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The Hausdorff dimension of Julia sets of entire functions II

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Abstract

Let $f$ be a transcendental entire function such that the finite singularities of $f^{-1}$ lie in a bounded set. We show that the Hausdorff dimension of the Julia set of such a function is strictly greater than one.

1. Introduction

Let $f: \mathbb{C} \to \mathbb{C}$ denote a nonlinear rational or entire function and $f^n$, $n \in \mathbb{N}$, the $n$th iterate of $f$. The set of normality, $N(f)$, is defined to be the set of points, $z \in \mathbb{C}$, such that the sequence $(f^n)$ forms a normal family in some neighbourhood of $z$. It is easy to see that $N(f)$ is open and has the property of complete invariance under $f$, that is $z \in N(f)$ if and only if $f(z) \in N(f)$. The complement, $J(f)$, of $N(f)$ is known as the Julia set. This set is clearly closed and completely invariant under $f$. More details of these and other basic properties of the sets $N(f)$ and $J(f)$ can be found in [7] and [8].

It was shown by Baker [3, corollary to theorem 1] that if $f$ is a transcendental entire function then $J(f)$ must contain continua and so the Hausdorff dimension of $J(f)$, $\dim J(f)$, lies in the range $1 \leq \dim J(f) \leq 2$. In [10] we constructed a family of transcendental entire functions, $f_K$, which satisfied the following result.

Theorem 1. Given $\delta > 0$, there exists $K_0(\delta)$ such that

$$\dim \text{J}(f_K) < 1 + \delta$$

for all $K \geq K_0$.

It remains open whether there exists a transcendental entire function whose Julia set has dimension equal to one. We put

$$S(f) = \{z: z \text{ is a finite singularity of } f^{-1}\},$$

$$B = \{f: f \text{ is transcendental entire and } S(f) \text{ is bounded}\}.$$  

In this paper we study the class $B$ and obtain the following result.

Theorem 2. If $f$ is a function in $B$ then $\dim J(f) > 1$.

When $J(f) = \mathbb{C}$ the result of Theorem 2 is obvious. If $f \in B$ and $J(f) \neq \mathbb{C}$ then, in order to prove Theorem 2, we construct a measure $\mu_\phi$ on a subset of $J(f)$. We do this by taking a point $z_0 \in J(f)$ and taking sets

$$I_n \subset \{z: f^n(z) = z_0\}.$$
For each \( z \in I_n \) we define
\[
d(z) = \frac{(1+|z_0|^2)/[(f^n)'(z)(1+|z|^2)])}{(1+|z|^2)(1+|z|^2)}
\]
and then put
\[
s = \inf \left\{ t: \sum_{n=1}^{\infty} \sum_{z \in I_n \cap B} d(z)^t < \infty \right\}.
\]

For \( s < t \leq 2 \) we define measures \( \mu_t \) by
\[
\mu_t(B) = c_t \sum_{n=1}^{\infty} \sum_{z \in I_n \cap B} d(z)^t,
\]
for each set \( B \subset \mathbb{C} \), where \( c_t \) is chosen so as to ensure that \( \mu_t(\mathbb{C}) = 1 \). We then take \( \mu_s \) to be a weak limit of the measures \( \mu_t \) as \( t \searrow s \). By choosing the point \( z_0 \) and the sets \( I_n \) carefully we can use the properties of the measure \( \mu_s \) to prove the following two results, which together imply Theorem 2.

**Theorem 3.** The value \( s \) defined by (1.1) lies in the range
\[1 < s \leq 2.\]

**Theorem 4.** The value \( s \) defined by (1.1) satisfies
\[\dim J(f) \geq s.\]

The motivation for looking at measures of this type comes from Sullivan's paper [11] where similar measures are used to obtain results on the Hausdorff dimension of Julia sets of rational functions.

In the last section of the paper we give an example of a function in \( B \) whose Julia set lies in a domain of finite Lebesgue measure.

## 2. Preliminary results

We begin by giving a formal definition of the Hausdorff dimension of a compact set \( E \). If, for each \( \mu > 0 \), we put
\[H_\mu(E) = \lim \inf_{\epsilon \to 0} \sum_i (r_i)^\mu,
\]
where the inf is taken over all possible covers of \( E \) with sets of diameter \( r_i < \epsilon \), then the Hausdorff dimension \( d \) of the set \( E \) is defined to be the unique value satisfying
\[H_\mu(E) = \begin{cases} \infty & \text{for } \mu < d \\ 0 & \text{for } \mu > d. \end{cases}\]

For more details see, for example, [9, p. 220].

We now give a list of the notation that will be used throughout this paper.

(i) \( d(z, w) = |z - w|/[(1 + |z|^2)^{1/2}(1 + |w|^2)^{1/2}] \),

(ii) \( B(w, r) = \{z: |w - z| < r\} \),

(iii) \( \text{diam } B = \sup_{z, w \in B} |z - w| \),

(iv) \( \text{Diam } B = \sup_{z, w \in B} d(z, w) \),

(v) \( f^*(z) = |f'(z)|(1 + |z|^2)/(1 + |f(z)|^2) \),
(vi) \( C(r) = \{ z : |z| = r \} \).
(vii) \( D(r) = \{ z : |z| \geq r \} \).

The next two results are due to Koebe. The first is known as Koebe’s quarter theorem and the second as Koebe’s distortion theorem.

**Lemma 2-1** (See, for example, [5, theorem 2-33]). If \( f \) is univalent in \( B(w, r) \) then \( f(B(w, r)) \supseteq B(f(w), |f'(w)| r/4) \).

**Lemma 2-2** (See, for example, [5, p. 32]). If \( f \) is univalent in \( B(w, r) \) then, for \( 0 < s < r \),

\[
\sup_{z_1, z_2 \in B(w, s)} |f'(z_1)|/|f'(z_2)| \leq L(s/r) = [(r+s)/(r-s)]^4.
\]

We put \( L(1/2) = L \) and apply this result to functions in the class \( B \).

**Lemma 2-3.** If \( S(f) \subset B(0, r_0/2) \), \( |w| = r \geq r_0 \) and \( 4|w|/5 < |z| < 5|w|/4 \) and if we take a branch \( g \) of \( f^{-1} \) that is defined at \( w \) and continue \( g \) analytically along a curve \( \gamma \) which lies within \( \{ t : 4|w|/5 < |t| < 5|w|/4 \} \) such that \( \gamma \) winds at most once around \( 0 \), then

\[
|g'(w)|/L^{26} \leq |g'(z)| \leq L^{26}|g'(w)|.
\]

**Proof.** If \( z = R \exp(i\phi) \) then the circle \( C(r) \) can be covered by the discs \( B_m = B(r \exp(i(\theta_m + \phi)), r/4) \), \( m = 0, 1, 2, \ldots, 24 \), where \( \theta_m = 2\pi m/25 \). We take a point \( z_{24} \in B_{24} \cap B_0 \) and, for each \( 0 \leq m \leq 23 \), we take a point \( z_m \in B_m \cap B_{m+1} \). As \( r \geq r_0 \), each branch \( g \) of \( f^{-1} \) is univalent in \( B(r \exp(i(\theta_m + \phi)), r/2) \). As \( z \in B_0 \) and \( w \in B_M \) for some \( 0 \leq M \leq 24 \), it follows from Lemma 2-2 that

\[
|g'(z)| \leq L|g'(z_0)| \leq L^2|g'(z_1)| \leq \cdots \leq L^{M+1}|g'(z_M)| \leq L^{M+2}|g'(w)| \leq L^{26}|g'(w)|.
\]

Similarly, we find

\[
|g'(z)| \geq |g'(w)|/L^{26}.
\]

**Lemma 2-4.** If \( f \) is a transcendental entire function with \( S(f) \subset B(0, r_0/2) \) and \( |f(0)| < r_0 \) then, if \( |f(z)| > r_0 \), we have

\[
|f'(z)| \geq |f(z)|(|\log |f(z)|| - \log(r_0)|)/[4\pi |z|].
\]

**Proof.** In [6, lemma 1], Eremenko and Lyubich show that, if \( A = \mathbb{C} \setminus B(0, r_0) \), \( G = f^{-1}(A) \) and \( U = \ln(G) \), then there exists a map \( F \) such that, for all \( w \in U \), \( f(e^w) = e^{F(w)} \) and \( |F'(z)| \geq [\Re(F(z)) - \log(r_0)]/(4\pi) \). Lemma 2-4 now follows.

As a direct consequence of this result we have

**Lemma 2-5.** If \( f \) is in \( B \) then there exists \( R_1(f) > 0 \) such that

(i) if \( |f(z)| > R_1(f) \) then \( |f'(z)| > |f(z)||\log |f(z)||/[8\pi |z|] \),
(ii) if \( |z| > R_1(f) \) then, for each branch \( g \) of \( f^{-1} \), \( |g'(z)| < 8\pi |g(z)|/|z||\log |z|\).

Using Lemma 2-2 together with Lemma 2-5 we are able to prove:

**Lemma 2-6.** If \( f \) is in \( B \) then there exists \( R_2(f) \geq R_1(f) \) such that, if \( |f^k(z)| \geq R_2(f) \) for each \( 0 \leq k \leq n \), the branch \( g \) of \( f^{-n} \) that maps \( f^n(z) \) to \( z \) satisfies:

(i) for each \( 0 \leq k \leq n-1 \) and each \( K \geq 4 \), \( f^kg(B(f^n(z), |f^n(z)|/|K|)) \subset B(f^k(z), |f^k(z)|/(4K)) \),
(ii) \( g \) is univalent in \( B(f^n(z), |f^n(z)|/4) \).
where  

(iii) if \( w \in B(f^n(z), |f^n(z)|/8) \) then \( |g'(f^n(z))|/L \leq |g'(w)| \leq L |g'(f^n(z))| \),

(iv) if \( w \in B(f^n(z), |f^n(z)|/8) \) then \( g^*(f^n(z))/(3L) \leq g^*(w) \leq 3Lg^*(f^n(z)) \).

