \\ &\le (\frac{2\pi C}{\log 2})\log\frac{1}{r}\sum_{k=1}^{\infty}(k+1)r^{\pi k} \\ &\le \frac{1}{4}r,\end{aligned}$$

if $r < r_0$ is small enough. Thus

$$\begin{aligned}\sum_{\omega(E_j)>Cr^{\pi}} \omega(E_j) &\ge \omega(D) - \frac{1}{4}r \\ &\ge \frac{3}{4}\omega(D),\end{aligned}$$

and there are at most $2\pi \log \frac{1}{r} / \log 2$ such j's. So if we take the σ_j to be a crosscut of Ω contained in U_j with

$$\beta_j = E_j \cup (\delta_j \cap \partial\Omega),$$

the lemma is proven. □

Now that the technical lemmas are finished, we can complete the proof of Makarov's theorem.

Proof of Theorem **??** By Theorem **??** we can choose a universal $a > 0$ so that

$$\liminf_{r \to 1} |f'(re^{i\theta})| \left(\frac{1-r}{\varphi_{-a}(1-r)}\right) = +\infty,$$

for almost every θ. So if we define

$$E_n = \{e^{i\theta} : |f'(re^{i\theta})| \geq \frac{\varphi_{-a}(1-r)}{1-r}, 1 - \frac{1}{n} \leq r < 1\},$$

then $\cup_n E_n$ has full measure. Thus it suffices to prove the following: If $E \subset f(E_n)$ and $\mathscr{H}_{\varphi_{2a}}(E) = 0$, then $\omega(E) = 0$.

Suppose $\{D_j\}$ is a covering of E by disks of radius $\{r_j\}$ such that $\max_j r_j \leq \delta$ and $\sum_j \varphi_{2a}(r_j) \leq \varepsilon$. By considering points of density of E (with respect to harmonic measure) and taking δ to be small enough we may suppose

$$\omega(D_j) \leq 2\omega(D_j \cap E).$$

We may also assume $\omega(D_j) \geq r_j$, for the remaining disks satisfy

$$\Omega(\cup_j D_j) \leq \sum_j r_j \leq \sum_j \varphi_{2a}(r_j) \leq \varepsilon.$$

Now for each j let $\{\sigma_k^j\}$ be the crosscuts given by the previous lemma which also satisfy $\beta_k^j \cap E \neq \emptyset$. Then

$$\sum_k \omega(\beta_k^j) \geq \frac{1}{4}\omega(D_j),$$

so by the two lemmas,

$$\begin{aligned}
\omega(E) &\leq \sum_j \omega(D_j) \\
&\leq 4 \sum_j \sum_k \omega(\beta_k^j) \\
&\leq \frac{8\pi}{\log 2} \sum_j \log \frac{1}{r_j} \varphi_{-a}^{-1}\left(\frac{2r_j}{C}\right).
\end{aligned}$$

Now observe $\varphi_{-a}^{-1}(t) \le \varphi_a(t)$. To prove this, note

$$\varphi_{-a}(\varphi_a(t)) = t\exp(a\sqrt{\log\frac{1}{r}\ldots} - \sqrt{\log\frac{1}{\varphi_a(t)}\ldots}).$$

Since $t \le \varphi_a(t)$, the power of the exponential is positive and so

$$\varphi_{-a}(\varphi(t)) \ge t.$$

Thus

$$\omega(E) \le \frac{2\pi}{\log 2}\sum_j \varphi_{2a}(\frac{2r_j}{C})(\log\frac{1}{r_j}\frac{\varphi_{-a}(2r_j/C)}{C/2r_j}).$$

Note,

$$(\log\frac{1}{t})(\frac{\varphi_{-a}(t)}{t}) = \exp(\log\log\frac{1}{t} - a\sqrt{\log\frac{1}{t}\log\log\log\frac{1}{t}}).$$

Since the left hand side tends to 0 as $t \to 0$, the left hand side is bounded for $t < \delta$. Hence

$$\omega(D) \le C\varphi_{2a}(|D|),$$

so

$$\omega(E) \le C\sum_j \varphi_{2a}(2r_j/C) \le C\sum_j \varphi_{2a}(r_j) \le C\varepsilon.$$

Since ε was arbitrary, we have proven the theorem. □

Corollary 2.5.3 *Harmonic measure for a general simply connected domain gives full measure to a set of σ-finite* 1*-measure, and Makarov's LIL holds.*

Proof If Ω is a simply connected domain, then any subset of $\partial\Omega$ of positive harmonic measure also has positive harmonic measure for some Jordan subdomain of Ω. Thus harmonic measure for Ω dominated by a countable sum of harmonic measures for Jordan domains and thus gives full measure to σ-finite length. On the the other hand if harmonic measure for Ω gave positive measure to a set of zero φ-measure (φ as in Makarov's LIL), then some Jordan domain would also give it positive measure, a contradiction. □

2.6 Converses

We have already seen that Makarov's LIL for harmonic measure is sharp except for the choice of the constant C. Here we give an alternate way to prove this, by proving converses of two results from earlier in the text: to each Bloch martigale there is an associated Bloch harmonic function and ach Bloch holomorphic function of sufficiently small norm is of the form $\log f'$ for some conformal map. Given these facts, we can start with a real-valued Bloch martigales which is sharp for the martigales LIL and produce a conformal map that is sharp for Makarov's LIL.

We saw in Lemma 1.18 that if f is conformal, then $\varphi = (\log f')'$ is Bloch with norm at most 6. This fact has a partial converse.

Theorem 2.6.1 *There is an $\varepsilon > 0$, so that if φ is in Bloch with norm at most ε, then $\varphi = \log f'$ for some conformal map f onto a quasidisk.*

Proof We will need the following inequality

$$\int_x^y ((\frac{1-x}{1-t})^\varepsilon - 1)dt \le \frac{\varepsilon}{1-\varepsilon}(y-x),$$

for $0 < \varepsilon < 1$, $0 \le x \le y \le 1$. This can be proved by observing that the left hand side is a convex function of y (for fixed x) and equals the the linear right hand side at $y = x$ and $y = 1$.

Given $z_1 \ne z_2$ in the disk we wish to show $f(z_1) \ne f(z_2)$. First consider the case when $z_1 = 0$ and $z_2 = r > 0$. Then for $0 < t < r$,

$$|\varphi(t)| = |\int_{r_1}^t \varphi'(s)ds| \le \varepsilon \int_0^t \frac{ds}{1-s} = \varepsilon \log \frac{1}{1-t}.$$

Thus,

$$\begin{aligned}
|f(r_2) - f(r_1) - (r_2 - r_1)| &= |\int_0^r (f'(t) - 1)dt| \\
&\le \int_0^r (e^{|\varphi(t)|} - 1)dt \\
&\le \int_0^r ((\frac{1-r_1}{1-t})^\varepsilon - 1)dt \\
&\le \frac{\varepsilon}{1-\varepsilon} r \\
&\le \frac{1}{2} r,
\end{aligned}$$

if $\varepsilon \ge 1/3$ is small enough. Thus $f(0) \ne f(r)$.

Now we consider the general case. It is easy to see that if f is not univalent then there are points z_1, z_2 such that $f(z_1) = f(z_2)$ and $|z_1| = |z_2|$. Without loss of generality we may take $z_1 = r$ and $z_2 = re^{i\theta}$ with $0 < \theta \le \pi$. If $r < \theta$, then the previous estimate gives

$$
\begin{aligned}
|f(z_1) - f(z_2)| &\geq |z_2 - z_1| - |f(z_2) - f(0) - z_2| - |f(z_1) - f(0) - z_1| \\
&\geq |z_2 - z_1| - \frac{2\varepsilon r}{1-\varepsilon} \\
&\geq \frac{2}{3}|z_2 - z_1|,
\end{aligned}
$$

if $\varepsilon < 1/4$.

Finally, if $r \geq \theta$ define a third point $z_3 = (r-\theta)e^{i\theta/2}$. This point is approximately "between" z_1 and z_2 and will play the role the origin did in the previous argument. See Figure 2.6.

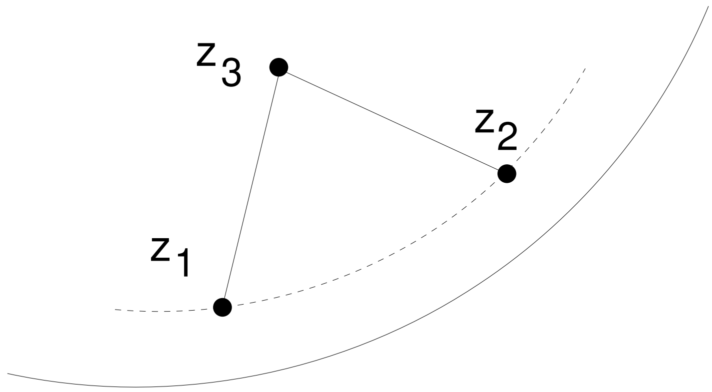


Figure 2.6.1 Proving f is univalent.

Without loss of generality we may assume $f(z_3) = 0$ and $f'(z_3) = 1$ (so $\varphi(z_3) = 0$). Then if w lies on the line segment between z_3 and z_1, i.e.,

$$w = (1-t)z_3 + tz_1,$$

then we have

$$
\begin{aligned}
|\varphi(w)| &= |\int_{z_3}^{w} \varphi'(\zeta) d\zeta| \\
&\leq 2\varepsilon \int_{|z_3|}^{|w|} \frac{d\zeta}{1-|\zeta|} \\
&\leq 2\varepsilon \log(\frac{1-|z_3|}{1-|w|}).
\end{aligned}
$$

Thus by repeating the argument from above,

$$\begin{aligned}|f(z_1) - f(z_3) - (z_3 - z_1)| &= |\int_{z_3}^{z_1} (f'(t) - 1)dt| \\ &\le 2\int_{|z_3|}^{|z_1|} ((\frac{1-r_1}{1-t})^{2\varepsilon} - 1)dt \\ &\le \frac{4\varepsilon}{1-2\varepsilon}(|z_1| - |z_3|) \\ &\le \frac{4\varepsilon}{1-2\varepsilon}|z_1 - z_3|.\end{aligned}$$

Of course, the same works with z_1 replaced by z_2. Thus

$$\begin{aligned}|f(z_1) - f(z_2)| &\ge |z_2 - z_1| - |f(z_2) - (z_2 - z_3)| - |f(z_1) - (z_1 - z_3)| \\ &\ge |z_2 - z_1| - \frac{8\varepsilon}{1-2\varepsilon}|z_1 - z_2| \\ &\ge \frac{1}{2}|z_2 - z_1|,\end{aligned}$$

if ε is sufficiently small. □

Lemma 2.1.1 has a converse:

Lemma 2.6.2 *Given any Bloch martingale $\{f_n\}$ on the circle, there is harmonic Bloch function u on the disk, such that $\|u\|_{\mathscr{B}} \le C\|\{f_n\}\|_{\mathscr{B}}$ and*

$$\sup_{I\in\mathscr{D}_n} |u(z_I) - f_n(I)| \le C\|\{f_n\}\|_{\mathscr{B}}.$$

Proof Suppose $\{f_n\}$ is a Bloch martingle of norm 1. Without loss of generality we may assume $f_0 = 0$ and hence all the elements have mean value zero. Let u_n is the harmonic extension of f_n to the unit disk. By our assumption $u_n(0) = 0$ for all n.

$$\begin{aligned}u_n(z) &= \int_{\mathbb{T}} P_z(e^{i\theta}) f_n(e^{i\theta}) d\theta \\ &= \sum_{k=0}^{n-1} \int_{\mathbb{T}} P_z(e^{i\theta})[f_{n+1}(e^{i\theta}) - f_n(e^{i\theta})]d\theta \\ &= \sum_{k=0}^{n-1} \int_{\mathbb{T}} P_z(e^{i\theta}) \Delta_n(e^{i\theta}) d\theta\end{aligned}$$

Note that Δ_n has means value zero over each dyadic interval of generation n, is bounded by 1 everywhere, and P_z differ from a constant by at most $O(\frac{2^{-n}}{1-|z|})$

on such an interval. Thus if $n < m$,

$$|u_n(z) - u_m(z)| \leq \sum_{k=n}^{m-1} \int_{\mathbb{T}} \frac{2^{-k}}{1-|z|} d\theta = O(\frac{2^{-n}}{1-|z|}),$$

which shows the sequence of harmonic functions converges uniformly on compact sets to a harmonic function u.

Next we want to prove

$$u(z) - f_I = O(1), z \in T(I).$$

This automatically proves that u is Bloch, since its variation over and $T(I)$ is uniformly bounded. Give a dyadic interval I we can form a disjoint collection $\mathscr{C}$ of dyadic intervals J so that $|I| \leq |J| \simeq \text{dist}(J,I)$ and there are only a bounded number of intervals of any given size. Note that $|J| = 2^k|I|$ and $\text{dist}(J,I) \leq 2^k|I|$ implies that $|f_J - f_I| = O(k)$ by the Bloch condition

$$\begin{aligned}
u_n(z) - f_I &= \int_{\mathbb{T}} P_z f_n - P_z f_I d\theta \\
&= \int_{\mathbb{T}} P_z(f_n - f_I) d\theta \\
&= \sum_{J \in \mathscr{C}} \int_J P_z(f_n - f_I) d\theta \\
&= \sum_{k=0}^{\infty} \sum_{J \in \mathscr{C}, |J| = 2^k|I|} \int_J P_z(f_n - f_I) d\theta \\
&= \sum_{k=0}^{\infty} \sum_{J \in \mathscr{C}, |J| = 2^k|I|} \int_J P_z(f_n - f_J) + (f_J - f_I) d\theta \\
&\leq \sum_{k=0}^{\infty} \sum_{J \in \mathscr{C}, |J| = 2^k|I|} \left(\int_J P_z(f_n - f_J) d\theta + \int_J P_z(f_J - f_I) d\theta \right)
\end{aligned}$$

The second term is bounded by

$$\sum_{k=0}^{\infty} \sum_{J \in \mathscr{C}, |J| = 2^k|I|} \int_J O(2^{-2k}|I|^{-1}k) d\theta \leq \sum_{k=0}^{\infty} \sum_{J \in \mathscr{C}, |J| = 2^k|I|} \sum_{k=0^\infty} O(2^{-k}k = O(1).$$

If we choose n so that $f_n - f_I$ mean zero on I, then $f_n - f_J$ also has mean zero on each $J \in \mathscr{C}$, since these are all at least as long as I. Let I be of generation m. Since P_z varies by less than $2^{-n-2k}|I|^{-1}$ on intervals of length 2^{-n} in J_k, we get

$$|\sum_{k=0}^{\infty}\sum_{J\in\mathscr{C},|J|=2^k|I|}\int_J P_z(f_n-f_J)d\theta| = |\sum_{k=0}^{\infty}\sum_{J\in\mathscr{C},|J|=2^k|I|}\int_J P_z\sum_{j=m-k}^{n-1}\Delta_j d\theta|$$
$$= |\sum_{k=0}^{\infty}\sum_{J\in\mathscr{C},|J|=2^k|I|}\int_J \sum_{j=m-k}^{n-1}|I|^{-1}2^{-2k-j}d\theta|$$
$$\leq \sum_{k=0}^{\infty}\sum_{J\in\mathscr{C},|J|=2^k|I|} O(2^m 2^{-2k-m+k})$$
$$= O(\sum_{k=0}^{\infty}\sum_{J\in\mathscr{C},|J|=2^k|I|} 2^{-k})$$
$$= O(1)$$

This completes the proof. □

Theorem 2.6.3 *There is a $c>0$, a quasidisk Ω and a set $E\subset\partial\Omega$ that has full harmonic measure, but zero Hausdorff φ-measure for*

$$\varphi(t) = t\exp(c\sqrt{\log\frac{1}{t}\log\log\frac{1}{t}}).$$

Proof CHRIS WILL FILL IN. USE 5-ADIC MARTINGALE THAT IS 0 ON FIRST, THIRD AND FIFTH INTERVALS AND -1, +1 ON SECOND AND FOURTH. DRAW GRAPH □

REMARK THAT MAXIMAL BLOCH FUNCTIONS LIKE :ACUNARY SERIES WORK AND PETER JONES GAVE A GEOMETRIC SUFFICIENT CONDITION

2.7 Notes

Makarov [**?**] proved that for any gauge function φ such that

$$\lim_{t\to 0}\frac{\varphi(t)}{t} = 0,$$

there is a subset $E\subset\partial\Omega$ so that $\omega(z,E,\Omega)=1$ and $\mathscr{H}^{\varphi}(E)=0$. Pommerenke [**?**] refined this to show that the set E can actually to take to have σ finite $\mathscr{H}^1$ measure. Wolff has pointed out that this can also be deduced directly from Makarov's theorem using a result of Besicovitch: if E has zero $\mathscr{H}^{\psi}$ measure for every guage function which satisfies

$$\lim_{t\to 0}\frac{\psi(t)}{\varphi(t)} = 0,$$

then E has σ-finite $\mathscr{H}^{\varphi}$ measure [**?**].

2.8 Exercise

Exercise 2.1 Show that a planar compact set is locally connected if and only if it is the continuous image of the unit circle.

3
Cone points and twist points

We saw in the previous chapter that Makarov's theorem split harmonic measure for a Jordan domain into two pieces: one supported on a set of σ-finite length and corresponding to points on the circle where the derivative of the conformal map has a radial limit, and another set of zero linear measure that corresponds to points on the circle where f' oscillates between $-\infty$ and $+\infty$. In this chapter we geometrically characterize these boundary points almost everywhere as either cone points or twist points. We also prove the F. and M. Riesz theorem and chacterize when harmonic measures corresponding to two sides of a closed Jordan curve are either mutually absolutely continuous or mutually singular.

3.1 The F. and M. Riesz theorem

In this section we will prove that if Ω is a Jordan domain bounded by a rectifiable curve, then harmonic measure is mutually absolutely continuous with respect to length measure on $\partial\Omega$, We start with Jensen's formula:

Lemma 3.1.1 *If f is analytic on the unit disk with zeros $\{z_n\}_1^N$ in $D(0,r)$, and suppose $f(0) \neq 0$ and f has no zeros on the circle $\{|z| = r\}$. Then*

$$|f(0)|\prod_{n=1}^{N}\frac{r}{|z_n|} = \exp(\frac{1}{2\pi}\int_0^{2\pi}\log|f(e^{i\theta})|d\theta).$$

Proof Let

$$B(z) = \prod_{n=1}^{N}\frac{r^2-\bar{z}_n z}{r(z_n-z)}.$$

then B and f have the same zeros and $|B| = 1$ on the circle $\{|z| = r\}$. Thus $g =$

f/B is analytic in $D(0,r)$, never vanishes in this disk. Thus $\log|g|$ is harmonic in $D(0,r)$, so by the mean value property

$$\log|g(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|g(re^{i\theta})|d\theta.$$

Since $|g| = |f|$ on the circle of radius r and $|f(0)| = |g(0)||B(0)| = |g(0)|\prod_{n=1}^{N}\frac{r}{|z_n|}$, we get the desired equality. □

The next fact is crucial in our study of harmonic measure.

Lemma 3.1.2 *If $f \in H^1$ is not the constant zero function, then the boundary values f^* satisfy $|f^*(e^{i\theta})| > 0$ for almost every θ.*

This is a generalization of the much simpler fact that a analytic function on the disk cannot vanish on an interval of the circle.

Proof Suppose $f \in H^1$ did have boundary values which vanish on a set of positive measure E on the boundary. By replacing $f(z)$ by $f(z)/z^k$ for some k, if necessary, we may assume $f(0) \neq 0$. Let

$$E_+\{|f^*| \geq 1\} = \{\log|f^*| \geq 0\},$$

$$E_- = \{|f| < 1\} = \{\log|f^*| < 0\}.$$

Then since $\log x \leq x$ for $x \geq 1$, for any $0 < r < 1$,

$$\int_{E_+} \log|f(re^{i\theta})|d\theta| \leq Mf(re^{i\theta})d\theta \leq C\|f\|_1.$$

On the other hand, for any $\varepsilon > 0$ there is an r_0 so that if $r > r_0$ then $|f(re^{i\theta})| < \varepsilon$ on a set of θ's of measure $> |E|/2$. Thus

$$\int_{E_-} \log|f(re^{i\theta})|d\theta \leq \frac{1}{2}H^1(E)\log\varepsilon$$

Combining the two estimates we get

$$\lim_{r\to 1}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta| = -\infty,$$

which implies $|f(0)| = 0$. This is a contradiction, so f^* cannot vanish on a set of positive measure. □

Theorem 3.1.3 (F. and M. Riesz, Version 1) *Suppose μ is a finite measure on the unit circle. Then*

$$\int f(e^{i\theta})d\mu(\theta) = 0,$$

for every analytic function on the disk with continuous boundary values, iff

$d\mu(\theta) = h(e^{i\theta})d\theta$ for some H^1 analytic function with $h(0) = 0$. In particular, μ is mutually absolutely continuous with respect to Lebesgue measure.

Proof Suppose μ annihilates analytic functions. Let h be the Poisson integral of μ, then h is clearly harmonic and satisfies

$$\|h\|_{H^1} = \sup_r \int_0^{2\pi} |h(re^{i\theta})|d\theta < \|\mu\|.$$

In fact, h must be analytic since

$$\begin{aligned}\int_0^{2\pi} h(re^{i\theta})(re^{i\theta})^n d\theta &= \int_0^{2\pi} [\int_0^{2\pi} P_r(e^{i(\theta-\psi)})d\mu(\psi)](re^{i\theta})^n d\theta \\ &= \int_0^{2\pi} [\int_0^{2\pi} P_r(e^{i(\theta-\psi)})(re^{i\theta})^n d\theta]d\mu(\psi) \\ &= r^n \int z^n d\mu(z) \\ &= 0,\end{aligned}$$

for all n. Thus h is in the Hardy space $H^1(\mathbb{D})$, and so by Theorem **??** h is the Poisson integral of its boundary values, i.e., $d\mu = h^* d\theta$, as desired. Since μ kills constants, it must have mean value 0, hence $h(0) = 0$. The other direction follows easily from the Cauchy integral formula. □

We say that a connected set is rectifiable if it has finite 1-dimensional measure. It is easy to check that if K is locally rectifiable, then it is locally connected. Thus if Ω is a simply connected domain with rectifiable boundary, $\partial\Omega$ is locally connected so by Carathéodory theorem any Riemann mapping of the disk onto Ω extends continuously to the boundary.

Theorem 3.1.4 *If Φ is univalent mapping of the unit disk onto a simply connected domain with rectifiable boundary, then $\Phi' \in H^1$. In particular, Φ' has finite, non-zero, non-tangential limits almost everywhere.*

Proof Since $\int_0^{2\pi} |\Phi(re^{i\theta})|d\theta$ is the length of the image of circle $\{|z| = r\}$ we only have to check that these lengths remain uniformly bounded as $r \to 1$. Since $\partial\Omega$ is rectifiable, it is locally connected, so Φ extends continuous to every boundary point. Thus every point in $\partial\Omega$ is the endpoint of a curve which is the image of a radius of the disk under Φ. By the Moore troid theorem (Theorem 3.1.7 only a countable subset of $\partial\Omega$ can be the endpoints of three or more such rays.

Now cover $\{|z| = r\}$ by intervals $\{I_j\}$ of length $1 - r$ and centered at points $\{z_j\}$. Let $\{J_j\}$ be the radial projections of these intervals onto the unit circle.

Since $\omega(z, J_j, \mathbb{D})$ is clearly bounded away from zero, Lemma **??** and Koebe's theorem implies

$$\mathscr{H}^1(\Phi(J_j)) \geq \operatorname{diam}(\Phi(J_j)) \geq C \operatorname{dist}(\Phi(z), \partial\Omega) \geq C(1-r)|\Phi'(z_j)|.$$

Moreover, Moore's theorem implies that $\sum_j \mathbf{1}_{\Phi(J_j)}(x) \leq 2$ except possibly on a countable set. Since $\log f'$ is a Bloch function (Lemma **??**) there is a uniform $C < \infty$ such that if f is univalent on the unit disk and $z_0 \in \mathbb{D}$, $D = D(z_0, \frac{1}{2}(1 - |z_0|)$, then

$$C^{1-} \leq \frac{\max_D |f'(z)|}{\min_D |f'(z)|} \leq C.$$

Therefore, if $d = 1 - |z_0|$ and I is the interval of length d centered at $z_0/|z_0|$,

$$\int_I |\Phi'(re^{i\theta})| d\theta \leq C|I||\Phi'(z_j)|.$$

Using this and summing over the points $\{z_j\}$, we get

$$\begin{aligned}\int_0^{2\pi} |\Phi'(re^{i\theta})| d\theta &\leq C(1-r)\sum_j |\Phi'(z_j)| \\ &\leq C \sum_j \mathscr{H}^1(\Phi(J_j)) \\ &\leq 2C\mathscr{H}^1(\partial\Omega). \quad \square\end{aligned}$$

Theorem 3.1.5 (F. and M. Riesz Theorem, Version 2) *Suppose that Φ is a univalent map of $\mathbb{D}$ onto a simply connected domain Ω with rectifiable boundary. Suppose $E \subset \mathbb{T}$. Then $\mathscr{H}^1(E) = 0$ iff $\mathscr{H}^1(\Phi(E)) = 0$. In other words, harmonic measure on $\partial\Omega$ is mutually absolutely continuous to* 1*-dimensional Hausdorff measure.*

Proof Since Φ is smooth inside the unit disk we have

$$\Phi(re^{i\theta_1}) - \Phi(re^{i\theta_2}) = \int_{\theta_1}^{\theta_2} \Phi'(re^{i\theta}) ire^{i\theta} d\theta,$$

for any $0 < r < 1$. Clearly the left hand side converges to

$$\Phi(e^{i\theta_1}) - \Phi(e^{i\theta_2}),$$

as $r \to 1$. By Theorem 3.1.4 $\Phi' \in H^1$, so the radial maximal function of Φ' in is L^1. Thus we may use the Lebesgue dominated convergence theorem to deduce the left hand side converges to

$$\int_{\theta_1}^{\theta_2} \Phi'(e^{i\theta}) ire^{i\theta} d\theta,$$

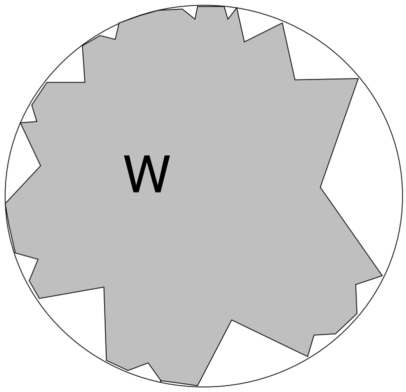


Figure 3.1.1 The sawtooth region W.

