THE HAUSDORFF DIMENSION OF JULIA SETS OF ENTIRE FUNCTIONS IV

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Abstract

It is known that, for a transcendental entire function f, the Hausdorff dimension of $J(f)$ satisfies $1 \leqslant \dim J(f) \leqslant 2$. For each $d \in (1, 2)$, an example of a transcendental entire function f with $\dim J(f) = d$ is given. It is then indicated how this function can be modified to produce a transcendental meromorphic function F with one pole with $\dim J(F) = d$. These appear to be the first examples of Julia sets with non-integer dimensions whose dimensions have been calculated exactly.

1. *Introduction*

Let f be a transcendental meromorphic function and denote by $f^n, n \in \mathbf{N}$, the nth iterate of f. The set of normality $N(f)$ is defined to be the set of points $z \in \mathbf{C}$ such that $(f^n)_{n\in\mathbf{N}}$ is well-defined and meromorphic, and forms a normal family in some neighbourhood of z. The complement, $J(f)$, of $N(f)$ is called the Julia set of f. An introduction to the properties of these sets can be found in, for example, [**2**].

It was shown by Baker [**1**, Corollary to Theorem 1] that, if f is a transcendental entire function, then $J(f)$ must contain continua and so the Hausdorff dimension of $J(f)$ satisfies $1 \leqslant \dim J(f) \leqslant 2$. McMullen [**5**] gave several examples of transcendental entire functions whose Julia sets have Hausdorff dimension equal to 2. In this paper we prove the following result.

Theorem 1.1. *For each* $d \in (1, 2)$*, there exists a transcendental entire function* f *with* $\dim J(f) = d$.

To do this we consider the family of functions defined by

$$E_p(z) = \frac{1}{2\pi i}\int_{L_p} \frac{\exp(e^{(\log t)^{1+p}})}{t-z}\,dt, \quad 0 < p < \infty,$$

where L_p is the boundary of the region

$$G_p = \left\{ z = x+iy : |y| < \frac{\pi x}{(1+p)(\log x)^p},\ x \geqslant 3 \right\}$$

described in a clockwise direction, for $z \in \mathbf{C}\backslash \bar{G}_p$, and by analytic continuation for $z \in \bar{G}_p$. We then define $f_{p,K}(z) = E_p(z) - K$. Here $(\log t)^{1+p}$ and $(\log x)^p$ are taken to be the values which are real for $t > e$.

In [**6**] we proved the following result.

Theorem 1.2. *There exists a positive real-valued increasing function* S *defined on* $(1, \infty)$ *such that*

(1) *if* $p \geqslant 2$ *and* $K > S(p)$*, then* $\dim J(f_{p,K}) \leqslant 1 + 2/p$;

(2) *if* $0 < p \leqslant \frac{1}{2}$ *and* $K > S(1/p)$*, then* $\dim J(f_{p,K}) \geqslant 2 - p$.

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In this paper we prove the following improvement to Theorem 1.2.

THEOREM 1.3. *For each* $p \in (0, \infty)$, *there exists* $C(p) > 0$ *such that, for each* $K > C(p)$,

$$\dim J(f_{p,K}) = 1 + \frac{1}{1+p}.$$

Since $\lim_{p\to 0} 1 + 1/(1+p) = 2$ and $\lim_{p\to\infty} 1 + 1/(1+p) = 1$, Theorem 1.1 follows directly from Theorem 1.3.

In the last section of the paper we consider the class M whose members are the transcendental meromorphic functions with at least one pole that are not conjugate to self-maps of the punctured plane. We indicate how the methods of the rest of the paper can be applied to functions of the form $F_{p,K}(z) = f_{p,K}(z) + b/(z-a)$ to obtain the following result.

THEOREM 1.4. *For each* $d \in (1, 2)$, *there exists* $f \in M$ *with* $\dim J(f) = d$.

2. *Preliminary results*

We begin this section with a formal definition for the Hausdorff dimension of a set E. If, for each $\mu > 0$, we put

$$H_\mu(E) = \lim_{\epsilon\to 0} \inf \sum_i r_i^\mu,$$

where the inf is taken over all possible covers of E with sets of diameter $r_i < \epsilon$, then the Hausdorff dimension $d = \dim E$ of the set E is defined to be the unique value satisfying

$$H_\mu(E) = \begin{cases} \infty & \text{for } \mu < d, \\ 0 & \text{for } \mu > d. \end{cases}$$

For more details see, for example, [**4**].

The next result is known as Koebe's distortion theorem. A proof of this result can be found in, for example, [**3**].

LEMMA 2.1. *If* f *is univalent in the disc* $B(z, r)$ *centred at* z *with radius* r *then, for* $0 < s < r$,

$$\sup_{v, w \in B(z,s)} \left|\frac{f'(v)}{f'(w)}\right| \leqslant L(s/r) = \left|\frac{r+s}{r-s}\right|^4.$$

For simplicity, we denote $L(1/2)$ by L.

The remaining results in this section concern the functions $f_{p,K}$ defined in Section 1. Most of these results were proved in [**6**]. We begin with a result concerning the singularities of the functions $f_{p,K}^{-n}$ which allows us to use Lemma 2.1 at various points in the proof of Theorem 1.3. For a proof of the next result, see [**6**, Lemma 3.2].

LEMMA 2.2. *For each* $p \in (0, \infty)$, *there exists* $C_1(p) > 0$ *such that, for each* $K > C_1(p)$, *we have*

$$\{z : z \text{ is a finite singularity of } f_{p,K}^{-n} \text{ for some } n \in \mathbf{N}\} \subset B(-K, C_1(p)) \subset \{z : \Re(z) < 0\}.$$

We now look at the behaviour of $f_{p,K}$ in $\mathbf{C}\backslash\bar{G}_p$ and in $\bar{G}_p$. For a proof of the next result, see [**6**, Lemma 3.4].

LEMMA 2.3. *For each $p\in(0,\infty)$, there exists $C_2(p)\geqslant C_1(p)$ such that, for each $K>C_2(p)$, we have*

$$f_{p,K}(\mathbf{C}\backslash\bar{G}_p)\subset B(-K,C_2(p))\subset\{z:\Re(z)<0\}\subset\mathbf{C}\backslash\bar{G}_p.$$

Thus $\mathbf{C}\backslash\bar{G}_p\subset N(f_{p,K})$ and there exists $z_0\in\{z:\Re(z)<0\}$ such that, for each z in the component of $N(f_{p,K})$ which contains $\mathbf{C}\backslash\bar{G}_p$, we have

$$f^n_{p,K}(z)\to z_0 \text{ as } n\to\infty.$$

We put

$$g_p(z)=e^{(\log z)^{1+p}}\quad\text{and}\quad h_{p,K}(z)=\exp(g_p(z))-K.$$

For a proof of the next result, see [**6**, Lemma 3.1].

LEMMA 2.4. *For each $p\in(0,\infty)$, there exists $M(p)>0$ such that, for each $z\in\bar{G}_p$, we have*

(1) $|f_{p,K}(z)-h_{p,K}(z)|<M(p)$;
(2) $|f'_{p,K}(z)-h'_{p,K}(z)|<M(p)$.

