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The Hausdorff dimension of Julia sets of entire functions III

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Abstract

It is known that, for a transcendental entire function \( f \), the Hausdorff dimension of the Julia set of \( f \) satisfies \( 1 \leq \text{dim} J(f) \leq 2 \). In this paper we introduce a family of transcendental entire functions \( f_{p,K} \) for which the set \( \{\text{dim} J(f_{p,K}) : 0 < p, K < \infty \} \) has infemum 1 and supremum 2.

1. Introduction

Let \( f: \mathbb{C} \to \mathbb{C} \) denote a transcendental entire function and \( f^n, n \in \mathbb{N} \), the \( n \)th iterate of \( f \). The set of normality, \( N(f) \) is defined to be the set of points, \( z \in \mathbb{C} \), such that the sequence \( (f^n)_{n \in \mathbb{N}} \) forms a normal family in some neighbourhood of \( z \). It is easy to see that \( N(f) \) is open and has the property of complete invariance under \( f \), that is \( z \in N(f) \) if and only if \( f(z) \in N(f) \). The complement, \( J(f) \), of \( N(f) \) is known as the Julia set. This set is clearly closed and completely invariant under \( f \). More details of these and other basic properties of the sets \( N(f) \) and \( J(f) \) can be found in [3] and [4].

It was shown by Baker [1, corollary to theorem 1] that, if \( f \) is a transcendental entire function, then \( J(f) \) must contain continua and so the Hausdorff dimension of \( J(f) \), \( \text{dim} J(f) \), lies in the range \( 1 \leq \text{dim} J(f) \leq 2 \).

If a function \( f \) has the property that \( J(f) = \mathbb{C} \), then clearly \( \text{dim} J(f) = 2 \). An example of such a function is \( f(z) = e^z \). McMullen has shown that, in fact, all functions of the form \( f(z) = \lambda e^z, \lambda \in \mathbb{R} \), have Julia sets of dimension 2 [6, theorem 1-2] even though, when \( 0 < \lambda < 1/e \), \( J(f) \) has area zero [6, theorem 1-3]. Similar arguments can be applied to the functions defined by

\[
F_n(z) = \frac{1}{2\pi i} \int_{\gamma_n} \frac{\exp(t^n)}{t-z} dt - K_n,
\]

where \( \gamma_n \) is the boundary of

\[
H_n = \{ z = re^{i\theta} : |\theta| < \pi/n, \ 0 < r < \infty \},
\]

described in a clockwise direction for \( z \in \mathbb{C} \setminus H_n \) and by analytic continuation for \( z \in H_n \). Provided \( K_n \) is sufficiently large, it can be shown that, for each \( n \in \mathbb{N}, n \geq 2 \), \( \text{dim} J(F_n) = 2 \).

In [7] we gave the first examples of transcendental entire functions whose Julia sets have been proved to have dimension less than 2. More precisely, we considered the function \( E \) defined by

\[
E(z) = \frac{1}{2\pi i} \int_L \frac{\exp(e^t)}{t-z} dt,
\]
where $L$ is the boundary of the region

$$G = \{ z : \text{Re}(z) > 0, \ -\pi < \text{Im}(z) < \pi \},$$

described in a clockwise direction, for $z \in \mathbb{C} \setminus \bar{G}$, and by analytic continuation for $z \in \bar{G}$. We proved the following result.

**Theorem A.** Given $\delta > 0$, there exists $K_0(\delta)$ such that, for all $K > K_0$, the function $f_K$ defined by $f_K(z) = E(z) - K$ satisfies $1 \leq \dim J(f_K) < 1 + \delta$.

In [8] we showed further that there exists $K_0$ such that, for all $K > K_0$, $\dim J(f_K) > 1$.

These results lead us to consider the intermediate family of functions defined by

$$E_p(z) = \frac{1}{2\pi i} \int_{L_p} \frac{\exp(e^{(\log t)^+p})}{t - z} dt,$$

where $L_p$ is the boundary of the region

$$G_p = \{ z = x + iy : |y| \leq \pi x / [(1 + p)(\log(x))^p], \ x \geq 3 \}$$

described in a clockwise direction, for $z \in \mathbb{C} \setminus G_p$, and by analytic continuation for $z \in G_p$. We then define $f_{p,K}(z) = E_p(z) - K$. By using a modification of McMullen’s method, together with the methods used in the proof of Theorem A, we are able to prove the following.

**Theorem B.** There exists a positive real-valued increasing function $S$ defined on $(1, \infty)$ such that

(i) for $p \geq 2$, if $K > S(p)$, then $\dim J(f_{p,K}) \leq 1 + 2/p$;
(ii) for $0 < p \leq 1/2$, if $K > S(1/p)$, then $\dim J(f_{p,K}) \geq 2 - p$.

In future work we hope to show that, for each $n \in \mathbb{N}$, $K > S(n)$, $\dim J(f_{p,K})$ is a continuous function of $p$ for $p \in [1/n, n]$ and hence that, for each $d \in (1, 2]$, there exists a transcendental entire function $f$ for which $\dim J(f) = d$.

2. Methods for obtaining estimates for Hausdorff dimension

We begin this section with a formal definition for the Hausdorff dimension of a compact set $E$. If, for each $\mu > 0$, we put

$$H_\mu(E) = \lim \inf_{\epsilon \to 0} \sum_i (r_i)^\mu,$$

where the inf is taken over all possible covers of $E$ with sets of diameter $r_i < \epsilon$, then the Hausdorff dimension $d = \dim E$ of the set $E$ is defined to be the unique value satisfying

$$H_\mu(E) = \begin{cases} \infty & \text{for} \quad \mu < d \\ 0 & \text{for} \quad \mu > d. \end{cases}$$

For more details see, for example, [5, p. 220].

The following result is part of Frostman’s Lemma and is a commonly used method for obtaining a lower bound for $\dim E$. We include the proof as it is so short.
Lemma 2.1. Let $E$ be a compact subset of $\mathbb{R}^n$ and $t$ be a real positive value. Suppose that there exist a measure $\mu$ supported on $E$ and constants $C, r_0 > 0$ such that every ball $B$ of radius $r < r_0$ satisfies $\mu(B) \leq C r^t$. Then $\dim E \geq t$.

Proof. Take a value $\varepsilon, 0 < \varepsilon < r_0$, and a countable collection of balls $B_i$ of radius $r_i < \varepsilon$ which gives a cover of $E$. Then

$$0 < M = \mu(E) \leq \sum_i \mu(B_i) \leq C \sum_i (r_i)^t$$

and so $\sum_i (r_i)^t \geq M/C$. Thus $H_t(E) \geq M/C > 0$ and hence $\dim E \geq t$.

By using Lemma 2.1, McMullen [6, proposition 2.2] was able to prove the following.

Lemma 2.2. For each $k \in \mathbb{N}$, let $\mathcal{A}_k$ be a finite collection of disjoint compact subsets of $\mathbb{R}^n$, each of which has positive $n$-dimensional measure, and define

$$\mathcal{A}_k = \bigcup_{A_k \in \mathcal{A}_k} A_k, \quad A = \bigcap_{k=1}^\infty \mathcal{A}_k.$$

Suppose also that, for each $A_k \in \mathcal{A}_k$, there exist $A_{k+1} \in \mathcal{A}_{k+1}$ and a unique $A_{k-1} \in \mathcal{A}_{k-1}$ such that $A_{k+1} \subset A_k \subset A_{k-1}$.

If $\Delta_k, d_k$ are such that, for each $A_k \in \mathcal{A}_k$,

$$\frac{\text{vol}(\mathcal{A}_{k+1} \cap A_k)}{\text{vol}(A_k)} \geq \Delta_k$$

and $\text{diam} A_k \leq d_k, d_k \to 0$ as $k \to \infty$, then

$$\lim_{k \to \infty} \sum_{i=1}^k \frac{\text{vol}(\mathcal{A}_{i+1} \cap A_i)}{\text{vol}(A_i)} \geq n - \dim A.$$

We will use some of the ideas of this result in our proof of Theorem B, part (ii).