**Proof.** In what follows we write \( R_t \) for \( R^f_t \) and \( R_2 \) for \( R_2(f) \). If \( S(f) \subset B(0, R_2/2) \) then, for each \( 1 \leq k \leq n \), the branch \( g_k \) of \( f^{-1} \) that maps \( f^k(z) \) to \( f^{k-1}(z) \) is univalent in \( B(f^k(z), |f^k(z)|/2) \) and so, from Lemma 2-2,

\[
g_k(B(f^k(z), |f^k(z)|/K)) \subset B(f^{k-1}(z), Lg_k(f^k(z))|f^k(z)|/K) \quad (2-1)
\]

for each \( K \geq 4 \). As \( R_2 \geq R_1 \), it follows from Lemma 2-5 that

\[
|g_k(f^k(z))| \leq 8\pi |f^{k-1}(z)|/(|f^k(z)| \log |f^k(z)|).
\]

If \( R_2 > \exp(32\pi L) \) then

\[
|g_k(f^k(z))| \leq |f^{k-1}(z)|/(4L|f^k(z)|)
\]

and so, from (2-1),

\[
g_k(B(f^k(z), |f^k(z)|/K)) \subset B(f^{k-1}(z), |f^{k-1}(z)|/(4K)).
\]

This proves part (i).

If \( S(f) \subset B(0, R_2/2) \) then \( g_k \) is univalent in \( B(f^k(z), |f^k(z)|/4) \) and so, from part (i), \( g = g_1 g_2 \ldots g_{n-1} g_n \) is univalent in \( B(f^n(z), |f^n(z)|/4) \). This proves part (ii), and part (iii) then follows from Lemma 2-2.

If \( w \in B(f^n(z), |f^n(z)|/4) \) then, from (i), \( g(w) \in B(z, |z|/4) \) and so

\[
\frac{[1 + (3|f^n(z)|/4)^2]}{[1 + (5|g(f^n(z))|/4)^2]} |g'(w)| < g^*(w) < \frac{[1 + (5|f^n(z)|/4)^2]}{[1 + (3|g(f^n(z))|/4)^2]} |g'(w)|.
\]

Combining this with part (iii) gives, for \( w \in B(f^n(z), |f^n(z)|/8) \),

\[
g^*(f^n(z))/(3L) < \frac{9(1 + |f^n(z)|^2)}{25(1 + |g(f^n(z))|^2)^2} |g'(f^n(z))| < g^*(w)
\]

\[
< \frac{25(1 + |f^n(z)|^2)}{9(1 + |g(f^n(z))|^2)} |g'(f^n(z))| \frac{|L|}{3} < 3Lg^*(f^n(z)).
\]

The next two results were proved by Eremenko and Lyubich [6, pp. 5, 6].

**Lemma 2-7.** If \( f \) is in \( B \) then there exists \( R_3(f) \geq R_3(f) \) such that, for each \( R \geq R_3 \), there exists an analytic curve \( \Gamma \) joining a point \( z_R \) to \( \infty \) such that \( |f(z)| = R \) for each \( z \in \Gamma \).

**Lemma 2-8.** If \( f \) is in \( B \) then \( J(f) = \overline{I(f)} \), where \( I(f) = \{ z : f^n(z) \to \infty \} \).

As a direct consequence of Lemma 2-8 we have:

**Lemma 2-9.** If \( f \) is in \( B \) then, for each \( r > 0 \), there exists a point \( z \in J(f) \cap D(r) \) for which

\[
|f^n(z)| \geq r, \text{ for } n \in \mathbb{N}.
\]

We now give a couple of results concerning the basic properties of Julia sets. Put

\[
O^-(w) = \{ z : f^n(z) = w \text{ for some } n \in \mathbb{N} \},
\]

\[
E(f) = \{ w : O^-(w) \text{ is finite} \}.
\]
If \( f \) is nonlinear entire then \( E(f) \) contains at most two points [8, p. 338].

**Lemma 2-10** ([8, p. 356]). If \( U \) is compact, \( U \cap E(f) = \emptyset \), \( z \in J(f) \) and \( V \) is an open neighbourhood of \( z \) then there exists \( N \in \mathbb{N} \) such that, for all \( n \geq N \), we have
\[
f^n(V) \supset U.
\]
As a simple consequence of Lemma 2-10 we have:

**Lemma 2-11.** If \( z \in \mathbb{C} \setminus E(f) \) then \( J(f) \subset O^-(z) \).

Using Lemma 2-10 we are able to prove:

**Lemma 2-12.** If \( f \) is in \( B \), \( z \in J(f) \) and \( |f^n(z)| \geq R_2 \) for each \( n \in \mathbb{N} \) then, given \( K > 0 \), there exists \( n \in \mathbb{N} \) such that
\[
|f^n(z)| \geq K |f^n(z)|.
\]

**Proof.** Suppose that \( |(f^n)'(z)| < K |f^n(z)| \) for each \( n \in \mathbb{N} \). From Lemma 2-6 we know that the branch \( g \) of \( f^{-n} \) that maps \( f^n(z) \) to \( z \) is univalent in \( B(f^n(z), |f^n(z)|/4) \) and so, from Lemma 2-1,
\[
g(B(f^n(z), |f^n(z)|/4) \supset B(z, |f^n(z)|/(16|f^n)'(z)|)) \supset B(z, 1/(16K)).
\]
As \( B(f^n(z), |f^n(z)|/4) \cap B(0, R_2/2) = \emptyset \), it follows that
\[
f^n(B(z, 1/(16K))) \cap B(0, R_2/2) = \emptyset
\]
for each \( n \in \mathbb{N} \), which contradicts Lemma 2-10.

We conclude this section with a result concerning the weak convergence of measures.

**Lemma 2-13** (See, for example, [1, theorem 4-5-1]). Let \( \{\mu_n\} \) be a sequence of finite measures on the Borel sets of a metric space \( \Omega \). If the measures \( \mu_n \) converge weakly to a measure \( \mu \) as \( n \to \infty \) then, for every closed set \( A \subset \Omega \),
\[
\mu(A) \geq \limsup_{n \to \infty} \mu_n(A).
\]

### 3. Construction of the measure \( \mu \)

We take a function \( f \in B \) with \( J(f) \neq \mathbb{C} \) and a value \( R' \) satisfying
\[
R' \geq R_3(f), \quad S(f) \subset B(0, R'/2), \quad \log(9R'/10) > 1600\pi^2L^{2\alpha}C, \tag{3.1}
\]
where \( R_3(f) \) is as defined in Lemma 2-7 and \( C > 4800L^3 \). Now take a point \( z_0 \in [J(f) \cap I(f) \cap D(4R')] \) with \( |f^n(z_0)| \geq 4R' \) for each \( n \in \mathbb{N} \) and \( B(z_0, |z_0|/4) \cap E(f) = \emptyset \). This is possible by Lemma 2-9. We put
\[
R = |z_0|, \quad A = B(z_0, R/C), \quad E = B(z_0, 2R/C).
\]
These definitions remain in force for the whole of Sections 3, 4 and 5.

**Lemma 3-1.** There exist \( w \in \mathbb{C}, \ r > 0 \) such that
(i) \( U = B(w, r) < N(f) \cap A \),
(ii) if, for \( i = 1, 2 \), \( g_i \) is a branch of \( f^{-n(i)} \) satisfying

\[ f^k g_i(U) \subset D(R') \]

for each \( 0 \leq k \leq n(i) \), and \( g_1|_U \neq g_2|_U \) then

\[ g_1(U) \cap g_2(U) = \emptyset. \]

Proof. As \( J(f) \neq \mathbb{C} \), there is a component \( N_0 \) of \( N(f) \). We claim that there exists an open set \( V \subset N(f) \) with \( f^n(V) \cap V = \emptyset \) for each \( n \in \mathbb{N} \). If \( f^n(N_0) \cap N_0 = \emptyset \) for each \( n \in \mathbb{N} \) then we can take \( V \) to be any open set in \( N_0 \). If \( N_0 \) is periodic (i.e. \( f^p(N_0) \subset N_0 \) for some \( p \in \mathbb{N} \)) then there are two possibilities (see, for example, [4, theorem 2-2]).

Case I. If \( f^{np}(z) \to c \), for some constant \( c \), as \( n \to \infty \), for all \( z \in N_0 \), then \( N_0 \) contains a set \( V \) of the required form.

Case II. If \( f^{np}(z) \nrightarrow c \) for all \( z \in N_0 \) then \( N_0 \) is a Siegel disc or a Herman ring and so there exists a component \( N_1 = f^{-p-1}(N_0) \) of \( N(f) \) with \( f(N_1) \subset N_0 \). Thus \( f^n(N_1) \cap N_1 = \emptyset \) for each \( n \in \mathbb{N} \) and so we can take \( V \) to be any open set in \( N_1 \).

We now take a point \( z \in V \setminus E(f) \). It follows from Lemma 2-11 that there exist \( w \in A, \; n' \in \mathbb{N} \) such that \( f^{n'}(w) = z \). It then follows that there exists \( r > 0 \) such that \( U = B(w, r) \subset A \) and \( f^n(U) \subset V \).

If \( f^{-(m+k)}(U) \cap f^{-k}(U) = \emptyset \) for some \( m, k \in \mathbb{N} \), then \( f^n(U) \cap f^{n+m}(U) = \emptyset \) and hence \( V \cap f^{m}(V) = \emptyset \) which is a contradiction. Thus

\[ f^{-(m+k)}(U) \cap f^{-k}(U) = \emptyset \]

for all \( k, m \in \mathbb{N} \).

Finally, suppose that, for some \( n \in \mathbb{N} \), there are two branches \( g_1 \) and \( g_2 \) of \( f^{-n} \) such that

\[ f^k(g_i(U)) \subset D(R') \subset D(R_n(f)), \]

for \( i = 1, 2, \; 0 \leq k \leq n \). As \( U = B(w, r) \subset A \) we must have \( r < |w|/4 \) and so it follows from Lemma 2-6 part (ii) that \( g_1 \) and \( g_2 \) are each univalent in \( U \). Thus

\[ g_1|_U = g_2|_U \quad \text{or} \quad g_1(U) \cap g_2(U) = \emptyset. \]

We put

\[ M(r) = \max_{|z|=r} |f(z)| \]

and define a sequence \( R'_n \) by putting

\[ R'_1 = 2R', \quad R'_{n+1} = M(R'_n). \]

As \( S(f) \subset B(0, R'_1) \) and (see, for example, [2, lemma 2])

\[ S(f^n) = \bigcup_{k=0}^{n-1} f^k(S(f)), \]

it follows that \( S(f^n) \subset B(0, R'_n) \).