Therefore,

$$\Phi(e^{i\theta_1}) - \Phi(e^{i\theta_2}) = \int_{\theta_1}^{\theta_2} \Phi'(re^{i\theta}) ie^{i\theta} d\theta,$$

for all θ_1, θ_2. This implies Φ is absolutely continuous on the unit circle. Thus if $E \subset \mathbb{T}$ has zero length we have

$$\mathscr{H}^1(\Phi(E)) \leq \int_E |\Phi'| d\theta = 0.$$

Conversely, if E has positive length, then the boundary values of Φ' are non-zero almost everywhere on E, so there is a subset $F \subset E$ so that Φ' only takes values in a ball $D_0 = D(x, |x|/2)$ on the set F. Let W be the union on Stolz cones with vertices on F (and angle close to π) and let Γ be the boundary of W (see Figure 3.1). Then using the existence of non-tangential limits we can find a subarc of γ of Γ which hits F in positive measure and on which Φ' on takes values in D_0. Then Φ is bi-Lipschitz on this arc and so F is mapped to a set of positive length. This completes the proof. □

Theorem 3.1.6 *Suppose $f : \mathbb{D} \to \Omega$ is a conformal map onto a simply connected domain. If $E \subset \mathbb{T}$ has positive length, then there is subset $F \subset E$ of positive length so that $f(W_F)$ is a Jordan domain.*

The proof of this requires a nice fact about planar topology due to R.L.Moore. A triod is a "Y" in the plane, i.e., is the union of three Jordan arcs which are pairwise disjoint except that they all share an endpoint. See Figure 3.1.

Theorem 3.1.7 (Moore's triod theorem) *Any pairwise disjoint collection of triods in the plane is countable.*

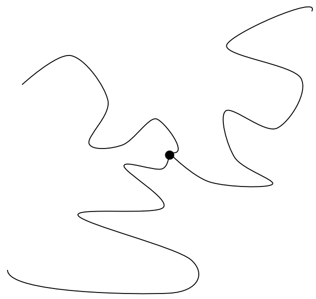


Figure 3.1.2 A triod.

Proof If there is an uncountable such collection. then must be integers n,m and distinct rationals r_1,r_2,r_3 with $|r_i - r_j| > 2/m$ all i,j and an uncountable subset of triods so that if x is the common endpoint of a triod T then the three arcs first intersect $\{|z-x| = 1/n\}$ within angle $1/m$ of the arguments r_1,r_2,r_3. Since any uncountable set has a finite accumulation point, it is easy to drive a contradiction. □

Proof of Theorem 3.1.6 Let $E_1 \subset$ *circle* be where f has a radial limit and define $d : E \times E \to [0,\infty)$ as $d(z,w) = |f(x) - f(y)|$. Let

$$E_1 = \{z \in E : d(z,w) = 0 \text{ only if } w = z\},$$

$$E_2 = \{z \in E : d(z,w) = 0 \text{ for exactly one } w \neq z\}.$$

By Moore's triod theorem, $E \setminus (E_1 \cup E_2)$ is a countable set, hence has linear measure zero. For each n let

$$E_2^n = \{z \in E_2 : d(z,w) = 0, |z-w| < \frac{1}{n} \text{ implies } z = w\}.$$

Since $\cup_n E_2^n = E_2$ and these sets are nested the length of E_2^n converges to the length of E_2.

If $X \subset \mathbb{T}$ has positive measure, then so does either $X \cap E_1$ or $X \cap E_2$. If the former has positive measure, then consider $X \cap E_1 \cap F_m$, where $\{F_m\}$ are of measure $\geq 1 - 1/m$ chosen using Lemma 1.4.8, and m is chosen so large that $X \cap E_1 \cap F_m$ has positive measure. If we take F to be a compact, positive length subset of this set, then f is continuous on W_F and is 1-to-1 on the whole boundary, so $f(W_F)$ is a Jordan domain.

If $X \cap E_2$ has positive length, then so does $X \cap E_2^n$ for some n. Fix such an n and choose an interval I of length $1/2n$ so that $I \cap X \cap E_2^n$ also has positive length. Then the radial limits of f are 1-to-1 restricted to this set, and the proof is finished exactly as above. □

Thus in any question about where harmonic measure on a simply connected domain is absolutely continuous to, or singular to, some other measure, it suffices to consider harmonic measure only on Jordan domains.

Corollary 3.1.8 *Suppose μ is a Borel measure on the plane and $\omega \ll \mu$ when ever ω is harmonic measure for a Jordan domain. Then this holds whenever ω is harmonic measure for a simply connected domain. The same principle holds for the relations $\mu \ll \omega$ and $\mu \perp \omega$.*

Proof If ω is harmonic measure for a simply connected domain and it gives positive mass to set E such that $\mu(E)$, then Lemma **??** gives a Jordan domain whose harmonic measure also gives E positive harmonic measure, by the F. and M. Riesz Theorem (Theorem 1.23). The claim for $\mu \ll \omega$ is similar, but only requires the maximum principle, in place of the Riesz theorem. Together, these two claims prove the third one.

□

3.2 Winding numbers

In this section we establish a simple fact about windings of a curve which is used in our proof of McMillan's twist point theorem. Suppose γ is an analytic Jordan curve defined on $[0,1]$ such that $\gamma(0) = 0$ and $\gamma(1) = 1$. If x is a point not on γ we can define the winding $w(x,\gamma)$ of γ around x by taking

$$\arg(0-x) - \arg(1-x),$$

where we take a continuous branch of $\arg(z-x)$ defined on γ. Since the curve is analytic it has a well defined tangent at each endpoint, so we can also define the windings at the endpoints by truncating the curve and taking limits. We can also define the change of argument of γ' as $\arg(\gamma'(0)) - \arg(\gamma'(1))$ where again we choose a continuous branch of arg.

Lemma 3.2.1

$$|2\pi[w(0,\gamma) + w(1,\gamma)] - [\arg(\gamma'(0) - \arg(\gamma'(1))]| \leq 4\pi.$$

Proof If γ is a line segment then there is nothing to do. Otherwise, because of analyticity we may assume γ hits $[0,1]$ only finitely often. Replace γ by a homotopic smooth curve which intersects $[0,1]$ the least number of times among all curves homotopic to γ by a homotopy which is the identity in some neighborhood of 0 and 1 (thus 0 and 1 are fixed and so are the tangent direction

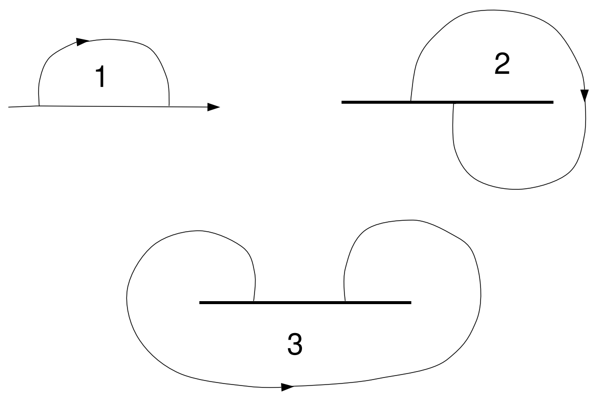


Figure 3.2.1 Three possible shapes for γ_i

at these points). The two quantities

$$w(0,\gamma)+w(1,\gamma),$$

$$\arg(\gamma'(0)-\arg(\gamma'(1)),$$

are invariant under such homotopies (since they can only take a discrete set of values, they can not be changed under continuous deformations), so it suffices to prove the result for the new curve.

So we assume γ has the minimum number of intersections with $[0,1]$, say $\{0=y_0,y_1,y_2,\dots y_n=1\}$, which map via γ^{-1} to points say $\{0=x_0,x_1,x_2,\dots x_n=1\}\subset[0,1]$. Divide γ into oriented subarcs $\gamma_i=\gamma|_{[x_i,x_{i+1}]}$. Then γ_i is a Jordan arc with endpoints on $[0,1]$, but otherwise disjoint from $[0,1]$. The three possible types of arcs (up to a homeomorphism of the plane mapping $[0,1]$ to itself) are shown in Figure 3.2. We denote the three types as 1,2 and 3.

Except at the points 0 and 1 its makes sense to say that γ_i approaches it endpoints from either "above" or "below" $[0,1]$. For each x_i with $0<i<n$ it is easy to see that γ_{i-1} and γ_i approach from different sides; otherwise there would be a smooth homotopy which removes the intersection at x_i, thus lowering the total number of intersections. Similarly, none of these subarcs can be of type 1 in Figure 3.2. Otherwise, using the fact that γ_{i-1} and γ_{i+1} approach x_i and x_{i+1} respectively from the opposite side we can homotopy γ_i across $[0,1]$ thus removing the intersections at both x_i and x_i+1. Thus the subarcs of γ must be either type 2 or 3.

We say that γ_i is "good" if $y_{i+1}>y_i$ and is "bad" if $y_{i+1}<y_i$. We first claim that the minimality of γ implies there are no bad subarcs. Suppose that there

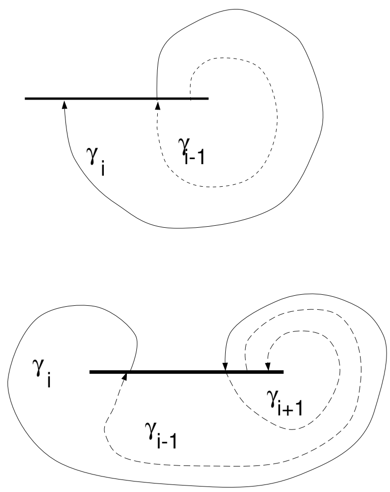


Figure 3.2.2 Bad arcs.

are bad arcs. We will consider two cases. First suppose there is a bad arc of type 2. Then there is a bad arc γ_i of type 2 with endpoint y_i as close to 1 as possible (i.e., farthest to the right among all bad type 2 arcs). See Figure 3.2. Then the preceding arc must be type 2 as well (see the figure), but this is only possible it is bad as well. This is a contradiction and implies that there are no bad arcs of type 2.

Now suppose all the bad arcs are type 3. Choose γ_i to be the last bad arc in the ordering of γ. Then γ_{i+1} is good and must be type 2. See Figure 3.2. Topologically, the only possibilities for γ_{i-1} are that it is type 1 or is a bad arc of type 2. Both are ruled out by hypothesis, so we deduce there are no bad arcs.

We can now finish the proof. Replace γ by a homotopic arc where the homotopy is the identity except in small neighborhoods of each intersection point y_i, $0 < 1 < n$ and in those neighborhoods the curve is changed so that γ' is horizontal and points to the right as γ crosses $[0,1]$. Then for each subarc γ_i the tangents points the same direction at either endpoint. Thus the change in argument of γ' along each γ_i is a multiple of 2π. There are only a few cases and it is each of them it is trivial to check that the change in argument of γ' is 2π times $w(0,\gamma_i)+w(1,\gamma_i)$ where $w(z,\gamma)$ denotes the change in $\arg(y-z)$ for some branch of the argument function defined on γ.

By summing over i we now get that the change of argument of γ' on $[x_1, x_{n-1}$ is equal to 2π times $w(0,\gamma|_{[x_1,x_{n-1}]}) + w(1,\gamma|_{[x_1,x_{n-1}]})$. Adding in the two end

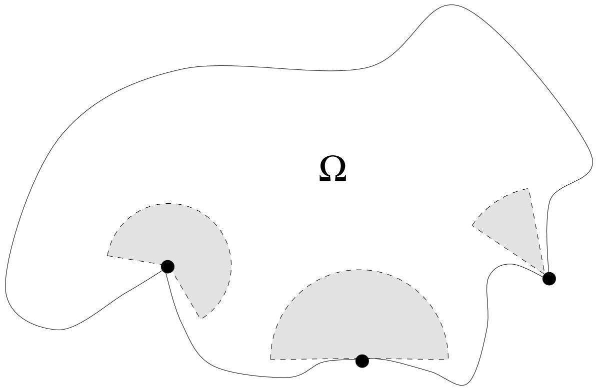


Figure 3.3.1 Cone points of $\partial\Omega$.

intervals γ_0 and γ_{n-1} can only alter the equality by a factor of at most 2π each so we obtain the lemma. □

3.3 McMillan's Twist Point Theorem

If Ω is simply connected we say $x \in \partial\Omega$ is an *inner tangent* point of Ω if for any $\varepsilon > 0$ x is the vertex of a cone in Ω with angle $\pi - \varepsilon$, but is the vertex of no cone with angle $> \pi$. We say that x is a *cone point* if it is the vertex of some cone in Ω.

Lemma 3.3.1 *If Ω is simply connected then the set of cone points has σ-finite 1-dimensional measure and almost every cone point is an inner tangent point.*

Proof By considering cones with rational angles and radius, we can write the set of cone points as a countable union of sets, each of each are the vertices of cones in Ω with fixed side directions and diameters. It clearly suffices to prove the claims for any such set.

Let $F \subset \partial\Omega$ be the set of points $x \in \partial\Omega$ so that

$$W_x = \{x + z : |z| < r, |\arg(-iz)| \le \varepsilon\}\} \subset \Omega.$$

By again dividing into a countable number of subsets we may assume that F is contained in the rectangle $R = \{z : |\mathrm{Im}(z)| < r/10, |\mathrm{Re}(z)| < r\varepsilon/10\}$. Let $W = \cup_{x\in F} W_x$. Then $R \cap \partial W$ is graph of a Lipschitz function (norm depending only on ε), and hence is rectifiable. Since it contains F, F has finite 1-dimensional measure and almost every point of F is a tangent point of the arc. Thus almost every point of F is an inner tangent of W and hence of Ω. □

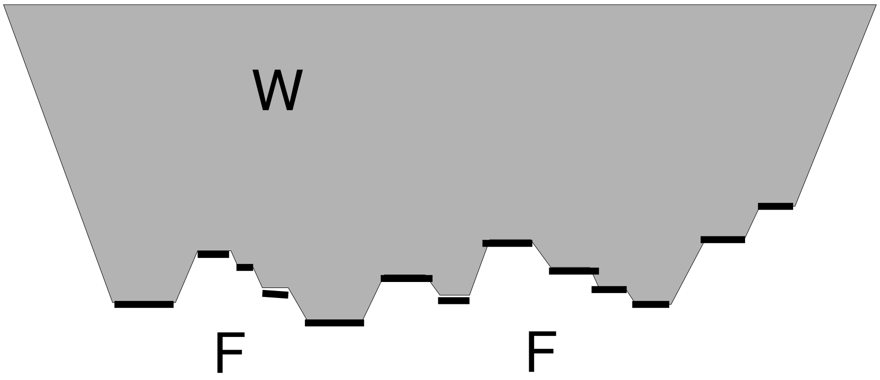


Figure 3.3.2 Cone points lie on Lipschitz graphs.

A point x is called a twist point for Ω if for any branch of $\arg(z-x)$ defined on Ω we have

$$\limsup_{z\to x, z\in\Omega} \arg(z-x) = \infty,$$

and

$$\liminf_{z\to x, z\in\Omega} \arg(z-x) = -\infty.$$

Thus to approach a twist point x through Ω we must "twist around" x arbitrarily far in both directions. It is difficult to draw a twist point on the boundary (see Figure 3.3 for a point with one twist), but we shall see later that such things can exist. For example, harmonic measure on the von Koch snowflake gives full measure to the twist points (see the exercises).

Thus inner tangent points and twist points represent two extremes of behavior for boundary points. What is surprising is that, with respect to harmonic measure, these are the only possibilities.

Theorem 3.3.2 (McMillan's Twist Point Theorem) *If Ω is a simply connected domain then almost every point on $\partial\Omega$ (with respect to harmonic measure) is either an inner tangent point or a twist point.*

Proof The proof is essentially Plessner's theorem. Let $\Phi : \mathbb{D} \to \Omega$ be a Riemann mapping and apply Plessner's theorem to the derivative Φ'. Plessner's theorem says that we can write $\mathbb{T} = E_0 \cup E_1 \cup E_2$ where E_0 has measure zero, Φ' has non-zero non-tangential limits at every point of E_1 and Φ' is non-tangentially dense at every point of E_2.

Clearly the set $\Phi(E_1) \cup \Phi(E_2)$ has full harmonic measure on $\partial\Omega$. Moreover,

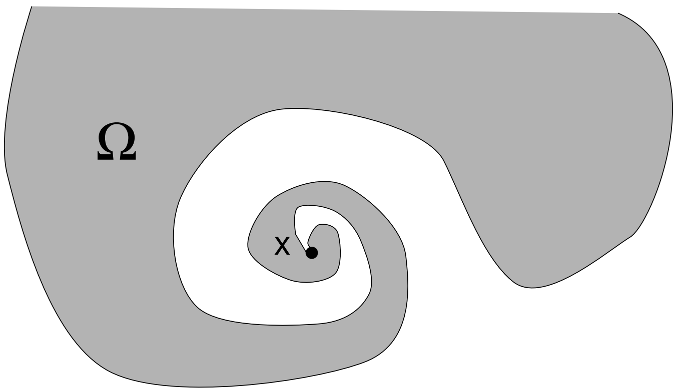


Figure 3.3.3 Boundary point with one "twist".

we saw in the last section that $\Phi(E_1) \subset \partial\Omega$ consists of inner tangents. Thus if we can show that $\Phi(E_2)$ consists of twist points almost everywhere (with respect to harmonic measure) we will be done. In fact all we have to do is produce a sequence of points $z_n \in \Omega$ with $\arg(z_n - x) \to +\infty$ and another with arguments tending tending to $-\infty$.

To prove this, suppose it fails. Then there is a set F of positive measure on $\mathbb{T}$ on which Φ has nontangential limits, Φ' is non-tangentially dense but

$$\arg[\Phi(re^{i\theta}) - \Phi(e^{i\theta})],$$

remains bounded above as $r \to 1$. We will show this is impossible. Since Φ' is non-tangentially dense on F, so is $\log \Phi' = \log|\Phi'| + i \arg \Phi'$. Hence $\arg \Phi'$ must be non-tangentially unbounded above and below by Plessner's theorem.

By Lemma **??** there is a M so that the images of the rays $[0, e^{i\theta})$ have length less than M except on a set of measure $|F|/2$. So be replacing F by a set of half the measure we may assume of the associated rays have bounded length.

Let $\theta_0 \in F$ be a point of density and consider a sequence $\{z_n\} \to 1$ so that $\arg \Phi(z_n) \to +\infty$. Fix a large N and choose an n so large that $\arg \Phi'(z_n) > 4\pi(N+1)$. Let $\gamma = \Phi([0, z_n)$ be the image of the radial segment from 0 to z_n. The change of argument of γ' from one endpoint to the other is $|\arg \Phi'(z_n)|$. By an elementary property of curves in the plane (Lemma 3.2.1 in Appendix C) the curve γ "winds around" one of its endpoints at least N times.

The winding of the curve is defined as follows. If x is not on γ we define the winding $w(x,\gamma)$ of γ around x by taking a continuous branch of $\arg(z-x)$ defined on γ and taking taking the difference $\arg(a-x) - \arg(b-x)$, where a, b are the endpoints of γ. If $x = a$ or $x = b$, then we can still define the winding by truncating the curve just short of the endpoint and taking the limit. Since γ is analytic it has a well defined angle at each endpoint, so this is no problem.

We prove in Appendix C that if the argument of γ' changes by more than $4\pi(N+1)$ between its two endpoints then

$$w(a,\gamma)+w(b,\gamma)\geq 2N,$$

By rescaling we may assume $\Phi'(0)=1$. So by Koebe's theorem (see Appendix C, Section **??**) There is a disk $D_0 \subset \Omega$ of diameter similar to 1 that γ never re-enters once it leaves. Moreover, γ does not wind around 0 inside this disk. In order to wind around a, γ must wind around the disk and since it has length at most M, it can wind around 0 at most $M/2\pi$ times. If N was chosen large enough, we see that most of the winding of γ must be around the point b.

We would like to deduce that the curve γ also winds around the point $\Phi(e^{i\theta_0})$, but this may not be true. Instead we will show that there is another point in F near b around which the curve does wind.

Recall that θ_0 was chosen to be a point of density of F. So if $|z_n|$ is close enough to 1, more that half the interval of length $1-|z_n|$ centered at $e^{i\theta_0}$ consists of point in F. By an application of Lemma **??** (where we now let z_n play the role of the origin) we can find a point x in F so that x can be connected to b in Ω by a curve of length at most $C\,\mathrm{dist}(b,\partial\Omega)$. Just as we argued for the origin above, this curve cannot wind around b more than a bounded number of times. This implies that the winding of γ around b and around x can differ by at most a bounded factor. Thus the winding of γ around x must be very large. This contradicts the assumption that $x \in F$, proving the theorem. □

The following is a local version of the F. and M. Riesz theorem.

Corollary 3.3.3 *Suppose Ω is simply connected and let E be a subset of the cone points on $\partial\Omega$. Then E has positive harmonic measure iff it has positive length.*

Proof First suppose E has positive length. Then pass to a subset of positive measure contained in a rectangle R exactly as in the proof of Lemma 3.3.1 and let W be the union of cones constructed there. Then $W_1 = W \cap R$ is a rectifiable subdomain of Ω which hits E in positive length. By the F. and M. Riesz theorem Theorem 3.1.5 E has positive harmonic measure in W_1 and hence in Ω by the maximum principle.

Next suppose E has positive harmonic measure. Let Φ be a Riemann mapping of $\mathbb{D}$ to Ω. Then $F=\Phi^{-1}(E)$ has positive length and Φ' has a non-zero non-tangential limit at almost every point of F. Therefore we can find a $\alpha>0$, $M<\infty$ and a subset $F_0 \subset F$ of positive measure so that $|\Phi'| \leq M$ on every

Stolz cone of angle α with vertex in F_0. Let W_2 be the union of these cones. Then W has rectifiable boundary, and $|\Phi'|$ is bounded on ∂W_2, so $\Phi(W_2)$ is a subdomain of Ω with rectifiable boundary. By Theorem 3.1.5 again, $\Phi(F_0)$ has positive length (since it has positive harmonic measure) and hence so does E. □

3.4 Mutually singular harmonic measures

Two measures μ and ν are called *mutually absolutely continuous* if they have the same null sets, i.e., $\mu(E) = 0$ if and only if $\nu(E) = 0$. The measures are called *mutually singular* if each is supported on a null set of the other, i.e., there is a set E with $\mu(E) = 0$ but $\nu(E^c) = 0$.

Suppose Γ is a closed Jordan curve which divides the Riemann sphere $\mathbb{C}^\infty$ into two simply connected domains Ω_1 and Ω_2. We know (Lemma **??**, Chapter **??**) that if we choose two points on the same side of Γ then the two harmonic measures will be mutually absolutely continuous with respect to each other. But what happens if we choose points from opposite sides of the curve? Can the two measures be mutually singular?

We have already see that harmonic measure for a simply connected domain Ω is mutually absolutely continuous with $\mathscr{H}^1$ on the set of inner tangents. A point of Γ is called a tangent point if it is an inner tangent for each of the two complementary domains. Thus the ω_1 and ω_2 are mutually absolutely continuous when restricted to the set tangent points of Γ. The following result says they are mutually singular on the rest of Γ.

Theorem 3.4.1 *Suppose $z_1 \in \Omega_1$, $z_2 \in \Omega_2$ and let ω_1, ω_2 denote the corresponding harmonic measures. Then ω_1 and ω_2 are mutually absolutely continuous on the set of tangent points of Γ and are mutually singular on the rest of Γ. In particular, $\omega_1 \perp \omega_2$ iff $\mathscr{H}^1(\text{tangent points}) = 0$.*

Proof This result follows from the proof of Makarov's theorem in Section **??** and an estimate of harmonic measure due to Beurling. The part of the proof of Makarov's theorem we need can be summarized as

Lemma 3.4.2 *Suppose Ω is simply connected and let ω be harmonic measure with respect to some point in Ω. If $T \subset \partial\Omega$ denotes the set of inner tangents then there is an $F \subset \partial\Omega \setminus T$ $\omega(F) = \omega(\partial\Omega \setminus T)$ such that for any $M > 0$ there is a disjoint covering of F by disks $\{D_j\}$ with $\omega(D_j) \geq M|D_j|$.*

The estimate of Beurling we want is

Lemma 3.4.3 *Suppose Γ is a closed Jordan curve dividing the sphere into two simply connected domains Ω_1, Ω_2. Let $z_i \in \Omega_i$ satisfy* $\mathrm{dist}(z_i, \partial\Omega_1)$ *for $i = 1,2$. Then there is a $C < \infty$ so that for any disk D,*

$$\omega_1(D)\omega_2(D) \leq C|D|^2.$$

Proof This follows from an estimate of harmonic measure known as the Ahlfors distortion theorem (Corollary 1.5.3, Appendix C). Suppose Ω is simply connected and $x \in \partial\Omega$. For each $t > 0$, let $\theta(t)$ denote the length of the longest arc in $\Omega \cap \{|z-x| = t\}$. Then if $\mathrm{dist}(z_0, \partial\Omega) \geq 1$, the distortion theorem says

$$\omega(z_0, D(x,r), \Omega) \leq C\exp(-\pi \int_r^1 \frac{dt}{\theta(t)}).$$

(A version of this also holds for multiply connected domains and for domains in higher dimensions. For example see [**?**] and its references.)

To apply this to our situation, let $x \in \Gamma$ and let $\theta_i(t)$ be the function corresponding to Ω_i for $i = 1,2$. The multiplying the estimates for each domain gives

$$\omega_1(D)\omega_2(D) \leq C\exp(-\pi \int_{|D|}^1 (\frac{1}{\theta_1(t)} + \frac{1}{\theta_2(t)})dt).$$

Since Ω_1 and Ω_2 are disjoint, $\theta_1 + \theta + 2 \leq 2\pi t$ and so a simple calculus exercise shows that $\theta_1^{-1} + \theta_2^{-1} \geq 2/\pi t$. Thus

$$\omega_1(D)\omega_2(D) \leq C\exp(-\pi \int_{|D|}^1 \frac{2\pi t}{d} t) = C|D|^2,$$

as desired. □

We can now prove the singularity of harmonic measures. We can divide Γ into 5 sets:

1. Tangent points,
2. Twist points,
3. Inner tangents for Ω_1 which are not inner tangents for Ω_2,
4. Inner tangents for Ω_2 which are not inner tangents for Ω_1,
5. Everything else.

We already know that the harmonic measures are mutually absolutely continuous on (1) and that (5) has zero harmonic measure from both sides. Moreover, ω_2 gives zero mass to (3), so the measures are singular on that set. Similarly for (4). Therefore all we need to show is that the measures are singular on the twist points.