3. Properties of the functions $E_p(z)$

Consider the functions $E_{p,n}(z), n \in \mathbb{N}, n \geq 3, 0 < p < \infty$, defined by

$$E_{p,n}(z) = \frac{1}{2\pi i} \int_{L_{p,n}} \frac{\exp(e^{(\log t)^{1+p}})}{t-z} dt,$$

where $L_{p,n}$ is the boundary of the region

$$G_{p,n} = \{z = x + iy: |y| \leq \pi x / [(1+p)(\log(x))^p], \quad x \geq n\}$$

described in a clockwise direction, for $z \in \mathbb{C} \setminus G_{p,n}$. It follows from Cauchy’s theorem that, for each $n \in \mathbb{N}, n \geq 3$, $E_{p,n+1}(z) = E_{p,n}(z)$ for $z \in \mathbb{C} \setminus G_{p,n}$ and so the functions $E_{p,n}, n \geq 3$, give an analytic continuation of $E_{p,3}$ to a function $E_p$ defined on the whole of $\mathbb{C}$. For simplicity, we write $G_p$ for $G_{p,3}$ and $L_p$ for $L_{p,3}$. 
Lemma 3.1. For each \( n \in \mathbb{N} \), there exist \( C(n), D(n) > 1 \) such that, for each \( p \in [1/n, n] \),

(i) if \( z \in \mathbb{C} \setminus G_p \), then \( |E_p(z)| < C(n) \);

(ii) if \( z \in G_p \), then \( |E_p(z)| - \exp (e^{\log z^{1+p}}) < C(n) \) and hence

\[
|E_p(z) - (1 + p)(\log z)^p e^{(\log z)^{1+p}} \exp (e^{(\log z)^{1+p}}/z) | < C(n);
\]

(iii) if \( z = x + iy \in G_p \) with \( x < D(n) \), then \( |E_p(z)| < C(n) \);

(iv) if \( z = x + iy \in G_p \) with \( x \geq D(n) \), then \(|1 + p)(\log z)^p e^{(\log z)^{1+p}}/z| > 8.
\]

Proof. If \( t = re^{i\theta} \), then there exists \( K(n) \) such that, for \( p \in [1/n, n] \), \( |\theta| < \pi/2 \), \( r \geq 3 \), we have

\[
\exp (\log t)^{1+p} = \exp (\log r + i\theta)^{1+p} = \exp \left( \log r \right)^{1+p} \left( 1 + \frac{i\theta}{\log r} \right)^{1+p} 
\]

\[
= \exp \left( \log r \right)^{1+p} \left( 1 + \frac{i\theta(1+p)}{\log r} + \epsilon \right) 
\]

\[
= \exp \left( \log r \right)^{1+p} + e_x + i(\theta(1+p)(\log r)^p + e_y). 
\]

where \( |\epsilon| \leq \frac{K(n) \theta^2}{(\log r)^2} \) and hence \( |e_x|, |e_y| \leq K(n) \theta^2(\log r)^{p-1} \).

Given \( \epsilon > 0 \), it is clear that there exists \( M(n, \epsilon) \) such that, for

\[
t = re^{i\theta} = x + iy, \quad x \geq M(n, \epsilon), \quad \frac{5\pi x}{6(\log x)^p(1+p)} \leq y \leq \frac{7\pi x}{6(\log x)^p(1+p)}.
\]

we have

\[
|\theta(1+p)(\log r)^p + e_y| \leq (1 + \epsilon) |\theta(1+p)(\log r)^p |
\]

\[
\leq (1 + \epsilon) \frac{y}{x}(1+p)(\log r)^p \leq (1 + \epsilon)^2 7\pi/6.
\]

and, similarly,

\[
|\theta(1+p)(\log r)^p + e_y| \leq (1 - \epsilon)^2 5\pi/6.
\]

Thus, by taking a sufficiently small value for \( \epsilon \), we find that there exists \( M(n) \) such that, for

\[
p \in [1/n, n], \quad t = re^{i\theta} = x + iy, \quad x \geq M(n), \quad \frac{5\pi x}{6(\log x)^p(1+p)} \leq y \leq \frac{7\pi x}{6(\log x)^p(1+p)}.
\]

we have

\[
\cos [\theta(1+p)(\log r)^p + e_y] \leq -1/2 \quad \text{and} \quad |e_x| \leq (\log r)^{p-1}/2
\]

so that

\[
\text{Re} [\exp (\log t)^{1+p}] \leq -\exp [((\log r)^{1+p}/2)/2 \leq -\exp [((\log x)^{1+p}/2]/2
\]

and hence

\[
|\exp [e^{(\log t)^{1+p}}]| \leq \exp [- \exp [(1/2)e^{\log x^{1+p}}/2]].
\]

(3.1)

If \( z \in \mathbb{C} \setminus G_p \), then it follows from (3.1) and Cauchy’s residue theorem that

\[
E_p(z) = \frac{1}{2\pi i} \int_{L_p} \frac{\exp(e^{(\log t)^+p})}{t - z} dt,
\]

where \( L_p \) is the boundary of

\[
G_p = \left\{ z = x + iy : x \geq M(n), \quad |y| \leq \frac{5\pi x}{6(\log x)^p(1+p)} \right\}
\]
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If we choose $M(n)$ sufficiently large to ensure that $M(n) \geq n$ and

$$\frac{\pi x}{6(\log x)^p(1+p)} \geq 2$$

for $x \geq M(n)$, $p \in [1/n, n]$, then it follows that

$$|E_p(z)| < \frac{1}{2\pi} \int_{L_p} |\exp (e^{(\log x)^p})| \, |dt|.$$ 

On the curves described by

$$|y| = \frac{5\pi x}{6(\log x)^p(1+p)}$$

we have

$$\left| \frac{dy}{dx} \right| = \left| \frac{5\pi}{6(1+p)} \left( 1/(\log x)^p - p/(\log x)^{1+p} \right) \right| < 2\pi$$

and hence $|dt| < 4\pi dx$. Thus, from (3.1), if $z \in \mathbb{C} \setminus \overline{G_p}$, then

$$|E_p(z)| < M(n) \exp \left[ e^{(\log z)^{1+p}} \right] + 2 \int_{M(n)}^{\infty} \exp \left[ -(1/2) e^{(\log x)^{1+p}/2} \right] \, dx.$$ 

If $z = x + iy \in \overline{G_p}$ and $x < M(n) + 1$, then it follows from Cauchy’s residue theorem and (3.1) that

$$E_p(z) = \frac{1}{2\pi i} \int_{L_p^*} \frac{\exp (e^{(\log t)^{1+p}})}{t-z} \, dt,$$

where $L_p^*$ is the boundary of

$$G_p^* = \left\{ z = x + iy : x \geq M(n) + 2, \, |y| \leq \frac{7\pi x}{6(\log x)^p(1+p)} \right\}$$

described in a clockwise direction and hence, by similar arguments to those used above,

$$|E_p(z)| < M(n) \exp \left[ e^{(\log z)^{1+p}} \right] + 2 \int_{M(n)}^{\infty} \exp \left[ -(1/2) e^{(\log x)^{1+p}/2} \right] \, dx$$

and hence

$$|E_p(z) - \exp [e^{(\log z)^{1+p}}]| < 2M(n) \exp \left[ e^{(\log z)^{1+p}} \right]$$

$$+ 2 \int_{M(n)}^{\infty} \exp \left[ -(1/2) e^{(\log x)^{1+p}/2} \right] \, dx.$$ 

Finally, if $z = x + iy \in \overline{G_p}$ and $x \geq M(n) + 1$, then it follows from Cauchy’s residue theorem and (3.1) that

$$E_p(z) = \exp [e^{(\log z)^{1+p}}] + \frac{1}{2\pi i} \int_{L_p} \frac{\exp (e^{(\log t)^{1+p}})}{t-z} \, dt,$$

where $L_p$ is the boundary of

$$G_p = \left\{ z = x + iy : x \geq M(n), \, |y| \leq \frac{7\pi x}{6(\log x)^p(1+p)} \right\}$$
described in a clockwise direction and so, arguing as before, we find that

$$|E_p(z) - \exp(e^{(\log z)^{1+p}})| < M(n) \exp(e^{(\log 2M(n))^{1+p}}) + 2 \int_{M(n)}^{\infty} \exp[-(1/2) e^{(\log x)^{1+p}/2}] \, dx.$$ 

Combined with (3-2) and (3-4) this shows that parts (i) and (ii) of Lemma 3-1 are satisfied if we take

$$C(n) = 2M(n) \exp(e^{(\log 2M(n))^{1+p}}) + 2 \int_{M(n)}^{\infty} \exp[-(1/2) e^{(\log x)^{1+p}/2}] \, dx < \infty.$$

If we take $D(n) = M(n) + 1$ then part (iii) of Lemma 3-1 follows from (3-3).

By using arguments similar to those used in proving (3-1), it can be seen that, provided $M(n)$ is chosen to be sufficiently large, part (iv) of Lemma 3-1 is also true.

We now consider the inverse function singularities of the functions $f_{p,K}$ defined by

$$f_{p,K}(z) = E_p(z) - K.$$ 

We put

$$S(f) = \{z : z \text{ is a finite singularity of } f^{-1}\},$$

$$P(f) = \{z : z \text{ is a finite singularity of } f^{-n} \text{ for some } n \in \mathbb{N}\} = \bigcup_{n=0}^{\infty} f^n(S(f)),$$

$$\hat{C}(p) = \begin{cases} C([p] + 1) & \text{if } p \geq 1, \\ C([1/p] + 1) & \text{if } 0 < p < 1, \end{cases}$$

$$B(z, r) = \{w : |w - z| < r\}.$$ 

**Lemma 3-2.** For each $p \in (0, \infty), K > 2\hat{C}(p)$, we have $P(f_{p,K}) \subset B(-K, 2\hat{C}(p))$.

**Proof.** The finite transcendental singularities of $(f_{p,K})^{-1}$ are the finite asymptotic values of $f_{p,K}$ on curves going to $\infty$. From Lemma 3-1, we see that these must be contained in the disc $B(-K, \hat{C}(p))$.