We now define two sequences \( R''_n, r_n \) by

\[ R''_n = \max \{ R'_n, r_1(f^n) \}, \quad r_n = |f^n(z_0)|, \]

where \( R_1 \) is as defined in Lemma 2-5. We also take an analytic curve \( \Gamma \) joining a point \( z_\Gamma \) to \( \infty \) such that \( |f(z)| = R = |z_\Gamma| \) for each \( z \in \Gamma \). Such a curve exists by Lemma 2-7.
LEMMA 3.2. There exist \( m, N \in \mathbb{N} \) such that:

(i) \( \Gamma \cap C(r_N) \neq \emptyset \),

(ii) \( G = \{ z : 9r_N/10 \leq |z| \leq 10r_N/9 \} \subset D(2R_m^\infty) \),

(iii) there exists a branch \( g_0 \) of \( f^{-1} \) such that

\[
w_0 = g_0(z_0) \in g_0(E) \subset G' = \{ z : 17r_N/18 \leq |z| \leq 19r_N/18 \},
\]

(iv) for each \( z \in G \) there exist two distinct points \( w_1, w_2 \in G' \) such that \( f^m(w_1) = f^m(w_2) = z \).

Proof. As \( z_0 \in A \subset B(z_0, R/4) \subset C \setminus E(f) \), it follows from Lemma 2.11 that there exist two distinct points \( z_1, z_2 \in O^{-}(z_0) \cap A \). As \( z_0 \in J(f) \), we must also have \( z_1, z_2 \in J(f) \) and so it follows from Lemma 2.10 that there exist \( r', m > 0 \) such that

\[
B(z_1, r') \cap B(z_2, r') = \emptyset, \quad B(z_i, r') \subset A, \quad f^m(B(z_i, r')) \supset A,
\]

for \( i = 1, 2 \).

As \( z_0 \in I(f) \), there exists \( N \in \mathbb{N} \) such that

\[
r_N = |f^N(z_0)| > \max \{ 20R_m^\infty, |z_R| \},
\]

and so these values of \( N, m \) satisfy (i) and (ii).

We now take a point \( z' \in \Gamma \cap C(r_N) \) so that \( w' = f(z') \in C(R) \). If we take \( \tilde{C}(R) \) to be the shortest segment of \( C(R) \) joining \( w' \) to \( z_0 \) and \( g_0 \) to be the branch of \( f^{-1} \) that maps \( w' \) to \( z' \) then we can continue \( g_0 \) univalently along \( \tilde{C}(R) \) to \( E \) and so, from Lemma 2.3, for \( w \in \tilde{C}(R) \cup E \) we have

\[
|g_0'(w)| \leq L^{26} |g_0(w')|.
\]

As each \( w \in E \) can be joined to \( w' \) by a curve of length less than \( 2\pi R \) which lies in \( \tilde{C}(R) \cup E \), we have, for \( w \in E \),

\[
|g_0(w) - z'| < 2\pi RL^{26} |g_0(w')|.
\]

As \( |w'| = R > R_1(f) \), it follows from Lemma 2.5 that

\[
|g_0'(w')| < 8\pi |g_0(w')|/\left( |w'| \log |w'| \right) = 8\pi |z'|/(R \log R) = 8\pi r_N/(R \log R)
\]

and so, as \( R > R' \), it follows from (3.1) that, for each \( w \in E \),

\[
|g_0(w) - z'| < 16\pi^{2}RL^{26}r_N/(R \log R) < r_N/36. \tag{3.2}
\]

Thus \( w_0 = g_0(z_0) \in g_0(E) \subset B(z', r_N/36) \subset G' \) which proves (iii).

We now take \( g_i \) to be the branch of \( f^{-m} \) that maps \( A \) univalently into \( B(z_i, r') \subset A \) and put \( h_i = g_0g_if \), for \( i = 1, 2 \). It follows from (3.2) that \( h_i \) is a branch of \( f^{-m} \) satisfying

\[
h_i(w_0) \in g_0(A) \subset B(z', r_N/36) \tag{3.3}
\]

for \( i = 1, 2 \).

If \( z \in G \) then there exists a simple curve \( C \subset G \) of length less than \( 2\pi r_N \) which joins \( w_0 = g_0(z_0) \) to \( z \). For each \( w \in C \) we have

\[
4|w_0|/5 < (9/10)R_m^{\infty} |w_0| \leq |w| \leq (10/9)R_m^{\infty} |w_0| < 5 |w_0|/4.
\]

As \( |w_0| > 2R_m^{\infty} > R_1(f^m) \) and \( S(f^m) \subset B(0, R_m^\infty) \subset B(0, R_m^\infty) \), it follows from Lemma 2.3 and Lemma 2.5 that we can continue \( h_i \) univalently along \( C \) and, for each \( w \in C \),

\[
|(h_i)'(w)| \leq L^{26} |(h_i)'(w_0)| \leq 8\pi L^{26} |h_i(w_0)|/|w_0| \log |w_0|.
\]
As $h_i(w_0), w_0 \in G$ and $r_N \geq R'$ it follows from (3.1) that
\[ |h_i(z) - h_i(w_0)| \leq 16\pi^2 L^2 r_N[10r_N/9]/[(9r_N/10)\log(9r_N/10)] < r_N/36. \]
Together with (3.3) this shows that
\[ h_i(z) \in B(z', r_N/18) \subset G' \]
for $i = 1, 2$. As $f^m h_i(z) = z$, this proves (iv).

We now put
\[ I_n = \{g(z_0) \in A \cup G : g \text{ is a branch of } f^{-n} \text{ with } f^k g(E) \subset D(R') \text{ for each } 0 \leq k \leq n\} \]
and
\[ I = \bigcup_{n=1}^{\infty} I_n. \]

**Lemma 3.3.** (i) If $|f^{mk}(z)| \geq R'_m$ for each $0 \leq k \leq n$ then $|f^k(z)| \geq 2R'$ for each $0 \leq k \leq nm$;
(ii) If, in addition, $z \in A \cup G$ and $f^{nm+1}(z) = z_0$ then $z \in I_{nm+1}$.

**Proof.** (i) If, for some $0 \leq k < n$, $0 < p < m$, we have $|f^{mk+p}(z)| < 2R'$ then $|f^{mk+k+1}(z)| < \max_{|w|=2R'} |f^{mk+p}(w)| \leq R'_m - p+1 \leq R'_m$, which is a contradiction. As $R'_m > 2R'$, we have
\[ |f^k(z)| \geq 2R' \]
for $0 \leq k \leq nm$.

(ii) If $f^{nm+1}(z) = z_0$ then, taking $g$ to be the branch of $f^{-(nm+1)}$ that maps $z_0$ to $z$ and noting that
\[ E \subset B(z_0, |z_0|/4), \quad |f^k(z)| \geq 2R' \geq R_z(f) \]
for $0 \leq k \leq nm + 1$, it follows from Lemma 2.6 part (i) that
\[ f^k g(E) \subset B(f^k g(z_0), |f^k g(z_0)|/16) = B(f^k(z), |f^k(z)|/16) \subset D(R') \]
and hence $z \in I_{nm+1}$ as claimed.

**Lemma 3.4.** For each $n, k \in \mathbb{N}$, $I_n \cap I_{n+k} = \emptyset$.

**Proof.** If, for some $z \in \mathbb{C}, n, k \in \mathbb{N}$, we have $z \in I_n \cap I_{n+k}$ then
\[ f^n(z) = f^{n+k}(z) = z_0 \]
and hence
\[ f^k(z_0) = f^{k+n}(z) = z_0 \]
which implies that $f^{nk}(z_0) = z_0$ for each $p \in \mathbb{N}$. This contradicts the fact that $f^p(z) \to \infty$ as $p \to \infty$.

For each $z \in I_n$ we put
\[ d(z) = \frac{1}{(f^n)^r(z)}. \]
It follows from Lemma 3.4 that $d(z)$ is a single-valued function. We then take $s$ to be the value defined by
\[ s = \inf\{t : \sum_{z \in I_n} d(z)^t < \infty\}. \quad (3.4) \]

**Lemma 3.5.** The value $s$ defined by (3.4) satisfies $0 < s \leq 2$. 
Proof. Take a set \( U \) which satisfies the conditions of Lemma 3.1. If \( z \in I_n \), for some \( n \in \mathbb{N} \), then \( z = g(z_0) \) for some branch \( g \) of \( f^{-n} \). We put
\[
U(z) = g(U).
\]

As \( R' \geq R_4(f) \) and \( U \subset E \subset B(z_0, |z_0|/8) \), it follows from part (iv) of Lemma 2.6 that
\[
\text{Diam}(U(z)) \leq 3Lg^n(z_0) \text{Diam}(U) = 3Ld(z) \text{Diam}(U)
\]
and so there exists \( K > 0 \) such that, for each \( z \in I \), the spherical area of \( U(z) \) is at most \( K(d(z))^2 \).

If \( z_1, z_2 \in I \) and \( z_1 \neq z_2 \), then it follows from Lemma 3.1 that \( U(z_1) \cap U(z_2) = \emptyset \).

As the area of the sphere is finite it follows that
\[
K \sum_{z \in I} d(z)^2 < \infty
\]
and so \( s \leq 2 \).

From Lemma 3.2 part (iii) we know that there exists \( w_0 \in G' \) with \( f(w_0) = z_0 \). We put
\[
J_n = \{ z : f^{mn}(z) = w_0, f^{mr}(z) \in G' \text{ for } 0 \leq r < n \}.
\]

As \( G' \subset D(R_m^n) \subset D(R'_m) \), it follows from Lemma 3.3 that \( J_n \subset I_{m_{n+1}} \). If we take
\[
K > \max \{ 1, \sup_{z \in G'} (f^m)^n(z) \}
\]
and put
\[
K' = f^s(w_0)
\]
then, for \( z \in J_n \), we have
\[
d(z) = \frac{1}{(f^{mn+1})^n(z)} = \frac{1}{f^s(w_0)} \prod_{k=0}^{n-1} \frac{1}{(f^m)^n(f^{mk}(z))} \geq 1/(K'K^n).
\]

From Lemma 3.2 part (iv) we know that \( J_n \) contains at least \( 2^n \) points and so
\[
\sum_{z \in I} d(z)^t \geq \sum_{n=0}^{\infty} \sum_{z \in I_{m_{n+1}}} d(z)^t \geq \sum_{n=0}^{\infty} \sum_{z \in J_n} d(z)^t \geq \sum_{n=0}^{\infty} 2^n/(K'K^n)^t.
\]

If \( t \leq \log 2/\log K \) then \( 2 \geq K' \) and so \( \sum_{z \in I} d(z)^t = \infty \). As \( K > 1 \), it follows that
\[
s \geq \log 2/\log K > 0.
\]

For \( 0 < t \leq 2 \), \( t \neq s \), we define functions \( \mu_t \) such that
\[
\mu_t(B) = c_t \sum_{z \in I \cap B} d(z)^t
\]
for each set \( B \subset \mathbb{C} \). If \( t < s \) we put \( c_t = 1 \) and if \( t > s \) we take \( c_t \) to be the value which gives \( \mu_t(\mathbb{C}) = 1 \). Clearly, for \( t > s \), \( \mu_t \) is a measure which is supported on \( A \cup G \). We now take a weak limit of the measures \( \mu_t \) as \( t \searrow s \) to give a measure \( \mu_s \) which will be supported on \( \overline{A} \cup G \) with \( \mu_s(\overline{A} \cup G) = 1 \).