Choose a large n and by the first lemma choose disjoint disks $\{D_j^n\}$ so that

$$\omega_1(D_j^n) \geq n|D_j^n|,$$

$$\omega(\cup_j D_j^n) = \omega_1(\text{twist points}).$$

Then if $F = \cap_n \cup_{k>n} \cup_j D_j^k$, we have

$$\omega_1(F) = \omega_1(\text{twist points}),$$

but by Beurling's estimate,

$$\begin{aligned}
\omega_2(F) &\leq \sum_j \frac{C|D_j^n|^2}{\omega_1(D_j^n)} \\
&\leq \frac{C}{n}\sum_j |D_j| \\
&\leq \frac{C}{n^2}\sum_j \omega(D_j) \\
&\leq \frac{C}{n^2} \\
&\to 0.
\end{aligned}$$

Thus the measures are singular on the twist points. □

The von Koch snowflake is an example where the harmonic measure for both sides lives on the twist points, so the harmonic measures are singular in this case. This is partly visible in Figure 3.4 which shows the images of Carleson grids on the unit disk under the mappings to both the interior and exterior of the snowflake. Note that the small boxes for the two sides seem to accumulate on different sets. Compare with Figure **??** which shows a Carleson grid in the disk and its image in the interior of the snowflake.

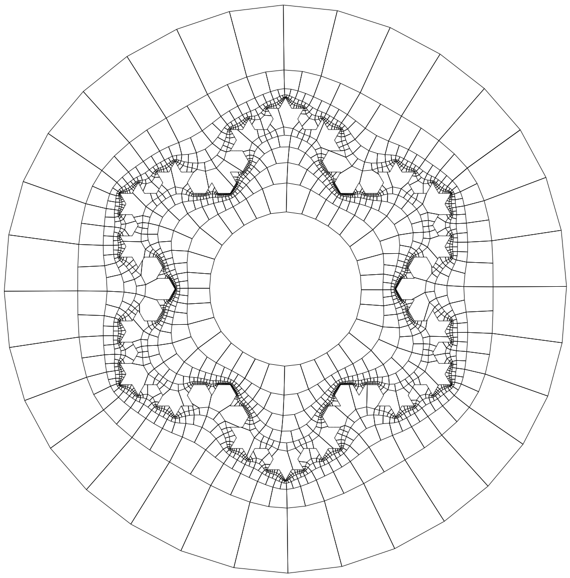


Figure 3.4.1 Conformal images of the Whitney grids for two sides of snowflake.

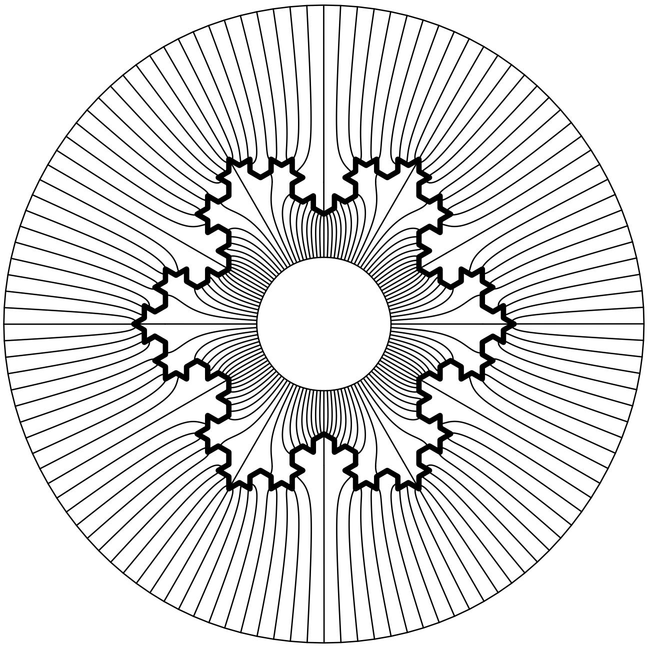


Figure 3.4.2 120 evenly spaced (for harmonic measure) radial lines mapped the the complementary components of the snowflake. The singularity of harmonic measure is evident in the distinct distributions of the endpoints on either side.

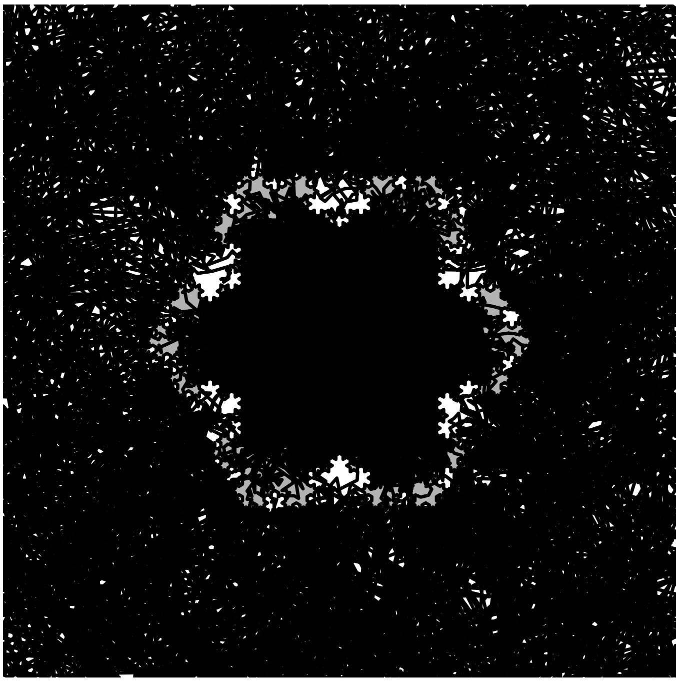


Figure 3.4.3 Another way of visualizing the singularity of harmonic measures. We have run 1000 random walks that step half-way to the boundary; this gives the same hitting probability as Brownian motion.

3.5 From fractals to space filling curves

By as space filling (or Peano) curve we mean a continuous map $f : [0,1] \to \mathbb{R}^2$ (or $\mathbb{R}^d$) which covers an open set. There are many explicit constructions of such maps. For example, see Figure 3.5 for a few steps in such a construction.

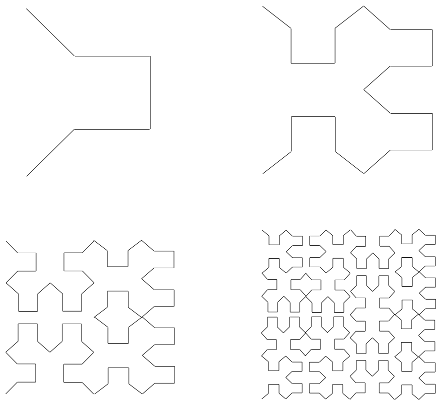


Figure 3.5.1 Approximation to a space filling curve.

Here we shall point out an indirect method for producing such examples using Frostman measures. We need the following easy consequence of the argument principle in complex analysis.

Lemma 3.5.1 *Suppose Γ is a simple Jordan arc in the plane and f is a bounded continuous function on the Riemann sphere $\mathbb{C}^\infty$ which is analytic off Γ. Then $f(\mathbb{C}^\infty) \subset f(\Gamma)$.*

Proof We need to show that if f takes a value off Γ it also takes it on Γ. Suppose not, e.g., suppose f is zero somewhere on $\mathbb{C}^\infty$ but not on Γ. Since f is continuous its zeros must be bounded away from Γ and since it is analytic off Γ there are only finitely many of them. By the argument principle, the number of zeros is counted (with multiplicity) by the winding number of a smooth curve which surrounds Γ and is sufficiently close to Γ. But since f is never zero on Γ, such a curve must have winding number zero, a contradiction □

If Γ is a line segment, it is a consequence of Morera's theorem that any bounded, continuous function analytic off Γ must be entire and thus constant by Liouville's theorem. In this case the above lemma holds, but only trivially. If Γ has dimension strictly larger than 1 however, there are many non-trivial examples of such functions, as we shall now show.

To get such an example, suppose Γ is any curve with dimension > 1, e.g., the von Koch snowflake. Let μ be a measure on Γ which satisfies

$$\mu(B) \leq C|B|^{\alpha},$$

for some $C < \infty$ and $\alpha > 1$. Then define

$$F(z) = \frac{1}{z} * \mu = \int \frac{1}{z-w} d\mu(w).$$

To estimate the integral we break it up into annuli of the form

$$A_n = \{w : 2^{-n} \leq |z-w| \leq 2^{-n+1}\}.$$

Then using the facts that $\mu(A_n) \leq \mu(D(z, 2^{-n+1}))$ and that the kernel $(z-w)^{-1}$ is bounded by 2^n on A_n, we get

$$|F(z)| \leq \sum_{n=0}^{\infty} 2^n \mu(D(z, 2^{-n})) \leq \sum_{n=1}^{\infty} 2^{-n(\alpha-1)} < \infty.$$

A similar calculation shows F is continuous, indeed, Hölder continuous. More precisely, suppose z_1 and z_2 are two points and choose N so that

$$2^{-N} \leq |z_1 - z_2| \leq 2^{-N+1}.$$

We now estimate

$$|F(z_1) - F(z_2)| = |\int (\frac{1}{z_1 - w} - \frac{1}{z_2 - w}) d\mu(w)|,$$

by breaking the integral to three types of pieces: annuli of size smaller than 2^{-N+2} around each of z_1 and z_2 and annuli of size greater than 2^{-N+1} around z_1. The estimates over the small annuli are exactly as above. For the large annuli we use the estimate that

$$|\frac{1}{z_1 - w} - \frac{1}{z_2 - w}| \leq \frac{|z_1 - z_2|}{|w - z_1||w - z_2|} \leq 2^{-N+2n},$$

for $|w - z_1| \geq 2^{-n} > 2^{-N}$. Thus

$$\begin{aligned}
|F(z_1) - F(z_2)| \leq & \sum_{n=N-2}^{\infty} 2^n \mu(D(z_1, 2^{-n})) + \sum_{n=N-2}^{\infty} 2^n \mu(D(z_2, 2^{-n})) + \\
& \sum_{n=-N}^{0} \mu(D(z_2, 2^n)) 2^{-N-2n} \\
& \leq C 2^{N(\alpha-1)} + 2^{-N} \sum_{n=-N}^{0} 2^{n\alpha+2n} \\
& \leq C 2^{-N(\alpha-1)} \\
& \leq C|z_1 - z_2|^{\alpha-1}.
\end{aligned}$$

It's easy to show using uniform convergence that

$$F'(z) = \int \frac{-1}{(z-w)^2} d\mu(w),$$

for $z \notin K$, so F is analytic off K. Furthermore,

$$\mathrm{Re}(F'(z)) = \int \frac{-1}{\mathrm{Re}((z-w)^2)} d\mu(w),$$

is strictly positive if w is a large real number, so F is non-constant.

Thus by the lemma

$$f(\mathbb{C}^\infty) \subset f(\Gamma).$$

But since f is a non-constant analytic function off Γ, the open mapping theorem for analytic functions says the left hand side is an open set. Thus $f : \Gamma \to \mathbb{R}^2$ covers an open set.

Since it is easy to parameterize Γ by $[0,1]$, this gives the desired function. In Figure 3.5.2 we show successive approximations to this curve when we start with the usual measure on the von Koch Snowflake which gives equal mass to each nth level piece. The figure shows the result of putting equal point masses on each vertex of the n generation snowflake, convolving this with $1/z$ and plotting the polygon with the image vertices.

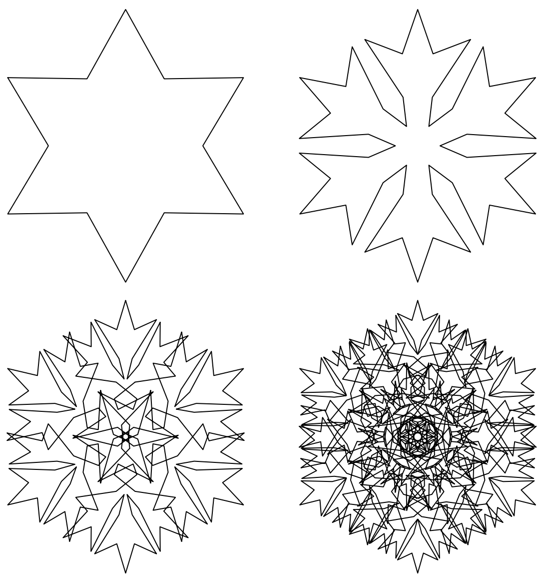


Figure 3.5.2 Constructing a Peano curve from a Frostman measure

From the figures it seems that the maps covers a hexagon. Is this correct? What is the dimension of the image of μ under the mapping? Is is absolutely continuous to or singular to area measure?

The singularity of harmonic measures arose as a natural condition in the study of function algebras, long before there was geometric characterization of when this occured.

Theorem 3.5.2 (Browder-Wermer) *Suppose Γ is a closed Jordan curve and the harmonic measures for the two sides of Γ are mutually singular. Then there are non-constant continuous functions on Γ which extend continuously to be bounded and analytic on $\mathbb{C}^\infty \setminus \Gamma$. In fact, the real parts of such functions are (uniformly) dense in all the real valued functions on Γ.*

Proof Let A_Γ be all the continuous functions on Γ which extend continuously to be bounded and analytic on $\mathbb{C}^\infty \setminus \Gamma$. This is a closed subset of the set of continuous functions on Γ $C(\Gamma)$ with the sup norm. Let Ω_1, Ω_2 denote the two simply connected complementary components of Γ and let A_j, $j = 1,2$ be the bounded analytic functions on Ω_j which extend continuously to Γ. Let W_j be the set of measures on Γ which annihilate A_j, $j = 1,2$. By conformal invariance and the F. and M. Riesz theorem (Theorem 3.1.3, Chapter **??**), measures in W_j are mutually absolutely continuous with respect to harmonic measure for Ω_j.

Clearly everything in $W_1 + W_2$ annihilates A_Γ. Conversely if μ annihilates A_Γ then it must be in $W_1 + W_2$, for otherwise the Hahn-Banach theorem provides a function $f \in C(\Gamma)$ so that $\int f d\mu = 1$ but $\int f d\nu = 0$ for every $\nu \in W_1 \cup W_2$. The second condition implies $f \in A_\Gamma$, a contradiction. Suppose for the moment that we knew $W_1 + W_2$ was closed. Then if μ annihilates A_Γ it must be in $W_1 + W_2$. Write

$$\mu = \mu_1 + i\mu_2 = (\nu_1 + \nu_2) + i(\tau_1 + \tau_2),$$

as a sum of real measures with $\nu_1, \tau_1 \in W_1$ and $\nu_2, \tau_2 \in W_2$. If μ is real then $\tau_1 = -\tau_2$. Since these measures are mutually singular, we deduce $\tau_1 = \tau_2 = 0$. But measures in W_j are boundary values of analytic functions, so cannot be real valued unless constant. Thus $\nu_1 = \nu_2 = 0$. Thus the only real valued measure on Γ which annihilates A_Γ is zero. Thus by the Hahn-Banach theorem $\mathrm{Re}(A_\Gamma)$ is dense in $C_{\mathbb{R}}(\Gamma)$.

Thus it only remains to verify that $W_1 + W_2$ is closed. Each of the two spaces is closed (since it is defined as the annihilator of a space of functions). Given $\mu \in W_1$ and $\nu \in W_2$, Note that

$$\|\mu + \nu\| = \|\mu\| + \|\nu\|,$$

since these measures are singular. Thus if $\{\lambda_n = \mu_n + \nu_n\} \in W_1 + W_2$ converges, we can find a subsequence along which $\mu_n \to \mu \in W_1$ and $\nu_n \to \nu \in W_2$. This proves $W_1 + W_2$ is closed. □

It is easy to construct curves of dimension 1, but with singular harmonic measures (so that this techniques applies but the Frostman measure construction does not). For example, we mimic the construction of the von Koch snowflake. For n not a power of 2 we simply replace each nth generation interval of length r, with four subintervals of length $r/4$. For n a power of 2 we replace it by four intervals of length $r/3$, arranged as in the usual construction of the snowflake. This set has dimension 1, but has not tangents since it osscilates on arbitrarily small scales.

A more concrete construction of functions in A_Γ is given in [**?**]. Theorem 3.4.1 can be generalized to characterize when harmonic measures on two general planar domains are mutually singular, [**?**], but the analogous question in $\mathbb{R}^3$ is open.

3.6 Notes

Carleson's proof for snowflake

Zdunik-Urbanski theorem for snowflake

Weierstrass graph , singular but 1-dimensional

Curves that have a.e. twists one one side, cones on the other

Dbar approach to Browder-Wermer

Curves with comparable harmonic measures; rectifiable if close to 1, high dimension posible Semmes, Bishop

continuous analytic capacity

local F and M Riesz therorem

Suhi Choi theorem - harmonic measure oscillates

Burdzy open problem - direction of approach of Brownian motion

Burdy thm - complementaty components of Brownian motion have twist points a.e.

β^2-theorem

ε^2 conjecture

progress in higher dimensions

Browder-Wermer analog for conformal maps,

Conformal welding exists if homeo has bi-Lipschitz extension to disk, flexible cuves, very singular harmonic measures log-singular

3.7 Exercises

Exercise 3.1 Construct a continuous real valued function f on the line such that the graph $\Gamma = \{(x, f(x))\}$, divides the plane into two components with mutually singular harmonic measures.

Exercise 3.2 Let $f(x) = \sum_{n=1}^{\infty} 2^{-n} \cos(2^n)$. Then f is differentiable almost nowhere, but the complementary domains of the graph of f have mutually absolutely continuous harmonic measures.

Exercise 3.3 Suppose Γ is a closed Jordan curve with complementary components Ω_1 and Ω_2 and harmonic measures ω_1 and ω_2. Let Φ_1, Φ_2 be Riemann mappings onto the two domains. Then $h = \Phi_2^{-1} \circ \Phi_1$ is a well defined homeomorphism of the circle to itself. Show that h is absolutely continuous (i.e., the integral of its derivative) iff ω_1 and ω_2 are mutually absolutely continuous. Show that h is singular (i.e., $h' = 0$ almost everywhere) iff $\omega_1 \perp \omega_2$. Also, h is singular iff there is a set $E \subset \mathbb{T}$ of full measure so that $h(E)$ has zero measure. The map h is called the conformal welding corresponding to Γ.

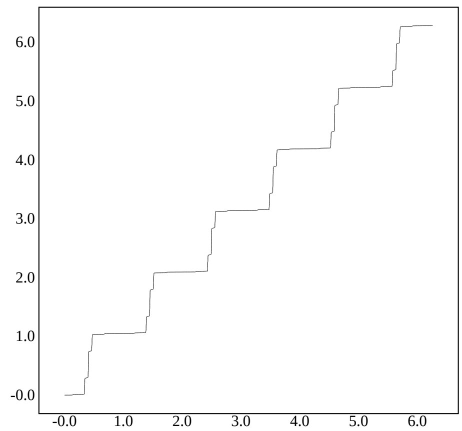


Figure 3.7.1 Welding map for von Koch snowflake graphed on $[0, 2\pi]$.

Exercise 3.4 Show there is a curve Γ with $\dim(\Gamma) > 1$ but the corresponding h is absolutely continuous. This is due to Garnett and O'Farrell [**?**].

Exercise 3.5 Show there is a curve Γ with $\dim(\Gamma) > 1$ but the corresponding h is Lipschitz. See [**?**]. It is interesting to note that the Lipschitz constant cannot be taken close to 1 (if h is Lipschitz with constant close to 1 then Γ must be rectifiable but the best constant is not known. See [**?**]).

Exercise 3.6 Suppose Γ is the von Koch snowflake and let ω_1 and ω_2 be harmonic measures for the two sides. Show that there is a $C < \infty$ and $\varepsilon > 0$ so that for any disk $D(x,r)$ centered on Γ,

$$\omega_1(D(x,r))\omega_2(D(x,r)) \leq Cr^{2+\varepsilon}.$$

(Hint: use the Ahlfors distortion theorem.)

Exercise 3.7 Let Φ_1, Φ_2 be Riemann mappings onto the two complementary domains of the von Koch snowflake. Then $h = \Phi_2^{-1} \circ \Phi_1$ has the following property: There is a $\delta > 0$ and a set $E \subset \mathbb{T}$ so that $\dim(E) < 1-\delta$ and $\dim(\mathbb{T} \setminus h(E)) < 1-\delta$. See [**?**].

Exercise 3.8 Prove that if Γ is the graph of the Weierstrass function $W_{a,b}$ with $ab = 1$, then harmonic measures for the two sides of Γ are mutually absolutely continuous.

Exercise 3.9 Suppose Ω_1, Ω_2 and Ω_3 are three disjoint simply connected domains with a common boundary point x. Show that there is a $C < \infty$ such that for every $r > 0$, there is a $i = 1,2,3$ so that

$$\omega_i(D(x,r)) \leq Cr^{3/2},$$

and $3/2$ is the largest value this is true for.

Exercise 3.10 Suppose Ω_1, Ω_2 and Ω_3 are three disjoint simply connected domains. Then there we can write $\partial\Omega_1 \cup \partial\Omega_2 \cup \Omega_3$ as a union of three sets E_1, E_2, E_3 so that $\omega_i(E_i) = 0$ for $i = 1,2,3$. (This is open in three or more dimensions, but is conjectured to be true. See [**?**], [**?**])

Exercise 3.11 Glicksberg [**?**] calls a simply connected domain Ω nicely connected if the Riemann mappings onto Ω are 1 to 1 almost everywhere on the unit circle. Show that Ω is nicely connected iff the set of double cone points has zero 1-dimensional measure ($x \in \partial\Omega$ is a double cone point for Ω is x is the vertex of two cones in Ω, each the reflection of the other through x.)

Exercise 3.12 A compact set K is called a Dirichlet algebra if A_K, the set of continuous functions on K which extend continuously to be analytic on $\overline{\mathbb{C}} \setminus K$, is a Dirichlet algebra. Prove K is Dirichlet iff K is connected, each component of $\Omega = \overline{\mathbb{C}} \setminus K$ is nicely connected and harmonic measures for different components of Ω are mutually singular. This is dues to Davie [**?**], see also [**?**]).

Exercise 3.13 Let A denote the disk algebra, the analytic functions on the unit disk which extend continuously to the boundary. Suppose $\varphi : \mathbb{T} \to \mathbb{T}$ is singular and define $A_\varphi = \{f \in A : f \circ \varphi \in A\}$. Then A_φ is a Dirichlet algebra. This is due to Browder and Wermer [**?**], [**?**].

Exercise 3.14 Suppose Ω is a simply connected domain on the Riemann sphere with harmonic measure ω. Given any function in $L^\infty(\omega)$ we obtain a bounded harmonic function in Ω if by solving the Dirichlet problem with these boundary values. Ω is called Poissonian if every bounded harmonic function on Ω is obtained in this way from something in $L^\infty(\omega)$. Show that the complement of a line segment is not Poissonian. (Hint: given a function of the segment the solution of the Dirichlet problem will be symmetric.)

Exercise 3.15 Let Γ be a subarc of the von Koch snowflake. Then the complement of Γ is Poissonian

Exercise 3.16 Ω is Poissonian iff for any two subdomains Ω_1, Ω_2 such that $\partial\Omega_1 \cap \partial\Omega_2 \subset \partial\Omega$ we have $\omega_1 \perp \omega_2$. [**?**].

Exercise 3.17 For every $0 < \alpha < 1$ there is a multiply connected domain Ω with $\dim(\partial\Omega) = \alpha$ and harmonic measure for Ω is mutually absolutely continuous with respect to α-dimensional measure on $\partial\Omega$.

Exercise 3.18 Given any $0 < \alpha \le 1$ and $\alpha \le \beta \le 2$ there is a set $E \subset \mathbb{T}$ and a univalent mapping F of the disk onto a Jordan domain so that $\dim(E) = \alpha$ and $\dim(F(E)) = \beta$.

Exercise 3.19 Let Γ be the von Koch snowflake and let ω be harmonic measure for the interior. Let F be the linear mapping $(x,y) \to (2x,y)$ and let $\Gamma' = F(\Gamma)$ and let ω' be the harmonic measure for the interior of Γ'. Let $F^*\omega$ be the push forward of ω by F onto Γ'. Prove $F^*\omega \perp \omega'$. (This holds for any linear map F which is not conformal.)

Exercise 3.20 Suppose Ω is a simply connected domain and for $z \in \Omega$ let $d(z) = \mathrm{dist}(z, \partial\Omega)$. For $t > 0$ let

$$d(z,t) = \sup_\theta d(z + te^{i\theta}).$$

Choose $z_0 \in \Omega$ with $d(z) \ge 1$ and let G be the Green's function with pole z_0. Then there is a $C < \infty$ so that for any $z \in \Omega$,

$$G(z) \le C \exp\left(\frac{1}{2}\int_0^1 \frac{dt}{d(z,t)}\right).$$

See [**?**] or [**?**], page 14.

Exercise 3.21 Suppose Ω is a simply connected domain whose boundary has positive area. Show that for almost every point $x \in \partial\Omega$ (with respect to area) we have

$$\limsup_{r\to 0} \frac{\omega(D(x,r))}{r^n} = 0,$$

for any $n > 0$. (Hint: by the Lebesgue density theorem, almost every point of $\partial\Omega$ is very hard to approach through Ω. The sharp result due to Jones and Makarov [**?**] is that

$$\limsup_{r\to 0} \frac{\omega(D(x,r))}{\exp(-M\log^2 \frac{1}{r})} = 0,$$

for every $M > 0$ and almost every (area) $x \in \partial\Omega$.)

Exercise 3.22 Show the Lemma 3.5.1 is still true if the curve Γ is replaced by any compact set [**?**].

Exercise 3.23 If $f : [0,1] \to [0,1]^2$ covers an open set, show there must be a point with three distinct preimages.

Exercise 3.24 Construct a $f : [0,1] \to [0,1]^2$ which covers an open set, and so that the set of points with a unique preimage has positive area.

4

The Jones-Wolff theorem

We saw in Chapter 2 that Makarov proved that harmonic measure on every simply connected planar domain has dimension exactly equal to 1. We can't expect the lower bound part of Makarov's theorem to hold for general domains, since the boundary of the domain might have dimension strictly less than 1. However, Peter Jones and Tom Wolff showed that the upper bound does hold in general:

Theorem 4.0.1 (Jones-Wolff) *For any compact planar set E with positive logaritmic capacity, harmonic measure for $\Omega = \mathbb{C} \setminus E$ with respect to ∞ gives full measure to a set of Hausdorff dimension at most* 1.