COROLLARY 2.5. *For each $p\in(0,\infty)$, there exists $C_3(p)\geqslant C_2(p)$ such that, if $z,f_{p,K}(z)\in\bar{G}_p$ and $|f_{p,K}(z)|>C_3(p)$, then*

(1) $|h_{p,K}(z)|/2<|f_{p,K}(z)|<2|h_{p,K}(z)|<2|e^{g_p(z)}|$;
(2) $2|h'_{p,K}(z)|>|f'_{p,K}(z)|>|h'_{p,K}(z)|/2=|g'_p(z)\,e^{g_p(z)}|/2>|g'_p(z)f_{p,K}(z)|/4.$

If, further, $|f_{p,K}(z)|>4K$, then

(3) $|f_{p,K}(z)|>|h_{p,K}(z)|/2>|e^{g_p(z)}|/4$;
(4) $|f'_{p,K}(z)|<2|h'_{p,K}(z)|=2|g'_p(z)\,e^{g_p(z)}|<8|g'_p(z)f_{p,K}(z)|.$

We now take a function $f=f_{p,K}$ with $p\in(0,\infty)$ and $K>2C_3(p)$. We also take a value $x_1\geqslant C_3(p)$ and define the real values x_n inductively by

$$x_{n+1}=x_n+r_n,\quad r_n=\frac{x_n}{(1+p)(\log x_n)^p}.$$

We define the sets B_n by

$$B_n=\{z=x+iy:x_n\leqslant x\leqslant x_n+R_n,|y|\leqslant R_n\},$$

where $r_n/(8L)\leqslant R_n\leqslant \pi r_n$, and define the following collection of sets inductively:

$$\mathscr{A}_1=\{A_1:A_1\text{ is a component of }f^{-1}(B_n)\text{ for some }n\in\mathbf{N},A_1\subset\bigcup_{n\in\mathbf{N}}B_n\}$$

$$\mathscr{A}_{k+1}=\{A_{k+1}:A_{k+1}\text{ is a component of }f^{-(k+1)}(B_n)\text{ for some }n\in\mathbf{N},A_{k+1}\subset\bigcup_{A_k\in\mathscr{A}_k}A_k\}.$$

We now put $g=g_p$ and $h=h_{p,K}$. For a proof of the next result, see [**6**, Lemma 4.4, Lemma 4.5].

LEMMA 2.6. *There exist $M_1>1$ and $C_4(p)\geqslant C_3(p)$ such that, if $x_1>C_4(p)$, $z\in\bigcup_{n=1}^{\infty}B_n$ and $r\leqslant|2\pi z/(1+p)(\log z)^p)|$, then*

(1) *for each* $w \in B(z,r)$,

$$\frac{z}{10(\log z)^p} \leqslant \frac{w}{(\log w)^p} \leqslant \frac{10z}{(\log z)^p}$$

$$\frac{|g(z)|}{M_1} \leqslant |g(w)| \leqslant M_1|g(z)|$$

$$\frac{|g'(z)|}{M_1} \leqslant |g'(w)| \leqslant M_1|g'(z)|;$$

(2) *for each* $n \in \mathbf{N}$,

$$|\{w : h(w) = x_n, w \in B(z,r)\}| \leqslant M_1|g'(z)|\, r.$$

Using these results we prove the following.

LEMMA 2.7. *There exist* $M_2 > 1$ *and* $C_5(p) \geqslant C_4(p)$ *such that, if* $x_1 > C_5(p)$, $z, f(z) \in \bigcup_{n=1}^{\infty} B_n$ *and*

$$\left|\frac{\sqrt{2}\pi z}{(1+p)(\log z)^p}\right| \geqslant r \geqslant \left|\frac{f(z)}{8L(1+p)(\log f(z))^p f'(z)}\right|,$$

then, for each $n \in \mathbf{N}$,

$$|\{A_1 : A_1 \text{ is a component of } f^{-1}(B_n), A_1 \cap B(z,r) \neq \varnothing\}| \leqslant M_2|g'(z)|\, r.$$

Proof. It follows from Lemma 2.4 that there exists $C_5(p) \geqslant C_4(p)$ such that, if $x_1 > C_5(p)$ and $f(A_1) = B_n$ then

$$h(A_1) \subset B\left(x_n, \frac{5x_n}{(1+p)(\log x_n)^p}\right) = H_n \subset B\left(x_n, \frac{x_n}{8}\right). \tag{2.1}$$

We let A_1' denote the component of $h^{-1}(H_n)$ that contains A_1 and note that it follows from Lemma 2.4 that, provided that $C_5(p)$ is sufficiently large,

$$f(A_1') \subset B\left(x_n, \frac{6x_n}{(1+p)(\log x_n)^p}\right) \subset B(x_n, x_n)$$

and hence, from Lemma 2.2, f is univalent in A_1'. Thus, for each $n \in \mathbf{N}$,

$$\begin{aligned} &|\{A_1 : A_1 \text{ is a component of } f^{-1}(B_n), A_1 \cap B(z,r) \neq \varnothing\}| \\ &\leqslant |\{A_1' : A_1' \text{ is a component of } h^{-1}(H_n), A_1' \cap B(z,r) \neq \varnothing\}|. \end{aligned} \tag{2.2}$$

Clearly the only transcendental singularity of h^{-1} is at $-K$ and, as $h'(z) = 0$ implies that $z = 1$, the only critical value of h is at $h(1) = e - K$. Thus, if $A_1' \cap B(z,r) \neq \varnothing$ and $h(A_1') = H_n$, then the branch of h^{-1} that maps H_n to A_1' is univalent in $B(x_n, x_n/2)$. It follows from (2.1) and Lemma 2.1 that, if $w \in B(z,r) \cap A_1'$, then

$$A_1' \subset B\left(w, \frac{10Lx_n}{(1+p)(\log x_n)^p\,|h'(w)|}\right) \tag{2.3}$$

and hence, if $r < (\operatorname{diam} A_1')/2$, then

$$B(z,r) \subset B\left(w, \frac{20Lx_n}{(1+p)(\log x_n)^p\,|h'(w)|}\right). \tag{2.4}$$

If $C_5(p)$ is sufficiently large, then

$$\frac{20L^2 x_n}{(1+p)(\log x_n)^p} < \frac{x_n}{8}$$

and so it follows from (2.1), (2.4) and Lemma 2.1 that, if $r < (\operatorname{diam} A_1')/2$, then

$$h(z) \in B(h(w), x_n/8) \subset B(x_n, x_n/4)$$

and hence, from Lemma 2.4, $f(z) \in B(x_n, x_n/2)$. Thus, if $r < (\operatorname{diam} A_1')/(3200L^3)$ and $C_5(p)$ is sufficiently large, then it follows from (2.3) and Corollary 2.5 that

$$\begin{aligned} r &< \left|\frac{20Lx_n}{3200L^3(1+p)(\log x_n)^p h'(w)}\right| < \left|\frac{200L^2 f(z)}{3200L^3(1+p)(\log f(z))^p h'(z)}\right| \\ &< \left|\frac{400L^2 f(z)}{3200L^3(1+p)(\log f(z))^p f'(z)}\right| = \left|\frac{f(z)}{8L(1+p)(\log f(z))^p f'(z)}\right|. \end{aligned}$$

This, however, is impossible and so

$$\operatorname{diam} A_1' \leqslant 3200L^3 r. \tag{2.5}$$

It follows from (2.1), (2.3) and Lemma 2.6 that, if $C_5(p)$ is sufficiently large, then

$$\begin{aligned} A_1' &\subset B\left(w, \frac{10Lx_n}{(1+p)(\log x_n)^p |h'(w)|}\right) \subset B\left(w, \left|\frac{100Lh(w)}{(1+p)(\log h(w))^p h'(w)}\right|\right) \\ &\subset B\left(w, \frac{1}{1+p}\right) \subset B\left(w, \left|\frac{z}{(1+p)(\log z)^p}\right|\right) \subset B\left(z, \left|\frac{2\pi z}{(1+p)(\log z)^p}\right|\right). \end{aligned}$$

Together with (2.5), (2.2) and Lemma 2.6, this shows that, if $C_5(p)$ is sufficiently large, then there exists

$$\hat{r} \leqslant \min\left(\left|\frac{2\pi z}{(1+p)(\log z)^p}\right|, 3300L^3 r\right)$$

such that

$$\begin{aligned} &|\{A_1 : A_1 \text{ is a component of } f^{-1}(B_n), A_1 \cap B(z,r) \neq \varnothing\}| \\ &\leqslant |\{A_1' : A_1' \text{ is a component of } h^{-1}(H_n), A_1' \subset B(z,\hat{r})\}| \\ &\leqslant |\{w : h(w) = x_n, w \in B(z,\hat{r})\}| \\ &\leqslant M_1 |g'(z)|\, \hat{r} \\ &\leqslant 3300L^3 M_1 |g'(z)|\, r. \end{aligned}$$

□

3. *An upper bound for* $\dim J(f_{p,K})$

In this section we will show that, if $p \in (0,\infty)$ and K is sufficiently large, then $\dim J(f_{p,K}) \leqslant 1 + 1/(1+p)$. Let f denote a function $f_{p,K}$ with $p, K \in (0,\infty)$ and g and h denote the corresponding functions g_p and $h_{p,K}$, and take sets B_n and A_k as defined in Section 2. For the remainder of this section, we take $R_n = \pi r_n$. We begin with the following result which follows from [**6**, Lemma 4.1, Lemma 4.2].