The remaining singularities of $(f_{p,K})^{-1}$ are the images of points $z$ such that $(f_{p,K})'(z) = 0$. If $z \in \mathbb{C}\setminus G_p$, then it follows from Lemma 3-1 that $f_{p,K}(z) \in B(-K, \hat{C}(p))$. If $z \in G_p$, then it follows from Lemma 3-1 that

$$|(1+p)(\log z)^p e^{(\log z)^{1+p}} \exp(e^{(\log z)^{1+p}/2})/z| < \hat{C}(p)$$

and either $f_{p,K}(z) \in B(-K, \hat{C}(p))$ or $|(1+p)(\log z)^p e^{(\log z)^{1+p}}|/z| > 1$. It follows that either $f_{p,K}(z) \in B(-K, \hat{C}(p))$ or $|\exp(e^{(\log z)^{1+p}})| < \hat{C}(p)$ and hence, from Lemma 3-1, $f_{p,K}(z) \in B(-K, 2\hat{C}(p))$.

Thus $S(f_{p,K}) \subset B(-K, 2\hat{C}(p))$. As $K > 2\hat{C}(p)$, we have $B(-K, 2\hat{C}(p)) \subset \mathbb{C}\setminus G_p$ and so it follows from Lemma 3-1 that $f_{p,K}(B(-K, 2\hat{C}(p))) \subset B(-K, \hat{C}(p))$. Thus, for each $n \in \mathbb{N}$, $(f_{p,K})^n(B(-K, 2\hat{C}(p))) \subset B(-K, \hat{C}(p))$ and hence $P(f_{p,K}) \subset B(-K, 2\hat{C}(p))$.

At several points in the proof of Theorem B we use the previous result together with the following result which is known as Koebe’s distortion theorem.

**Lemma 3-3** (See, for example, [2, p. 32]). If $g$ is univalent in the disc $B(z, r)$ then, for $0 < s < r$,

$$\sup_{v, w \in B(z, s)} |g'(v)/g'(w)| \leq L(s/r) = [(r+s)/(r-s)]^4.$$
For simplicity we denote $L(1/2)$ by $L(=81)$. We conclude this section with some results concerning the set of normality and the Julia set of $f_{p,K}$.

**Lemma 34.** If $K > \tilde{C}(p)$ then $\mathbb{C} \setminus \mathbb{G}_p \subset N(f_{p,K})$.

**Proof.** It follows from Lemma 31 that, if $K > \tilde{C}(p)$, then

$$f_{p,K}(\mathbb{C} \setminus \mathbb{G}_p) \subset B(-K, \tilde{C}(p)) \subset \{z : \text{Re}(z) < 0\} \subset \mathbb{C} \setminus \mathbb{G}_p$$

and hence $\mathbb{C} \setminus \mathbb{G}_p \subset N(f_{p,K})$.

We note that it is clear that there exists $z_0 \in B(-K, \tilde{C}(p))$ such that, for each $z$ in the component of $N(f_{p,K})$ which contains $\mathbb{C} \setminus \mathbb{G}_p$,

$$(f_{p,K})^n(z) \rightarrow z_0 \text{ as } n \rightarrow \infty.$$  

We put

$$E_{R,N} = \{z : (f_{p,K})^n(z) \in \mathbb{G}_{p,R} \text{ for } 0 \leq n \leq N\},$$

$$E_R = \bigcap_{N=1}^{\infty} E_{R,N},$$

$$\tilde{D}(p) = \begin{cases} D([p]+1) & \text{if } p \geq 1 \\ D([1/p]+1) & \text{if } 0 < p < 1, \end{cases}$$

and take $D'(p)$ to be a value satisfying

$$D'(p) \geq 24L\tilde{C}(p).$$

Finally, we put

$$\tilde{C}(p) = \max \{\exp\left((p+1/p)8 \cdot 3 \times 10^8\right), \tilde{C}(p), \exp\left((5 \times 10^5)/p\right)^{1/p}, \tilde{D}(p), D'(p)\}.$$  

**Lemma 35.** If $R \geq \tilde{C}(p)$ and $K > 2\tilde{C}(p)$ then

(i) for each $z \in E_{R,N}, 1 \leq n \leq N$, we have $\left| (f_{p,K}^n)'(z) \right| > 3^n \left| f_{p,K}^n(z) \right|$, 

(ii) $E_R \subset J(f_{p,K})$.

**Proof.** (i) If $R \geq \tilde{C}(p)$ and $z = x + iy \in \mathbb{G}_{p,R}$, then

$$y \leq \frac{\pi x}{(1 + p)(\log x)^p} \leq \frac{\pi x}{(1 + p)(\log R)^p} \leq \frac{\pi x}{(1 + p)4\pi} < x/4 < |z|/4$$  \hspace{1cm} (3-5)$$

and hence

$$|z| < 2x.$$  \hspace{1cm} (3-6)$$

If, further, $K > \tilde{C}(p)$, then it follows from Lemma 31 that

$$\left| (f_{p,K}^n)'(z) \right| > \left| (1 + p)(\log z)^{p+1} e^{(\log z)^{p+1}}\exp(e^{(\log z)^{p+1}})/z \right| - \tilde{C}(p)$$

$$> 8\left| e^{(\log z)^{p+1}} \right| - \tilde{C}(p)$$

$$\geq 8 \left| \exp(e^{(\log z)^{p+1}}) \right| - \tilde{C}(p) > 8 \left| f_{p,K}(z) \right| - \tilde{C}(p).$$

If, in addition, $f_{p,K}(z) \in \mathbb{G}_{p,R}$, then it follows from (3-6) that

$$\left| (f_{p,K}^n)'(z) \right| > 4|f_{p,K}(z)| - \tilde{C}(p) > 3|f_{p,K}(z)|.$$  

Thus, for each $z \in E_{R,N}, 1 \leq n \leq N$,

$$\left| (f_{p,K}^n)'(z) \right| = \prod_{r=0}^{n-1} \left| (f_{p,K})'(f_{p,K}^r(z)) \right| > 3^n \prod_{r=0}^{n-1} \left| (f_{p,K}^r)'(z) \right| > 3^n |f_{p,K}(z)|.$$
(ii) As $K > 2\hat{C}(p)$, it follows from Lemma 3.2 that, for each $z \in E_p, n \in \mathbb{N}$, the branch $g_n$ of $f_{p,K}^n$ that maps $f_{p,K}^n(z)$ to $z$ is univalent to $B(f_{p,K}^n(z), |f_{p,K}^n(z)|)$ and hence, from Lemma 3.3,

$$g_n(B(f_{p,K}^n(z), |f_{p,K}^n(z)|/2)) \subseteq B(z, 81 |f_{p,K}^n(z)|/|2(f_{p,K}^n)'(z)|). \quad (3.7)$$

If $z \in N(f_{p,K})$ then there exists $r > 0$ such that $B(z, r) \subseteq N(f_{p,K})$. From (i) and (3.7) we see that, for sufficiently large values of $n$,

$$g_n(B(f_{p,K}^n(z), |f_{p,K}^n(z)|/2)) \subseteq B(z, r)$$

and hence

$$f_{p,K}^n(B(z, r)) = B(f_{p,K}^n(z), |f_{p,K}^n(z)|/2).$$

It follows from (3.5) that

$$B(f_{p,K}^n(z), |f_{p,K}^n(z)|/2) \cap C \setminus G_p = \emptyset$$

and so, from the comment after Lemma 34, if $B(z, r) \subseteq N(f_{p,K})$, then

$$f_{p,K}^n(z) \to z_0 \in \{z: \text{Re}(z) < 0\} \quad \text{as \quad } n \to \infty.$$

This, however, is impossible as $z \in E_p$ and hence $f_{p,K}^n(z) \in G_p \subseteq \{z: \text{Re}(z) \geq 3\}$ for each $n \in \mathbb{N}$.

4. Proof of Theorem B part (i)

We take a function $f = f_{p,K}$ with $K > 2\hat{C}(p)$, a value $x_1 \geq \hat{C}(p)$ and define the real values $x_n$ inductively by

$$x_{n+1} = x_n + r_n, \quad r_n = x_n/[(1 + p)(\log x_n)^p].$$

We define the sets $B_n$ by

$$B_n = \{z = x + iy: x_n \leq x \leq x_n + R_n, |y| \leq R_n\},$$

where $r_n/(8L) \leq R_n \leq \pi r_n$, and define the following collection of sets inductively:

$$\mathcal{A}_k = \{A_k: A_k \text{ is a component of } f^{-1}(B_n) \text{ for some } n \in \mathbb{N}, A_k \subseteq \bigcup_{n \in \mathbb{N}} B_n\},$$

and

$$\mathfrak{A}_k = \{A_{k+1}: A_{k+1} \text{ is a component of } f^{-1}(B_n) \text{ for some } n \in \mathbb{N}, A_{k+1} \subseteq \mathfrak{A}_k\}.$$

**Lemma 4.1.** For each $k \in \mathbb{N}$, $A_k \in \mathfrak{A}_k$.

**Proof.** If $A_k \in \mathfrak{A}_k$, then there exists $n \in \mathbb{N}$ such that $f^k(A_k) = B_n$. As $K > 2\hat{C}(p)$, $\mathfrak{A}_k \subseteq E_{x_1,k}$ and $x_1 \geq \hat{C}(p)$, it follows from Lemma 3.5 that

$$\text{diam} A_k < 1/3^k \to 0 \quad \text{as \quad } k \to \infty.$$

We put

$$\hat{h}(z) = \exp (e^{\log x} z^{1/p}),$$

$$\hat{G}_p = \{z = x + iy: z \in \hat{G}_p, x \leq \hat{C}(p)\},$$

and

$$S(p) = 2\hat{C}(p) + \max_{z \in \hat{G}_p} |\hat{h}(z)|.$$
**Lemma 4.2.** If $K > S(p)$, $x = \hat{C}(p)$ and $B_n = \pi r_n$ then, for each $k \in \mathbb{N}$, the sets in $\mathfrak{A}_k$ form a cover of $J(f_{p,K})$.