4.1 Green's function

As noted in Lemma 1.2.8, given a compact set E of positive logarithmic capacity, there exits a positive measure μ on E so that the potential

$$U_\mu(z) = \int_E \log \frac{2}{|z-w|} d\mu(w),$$

is bounded above by 1 everywhere in the plane and is equal to 1 everywhere on E, except possibly on a subset of zero logarithmic capacity. Let $\omega = \mu / \|\mu\|$. The notation is not a coincidence; ω is the harmonic measure for $\Omega = \mathbb{C} \setminus E$. Define

$$G(z) = \frac{1}{\mathrm{Cap}_{\log}(\mathrm{E})} - U_\omega(z).$$

This is the Green's function for the domain Ω with pole at ∞, so we might write this as $G(z,\infty)$, but since we not use any other poles in this chapter, we prefer the more compact notation. Green's function is a positive harmonic function on

Ω that has a logarithmic pole at infinity; more precisely $h(z) = G(z) - \log|z|$ is a bounded harmonic function in a neighborhood of ∞. Also, $G = 0$ on E, except possibly for a subset of zero logarithmic capacity.

If E is a union of smooth curves, then it is not hard to show that

$$\omega = \frac{\partial G}{\partial n} ds = |\nabla G| ds$$

where $\partial/\partial n$ is the normal derivative of G pointing into the domain Ω and ∇G is the gradient of G (on a smooth boundary $G = 0$ so the normal derivative and $|\nabla G|$ are equal there). Since ω is a probability measure it will be supported on a set of small length if $|\nabla G|$ is large but is spread over large length if $|\nabla G|$ is small. One way to make this precise is to consider

$$I(E) = \int_E \frac{\partial G}{\partial n} \log|\nabla G| ds = \int_E \log|\nabla G| d\omega. \tag{4.1.1}$$

Roughly speaking, if this integral is large, then ω should give most of its mass to the set where $|\nabla G|$ is large, and hence most of harmonic measure lives on a small set. To make this argument precise for general sets E, where it does not make sense to take normal derivatives, we will replace E by an approximating set consisting of a union of disks and apply the idea above to the new set. Moreover, the Green's integral above will not be take over the boundary, but over a collection of closed curves that surround the set and so that G is constant on each curve; these contours may include boundary components (where $G = 0$, but may also include curves γ that surround several boundary components and such that $G|_\gamma = c > 0$. Thus the precise estimate of (4.1.1) that we will use is

Lemma 4.1.1 *Suppose $E \subset \overline{\mathbb{D}}$ is a compact set of positive logarithmic capacity and G is the Green's function for $\Omega = \mathbb{C} \setminus E$. Suppose Γ is a union of closed curves that separates E from ∞ and that G is constant on each component of Γ. Then*

$$I(\Gamma) \equiv \frac{1}{2\pi} \int_\Gamma |\nabla G| \log|\nabla G| ds \geq 0. \tag{4.1.2}$$

Proof Since G is harmonic, ∇G is a holomorphic function on Ω and so its zeros are discrete and can only accumulate on $\partial\Omega \subset E$. Given $R > 0$ let $\Gamma_R = \{|z| = R\}$ and $\Omega_R = \Omega \cap \|z| < R\}$. Note that $\partial\Omega_R = \Gamma \cup \Gamma_R$.

Let $\{z_j\}$ be the finitely many zeros of ∇G that lie outside Γ, i.e., $\{z_j\}$ are the critical points of G in the unbounded complementary component of Γ. All these critical points lie in the convex hull of E (Exercise 4.1), so they are also in Ω_R.

Green's formula states that

$$\int_{\Omega_R} u\Delta v - v\Delta u dxdy = \int_{\partial\Omega_R} dsu\frac{\partial v}{\partial n} - \int_{\partial\Omega_R} v\frac{\partial u}{\partial n}ds, \tag{4.1.3}$$

where we use the outward pointing normal derivative. We apply this with $u = G$ and $v = \log|\nabla G|$, and analyse each of the various pieces of the formula.

First consider the area integral (the left side of (4.1.3). Then $\Delta u = 0$ on Ω and Δv is a sum of point masses at the zeros of ∇G (with mass d at a critical point of multiplicity d). Thus the left side of (4.1.3) equals $\sum_j G(z_j)$, summed with multiplicity.

Let $\{\Gamma_k\}$ denote the connected components of Γ, and suppose $G = c_k$ on Γ_k. Consider the first boundary integral over Γ_k. Here $u = G$ is constant, so

$$\int_{\Gamma_k} u\frac{\partial v}{\partial n}ds = c_k\int_{\Gamma_k}\frac{\partial}{\partial n}\log|\nabla G|ds.$$

The second boundary integral over Γ_k is

$$\int_{\Gamma_k} v\frac{\partial u}{\partial n} = \int_{\Gamma_k}|\nabla G|\log|\nabla G|ds \equiv I(\Gamma_k).$$

Thus $I(\Gamma)$ in (4.1.2) equals the second boundary integral (4.1.3) taken over Γ.

Finally consider the boundary integrals over Γ_R. We know Green's function has the form

$$G(z) = \frac{1}{\mathrm{Cap}_{\log}(\mathrm{E})} + H(z) + \log|z|$$

where H is a harmonic function that tends to zero at infinity, so the first boundary integral is

$$\begin{aligned}\int_{\Gamma_R} u\frac{\partial v}{\partial n}ds &= \int_{\Gamma_k}(\gamma + H(z) + \log|z|)(|\nabla H(z)| + \frac{1}{|}z|)ds\\ &= \int_{\Gamma_k}(\gamma + H(z) + \log|z|)(O(|z|^{-2}) + \frac{1}{|}z|)ds\\ &= \frac{2\pi}{\mathrm{Cap}_{\log}(\mathrm{E})} + 2\pi\log R + o(1)\end{aligned}$$

and the second boundary integral is

$$\int_{\Gamma_R} v\frac{\partial u}{\partial n}ds = \int_{\Gamma_k}\log|\nabla G|(O(|z|^{-2} + \log|z|)ds = 2\pi\log R + o(1)$$

so their difference tends to $2\pi/\mathrm{Cap}_{\log}(\mathrm{E})$. Combining our evaluations of the terms in (4.1.3) gives

$$I(\Gamma) = \sum_j G(z_j) + \sum_k \frac{1}{2\pi}\int_{\Gamma_k} G\log|\nabla G|ds + \frac{1}{\mathrm{Cap}_{\log}(\mathrm{E})}.$$

We now have to show that this is positive.

We start by noting that since ∇G is a holomorphic function, the functions $\log|\nabla G|$ and $\arg\nabla G$ are harmonic conjugates of each other (away from the critical points of G) and hence the normal derivative of one on Γ_k is the tangtial derivative of the other. Thus

$$\frac{1}{2\pi}\int_{\Gamma_k}\frac{\partial}{\partial n}\log|\nabla G|ds = -\frac{1}{2\pi}\int_{\Gamma_k}\frac{\partial}{\partial t}\arg\nabla G ds,$$

where $\partial/\partial t$ denotes the partial derivative in the tangential direction along Γ_k. We have $\partial G/\partial n = -|\nabla G|$ since we are using the outward pointing normal derivative in Green's formula (on Γ_k this points in a direction where G is decreasing, hence is the negative of the gradient). Since Γ_k is a level curve of G, the vector ∇G is perpendicular to Γ_k and so $\arg\nabla G$ changes by exactly 2π as we make a complete loop around Γ_k. Thus taking the limit as $R \nearrow \infty$ gives

$$\frac{1}{2\pi}\int_{\Gamma_k} G\frac{\partial}{\partial n}\log|\nabla G|ds = \frac{c_k}{2\pi}\int_{\Gamma_k}\frac{\partial}{\partial n}\log|\nabla G|ds = -c_k.$$

Therefore, summing over the components $\{\Gamma_k\}$ gives

$$I(\Gamma) = \sum_j G(z_j) - \sum_k c_k + \frac{1}{\mathrm{Cap}_{\log}(\mathrm{E})}.$$

Now let $n(c)$ be the number of components of $\{G = c\}$ that lie outside Ω, i.e., between E and Γ. Each such component is contained in the bounded region bordered by some Γ_k where $c_k \geq c$ (perhaps more than one component of $\{G = c\}$ is contain in this region)), so $n(c) \geq \#(k : c_k \geq c)$, the number of curves Γ_k on which $G \geq c$. Let $m = \max_j G(z_j)$ and let $\mathbf{1}_{t<a}$ be the function which equals 1 for $t < a$ and equals 0 for $t \geq a$. Then

$$\begin{aligned}\sum_k c_k &= \sum_k \int_0^\infty \mathbf{1}_{t<c_k} dt \\ &= \int_0^\infty \sum_k \mathbf{1}_{t<c_k} dt \\ &= \int_0^\infty \#(k : c_k > t) dt \\ &\leq \int_0^m n(t) dt,\end{aligned}$$

and hence

$$\sum_j G(z_j) = \int_0^\infty (n(t)-1)dt = \int_0^m n(t)dt - m \geq \sum_k c_k - m.$$

Thus

$$I(\Gamma) = \sum_j G(z_j) - \sum_k c_k + \gamma \geq (\sum_k c_k - m) + \sum_k c_k + \frac{1}{\mathrm{Cap_{log}(E)}} = \frac{1}{\mathrm{Cap_{log}(E)}} - m.$$

For points $w, z \in \mathbb{D}$, $|z-w| \leq 2$, and we have assumed $E \subset \mathbb{D}$, so for $z \in \mathbb{D}$,

$$G(z) = \gamma - U_\omega(z) = \gamma - \int \log \frac{2}{|z-w|} d\omega(w) \leq \gamma.$$

Since $\{z_j\} \subset \mathbb{D}$ (Exercise 4.1) we get $m \leq \gamma$, and hence $I(\Gamma) \geq 0$. □

4.2 Capacity and harmonic measure

A set E has positive logarithmic capacity if and only if it has a positive probability of being hit by a Brownian motion (started off the set). If we run the Brownian motion forever, then it will eventually hit any set of positive capacity. However, if we stop the Brownian motion, say when it hits some second set X, then we expect the probability that it hits E before hitting E to decrease to zero as the logarithmic capacity of E decreases to zero. In this section we shall formulate and prove several results that make this idea precise.

Lemma 4.2.1 *Suppose* $0 < r < 1$ *and* $t = (1+r)/2$. *Suppose* $E \subset r \cdot \mathbb{D}$ *has positive capacity. Then*

$$\omega(t) = \omega(E, t, \mathbb{D} \setminus E) \simeq \mathrm{Cap_{log}(E)},$$

with constants that only depend on r.

Proof Let μ be the equilibrium measure for E. The potential function

$$U_\mu(z) = \int \frac{2}{\log|z-w|} d\mu(w)$$

is harmonic off E, equals 1 on E and is positive on $\mathbb{T}$, so by the maximum principle

$$\omega(t) \leq U_\mu(t) \leq \|\mu\| \cdot \log \frac{2}{r-t} = \mathrm{Cap_{log}(E)} \cdot \log \frac{4}{1-\mathrm{r}},$$

which gives one direction.

By Harnack's inequality $\omega(z)$ takes comparable values at every point of $\{|z| = t\}$ (with a constant only depending on r) so it is enough to bound $\omega(z)$ any such point. We will do the estimate for $z = tx$ where $x \in \mathbb{T}$ is the point where U_μ takes its maximum value m on $\mathbb{T}$. By rotating we may assume $x = 1$. Then $U_\mu(1) = m$ and

$$U_\mu(t) = \int_E 1 d\omega_t(w) + \int_{\mathbb{T}} U_\mu(w) d\omega_t(w) \le \omega(t) + m(1-\omega(t)) = m + (1-m)\omega(t).$$

Note that

$$|z-t| \le \lambda(r)|z-1|$$

for every $z \in E$ for some constant $\lambda(r) < 1$. Plugging this into the integral formula for the potential function gives

$$U_\mu(t) \ge U_\mu(1) + \|\mu\| \log \frac{1}{\lambda(r)} > U_\mu(1).$$

Therefore, combining our estimates gives

$$\|\mu\| \log \frac{1}{\lambda(r)} \le U_\mu(t) - U_\mu(1) \le \omega(t) + m(1-\omega(t)) - m = \omega(t)(1-m).$$

Note that $m > 0$, so we get

$$\omega(t) \ge \|\mu\| \cdot \frac{1}{\lambda(r)} = \mathrm{Cap}_{\log}(\mathrm{E}) \cdot \frac{1}{\lambda(\mathrm{r})}.$$

Thus $\omega(z) \simeq \mathrm{Cap}_{\log}(\mathrm{E})$ with constants depending only on r. □

Lemma 4.2.2 *In Lemma 4.2.1, if r small and $|z| = 1/2$, then*

$$\mathrm{Cap}_{\log}(\mathrm{E})(\log 4 - \mathrm{O}(\mathrm{r})) \le \omega(\mathrm{z}) \le \mathrm{Cap}_{\log}(\mathrm{E})(\log 4 + \mathrm{O}(\mathrm{r})), \quad (4.2.1)$$

with a uniform bound.

Proof We just rework the previous proof, being more careful about the constants. As before the maximum principle implies $\omega(w) \le U_\mu(w)$ on $\mathbb{D} \setminus E$, so for $|w| = 1/2$,

$$0 < \omega(w) \le U_\mu(w) \le \|\mu\| \log \frac{2}{(1/2) - r} = \mathrm{Cap}_{\log}(\mathrm{E})(\log 4 + \mathrm{O}(\mathrm{r})).$$

This gives (**??**).

To prove the lower bound, note that when we move from $w \in \mathbb{T}$ to $w/2$ the distance from w to each point z of E decreases by a multiplicative factor of $2 + O(r)$ (the exact change depends on w and z. Thus the integral defining U_μ increases by an additive factor of $\|\mu\|(\log 2 + O(r))$. In other words, for $|w| = 1$,

$$U_\mu(\frac{w}{2}) \ge U_\mu(w) + \|\mu\|(\log 4 - O(r)) \ge \|\mu\|(\log 2 - O(r)).$$

On the other hand, since U_μ is a harmonic function, $U_\mu(\frac{w}{2})$ can be evaluated by integrating its boundary values against harmonic measure. Let $x = w/2$ and $m = \max_{\mathbb{T}} U_\mu$. Then

$$\begin{aligned} U_\mu(x) &= \int_E U_\mu(z) d\omega_x(z) + \int_{\mathbb{T}} U_\mu(z) \omega_x(z) \\ &\le \int_E 1\, d\omega_x(z) + m \int_{\mathbb{T}} 1\, \omega_x(z) \\ &= \omega(x) + m(1-\omega(x)). \end{aligned}$$

Thus

$$\begin{aligned} \|\mu\|(\log 4 + O(r)) &\le U_\mu(x) - U_\mu(w) \\ &\le \omega(x) + m(1-\omega(x))) - U_\mu(w) \\ &= m + (1-m)\omega(x) - U_\mu(x) \\ &\le (m - U_\mu(w)) + \omega(x) \\ &= O(r)\|\mu\| + \omega(x) \end{aligned}$$

so,

$$\omega(x) \ge \|\mu\|(\log 4 - O(r)). \quad \square$$

Lemma 4.2.3 *Suppose $1 < r \le 2$ and let $A = \{1 < |z| < r\}$. Suppose X, Y are each compact sets of positive capacity that lie inside and outside A respectively. Then for any z with $|z| = (1+r)/2$,*

$$\omega(z, X, \mathbb{C} \setminus (X \cup Y)) \simeq \frac{\mathrm{Cap}_{\log}(\mathrm{X})}{\mathrm{Cap}_{\log}(\mathrm{X}) + \mathrm{C}(\mathrm{Y})},$$

where $C(Y) > 0$ is positive constant depending on r and Y, but not on X.

Proof Again by Harnack's inequality $\omega(z)$ takes comparable values at all points of the circle $\{|z| = \sqrt{r}\}$ so we only need to prove the estimate at one such point. Choose s, t so that $1 < s < \frac{1+r}{2} < t < r$ and with spacing between these points all comparable to $r-1$. Let $\Omega_X = t\mathbb{D} \setminus X$ and $\Omega_Y = \mathbb{C} \setminus (Y \cup s\overline{\mathbb{D}})$. Let ω_X and ω_Y denote the harmonic measure functions for X and Y in Ω_X and Ω_Y respectively.

Let $p = \omega_X(s)$ and $q = \omega_Y(t)$. By Lemma 4.2.1 $p \simeq \mathrm{Cap}_{\log}(\mathrm{X})$. Without loss of generality we can assume that s and t are so close that $p, q \ll 1/2$. Consider a Brownian path started at $s \in C_s$. The path's nth visit to C_s is the terminal point of the nth subarc in $\{s < |z| < t\}$ that starts on C_t and ends on C_s. Similarly for visits to C_t. The probability that a path visits C_s more than k times decays exponentially in k, since the probability of hitting hitting either X or Y first is bounded uniformly away from zero.

The probability of hitting X before hitting Y, starting from a point of C_s is given by

$$\mathbb{P}(X) = \sum_{k=1}^{\infty} \mathbb{P}(K \le k) p_{k+1} \simeq p \sum_{k=1}^{\infty} \mathbb{P}(K \le k),$$

where K is number of times the Brownian motion path visits C_s and p_{k+1} is the probability that it hits X before C_t after making the kth visit (the zeroth visit is the starting point). The infinite sum converges since the terms in it decay geometrically fast.

Similarly, the probability of hitting Y before X is

$$\mathbb{P}(Y) = \sum_{k=1}^{\infty} \mathbb{P}(K \le k)(1 - p_{k+1}) q_{k+1} \simeq q \sum_{k=1}^{\infty} \mathbb{P}(K \le k),$$

where $q_{k+1} = \simeq q$ is the probability of hitting Y before C_s after the kth visit to C_t. The sums converges since the terms in it decay geometrically fast (the probability of making a crossing in starting at either circle is bounded strictly below 1).

Then

$$\mathbb{P}(X) = \frac{\mathbb{P}(X)}{\mathbb{P}(X) + \mathbb{P}(Y)} \simeq \frac{p}{p+q} \simeq \frac{\mathrm{Cap_{log}(X)}}{\mathrm{Cap_{log}(X) + q}}.$$

Thus we can choose $C(y) = q$, the probability of hitting Y before C_s, starting from t. □

Corollary 4.2.4 *Using the notation of Lemma 4.2.3, suppose X' is another set satisfying the same conditions as X' and that* $\mathrm{Cap_{log}(X) \simeq Cap_{log}(X')}$. *Then for any z with $|z| = (1+r)/2$, we have*

$$\omega(z, X, \mathbb{C} \setminus (X \cup Y)) \simeq \omega(z, X', \mathbb{C} \setminus (X' \cup Y))$$

with bounds that only depend on r.

Proof By Lemma 4.2.3, the ratio of the two harmonic measures is comparable to

$$\frac{\mathrm{Cap_{log}(X)(Cap_{log}(X) + C(Y))}}{\mathrm{Cap_{log}(X')(Cap_{log}(X') + C(Y))}} \simeq \frac{\mathrm{(Cap_{log}(X) + C(Y))}}{\mathrm{(Cap_{log}(X') + C(Y))}} \simeq \frac{\mathrm{Cap_{log}(X)}}{\mathrm{Cap_{log}(X')}} \simeq 1.$$

□

Lemma 4.2.5 *Given $A \ge 1$ there is a $B = B(A) < \infty$ so that the following holds. Suppose* $\mathrm{Cap_{log}(X') = ACap_{log}(X)}$ *and $X, X' \subset \{|z| < 1/B\}$. Then*

$$\omega(z, E, \mathbb{C} \setminus (\mathbb{D} \cup X')) \le \omega(z, E, \mathbb{C} \setminus (\mathbb{D} \cup X))$$

for any $E \subset \mathbb{T}$ and $|z| = 1/2$.

Proof Thinking of harmonic measure as the hitting distribution of Brownian motion, we can omit paths starting at z that never hit X or X'. Thus the two distributions on the unit circle are just the Poisson distribution corresponding to z, minus the hitting distribution of paths started on X and X' according to the first hitting distributions on these sets starting from z. The total masses of these distributions are $m = \omega(z, X, \mathbb{D} \setminus X)$ and $m' = \omega(z, X', \mathbb{D} \setminus X')$ respectively. If r is small then the hitting distribution from a point w with $|w| \le r$ is between $1 - O(r)$ and 1 on the whole unit circle. Thus the claim follows if

$$m \le m'(1-r).$$

By Lemma 4.2.2

$$m' = \mathrm{Cap_{log}(X')(\log 4 + o(r)) = ACap_{log}(X)(\log 4 + o(r))},$$

$$m = \mathrm{Cap_{log}(X)(\log 4 + o(r))},$$

and since $A > 1$, we see that the claim is true for sufficiently small r. □

Lemma 4.2.6 *Given $A \ge 1$ there is a $B = B(A) < \infty$ so that the following holds. Suppose Q is the square of side length* 1 *centered at the origin and $X, X' \subset Q$ and $\cap BQ = \emptyset$. Assume* $\mathrm{Cap_{log}(X') = ACap_{log}(X)}$. *Then*

$$\omega(z, Z, \mathbb{C} \setminus (Y \cup X')) \le \omega(z, Z, \mathbb{C} \setminus (Y \cup X))$$

for any $Z \subset Y$ and $|z| = B/4$.

Proof The idea is that since X' has larger capacity, it should absorb more Brownian paths than X, leaving fewer to hit $Z \subset Y$.

To make this precise, consider the regions $D_1 = \{|z| > B/4\}$ and $D_2 = \{|z| < B/2\}$ and their boundaries C_1, C_2. As in the proof of Lemma 4.2.3 we consider Brownian paths starting on C_1 and its subsequent visits to C_1 and C_2. At each visit to C_2 the probabilities of hitting Z, Y or C_1 obviously don't depend on X or X'. By Lemma 4.2.5 the exit distribution on C_2 for $\Omega_X = D_2 \setminus X'$ is strictly less than the corresponding distribution for X with the same starting point. This inequality remains valid under iteration, so the probability of eventually hitting Z in $\mathbb{C} \setminus (Y \cup X')$ is less than the probability of hitting Z in $\mathbb{C} \setminus (Y \cup X)$. □

4.3 The modification algorithm

Our goal in the next two sections is to prove

Theorem 4.3.1 *Suppose $E \subset \mathbb{D}$ is compact and has positive logarithmic capacity. Suppose $\varepsilon, \delta > 0$. Then there are two collections of disjoint dyadic squares $\{Q_j^1\}$ and $\{Q_j^2\}$ so that*

$$E \subset \cup Q_j^1 \bigcup \cup Q_j^2, \tag{4.3.1}$$

$$\sum \ell(Q_j^1)^{1+\varepsilon} < \delta, \tag{4.3.2}$$

$$\sum \omega(E \cap Q_j^2) < \delta. \tag{4.3.3}$$

If we can prove this, then the Jones-Wolff theorem (Theorem **??** is easily deduced as follows.

Proof of Theorem **??** Fix ε and for $n = 1, 2, \ldots$ take $\delta = 2^{-n}$. Let $E_n^1 = E \cap \cup Q_j^1$, $E_n^2 = E \cap \cup Q_j^2$, for the squares obtained from Theorem 4.3.1. Then

$$E^1 = \cap_n \cup_{k>n} E_1^k, \qquad E^2 = \cap_n \cup_{k>n} E_2^k$$

are subsets of E that satisfy $E = E^1 \cup E^2$, $\mathscr{H}_{1+\varepsilon}(E^1) = 0$ and $\omega(E^2) = 0$ (ω is harmonic measure for the complement of E). Thus for every $\varepsilon > 0$ we have $\dim(\omega) \leq 1 + \varepsilon$. Taking $\varepsilon \to 0$ proves the result. □

We will next describe an algorithm for constructing the collections of squares described in Theorem 4.3.1. Given $\varepsilon > 0$ let $A = 1 + \varepsilon$ and let $B = B(A)$ be the constant given by Lemma 4.2.6. Cover the plane by a grid $\mathscr{D}$ of dyadic squares of side length 2^{-N} and partition this collection into $O(B)$ periodic sub-collections $\mathscr{D}_k$ so that any two squares in the same sub-collection are separated by at least $B \cdot 2^{-n}$. Give the compact set $E \subset \mathbb{D}$ we define $O(B^2)$ subsets by $\cup_{Q \in \mathscr{D}_k} E \cap Q$. If we can prove Theorem 4.3.1 for each E_k but with δ divided by the number of E_k's, then union of resulting squares will satisfy Theorem 4.3.1 for E.

Therefore, from this point on, we may assume E is covered by a collection $\mathscr{C}$ of dyadic squares Q of side length 2^{-n}, with the property that $E \cap BQ = E \cap Q$. In other words, $BQ \setminus Q$ contains no points of E.

To obtain the desired covering of E, we are going to modify E in a number of stages, but to keep notation simply we will also denote the sets by E; when we want to refer to the set we are starting with, we will call it the "original E". The modifications are necessary to make E smooth. To start with, E is simply compact and has positive logarithmic capacity. To apply our Green's function estimates, we have to replace E by something with a smooth boundary; our final version of E will be a finite union of disks.

The two basic modification we make are:

A-**Disk construction:** Given a compact set E, $A \geq 1$ and a square Q of side length 1, we remove $E \cap Q$ from E and replace it by a disk D, concentric with Q and having logarithmic capacity $\mathrm{Cap}_{\log}(\mathrm{D}) = \mathrm{Cap}_{\log}(\mathrm{E} \cap \mathrm{Q})/\mathrm{A}$. If Q has some other side length, then we rescale E and Q so Q has side length 1, perform this construction, and then rescale back to the original size.

B-**Annulus construction:** Given a compact E, a square Q and $B < \infty$, we remove $E \cap (BQ \setminus Q)$ from E, where BQ denotes the concentric square with side length B times that of Q.

When we do the B-annulus contruction, it is clear that the harmonic measure of every remaining subset of E increases; we have increased the collection of Brownian path that can hit these sets. It is not immediately obvious what effect the A-disk construction has, since we are both adding and removing boundary, but it clarified by the following:

Lemma 4.3.2 *Suppose $A > 1$. Suppose $Q \cap E \neq \emptyset$ and assume E does not hit $BQ \setminus Q$. If $B = B(A)$ is the constant from Lemma 4.2.6, then when we perform an A-disk construction on E, the harmonic measure of any subset of $E \setminus Q$ increases and the harmonic measure of $E \cap Q$ decreases by at most a bounded factor (depending on A).*

Proof The fact that harmonic measure in $E \cap BQ$ increases is Lemma 4.2.6. The fact that harmonic measures inside Q decrease by at most a bounded factor is Corollary 4.2.4. □

Now suppose we are given a compact $E \subset \mathbb{D}$ and $\varepsilon, \delta > 0$. As above, set $A = 1 + \varepsilon$, let $B = B(A)$ as in Lemma 4.2.6, and suppose that N is a large integer, chosen later depending on ε and δ. Let $M = \sqrt{N}$. Assume that we have "thinned out" the set E as described earlier, so that E is covered by a collection of $\mathscr{C}$ of dyadic squares with side length 2^{-N} and so that $E \cap (BQ \setminus Q) = \emptyset$.