LEMMA 3.1. *There exist* $C_6(p) \geqslant C_5(p)$ *and* $C(x_1, p) \geqslant C_2(p)$ *such that, if* $x_1 > C_6(p)$ *and* $K > C(x_1, p)$, *then*

(1) $J(f) \subset \bigcup_{n=1}^{\infty} B_n$, *and hence the sets in* $\mathscr{A}_k$ *form a cover of* $J(f)$ *for each* $k \in \mathbf{N}$;

(2) *for each* $k \in \mathbf{N}$, $\sup_{A_k \in \mathscr{A}_k} \operatorname{diam} A_k < 1/3^k \to 0$ *as* $k \to \infty$.

The next result follows directly from Lemma 3.1 and the definition of Hausdorff dimension.

LEMMA 3.2. *If* $x_1 > C_6(p)$, $K > C(x_1, p)$ *and*

$$\lim_{k\to\infty} \sum_{A_k \in \mathscr{A}_k} (\operatorname{diam} A_k)^t < \infty,$$

then $\dim J(f) \leqslant t$.

The next two results will enable us to show that, if x_1 is sufficiently large and $K > C(x_1, p)$, then

$$\lim_{k\to\infty} \sum_{A_k \in \mathscr{A}_k} (\operatorname{diam} A_k)^t < \infty$$

for each $t \in (1+1/(1+p), 2)$ and hence $\dim J(f) \leqslant 1+1/(1+p)$ as claimed.

LEMMA 3.3. *If* $t > 1$, *then there exists* $C_7(p,t) \geqslant C_6(p)$ (*where* $C_7(p,t)$ *is a decreasing function of t on* $(1,2)$) *such that, if* $x_1 > C_7(p,t)$ *and* $K > C(x_1, p)$, *then, for each* $b \in J(f)$,

$$\sum_{f(z)=b} \frac{1}{|f'(z)|^t} < \frac{1}{2}.$$

Further, if $0 < \epsilon < t-1$, *then, provided that* $|b|$ *is sufficiently large, we have*

$$\sum_{f(z)=b} \frac{1}{|f'(z)|^t} < \frac{(2M_1)^{t-1-\epsilon}}{(\log |b|)^{t-1-\epsilon} |b|^t}.$$

Proof. Suppose that $t > 1$, $x_1 > C_6(p)$ and $K > C(x_1, p)$. Take $b \in J(f)$ and a point $z \in B_n$ such that $f(z) = b$. It follows from Lemma 3.1 that $|b| \geqslant x_1 \geqslant C_4(p)$. Since we know from Lemma 2.3 that $J(f) \subset \bar{G}_p$, it follows from Corollary 2.5 and Lemma 2.6 that

$$|f'(z)| > \frac{|g'(z)\, b|}{4} > \frac{g'(x_n)\, |b|}{4M_1}. \tag{3.1}$$

We note from Lemma 2.2 that f is univalent in each $A_1 \in \mathscr{A}_1$ and so, since $b \in B_m$ for some $m \in \mathbf{N}$, and $x_1 > C_5(p)$, it follows from Lemma 2.7 that

$$\begin{aligned} &|\{z : f(z) = b, z \in B_n\}| \\ &\qquad \leqslant |\{A_1 : A_1 \text{ is a component of } f^{-1}(B_m), A_1 \cap B(x_n, \sqrt{2}\pi r_n) \neq \varnothing\}| \\ &\qquad \leqslant M_2\, g'(x_n)\, \sqrt{2}\pi r_n. \end{aligned} \tag{3.2}$$

Together with (3.1), this shows that, for each $n \in \mathbf{N}$,

$$\begin{aligned} \sum_{f(z)=b,\, z \in B_n} \frac{1}{|f'(z)|^t} &< \sqrt{2}\pi r_n\, M_2\, g'(x_n) \left(\frac{4M_1}{g'(x_n)\,|b|}\right)^t \\ &< \frac{100 M_1^2\, M_2\, r_n}{g'(x_n)^{t-1}\, |b|^t}. \end{aligned} \tag{3.3}$$

It follows from Lemma 3.1 that, if $f(z) = b$, then $z \in \bigcup_{n=1}^{\infty} B_n$. If we take N to be the smallest value of $n \in \mathbf{N}$ for which $\{z : f(z) = b\} \cap B_n \neq \varnothing$, then it follows from Lemma 2.6 and (3.3) that

$$\sum_{f(z)=b} \frac{1}{|f'(z)|^t} < \frac{100 M_1^3 M_2}{|b|^t} \int_{x_N}^{\infty} \frac{1}{g'(x)^{t-1}} \, dx. \tag{3.4}$$

Now

$$\begin{aligned}
\int_{x_N}^{\infty} \frac{1}{g'(x)^{t-1}} \, dx &= \int_{x_N}^{\infty} \left[\left(\frac{x^t}{(1-t)(1+p)^t (\log x)^{pt}} \right) \frac{\partial}{\partial x} \left(\frac{1}{e^{(\log x)^{1+p}(t-1)}} \right) \right] dx \\
&= \left[\frac{x^t}{(1-t)(1+p)^t (\log x)^{pt} e^{(\log x)^{1+p}(t-1)}} \right]_{x_N}^{\infty} \\
&\quad + \int_{x_N}^{\infty} \frac{t x^{t-1}}{(t-1)(1+p)^t (\log x)^{pt} e^{(\log x)^{1+p}(t-1)}} \left[1 - \frac{p}{\log x} \right] dx \\
&= \frac{x_N^t}{(t-1)(1+p)^t (\log x_N)^{pt} e^{(\log x_N)^{1+p}(t-1)}} \\
&\quad + \frac{t}{(t-1)(1+p)} \int_{x_N}^{\infty} \frac{1}{g'(x)^{t-1} (\log x)^p} \left[1 - \frac{p}{\log x} \right] dx.
\end{aligned}$$

It follows that there exists $C_7(p, t) \geqslant C_6(p)$ where $C_7(p, t)$ is a decreasing function of t on $(1, 2)$ such that, if $x_1 \geqslant C_7(p, t)$, then

$$\int_{x_N}^{\infty} \frac{1}{g'(x)^{t-1}} \, dx < \frac{x_N^t}{(t-1)(1+p)^t (\log x_N)^{pt} e^{(\log x_N)^{1+p}(t-1)}} + \frac{1}{2} \int_{x_N}^{\infty} \frac{1}{g'(x)^{t-1}} \, dx$$

and hence, from (3.4),

$$\sum_{f(z)=b} \frac{1}{|f'(z)|^t} < \frac{200 M_1^3 M_2 x_N^t}{(t-1)|b|^t (\log x_N)^{pt} e^{(\log x_N)^{1+p}(t-1)}} < \frac{1}{2}.$$

It follows that, if $0 < \epsilon < t-1$ and if $|b|$ and hence x_N is sufficiently large, then

$$\sum_{f(z)=b} \frac{1}{|f'(z)|^t} < \frac{1}{e^{(\log x_N)^{1+p}(t-1-\epsilon)} |b|^t} = \frac{1}{g(x_N)^{t-1-\epsilon} |b|^t}. \tag{3.5}$$

We now take a point $z \in B_N$ such that $f(z) = b$. It follows from Corollary 2.5 and Lemma 2.6 that

$$g(x_N) > \frac{|g(z)|}{M_1} > \frac{\log |b|}{2M_1}.$$

It now follows from (3.5) that

$$\sum_{f(z)=b} \frac{1}{|f'(z)|^t} < \frac{(2M_1)^{t-1-\epsilon}}{(\log |b|)^{t-1-\epsilon} |b|^t}. \qquad \square$$

We now fix $t \in (1 + 1/(1+p), 2)$. Since $C(x_1, p) \geqslant C_1(p)$, the next result follows directly from Lemma 2.1 and Lemma 2.2.