**Proof.** If $z \in G_p$ then, as $K > S(p)$,

$$\text{Re} [f(z)] \leq \hat{k}(z) - K + \hat{C}(p) < 0$$

and hence, from Lemma 3.4, $f(z) \in \mathbb{C} \setminus G_p \subset N(f)$. So $J(f) \subset G_{p,x_i}$.

If

$$z = x + iy \in G_{p,x_i} \setminus \bigcup_{n=1}^{\infty} B_n,$$

then there exists $n \in \mathbb{N}$ such that

$$x_n \leq x < x_{n+1}, \quad \frac{\pi x}{(1 + p)(\log x)^p} \geq |y| \geq \frac{\pi x_n}{(1 + p)(\log x_n)^p} \geq \frac{5\pi x}{6(1 + p)(\log x)^p}$$

and so it follows from (3.1) that

$$\text{Re} [\exp (\log z)^{1+p}] \leq -\exp [(\log x)^{1+p}/2] < -1/2.$$ 

Thus

$$\text{Re} (f(z)) \leq |\hat{k}(z)| - K + \hat{C}(p) < e^{-1/2} - K + \hat{C}(p) < 0$$

and so, as previously, $z \in N(f)$.

So

$$J(f) \subset \bigcup_{n=1}^{\infty} B_n$$

and hence, as $J(f)$ is forward invariant under $f$,

$$J(f) \subset \bigcup_{n=1}^{\infty} f^{-k}(B_n)$$

for each $k \in \mathbb{N}$. It follows from the arguments given above that

$$f \left( \mathbb{C} \setminus \bigcup_{n=1}^{\infty} B_n \right) \subset \{ z : \text{Re} (z) < 0 \}$$

and so, if

$$f^k(z) \in \bigcup_{n=1}^{\infty} B_n,$$

we must have

$$f^{k-1}(z) \in \bigcup_{n=1}^{\infty} B_n.$$

Thus, for each $k \in \mathbb{N}$, we have

$$J(f) = \bigcup_{n=1}^{\infty} f^{-k}(B_n) = \mathfrak{A}_k.$$

If, for some $0 < t \leq 2$,

$$\sum_{A_{n+1} \cap A_n + \mathcal{O}} \text{diam} A_{n+1}^t \leq \text{diam} A_n^t$$

then

$$\sum_{A_{n+1} \cap A_n \neq \emptyset} \text{diam} A_{n+1}^t \leq \text{diam} A_n^t$$
for each $n \in \mathbb{N}, A_n \in \mathcal{A}$, then it follows from Lemma 4.1 that, for each $z \in \mathbb{C}, n \in \mathbb{N},$

$$
\sum_{A_n \in \mathcal{A}} (\text{diam } A_{n+1})^I \leq \sum_{A_n \in \mathcal{A}} (\text{diam } A_n)^I
$$

$$
\leq \sum_{A_n \in \mathcal{A}} (\text{diam } A_1)^I \leq \sum_{A_n \in \mathcal{A}} (\text{diam } A_1)^I.
$$

If, further, we have $K > S(p)$ and $R_n = \pi r_n$, then it follows from Lemma 4.2 that, for each $z \in \mathbb{C},$

$$
\dim [J(f) \cap B(z, 1)] \leq t
$$

and hence $\dim J(f) \leq t$.

**Lemma 4.3.** If $A_k \in \mathcal{A}$ and $f^k(A_k) = B_n$, then

$$
\sum_{A_k \in \mathcal{A}} (\text{diam } A_k)^I \leq L
$$

$$
\sum_{A_k \in \mathcal{A}} (\text{diam } A_k)^I \leq \sum_{A_k \in \mathcal{A}} (\text{diam } A_1)^I/\text{diam } B_n.
$$

**Proof.** Clearly

$$
\text{diam } A_k \leq \text{diam } f^k(A_k) \leq \inf_{z \in A_k} |(f^k)'(z)|
$$

$$
\text{diam } A_k \geq \text{diam } B_n/\sup_{z \in A_k} |(f^k)'(z)|.
$$

If $A_{k+1} \in \mathcal{A}_{k+1}$, then $f^k(A_{k+1}) \in \mathcal{A}_k$ and so it follows from Lemma 4.1 that

$$
f^k(A_k \cup \bigcup_{A_k \in \mathcal{A}} \cup_{A_k \in \mathcal{A}_k} ) \subset B(x_n, x_n/4).
$$

Thus, from Lemma 3.2 and Lemma 3.3, if $A_{k+1} \cap A_k \neq \emptyset$ and $A_{k+1} \in \mathcal{A}_{k+1}$, then

$$
\sup_{v, w \in A_k \cup A_{k+1}} |(f^k)'(v) - (f^k)'(w)| \leq L.
$$

Combined with (4.1) this gives the desired result.

Thus, if $K > S(p)$, in order to show that $\dim J(f) \leq t$, it is sufficient to show that, for each $n \in \mathbb{N},$

$$
\sum_{A_k \in \mathcal{A}} (\text{diam } A_1)^I \leq (\text{diam } B_n)^I/L^I,
$$

when $R_n = \pi r_n$. In order to do this we need the following results. We put

$$
g(z) = \exp (\log z)^1 P.
$$

**Lemma 4.4.** If

$$
z \in \bigcup_{n=1}^{\infty} B_n \quad \text{and} \quad r \leq \frac{2\pi z}{|1 + p| (\log z)^p}
$$

then, for each $w \in B(z, r),$

(i) $1/(6 \times 10^{21}) \leq |g(w)/g(z)| \leq 6 \times 10^{21},$

(ii) $1/2 \leq |w/z| \leq 2,$

(iii) $1/5 \leq |(\log w)/(\log z)|^p \leq 5,$

(iv) $1/(6 \times 10^{22}) \leq |g'(w)/g'(z)| \leq 6 \times 10^{22}.
Lemma 4.5. If

\[ z \in \bigcup_{n=1}^{\infty} B_{2n} \quad \text{and} \quad r \leq \frac{2\pi z}{(1+p)(\log z)^p} \]

then, for each \( n \in \mathbb{N} \),

\[ |\{w : h(w) = x_n, w \in B(z, r)\}| \leq 2.8 \times 10^{26}|g'(z)|r. \]

Proof. We put

\[ \Gamma_m = \{ t = x + iy : \log(x_1 + K) \leq x < \infty, y = 2\pi m \}, \quad \tilde{h}(t) = e^t - K. \]

As \( \tilde{h}(t) = x_n \) implies that \( t \in \Gamma_m \) for some \( m \in \mathbb{Z} \) and \( \tilde{h}(t) \) is a strictly increasing real-valued function of \( t \) on each \( \Gamma_m, m \in \mathbb{Z} \), it follows that

\[ |\{w : h(w) = x_n, w \in B(z, r)\}| \leq |\{m \in \mathbb{Z} : g(B(z, r)) \cap \Gamma_m \neq \emptyset\}| \times \max_{t \in C} |\{w : g(w) = t, w \in B(z, r)\}|. \quad (4.3) \]

From Lemma 4.4 we know that, if \( z, r \) satisfy the given conditions, then

\[ g(B(z, r)) \subset B(g(z), 6.8 \times 10^{26}|g'(z)|r) \]

and so

\[ |\{m \in \mathbb{Z} : g(B(z, r)) \cap \Gamma_m \neq \emptyset\}| < 6.8 \times 10^{26}|g'(z)|r. \quad (4.4) \]

We note that it follows from Lemma 4.4 that, if \( w = x + iy = se^{i\theta} \in B(z, r) \), then

\[ w \in B\left( x, \frac{5\pi z}{(1+p)(\log z)^p} \right) \subset B\left( x, \frac{50\pi x}{(1+p)(\log x)^p} \right). \]
So \( \log w = \log (s) + i\theta = t e^{i\delta} \), where
\[
|\phi| < |\tan \phi| = |\theta|/(\log s) < |\tan \theta|/(\log s)
\]
\[
= \frac{|y|}{x (\log s)} < \left| \frac{50\pi}{(1 + p) (\log x)^{1+p}} \right| < \pi/[2(1 + p)], \quad (4.5)
\]
and hence \((\log w)^{1+p}\) is univalent in \(B(z, r)\).

We also note that, if \(w \in B(z, r)\), then it follows from (4.5) that
\[
|\Im[(\log w)^{1+p}] = |t^{1+p}\sin((1 + p)\phi)| \leq |\log w|^{1+p}2(1 + p)|\phi|
\]
\[
< |\log(x) + 2|^{1+p}2(1 + p)|\phi| = (\log x)^{1+p}|1 + 2/(\log x)|^{1+p}2(1 + p)|\phi|
\]
\[
< 10(\log x)^{1+p}2(1 + p) \left| \frac{50\pi}{(1 + p) (\log x)^{1+p}} \right| < 1000\pi < 4 \times 10^2
\]
and so
\[
\max_{t \in \mathbb{C}} |w: g(w) = \exp(\log w)^{1+p} = t, w \in B(z, r)| \leq 4 \times 10^2.
\]

Combining this with (4.3) and (4.4) gives the desired result.