From this initial set, we perform the following operations:

Step 0: Apply the A-disk construction to all $Q \in \mathscr{C}$. Thus E is now a union of tiny disks.

Step 1: Choose the largest dyadic square Q so that $\ell(Q) \geq 2^{-N}$ and both

(1) $\omega(Q) \geq M\ell(Q)$, and

(2) Q is not contained in any previously chosen square.

If this is jth time Step 1 has been performed, label this square Q_j. If no such square exists, the algorithm stops here. Otherwise proceed to Step 2.

Step 2: Perform the B-annulus construction for the square Q_j found in Step 1,i.e., delete $BQ \setminus Q$ from E. However, if $k < j$ and $BQ_k \not\subset BQ_j$ then do not remove any part of $E \cap Q_k$. In other words, previously chosen squares so that

$BQ_k \subset BQ_j$ are removed, but the parts of E in all other previously chosen squares are left alone. Go to Step 3.

Step 3: Perform the A-disk construction for Q_j. Go to Step 1.

Lemma 4.3.3 *The algorithm stops.*

Proof By condition (2) in Step 1 we never choose a sub-square of a previously chosen square. Thus the chosen dyadic squares are pairwise disjoint and each contains at least one square in $\mathscr{C}$. This is a finite collection, so Step 1 is performed only finitely often. □

When the algorithm stops we are left with three types of squares. Type 0 squares are those selected at some stage in Step 1, and were never removed at later stages of Step 2. The remaining squares all elements of $\mathscr{C}$ that were never removed. We subdivide these surviving squares into two subclasses as follows: if

$$\omega(E \cap Q) \geq \mathrm{r}(\Delta) 2^{-\mathrm{N}\varepsilon/2},$$

then we call Q Type 1. Here Δ is the disk that Q was replaced by in Step 0 of the algorithm and $\mathrm{r}(\Delta)$ is its radius. Otherwise we say Q is Type 2. The union of all three types we will call the "remaining squares". Eventually we will show that the union of Type 0 and Type 1 squares has small $1+3\varepsilon$ Hausdorff content and that the union of Type 2 squares has small harmonic measure. First we gather some facts about the output of the algorithm.

Lemma 4.3.4 *Each Type 0 square satisfies* $E \cap (BQ_j \setminus Q_j) = \emptyset$.

Proof This is obvious when Step 2 is performed on Q_j, and in later stages, $E \cap Q_j$ is either completely removed or left alone. □

Lemma 4.3.5 *All Type 1 and 2 squares have larger harmonic measure than before the algorithm was run.*

Proof This follows from Lemma 4.3.2. □

Lemma 4.3.6 *Let E^* denote the original set (pre-algorithm) and E the modified version (post-algorithm). For any $Q \in \mathscr{C}$, that is not Type 1 or 2, we have $Q \subset 2BQ_j$ for some Type 0 square Q'.*

Proof If Q is no longer present, it must have been removed by Step 2 being applied to some chosen square Q_{j_1}, so $Q \subset BQ_{j_1}$. Clearly $\ell(Q_{j_1}) \geq \ell(Q)$ since we never select squares with side length smaller than 2^{-N}. However Q_{j_1} may itself have been removed at a later stage. If so, BQ_{j_1} must have been contained in BQ_{j_2} for some Q_{j_2} that was selected later, i.e., $j_2 > j_1$. Since $Q_{j_1} \neq Q_{j_2}$, this is only possible if $\ell(Q_{j_1}) < \ell(Q_{j_2})$ and since the squares are dyadic we must have $\ell(Q_{j_1}) \leq 2\ell(Q_{j_2})$. Continuing in this way, we get a finite chain of squares $\{Q_{j_n}\}$ so each square is at least twice as large as the previous one and its B-fold expansion contains B-fold expansion of the previous square. This implies the first square is inside the $B+B/2+B/4+\cdots = 2B$ expansion of the last square, as claimed. □

Corollary 4.3.7 *The collection BQ where Q ranges over all Type 0, 1 and 2 squares covers E.*

Lemma 4.3.8 *Each Type 0 square Q satisfies $\omega(Q) \geq CM\ell(Q)$ for some constant C depending on A.*

Proof By construction, $\omega(Q) \geq M\ell(Q)$ when it was chosen in Step 1, its harmonic measure increases in Step 2, and the harmonic measure only decreases by a bounded factor in Step 3. In later stages, Lemma 4.3.2 implies its harmonic measure only increases. □

Lemma 4.3.9 *If Q is Type 0, 1 or 2, $z \in Q$ and $r > \ell(Q)$, then $\omega(D(z,r)) \leq CMr$.*

Proof If $\omega(D(z,r)) > CMr$, then $D(z,r)$ is covered by at most 4 dyadic squares with side length between $2r$ and $4r$ and at one of these, say Q' must satisfy $\omega(Q') \geq \omega(D)z,r))/4$ and hence $\omega(Q') \geq M\ell(Q)$, if C is large enough. Such a square would have been chosen in Step 1. If this had happened, we claim that Q would have been removed in Step 2. Note that $\ell(Q) \leq r \leq\leq \ell(Q')/2$ and $Q \subset D(z,r) \subset 3Q'$. This implies $BQ \subset BQ'$, so Q would have been removed, as claimed. This contradiction proves the lemma. □

Lemma 4.3.10 *The Hausdorff 1-content of the union of the Type 0 squares tends to zero as $N \nearrow \infty$, uniform in ε.*

Proof These squares were chosen to satisfy $\omega(Q) \geq M\ell(Q)$ and by Lemma 4.3.2 their harmonic measure only increases at later stages. Therefore summing over Type 0 squares gives

$$\sum \ell(Q_j) \leq \frac{1}{M} \sum \omega(Q_j) \leq \frac{1}{M},$$

since ω is a probability measure. Since $M^2 = N$, the lemma follows. □

Lemma 4.3.11 *For every Type 0, 1 or 2 square*

$$|\nabla G| \simeq \omega(\Delta)/\mathrm{r}(\Delta)$$

on $\partial\Delta$ (since $G = 0$ on $\partial\Delta$ this is the same as estimating the normal derivative).

Proof Before Step 0, the Type 1 and 2 squares where chosen to have disjoint B-fold expansions, so if we set $A = \{\mathrm{r}(\Delta) < |\mathrm{z} - \mathrm{z}_0| < 4\mathrm{r}(\Delta)\}$, then it is clear that $A \cap E = \emptyset$. Thus Harnack's inequality applies on the circle $|z - z_0| = 2\mathrm{r}(\Delta)$ and we deduce G has values comparable to $D = G(z_0 + 2\mathrm{r}(\Delta))$. We then easily deduce that the normal derivative is comparable to $D/\mathrm{r}(\Delta)$ at every point of $\partial\Delta$. Since integrating this around $\partial\Delta$ gives $\omega(\Delta)$, we must have $D \simeq \omega(\Delta)$ and the lemma follows. □

4.4 Constructing the contours

At the beginning of this chapter we proved Lemma 1.23, giving an estimate for a certain integral involving $|\nabla G|$. The integral was over certain contours $\{\Gamma_k\}$ that surrounded the set E and such that G was constant on each Γ. In this section we select these contours for the modified set E constructed in the last section. There will be one contour Γ_Q for each Type 0, 1 and 2 square Q. Moreover, we will want the estimate

$$\omega(\Delta) = O(N^2 S),$$

where Δ is the disk associated to Q and $S = S(Q)$ is the diameter of the contour associated to Q.. Types 0 and 2 will be easy to describe; Type 1 will take more work. We will let $S(Q)$ be the diameter of the contour associated to Q.

Lemma 4.4.1 *For each Type 0 square Q let the contour Γ_Q be the circle of radius $\ell(Q)$ and let $S(Q) = \ell(Q)$. Then $\omega(\Delta) \leq M^2 NS = N^2 S$.*

Proof Lemma 4.3.9 implies

$$\omega(\Delta) \leq CM\ell(Q) \leq M^2 N\ell(Q).$$

□

Lemma 4.4.2 *For each Type 2 square Q, let the contour γ surrounding Δ just be $\partial\Delta$. Then $\omega(\Delta) \leq M^N S(\Delta)$.*

Proof This is obvious, since the Type 2 condition implies

$$\omega(\Delta) \leq \mathrm{r}(\Delta) 2^{-\mathrm{N}\varepsilon/2} \ll \mathrm{r}(\Delta).$$

□

Lemma 4.4.3 *If Q is Type 0 or Type 1 with associated disk Δ, then there is* $\mathrm{r}(\Delta) \leq \mathrm{S} \leq \ell(\mathrm{Q})$, *and a contour Γ_Q surrounding Δ on which G is constant and* $S = \mathrm{diam}(\Gamma_Q) \simeq \mathrm{dist}(\Gamma_Q, \Delta)$. *Moreover we have $\omega(\Delta) = O(M^2 NS) = O(N^2 S)$.*

Proof For the first part of the proof we let Q be either Type 0 or Type 1 and let Δ be its corresponding disk. Later we will specialize to Type 1.

Define $T = \omega(\Delta)/M^2N$. If $T \le \mathrm{r}(\Delta)$ then we take $S = \mathrm{r}(\Delta)$ and the contour is $\partial\Delta$. In this case

$$\omega(\Delta) \le M^2NT \le M^2N\mathrm{r}(\Delta) = \mathrm{M^2NS}.$$

Next suppose $T \ge \ell(Q)$. Then we take $S = \ell(Q)$ and by Lemma 4.3.9

$$\omega(\Delta) \le CM\ell(Q) = 2CMS \le M^2NS,$$

since M and N are both large.

Finally, assume $\mathrm{r}(\Delta) < \mathrm{T} < \ell(\mathrm{Q})$. For $z \in Q \setminus \Delta$ we can write

$$\begin{aligned} G(z) &= \int_\Delta \log|z-w|d\mu(w) + \int_{E\setminus\Delta} \log|z-w|d\mu(w) + \frac{1}{\mathrm{Cap_{log}(E)}} \\ &= u(z) + v(z) + \frac{1}{\mathrm{Cap_{log}(E)}}, \end{aligned}$$

where μ is the equilibrium measure for E (it agrees with the harmonic measure for E with respect to ∞ up to renormalization). Then using Lemma 4.3.9,

$$|\nabla v(z)| \le \int_{E\setminus\Delta} \frac{d\mu(w)}{|z-w|} \le CM \int_{\ell(Q)} \frac{dr}{r} = CM \log\frac{1}{\ell(Q)}.$$

Also note

$$\nabla u(z) = \int_\Delta \frac{z-z_0}{|z-w|^2} d\mu(w),$$

and that for $r \ge \mathrm{r}(\Delta)$ the vectors we are integrating all lie in a cone of angle $O(\mathrm{r}(\Delta)/\mathrm{r})$ around the direction $z - z_0/|z-z_0|$. Projecting these vectors onto this fixed vector only shortens their length by a factor of $\cos(\mathrm{r}(\Delta)/\mathrm{r}) = 1 - \mathrm{O}((\mathrm{r}(\Delta)/\mathrm{r})^2)$, so

$$-\frac{\partial u}{\partial r}(z) \ge \frac{\omega(\Delta)}{|z-z_0|} - O(\frac{\mathrm{r}(\Delta)\omega(\Delta)}{|z-z_0|^2}).$$

If we take $r = |z-z_0| \simeq S$ then this becomes

$$-\frac{\partial u}{\partial r}(z) \ge \frac{\omega(\Delta)}{S} - O(\frac{\mathrm{r}(\Delta)\omega(\Delta)}{S^2}). \tag{4.4.1}$$

This equation holds for both Type 0 and Type 1 squares. We now assume that Q is Type 1 and we shall return to the Type 2 case below.

Using the definition of $S = T$ and of Type 1 squares, this becomes

$$-\frac{\partial u}{\partial r}(z) \geq \frac{\omega(\Delta)M^2N}{\omega(\Delta)} - O(\frac{\mathrm{r}(\Delta)\omega(\Delta)\mathrm{M}^4\mathrm{N}^2}{\omega(\Delta)^2})$$
$$\geq M^2N - O(\frac{\mathrm{r}(\Delta)\mathrm{M}^4\mathrm{N}^2}{\omega(\Delta)})$$
$$\geq M^2N - O(\frac{\mathrm{r}(\Delta)\mathrm{M}^4\mathrm{N}^2}{\mathrm{r}(\Delta)2^{\mathrm{N}\varepsilon/2}})$$
$$\geq M^2N - O(\frac{M^4N^2}{2^{N\varepsilon/2}})$$

Since $M^2 = N$ this becomes

$$-\frac{\partial u}{\partial r}(z) \geq CN^2 - O(\frac{N^4}{2^{N\varepsilon/2}}) \geq \frac{1}{2}CN^2,$$

if N is large enough. This lower bound is much larger than our upper bound for $|\nabla v| = O(MN) = O(N^{3/2})$. This means that the gradient vector of G at z is within a small angle of the radial direction. Thus the level sets of G have tangents that are close to perpendicular to this direction. Following such a curve around Δ, it must stay in the annulus $\{S/2 < |z| < 2S\}$. Moreover, it must form a closed loop, since our estimates show G is strictly decreasing in the radial direction.

Finally, by the definition of T $\omega(\Delta) = M^2NT = N^2S$ as desired.

Now assume Q is Type 0. If $\mathrm{r}(\Delta) \simeq \ell(\mathrm{Q})$, then we just take the contour to be $\partial\Delta$ and $S = 2\mathrm{r} * \Delta$. Then by Lemma 4.3.9 $\omega(\Delta) = O(M\ell(Q))) = O(MS) = O(N^2S)$.

Otherwise we can assume $\mathrm{r}(\Delta) \ll \ell(\mathrm{Q})$. In this case (4.4.1) becomes

$$-\frac{\partial u}{\partial r}(z) \geq C\frac{\omega(\Delta)}{|z-z_0|}(1 - O(\frac{\mathrm{r}(\Delta)}{|z-z_0|})).$$

If we take $S = \ell(Q) \gg \mathrm{r}(\Delta)$, then the second term is comparable to 1, so

$$-\frac{\partial u}{\partial r}(z) \geq C\frac{\omega(\Delta)}{S} \simeq M.$$

□

4.5 Finishing the proof

We saw in Lemma 4.1.1

$$I(\Gamma) = \frac{1}{2\pi}\int_\Gamma |\nabla G| \log |\nabla G| ds \geq 0.$$

The integrand has negative and positive parts, depending on whether $|\nabla G| > 1$ or $|\nabla G| <$. The estimate above says that the positive part dominates the negative part, so we can obtain a bound on the negative part by bounded just the positive part. Let $\log^+ = \max(\log, 0)$ and $\log^- = \min(\log, 0)$.

Lemma 4.5.1 *If Γ consists of the contours constructed in Lemmas **??**, and 4.4.2, then*

$$\frac{1}{2\pi}\int_\Gamma |\nabla G| \log^- |\nabla G| ds \geq -C\log N.$$

$$\frac{1}{2\pi}\int_\Gamma |\nabla G| \log^+ |\nabla G| ds \leq C\log N.$$

Proof As noted above only have to prove the second inequality. Note that

$$\begin{aligned}\frac{1}{2\pi}\int_\Gamma |\nabla G| \log^+ |\nabla G| ds &\leq C\int_\Gamma |\nabla G| \log^+\left(\frac{\omega(\Delta)}{S(Q)}\right) ds \\ &\leq C\int_\Gamma |\nabla G| \log^+ (N^2) ds \\ &\leq C\log N \sum \omega(\Delta) \\ &= O(\log N) \quad \square\end{aligned}$$

Lemma 4.5.2 *The union of the Type 2 squares has harmonic measure that tends to zero as $N \nearrow \infty$.*

Proof Suppose Q and Δ are Type 2. Since $\omega(\Delta) \geq \mathrm{r}(\Delta)2^{-\mathrm{N}\varepsilon/2}$, and using Lemma 4.3.11, on $\partial\Delta$ we have

$$\begin{aligned}\log\frac{1}{|\nabla G|} \geq \log\frac{C\mathrm{r}(\Delta)}{\omega(\Delta)} &\geq \log\frac{C\mathrm{r}(\Delta)}{\mathrm{r}(\Delta)2^{-\mathrm{N}\varepsilon/2}} \\ &= \log C 2^{N\varepsilon/2} = N\frac{\varepsilon}{2}\log 2 + \log C.\end{aligned}$$

For N large enough the rightmost term is larger than $N\varepsilon/8$. Summing over all Type 2 squares gives

$$\sum \omega(Q_j) = \sum \frac{1}{2\pi}\int_{\partial\Delta_j} \frac{\partial G}{\partial n} ds \leq \sum \frac{C}{\varepsilon N}\int_{\partial\Delta_j} \frac{\partial G}{\partial n} \log\frac{1}{|\nabla G|} ds \leq \sum \frac{C\log N}{\varepsilon N}$$

where we used Lemma 4.5.1 in the last step. $\square$

Lemma 4.5.3 *Suppose $\alpha > 0$ and $E \subset \mathbb{D}$ has positive Hausdorff α-content. Then $\mathscr{H}^\alpha_\infty(E) \leq C_\alpha \exp(-\alpha/\mathrm{Cap}_{\log}(\mathrm{E}))$.*

Proof By Frostman's (e.g., Lemma 3.1.1 of [**?**] E supports measure μ such that $\mu(D(x,r)) \le Cr^\alpha$ and $\|\mu| \ge \mathscr{H}^\alpha_\infty$. Fix z and let $A_n = \{e^{-n} < |z-w| < e^{-n+1}\}$. Then

$$U_\mu(z) = \int_E \log \frac{1}{|z-w|} d\mu(w) \le C \sum_{n=0}^{\infty} n\mu(A_n)$$

For a given amount of mass, this sum will be maximized if we give the most mass to the smallest annuli, i.e., if $m = \frac{1}{\alpha} \log \frac{1}{\|\mu\|}$, then

$$U_\mu(z) \le \sum_{n=m}^{\infty} ne^{-\alpha n} \le m \exp(-\alpha m) = \frac{1}{\alpha} \|\mu\| \log \frac{1}{\|\mu\|}.$$

Hence the capacity of E is at least $C_\alpha / \log 1/\|\mu\|$, or $\|\mu\| \le \exp(-C_\alpha/\mathrm{Cap}_{\log}(\mathrm{E}))$. Since $\|\mu\| \ge \mathscr{H}^\alpha_\infty(E)$, this proves the lemma. □

Lemma 4.5.4 *The Hausdorff* $(1+\varepsilon)$*-content of the union of Type 1 squares tends to* 0 *as* $N \to \infty$*.*

Proof Let Q_j be a Type 1 square and let $E_j = E \cap Q_j$. Using $\alpha = 1+\varepsilon = A$ in Lemma 4.5.3 we get

$$\mathscr{H}^\alpha_\infty(E_j) \le \exp(-\alpha/\mathrm{Cap}_{\log}(\mathrm{E_j})) = \exp(-\alpha/\mathrm{Cap}_{\log}(\Delta_\mathrm{j})(1+\varepsilon)) = \exp(-1/\mathrm{Cap}_{\log}(\Delta_\mathrm{j}))^{\alpha/(1+\varepsilon)} = \mathrm{r}(\Delta).$$

Summing over all the Type 1 squares gives

$$\begin{aligned} \mathscr{H}^\alpha_\infty(E) &\le \sum \mathscr{H}^\alpha_\infty(E_j) \\ &\le \sum \mathrm{r}(\Delta) \\ &\le \sum \omega(\Delta) 2^{-N\varepsilon/2} \\ &\le 2^{-N\varepsilon/2} \sum \omega(\Delta) \to 0. \quad \square \end{aligned}$$

Notes

Exercises

Exercise 4.1 All the critical points of G, Green's function for $\mathbb{C} \setminus E$ with pole at ∞, are within the convex hull of E.

Solution 4.1 The gradient of G is non-zero iff the gradient of U_μ is and for z is outside the convex hull of E this gradient is given by the convolution of ω and $1/(z-w)$. Since E lies in a half-plane not containing z, this convolution is clearly non-zero.

Exercise 4.2 If $G(z,w)$ is Green's function for a domain Ω with pole at w then $G(z,w) = G(w,z)$.

Solution 4.2 This follows taking Green's theorem with $u(x) = G(x,w)$ and $v(x) = G(x,z)$ and fact that $\Delta G(z,w)$ is a δ-mass at w.

5

Wolff snowflakes

6
Analytic capacity

6.1 Definitions

Suppose E is a compact set in the complex plane and let Ω denote its complement. Recall that $H^\infty(\Omega)$ denotes the algebra of bounded holomorphic functions on Ω. We define the analytic capacity of E as

$$\gamma(E) = \sup\{|f'(\infty)| : f \in H^\infty(\Omega), \|f\|_\infty \leq 1\}.$$

The derivative of f at ∞ can either be defined by as $f'(\infty) = g'(0)$ where $g(z) = f(1/z)$, or as the limit

$$f'(z) = \lim_{z\to\infty} z(f(z) - f(\infty)).$$

The capacity is a monotone set function, but is not well understood at all. For example, it is unknown whether the union of two sets of analytic capacity zero also has capacity zero (but it is true for disjoint compact sets). This question is a special case of the subadditivity conjecture: Does there exist a constant $C < \infty$ so that

$$\gamma(E \cup F) \leq C[\gamma(E) + \gamma(F)],$$

for any two compact sets E and F? This is only known in some special cases, e.g., if E and F are separated by a straight line.

Analytic capacity is interesting because it characterizes which domains support a non-constant bounded analytic function. Any proper open set of the Riemann sphere has a non-constant holomorphic function defined on it, but if we consider bounded analytic functions then this is not always true. Riemann's theorem says that an isolated singularity of a bounded holomorphic function is removable. Therefore if f is bounded and holomorphic on the complement of a finite number of of points, then it can be extended to be bounded and holomorphic on the whole plane. Hence it is constant by Liouville's Theorem.

It follows easily from the definition of analytic capacity that $H^\infty(\Omega)$ has non-constant functions iff $\gamma(\Omega^c) \neq 0$.

A related capacity is the continuous analytic capacity, defined by

$$\alpha(E) = \sup\{|f'(\infty)| : f \in A(\Omega), \|f\|_\infty \leq 1\},$$

where $A(\Omega) \subset H^\infty(\Omega)$ are those elements of $H^\infty(\Omega)$ which extend continuously to the closure of Ω. The continuous analytic capacity is even less well understood than analytic capacity.

Clearly we always have $\gamma(E) \geq \alpha(E)$. To see that equality need not hold, note that if $E = [0,1]$ is a line segment then the Riemann mapping from E^c to the unit disk is a non-constant bounded analytic function and so $\gamma(E).0$ However, an easy application of Morera's theorem shows that if f is holomorphic on the complement of a line segment and extends continuously across the segment, then f is entire. If f is in addition bounded then it must be constant by Liouville's theorem, which implies $\alpha(E) = 0$.

6.2 Capacity and Length

Theorem 6.2.1 *If $\mathscr{H}_1(E) = 0$ then $\gamma(E) = 0$. If $dim_H(E) > 1$ then $\gamma(E) \geq \alpha(E) > 0$.*

Proof If E is compact and $\mathscr{H}_1(E) = 0$, then we can cover E by a finite union of disks whose radii sum to less than ε, for any ε we choose. Let Γ be the rectifiable curve bounding this union of disks. Its length is at most $2\pi\varepsilon$. Suppose $f \in H^\infty(\Omega)$ with $\|f\|_\infty \leq 1$ and apply Cauchy's formula for the derivative of f at some point $z \notin E$. Then

$$|f'(z)| = |\frac{1}{2\pi i}\int_\Gamma \frac{f(w)dw}{(w-z)^2}| \leq \frac{\varepsilon}{\mathrm{dist}(z,E)}.$$

Taking $\varepsilon \to 0$ shows f must be constant on Ω, so $\gamma(E) = 0$.

If $\dim_H(E) > 1$ then by Frostman's lemma there is an $\varepsilon > 0$ and a positive measure μ supported on E which satisfies

$$\mu(B(x,r)) \leq r^{1+\varepsilon},$$

for every ball. If we convolve this measure with the function $1/z$ the result is a bounded, continuous function

$$f(z) = \int \frac{d\mu(w)}{w-z},$$

on the plane which is holomorphic off the support of μ. Its easy to check that

$$f'(z) = \lim_{z\to\infty} z(f(z) - f(\infty)) = \int \frac{zd\mu(w)}{z-w} = \mu(E) > 0,$$

so that $\alpha(E) > 0$. □

For the contnuous analytic capacity we have

Theorem 6.2.2 *If E is compact and has σ finite $\mathscr{H}_1$ measure then $\alpha(E) = 0$. For any gauge function $\varphi(t) = o(t)$ there is a set E with $\alpha(E) > 0$ and $\mathscr{H}^\varphi(E) = 0$.*

Proof Suppose $E = \cup E_n$ where E_n have finite 1-dimensional measure. Assume f is analytic off E and extends continuously to E. Then since E is compact there f is uniformly continuous on E, i.e., there is a function $h(t) = o(1)$, such that $|f(x) - f(y)| \le h(|x-y|)$ for x, y in neigbothood of E. Cover E_n by open squares $\{Q_j\}$ such that $\sum |Q_j| h(|Q_j|) \le \varepsilon 2^{-n}$. Taking the union over n gives a covering of E. Since E is compact there is a finite subcovering. By the Cauchy integral theorem we have (for z outside the union of squares)

$$|f'(z)| \le \frac{1}{2\pi} \sum_n \sum_j \int_{\partial Q_{jn}} \frac{f(w) - c_{jn}}{(w-z)^2} dw,$$

where Q_{jn} are squares of the covering coming from the cover of E_n and c_{jn} is any constant we choose (since the Cauchy integral of constants is zero). We now choose c_{jn} so that

$$|f(w) - c_{jn}| \le h(|Q_{jn}|),$$

for $w \in \partial Q_{jn}$. Then

$$|f'(z)| \le \frac{C}{\text{dist}}(z, E)^{-2} \sum_n \sum_j |Q_{jn}| h(|Q_{jn}|) \le C\varepsilon,$$

for any ε we wish. Thus E has zero continuous analytic capacity.