LEMMA 3.4. *If $K > C(x_1, p)$ then*

$$\sum_{A_m \in \mathcal{A}_m} (\operatorname{diam} A_m)^t \leqslant (81)^2 \sum_{n=1}^{\infty} \sum_{f^m(z)=x_n} \frac{(\operatorname{diam} B_n)^t}{|(f^m)'(z)|^t}.$$

We are now in a position to prove the main result of this section.

LEMMA 3.5. *There exists $C_8(p) \geqslant C_7(p, 1+1/(1+p))$ such that, if $x_1 > C_8(p)$ and $K > C(x_1, p)$, then*

$$\lim_{k\to\infty} \sum_{A_k \in \mathcal{A}_k} (\operatorname{diam} A_k)^t = 0.$$

Proof. Let $\epsilon = \frac{1}{2}(t-(1+1/(1+p)))$. If $x_1 > C_8(p)$ and $K > C(x_1, p)$ then, provided that $C_8(p)$ is sufficiently large, it follows from Lemma 3.3 and Lemma 3.4 that there exists $N(p,t) \in \mathbf{N}$ such that, for each $m \in \mathbf{N}$,

$$\begin{aligned}
&\sum_{A_m \in \mathcal{A}_m} (\operatorname{diam} A_m)^t \\
&\quad \leqslant (81)^2 \sum_{n=1}^{\infty} (\operatorname{diam} B_n)^t \sum_{f(b)=x_n} \frac{1}{|f'(b)|^t} \sum_{f^{m-1}(z)=b} \frac{1}{|(f^{m-1})'(z)|^t} \\
&\quad \leqslant (81)^2 \sum_{n=1}^{N(p,t)} \frac{(\operatorname{diam} B_n)^t}{2^m} \\
&\qquad + \frac{(81)^2 (2M_1)^{t-1-\epsilon}}{2^{m-1}} \sum_{n=N(p,t)}^{\infty} \frac{(\operatorname{diam} B_n)^t}{(\log x_n)^{t-1-\epsilon} x_n^t}.
\end{aligned}$$

Now, if $C_8(p)$ is sufficiently large, then

$$\begin{aligned}
&\frac{(81)^2 (2M_1)^{t-1-\epsilon}}{2^{m-1}} \sum_{n=N(p,t)}^{\infty} \frac{(\operatorname{diam} B_n)^t}{(\log x_n)^{t-1-\epsilon} x_n^t} \\
&\quad < \frac{(81)^2 (2M_1)^{t-1-\epsilon}}{2^{m-1}} \sum_{n=N(p,t)}^{\infty} \frac{r_n x_n^{t-1} (3\pi)^t}{(\log x_n)^{t-1-\epsilon} x_n^t (1+p)^{t-1} (\log x_n)^{p(t-1)}} \\
&\quad < \frac{(81)^2 (2M_1)^{t-1-\epsilon} \pi^2}{2^{m-5} (1+p)^{t-1}} \int_{x_{N(p,t)}}^{\infty} \frac{1}{x (\log x)^{(p+1)(t-1)-\epsilon}}\, dx \\
&\quad < -\frac{(81)^2 (2M_1)^{t-1-\epsilon} \pi^2}{2^{m-5}[(p+1)(t-1)-\epsilon-1]} \left[\frac{1}{(\log x)^{(p+1)(t-1)-\epsilon-1}}\right]_{x_{N(p,t)}}^{\infty}.
\end{aligned}$$

Since $\epsilon = \frac{1}{2}(t-(1+1/(1+p)))$, it follows that

$$(p+1)(t-1)-\epsilon-1 = (p+1)\left(2\epsilon + \frac{1}{1+p}\right) - \epsilon - 1 = 2\epsilon(p+1) - \epsilon > \epsilon$$

and so

$$\sum_{A_m \in \mathcal{A}_m} (\operatorname{diam} A_m)^t < (81)^2 \sum_{n=1}^{N(p,t)} \frac{(\operatorname{diam} B_n)^t}{2^m} + \frac{(81)^2 (2M_1)^{t-1-\epsilon} \pi^2}{2^{m-5} \epsilon (\log x_{N(p,t)})^{\epsilon}} \to 0$$

as $m \to \infty$. □

As noted after Lemma 3.2, this is sufficient to show that, if K is sufficiently large, then

$$\dim J(f) \leqslant 1+\frac{1}{1+p}.$$

4. *A lower bound for* $\dim J(f_{p,K})$

Let f denote a function $f_{p,K}$ with $p, K \in (0,\infty)$ and let g and h denote the corresponding functions g_p and $h_{p,K}$. We will show that, if K is sufficiently large, then $\dim J(f) \geqslant 1+1(1+p)$. To do this, we take sets B_n and A_k defined as in Section 2 with $R_n = r_n/8L$ and $x_1 = 4K^2$. We show that, for large K, the set $A = \bigcap_{k=1}^{\infty} \bigcup_{A_k \in \mathcal{A}_k} A_k$ is contained in $J(f)$ and satisfies $\dim A \geqslant 1+1/(1+p)$. We begin by proving the following preliminary results.

LEMMA 4.1. *There exists $C_9(p) \geqslant C_5(p)$ such that, if $K > C_9(p)$, $z \in A_1$ for some $A_1 \in \mathcal{A}_1$ and $f(A_1) = B_m$, then*

(1) $x_m/4 < |f(z)|/2 < |\exp g(z)| < 3|f(z)| < 4x_m$;

(2) $\Re(g(z)) = \log x_m + e(z)$, *where* $|e(z)| < 2/(\log x_m)^p < 1$;

(3) $1/(64L^2|g'(z)|(\log x_m)^p(1+p)) < \operatorname{diam} A_1 < 3/(|g'(z)|(\log x_m)^p(1+p))$.

Proof. Let $z \in A_1$, $f(A_1) = B_m$ and $K > C_3(p)$. Since $A_1 \cup B_m \subset \bar{G}_p$ and $|f(z)| \geqslant x_1 = 4K^2 > C_3(p)$, it follows from Corollary 2.5 that

$$\frac{x_m}{4} < \frac{|f(z)|}{2} < |\exp g(z)| = |h(z)+K| < 2|f(z)|+K < 3|f(z)| < 4x_m$$

and so the first part of Lemma 4.1 is true.

It follows from Lemma 2.4 that $|e^{g(z)}| = x_m + e_1(z)$, where $|e_1(z)| < K + M(p) + x_m/(\log x_m)^p$. Thus

$$\Re(g(z)) = \log x_m + e(z),$$

where

$$|e(z)| < \log\left(1+\frac{K+M(p)}{x_m}+\frac{1}{(\log x_m)^p}\right) < \frac{2}{(\log x_m)^p} < 1$$

if K is sufficiently large. Thus the second part of Lemma 4.1 is true.