We conclude this section with some results which will be useful for the proofs of Theorems 3 and 4.

**Lemma 3.6.** (i) There exists \( \epsilon > 0 \) such that, for \( 0 < t \leq 2 \), \( \mu_t(\overline{A}) \geq \epsilon \mu_t(G) \),
(ii) for \( s \leq t \leq 2 \), \( \mu_t(\overline{A}) \geq \epsilon/(1 + \epsilon) \).
Proof. We claim that, for each \( z \in G \), there exists \( w \in A \) such that \( z = f^N(w) \) and \( |f^p(w)| \geq 2R' \) for each \( 0 \leq p \leq N \). Let \( g \) be the branch of \( f^{-1} \) that maps \( f^N(z_0) \) to \( f^{N-1}(z_0) \) and continue \( g \) univalently to \( z \in G \) along a simple curve \( C \) lying in \( G \) of length less than \( 2\pi r_N \). From Lemma 3-2 part (ii) we know that \( G \subset D(2R'_m) \subset D(2R') \). As \( S(f) \subset B(0,R'/2) \) and \( R' \geq R_1(f) \), it follows from Lemma 2-3 and Lemma 2-5 that, for each \( w \in C \),

\[
|g'(w)| \leq L^{26} \frac{|f^{N-1}(z_0)|}{|f^N(z_0)|} \leq \frac{8\pi L^{28} r_{N-1}}{|f^N(z_0)| \log r_N} = \frac{8\pi L^{28} r_{N-1}}{r_N \log r_N}
\]

and so, as \( \log r_N > \log R' > 16\pi^2 L^{28} \),

\[
g(z) \in B \left( f^{N-1}(z_0), \frac{16\pi^2 L^{28} r_{N-1} r_N}{r_N \log r_N} \right) \subset B( f^{N-1}(z_0), r_{N-1} / C). \tag{3-5}
\]

As \( |f^p(z_0)| \geq 4R' \geq R_4(f) \) for \( 0 \leq p \leq N-1 \), it follows from Lemma 2-6 that, taking \( h \) to be the branch of \( f^{-(N-1)} \) that maps \( f^{N-1}(z_0) \) to \( z_0 \),

\[
f^p h(B( f^{N-1}(z_0), r_{N-1} / C) \subset B(f^p(z_0), r_p / C) \subset B(f^p(z_0), r_p / 4) \tag{3-6}
\]

for \( 0 \leq p \leq N-1 \).

From (3-5) together with (3-6) in the case \( p = 0 \) we see that there exists \( w \in A \) with \( f^N(w) = z \). As \( |f^p(z_0)| \geq 4R' \) we see from (3-6) that \( |f^p(w)| > 2R' \) for \( 0 \leq p \leq N-1 \). As \( f^N(w) = z \in G \subset D(2R') \) our claim is proved.

Now suppose that \( z \in I_n \cap G \) for some \( n \in \mathbb{N} \). Take \( w \in A \) such that \( f^N(w) = z \) and \( |f^p(w)| > 2R' \) for \( 0 \leq p \leq N \) and let \( g \) be the branch of \( f^{-(n+N)} \) that maps \( z_0 \) to \( w \). As \( z \in I_n \) we know that

\[
f^{N+k}(z_0) \in f^{N+k}(E) \subset D(R') \tag{3-7}
\]

for \( 0 \leq k \leq n \) and so, as \( E \subset B(z_0, |z_0|/4) \), it follows from Lemma 2-6 that

\[
f^N g(E) \subset B(f^N g(z_0), |f^N g(z_0)|/4) = B(f^N(w), |f^N(w)|/4).
\]

As \( |f^p(w)| > 2R' \) for \( 0 \leq p \leq N \), it follows from Lemma 2-6 that

\[
f^p g(E) \subset B(f^p(w), |f^p(w)|/4) \subset D(R')
\]

for \( 0 \leq p \leq N \). Together with (3-7) this shows that \( w \in I_{n+N} \).

Clearly

\[
d(w) = d(z)/(f^N)^t(w)
\]

and so, taking

\[
K = \max \{ 1, \sup_{w \in A} (f^N)^t(w) \},
\]

we have, for \( 0 < t \leq 2, t \neq s \),

\[
\mu_t(\overline{A}) = \mu_t(A) = c_t \sum_{w \in I \cap A} d(w)^t \geq c_t \sum_{f^N(w) \in I \cap G} d(w)^t \geq c_t \sum_{z \in I \cap G} d(z)^t/K^t = \mu_t(G)/K^t \geq \mu_t(G)/K^2.
\]

This proves part (i) for \( t \neq s \).
If $2 > t > s$ then

$$\mu_t(C) = \mu_t(A) + \mu_t(G) = 1$$

and so, from (i),

$$1 \leq \mu_t(A) + \mu_t(A)/\epsilon = \mu_t(A)/(\epsilon + 1)/\epsilon.$$ 

This proves part (ii) for $t > s$. It now follows from Lemma 2.13 that the results must also be true for $t = s$.

**Lemma 3.7.** There exist $K_1 = K_1(f) > 0, K_2 = K_2(f) > 1$ such that, for each $n \in \mathbb{N}$ and each $z \in I_n$,

$$d(z) < K_1/(K_2)^n.$$ 

**Proof.** If $z \in I_n$ then $|f^k(z)| \geq R' \geq R_1(f)$, for $0 \leq k \leq n$, and so, from Lemma 2.5,

$$1/d(z) = (f^n)^*(z) = \frac{1 + |z|^2}{1 + |f^n(z)|^2}|(f^n)'(z)|$$

$$\geq \frac{(1 + |z|^2)|f^n(z)|}{(1 + |f^n(z)|^2)} \prod_{k=1}^{n} \frac{|\log |f^k(z)||}{8\pi} \geq \frac{(1 + |z|^2)|z_0|}{(1 + |z_0|^2)|z|} \left(\frac{\log R'}{8\pi}\right)^n.$$ 

As $I_n \subset G \cup A$ we must have $|z| > R/2$ and so, as $\log R' > 8\pi$, the result follows.

**Lemma 3.8.** Given $K > 0, 0 < a < s$, there exist infinitely many values of $n \in \mathbb{N}$ for which

$$\sum_{z \in I_n \cap A} d(z)^{t-a} \geq K.$$ 

**Proof.** We note that

$$\mu_{s-a/2}(A) + \mu_{s-a/2}(G) = \sum_{z \in I \cap A} d(z)^{t-a/2} + \sum_{z \in I \cap G} d(z)^{t-a/2} = \sum_{z \in I} d(z)^{t-a/2} = \infty.$$ 

From Lemma 3.6 we know that

$$\mu_{s-a/2}(G) \leq \mu_{s-a/2}(\overline{A})/\epsilon = \mu_{s-a/2}(A)/\epsilon$$

and so we must have

$$\mu_{s-a/2}(A) = \sum_{n=1}^{\infty} \sum_{z \in I_n \cap A} d(z)^{t-a/2} = \infty. \quad (3.8)$$

If there are only finitely many $n \in \mathbb{N}$ for which

$$\sum_{z \in I_n \cap A} d(z)^{t-a} \geq K$$

then there must exist $K' > K$ such that

$$\sum_{z \in I_n \cap A} d(z)^{t-a} < K'$$

for each $n \in \mathbb{N}$. It follows from Lemma 3.7 that, for each $n \in \mathbb{N}$,

$$\sum_{z \in I_n \cap A} d(z)^{t-a/2} \leq \sup_{z \in I_n \cap A} d(z)^{a/2} \sum_{z \in I_n \cap A} d(z)^{t-a} < K'[K_1/(K_2)^n]^{a/2},$$
where $K_1 > 0$, $K_2 > 1$, and so

$$\sum_{n=1}^{\infty} \sum_{z \in f^n \cap A} d(z)^{s-a/2} < K'(K_1)^{a/2} \sum_{n=1}^{\infty} (1/K_2)^{a/2} < \infty$$

which contradicts (3·8).

We put

$$\mathcal{A}_n = \{g(A) : g \text{ is a branch of } f^{-n} \text{ with } g(z_0) \in I_n\}$$

and

$$\mathcal{E}_n = \{g(E) : g \text{ is a branch of } f^{-n} \text{ with } g(z_0) \in I_n\}$$

**Lemma 3·9.** There exists $n_0 \in \mathbb{N}$ such that

$$\text{diam}(A_n) < \text{diam}(E_n) < 1$$

for each $A_n \in \mathcal{A}_n$, $E_n \in \mathcal{E}_n$, with $n \geq n_0$.

**Proof.** Let $g$ denote the branch of $f^{-n}$ which maps $E$ to $E_n$ and $A$ to $A_n$. As $g(z_0) = z' \in I_n$ we have

$$|f^kg(z_0)| > R'$$

for $0 \leq k \leq n$ and so, from Lemma 2·6,

$$\text{Diam}(A_n) < \text{Diam}(E_n) \leq \sup_{z \in E} g^\chi(z) \text{ Diam}(E)$$

$$\leq 3Lg^\chi(z_0) \text{ Diam}(E) = 3Ld(z') \text{ Diam}(E).$$

From Lemma 3·7 we have

$$d(z') < K_1/(K_2)^n$$

and so

$$\text{Diam}(A_n) < \text{Diam}(E_n) < 3LK_1 \text{ Diam}(E)/(K_2)^n. \tag{3·9}$$

As $E_n \cap I_n \neq \emptyset$ and $I_n \subset A \cup G \subset \{z : |z| \leq 10r_N/9\}$ we see from (3·9) that, for large $n$,

$$E_n \subset \{z : |z| \leq 11r_N/9\}$$

and hence, for these values of $n$, there exists $K > 0$ such that

$$\text{diam}(A_n) < \text{diam}(E_n) < K \text{ Diam}(E_n) < 3KLK_1 \text{ Diam}(E)/(K_2)^n$$

and, as $K_2 > 1$, the result follows.