To prove the other statement, we simply note that it is easy to build closed curves Γ with $\mathscr{H}^\varphi(\Gamma) = 0$ for a given guage φ and which have tangents almost nowhere. Thus by the results of Chapter 3 the harmonic measures for the two sides of Γ are mutually singular and so By the Browder-Wermer theorem Theorem **??**, there are many functions analytic off Γ and continuous on the whole sphere. □

The results above for analytic capacity show that the problem of geometrically characterizing the sets of zero analytic capacity comes down to the sets of dimension 1. To state the main conjecture we need to define a few terms.

that for $0 < \mathscr{H}_1(E) < \infty$, $\gamma(E) = 0$ iff E has zero Favard length. For a set E define the upper and lower densities

$$\overline{D}(E,x) = \limsup_{r\to 0} \frac{\mathscr{H}_1(E \cap B(x,r))}{2r},$$

$$\underline{D}(E,x) = \liminf_{r\to 0} \frac{\mathscr{H}_1(E \cap B(x,r))}{2r}.$$

A point x of E is called regular if $\overline{D}(E,x) = \underline{D}(E,x) = 1$ and is called irregular otherwise. A set E is called (Besicovitch) regular if a.e. point of E (w.r.t. $\mathscr{H}_1$) is regular. Similarly, E is called (Besicovitch) irregular if a.e. point is irregular. It is a theorem of Besicovitch that every set E with $0 < \mathscr{H}_1(E) < \infty$ can be written as the union of a regular and irregular set [**?**], [**?**]. Moreover, E is regular iff it is contained in a countable union of rectifiable curves. On the other hand, if E is irregular then $\mathscr{H}_1(E \cap \Gamma) = 0$ for any rectifiable curve Γ. Moreover, a set of finite 1 dimensional measure is Besicovitch irregular iff its Favard length

$$\text{Fav}(E) = \int_0^{\pi} |P_\theta(E)| d\theta,$$

is zero (P_θ is the orthogonal projection of E on to a line of angle θ to the x-axis). See Section 1.23. All these results can be found in Falconer's book [**?**]

Vitushkin's conjecture states that if that $0 < \mathscr{H}_1(E) < \infty$, then $\gamma(E) = 0$ iff E is Besicovitch irregular. One direction known; if $\gamma(E) = 0$ then E is iregular. This follows from the Denjoy conjecture: a positive length subset of a rectifable curve has positive analytic capacity. This follows from the deep result of Calderón [**?**] that the Cauchy integral corresponding to a Lipschitz graph is a bounded operator on L^2 . Although there are by now many proofs of Calderón's theorem, all require a fair amount of work. The shortest proof is probably in [**?**]. See also the excellent text of Christ [**?**] for another proof of Calderón's theorem and how the Denjoy conjecture follows (as well as a survey of more recent work).

It is also known that there exist irregular sets of positive $\mathscr{H}_1$ measure but zero analytic capacity [**?**] , [**?**], [**?**] We shall give an example of such a set in the next section.

A few special cases of the Vitushkin conjecture are known, when the set E satisfies additional geometric assumptions. We shall state two such cases in order to give a flavor of what is known. We require yet more notation.

Suppose $x \in E$ is such that $\overline{D}(E,x) > 0$ and L is a line passing through E. Let $S(x,L,\theta)$ be the union of lines passing through x making angle less than θ with

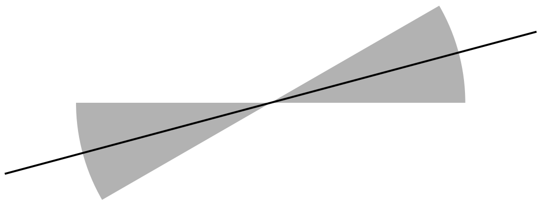


Figure 6.2.1 $S(x,L,\theta)$

L. Clearly $S(x,L,\theta)$ is the union of $\{x\}$ with two open cones centered along L. When needed, we will refer to the two cones as $S^+(x,L,\theta)$ and $S^-(x,L,\theta)$. We also let $S^\pm(x,L,\theta,r) = S^\pm(x,L,\theta)\cap B(x,r)$ denote the truncated cone.

The line L is called a tangent of E if

$$\lim_{r\to 0}\mathscr{H}_1(E\cap D(x,r))\backslash S(x,L,\theta) = 0,$$

for every $\theta > 0$. The line L is called a weak tangent if

$$\liminf_{r\to 0}\mathscr{H}_1(E\cap D(x,r))\backslash S(x,L,\theta) = 0,$$

for every $\theta > 0$.

A very useful result in this connection is the following estimate of Besicovitch [**?**]: suppose E is Besicovitch irregular and $\mathscr{H}_1(E) < \infty$. Then for a.e. $x \in E$, and a line L passing through x,

$$\limsup_{r\to 0}\frac{\mathscr{H}_1(E\cap S^+(x,L,\theta,r))}{r} + \limsup_{r\to 0}\frac{\mathscr{H}_1(E\cap S^-(x,L,\theta,r))}{r} \geq \frac{1}{6}\sin(\theta).$$

Because of this an irregular set cannot have tangents except on a set of measure 0. They can have weak tangents however. Indeed, there are irregular sets with weak tangents in all directions at every point (e.g., locally flat sets; see Section ?). On the other hand, Besicovitch regular sets have tangents at a.e. point since they are contained in countable unions of rectifiable curves.

Our first special case is due to Mattila [**?**]:

Theorem 6.2.3 *Let E be a compact Besicovitch irregular set in the plane with $\mathscr{H}_1(E) < \infty$. Suppose that for $\mathscr{H}_1$ almost every $x \in E$, $\underline{D}(E,x) > 0$ and there is a $\theta \in [0,\pi)$ which is not a weak tangent direction for E. Then $\gamma(E) = 0$.*

If E is a self-similar set satisfying the the open set condition (see Section **??**)

with finite $\mathscr{H}_1$ measure, then either E is a line segment or E has no weak tangent directions at all ([**?**]). Thus

Corollary 6.2.4 *Suppose E is a compact, self-similar set satisfying the open set condition and that $\mathscr{H}_1(E) < \infty$. Then either E is a line segment or $\gamma(E) = 0$.*

Another special case of the conjecture is due to Fang [**?**] Recall that a set E is called local flat if $\beta_E(Q) \to 0$ as $\ell(Q) \to 0$ (see Section **??**) and that E is called Ahlfors-David regular if there are constants so that

$$C^{-1}r \leq \mathscr{H}^1(E \cap D(x,r)) \leq Cr,$$

for every $x \in E$ and $0 < r \leq |E|$. Fang shows that

Proposition 6.2.5 *[**?**] If E Ahlfors-David regular and locally flat then $\gamma(E) = 0$ iff E is Besicovitch irregular.*

This is a complement of Mattila's result because an Ahlfors-David regular, Besicovitch irregular, locally flat set has weak tangents in every direction [**?**]. Therefore Mattila's result does not apply to such a set.

Because a set of finite 1-dimensional measure is Besicovitch irregular iff its Favard length is zero, Vitushkin's conjecture can be restated as saying $\gamma(E) = 0$ iff $\mathrm{Fav}(E) = 0$. The latter statement has the advantage that we do not have to assume that $\mathscr{H}^1(E) < \infty$ in order to state it.

However, for sets of infinite 1-dimensional measure it is known that the conjecture fails. In [**?**] Mattila constructed a set E and a Möbius transformation τ such that $\mathrm{Fav}(E) = 0$ but $\mathrm{Fav}(\tau(E)) > 0$. Since vanishing analytic capacity is clearly invariant under conformal transformations, either E or $\tau(E)$ must be a counterexample to the conjecture (but the proof does not say which is)! Later Murai [**?**] constructed a sequence of sets E_n with $\mathrm{Fav}(E_n) \to 0$, but $\gamma(E_n)$ bounded away from zero using delicate estimates of the bounds on the L^2 norm of the Cauchy integral. Unfortunately the limit of his sets E_n is a line segment, which is not a counterexample. However, he and Jones [**?**] eventually constructed an example of a set E with $\gamma(E) > 0$ and $\mathrm{Fav}(E) = 0$. Simpler examples have since been discovered and will be described in a later section. The question of whether there is a set E with $\gamma(E) = 0$ and $\mathrm{Fav}(E) > 0$ remains open.

6.3 Positive Length but Zero Capacity

First we show that there is a set E with $0 < \mathscr{H}_1(E) < \infty$ and γ. The first example of such a set is due to Vitushkin [**?**]. The example we present is due to Garnett

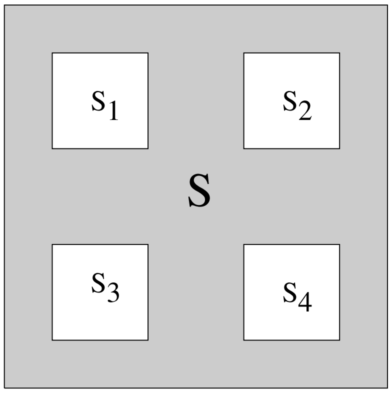


Figure 6.3.1 The square S and its 4 subsquares.

and is the product of the $1/4$ Cantor set $[\sum_{n=1}^{\infty} a_n 4^{-n} : a_n \in \{0,3\}]$ with itself. This set clearly has the correct Hausdorff measure, so we need only show it has zero analytic capacity.

It is convenient to think of this set as formed by an iterative construction in which a square S of size r is replaced by 4 disjoint, closed subsquares S_i, $i = 1,2,3,4$, each of size $r/4$. We may also assume that the subsquares satisfy

$$\mathrm{dist}(S_i, S_j) \sim \mathrm{dist}(S_i, S^c) \sim \mathrm{diam}(S).$$

See Figure 6.3.1 Thus $W = \mathrm{int}(S) \setminus (\cup_i S_i)$ is a 4-connected domain (this is the shaded region in Figure 6.3.1. Let $\Omega = E^c$ and suppose f were an analytic function on Ω bounded by 1 and $f(\infty) = 0$. Let $E_n = \cup_j S_{n,j}$ be the nth stage of constructing E, $\Omega_n = E_n^c$ and write $\Omega_n = \cup_{n,j} W_{n,j}$. If $S_{m,k} \subset S_{n,j}$ we will write $m,k << n,j$.

Jones' Proof: Our first proof is based on the proof given by Jones in [**?**]. Let $b_{n,j} = \sup_{W_{n,j}} 4^{-n} |f'(z)|$. We claim that

Lemma 6.3.1 *Fix a $\delta > 0$. For each n,j there is a m,k with $S_{m,k} \subset S_{n,j}$ with $b_{m,k} \le \delta$ and $m \le n + M\delta^{-2}$.*

First we will show how to deduce $\gamma(E) = 0$ and then we will prove the lemma. Let $\delta > 0$ be small. By iterating the lemma we can cover E by two finite families of squares $\mathscr{F}_1$ and $\mathscr{F}_2$; the first with diameters summing to less than δ and the second with consisting of squares $S_{n,j}$ were f differs from some constant $c_{n,j}$ by at most δ. Then by the Cauchy integral formula applied to f

$$\begin{aligned}|2\pi i f(w)| &\leq |\sum_{\mathscr{F}_1}\int_{\partial S}\frac{f(z)dz}{z-w}|+|\sum_{\mathscr{F}_2}\int_{\partial S}\frac{f(z)-c_{n,j}dz}{z-w}| \\ &\leq C\sum_{\mathscr{F}_1}\ell(S_{n,j})+\delta\sum_{\mathscr{F}_2}\ell(S_j) \\ &\leq C\delta.\end{aligned}$$

Taking $\delta \to 0$ shows f must be the constant zero.

Now we prove the lemma. Let G denote the Green's function for Ω with pole at ∞ and G_n the corresponding Green's function for Ω_n. Note that by the Maximum principle, $G \leq G_n$. By Greens Theorem and using the equality $\Delta|f|^2 = 4|f'|^2$,

$$\begin{aligned}\iint_{\Omega_n}|f'(z)|^2G(z)dxdy &\leq \iint_{\Omega_n}|f'(z)|^2G_n(z)dxdy \\ &= C\int_{\partial\Omega_n}|f|^2\frac{\partial G}{\partial n}ds \leq C\|f\|_\infty^2 \\ &< \infty.\end{aligned}$$

Thus if we break up the sum into the integrals over all the regions $W_{n,j}$ we get

$$\sum a_{n_j} = \sum \iint_{W_{n,j}}|f'(z)|^2G(z)dxdy < \infty.$$

By Harnack's inequality G is approximately constant on W_{n_j}, and by Green's theorem

$$G(z) \sim \int_{\partial W_{n,j}}\frac{\partial G(z)}{\partial n}ds \sim \omega_{n,j},$$

where $\omega_{n,j} = \omega(S_{n,j}\cap E)$ denotes the harmonic measure on E with respect to ∞. Its easy to check that $b_{n,j}^2 \leq Ca_{n,j}$, so we deduce

$$\sum_{m,k<<n,j} b_{n,j}^2\omega_{n,j} \leq C < \infty.$$

Let $\mathscr{F}$ be the subsquares $S_{m,k}$ of $S_{n,j}$ with $n \leq m \leq n+M\delta^{-2}$. If the lemma fails then $b_{m,k} \geq \delta$ for all squares in $\mathscr{F}$. Thus

$$\begin{aligned}\sum_{\mathscr{F}} b_{m,k}^2\omega_{m,k} &\geq \delta^2\sum_{\mathscr{F}}\delta^2\omega_{n,j} \\ &\geq \delta^2\sum_{m=n}^{n+M\delta^{-2}}\omega(E) \\ &\geq \delta^2M\delta^{-2} \\ &\geq M,\end{aligned}$$

which is a contradiction if M is large enough. Hence the lemma is true.

6.4 Mattila's counterexample to Vitushkin conjecture

In [**?**] Mattila proved

Proposition 6.4.1 *Suppose f is a C^2 diffeomorphism of the plane which is not segmented. Then there is a compact set E so that Fav$(E) = 0$, but such that $|P_\theta(f(E))| > 0$ for every direction.*

We shall sketch his construction with the stronger assumption that f maps no segment to a another segment (e.g., a Möbius transformation which moves ∞ to a finite point has this property), and we will only show that $\mathrm{Fav}(f(E)) > 0$. In what follows parallelogram always means a parallelogram with short sides parallel to the x-axis. $\Gamma(P)$ denotes the C^1 curves so that some component of $C \cap P$ connect the two long sides, and $\mathrm{dir}(C,P) \subset [0,\pi]$ denotes the set of tangent directions taken along that arc.

Lemma 6.4.2 *Suppose P is parallelogram and suppose we are given angles $0 < \zeta < \zeta + 2\alpha < \pi$ and $\varepsilon > 0$. Then there is a finite family $\mathscr{P} = \mathscr{P}(P,\zeta,\alpha,\varepsilon)$ such that*

(1) $|P_\theta(\cup\mathscr{P})| \le \varepsilon$ for all $\zeta \le \theta \le \zeta + \alpha$.

(2) If $C \in \Gamma(P)$ with dir$(C,P) \cap (\zeta - \alpha, \zeta + 2\alpha) = \emptyset$, then $C \in \Gamma(P')$ for some $P' \in \mathscr{P}$.

Proof If the long sides of P are not in direction ζ then replace P be a finite collection of subparallelograms with this property and such that $C \in \Gamma(P)$ implies $C \in \Gamma(P')$ for at least on of the new parallelograms. Thus we may assume P has longer sides in direction ζ. Let p be a large enough integer so that

$$\beta \equiv \frac{\alpha}{p} \le \frac{\varepsilon \sin(\alpha)}{2\,\mathrm{diam}(P)}.$$

Let J be the left hand longer side of P and subdivide it into m equal subintervals with endpoints $\{x_0,\dots x_m\}$. Replace J by the m segments $J_i = [x_i, y_i] = [x_i, x_i + te^{i(\zeta+\beta)}]$ where t is chosen so that $\arg(y_i - x_{i+1}) = \zeta + 2\alpha$. Let $K_1 = \cup_i J_i$. See Figure 6.4.1

Obviously $|P_{\zeta+\beta}|(K) = 0$ and $|P_\zeta|(K_1) = \sin(\beta)t \le \beta\,\mathrm{diam}(P) \le \varepsilon/2$. Similarly, for $\zeta \le \theta \le \zeta + \beta$ we get

$$|P - \theta(K_1)| \le |P_\theta(J)| + \frac{\varepsilon}{2} \le \varepsilon,$$

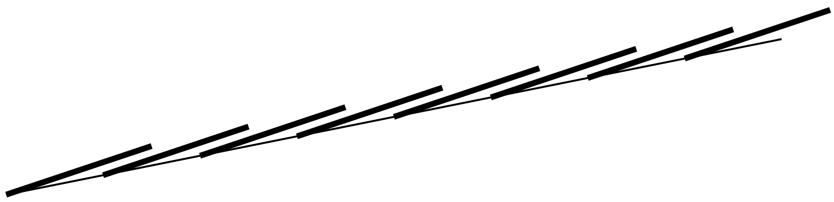


Figure 6.4.1 The segments J and $\{J_i\}$

if m is large enough. For angles $\zeta+\beta \le \theta \le \zeta+2\beta$ we have

$$|P_\theta(K_1)| \le \sum_i \frac{\sin(\beta)}{\sin(\alpha)} |P_{\zeta+\alpha}(J_i)| = \frac{\sin(\beta)}{\sin(\alpha)} \sum_i |J|/m \le \frac{\varepsilon}{2}.$$

Also note that any curve in $\Gamma(P)$ which misses all the J_i's must pass between two of them and therefore (by the mean value theorem) takes a direction in $\cap(\zeta-\alpha, \zeta+2\alpha)$. Thus any curve omitting these directions must hit one of the J_i.

Now repeat the construction by replacing each J_i with segments $[x_i, y_i]$ in direction $\zeta+2\beta$, with lengths chosen so that $\arg(y_1 - x_{i+1}) = \zeta+2\alpha$. If m is large enough the resulting set K_2 still has projections less than ε in directions $\zeta \le \theta \le \zeta+2\beta$, while the previous argument shows that it projections in directions $\zeta+2\beta \le \theta\zeta+4\beta$ may be made less than ε.

Continue the replacements for p generations. The resulting set K_p consists of a finite union of line segments and has projection of measure less than ε is all directions $\zeta \le \theta\zeta+\alpha$, as desired. Moreover, our remarks above plus induction show that any $C \in \Gamma(P)$ omitting directions in $(\zeta-\alpha, \zeta+2\alpha)$ must hit some segment in K. Finally we truncate the segments as necessary so that they lie in P and replace each of them by a very thin parallelogram. □

Lemma 6.4.3 *Let p be a parallelogram, $\delta > 0$ and $\alpha = \pi/(5(k+1))$ for some large integer k. Then there is a finite family of subparallelograms $\mathscr{R}$ such that if $R = \cup_{\mathscr{R}} P_i$, then*

(1) $|P_\theta(R)| \le \delta$ for $\theta \in E = E_\alpha = \cup_{j=1}^k [5j\alpha, (5j+1)\alpha]$.

(2) If $C \in \Gamma(P)$ omits all directions in $F = \cup_{j=1}^k [(5j-1)\alpha, (5j+2)\alpha]$, then $C \in \Gamma(P_j)$ for some $P_j \in \mathscr{R}$.

Proof Simply apply the previous lemma k times. If at the jth stage we have constructed a collection of n_j parallelograms then the next time we apply the lemma to each of these with $\zeta = 5j\alpha$ and $\varepsilon = \delta/n_j$. □

We will construct S so that $\mathrm{Fav}(S) = 0$ but $\mathrm{Fav}(f(S)) > 0$ as $S = \cap_n S_n$ where

S_n is a finite union of parallelograms and S_{n+1} is obtained from E_n by applying the second lemma to each parallelogram in E_n. Let $\Gamma_L(P) \subset \Gamma(P)$ denote those curves C such that $f(C)$ is a line. Start with a parallelogram S_0 so that $\{(r,\theta) : f(L(r,\theta)) \in \Gamma(S_0)\}$ has positive measure in the space of lines.

Suppose in general that S_{n-1} consists of M parallelograms $\{P_1 \ldots, P_M\}$. Choose α_n so small that $|\text{dir}(C,P)| \geq \alpha_n$ for every $C \in \Gamma_L(P_j)$ and every P_j. Then subdivide each P_j into subparallelograms $\{P_{j,k}\}$ (with sides parallel to P_j) so that $|\text{dir}(C,P_{j,k})| \leq \alpha_n/20$ for every $C \in \Gamma_L(P_{j,k})$ (which we can do since $f \in C^2$). Let N be the total number of such subparallelograms and apply the previous lemma to each $P_{j,k}$ with $\alpha_n/5 < \alpha \leq \alpha_n$, $\delta = 1/(nN)$, and let S_n be the union of the resulting parallelograms. By Lemma 6.4.3

$$|P_\theta(S_n)| \leq \frac{1}{n},$$

for $\theta \in E_{\alpha_n}$, so $|P_\theta(E)| = 0$ for

$$\theta \in E \equiv \cap_n \cup_{k>n} E_{\alpha_k}.$$

Taking $\alpha_n \to 0$ fast enough (which we may clearly do), and applying the second Borel-Cantelli lemma proves $|E|$ has full measure, as desired.

We will construct S so that $\text{Fav}(S) = 0$ but $\text{Fav}(f(S)) > 0$ as $S = \cap_n S_n$ where S_n is a finite union of parallelograms and S_{n+1} is obtained from E_n by applying the second lemma to each parallelogram in E_n. Let $\Gamma_L(P) \subset \Gamma(P)$ denote those curves C such that $f(C)$ is a line. Start with a parallelogram S_0 so that $\{(r,\theta) : f(L(r,\theta)) \in \Gamma(S_0)\}$ has positive measure in the space of lines.

Suppose in general that S_{n-1} consists of M parallelograms $\{P_1 \ldots, P_M\}$. Choose α_n so small that $|\text{dir}(C,P)| \geq \alpha_n$ for every $C \in \Gamma_L(P_j)$ and every P_j. Then subdivide each P_j into subparallelograms $\{P_{j,k}\}$ (with sides parallel to P_j) so that $|\text{dir}(C,P_{j,k})| \leq \alpha_n/20$ for every $C \in \Gamma_L(P_{j,k})$ (which we can do since $f \in C^2$). Let N be the total number of such subparallelograms and apply the previous lemma to each $P_{j,k}$ with $\alpha_n/5 < \alpha \leq \alpha_n$, $\delta = 1/(nN)$, and let S_n be the union of the resulting parallelograms. By Lemma 6.4.3

$$|P_\theta(S_n)| \leq \frac{1}{n},$$

for $\theta \in E_{\alpha_n}$, so $|P_\theta(E)| = 0$ for

$$\theta \in E \equiv \cap_n \cup_{k>n} E_{\alpha_k}.$$

Taking $\alpha_n \to 0$ fast enough (which we may clearly do), and applying the second Borel-Cantelli lemma proves $|E|$ has full measure, as desired.

Suppose $C \in \Gamma_L(P_j)$. Finally, because each element of $|\text{dir}\Gamma_L(P_j)| \geq 5\alpha_n$, but $|\text{dir}\Gamma_L(P_{j,k})| \leq \alpha_n/2$, for every $P_{j,k} \subset P_j$, there must be some $P_{j,k} \subset P_j$ where dir(C) omits all values in F_{α_n}. Thus $C \in \Gamma_L(P_{j,k})$ by the lemma. By induction we see that $C \in \Gamma_L(S_0)$ implies C hits S. Thus $f(S)$ is hit by a positive measure set of lines, as desired.

We already know that $\mathrm{Fav}(K) = 0$ implies $\mathrm{Cap}_1(K) = 0$. If we choose a smooth f and K so $\mathrm{Fav}(K) = 0$, $\mathrm{Fav}(f(K)) \neq 0$, then $\mathrm{Cap}_1(f(K)) = \mathrm{Cap}_1(K) = 0$. Thus we have proven

Corollary 6.4.4 *There is compact set K so that* $\mathrm{Fav}(K) > 0$ *but* $Cap_1(K) = 0$.

6.5 The venitian blind example

6.6 exercises

Exercise 6.1 Prove Mattila's theorem.

Exercise 6.2 This and following two exercises complete Garnett's proof. With $a_{n,j}$ as defined in Garnett's proof, show $|a_{n,j}| \leq M4^{-n}$ for some uniform $M < \infty$.

Exercise 6.3 Let $h_{n,j}(z) = a_{n,j}4^{2n} \iint_{E_{n,j}} \frac{dxdy}{x+iy-z}$. Show that $h_n = \sum_j h_{n,j}$ are uniformly bounded.

Exercise 6.4 Show that for some n, j, $a_{n,j} \neq a4^{-n}$. Deduce the claim used in this section that for any $\varepsilon > 0$ and $M < \infty$ there is a $\delta > 0$ such that if f is holomorphic on E^c, $\|f\|_\infty < M$ and $|f'(\infty)| \geq \varepsilon$ then there is a n, j so that $|a_{n,j}| \geq (1+\delta)4^{-n}|f'(\infty)|$.

Exercise 6.5 Vitushkin's example. Start with a line segment of unit length. Divide it into two equal segments and rotate each around its center by 90 degrees. In general, given an interval in the nth stage of the construction, subdivide it and rotate around its center by 90 degrees. Let E the limit of the resulting sets. Show $0 < \mathscr{H}_1(E) < \infty$ but $\gamma(E) = 0$.

Exercise 6.6 Rising sun lemma: Suppose f is continuous on $[a,b] \to \mathbb{R}$ and has total variation V. Show that for any $\lambda > 0$ there is a collection of closed intervals $I_j = [a,b_j] \subset [a,b]$ such that (1) $f(x) - f(y) \leq \lambda(y-x)$ for $x \leq y$, $y \notin \cup_j I_j$, (2) $f(b_j) - f(a_j) = \lambda(b_j - a_j)$ and (3) $\sum_j |I_j| \leq V/\lambda$.

Exercise 6.7 Let $f : [a,b] \to \mathbb{R}$ be continuous and bounded variation. Prove there exists a Lipschitz h on $[a,b]$ so that $\{x : f(x) = h(x)\} \geq (b-a)/2$. (Hint: use previous exercise twice.)

Exercise 6.8 Show that if Γ is rectifiable, then there is a Lipschitz graph Γ' so that $\mathscr{H}_1(\Gamma \cap \Gamma') > 0$.

7
Limit sets of Kleinian groups

7.1 Definitions

A Möbius transformation is a mapping of the form

$$\frac{az+b}{cz+d},$$

which map $\overline{\mathbb{C}}$ 1-1 and onto itself and they are the only analytic 1-1 maps of the sphere to itself. These are also called linear fractional transformations. Since

$$\frac{az+b}{cz+d} = \frac{\lambda az+\lambda b}{\lambda cz+\lambda d},$$

for any $\lambda \in \mathbb{C}$, these mappings can be identified with elements of the group

$$\mathrm{PSL}(2,\mathbb{C}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad-bc = 1\}$$

by the identification

$$\frac{az+b}{cz+d} \leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

This gives us a topology on the set of Möbius transformations. A Kleinian group is a discrete group of Möbius transformations acting on the Riemann sphere $\overline{\mathbb{C}}$. This means that there is no sequence $\{g_n\} \subset G$ of distinct elements which converges to the identity.