The next result is the main result of this section and has two parts. The proof of Theorem B part (i) will follow easily from the first part and the second part will be of use in the proof of Theorem B part (ii).

**Lemma 4.6.** Let \( H = \{A_1: A_1 \in \mathcal{A}_1, A_1 \cap B(z, r) \neq \emptyset\}. \)

(i) If \( z \in \mathbb{R}, |z| \geq \hat{C}(p), R_n = \pi r_n \)
\[
r = \left| \frac{(2)^{1/2}nz}{(1 + p)(\log z)^{1/p}} \right|,
\]
\( p \geq 2 \) and \( 2 \geq t > 1 + 2/p, \) then
\[
\sum_{A_1 \in H} (\text{diam } A_1)^t \leq r/L^2 < r/L'.
\]

(ii) If
\[
z, f(z) \in \bigcup_{n=1}^{\infty} B_n, \quad p \leq 1/2, \quad R_n = r_n/(8L), \quad x_1 \geq 2\hat{C}(p) + K
\]
and
\[
\left| \frac{f(z)}{8L(1 + p)(\log f(z))^{1/p} f'(z)} \right| \leq r \leq \left| \frac{z}{8L(1 + p)(\log z)^{1/p}} \right|
\]
then
\[
\sum_{A_1 \in H} (\text{diam } A_1)^2 \leq 9.6 \times 10^{112}r^{2-2/p}/|g'(z)|^p.
\]

**Proof.** As \( x_1 \geq \hat{C}(p) \), it follows from Lemma 3.1 that, if \( f(A_1) = B_n \), then
\[
h(A_1) \subset B\left(x_n, \frac{4x_n}{(1 + p)(\log x_n)^p}\right) = H_n.
\]
(4.6)
We let \( \lambda_1 \) denote the component of \( h^{-1}(H_n) \) that contains \( A_1 \) and note that it follows from Lemma 3.1 that
\[
f(A_1) \subset B\left(x_n, \frac{5x_n}{(1 + p)(\log x_n)^p}\right) \subset B(x_n, x_n)
\]
(4.7)
and hence, from Lemma 3.2, \( f \) is univalent in \( A'_1 \). Thus, for each \( n \in \mathbb{N} \),

\[
||A_1 : A_1 \text{ is a component of } f^{-1}(B_n), A_1 \in H|| \\
\leq ||A'_1 : A'_1 \text{ is a component of } h^{-1}(H_n), A'_1 \cap B(z, r) \neq \emptyset||. \tag{4.8}
\]

Clearly the only transcendental singularity of \( h^{-1} \) is at \(-K\) and, as \( h'(z) = 0 \) implies that \( z = 1 \), the only critical value of \( h \) is at \( h(1) = e - K \). Thus, if \( A'_1 \cap B(z, r) \neq \emptyset \) and \( f(A'_1) = H_n \), the branch of \( h^{-1} \) that maps \( H_n \) to \( A'_1 \) is univalent in \( B(x_n, x_n/2) \). It follows from Lemma 3.3 and Lemma 4.4 that, if \( w \in B(z, r) \cap A'_1 \),

\[
A'_1 \subseteq B \left( w, \frac{8Lx_n}{(1 + p)(\log x_n)^p |h'(w)|} \right) \subseteq B \left( w, \frac{80L_h(w)}{(1 + p)(\log h(w))^p |h'(w)|} \right) \\
\subseteq B(w, 5/(1 + p)) \subseteq B \left( w, \frac{|z|}{(1 + p)(\log z)^p} \right) \subseteq B \left( z, \frac{2\pi z}{(1 + p)(\log z)^p} \right). \tag{4.9}
\]

If \( r < \text{diam}(A'_1)/2 \), then it follows from (4.9) that

\[
B(z, r) \subseteq B \left( w, \frac{16Lx_n}{(1 + p)(\log x_n)^p |h'(w)|} \right).
\]

As \( 16L^2x_n/[(1 + p)(\log x_n)^p] < x_n/8 \), it follows from Lemma 3.3 that

\[
h(z) \in B(h(w)), x_n/8 \subseteq B(x_n, x_n/4)
\]

and hence, from Lemma 3.1, we have \( f(z) \in B(x_n, x_n/2) \).

Thus, if \( r < \text{diam}(A'_1)/[2560L^2] \), then it follows from (4.9) that

\[
r < \frac{16Lx_n}{2560L^2(1 + p)(\log x_n)^p |h'(w)|} < \frac{160L^2f(z)}{2560L^2(1 + p)(\log f(z))^p |h'(w)|}. \tag{4.10}
\]

It follows from Lemma 3.1 that

\[
|f'(z)| + \tilde{C}(p) > |h'(z)| > |f'(z)| - \tilde{C}(p).
\]

As

\[
z, f(z) \in \bigcup_{n=1}^{\infty} B_n,
\]

it follows from Lemma 3.5 that

\[
|f'(z)| > 3|f(z)| > 3x_1 > 3\tilde{C}(p)
\]

and hence \( |h'(z)| > |f'(z)|/2 \).

So, if \( r < \text{diam}(A'_1)/[2560L^2] \), it follows from (4.10) that

\[
r < \frac{320L^2f(z)}{2560L^2(1 + p)(\log f(z))^p |f'(z)|} = \frac{f(z)}{8L(1 + p)(\log f(z))^p |f'(z)|}.
\]

This is clearly impossible and so we must have

\[
\text{diam } A'_1 \leq 2560L^2r < 1.4 \times 10^9r. \tag{4.11}
\]

It follows from (4.8), (4.9) and (4.11) that there exists

\[
\hat{r} \leq \min \left\{ \frac{2\pi z}{(1 + p)(\log z)^p}, 1.5 \times 10^9r \right\} \tag{4.12}
\]
such that, for each \( n \in \mathbb{N} \),
\[
\begin{align*}
|\{A_1 : A_1 \text{ is a component of } f^{-1}(B_n), A_1 \in H\}| \\
&\leq |\{A'_1 : A'_1 \text{ is a component of } h^{-1}(H_n), A'_1 \subset B(z, r)\}| \\
&\leq |\{w : h(w) = x_n, w \in B(z, \hat{r})\}|
\end{align*}
\]
and so, from Lemma 4.5 and (4.12),
\[
|\{A_1 : A_1 \text{ is a component of } f^{-1}(B_n), A_1 \in H\}| \\
\leq 2.8 \times 10^{26} |g'(z)|r < 2.4 \times 10^{26} |g'(z)|r. \tag{4.13}
\]
If \( A_1 \) is a component of \( f^{-1}(B_n) \) and \( w \in A_1 \cap B(z, r) \), then it follows from Lemma 3.3 and Lemma 3.4 that
\[
diam A_1 < L \text{diam}(B_n)/|f'(w)|.
\]
Arguing as before, we see that \(|f'(w)| > |h'(w)}/2\) and hence, as \(|f'(w)| \geq x \geq 2\hat{C}(p)\), it follows from Lemma 3.1 and Lemma 4.4 that
\[
diam A_1 < \frac{4\pi L x_n}{(1 + p)(\log x_n)^p h'(w)} < \frac{8L\pi}{(1 + p)(\log x_n)^p |g'(w)|} < \frac{1.4 \times 10^{26}}{(1 + p)(\log x_n)^p |g'(z)|}.
\]
Now let \( m \) denote the smallest and \( M \) the largest \( n \in \mathbb{N} \) for which there exists a set \( A_1 \in H \) with \( f(A_1) = B_n \). It follows from (4.13), (4.14) and Lemma 4.4 that
\[
\sum_{A_1 \in H} (\text{diam } A_1)^f \leq \frac{4.2 \times 10^{32} |g'(z)| r (1.4 \times 10^{26})^2}{(1 + p)^f |g'(z)|^f} \times \sum_{n = m}^M 1/(\log x_n)^p t \\
< \frac{8.3 \times 10^{45} r}{(1 + p)^f |g'(z)|^f} \int_{x_m}^{x_n} \frac{10(1 + p)}{x \log x} (p-1) dx \\
= \frac{8.3 \times 10^{88} r}{(1 - p(t-1))(1 + p)^{t-1} |g'(z)|^{t-1} (\log x)^{1-p(t-1)} x_m^{t-1}}. \tag{4.15}
\]
If we are in case (i) then, as \( 2 \geq t > 1 + 2/p \), we have \( 1 - p(t-1) < -1 \) and so, as \( \log x_m \geq \log \hat{C}(p) > 1 \) and \( z \in \mathbb{R} \), it follows from (4.15) that
\[
\sum_{A_1 \in H} (\text{diam } A_1)^f \leq 8.3 \times 10^{88} r/|g'(z)|t-1 < \frac{8.3 \times 10^{88} r^{t-1}}{|g'(z)|^{t-1} (\log z)^{t-1}} < 8.3 \times 10^{88} r/(\log z)^2.
\]
As \( |z| \geq \hat{C}(p) \geq \exp(8.3 \times 10^{88}) \), it follows that
\[
\sum_{A_1 \in H} (\text{diam } A_1)^f \leq r/L^2 < r'/L^f
\]
as claimed.