**Lemma 3·10.** There exists $K_3 > 0$ such that, for each $A_n \in \mathcal{A}_n$ with $n \geq n_0$,

$$K_3 \mu_3(\mathcal{A}_n) > (\text{diam}(A_n))^s.$$

**Proof.** If $A_n \in \mathcal{A}_n$ then, for each $z \in I_p \cap A$, there exists $w \in A_n$ such that $f^n(w) = z$. We claim that $w \in I_{n+p}$. If $h$ denotes the branch of $f^{-p}$ that maps $z_0$ to $z \in A$ and $g$ denotes the branch of $f^{-n}$ that maps $z_0$ to $z' \in A_n$, then $gh$ is the branch of $f^{-(n+p)}$ that maps $z_0$ to $w$. As

$$f^{n+k}gh(z_0) \in f^{n+k}g(E) = f^kh(E) \subset D(R')$$

for $0 \leq k \leq p$ and as $z \in A$, it follows from Lemma 2·6 that

$$f^kg(E) = h(E) \subset B(h(z_0), 2|h(z_0)|/(4C)) = B(z, |z|/(2C)) \subset E.$$

As $g(z_0) \in I_n$ it follows that, for $0 \leq k \leq n$,

$$f^kg(E) \subset f^k g(E) \subset D(R').$$
Hausdorff dimension of Julia sets of entire functions II

Together with (3.10) this shows that \( w \in I_{n+p} \).

As \( f^n(A_n) = A \subset B(z_0, |z_0|/8) \) and \( g(z_0) = z' \in I_n \), it follows from Lemma 2-6 that, for each \( w \in A_n \),

\[
(f^n)^\prime(z')/(3L) \leq (f^n)(z) \leq 3L(f^n)^\prime(z')
\]

and so

\[
\text{Diam}(A) \geq (f^n)^\prime(z') \text{Diam}(A_n)/(3L).
\]

If \( w \in A_n \) and \( f^n(w) = z \in I \) then, from (3.11) and (3.12),

\[
d(w) = d(z)/(f^n)^\prime(z) \geq d(z)/[3L(f^n)^\prime(z')] \geq d(z) \text{Diam}(A_n)/[9L^2 \text{Diam}(A)]
\]

and so, for \( t > s \),

\[
\mu_t(A_n) = c_t \sum_{w \in I_n} d(w)^t \geq c_t \sum_{w \in I_n} d(w)^t \geq \left( \frac{\text{Diam}(A_n)}{9L^2 \text{Diam}(A)} \right)^t c_t \sum_{w \in I_n} d(w)^t
\]

\[
= (\text{Diam}(A_n))^t \mu_t(A)/[9L^2 \text{Diam}(A)]^t.
\]

From Lemma 3-6 we have

\[
\mu_t(A_n) \geq (\text{Diam}(A_n))^t \epsilon/[9L^2 \text{Diam}(A)]^t(1+\epsilon). \tag{3.13}
\]

If \( n \geq n_0 \) then, from Lemma 3-9, \( \text{diam}(A_n) < 1 \). As \( A_n \cap I_n \neq \emptyset \) and \( I_n \subset A \cup G \), it follows that, for some \( K' > 0 \), we have

\[
\text{Diam}(A_n) \geq \text{diam}(A_n)/K'
\]

and so, taking the limit of (3.13) as \( t \searrow s \), we see from Lemma 2-13 that

\[
\mu_s(A_n) \geq (\text{diam}(A_n))^s \epsilon/[9L^2 K' \text{Diam}(A)]^s(1+\epsilon)].
\]

4. Proof of Theorem 3

Recall that \( \Gamma \) is an analytic curve joining a point \( z_R \) to \( \infty \) and that \( |f(z)| = R \) for each \( z \in \Gamma \).

Lemma 4.1. For each \( r > 0 \), the length of \( \Gamma \cap B(0, r) \) is finite.

Proof. It is clear that there are only a finite number of branches \( g_1, g_2, \ldots, g_n \) of \( f^{-1} \) satisfying \( g_i(z_0) \in B(0, 2r) \). We cut \( C(R) \) at \( z_0 \) and, for \( 1 \leq i \leq n \), continue \( g_i \) univalently in an anticlockwise direction around the cut curve \( C(R) \). The length of \( \bigcup_{i=1}^n g_i(C(R)) \) is clearly finite.

Now suppose that \( z \in \Gamma \cap B(0, r) \). There exists \( w' \in C(R) \) and a branch \( g \) of \( f^{-1} \) such that \( z = g(w') \). We continue \( g \) analytically in a clockwise direction from \( w' \) along \( C(R) \) to \( z_0 \). As \( R > 4R' \geq R(f) \) and \( S(f) \subset B(0, R'/2) \), it follows from Lemma 2-3 and Lemma 2-5 that, for each point \( w \) on this segment of \( C(R) \),

\[
|g'(w)| \leq L^{26} |g'(w')| \leq \frac{8\pi L^{26} |g(w')|}{|w'| \log |w'|} \leq \frac{8\pi L^{26} R}{R \log R}
\]

and so, as \( \log R > \log R' > 16\pi^2 L^{26} \),

\[
|g(w') - g(z_0)| \leq \frac{16\pi^2 L^{26} R}{R \log R} < r.
\]
As \( z = g(w') \in B(0, r) \), it follows that \( g(z_0) \in B(0, 2r) \) and hence \( g = g_i \) for some \( 1 \leq i \leq n \). Thus \[
\Gamma \cap B(0, r) \subseteq \bigcup_{i=1}^{n} g_i(C(R))
\]
and hence has finite length.

We take \( m, N \) to be values which satisfy Lemma 3-2.

**Lemma 4-2.** For each \( n \in \mathbb{N} \cup \{0\} \) there exist \( 2^n \) curves \( \gamma_{n,i} \), \( 1 \leq i \leq 2^n \), each of which joins \( C(19r_N/18) \) to \( \infty \) and lies in \( D(17r_N/18) \) with

(i) \( \gamma_{0,1} \subseteq \Gamma \),
(ii) for each \( 0 \leq r \leq n \),
\[
f^m(\gamma_{n,i}) \subseteq \gamma_{n-r,j}
\]
for some \( 1 \leq j \leq 2^{n-r} \),
(iii) \( \gamma_{n,i} \cap \gamma_{n,j} = \emptyset \), if \( i \neq j \).

**Proof.** We begin by considering the case \( n = 0 \). From Lemma 3-2 part (i) we have \( \Gamma \cap C(r_N) \neq \emptyset \) and so \( \Gamma \) joins \( C(19r_N/18) \) to \( \infty \). If there does not exist a segment \( \gamma_{0,1} \subseteq \Gamma \cap N(19r_N/18) \) joining \( C(19r_N/18) \) to \( \infty \) then the length of
\[
\Gamma \cap \{ z: 17r_N/18 \leq |z| \leq 19r_N/18 \}
\]
must be infinite which contradicts Lemma 4-1.

We now assume that the result is true for \( n - 1 \) and, for some \( 1 \leq i \leq 2^{n-1} \), consider the curve \( \gamma_{n-1,i} \). We take \( z' \in C(19r_N/18) \) and note from Lemma 3-2 part (iv) that there exist two points \( w_1, w_2 \in G' \) such that \( f^m(w_1) = f^m(w_2) = z' \). Let \( h_k \) denote the branch of \( f^{-m} \) that maps \( z' \) to \( w_k \). As we know from Lemma 3-2 part (ii) that \( S(f^m) \subseteq B(0, R'_m) \subseteq B(0, R''_m) \subseteq B(0, 9r_N/10) \) we can continue \( h_k \) univalently along \( \gamma_{n-1,i} \). As \( w_1 \in G' \), the curve \( \Gamma_{n,2i} = h_1(\gamma_{n-1,i}) \) must join \( C(19r_N/18) \) to \( \infty \). If \( \Gamma_{n,2i} \) does not contain a curve \( \gamma_{n,2\ell} \subset D(17r_N/18) \) which joins \( C(19r_N/18) \) to \( \infty \) then the length of
\[
\Gamma_{n,2i} \cap \{ z: 17r_N/18 \leq |z| \leq 19r_N/18 \}
\]
must be infinite. As \( f^{nm}(\Gamma_{n,2i}) \subset \gamma_{0,1} \subset \Gamma \), it follows that the length of \( \Gamma \cap B(0, r) \), where
\[
r = \sup_{|z| \leq 19r_N/18} |f^{nm}(z)|,
\]
is infinite, which contradicts Lemma 4-1. In the same way we can show that \( \Gamma_{n,2i-1} = h_2(\gamma_{n-1,i}) \) contains a curve \( \gamma_{n,2i-1} \subset N(17r_N/18) \) which joins \( C(19r_N/18) \) to \( \infty \).

Recall that, for each \( n \in \mathbb{N} \),
\[
\mathcal{A}_n = \{ g(A) : g \text{ is a branch of } f^{-n} \text{ with } g(z_0) \in I_n \}.
\]
For each \( n \in \mathbb{N} \), \( 1 \leq i \leq 2^n \), we put
\[
F_{n,i} = \{ z : z \in \gamma_{n,i} \cap G, f^{nm+1}(z) = z_0 \},
\]
\[
H_{n,i} = \{ g(A) : g \text{ is a branch of } f^{-(nm+1)} \text{ with } g(z_0) \in F_{n,i} \}.
\]

**Lemma 4-3.** For each \( n \in \mathbb{N} \), \( 1 \leq i \leq 2^n \), \( F_{n,i} \subset I_{mn+1} \) and hence \( H_{n,i} \subset \mathcal{A}_{mn+1} \).
Hausdorff dimension of Julia sets of entire functions II

Proof. If \( z \in F_{n, t} \) then, from Lemma 4-2 and Lemma 3-2, for \( 0 \leq r \leq n \) there exists \( 1 \leq j \leq 2^{n-r} \) such that

\[
\gamma_{n-r, j} \subset D(17r_N/18) \subset D(R_m) \subset D(R'_m)
\]

and so, from Lemma 3-3, \( z \in I_{mn+1} \).