It follows from Lemma 2.1, Lemma 2.2 and Corollary 2.5 that

$$\begin{aligned}\operatorname{diam} A_1 &\leqslant \frac{L \operatorname{diam} B_m}{|f'(z)|} < \frac{4L \operatorname{diam} B_m}{|g'(z)|\,|f(z)|}\\ &< \frac{8L \operatorname{diam} B_m}{|g'(z)|\,x_m} < \frac{3}{|g'(z)|\,(\log x_m)^p(1+p)}\end{aligned}$$

and

$$\begin{aligned}\operatorname{diam} A_1 &\geqslant \frac{\operatorname{diam} B_m}{L|f'(z)|} > \frac{\operatorname{diam} B_m}{8L|g'(z)|\,|f(z)|}\\ &> \frac{\operatorname{diam} B_m}{16L|g'(z)|\,x_m} > \frac{1}{64L^2|g'(z)|\,(\log x_m)^p(1+p)}.\end{aligned}$$

□

LEMMA 4.2. *There exists $C_{10}(p) \geqslant C_9(p)$ such that, if $K > C_{10}(p)$, then*

$$|(f^n)'(z)| > 2^{n^2/2}|f^n(z)|$$

for each $z \in A$, $n \in \mathbf{N}$.

Proof. If $z \in A$ and K is sufficiently large, then it follows from Corollary 2.5 that

$$|f'(z)| > \frac{|g'(z)|\,|f(z)|}{4} > 2|f(z)| \tag{4.1}$$

and, from Lemma 4.1, that

$$|f(z)| > \frac{|\exp g(z)|}{3} = \frac{\exp \Re(g(z))}{3}. \tag{4.2}$$

If $z = re^{i\theta}$, then

$$g(z) = e^{(\log r + i\theta)^{1+p}} = e^{(\log r)^{1+p}(1+i\theta/\log r)^{1+p}}.$$

If $x_1 = 4K^2$ is sufficiently large, it is not difficult to show that

$$|\theta| < 2|\tan\theta| < \frac{1}{2L(1+p)(\log r)^p}$$

and hence

$$\left(1 + \frac{i\theta}{\log r}\right)^{1+p} = 1 + \hat{e}(z),$$

where $|\hat{e}(z)| < 1/(L(\log r)^{1+p})$.

Thus, for large K,

$$\Re(g(z)) > e^{((\log r)^{1+p})/2}\cos(1/L) > 2\log r.$$

Putting this into (4.2) gives, for large K,

$$|f(z)| > \frac{e^{2\log r}}{3} = \frac{r^2}{3} > 2r = 2|z|. \tag{4.3}$$

If $z \in A$, then $f^m(z) \in A$ for each $m \in \mathbf{N}$ and so it follows from (4.1) and (4.3) that, for each $n \in \mathbf{N}$ and large K,

$$\begin{aligned} |(f^n)'(z)| &= \prod_{m=0}^{n-1} |f'(f^m(z))| > 2^n \prod_{m=0}^{n-1} |f^{m+1}(z)| \\ &> 2^n |f^n(z)| \prod_{m=0}^{n-2} 2^{m+1}|z| > 2^n 2^{((n-1)\,n)/2}\,|f^n(z)| \\ &> 2^{n^2/2}|f^n(z)|. \end{aligned}$$

□

LEMMA 4.3. *There exists $C_{11}(p) \geqslant C_{10}(p)$ such that, if $K > C_{11}(p)$, then $A \subset J(f)$.*

Proof. If $K > C_1(p)$, then it follows from Lemma 2.2 that, for each $z \in A$, $n \in \mathbf{N}$, the branch F_n of f^{-n} that maps $f^n(z)$ to z is univalent in $B(f^n(z), |f^n(z)|/2)$ and hence, from Lemma 2.1,

$$F_n\left(B\left(f^n(z), \frac{|f^n(z)|}{4}\right)\right) \subset B\left(z, \frac{L|f^n(z)|}{4|(f^n)'(z)|}\right). \tag{4.4}$$

If $z \in N(f)$, then there exists $r > 0$ such that $B(z, r) \subset N(f)$. It follows from Lemma 4.2 and (4.4) that, for sufficiently large values of n, K,

$$F_n\left(B\left(f^n(z), \frac{|f^n(z)|}{4}\right)\right) \subset B(z, r)$$

and hence

$$f^n(B(z, r)) \supset B\left(f^n(z), \frac{|f^n(z)|}{4}\right).$$

Clearly $B(f^n(z), |f^n(z)|/4) \cap \mathbf{C} \backslash \bar{G}_p \neq \varnothing$ and so it follows from Lemma 2.3 that, if $K > C_2(p)$, then, for some $w \in B(z, r)$, we have $f^m(w) \to z_0 \in \{u : \Re(u) < 0\}$ as $m \to \infty$. This contradicts the fact that $f^m(z) \in \bigcup_{k=1}^{\infty} B_k$ for each $m \in \mathbf{N}$ and so we must have $z \in J(f)$ as claimed. □

By using Lemma 4.1, we prove the following. For simplicity, we put $s = 1 + 1/(1+p)$.

LEMMA 4.4. *There exists $M_3(p) > 0$, $C_{12}(p) \geqslant C_{11}(p)$ such that, if $K > C_{12}(p)$, then, for each $m \in \mathbf{N}$ and each $A_m \in \mathscr{A}_m$, we have*

$$\sum_{\substack{A_{m+1} \subset A_m \\ A_{m+1} \in \mathscr{A}_{m+1}}} (\operatorname{diam} A_{m+1})^s > M_3(p)\,(\operatorname{diam} A_m)^s.$$

Proof. We begin by noting that, if $A_m \in \mathscr{A}_m$, then $f^m(A_m) = B_n$ for some $n \in \mathbf{N}$. It is not difficult to see that, if $A_1 \in \mathscr{A}_1$ and $A_1 \subset B_n$, then there exists $A_{m+1} \in \mathscr{A}_{m+1}$ with $A_{m+1} \subset A_m$ and $f^m(A_{m+1}) = A_1$. Clearly

$$\operatorname{diam} A_{m+1} \geqslant \frac{\operatorname{diam} A_1}{\sup_{z \in A_m} |(f^m)'(z)|}$$

and

$$\operatorname{diam} A_m \leqslant \frac{\operatorname{diam} B_n}{\inf_{z \in A_m} |(f^m)'(z)|}$$

and so, if $K > C_{11}(p)$, it follows from Lemma 2.1 and Lemma 2.2 that

$$\begin{aligned} \sum_{\substack{A_{m+1} \subset A_m \\ A_{m+1} \in \mathscr{A}_{m+1}}} \frac{(\operatorname{diam} A_{m+1})^s}{(\operatorname{diam} A_m)^s} &\geqslant \inf_{w, z \in A_m} \left|\frac{(f^m)'(w)}{(f^m)'(z)}\right|^s \sum_{\substack{A_1 \subset B_n \\ A_1 \in \mathscr{A}_1}} \frac{(\operatorname{diam} A_1)^s}{(\operatorname{diam} B_n)^s} \\ &\geqslant \frac{1}{L^s} \sum_{\substack{A_1 \subset B_n \\ A_1 \in \mathscr{A}_1}} \frac{(\operatorname{diam} A_1)^s}{(\operatorname{diam} B_n)^s}. \end{aligned} \tag{4.5}$$