If we think of the Riemann sphere $\overline{\mathbb{C}}$ as the unit sphere S^2 in $\mathbb{R}^3$ then any Möbius transformation acting on S^2 has a canonical extension to the unit ball $\mathbb{B} \subset \mathbb{R}^3$. One way to see this is to note that the Möbius group and its conjugates are generated by reflections through circles on S^2. These reflections can be extended to reflections through spheres and these map $\mathbb{B}$ to itself.

The limit set, $\Lambda(G)$, is the accumulation set (on S^2) of the orbit of the origin in $\mathbb{B}$. It is easy to see that the accumulation set of any other point gives the same set. The ordinary set of G, $\Omega(G) = S^2 \setminus \Lambda(G)$, is the subset of S^2 where G acts discontinuously, e.g., $\Omega(G)$ is the set of points z such that there exists a disk around z which hits itself only finitely often under the action of G. The limit set $\Lambda(G)$ has either 0,1,2 or infinitely many points and G is called elementary if $\Lambda(G)$ is finite. In some sources, the term "Kleinian group" is reserved for discrete Möbius groups such that $\Omega(G)$ is not empty.

A few examples of limits sets are shown in the figures. In each case the figures show a collection of light gray circles. The group is formed by taking the reflections through these circles as generators. (This gives a group of transformations which includes orientation reversing maps. To get a true Kleinian group we should pass to the index two subgroup of orientation preserving transformations. However, this does not effect the limit sets). The black set is a approximation of the limit set. The top left shows a totally disconnected limit set and the top right a Jordan curve. On the lower left is an example where Ω has infinitely many components, all of which are disks. In the example on the lower right there is a single distinguished component which is not a disk (the "outside").

In addition to these examples, is fairly easy to construct groups where the limit set is a circle or line. Such groups are called Fuchsian or extended-Fuchsian (depending on whether each side of the circle is invariant or not). It is also possible for the limit set to be the whole sphere.

There is another possibility as well. G is called degenerate (or totally degenerate) if $\Omega(G)$ is connected and simply connected, i.e., Λ is connected and does not divide the plane. Such sets are called "dendrites". Although Bers [**?**] proved that such groups exist, there is no good procedure for drawing their limit sets.

Every Möbius transformations is conjugate to one of the following types of maps

$$z \to z+1,$$

$$z \to ze^{i\theta}, \text{ for some } \theta,$$

$$z \to z\lambda e^{i\theta} \text{ for some } \theta \text{ and some } \lambda > 0.$$

These three types are called parabolic, elliptic and loxodromic respectively. The study of Kleinian groups simplifies somewhat if we assume that there are no elliptics or parabolics in the group and we shall make this assumption from now on (except where we explicitly state otherwise). We shall refer to a group with no elliptics or parabolics as a loxodromic group.

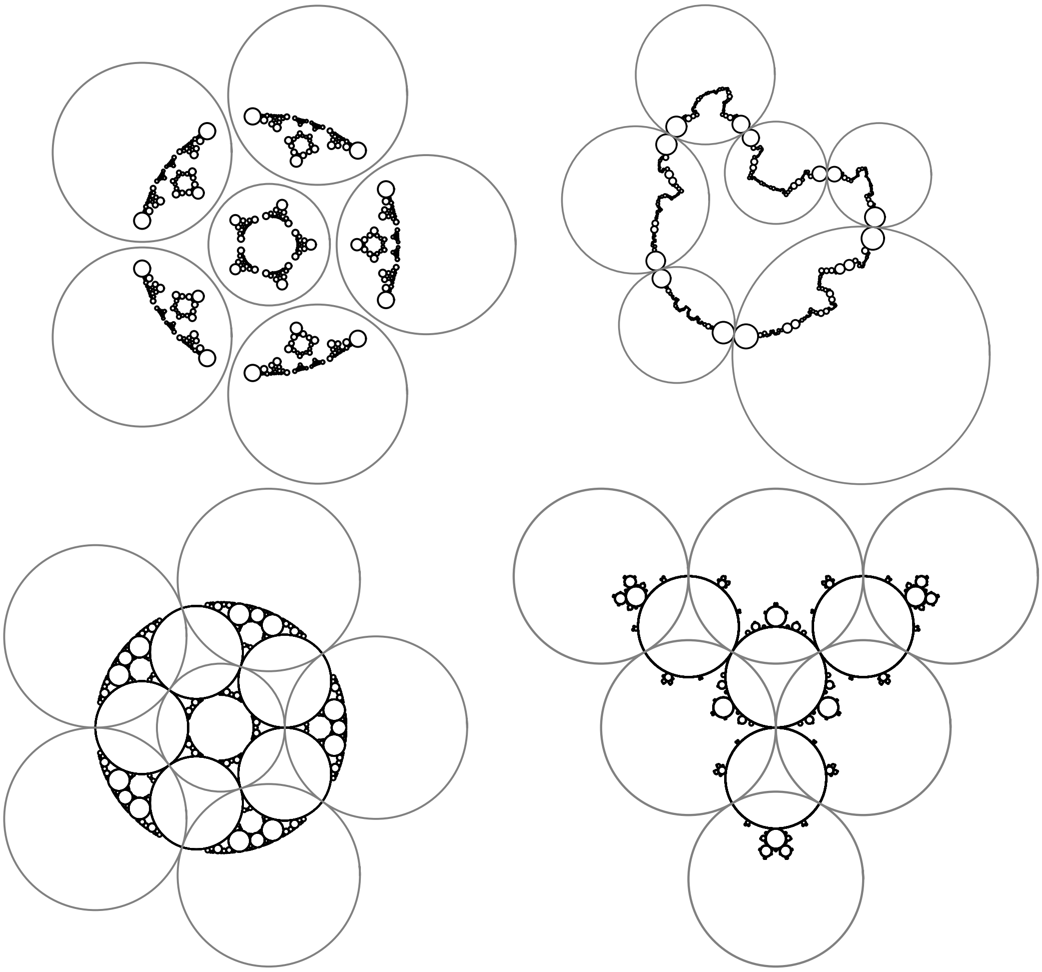


Figure 7.1.1 Some examples of limit sets

The unit ball $\mathbb{B} \subset \mathbb{R}^3$ comes with a hyperbolic metric ρ (just as the disk in the plane does) defined as

$$d\rho = \frac{d|z|}{1-|z|^2}.$$

We define the Poincaré series for a discrete group G acting on $\mathbb{B}^3$ to be

$$\sum_G \exp(-s\rho(0,g(0))),$$

where $s > 0$. Because

$$\rho(0,z) = \log \frac{1+|z|}{1-|z|} \sim \log \frac{1}{1-|z|},$$

the sum in the Poincaré series is comparable to

$$\sum_G (1-|g(0|)^s.$$

One can easily show the series converges for all large enough values of s (e.g., all $s > 2$) and we define

$$\delta(G) = \inf\{s : \sum_G \exp(-s\rho(0,g(0))) < \infty\}.$$

This is called the *Poincaré exponent or critical exponent* of the group.

A point $x \in \Lambda(G)$ is called a *conical limit* point if there is a sequence of orbit points which converges to x inside a (Euclidean) non-tangential cone with vertex at x (such points are also called radial limit points or points of approximation). The set of such points is the *conical limit set* and is denoted $\Lambda_c(G)$.

G is called *geometrically finite* if there is a finite sided fundamental polyhedron for the G action on $\mathbb{B}$. Otherwise G is called geometrically infinite. A result of Beardon and Maskit [**?**] implies that if G is loxodromic then it is geometrically finite iff $\Lambda(G) = \Lambda_c(G)$. An alternate characterization that we will want to use involves the convex hull of the limit set. If K is a compact set on $S^2 = \partial\mathbb{B}$ we will let $C(K) \subset \mathbb{B}$ denote its convex hull with respect to the hyperbolic metric on $\mathbb{B}$. We let $M = \mathbb{B}/G$ be the 3-manifold associated to G. A manifold of this form is called hyperbolic. Then $C(M) = C(\Lambda(G))/G \subset M$ is called the convex core of M. Much of the interesting topology of M is associated to the topology of the convex core. In particular, for loxodromic groups, G is geometrically finite iff $C(M)$ is compact. (For a general finitely generated Kleinian group, G is geometrically finite iff $C(M)$ has finite volume.)

In this chapter we will prove the following results

Theorem 7.1.1 *Suppose G is a finitely generated (loxodromic) Kleinian group. Then* $\Lambda(G)$ *is either uniformly wiggley for it is a circle. In particular* $\dim(\Lambda(G)) = 1$ *iff* $\partial\Omega$ *is a circle.*

Theorem 7.1.1 was first formulated by Bowen in [**?**] in the case of quasi-Fuchsian, loxodromic groups. The geometrically finite, loxodromic Kleinian case is proven in [**?**] and [**?**]. This case is sometimes called "convex co-compact". See also [**?**]. The general geometrically finite case is proven by Canary and Taylor in [**?**] and the result for all finitely generated Kleinian groups is in [**?**].

Theorem 7.1.2 *If G is a non-elementary, discrete Möbius group on $\mathbb{B}$ then* $\delta(G) = \dim(\Lambda_c(G))$.

The direction

$$\dim(\Lambda_c(G)) \leq \delta(G),$$

of Theorem 7.1.2 is easy and well known. The opposite direction with the additional assumptions that G is geometrically finite or Fuchsian is found in [**?**], [**?**]. The general case is proven in [**?**].

Theorem 7.1.3 *If G is a (loxodromic) finitely generated Kleinian group then the Minkowski dimension of Λ exists and equals the Hausdorff dimension.*

Theorem 7.1.3 was proven in the geometrically finite case by Stratmann and Urbanski [**?**]. The case with general, finitely generated Kleinian groups follows from this and the general case of Theorem 7.1.4 (however, there is also an "elementary" proof of the general case of Theorem 7.1.3 which does not use Theorem 7.1.4).

Theorem 7.1.4 *If G is a (loxodromic) finitely generated Kleinian group and the injectivity radius of the manifold $M = \mathbb{B}/G$ is bounded away from zero then* $\dim(\Lambda(G)) < 2$ *iff G is geometrically finite.*

The assumption on the injectivity radius is not essential and is made only to avoid having to quote a well known result about Kleinian groups (the Margulis lemma). Examples of groups with $\dim(\Lambda(G)) = 2$ were constructed by Sullivan in [**?**], and Canary [**?**] proved Theorem 7.1.4 holds if $M = \mathbb{B}/G$ is a certain topological condition (that it is "topologically tame", i.e., homeomorphic to the interior of a compact manifold with boundary) and a certain geometric condition (a special case is that the injectivity radius of the manifold is bounded away from zero). The proof for all finitely generated Kleinian groups is in [**?**]. Sullivan [**?**] and Tukia [**?**] independently showed that if G is a geometrically finite group then $\dim(\Lambda(G)) < 2$. Thus a finitely generated group is geometrically finite iff $\dim(\Lambda(G)) < 2$.

In each theorem stated above, the assumption that there are only loxodromic elements is completely unnecessary and is only made to simplify the exposition of the proofs. Selberg's lemma [**?**] says that any finitely generated matrix group contains a normal subgroup of finite index which contains no elements of finite order. It is easy to show that a discrete group can only contain elliptic elements if they have finite order, so Selberg's lemma can be used to reduce the general case to the elliptic-free case. The presence of parabolics causes more substantial difficulties, but these can always be handled.

The way that we will make use of the loxodromic hypothesis is by quoting the following two results:

Theorem 7.1.5 *If G is a finitely generated Kleinian group with no parabolic or elliptic elements, then $\Omega(G)/G$ is a a finite union of compact Riemann surfaces. In particular, if Ω_0 is a component of $\Omega(G)$ and $z_0 \in \Omega_0$ is any point. Then every point of Ω is within a bounded hyperbolic distance of the orbit of z_0.*

Theorem 7.1.6 *If G is a finitely generated Kleinian group then $\Lambda(G)$ is uniformly perfect. In other words, if $d\rho$ is hyperbolic metric on a component of $\Omega(G)$ then*

$$|d\rho(z)| \sim \frac{|dz|}{d(z)}.$$

The first is a version of the Ahlfors' finiteness theorem [**?**]. In general, this says that if G is any finitely generated group then $\Omega(G)/G$ is a finite union of finite type surfaces (i.e., compact with a finite number of punctures. The type of the surfaces can be explicitly bounded using the "area estimate" of Bers [**?**]. The second proposition is easier and can be found in [**?**] or [**?**].

A Whitney square for a domain Ω is a square $Q \subset \Omega$ such that $\ell(Q) \sim \text{dist}(Q, \partial\Omega)$. If G is a finitely generated Kleinian group then using these two results we can easily see that there is a finite set $E \subset \Omega(G)$ so that every Whitney square in Ω contains at least one point of E and at most a bounded number (the bound depending on G). This is property of loxodromic groups that we shall actually use in our proofs.

7.2 Limit sets are uniformly wiggley

In this section we prove Theorem 7.1.1. Suppose G is a finitely generated loxodromic Kleinian group. These means that there is a $C < \infty$ and a finite set of points $\{z_1, \dots, z_n\}$ so that for any point $z \in \Omega(G)$ there is a z_j and a $g \in G$ so that $g(z_j)$ is in the same component of $\Omega(G)$ as z and the hyperbolic distance (in this component) between z and $g(z_j)$ is less than C.

Now suppose $\Lambda(G)$ is not uniformly wiggley and choose a square Q hitting E for which $\beta = \beta_E(Q)$ is very, very small. Let S denote the strip of width $\beta\ell(Q)$ which contains $E \cap 3Q$. Choose a point $z \in \Omega \cap 3Q$ with $\text{dist}(z, E) \sim \beta^{1/2}\ell(Q)$. Because Λ is connected, each component of Ω is simply connected and so the hyperbolic metric on each component of Ω is comparable to $\text{dist}(z, \Lambda)^{-1} ds$. Therefore our hypothesis gives us z_j and $g \in G$ so that $|z - g(z_j)| \le C\beta^{1/2}\ell(Q)$.

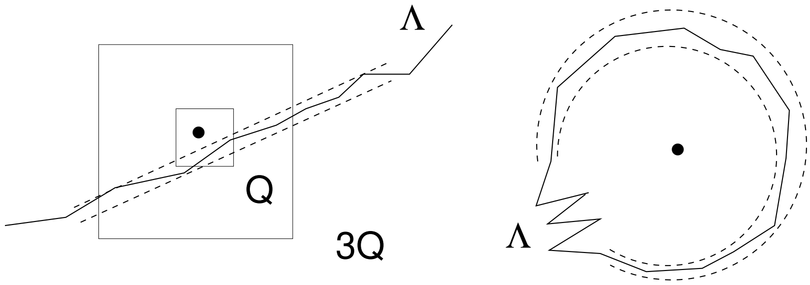


Figure 7.2.1 Proof that limit sets are uniformly wiggley.

Now map the picture back to z_j by the mapping g^{-1} and consider the picture on the sphere S^2. The limit set is mapped to itself and the strip S is mapped to a region between two circles. The distance between the circles is approximately $\beta^{1/2}\,\mathrm{dist}(z_j,\Lambda)$ and the part of Λ outside of $3Q$ is mapped into a region of diameter at most $C\beta^{1/2}\,\mathrm{dist}(z_j,\Lambda)$. Thus Λ is contained in a $S\beta^{1/2}$ neighborhood of circle on the sphere.

Taking limits as $\beta \to 0$, we deduce that Λ must be a circle, as desired.

7.3 The conical limit set

In this section we prove Theorem 7.1.2.

Proof We start with the easy direction

$$\dim(\Lambda_c(G)) \le \delta(G).$$

Let $\mathscr{G} = G(0)$ denote the orbit of the origin of the ball $\mathbb{B}$ under G. Fix a large number M and for each $g \in G$ let B_g be the ball centered at $g(0)/|g(0)|$ (the radial projection of the orbit point onto the sphere) and radius $M(1-|g(0)|)$. Let E_M be the set of points which are in infinitely many of the balls B_g. Since

$$\sum_g |B_g|^{\delta+\varepsilon} < \infty,$$

for any $\varepsilon > 0$ we see that $\dim(E_M) \le \delta(G)$ for any M. On the other hand, and point of $\Lambda_c(G)$ is in E_M for some M. Thus $\dim(\Lambda_c(G)) \le \delta(G)$, as desired.

Now we prove the opposite inequality. Let $\{z_n\}$ denote the orbit of 0 under G in the hyperbolic 3-ball, $\mathbb{B}$. Let $\delta = \delta(G)$ be the critical exponent for the

Poincaré series of G and let $\varepsilon > 0$. Choose a point $x \in \overline{\mathbb{C}} = \partial\mathbb{B}$ so that

$$\sum_{j:|z_j-x|<r} (1-|z_j|)^{\delta-\varepsilon} = \infty,$$

for every $r > 0$ (here $|z-x|$ denotes the spherical metric). We can do this by a simple compactness argument. Since G is non-elementary, x is not fixed by every element of G. Therefore we can choose elements $\{g_1,\dots,g_4\} \in G$ so that $x_i = g_i(x)$ are all distinct. Fix $r > 0$ to be so small that the balls B_i on $\overline{\mathbb{C}}$ (in the spherical metric) of radius $2r$ around the points $\{x_1,\dots x_4\}$ are pairwise disjoint.

Suppose M,N are large numbers (to be chosen below depending only on G and r). Let $A_n = \{z \in \mathbb{B} : 2^{-n-1} \le 1-|z| < 2^{-n}\}$. If it were true that

$$\sum_{j:z_j\in B_i\cap A_n} (1-|z_j|)^{\delta-2\varepsilon} \le M,$$

for all large enough n, then

$$\begin{aligned}\sum_{j:z_j\in B_i} (1-|z_j|)^{\delta-\varepsilon} &\le C\sum_n 2^{-n\varepsilon} \sum_{j:z_j\in B_i\cap A_n} (1-|z_j|)^{\delta-2\varepsilon}\\ &\le CM\sum_n 2^{-n\varepsilon}\\ &< \infty.\end{aligned}$$

This is a contradiction, so we must have

$$\sum_{j:z_j\in B_i\cap A_n} (1-|z_j|)^{\delta-2\varepsilon} \ge M,$$

for infinitely many values of n. For each $i = 1,2,3,4$, fix a value of n_i (depending on M and hence of r) for which this inequality holds.

Since the z_j's make up the orbit of a single point, they are uniformly separated in the hyperbolic metric of $\mathbb{B}$. Thus for any $A < \infty$ we may split the sequence into a finite number B of sequences (depending on A) each of which is separated by at least A in the hyperbolic metric. Therefore, to each point x_i we may associate a collection of points $\mathcal{G}_i(0) \subset \{z_j\}$ such that

$$\mathcal{G}_i(0) \subset B_i \cap A_{n_i},$$

$$z,w \in \mathcal{G}_i(0) \text{ implies } |z-w| \ge 3N2^{-n_i},$$

$$\sum_{j:z_j\in\mathcal{G}_i(0)} (1-|z_j|)^{\delta-2\varepsilon} \ge M/B.$$

For each $z \in \mathscr{G}_i(0)$ let $z^* = z/|z|$ denote its radial projection onto the sphere $\overline{\mathbb{C}} = \partial \mathbb{B}$. For $z \in \mathbb{B}$, let

$$B(z_j) = B(z_j^*, N(1-|z|)).$$

By hypothesis, the balls $B(z_j)$ are disjoint for all $z_j \in \cup_{i=1}^4 \mathscr{G}_i(0)$.

Since the balls $\{B_1, \dots, B_4\}$ have disjoint doubles, any sufficiently small disk (depending on r) can intersect at most one of the balls. For any point $z = g(0)$ in the orbit of 0 consider the four balls $\{g(B_1), \dots, g(B_4)\}$. The preceding statement implies that if N is sufficiently large (depending only on r) then at most one of these balls can intersect $\mathbb{B} \setminus B(z)$. This determines our choice of N. Therefore, at least three of the balls are contained in $B(z)$. Without loss of generality, assume they are $g(B_1), g(B_2)$ and $g(B_3)$.

The Möbius transformation g has bounded distortion as a map from S^2 to itself except possibly at one point. More precisely,

Lemma 7.3.1 *Suppose $r > 0$. There is a $C < \infty$ (depending only on r) such that given any Möbius transformation g of $S^2 = \overline{\mathbb{C}}$ to itself we have*

$$C^{-1}(1-|g(0)|) \le |g'| \le C(1-|g(0)|),$$

except possibly on a disk D of radius r (both the derivatives and the disk are taken with respect to the spherical metric).

Proof We may assume $g(0) \neq 0$ since otherwise the lemma is easy. Let z be the radial projection of $g(0)$ onto S^2 and choose R so big (depending only on r) so that

$$\omega(g(0), S^2 \setminus B(z, R(1-|g(0)|), \mathbb{B}) \le r^2.$$

Then $D = g^{-1}(B(z, R(1-|g(0)|))$ is a disk of radius less than r and $|g'|$ is comparable to $1-|g(0)|$ on its complement. □

So for any $g \in G$, at least two of the disks (say B_1 and B_2) are bounded away from this point so we get

$$C^{-1}(1-|g(0)|) \le |g'| \le C(1-|g(0)|),$$

on B_1 and B_2 with constants depending only on r.

Note that if we choose n_i large enough (depending only on N) we may assume

$$\frac{1-|z|}{2N} \ge 1-|z_j| \ge \frac{1-|z|}{CN},$$

for some uniform C depending only on G and r.

Now for the orbit point $z = g(0)$ define $\mathscr{G}(z) = g(\mathscr{G}_1(0))$. Thus

$$\begin{aligned}\sum_{z_j \in \mathscr{G}(z)} (1-|z_j|)^{\delta-2\varepsilon} &\geq C^{-\delta+2\varepsilon}(1-|z|)^{\delta-2\varepsilon} \sum_{z_k \in \mathscr{G}_1(0)} (1-|z_k|)^{\delta-2\varepsilon} \\ &\geq C^{-\delta+2\varepsilon}(1-|z|)^{\delta-2\varepsilon}\frac{M}{B} \\ &\geq C^{-2}\frac{M}{B}(1-|z|)^{\delta-2\varepsilon} \\ &\geq (1-|z|)^{\delta-2\varepsilon},\end{aligned}$$

where the last line holds if M is large enough. Since C_1 depends only on r and B depends only on group G (more precisely it only depends on the injectivity radius of G at 0), this determines our choice of M.

We have now constructed a set of points $z_j = \mathscr{G}(z)$ which satisfy the following conditions:

$$z_j \in B(z^*, N(1-|z|)),$$

$$B(z_j^*, 2N(1-|z_j|)) \cap B(z_k^*, 2N(1-|z_k|)) = \emptyset \text{ for } j \neq k,$$

$$\sum_j (1-|z_j|)^{\delta-2\varepsilon} \geq (1-|z|)^{\delta-2\varepsilon},$$

$$\frac{1-|z|}{2N} \geq 1-|z_j| \geq \frac{1-|z|}{CN},$$

for some uniform C depending only on G and r (because the points in $G_1(0)$ do and the map g has uniformly bounded distortion on $G_1(0)$).

Construct generations of points starting with $\mathscr{G}_0 = \{0\}$, and for each $z \in \mathscr{G}_n$, define points $\{z_j\}$ in $\mathscr{G}_{n+1}$ as above. To each point $z \in \mathscr{G} = \cup_n \mathscr{G}_n$, associate the disk $B_z = B(z^*, 2N(1-|z|))$. Then let

$$E_n = \cup_{z \in \mathscr{G}_n} B_z,$$

$$E = \cap_n E_n.$$

It is easy to see that $E \subset \Lambda_c(G)$.

Define a probability measure μ on E by setting $\mu(E_0) = 1$, and for $z \in \mathscr{G}_n$ with "parent" $z' \in \mathscr{G}_{n-1}$, set

$$\mu(B_z) = \frac{(1-|z|)^{\delta-2\varepsilon}}{\sum_{w \in \mathscr{G}(z')} (1-|w|)^{\delta-2\varepsilon}} \mu(B_{z'}).$$

It is easy to see by induction that

$$\mu(B_z) \leq (1-|z|)^{\delta-2\varepsilon} \leq C \operatorname{diam}(B_z)^{\delta-2\varepsilon},$$

for each z in $\mathscr{G}$. We want to show this inequality is true for any disk D. Let D be any disk and let $D_0 = B_z$ be the lowest generation disk in our construction so that $D_0 \cap D \neq \emptyset$ but $D_0 \subset 2D$. Let D_1 be the parent of D_0. By the maximality of D_0 we have $D \subset 2D_1$. Since $2D_1$ is disjoint from any other balls of the same generation,

$$\begin{aligned}\mu(D) &\leq \mu(D_1) \\ &\leq C\,\mathrm{diam}(D_1)^{\delta-2\varepsilon} \\ &\leq C(NC)^{\delta-2\varepsilon}\,\mathrm{diam}(D_0)^{\delta-2\varepsilon} \\ &\leq 2C(NC)^{\delta-2\varepsilon}\,\mathrm{diam}(D)^{\delta-2\varepsilon}.\end{aligned}$$

This is the desired inequality (the constant in front is larger, but is uniform over all disks; the power is the same). The mass distribution principle Lemma **??** now implies $\dim(\Lambda_c(G)) \geq \delta(G) - 2\varepsilon$. Since ε was arbitrary, we get Theorem 7.1.2. □

We mentioned in the introduction that for geometrically finite loxodromic groups, $\Lambda_c(G) = \Lambda(G)$. Thus

Corollary 7.3.2 *If G is a geometrically finite loxodromic group then* $\delta(G) = \dim(\Lambda(G))$.

Beardon and Maskit [**?**] show that for a general geometrically finite group, $\Lambda(G)$ is the union of $\Lambda_c(G)$ and a countable set (the parabolic fixed points of G), so the corollary is true for all geometrically finite groups. It is not known if it is true for all finitely generated groups, but any counterexample would have a limit set of positive area (see Section 7.5).

7.4 The Besicovitch-Taylor index

Given a compact set K in the plane let $\Omega = \mathbb{R}^2 \setminus K$ be its complement. A Whitney decomposition of Ω is a collection of squares $\{Q_j\}$ which are disjoint, except along their boundaries, and such that

$$\frac{1}{10}\,\mathrm{dist}(Q_j, \partial\Omega) \leq \ell(Q_j) \leq 10\,\mathrm{dist}(Q_j, \partial\Omega).$$

The existence of Whitney decomposition for any open set is a standard fact in real analysis (e.g., [**?**]). The squares $\{Q_j\}$ may also be taken to be dyadic.