**Lemma 4-4.** There exists \( K_4 > 0 \) such that, for each \( n \in \mathbb{N} \), \( 1 \leq i \leq 2^n \),

\[
\sum_{A} \text{diam}(A_{mn+1}) \geq K_4.
\]

**Proof.** From Lemma 4-2 we know that, for each \( n \in \mathbb{N} \), \( 1 \leq i \leq 2^n \), \( \gamma_{n, i} \) contains a segment \( \gamma_{n, i} \subset C(19r_N/18) \) to \( C(39r_N/36) \) such that

\[
\gamma_{n, i} \subset \{ z : 17r_N/18 < |z| < 39r_N/36 \}.
\]

We take a point \( z \in \gamma_{n, i} \) and let \( h \) denote the branch of \( f^{-1} \) that maps \( w' \subset C(R) \) to \( f^{mn}(z) \). We cut \( C(R) \) at \( -z_0 \) and continue \( h \) univalently to the whole of the cut curve \( C(R) \) and to \( A \). If \( -z_0 = R \exp(i\phi) \) then we define \( h(-z_0) \) by

\[
h(-z_0) = \lim_{\theta \to \phi} h(R \exp(i\theta)).
\]

It follows from Lemma 2-3 that, for each \( w \in C(R) \cup A \),

\[
|h'(w)| \leq L^{26} |h'(w')|
\]

and hence

\[
h(C(R) \cup A) \subset B(h(w'), 2\pi R L^{26} |h'(w')|).
\] (4-1)

From Lemma 2-5 we have

\[
|h'(w')| < \frac{8\pi |h(w')|}{|w'| \log |w'|}
\]

and so, as \( \log R > \log R' > 1600n^2 L^{26} \),

\[
2\pi R L^{26} |h'(w')| < 16\pi^2 L^{26} |h'(w')|/\log R < |h(w')|/100 < |h(w')|/8.
\] (4-2)

As \( z \in \gamma_{n, i} \) we know that, for \( 0 \leq p \leq n \),

\[
f^{mp}(z) \in \gamma_{n-p, j} \subset D(R'_m),
\]

for some \( 1 \leq j \leq 2^{n-p} \), and so it follows from Lemma 3-3 that

\[
|f^p(z)| > R'
\] (4-3)

for \( 0 \leq p \leq mn+1 \). Thus, from Lemma 2-6, the branch \( H \) of \( f^{-mn} \) that maps \( h(w') \) to \( z \) is univalent in \( B(h(w'), |h(w')|/4) \). It follows from Lemma 2-2, (4-1) and (4-2) that

\[
g(C(R)) = H h(C(R)) \subset B(z, 2\pi R L^{27} |g'(w')|)
\] (4-4)

and, from Lemma 2-6, (4-1), (4-2) and (4-3), that

\[
g(C(R)) \subset B(z, |z|/100) \subset G.
\] (4-5)

Also, from (4-3) and Lemma 2-6, we know that \( g \) is univalent in \( A \) and hence, from Lemma 2-1, (4-1) and (4-2),

\[
g(A) \supset B(g(z_0), |g'(z_0)| R/(4C)) \supset B(g(z_0), |g'(w')| R/(4C L^{27})).
\] (4-6)
We now take a collection \( G_{n,t} \) of disjoint curves \( \gamma_k \) such that \( \gamma_k \subset \gamma_{n,t}, \gamma_k \cap \gamma_{n',t} \neq \emptyset \), \( f^{mn+1} \) maps \( \gamma_k \) univalently onto \( C(R) \) and
\[
\bigcup_{\gamma_k \in G_{n,t}} \gamma_k \supseteq \gamma'_{n,t}. \tag{4.7}
\]

From (4.5) we know that \( \gamma_k \subset G \) and so, if \( \gamma_k = g_k(C(R)) \), then \( g_k(A) \in H_{n,t} \). From (4.4), (4.6) and (4.7) we see that
\[
\sum_{A_{mn+1} \in H_{n,t}} \text{diam}(A_{mn+1}) \geq \sum_{g_k(C(R)) \in G_{n,t}} \text{diam}(g_k(A))
\geq \frac{1}{(8\pi L^{54}C)} \sum_{g_k(C(R)) \in G_{n,t}} \text{diam}(g_k(C(R)))
\geq \text{diam}(\gamma'_{n,t})/(8\pi L^{54}C) > r_n/(288\pi L^{54}C).
\]

We are now in a position to prove Theorem 3. If \( A_{mn+1} = g(A) \in A_{mn+1} \), then it follows from Lemma 2.6 that \( g \) is univalent in \( E \) and so all the closures of the sets in \( \bigcup_{i=1}^{2^n} H_{n,i} \) are disjoint. It therefore follows from Lemma 3.10 that, for each \( n \in \mathbb{N} \) with \( n \geq n_0 \),
\[
\sum_{i=1}^{2^n} \sum_{A_{mn+1} \in H_{n,t}} \text{diam}(A_{mn+1}) \leq K_3 \sum_{i=1}^{2^n} \sum_{A_{mn+1} \in H_{n,t}} \mu_2(A_{mn+1}) \leq K_3. \tag{4.8}
\]

If \( n \geq n_0 \) then it follows from Lemma 3.9 that, for \( 1 \leq i \leq 2^n \) and each \( A_{mn+1} \in H_{n,t} \), \( \text{diam}(A_{mn+1}) < 1 \) and so, if \( s \leq 1 \), it follows from (4.8) that
\[
\sum_{i=1}^{2^n} \sum_{A_{mn+1} \in H_{n,t}} \text{diam}(A_{mn+1}) \leq \sum_{i=1}^{2^n} \sum_{A_{mn+1} \in H_{n,t}} \text{diam}(A_{mn+1})^s \leq K_3. \tag{4.9}
\]

Finally, it follows from Lemma 4.4 that, for each \( n \in \mathbb{N} \),
\[
\sum_{i=1}^{2^n} \sum_{A_{mn+1} \in H_{n,t}} \text{diam}(A_{mn+1}) \geq 2^n K_4. \tag{4.10}
\]

Combining (4.9) and (4.10) we see that, for \( n \geq n_0 \),
\[
2^n K_4 \leq K_3.
\]

As there are arbitrarily large values of \( n \in \mathbb{N} \) satisfying \( n \geq n_0 \) this is clearly a contradiction and so we must have \( s > 1 \) as claimed.

5. Proof of Theorem 4

We take a value \( a \in (0, s) \) and recall that
\[
\mathcal{E}_{n} = \{ g(E) : g \text{ is a branch of } f^{-n} \text{ with } g(z_0) \in I_n \}.
\]

Lemma 5.1. There exists \( M > 0 \) such that
\[
\sum_{E_M \in \mathcal{E}_{M}} (\text{diam}(E_M))^s \geq L^{2(s-a)}(\text{diam}(E))^s. \]
Hausdorff dimension of Julia sets of entire functions II

529

Proof. Take a value \( n \geq n_0 \) and a set \( E_n = g(E) \in \mathcal{E}_n \) with \( g(z_0) \in A \). We know from Lemma 3-9 that \( \text{diam}(E_n) < 1 \) and hence \( E_n \subset E \). As \( g(z_0) \in I_n \), we know that \( |f^r g(z_0)| \geq R' \geq R_2(f) \), for \( 0 \leq r \leq n \) and so, from Lemma 2-6,

\[
\text{Diam}(E_n) \geq \text{inf}_{zeE} g^\alpha(z) \text{Diam}(E) \geq g^\alpha(z_0) \text{Diam}(E)/(3L).
\]

As  \( E_n \subset E \) it follows that there exists \( K > 1 \) such that

\[
\text{diam}(E_n) \geq g^\alpha(z_0) \text{diam}(E)/(3LK) = d(z) \text{diam}(E)/(3LK),
\]

where \( z = g(z_0) \in I_n \). Thus, for \( n \geq n_0 \),

\[
\sum_{E_n \in \mathcal{E}_n} (\text{diam}(E_n))^{s-a} \geq (\text{diam}(E)/(3LK))^{r-a} \sum_{z \in A \cap I_n} d(z)^{r-a}. \tag{5.1}
\]

From Lemma 3-8 we know that there exists \( M \geq n_0 \) such that

\[
\sum_{z \in A \cap I_M} d(z)^{s-a} \geq (3L^3K)^{s-a}. \tag{5.2}
\]

Combining (5.1) with (5.2) we see that

\[
\sum_{E_n \in \mathcal{E}_n} (\text{diam}(E_n))^{s-a} \geq L^{a(s-a)}(\text{diam}(E))^{s-a}.
\]

We now put

\[
\mathcal{F}_n = \{g(E M) : E M \in \mathcal{E}_M, E M \subset E, g \text{ is a branch of } f^{-n} \text{ with } g(z_0) \in I_n\}.
\]

**Lemma 5.2.** There exists a set \( F \in \mathcal{F}_1 \) with \( F \subset G \).

**Proof.** From Lemma 3-2 part (iii) we know that there exists a branch \( g \) of \( f^{-1} \) with \( g(E) \subset G' \subset G \). Clearly \( g(z_0) \in I_1 \) and so, for each \( E M \in \mathcal{E}_M \) with \( E M \subset E \), we have \( g(E M) \in \mathcal{F}_1 \) and \( g(E M) \subset g(E) \subset G \).

**Lemma 5.3.** If \( F_n \in \mathcal{F}_n \) and \( F_n \subset G \) for some \( n \in \mathbb{N} \) then

\[
\sum_{F_n \in \mathcal{F}_n} (\text{diam}(F_{n+M}))^{s-a} > (\text{diam}(F_{n}))^{s-a}. \tag{5.3}
\]

**Proof.** Take a set \( F_n = g(E M) \in \mathcal{F}_n \) and a set \( E M \in \mathcal{E}_M \) such that \( E M \subset E \). Taking \( h \) to be the branch of \( f^{-M} \) that maps \( E \) to \( E M \), we note that

\[
gh(E M) \subset gh(E) = g(E M) = F_n \subset G. \tag{5.4}
\]

As \( h(z_0) \in I_M \) we see that, for \( 0 \leq k \leq M \),

\[
f^n k gh(E) = f^k h(E) \subset D(R').
\]

As \( g(z_0) \in I_n \) we see that, for \( 0 \leq k \leq n \),

\[
f^k gh(E) = f^k g(E M') \subset f^k g(E) \subset D(R').
\]

Thus, for \( 0 \leq k \leq n + M \),

\[
f^k gh(E) \subset D(R'). \tag{5.5}
\]
From (5·3) we know that $gh(z_0) \in gh(E) \subset G$ and so, together with (5·4), we see that $gh(z_0) \in I_{n+M}$ and hence $gh(E_M) \in \mathcal{F}_{n+M}$.