We now note that it follows from Lemma 2.4 that, if K is sufficiently large and

$$h(z) = x'_m = x_m + R_m/2$$

for some $m \in \mathbf{N}$, then $f(z) \in B_m$. Thus, for each point z such that $h(z) = x'_m$, there exists a set $A_1(z) \in \mathscr{A}_1$ such that $z \in A_1(z)$ and $f(A_1(z)) = B_m$. We note further that $h(A_1(z)) \subset B(x_m, x_m/4)$ and hence, as the singularities of h^{-1} are contained in $\{z : \Re(z) < 0\}$, h is univalent in $A_1(z)$. Thus, if $z_1 \neq z_2$, then $A(z_1) \neq A(z_2)$. We note further that, if $A_1 \cap B(x'_n, R_n/4) \neq \varnothing$, then, since the sets B_k, $k \in \mathbf{N}$, are disjoint, it follows that $A_1 \subset B_n$. For each $M \in \mathbf{Z}$, we take Γ_M to be the curve defined by

$$\Gamma_M = \{z = x + iy : x \geqslant \log(K + x_1), y = 2\pi M\}.$$

If $g(z) \in \Gamma_M$, for some $M \in \mathbf{Z}$, then $h(z) \in [x_1, \infty)$. Now let γ_M be a curve which is mapped onto Γ_M by g. If $\gamma_M \cap B(x_n', R_n/8) \neq \varnothing$, then there exists a segment of $\gamma_M \cap B(x_n', R_n/4)$ of length at least $R_n/8$ and hence, from Lemma 2.6, there exists a segment of $\Gamma_M \cap g(B(x_n', R_n/4))$ of length at least $g'(x_n) R_n/(8M_1)$. It follows that there exist $z_1, z_2 \in \gamma_M \cap B(x_n', R_n/4)$ such that

$$h(z_1) = x'_{n(1)}, \quad h(z_2) = x'_{n(2)}, \quad h(\gamma_M \cap B(x_n', R_n/4)) \supset [x'_{n(1)}, x'_{n(2)}]$$

and

$$\Re(g(z_2)) - \Re(g(z_1)) \geqslant \frac{g'(x_n) R_n}{8M_1} - 2 = \frac{g(x_n)}{64LM_1} - 2 > \frac{g(x_n)}{100LM_1}. \tag{4.6}$$

Thus, if $\gamma_M \cap B(x_n', R_n/8) \neq \varnothing$, then it follows from Lemma 4.1 and Lemma 2.6 that, if K is sufficiently large, then

$$\begin{aligned}\sum_{\substack{A_1 \subset B_n \\ A_1 \cap \gamma_M \neq \varnothing}} \frac{(\operatorname{diam} A_1)^s}{(\operatorname{diam} B_n)^s} &\geqslant \frac{1}{(24L(1+p) M_1 g'(x_n) r_n)^s} \sum_{r=n(1)}^{n(2)} \frac{1}{(\log x_r)^{ps}} \\ &> \frac{1}{(50L(1+p) M_1 g(x_n))^s} \int_{x'_{n(1)}}^{x'_{n(2)}} \frac{1+p}{x(\log x)^{p(s-1)}} \, dx \\ &> \frac{1}{(50L(1+p) M_1 g(x_n))^s (\log x'_{n(2)})^{p(s-1)}} [\log x'_{n(2)} - \log x'_{n(1)}].\end{aligned} \tag{4.7}$$

Since $\exp(g(z_1)) = x'_{n(1)} + K$ and $\exp(g(z_2)) = x'_{n(2)} + K$, it follows from Lemma 2.6 that, if K is sufficiently large, then

$$\Re(g(z_2)) - 1 \leqslant \log x'_{n(2)} \leqslant \Re(g(z_2)) \leqslant |g(z_2)| \leqslant M_1 g(x_n)$$

and

$$\log x'_{n(1)} \leqslant \Re(g(z_1)).$$

Thus, if $\gamma_M \cap B(x_n', R_n/8) \neq \varnothing$, it follows from (4.6) and (4.7) that

$$\begin{aligned}\sum_{\substack{A_1 \subset B_n \\ A_1 \cap \gamma_M \neq \varnothing}} \frac{(\operatorname{diam} A_1)^s}{(\operatorname{diam} B_n)^s} &> \frac{\Re(g(z_2) - g(z_1)) - 1}{(50L(1+p))^s M_1^{s+p(s-1)} (g(x_n))^{s+p(s-1)}} \\ &> \frac{g(x_n)}{200LM_1(50L(1+p))^s M_1^2 (g(x_n))^2} \\ &> \frac{1}{(50L(1+p))^{s+2} M_1^3 g(x_n)}.\end{aligned} \tag{4.8}$$

Clearly $g(x_n') \in \Gamma_0$. Since we know from Lemma 2.6 that

$$g(B(x_n', R_n/8)) \supset B(g(x_n'), g'(x_n) R_n/(8M_1)),$$

it follows that there are at least $g'(x_n) R_n/(16M_1 \pi) = g(x_n)/(128LM_1 \pi)$ values of $M \in \mathbf{Z}$ for which there exists a curve γ_M satisfying $\gamma_M \cap B(x_n', R_n/8) \neq \varnothing$. If a set $A_1 \in \mathscr{A}_1$ meets two curves $\gamma_{M(1)}$ and $\gamma_{M(2)}$, with $M(1) \neq M(2)$, then there exist $w \in A_1$, and $M \in \mathbf{N}$ such that $\mathfrak{F}(g(w)) = (2M+1)\pi$ and hence $\Re(h(w)) < 0$. This, however, is impossible and so it follows from (4.5) and (4.8) that, if K is sufficiently large, then

$$\sum_{\substack{A_{m+1} \subset A_m \\ A_{m+1} \in \mathscr{A}_{m+1}}} \frac{(\operatorname{diam} A_{m+1})^s}{(\operatorname{diam} A_m)^s} > \frac{g(x_n)}{(50L(1+p))^{2s+3} M_1^4 g(x_n)} = \frac{1}{(50L(1+p))^{2s+3} M_1^4}. \qquad \square$$

LEMMA 4.5. *There exist $C_{13}(p) \geqslant C_{12}(p)$, $M_4(p) > 1$ such that, if $K > C_{13}(p)$, $z, f(z) \in \bigcup_{n=1}^{\infty} B_n$,*

$$\left|\frac{f(z)}{8L(1+p)(\log f(z))^p f'(z)}\right| \leqslant r \leqslant \left|\frac{z}{8L(1+p)(\log z)^p}\right|$$

and

$$H = \{A_1 : A_1 \in \mathscr{A}_1, A_1 \cap B(z,r) \neq \varnothing\},$$

then

$$\sum_{A_1 \in H} (\operatorname{diam} A_1)^s < M_4(p)(\operatorname{diam} B(z,r))^s.$$

Proof. If the conditions of the lemma are satisfied and $K > C_9(p)$, then it follows from Lemma 2.6, Lemma 2.7 and Lemma 4.1 that, for each $n \in \mathbf{N}$,

$$\sum_{\substack{A_1 \in H \\ f(A_1)=B_n}} (\operatorname{diam} A_1)^s < \frac{M_2 |g'(z)|\, r(3M_1)^s}{|g'(z)|^s (\log x_n)^{ps}}. \tag{4.9}$$

Letting m denote the smallest and M the largest values of $n \in \mathbf{N}$ for which there exists a set $A_1 \in H$ with $f(A_1) = B_n$, we deduce from (4.9) that, if K is sufficiently large, then

$$\begin{aligned}\sum_{A_1 \in H} (\operatorname{diam} A_1)^s &< \frac{9M_2 M_1^2 r}{|g'(z)|^{s-1}} \sum_{n=m}^{M} \frac{1}{(\log x_n)^{ps}} \\ &< \frac{10M_2 M_1^2 r}{|g'(z)|^{s-1}} \int_{x_m}^{x_{M+1}} \frac{(1+p)}{x(\log x)^{p(s-1)}}\,dx \\ &= \frac{10M_2 M_1^2 r(1+p)^2}{|g'(z)|^{s-1}} [(\log x_{M+1})^{1/(1+p)} - (\log x_m)^{1/(1+p)}]. \end{aligned} \tag{4.10}$$

We note that

$$\log x_{M+1} = \log\left(x_M + \frac{x_M}{(1+p)(\log x_M)^p}\right) < \log x_M + \frac{2}{(\log x_M)^p},$$

if K is sufficiently large. Also, we know from Lemma 4.1 that, if $z_1, z_2 \in B(z,r)$ with $f(z_1) \in B_m$ and $f(z_2) \in B_M$, then

$$\Re(g(z_1)) = \log x_m + e_1 \quad \text{and} \quad \Re(g(z_2)) = \log x_M + e_2,$$

where

$$|e_1| < \frac{2}{(\log x_m)^p} < 1 \quad \text{and} \quad |e_2| < \frac{2}{(\log x_M)^p} < 1$$

if $K > C_9(p)$. Thus, for large values of K, it follows from Lemma 2.6 that

$$\begin{aligned}\log x_{M+1} &< \log x_m + \Re(g(z_2) - g(z_1)) + \frac{6}{(\log x_m)^p} \\ &\leqslant \log x_m + \sup_{w \in B(z,r)} |g'(w)|\, r + \frac{6}{(\log x_m)^p} \\ &\leqslant \log x_m + M_1 |g'(z)|\, r + \frac{6}{(\log x_m)^p}. \end{aligned} \tag{4.11}$$