If we are in case (ii) then, as \( p \leq 1/2 \), it follows from (4.15) that
\[
\sum_{A_1 \in H} (\text{diam } A_1)^2 \leq \frac{1.7 \times 10^{89} r}{|g'(z)| (\log x_m)^p} |\log (x_{M+1}) - \log (x_m)|. \tag{4.16}
\]
As \( x_m \geq 2(\hat{C}(p) + K) \) and \( K \geq \hat{C}(p) \), it follows from Lemma 3.1 that, if \( z_1, z_2 \in B(z, r) \) with \( f(z_1) \in B_m \) and \( f(z_2) \in B_M \), then
\[
|\exp(e^{(\log z_1)^p})| \leq 2x_m, \quad |\exp(e^{(\log z_2)^p})| \geq x_M > x_{M+1}/2. \tag{4.17}
\]
So it follows from Lemma 4.4 that
\[ |\log (x_{M+1}) - \log (x_m)| \leq \text{Re}(g(z_2)) - \text{Re}(g(z_1)) + \log 4 \leq \log (4 + |g(z_2) - g(z_1)|) \leq \log (4 + 2 \times 6.8 \times 10^{22}|g'(z)| r). \]
Thus, if \( r |g'(z)| > 1 \), then
\[ |\log (x_{M+1}) - \log (x_m)| < 1.4 \times 10^{23}|g'(z)| r \]
and so, from (4.16) and (4.17),
\[ \sum_{A_i \in H} (\text{diam} A_i)^2 < \frac{1.7 \times 10^{89} r}{|g'(z)| \langle \log x_m \rangle^p} \times 1.4 \times 10^{23}|g'(z)| r < 4.8 \times 10^{112} r^2 / [\text{Re}(g(z_1))]^p. \]

(4.18)

It is not difficult to deduce from the conditions on \( p, z \) and \( r \) that there exists \( w \in \mathbb{R} \) with \( w \geq \tilde{C}(p) \) such that
\[ z_1 \in B(z, r) \subseteq B \left( w : \frac{w}{3L(1+p)(\log w)^p} \right). \]
As
\[ g'(w) L \frac{w}{3L(1+p)(\log w)^p} = g(w)/3, \]
it follows from Lemma 3.3 that \( g(B(z, r)) \subseteq B(g(w), g(w)/3) \) and hence
\[ \text{Re}(g(z_1)) > 2g(w)/3 > |g(z)|/2. \]
So, if \( r |g'(z)| > 1 \), it follows from (4.18) that
\[ \sum_{A_i \in H} (\text{diam} A_i)^2 < 2^p \times 4.8 \times 10^{112} r^2 / |g(z)|^p < 9.6 \times 10^{112} r^2 / |g(z)|^p \]
\[ \leq 9.6 \times 10^{112} r^2 / |g(z)|^p \times \left| \frac{z}{8L(1+p)(\log z)^p} \right|^p \]
\[ = 9.6 \times 10^{112} r^2 / |8L| |g'(z)|^p < 9.6 \times 10^{112} r^2 / |g'(z)|^p. \]
Finally, if \( r |g'(z)| \leq 1 \) then, from (4.12),
\[ \sum_{A_i \in H} (\text{diam} A_i)^2 \leq (1.5 \times 10^9)^2 r^2 < 2.3 \times 10^{18} r^2 < 2.3 \times 10^{18} r^2 / |g'(z)|^p. \]

This completes the proof of Lemma 4.6.

We are now in a position to prove Theorem B part (i). If \( B_n = \pi r_n \) then, for each \( n \in \mathbb{N}, \)
\[ \text{diam} B_n = (2)^{1/2} \pi r_n \]
and
\[ B_n \subseteq B(x_n, (2)^{1/2} \pi r_n) = B \left( x_n, \frac{(2)^{1/2} \pi x_n}{(1+p)(\log x_n)^p} \right). \]
Thus, if \( H = \{ A_1 : A_1 \in \mathcal{A}, A_1 \cap B(x_n, (2)^{1/2} \pi r_n) \neq \emptyset \} \), it follows from Lemma 4.6 part (i) and (4.19) that, if \( p \geq 2, 2 \geq t > 1 + 2/p \), then, for each \( n \in \mathbb{N}, \)
\[ \sum_{A_i \in H} (\text{diam} A_i)^t \leq \sum_{A_i \in H} (\text{diam} A_i)^t < [(2)^{1/2} \pi r_n]^t / L^t = (\text{diam} B_n)^t / L^t. \]
It now follows from (4.2) that, if \( p \geq 2, K \geq S(p) \), then \( \dim J(f) \leq 1 + 2/p. \)
5. Proof of Theorem B part (ii)

Take a function $f = f_{n,K}$, where $p \leq 1/2, K \geq 2\hat{C}(p)$ and consider the sets $A_k$ defined at the beginning of section four with $x_1 \geq 2(K+\hat{C}(p))$ and $R_n = r_n/(8L)$. We note that, for each $k \in \mathbb{N}$, the sets in $A_k$ are disjoint and so, if $A_{k+1} \in \mathcal{A}_{k+1}$, there exists a unique $A_k \in \mathcal{A}_k$ such that $A_{k+1} \subset A_k$. We choose a set $\hat{A}_k \in \mathcal{A}_k$ and put $\mathcal{B}_k = \mathcal{A}_k \cap \hat{A}_1$. If $\mathcal{A} = \bigcap_{k=1}^{\infty} \mathcal{B}_k$, then $A \subset E_{\mathcal{C}(p)}$ and so, from Lemma 3.5, $A \subset J(f)$.

Following the ideas used by McMullen in his proof of Lemma 2-2, we construct a measure $\mu$ which is supported on $\hat{A}$. We first construct a sequence of measures $\mu_k$ supported on $\mathcal{B}_k$ in the following way. If $\mu_k$ is the 2-dimensional volume measure, we have $\mu_k = 0$ on $\mathbb{C} \setminus \hat{A}_1$ and $\mu_k = \mu_{k+1}|_{\hat{A}_1}$ elsewhere. Then, given $\mu_k$, we define $\mu_{k+1}$ by letting $\mu_{k+1} = 0$ on $\mathcal{B}_k \setminus \mathcal{B}_{k+1}$ and elsewhere taking it to be the restriction of $\mu_k$ to $\mathcal{B}_{k+1}$ scaled within each $A_k \in \mathcal{A}_k$ so that $\mu_{k+1}(A_k) = \mu_k(A_k)$. There is then a subsequence of measures $\mu_k$ which converge weakly to a measure $\mu$ supported on $\hat{A}$ (see, for example, [9, theorem 5-13]).

We note some properties of these measures. First, for each $k \in \mathbb{N}$ and each $A_k \in \mathcal{A}_s$, $\mu_k(A_k) = \mu_k(A_k)$ for $n \geq k$ and so

$$\mu(A_k) = \mu_k(A_k).$$

Secondly, if

$$\Delta(A_k) = \text{vol}(\mathcal{A}_{k+1} \cap A_k)/\text{vol}(A_k), \quad \forall(A_k) = \prod_{r=1}^{k} \Delta(A_r),$$

where

$$A_k \subset A_{k-1} \subset \ldots \subset A_1$$

then for each $r \in \mathbb{N}, A_r \in \mathcal{A}_r$ with $A_r \subset \hat{A}_1$ we have

$$\mu_{r+1}|_{\mathcal{A}_{r+1} \cap A_r} = (1/\Delta(A_r)) \mu_r|_{\mathcal{A}_{r+1} \cap A_r}$$

and so, for each $k \in \mathbb{N}, A_k \in \mathcal{A}_k$ with $A_k \subset \hat{A}_1$,

$$\mu_{k+1}|_{\mathcal{A}_{k+1} \cap A_k} = (1/\forall(A_k)) \mu_k|_{\mathcal{A}_{k+1} \cap A_k}. \quad (5\cdot2)$$

In the following results we obtain enough information on the sets $A_n$ (and hence on the measure $\mu$) to enable us to apply Lemma 2-1 to show that

$$\text{dim}(J(f)) \geq \text{dim}(\hat{A}) \geq 2 - p.$$
and 
\[ \operatorname{diam} A_k \leq \operatorname{diam} (B_n)/\inf_{z \in A_k} |(f^k)'(z)|. \]

As \( f^k(A_k) = B_n \subset B(x_n, x_n/2) \), it follows from Lemma 3.2 and Lemma 3.3 that
\[ \sup_{u, v \in A_k} |(f^k)'(u)/(f^k)'(v)| \leq L. \]

The result now follows.

**Lemma 5.2.** For each \( n \in \mathbb{N} \),
\[ \sum_{A_1 \subset B_n} (\operatorname{diam} A_1)^2/(\operatorname{diam} B_n)^2 \geq 1/[6 \cdot 3 \times 10^{126}(g(x_n))^p]. \]

**Proof.** We begin by noting that it follows from Lemma 3.1 that, if
\[ h(z) = x'_m = x_m + R_m/2, \]
for some \( m \in \mathbb{N} \), then \( f(z) \in B_m \). Thus, for each point \( z \) such that \( h(z) = x'_m \), there exists a set \( A_1(z) \subset A_k \) such that \( z \in A_1(z) \) and \( f(A_1(z)) = B_m \). We note further that \( h(A_1(z)) \subset B(x_m, x_m/4) \) and hence, as the singularities of \( h^{-1} \) are contained in \( \{z : \operatorname{Re}(z) < h \} \), \( h \) is univalent in \( A_1(z) \). Thus, if \( z_1 \neq z_2, A(z_1) \neq A(z_2) \).