As $z_0 \in E$, it follows from (5·4) together with Lemma 2·6 that

$$
\text{diam}(F_n) = \text{diam}(g(E_M)) = \text{diam}(gh(E)) \leq L |(gh)'(z_0)| \text{diam}(E)
$$

and, as $E_M \subset E$,

$$
\text{diam}(gh(E_M)) \geq |(gh)'(z_0)| \text{diam}(E_M)/L.
$$

Together with Lemma 5·1 this gives

$$
\sum_{F_{n+M} \in \mathcal{F}_{n+M}} (\text{diam}(F_{n+M}))^{s-a} \geq \sum_{E_M \in E} (\text{diam}(gh(E_M)))^{s-a}
$$

$$
\geq \left( |(gh)'(z_0)|/L \right)^{s-a} \sum_{E_M \in E} (\text{diam}(E_M))^{s-a}
$$

$$
\geq (L |(gh)'(z_0)| \text{diam}(E))^{s-a} \geq (\text{diam}(F_n))^{s-a}.
$$

We are now in a position to prove one of the two main results of this section. We put

$$
J' = \{ z : z \in J(f) \cap G, |f^n(z)| \geq R' \text{ for each } n \in \mathbb{N} \}.
$$

**Lemma 5·4.** There exists $r_0 > 0$, $K_5 > 0$ such that, if $r < r_0$ and $z \in J'$, there exists $P \in \mathbb{N}$ for which

$$
2 \sum_{F_{PM+1} \in \mathcal{F}_{PM+1} \cap B(z,r) \neq \emptyset} (\text{diam}(F_{PM+1}))^{s-a} < K_5 (\text{diam}(B(z,r)))^{s-a}.
$$

**Proof.** We take $r > 0$ and $z \in J'$. Let $q$ be the smallest value of $n \in \mathbb{N}$ for which

$$
|f^n(z)| r > |f^n(z)|/50
$$

and take $P$ to be the integer satisfying

$$
q - M \leq PM + 1 < q.
$$

Note that the existence of $q$ follows from Lemma 2·12. As $|f^q(z)| \geq R'$, it follows from (5·5) that $|(f^q)'(z)| r > R'/50$. If

$$
r < r_0 = \inf_{z \in G} \{ R'/(50 |f'(z)|), R'/50 |(f^2)'(z)|, \ldots, R'/50 |(f^{M+1})'(z)| \},
$$

then we must have $q > M+1$ and hence $P \geq 1$.

For each $E_M \in \mathcal{S}_M$ with $E_M \subset E$ we put

$$
H(E_M) = \{ F_{PM+1} : F_{PM+1} \in \mathcal{F}_{PM+1}, f^{PM+1}(F_{PM+1}) = E_M, F_{PM+1} \cap B(z, r) \neq \emptyset \}.
$$

We note from (5·6) that $0 \leq q - 2 - PM \leq M - 1$ and consider two separate cases.

**Case I.**

$$
f^{q-2-PM}(E_M) \cap f^{q-1}(B(z, r)) = \emptyset
$$

for some $E_M \in \mathcal{S}_M$ satisfying $E_M \subset E$. 


Suppose that there exists a set \( F_{PM+1} \in \mathcal{F}_{PM+1} \) with \( f^{PM+1}(F_{PM+1}) = E_M \) such that \( F_{PM+1} \cap B(z, r) \neq \emptyset \)). Then \( f^{q-1}(F_{PM+1}) \cap f^{q-1}(B(z, r)) \neq \emptyset \). But \( f^{q-1}(F_{PM+1}) = f^{q-2-PM}(E_M) \) and so we have a contradiction. Thus

\[
\sum_{F_{PM+1} \in \mathcal{H}(E_M)} \left( \text{diam}(F_{PM+1}) \right)^{s-a} = 0.
\]

**Case II.**

\( f^{q-2-PM}(E_M) \cap f^{q-1}(B(z, r)) \neq \emptyset \)

for some \( E_M \in \mathcal{H}_M \) satisfying \( E_M \subset E \).

We begin by showing that

\[
\text{diam}(f^{q-2-PM}(E_M)) < \text{diam}(f^{q-1}(B(z, r))).
\] (5-7)

Let \( g \) denote the branch of \( f^{-M} \) that maps \( E \) to \( E_M \). As \( g(z_0) \in I_M \) we know that

\[
|f^p g(z_0)| \geq R' \geq R_2(f),
\] (5-8)

for \( 0 \leq p \leq M \). As \( 0 < q - 1 - PM \leq M \), it follows from Lemma 2.6 that

\[
B(f^{q-1-PM}(g(z_0)), |f^{q-1-PM}(g(z_0))|/(200L))
\]

\[
\supset B(f^{q-1-PM}(g(z_0)), R |(f^{q-1-PM}g)'(z_0)|/(200L^2)).
\] (5-9)

Let \( h \) denote the branch of \( f^{-1} \) that maps \( f^{q-1-PM}(z_0) \) to \( f^{q-2-PM}(z_0) \). From (5-8) we have \( |f^{q-1-PM}g(z_0)| \geq R' \) and so \( h \) is univalent in \( B(f^{q-1-PM}g(z_0), |f^{q-1-PM}g(z_0)|/(200L)) \).

It follows from (5-9) and Lemma 2.1 that

\[
h(B(f^{q-1-PM}g(z_0), |f^{q-1-PM}g(z_0)|/(200L)))
\]

\[
\supset h(B(f^{q-1-PM}g(z_0), R |(f^{q-1-PM}g)'(z_0)|/(200L^2)))
\]

\[
\supset B(f^{q-2-PM}g(z_0), R |(f^{q-2-PM}g)'(z_0)|/(800L^2)).
\] (5-10)

As \( 0 \leq q - 2 - PM < M \) and \( C > 4800L^3 \), it follows from Lemma 2.6 and (5-8) that

\[
f^{q-2-PM}(E_M) = f^{q-2-PM}(E) = f^{q-2-PM}(B(z_0, 2R/C))
\]

\[
\subset B(f^{q-2-PM}(g(z_0), 2RL |(f^{q-2-PM}g)'(z_0)|/C)
\]

\[
\subset B(f^{q-2-PM}g(z_0), R |(f^{q-2-PM}g)'(z_0)|/(2400L^2)).
\]

Thus, if \( \text{diam}(f^{q-1}(B(z, r))) \leq \text{diam}(f^{q-2-PM}(E_M)) \), we have

\[
f^{q-1}(B(z, r)) \subset B(f^{q-2-PM}g(z_0), R |(f^{q-2-PM}g)'(z_0)|/(800L^2))
\]

and hence, from (5-10),

\[
f^q(B(z, r)) \subset B(f^{q-1-PM}g(z_0), |f^{q-1-PM}g(z_0)|/(200L)).
\]

Thus

\[
f^q(B(z, r)) \subset B(f^q(z), |f^q(z)|/(50L)).
\]

As \( |f^p(z)| \geq R' \geq R_2(f) \) for \( 0 \leq p \leq q \), it follows from Lemma 2.6 that

\[
B(z, r) \subset B(z, |f^q(z)|/(50 |(f^q)'(z)|))
\]

and so \( |(f^q)'(z)| r \leq |f^q(z)|/50 \) which contradicts (5-5). Thus (5-7) must indeed be true.

As \( |f^p(z)| \geq R' \leq 0 \leq p \leq q - 1 \) and

\[
|(f^{q-1})'(z)| r \leq |f^{q-1}(z)|/50,
\] (5-11)
it follows from Lemma 2-6 that the branch $h_1$ of $f^{-(q-1)}$ that maps $f^{q-1}(z)$ to $z$ is univalent in $B(f^{q-1}(z), 4|f^{q-1}(z)|r)$ and so, from Lemma 2-1,
\[ f^{q-1}(B(z, r)) \subset B(f^{q-1}(z), 4|f^{q-1}(z)|r). \]  
(5.12)

It follows from (5.7) and (5.11) that
\[ f^{q-2-PM}(E_M) \subset B(f^{q-1}(z), 12|f^{q-1}(z)|r) \subset B(f^{q-1}(z), |f^{q-1}(z)|/4) \]
and so, as $|f^{q-1}(z)| \geq R'$, it follows from Lemma 2-6, (5.7) and (5.12) that
\[
\sum_{F_{PM+1} \in H(E_M)} (\text{diam}(F_{PM+1}))^{s_{-a}} = (\text{diam}(h_1 f^{q-2-PM}(E_M)))^{s_{-a}} < (L \text{diam}(f^{q-2-PM}(E_M))/|f^{q-1}(z)|)^{s_{-a}} < (L \text{diam}(f^{q-1}(B(z, r))/|f^{q-1}(z)|)^{s_{-a}} \leq (4L)^{s_{-a}}(\text{diam}(B(z, r)))^{s_{-a}} < 16L^2(\text{diam}(B(z, r)))^{s_{-a}}.
\]

As there are only a finite number of sets $E_M \in \mathcal{E}_M$ with $E_M \subset E$, combining the results of Case I and Case II gives the desired result.

**Lemma 5-5.** If, for some $n \in \mathbb{N}$, $F_n \in \mathcal{F}_n$ and $F_n \subset G$ then $F_n \cap J' \neq \emptyset$.

*Proof.* Take a set $F_n \in \mathcal{F}_n$ with $F_n \subset G$ and the set $E_M = f^n(F_n) \in \mathcal{E}_M$ with $E_M \subset E$. Let $w'$ denote the point in $I_M \cap E_M$ and $z'$ denote the point in $F_n$ which satisfies $f^n(z') = w'$. As $z_0 \in J_f$ and $|f^n(z_0)| \geq R'$ for each $p \in \mathbb{N}$, it follows that $z' \in J_f$ and
\[ |f^{p+n}(z')| = |f^p(z_0)| \geq R', \quad (5.13) \]
for each $p \in \mathbb{N}$.

Let $h$ denote the branch of $f^{-M}$ that maps $E$ to $E_M$ and $g$ denote the branch of $f^{-n}$ that maps $E_M$ to $F_n$. Then
\[ f^{p+n}(z') \in f^{p+n}(F_n) = f^{p+n}gh(E) = f^p h(E) \subset D(R'), \]
for $0 \leq p \leq M$, and
\[ f^p(z') \in f^p(F_n) = f^pg(E_M) \subset f^pg(E) \subset D(R'), \]
for $0 \leq p \leq n$. Together with (5.13) this shows that $|f^p(z')| \geq R'$ for each $p \in \mathbb{N}$ and so $z' \in F_n \cap J'$.

We are now in a position to prove the other main result of this section.