Together with (4.10), this implies that, if K is sufficiently large, then

$$\sum_{A_1 \in H} (\operatorname{diam} A_1)^s < \frac{10 M_2 M_1^2 r(1+p)^2}{|g'(z)|^{s-1}} \left[\left(\log x_m + M_1 |g'(z)| r + \frac{6}{(\log x_m)^p} \right)^{1/(1+p)} - (\log x_m)^{1/(1+p)} \right]. \quad (4.12)$$

If $|g'(z)| r > \log x_m$, then it follows from (4.12) that, if K is sufficiently large, then

$$\sum_{A_1 \in H} (\operatorname{diam} A_1)^s < \frac{10 M_2 M_1^2 r(1+p)^2}{|g'(z)|^{s-1}} (3M_1 |g'(z)| r)^{1/(1+p)} < 30 M_2 M_1^3 (1+p)^2 r^s. \quad (4.13)$$

Now suppose that $|g'(z)| r \leqslant \log x_m$. In this case, it follows from (4.11) that

$$|\log f(z)| \leqslant 2 \log x_M < 6 M_1 \log x_m. \quad (4.14)$$

It follows from (4.14) and Corollary 2.5 that

$$\begin{aligned} |g'(z)| r &\geqslant \left| \frac{g'(z) f(z)}{8L(1+p)(\log f(z))^p f'(z)} \right| \\ &> \left| \frac{g'(z)\, e^{g(z)}}{(6M_1)^p\, 64 L(1+p)\, g'(z)\, e^{g(z)} (\log x_m)^p} \right| \\ &= \frac{1}{(6M_1)^p\, 64 L(1+p)(\log x_m)^p} \end{aligned}$$

and so it follows from (4.12) that

$$\begin{aligned} &\sum_{A_1 \in H} (\operatorname{diam} A_1)^s \\ &< \frac{10 M_2 M_1^2 r(1+p)^2}{|g'(z)|^{s-1}} [(\log x_m + 400(6M_1)^{p+1} L(1+p)\, |g'(z)| r)^{1/(1+p)} - (\log x_m)^{1/(1+p)}] \\ &< \frac{10 M_2 M_1^2 r(1+p)^2}{|g'(z)|^{s-1}} (\log x_m)^{1/(1+p)} \left[1 + \frac{400(6M_1)^{p+1} L(1+p)\, |g'(z)| r}{\log x_m} - 1 \right] \\ &= \frac{10 M_2 M_1^2 r^2 (1+p)^3 |g'(z)|^{p/(1+p)}\, 400(6M_1)^{p+1} L}{(\log x_m)^{p/(1+p)}} \\ &< 4000 M_2 (6M_1)^{3+p} (1+p)^3 L \left(\frac{r g'(z)}{\log x_m} \right)^{p/(1+p)} r^s \\ &\leqslant 4000 M_2 (6M_1)^{3+p} (1+p)^3 L r^s. \end{aligned}$$

Together with (4.13), this gives the desired result. □

Using Lemma 4.5, we prove the following result.

LEMMA 4.6. *There exists $C_{14}(p) \geqslant C_{13}(p)$ such that, if $K > C_{14}(p)$, $z \in A$, $r > 0$ and $n \geqslant 2$ is the smallest integer for which*

$$|(f^n)'(z)|\, r > \left|\frac{f^n(z)}{8L^2(1+p)(\log f^n(z))^p}\right|,$$

then

$$\sum_{\substack{A_n \cap B(z,r) \neq \varnothing \\ A_n \in \mathscr{A}_n}} (\operatorname{diam} A_n)^s < M_4(p)\, L^{2s} (\operatorname{diam} B(z,r))^s.$$

Proof. If $z \in A$, $r > 0$ and $n \geqslant 2$ is the integer described above, then

$$\left|\frac{f^n(z)}{8L(1+p)(\log f^n(z))^p f'(f^{n-1}(z))}\right| < |(f^{n-1})'(z)|\, rL \leqslant \left|\frac{f^{n-1}(z)}{8L(1+p)(\log f^{n-1}(z))^p}\right| \tag{4.15}$$

and

$$f^m(z) \in A \subset \bigcup_{k=1}^{\infty} B_k \tag{4.16}$$

for each $m \in \mathbf{N}$. It follows from Lemma 2.1, Lemma 2.2, (4.15) and (4.16) that, if $K > C_1(p)$, then

$$f^{n-1}(B(z,r)) \subset B(f^{n-1}(z), |(f^{n-1})'(z)|\, rL). \tag{4.17}$$

It follows from (4.15)–(4.17) and Lemma 4.5 that, if $K > C_{13}(p)$, then

$$\sum_{\substack{A_1 \cap f^{n-1}(B(z,r)) \neq \varnothing \\ A_1 \in \mathscr{A}_1}} (\operatorname{diam} A_1)^s < M_4(p)\, (2|(f^{n-1})'(z)|\, rL)^s. \tag{4.18}$$

We note that it follows from Lemma 4.1 that, if K is sufficiently large, then

$$\sup_{A_1 \in \mathscr{A}_1} \operatorname{diam} A_1 < 1. \tag{4.19}$$

If $A_n \in \mathscr{A}_n$ and $A_n \cap B(z,r) \neq \varnothing$, then it is clear that there exists $A_1 \in \mathscr{A}_1$ with $A_1 \cap f^{n-1}(B(z,r)) \neq \varnothing$ and $f^{n-1}(A_n) = A_1$. It follows from (4.15)–(4.17), (4.19), Lemma 2.1 and Lemma 2.2 that

$$\operatorname{diam} A_n \leqslant \frac{L \operatorname{diam} A_1}{|(f^{n-1})'(z)|}$$

and so it follows from (4.18) that

$$\sum_{\substack{A_n \cap B(z,r) \neq \varnothing \\ A_n \in \mathscr{A}_n}} (\operatorname{diam} A_n)^s \leqslant \frac{L^s}{|(f^{n-1})'(z)|^s} M_4(p)\, (2|(f^{n-1})'(z)|\, rL)^s = M_4(p)\, L^{2s} (\operatorname{diam} B(z,r))^s. \qquad \square$$

We are now in a position to prove the main result of this section.

Lemma 4.7. *Let $K > C_{14}(p)$. For each $\delta > 0$, there exists $r(\delta) > 0$ such that, if the sets $B(z_i, r_i)$, $1 \leqslant i \leqslant N$, form a cover of $A \cap B_1$ with $z_i \in A \cap B_1$ and $r_i < r(\delta)$ for each $1 \leqslant i \leqslant N$, then*

$$\sum_{i=1}^{N} (2r_i)^{s-\delta} > \sum_{\substack{A_1 \subset B_n \\ A_1 \in \mathscr{A}_1}} (\operatorname{diam} A_1)^s.$$

Proof. Take $M \in \mathbf{N}$ with $M \geqslant 2$, $\delta > 0$ and a collection of sets $D_i = B(z_i, r_i)$, $1 \leqslant i \leqslant N$, which form a cover of $A \cap B_1$ with $z_i \in A \cap B_1$ and

$$r_i < \inf_{\substack{z \in J(f) \cap B_1 \\ 1 \leqslant m \leqslant M}} \left| \frac{f^m(z)}{8L^2(1+p)(\log f^m(z))^p (f^m)'(z)} \right| = R_M \tag{4.20}$$

for each $1 \leqslant i \leqslant N$. Denote by $n(i)$ the smallest integer n for which

$$r_i |(f^n)'(z_i)| > \left| \frac{f^n(z_i)}{8L^2(1+p)(\log f^n(z_i))^p} \right|. \tag{4.21}$$