For any compact set $K \subset \mathbb{C}$ we can define an exponent of convergence similar to the exponent of convergence of a Poincaré series,

$$\kappa = \kappa(K) = \inf\{\alpha : \sum_j \ell(Q_j)^\alpha < \infty\},$$

where the sum is taken over all squares in Whitney decomposition of $\Omega = K^c$ which are within distance 1 of K (we have to drop the "far away" squares or the series might not converge). It is easy to check that this does not depend on the particular choice of Whitney decomposition. This number has been rediscovered many times in the literature, but seem to have been first used by Besicovitch and Taylor in [**?**]. See also Tricot's paper[**?**] where he shows that it agrees with the upper Minkowski dimension.

Lemma 7.4.1 *For any compact set K, $\kappa \le \overline{\dim}_{\mathscr{M}}(K)$. If K also has zero area then $\kappa = \overline{\dim}_{\mathscr{M}}(K)$.*

Proof We start with the easy direction, $\kappa \le \overline{\dim}_{\mathscr{M}}(K)$. Let $D = \overline{\dim}_{\mathscr{M}}(K)$ and choose $\varepsilon > 0$. Then by the definition of $\overline{\dim}_{\mathscr{M}}(K)$ we have that

$$N(K, 2^{-n}) \le C2^{D+\varepsilon}.$$

Let $\mathscr{S}_n$ be a covering of K by fewer than $C2^{D+\varepsilon}$ squares of size 2^{-n}. If Q is a Whitney square with $2^{-n-1} \le \ell(Q) < 2^{-n}$, then choose a point $x \in K$ with $\mathrm{dist}(x, Q) \le \ell(Q)$. Let $S(Q) \in \mathscr{S}_n$ be the square containing Q. Since $S(Q)$ and Q have comparable sizes and there distance apart is at most $\ell(Q)$, we easily see that each $S \in \mathscr{S}_n$ can only be associated to a uniformly bounded number of Q's in the Whitney decomposition, say A.

Let W_n be the number of Whitney squares with size $2^{-n-1} \le \ell(Q) < 2^{-n}$. Then

$$\begin{aligned}
\sum_j \ell(Q_j)^{D+2\varepsilon} &\le \sum_{n=0}^{\infty} W_n 2^{-n(D+2\varepsilon)} \\
&\le \sum_{n=0}^{\infty} CN(K, 2^{-n}) 2^{-n(D+2\varepsilon)} \\
&\le C \sum_{n=0}^{\infty} 2^{n(D+\varepsilon)} 2^{-n(D+2\varepsilon)} \\
&\le C \sum_{n=0}^{\infty} 2^{-n\varepsilon} \\
&< \infty,
\end{aligned}$$

which proves $\kappa \le D + 2\varepsilon$. Taking $\varepsilon \to 0$ gives $\kappa \le D = \overline{\dim}_{\mathscr{M}}(K)$.

Now we assume K has zero area and will prove $\kappa \ge D = \overline{\dim}_{\mathscr{M}}(K)$. As above, let $\varepsilon > 0$ and suppose $\{Q_j\}$ is a Whitney decomposition of $\Omega = K^c$. By the definition of $\overline{\dim}_{\mathscr{M}}(K)$ we have

$$N(K, 2^{-n}) \ge 2^{n(D-\varepsilon)},$$

for infinitely many n. Suppose n_0 is a value where this occurs and let $\mathscr{S} = \{S_k\}$

be a covering of K with dyadic squares of size 2^{-n_0}. Let for each $S_k \in \mathscr{S}$, $\mathscr{C}_k = \{Q_{jk}\}$ be the collection of Whitney squares which intersect S_k. If we assume the Q_j are dyadic, then every square hitting S_k is contained in S_k. Since the area of K is zero, this gives

$$2^{-2n_0} = \text{area}(S_k) = \text{area}(S_k \setminus K) = \text{area}(S_k \cap \Omega) = \sum_{\mathscr{C}_k} \ell(Q_{jk})^2.$$

Therefore,

$$\begin{aligned}
\sum_{\mathscr{C}_k} \ell(Q_{jk})^{D-2\varepsilon} &= \sum_{\mathscr{C}_k} \ell(Q_{jk})^2 \ell(Q_{jk})^{-2+D-2\varepsilon} \\
&\geq \sum_{\mathscr{C}_k} \ell(Q_{jk})^2 \ell(S_k)^{-2+D-2\varepsilon} \\
&= \ell(S_k)^{-2+D-2\varepsilon} \sum_{\mathscr{C}_k} \ell(Q_{jk})^2 \\
&= \ell(S_k)^{-2+D-2\varepsilon} \ell(S_k)^2 \\
&= \ell(S_k)^{D-2\varepsilon} \\
&= 2^{-n_0(D-2\varepsilon)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_j \ell(Q_j)^{D-2\varepsilon} &\geq \sum_k \sum_{\mathscr{C}_k} \ell(Q_{jk})^{D-2\varepsilon} \\
&\geq \sum_k \ell(S_k)^{D-2\varepsilon} \\
&\geq N(K, 2^{-n_0}) 2^{-n_0(D-2\varepsilon)} \\
&\geq 2^{n_0(D-\varepsilon)} 2^{-n_0(D-2\varepsilon)} \\
&= 2^{n_0\varepsilon}.
\end{aligned}$$

Taking $n_0 \to \infty$, we get $\sum_j \ell(Q_j)^{D-2\varepsilon} = \infty$, and hence $\kappa \geq D - 2\varepsilon$. Taking $\varepsilon \to 0$ gives the desired result. □

7.5 Minkowski dimension equals Hausdorff dimension

In this section we will prove Theorem 7.1.3. We will deduce it from

Theorem 7.5.1 *Suppose G is a finitely generated, loxodromic Kleinian group. If area*$(\Lambda(G)) = 0$ *then* $\delta(G) = \overline{\dim}_{\mathscr{M}}(\Lambda(G))$.

As usual, the assumption that G contains no parabolics is not essential for this theorem. It is not known whether the assumption that $\text{area}(\Lambda(G)) = 0$ is necessary. The Ahlfors conjecture states that the limit set of finitely generated Kleinian group is either the whole sphere or has zero area, but so far this has only been proven in special cases (see Canary's paper [**?**] and its references). The following lemma says that to check the convergence or divergence of the Poincaré series, it is enough to consider points in $\Omega(G)$ (rather than points in $\mathbb{B}$).

Lemma 7.5.2 *If G is a Kleinian group, 0 is the center of $\mathbb{B}$ and $z \in \Omega(G) \subset \partial\mathbb{B}$, then*

$$d(g(z)) \sim 1 - |g(0)|, \quad \textit{for all } g \in G$$

with constants that depend on z and G, but not on g.

Proof Choose a ball $B \subset \Omega$ centered at z_0 so that $\text{diam}(B) \leq \frac{1}{2}\text{dist}(z_0, \partial\Omega$. Let $\omega_1 = \omega(0, B, \mathbb{B})$ be the harmonic measure of this ball in $\mathbb{B}$ with respect to the point zero and let ω_2 be the harmonic measure of $\frac{1}{2}B$ with respect to 0. Then by the conformal invariance of harmonic measure, for any $g \in G$, $g(0)$ is the unique point z so that

$$\omega(z, g(B), \mathbb{B}) = \omega_1,$$

$$\omega(z, g(\frac{1}{2}B), \mathbb{B}) = \omega_2.$$

By our choice of B and the Koebe $1/4$ theorem, $g(\frac{1}{2}B) \subset \lambda g(B)$ for some $\lambda < 1$ independent of g. Therefore any $z \in \mathbb{B}$ which satisfies the two equalities above must satisfy

$$|g(z_0) - z| \leq C\,\text{diam}(g(B)),$$

$$1 - |z| \geq \frac{1}{C}\,\text{diam}(g(B)).$$

This proves the lemma. □

Proof of Theorem 7.5.1: We can choose a finite number of points

$$E = \{z_1, \ldots, z_M\} \subset \Omega(G),$$

so that the G-orbit of E is ε-dense in the hyperbolic metric on $\Omega(G)$, i.e., every point of Ω is within distance ε of some point of $G(E) = \cup_{i=1}^{M} \cup_{g \in G} g(z_i)$. Thus each Whitney square Q_j contains at least one point of $G(E)$. Therefore

$$\sum_j \ell(Q_j)^\alpha \leq C \sum_{z \in G(E)} d(z)^\alpha.$$

By Lemma 7.5.2, the infinite series on the right hand side converges for $\alpha > \delta(G)$, hence so does the left hand side, i.e., $\kappa(\lambda) \leq \delta(G)$. Since we already proved

$$\delta(G) = \dim(\Lambda_c(G)) \leq \dim(\Lambda(G)) \leq \overline{\dim}_{\mathscr{M}}(\Lambda(G)),$$

we have the desired equality the desired equality $\kappa(\Lambda) = \delta(G)$.

If $\Lambda(G)$ has zero area, then Lemma 7.4.1 implies

$$\overline{\dim}_{\mathscr{M}}(\Lambda) = \kappa(\Lambda) = \delta(G),$$

as desired. □

Proof of Theorem 7.1.3: Consider two cases. First, if $\dim(\Lambda) = 2$ then

$$2 = \dim(\Lambda) \leq \underline{\dim}_{\mathscr{M}}(\Lambda) \leq \overline{\dim}_{\mathscr{M}}(\Lambda) \leq 2,$$

so all are equal to 2. Secondly, if $\dim(\Lambda) < 2$, then Λ has zero area, so Theorem 7.5.1 applies and gives

$$\dim(\Lambda) \leq \underline{\dim}_{\mathscr{M}}(\Lambda) \leq \overline{\dim}_{\mathscr{M}}(\Lambda) = \delta(G) = \dim(\Lambda_c) \leq \dim(\Lambda),$$

so again, all these numbers are equal. Thus in both cases the Minkowski dimension exists and equals the Hausdorff dimension. □

7.6 Geometrically finite groups

Here is where we need to use the fact stated in Section 7.1 that a loxodromic group is geometrically finite iff the convex core $C(M)$ is compact. We will also want to use the observation that the convex hull of the limit set is the complement in $\mathbb{B}$ of all the hemispheres perpendicular to S^2 whose bases lie in $\Omega(G)$ (this is just the characterization of convex sets in terms of supporting half-spaces).

The important distinction between geometrically finite and geometrically infinite groups is that (at least for loxodromic groups) the convex core $C(M)$ of $M = \mathbb{B}/G$ is compact in the first case and non-compact in the second case. The ordinary set $\Omega(G)$ corresponds to exponentially expanding ends of the manifold M. Think of a Brownian motion on the manifold starting somewhere in the convex core (this is defined by taking the image of a Brownian path on $\mathbb{B}$ under the quotient map). If the convex core is compact, the Brownian path will almost surely leave the core eventually and enter one of the exponentially expanding ends corresponding to a component of $\Omega(G)$. Once in such an end it has a positive chance of going to infinity in this end. Thus by Borel-Cantelli

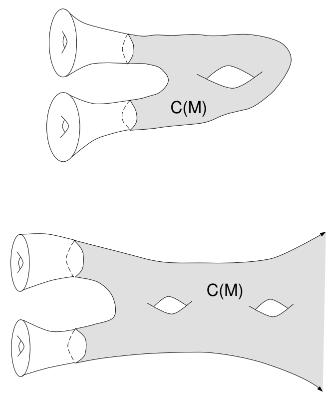


Figure 7.6.1 Geometrically finite and infinite 3-manifolds

almost all Brownian paths are transient, tending to infinity in one of ends corresponding to $\Omega(G)$. See Figure 7.6.

Lifting to the disk, this argument implies that $\Omega(G)$ has full harmonic measure from inside $\mathbb{B}$. Thus $\Lambda(G)$ must have zero area for any geometrically finite group. We shall sharpen this below to show that the limit set actually has dimension less than 2.

If G is geometrically infinite, can a Brownian motion remain in the convex core forever (with positive probability). Using results of Davies and Sullivan, we shall see that this does happen if $\delta(G) < 2$. Thus if G is geometrically infinite and $\delta(G) < 2$ we shall be able to prove that the limit set must have positive area. If $\delta(G) = 2$ then Theorem 7.1.1 says $\dim(\Lambda(G)) = 2$ as well, so we get that $\dim(\Lambda(G)) = 2$ for any (loxodromic) geometrically infinite group.

Theorem 7.6.1 *Suppose G is a finitely generated, loxodromic group and $\Lambda(G) \neq S^2$. If G is geometrically finite then* $\dim(\Lambda(G)) < 2$.

Proof For $z \in \mathbb{B}$ define

$$w(z) = \max_{D \subset \Omega(G)} \omega(z, D, \mathbb{B}),$$

where the max is over all round disks in $\Omega(G)$. Then $C(\Lambda) = \{z : w(z) \leq 1/2\}$. It is also G invariant, so defines a function on M.

Since w never vanishes and $C(M)$ is compact it takes a positive minimum w_0 on $C(M)$. Thus $w(x) \geq w_0$ on all of M (since it is larger than $1/2$ off $C(M)$). Therefore at any point $z \in \mathbb{B}$ there is a disk in Ω with size $\sim 1 - |z|$ and with distance to $z \sim 1 - |z|$.

To see why this implies $\dim(\Lambda(G)) < 2$ consider any square in the plane.

Choose N very large and divide Q in N^4 equal subsquares of side length N^{-2}. Let c be the center of Q and consider the point z in the upper half space distance N^{-1} above c. Each of the subsquares has small harmonic measure from z if N is large. Since $w(z)$ is bounded away from zero, we can deduce that at least one of the subsquares is completely contained in $\Omega(G)$. Throw away this square and repeat the argument on the rest. By induction we see that $\Lambda(G) \cap Q$ can be covered by $(N^4-1)^n$ squares of side length N^{-2n}. This implies

$$\dim(\Lambda(G) \cap Q) \leq \frac{\log N^4 - 1}{\log N^2} < 2.$$

This easily implies $\dim(\Lambda(G)) < 2$. □

7.7 Geometrically infinite groups

In this section we will prove that if G is a finitely generated, geometrically infinite group then the limit set has dimension 2. If $\delta(G) = 2$ the this follows immediately from Theorem 7.1.2.

Therefore we may assume that $\delta(G) < 2$. In this case we will show Λ must have positive area (hence dimension 2).

The heat kernel $K(x,y,t)$ is a positive function on M which gives the distribution of Brownian paths started at x at time t. In other words, if $B(t)$ is a Brownian motion started at x then at time t the probability that the Brownian path is in a set $E \subset M$ is

$$\int_E K(x,y,t)dy.$$

The expected time that a Brownian path spends in the set E is

$$\int_0^\infty \int_E K(x,y,t)dydt.$$

We will apply this by taking a unit neighborhood of $\partial C(M)$ and showing that if we start Brownian motion far enough away from $\partial C(M)$ then the expect time spent in this neighborhood is as small as we wish. We will then deduce that the chance of ever hitting $\partial C(M)$ must also be small.

Proposition 7.7.1 *If $\delta(G) < 2$ and $M = \mathbb{B}/G$ then there exists $\lambda > 0$ so that*

$$0 \leq K(x,y,t) \leq Cvol(B(x,1))^{-1/2} vol(B(y,1))^{-1/2} e^{-\lambda t}.$$

This follows by combining a result of Sullivan [**?**] which relates $\delta(G)$ to the base eigenvalue for the Laplacian of M with a heat kernel estimate of Davies [**?**] with gives a Gaussian upper bound for the heat kernel in terms of the base eigenvalue. However, there is a also a simple direct proof of the proposition above by summing over the group G.

Using this proposition we will now prove:

Theorem 7.7.2 *Suppose G is a finitely generated, loxodromic group and $\delta(G) < 2$. If G is geometrically infinite then $\Lambda(G)$ has positive area.*

Proof Since $\partial C(M)$ is compact, its unit hyperbolic neighborhood U has finite volume. Choose R large and by the first proposition above choose $x \in C(M)$, so that $\text{dist}(x, \partial C(M)) > R$. Suppose $T > 0$ is given. Since Brownian motion on M moves no faster than on $\mathbb{B}$, we can choose R so large that the probability that a Brownian motion started at x every hits U before time T is less than $1/4$.

Since U is compact $\text{vol}(B(y,1))^{-1/2}$ is bounded on U. Thus by the proposition, the expected time that a Brownian motion starting at x will spend in U after time T is

$$C\text{vol}(B(x,1))^{-1/2} \int_T^\infty e^{-\lambda t} dt.$$

Using the fact that the injectivity radius is bounded below we get that the expected time spent in U is at most

$$C \int_T^\infty e^{-\lambda t} dt.$$

By choosing T large we can make this as small as we wish.

Note that the expected time it takes a Brownian motion started at a point y of $\partial C(M)$ to first leave U (i.e., to travel unit distance from $\partial C(M)$) is bounded away from zero independent of the starting point y. This is because the expected time to travel distance 1 in M is greater than or equal the expected time to travel this distance in the covering space $\mathbb{B}$, and this is bounded away from zero. Let $t_0 > 0$ be a lower bound for the expected time to travel distance 1.

Thus the expected time a Brownian motion started at x spends in U is at least the probability that it every hits $\partial C(M)$ times the bound t_0. Now choose T so that the expected time spent in U after time T is less than $t_0/2$. Then the probability that a Brownian motion starting at x_0 hits $\partial C(M)$ after T is $< 1/2$.

Given this T now choose $R = \text{dist}(x, \partial C(M))$ to be so large that the probability of hitting $\partial C(M)$ before time T is $< 1/4$. Thus the chance that a Brownian motion started at x every hits $\partial C(M)$ is $< 3/4$. Thus with probability $> 1/4$ such paths stay in $C(M)$ forever.

Lifting to the ball, $\mathbb{B}$, we see that we can start Brownian motion at a point somewhere in the convex hull of the limit set and that with probability $> 1/4$ the path never leaves the convex hull. Such a path first hits the boundary of the ball in the limit set. Hence the limit set has positive area. □

Exercises

Exercise 7.1 Prove a discrete group G acts discontinuously on $\Omega(G)$.

Exercise 7.2 Prove that the limit set of a Kleinian group contains either $0,1,2$ or infinitely many points.

Exercise 7.3 Prove that if the limit set has interior then it is the whole sphere.

Exercise 7.4 Prove that the limit set is either finite or perfect.

Exercise 7.5 Prove that the limit set of a finitely generated Kleinian group is uniformly perfect [**?**], [**?**].

Exercise 7.6 K is uniformly perfect iff the hyperbolic metric ρ on every component of $\Omega = \overline{\mathbb{C}} \setminus K$ satisfies [**?**], [**?**]

$$d\rho \sim \frac{ds}{\operatorname{dist}(z,K)}.$$

Exercise 7.7 Use Selberg's lemma [**?**] to show that for any finitely generated Kleinian group G there is a subgroup G' with no elliptic elements so that $\Lambda(G) = \Lambda(G')$.

Exercise 7.8 If x is a point which is fixed by a parabolic element of a Kleinian group G then the set of all parabolic elements in G which fix x form a subgroup which is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}^2$. The point x is called a fixed point of rank 1 or rank 2 respectively. Show that if G is non-elementary and $\Lambda(G)$ contains a rank 1 fixed point then $\dim(\Lambda(G)) \geq 1/2$. If G is non-elementary and $\Lambda(G)$ contains a rank 2 fixed point then $\dim(\Lambda(G)) \geq 1$.

Exercise 7.9 Suppose $r > 0$ and let $C = C(r)$ and $N = N(r)$ be the constants in the proof of Theorem 7.1.2. Suppose G is a group of Möbius transformations on $\mathbb{B}$ and suppose there are three disjoint balls B_1, B_2, B_3 of (spherical) radius r and a collection of points $F \subset G(0) \cap A_n$ which satisfy

$$z, w \in F \text{ implies } |z-w| \geq N2^{-n},$$

$$\sum_{z \in F \cap B_i} (1-|z|)^\alpha \geq C^{-2}.$$

Prove $\delta(G) \geq \alpha$.

Exercise 7.10 Suppose $\{G_n\}$ is a sequence of m-generated Möbius groups each with a specific listing of its generators $G_n = \{g_{1n}, \dots g_{mn}\}$. We say that G_n converges *algebraically* to a Kleinian group G with generators $\{g_1, \dots, g_m\}$

if $g_{jn} \to g_j$ for each $1 \le j \le m$, as elements of $\mathrm{PSL}(2,\mathbb{C})$. See [**?**]. If we identify groups with points in $\mathrm{PSL}(2,\mathbb{C})^m$, this is just convergence in the product topology. Use the previous exercise to show:

If $\{G_n\}$ is a sequence of discrete Möbius groups converging algebraically to G, then

$$\liminf_{n\to\infty} \delta(G_n) \ge \delta(G).$$

This says $\delta(G)$ is lower semi-continuous with respect to algebraic convergence.

Strict inequality can occur. For example, it is possible to have a sequence of geometrically infinite groups (which have dimension 2) converge algebraically to a geometrically finite group (which has dimension < 2).

Exercise 7.11 Suppose K is a compact set in the plane. Define the conical limit set $K_c \subset K$ as the set of points $z \in K$ such that there exists $\varepsilon > 0$ and a sequence $z_n \in \Omega = \mathbb{R}^2 \setminus K$ converging to z and with $\mathrm{dist}(z_n, K) \ge \varepsilon |z - z_n|$. Show that $\mathscr{H}^2(K_c) = 0$ for any K.

Exercise 7.12 Show that if $K = 0$ has zero area, then $\dim(K) = \dim(K_c)$. Give an example to show this can fail if K has positive area.

Exercise 7.13 The circle of inversion of a Möbius transformation g is defined to be

$$I(g) = \{z : |g'(z)| = 1\}.$$

Show that g can be written as inversion through this circle followed by a Euclidean similarity.

Exercise 7.14 Let $r(g)$ denote the radius of the circle of inversion of g. If G is a discrete group show

$$\sum_{g\in G} r(g)^4 < \infty.$$

Exercise 7.15 Let $\{C_1, \dots C_n\}$ be disjoint circles with disjoint interiors and let $\{g_1, \dots, g_n\}$ be the reflections through each circle. Let G be the group of transformations generated by these maps. Show that G is discrete and that the limit set is totally disconnected. (G is not Kleinian since it contains orientation reversing elements, but it does have a Kleinian subgroup of index 2. This kind of group is called a classical Schottky group.)

Exercise 7.16 Let $\{C_1, C_2, C_3\}$ be three circles with disjoint interiors so that each is tangent to the other two. let $\{g_1, \dots, g_n\}$ be the reflections through each circle. Let G be the group of transformations generated by these maps. Show that the limit set is a circle.

Exercise 7.17 Let $\{C_1, \dots C_n\}$ be circles with disjoint interiors so that C_j is tangent to C_{j-1} and C_{j+1} (modulo n). let $\{g_1, \dots, g_n\}$ be the reflections through each circle. Let G be the group of transformations generated by these maps. Show that G is discrete and that the limit set is a closed Jordan curve.

Exercise 7.18 Let C_0 be the unit circle and $\{C_1, C_2, C_3, C_4\}$ be circles of radius $1/(\sqrt{2}-1)$ centered at $\pm 1, \pm i$. Let G be the group generated by reflections through these circles. Show that the ordinary set has infinitely many components, all of which are disks (on the sphere).

Exercise 7.19 Show that for every $0 < \alpha \leq 1$ there is a finitely generated Kleinian group with $\dim(\Lambda(G) = \alpha$. (Hint: use reflections through disjoint circles to get totally disconnected limit sets whose dimension varies continuously with the circles. There are limit sets with all possible dimensions in $(0, 2]$, but a theorem of Doyle [**?**] says that one cannot get dimensions close to 2 just by reflecting through disjoint circles.)

Exercise 7.20 Suppose G is a finitely generated Kleinian group with a simply connected invariant component Ω. Prove the following are equivalent (you may assume the group is loxodromic, but this is not necessary):

1. G is not an extended Fuchsian group.
2. $\partial\Omega$ is not a circle.
3. $\partial\Omega$ has infinite 1-dimensional measure.
4. $\dim(\partial\Omega) > 1$.
5. $\dim(\Lambda_c(G)) > 1$.
6. $\delta(G) > 1$.
7. $\partial\Omega$ fails to have a tangent somewhere.
8. $\partial\Omega$ fails to have a inner tangent and all points except possibly the rank 1 parabolic fixed points.
9. The set of inner tangent points of $\partial\Omega$ has zero 1-dimensional measure.
10. Harmonic measure for Ω is singular to 1-dimensional measure (i.e., there is a subset of $\partial\Omega$ of full harmonic measure and 1-dimensional measure zero).
11. Almost every (with respect to harmonic measure) point of $\partial\Omega$ is a twist point.
12. Harmonic measures for distinct components of $\Omega(G)$ are mutually singular.

Exercise 7.21 A finitely generated Kleinian group is called degenerate if $\Omega(G)$ is non-empty, connected and simply connected. prove that if G is degenerate then $\Lambda(G)$ is uniformly wiggley (even if we allow parabolics). (A degenerate group must be geometrically infinite by a theorem of Greenberg [**?**], hence have dimension 2).)

Exercise 7.22 Prove that if G is finitely generated and $\Omega(G)$ is a union of disks, then $\dim(\Lambda(G)) > 1$.

Exercise 7.23 Prove Larman's theorem [**?**]: there is an $\varepsilon > 0$ so that if $\{D_j\}$ is a collection of three or more disjoint open disks and $K = \mathbb{R}^2 \setminus \cup_j D_j$, then $\dim(K) > 1 + \varepsilon$.

Exercise 7.24 Prove that if $\Lambda(G)$ has zero area then $\delta(G) = \dim(\Lambda(G))$.

Exercise 7.25 Show the set of conical limit points of a Kleinian groups either has full measure on the sphere or zero measure. The conical limit set has zero measure iff the Poincaré series converges when $s = 2$. [**?**].

Exercise 7.26 Show that a point $x \in \Lambda(G)$ is in the conical limit set iff the geodesic ray which connects $0 \in \mathbb{B}$ to x returns to some compact subset (in $M = \mathbb{B}/G$) infinitely often.

Exercise 7.27 If $M = \mathbb{B}/G$ and $x \in M$, let E_x denote the set of directions at x which correspond to geodesic rays starting at x which remain bounded forever. Show that $\dim(E_x) = \delta(G)$. (Hint: this is contained in the proof of Theorem 7.1.2.)

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Julia sets and the Mandelbrot set

9

Transcendental dynamics

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