We note from (4.14) that, if \( A \cap B(x'_m, R_n/4) \neq \emptyset \), then
\[ \operatorname{diam} A_1 < 1 \cdot 4 \times 10^{26}/|g'(x_n)| = 1 \cdot 4 \times 10^{26} r_n/g(x_n) < 1 \cdot 4 \times 10^{26} r_n/x_n < R_n/4 \]
and hence \( A_1 \subset B_n \).

For each \( M \in \mathbb{Z} \), we take \( \Gamma_M \) to be the curve defined by
\[ \Gamma_M = \{z = x + iy : x \geq \log(K + x_2), y = 2\pi M\}. \]

If \( g(z) \in \Gamma_M \), for some \( M \in \mathbb{Z} \), then \( h(z) \in [x_1, \infty) \). Now let \( \gamma_M \) be a curve which is mapped onto \( \Gamma_M \) by \( g \). If \( \gamma_M \cap B(x'_n, R_n/8) \neq \emptyset \), then there exists a segment of \( \gamma_M \cap B(x'_n, R_n/4) \) of length at least \( R_n/8 \) and hence, from Lemma 4.4, there exists a segment of \( \Gamma_M \cap g(B(x'_n, R_n/4)) \) of length at least
\[ g'(x_n) R_n/(8 \times 6 \cdot 8 \times 10^{22}) \geq g'(x_n) R_n/(5 \cdot 5 \times 10^{23}). \]

It follows that there exist \( z_1, z_2 \in \gamma_M \cap B(x'_n, R_n/4) \) such that
\[ h(z_1) = x_{n(1)}, \quad h(z_2) = x_{n(2)}, \quad h(\gamma_M \cap B(x'_n, R_n/4)) \supset [x_{n(1)}, x_{n(2)}] \]
and
\[ \operatorname{Re} g(z_2) - \operatorname{Re} g(z_1) \geq g'(x_n) R_n/(5 \cdot 5 \times 10^{23}) - 4. \tag{5.3} \]

As \( B_m \subset B(x_m, x_m/4) \), for each \( m \in \mathbb{N} \), it follows from Lemma 3.2 and Lemma 3.3 that, if \( f(A_1) = B_m \), \( z \in A_1 \) and \( h(z) = x'_m \), then
\[ \operatorname{diam} A_1 \geq \operatorname{diam} (B_m)/(L_2 f'(z)) = x_m/\{4(2)^{1/2}(1 + p) (\log x_m)^p L_2 f'(z)\}. \]

As \( x_m > 2(K + \bar{c}(p)) \), it follows from Lemma 3.1 and Lemma 4.4 that
\[ \operatorname{diam} A_1 > x_m/[16(1 + p) (\log x_m)^p L_2 h'(z)] \]
\[ = x_m/[16(1 + p) (\log x_m)^p L_2 g'(z) (h(z) + K)] \]
\[ > 1/[32 \times 6 \cdot 8 \times 10^{22}(1 + p) (\log x_m)^p L_2 g'(x_n)] \]
\[ > 1/[1 \cdot 5 \times 10^{28}(1 + p) (\log x_m)^p g'(x_n)]. \]
Combined with earlier observations, this shows that, if $\gamma_M \cap B(x_n, R_n/8) \neq \emptyset$, then

$$\sum_{A_i \subset B_n, A_i \cap \gamma_m \neq \emptyset} (\text{diam } A_i)^2 \geq 1/[1 \cdot 5 \times 10^{28}(1 + p) g'(x_n)]^2 \sum_{x_n^{(2)}} 1/[\log x_m]^{2p}$$

$$\geq 1/[1 \cdot 5 \times 10^{28}(1 + p) g'(x_n)]^2 \int_{x_n^{(2)}}^x (1 + p)/[\log x]^p \, dx$$

$$> 1/[2 \cdot 3 \times 10^{36}(1 - p^2)(g'(x_n))^2][(\log x)^{1-p}]^{x_n^{(2)}}$$

$$> [\log (x_n^{(2)} - \log (x_n^{(1)}))/[2 \cdot 3 \times 10^{36}(g'(x_n))^2(\log x_n^{(2)})]^p].$$

As $\exp(g(z_1)) = x_n^{(1)} + K$ and $\exp(g(z_2)) = x_n^{(2)} + K < 2x_n^{(2)}$, it follows from Lemma 4.4 that

$$\text{Re}(g(z_2)) - 2 \leq \log x_n^{(2)} \leq \text{Re}(g(z_2)) \leq |g(z_2)| \leq 6 \cdot 8 \times 10^{21}|g(x_n)|$$

and

$$\log (x_n^{(1)}) \leq \text{Re}(g(z_1)).$$

Thus, if $\gamma_M \cap B(x_n, R_n/8) \neq \emptyset$, it follows from (5.3) that

$$\sum_{A_i \subset B_n, A_i \cap \gamma_M \neq \emptyset} (\text{diam } A_i)^2 > [\text{Re}(g(z_2) - g(z_1)) - 2]/[2 \cdot 3 \times 10^{36}(g'(x_n))^2 \cdot 6 \cdot 8 \times 10^{21}(g(x_n))^p]$$

$$> |g'(x_n)| R_n^{(5 \cdot 5 \times 10^{23})} - 4/[(1 \cdot 6 \times 10^{37}(g'(x_n))^2(g(x_n))^p].$$

As

$$g'(x_n) R_n^{(5 \cdot 5 \times 10^{23})} = g(x_n)/(8L \times 5 \cdot 5 \times 10^{23}) > x_n/(8L \times 5 \cdot 5 \times 10^{23}) > 8,$$

it follows that

$$\sum_{A_i \subset B_n, A_i \cap \gamma_M \neq \emptyset} (\text{diam } A_i)^2 > R_n/[2 \times 5 \cdot 5 \times 10^{23} \times 1 \cdot 6 \times 10^{37}g'(x_n)(g(x_n))^p]$$

$$> R_n/[1 \cdot 8 \times 10^{102}g'(x_n)(g(x_n))^p].$$

(5.4)

Clearly $g(x_n) \in \Gamma_0$. As we know from Lemma 4.4 that

$$g(B(x_n, R_n/8)) \supset B(g(x_n), g'(x_n) R_n^{(5 \cdot 5 \times 10^{23})}),$$

it follows that there are at least $g'(x_n) R_n^{(\pi \times 5 \cdot 5 \times 10^{23})}$ values of $M \in \mathbb{Z}$ for which there exists a curve $\gamma_M$ satisfying $\gamma_M \cap B(x_n, R_n/8) \neq \emptyset$. If a set $A_i \in \alpha_i$ meets two curves $\gamma_M^{(1)}, \gamma_M^{(2)}$, where $M(1) \neq M(2)$, then there exist $w \in A_i, M \in \mathbb{N}$ such that $\text{Im}(g(w)) = (2M + 1)\pi$ and hence $\text{Re}(h(w)) < 0$. This, however, is impossible and so it follows from (5.4) that

$$\sum_{A_i \subset B_n} (\text{diam } A_i)^2 \geq (R_n)^2/[1 \cdot 8 \times 10^{102}\pi \times 5 \cdot 5 \times 10^{23}(g(x_n))^p]$$

$$> (\text{diam } B_n)^2/[6 \cdot 3 \times 10^{125}(g(x_n))^p].$$

It follows from Lemma 5.1 and Lemma 5.2 that, if $f^k(A_k) = B_n$, then

$$\Delta(A_k) \geq 1/[4 \cdot 2 \times 10^{125}(g(x_n))^p].$$

(5.5)

Before we can complete the proof of Theorem B part (ii), we need a few more preliminary results. Recall that $h(z) = b(z) + K$. 

Lemma 5.3. If \( z, f(z) \in \bigcup_{n=1}^{\infty} B_n \), then

(i) \( |f(z)| > |f(z)| > |\hat{h}(z)|/2 \).

(ii) \( |\log |f(z)||^p > 4(2.9 \times 10^{152})^{1/p} |z| \).

Proof. (i) It is not difficult to see that, if \( z, f(z) \in \bigcup_{n=1}^{\infty} B_n \), then \( |\hat{h}(z)| > |f(z)| \). As \( |f(z)| > x_1 > 2(K + \hat{C}(p)) \), it follows from Lemma 3.1 that

\[ |f(z)| > |\hat{h}(z)| - K + \hat{C}(p) > |\hat{h}(z)|/2. \]

(ii) It follows from part (i) that

\[ \log |f(z)| > \log |\hat{h}(z)|/2 > |\log |\hat{h}(z)||/2 = |\Re (g(z))|/2. \]

As \( z \in B_n \subset B(x_n, r_n/(4L)) \), for some \( n \in \mathbb{N} \), and \( g'(x_n) r_n L/(4L) = (g(x_n))/4 \), it follows from Lemma 3.3 that \( g(z) \in B(g(x_n), (g(x_n))/4) \) and hence, as \( x_n \geq \hat{C}(p) \),

\[ |\log (f(z))|^p > |\Re (g(z))/2|^p > (g(x_n))/4 \]

\[ = \exp [p(\log x_n)^2]/4 > (x_n)^2/4 > x_n |z|/8 \]

\[ > \exp [8.3 \times 10^{57}/p] |z|/8 > 4(2.9 \times 10^{152})^{1/p} |z|. \]

Lemma 5.4. If \( z \in B_n \) and \( r \leq |z|/[(4L + 1)(\log z)]^2 \), then \( B(z, r) \cap B_m = \emptyset \), for each \( m \in \mathbb{N}, m \neq n \).