The existence of $n(i)$ follows from Lemma 4.2. It follows from Lemma 4.3 that $A \subset J(f)$ and so, from (4.20), that $n(i) > M \geqslant 2$. We may assume that $n(1) \leqslant n(2) \leqslant \ldots \leqslant n(N)$. It follows from Lemma 4.4 and Lemma 4.6 that

$$\begin{aligned}
&\sum_{\substack{A_1 \subset B_1 \\ A_1 \in \mathscr{A}_1}} (\operatorname{diam} A_1)^s \\
&\quad < \frac{1}{M_3^{n(1)-1}(p)} \sum_{A_{n(1)} \subset B_1} (\operatorname{diam} A_{n(1)})^s \\
&\quad \leqslant \frac{1}{M_3^{n(1)-1}(p)} \sum_{A_{n(1)} \cap D_1 \neq \varnothing} (\operatorname{diam} A_{n(1)})^s + \frac{1}{M_3^{n(1)-1}(p)} \sum_{A_{n(1)} \subset B_1 \setminus D_1} (\operatorname{diam} A_{n(1)})^s \\
&\quad < \frac{M_4(p)\, L^{2s}}{M_3^{n(1)-1}(p)} (\operatorname{diam} D_1)^s + \frac{1}{M_3^{n(2)-1}(p)} \sum_{A_{n(2)} \subset B_1 \setminus D_1} (\operatorname{diam} A_{n(2)})^s \\
&\quad < M_4(p)\, L^{2s} \left[\frac{(\operatorname{diam} D_1)^s}{M_3^{n(1)-1}(p)} + \frac{(\operatorname{diam} D_2)^s}{M_3^{n(2)-1}(p)} \right] + \frac{1}{M_3^{n(2)-1}(p)} \sum_{A_{n(2)} \subset B_1 \setminus (D_1 \cup D_2)} (\operatorname{diam} A_{n(2)})^s \\
&\quad \leqslant \\
&\quad\ \ \vdots \\
&\quad \leqslant M_4(p)\, L^{2s} \sum_{i=1}^{N} \frac{(\operatorname{diam} D_i)^s}{M_3^{n(i)-1}(p)} + \frac{1}{M_3^{n(N)-1}(p)} \sum_{A_{n(N)} \subset B_1 \setminus \bigcup_{i=1}^{N} D_i} (\operatorname{diam} A_{n(N)})^s.
\end{aligned} \tag{4.22}$$

(Note that the expression $A_{n(i)}$ should be taken to denote a member of $\mathscr{A}_{n(i)}$ wherever it appears in the above equation.) As it follows from Lemma 4.4 that $A_{n(N)} \cap A \neq \varnothing$ for each $A_{n(N)} \in \mathscr{A}_{n(N)}$ and the sets D_i form a cover of $B_1 \cap A$, we see that there are no sets $A_{n(N)} \in \mathscr{A}_{n(N)}$ for which $A_{n(N)} \subset B_1 \setminus \bigcup_{i=1}^{N} D_i$ and so it follows from (4.22) that

$$\begin{aligned}
\sum_{\substack{A_1 \subset B_1 \\ A_1 \in \mathscr{A}_1}} (\operatorname{diam} A_1)^s &< M_4(p)\, L^{2s} \sum_{i=1}^{N} \frac{(\operatorname{diam} D_i)^s}{M_3^{n(i)-1}(p)} \\
&\leqslant \max_{1 \leqslant i \leqslant N} \frac{(\operatorname{diam} D_i)^\delta}{M_3^{n(i)-1}(p)} M_4(p)\, L^{2s} \sum_{i=1}^{N} (\operatorname{diam} D_i)^{s-\delta}.
\end{aligned} \tag{4.23}$$

It follows from (4.21) and Lemma 4.2 that, for each $1 \leqslant i \leqslant N$,

$$\begin{aligned}\frac{(\operatorname{diam} D_i)^{\delta}}{M_3^{n(i)-1}(p)} M_4(p)\, L^{2s} &\leqslant \frac{M_4(p)\, L^{2s}}{M_3^{n(i)-1}(p)} \left| \frac{f^{n(i)-1}(z_i)}{4L^2(1+p)\,(\log f^{n(i)-1}(z_i))^p\,(f^{n(i)-1})'(z_i)} \right|^{\delta} \\ &< \frac{M_4(p)\, L^{2s}}{M_3^{n(i)-1}(p)\, 2^{(n(i)-1)^2 \delta/2}} \\ &\leqslant \left(\frac{M_4(p)\, L^{2s}}{M_3(p)\, 2^{(n(i)-1)\delta/2}} \right)^{n(i)-1}.\end{aligned}$$

It is clear that

$$\frac{M_4(p)\, L^{2s}}{M_3(p)\, 2^{(n(i)-1)\delta/2}} < 1$$

provided that $n(i)$ is sufficiently large. As $n(i) > M$ for each $1 \leqslant i \leqslant N$, it follows from (4.23) that, provided that M is sufficiently large,

$$\sum_{\substack{A_1 \subset B_1 \\ A_1 \in \mathscr{A}_1}} (\operatorname{diam} A_1)^s < \sum_{i=1}^{N} (2r_i)^{s-\delta}.$$

By taking $r(\delta) = R_M$ for a sufficiently large value of M, we obtain the desired result. □

It follows from Lemma 4.7 that, if $K > C_{14}(p)$, then, for each $\delta > 0$, $\dim(A \cap B_1) \geqslant s-\delta$ and hence $\dim A \geqslant s$. Since $A \subset J(f)$, it follows that, for large values of K, $\dim J(f) \geqslant s = 1+1/(1+p)$ as claimed. Together with the results of Section 3, this is sufficient to prove Theorem 1.3.

5. *Outline proof of Theorem* 1.4

Let $F_{p,K,a,b}$ denote the function defined by

$$F_{p,K,a,b}(z) = f_{p,K}(z) + \frac{b}{z-a},$$

where $p, K > 0$, $0 < b < 1$ and $a \in \mathbf{C}$. It is clear that $F_{p,K,a,b}$ is a transcendental meromorphic function with one pole and that, for $|z-a| > \sqrt{b}$, we have

$$|F_{p,K,a,b}(z) - f_{p,K}(z)| < 1 \quad \text{and} \quad |F'_{p,K,a,b}(z) - f'_{p,K}(z)| < 1. \tag{5.1}$$

Given $p \in (0,\infty)$, we choose $a \in G_p$ such that $g_p(a)$ is large and negative. It is not difficult to show that, if z is a critical point of $F_{p,K,a,b}$ and $|z-a| < \sqrt{b}$, then, provided that b is sufficiently small and K is sufficiently large, we have $F_{p,K,a,b}(z) \in \{z : \Re(z) < 0\}$ and so it follows from (5.1) that similar arguments to those used in the proof of Lemma 2.2 can be used to show that, if K is sufficiently large, then

$$\{z : z \text{ is a singularity of } F^{-n}_{p,K,a,b} \text{ for some } n \in \mathbf{N}\} \subset \{z : \Re(z) < 0\}.$$

We also choose b sufficiently small to ensure that, for each $n \in \mathbf{N}$, there is at most one pre-image of B_n under $F_{p,K,a,b}$ in $B(a, \sqrt{b})$, the size of which can be kept arbitrarily small by taking an appropriately small value of b.

The above comments are sufficient to indicate that, given a value of $p \in (0, \infty)$, we can choose $a(p) \in \mathbf{C}$ and $b(p) \in (0, 1)$ in such a way as to allow similar arguments to those used in Sections 2–4 to be applied to the functions $F_{p,K} = F_{p,K,a(p),b(p)} \in M$, thus proving Theorem 1.4.

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