**Lemma 5-6.** $\dim J' \geq s_{-a}$.

*Proof.* Take a set $F \in \mathcal{F}_L$ such that $F \subset G$. The existence of such a set follows from Lemma 5-2. If $\dim J' < s_{-a}$ then there exists a cover of $J'$ with sets $B_i = B(z_i, r_i)$, $i \in I$, such that, for each $i \in I$, we have $z_i \in J'$, $r_i < r_0$ and
\[ \sum_{i \in I} (\text{diam}(B_i))^{s_{-a}} < (\text{diam}(F))^{s_{-a}}/(2K_\delta). \]  
(5.14)

From Lemma 5-4 we know that, for each $i \in I$, there exists $P(i) \in \mathbb{N}$ such that
\[ \sum_{F_{PM+1} \in H(E_M, B_i \neq \emptyset)} (\text{diam}(F_{PM}))^{s_{-a}} < K_\delta (\text{diam}(B_i))^{s_{-a}}. \]  
(5.15)
As $J'$ is compact we can cover $J'$ with a finite number of the sets $B_i$. We label these sets $B_1, B_2, \ldots, B_n$ in such a way that $P(i) \leq P(i+1)$, for $1 \leq i \leq n$. It follows from Lemma 5·3 and (5·15) that

$$(\text{diam}(F))^{s-a} \leq \sum_{F_{P(i)}M+1 \in C \setminus B_i} (\text{diam}(F_{P(i+1)}))^{s-a} \leq \sum_{F_{P(i)}M+1 \in C \setminus (B_i \cup B_{i+1})} (\text{diam}(F_{P(i)}))^{s-a} \leq \sum_{F_{P(i)}M+1 \in (C \setminus B_i)} (\text{diam}(F_{P(i)}))^{s-a} + K_s (\text{diam}(B_i))^{s-a}$$

where $F_{P(i)}M+1$ denotes a set in $\mathcal{F}_{P(i)}M+1$ in each of the above statements. As $J' \subset \bigcup_{i=1}^n B_i$, it follows from Lemma 5·5 that

$$\left\{ F_{P(n)}M+1 : F_{P(n)}M+1 \in \mathcal{F}_{P(n)}M+1, F_{P(n)}M+1 \subset \left( C \setminus \bigcup_{i=1}^n B_i \right) \right\} = \emptyset$$

and so, from (5·14) and (5·16) we have

$$(\text{diam}(F))^{s-a} \leq K_s \sum_{i=1}^n (\text{diam}(B_i))^{s-a} < (\text{diam}(F))^{s-a}/2$$

which is a contradiction.

As $a$ was an arbitrary value in $(0, s)$ it follows from Lemma 5·6 that $\dim J' \geq s$. As $J' \subset J(f)$, this proves Theorem 4.

6. Examples

As mentioned in the introduction, in [10] we studied a family of functions $f_k$ satisfying Theorem 1. To obtain the functions $f_k$ we used the function defined by

$$E(z) = \frac{1}{2\pi} \int_L \frac{\exp(e^t)}{t-z} dt,$$

where $L$ is the boundary of the region

$$G = \{ z : \text{Re}(z) > 0, -\pi < \text{Im}(z) < \pi \}$$
described in a clockwise direction, for \( z \in \mathbb{C} \setminus \overline{G} \), and by analytic continuation for \( z \in G \). The functions \( f_k \) were then defined by

\[
f_k(z) = E(z) - K.
\]

For large \( K \) we showed that \( J(f_k) \subseteq G \). From the properties of the functions \( f_k \) given in [10, section 3], it can easily be seen that, for large \( K \), \( f_k \in B \) and hence \( \dim J(f_k) > 1 \).

As \( \dim J(f_k) \) is very close to one for large \( K \), it has been suggested that if a function \( f \) could be constructed in such a way that its Julia set, \( J(f) \), was contained in a domain of finite Lebesgue measure then \( J(f) \) might be expected to have dimension equal to one. We construct a family of such functions, \( F_k \), which belong to \( B \) and hence satisfy \( \dim J(F_k) > 1 \).

Consider the functions defined by

\[
E_n(z) = \frac{1}{2\pi i} \int_{L_n} \frac{\exp(e^t)}{t - z} dt,
\]

where \( L_n \) is the boundary of the region

\[
G_n = \{ z = x + iy : x > n, |y| < \pi e^{-x} \}
\]

described in a clockwise direction, for \( z \in \mathbb{C} \setminus \overline{G_n} \). By the residue theorem we see that \( E_n(z) = E_1(z) \) for each \( z \in \mathbb{C} \setminus \overline{G_1} \) and each \( n > 1 \). Thus the functions \( E_n, n > 1 \), give an analytic continuation of \( E_1 \) to a transcendental entire function \( F \).

**Lemma 6.1.** There exists \( C_1 > 0 \) such that \( |F(z)| < C_1 \) for each \( z \in \mathbb{C} \setminus \overline{G_1} \).

**Proof.** It follows from the residue theorem that, for \( z \in \mathbb{C} \setminus \overline{G_1} \),

\[
F(z) = \frac{1}{2\pi i} \int_{L_1} \frac{\exp(e^t)}{t - z} dt.
\]

Let

\[
\Gamma_1(x) = \{ z = x + iy : 5\pi e^{-x}/6 \leq y \leq \pi e^{-x} \},
\]

\[
\Gamma_2(x) = \{ z = x + iy : -\pi e^{-x} \leq y \leq -5\pi e^{-x}/6 \}.
\]

As

\[
\left| \frac{1}{2\pi i} \int_{\Gamma_i(x)} \frac{\exp(e^t)}{t - z} dt \right| \to 0
\]

as \( x \to \infty \), for \( i = 1, 2 \), it follows from the residue theorem that, for \( z \in \mathbb{C} \setminus \overline{G_1} \),

\[
F(z) = \frac{1}{2\pi i} \int_{L_2'} \frac{\exp(e^t)}{t - z} dt,
\]

where \( L_2' \) is the boundary of

\[
G_2' = \{ z = x + iy : x > 2, |y| < 5\pi e^{-x}/6 \}.
\]

It is not difficult to show that there exists \( C_1 > 0 \) such that, for \( z \in \mathbb{C} \setminus \overline{G_1} \),

\[
\left| \frac{1}{2\pi i} \int_{L_2'} \frac{\exp(e^t)}{t - z} dt \right| < C_1/2
\]

and so, for \( z \in \mathbb{C} \setminus G_1, |F(z)| \leq C_1/2 < C_1 \).
We define the functions $F_k$ by

$$F_k(z) = F(z) - K.$$ 

If $K > C_1$ then it follows from Lemma 6.1 that, for $z \in \mathbb{C} \setminus \mathcal{G}_1$,

$$F_k(z) \in B(-K, C_1) \subset \{z: \text{Re}(z) < 0\} \subset \mathbb{C} \setminus \mathcal{G}_1$$  \hspace{1cm} (6.1)

and so $(\mathbb{C} \setminus \mathcal{G}_1) \subset N(F_k)$. Thus $J(F_k) \subset \mathcal{G}_1$. It is not difficult to see that the plane Lebesgue measure of $\mathcal{G}_1$ is equal to $2\pi/e < \infty$. It remains to show that, for large $K$, $F_k \in B$.

**Lemma 6.2.** There exists $g_1(z)$ such that $F(z) = \exp(\epsilon^x) + g_1(z)$ for each $z \in \mathcal{G}_1$ and $|g_1(z)| \to 0$ as $z \to \infty$.

**Proof.** If $z \in \mathcal{G}_1$ and $\text{Re}(z) < n$ then

$$F(z) = E_n(z) = \frac{1}{2\pi i} \int_{L_n} \frac{\exp(\epsilon^t)}{t-z} \, dt.$$  \hspace{1cm} (6.2)

By the residue theorem,

$$\frac{1}{2\pi i} \int_{L_n} \frac{\exp(\epsilon^t)}{t-z} \, dt - \frac{1}{2\pi i} \int_{L_0} \frac{\exp(\epsilon^t)}{t-z} \, dt = \exp(\epsilon^x).$$  \hspace{1cm} (6.3)

Let

$$\gamma_1(x) = \{z = x + iy: \pi e^{-x} \leq y \leq 5\pi e^{-x}/4\},$$

$$\gamma_2(x) = \{z = x + iy: -5\pi e^{-x}/4 \leq y \leq -\pi e^{-x}\}.$$ 

As

$$\left| \frac{1}{2\pi i} \int_{\gamma_i(x)} \frac{\exp(\epsilon^t)}{t-z} \, dt \right| \to 0$$

as $x \to \infty$, for $i = 1, 2$ it follows from the residue theorem that, for $z \in \mathcal{G}_1$,

$$\frac{1}{2\pi i} \int_{L_0} \frac{\exp(\epsilon^t)}{t-z} \, dt = \frac{1}{2\pi i} \int_{L'_0} \frac{\exp(\epsilon^t)}{t-z} \, dt,$$  \hspace{1cm} (6.4)

where $L'_0$ denotes the boundary of

$$\mathcal{G}_0' = \{z = x + iy: x > 0, |y| < 5\pi e^{-x}/4\}.$$ 

Combining the results of (6.2), (6.3) and (6.4) we see that, for each $z \in \mathcal{G}_1$,

$$F(z) = \exp(\epsilon^x) + g_1(z),$$

where

$$g_1(z) = \frac{1}{2\pi i} \int_{L'_0} \frac{\exp(\epsilon^t)}{t-z} \, dt.$$ 

It is not difficult to show that $g_1(z) \to 0$ as $z \to \infty$ in $\mathcal{G}_1$.

Thus, for $z \in \mathcal{G}_1$,

$$F'_k(z) = \exp(\epsilon^x) - K + g_1(z),$$  \hspace{1cm} (6.5)

where $|g_1(z)| < C_2$, and so from Cauchy's inequalities

$$F'_k(z) = \epsilon^x e^{ix} \exp(\epsilon^x) + g_2(z),$$  \hspace{1cm} (6.6)

where $|g_2(z)| < C_2$. 


LEMMA 6·3. For $K > C_1$ we have $F_k \in B$.

Proof. The transcendental singularities of $F_k^{-1}$ are the asymptotic values of $F_k$. The only finite asymptotic value of $\exp(e^z)$ in $G_1$ is 0 and so from Lemma 6·1 and Lemma 6·2 we see that, for $K > C_1$, the finite transcendental singularities of $F_k^{-1}$ are contained in $B(-K, C_1 + C_2)$.

The remaining singularities of $F_k^{-1}$ are the images of points $z$ such that $F_k'(z) = 0$. If $z \in G_1$ and $F_k'(z) = 0$ then, from (6·6),

$$|\exp(e^z)| \leq |e^z e^z \exp(e^z)| < C_2$$

and hence, from (6·5), $F_k(z) \in B(-K, 2C_2)$. Together with Lemma 6·1 and the results of the first paragraph this shows that, for $K > C_1$,

$$S(F_k) \subset B(-K, 2C_2 + C_1)$$

and so $F_k \in B$.

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REFERENCES