Proof. If \( z \in B_n \), then \( x_n \leq |z| < 2x_n \) and hence, from Lemma 4.4,

\[ r \leq |z|/[(4L + 1)(\log |z|)^2] < 2x_n/[(4L + 1)(\log x_n)^2] \]

\[ = x_n/(2L) < 10 x_n/[2L(1 + p)(\log x_n)] \leq r_n - x_n \leq r_n/2. \]

Thus

\[ \Re (z) > x_n + r_n/(8L) + r_n/(2L) < x_n + r_n = x_n + r \]

and

\[ \Re (z) > x_n - r_n - 1/2 = x_n - r_n - 1/2 \]

and hence \( z \notin \bigcup_{m=n} B_m \).

By using Lemmas 5.4 and 4.6, we are able to prove the following.

Lemma 5.5. There exists \( r_n > 0 \) such that, for each \( z \in \hat{A}, r < r_0 \), there exists \( n \in \mathbb{N} \) and a set \( A_n \in \mathcal{A}_n \) such that

(i) if \( A_n+1 \in H = \{A_{n+1} \in \mathcal{A}_{n+1} : A_{n+1} \cap B(z, r) \neq \emptyset \} \), then \( A_{n+1} \subset A_n \),

(ii) \( \sum_{A_{n+1} \in H} (\text{diam } A_{n+1})^2 \leq 4.2 \times 10^{129} r^{2-p} / |(f^{n+1})'(z) (f^n)'(z)|^p. \)

Proof. Take \( z \in \hat{A}, r < \sup_{z \in \hat{A}} |f(z)|/[(8L^2 + 1)(\log |f(z)|)|f'(z)|] \) and the smallest value of \( n \in \mathbb{N} \cup \{0\} \) for which

\[ |(f^{n+1})'(z)| \geq |f^{n+1}(z)|/[(8L^2 + 1)(\log |f^{n+1}(z)|)|f'(z)|]. \]  \hspace{1cm} (5.6)

The existence of such an \( n \) follows from Lemma 3.5 and, due to the choice of \( r \), we must have \( n \geq 1 \).

As \( z \in \hat{A} \), we know that \( f^n(z) \in \bigcup_{m=1}^{\infty} B_m \) and so, from Lemma 3.2, the branch of \( f^{-n} \) that maps \( f^n(z) \) to \( z \) is univalent in \( B(f^n(z), |f^n(z)|/2) \). As \( |(f^n)'(z)| r L < |f^n(z)|/8 \), it follows from Lemma 3.3 that

\[ f^n(B(z, r)) \subset B(f^n(z), |(f^n)'(z)| r L) \subset B(f^n(z), |f^n(z)|/8). \]  \hspace{1cm} (5.7)

It follows from Lemma 4.1 that

\[ \sup_{z \in \hat{A}} \text{diam } A_1 \leq 1/3 \]

for each \( A_1 \in \mathcal{A}_1 \).
and hence, if \( A_1 \in \mathcal{A} \) and \( A_1 \cap B(f^n(z), |(f^n)'(z)| r L \) \( \neq \emptyset \), it follows from (5.6) that

\[
A_1 \subset B(f^n(z), |(f^n)'(z)| r L + 1/3)
\]

\[
\subset B(f^n(z), |f^n(z)/[4L(1 + p)(\log f^n(z))^p]|) \subset B(f^n(z), |f^n(z)/4|). \tag{5.8}
\]

Clearly \( f^n(z) \in B_m \) for some \( m \in \mathbb{N} \). It follows from (5.8) and Lemma 5.4 that, if \( A_1 \cap B(f^n(z), |(f^n)'(z)| r L) \neq \emptyset \), then \( A_1 \subset B_m \).

If \( A_{n+1} \in H \), then \( f^n(A_{n+1}) \in \mathcal{A} \) and it follows from (5.7) that,

\[
f^n(A_{n+1}) \cap B(f^n(z), |(f^n)'(z)| r L) \neq \emptyset.
\]

Thus, if \( f_n \) is the branch of \( f^{-n} \) that maps \( f^n(z) \) to \( z \), then \( A_{n+1} \subset f_n(B_m) \in \mathcal{A}_n \). This shows that part (i) is true and it now follows from Lemma 3.2 and Lemma 3.3 that

\[
\sum_{A_{n+1} \in H} (\text{diam } A_{n+1})^2 \leq (L^2/|(f^n)'(z)|^2) \sum_{A_n \in \mathcal{A}} (\text{diam } A_n)^2.
\]

It follows from (5.6) that

\[
|(f^n)'(z)| r L < |f^n(z)/[8L(1 + p)(\log f^n(z))^p]| \tag{5.9}
\]

and that

\[
|(f^n)'(z)| r L = |(f^n+1)'(z)| r L/|(f^n)'(z)|
\]

\[
\geq |(f^n+1)'(z)|/[8L(1 + p)(\log f^{n+1}(z))^p/(f^n(z))^p]|. \tag{5.10}
\]

As \( z \in \hat{A} \), we know that \( f^k(z) \in \hat{A} \subset \bigcup_{m=1}^\infty B_m \) for each \( k \in \mathbb{N} \) and so it follows from (5.9) and (5.10) that \( B(f^n(z), |(f^n)'(z)| r L) \) satisfies the conditions of Lemma 4.6 part (ii).

Thus

\[
\sum_{A_{n+1} \in H} (\text{diam } A_{n+1})^2 \leq (L^2/|(f^n)'(z)|^2) 9.6 \times 10^{112} |(f^n)'(z)| r L)^{2-p}/|g'(f^n(z))|^p
\]

\[
< 4.2 \times 10^{129} r^{2-p}/|(f^n)'(z)| g'(f^n(z))|^p.
\]

We are now in a position to complete the proof of Theorem B part (ii).

**Lemma 5.6.** There exists \( r_0 > 0 \) such that, for each \( z \in \hat{A}, r < r_0 \),

\[
\mu(B(z, r)) < 1.9 \times 10^{289} r^{2-p}
\]

and hence, from Lemma 2.1, \( \dim \hat{A} \geq 2 - p \).

**Proof.** We take a value of \( r_0 \) which satisfies the conditions of Lemma 5.5 and then take \( z \in \hat{A}, r < r_0 \) and a value \( n \in \mathbb{N} \) which satisfies the conclusions of Lemma 5.5. As \( \mu \) is supported on \( \hat{A} \subset \mathfrak{W}_{n+1} \), it follows from (5.1) that

\[
\mu(B(z, r)) \leq \sum_{A_{n+1} \in H} \mu(A_{n+1}) = \sum_{A_{n+1} \in H} \mu_{n+1}(A_{n+1}).
\]

We know from Lemma 5.5 that there exists a unique set \( A_n \in \mathcal{A}_n \) such that, if \( A_{n+1} \in H \), then \( A_{n+1} \subset A_n \). If \( A_n \cap \hat{A} = \emptyset \), then \( \mu(A_n) = 0 \) and hence \( \mu(B(z, r)) = 0 \). If \( A_n \subset \hat{A} \), then it follows from (5.2) that

\[
\mu(B(z, r)) \leq \sum_{A_{n+1} \in H} \mu_{n+1}(A_{n+1}) = [1/\nu(A_n)] \sum_{A_{n+1} \in H} (\text{diam } A_{n+1})^2. \tag{5.11}
\]

Now

\[
\nu(A_n) = \prod_{r=1}^n \Delta(A_r) = \prod_{r=1}^n \text{vol}(\mathfrak{W}_{r-1} \cap A_r)/\text{vol}(A_r).
\]
where $A_n \subset A_{n-1} \subset \ldots \subset A_r \subset \hat{A}_1$. We denote $f^r(A_r), 1 \leq r \leq n$, by $B_m(r)$. It follows from (5·5) that

$$\Delta(A_r) \geq 1/[4 \cdot 2 \times 10^{130}(g(x_{m(r)}))^p].$$

As $z \in \hat{A}$, we must have $z \in A_n \subset \ldots \subset A_r \subset \hat{A}_1 \subset B_m(r)$ and so it follows from Lemma 4·4 that, for $0 \leq r \leq n$,

$$|g(f^r(z))| \geq g(x_{m(r)}/(6.8 \times 10^{21})$$

and hence

$$\Delta(A_r) \geq 1/[2 \cdot 9 \times 10^{152}|g(f^r(z))|^p].$$

Thus

$$\nabla(A_n) \geq \prod_{r=1}^{n} 1/[2 \cdot 9 \times 10^{152}|g(f^r(z))|^p].$$

and so, from (5·11), Lemma 5·5 and Lemma 3·5,

$$\mu(B(z, r)) \leq \frac{4 \cdot 2 \times 10^{120}r^2 - p}{(f^n)^2(z) g(f^n(z))} \prod_{r=1}^{n} 2 \cdot 9 \times 10^{152}|g(f^r(z))|^p$$

$$< 1 \cdot 3 \times 10^{273}r^{p-2} - p \left|\frac{|f^r(z)|^p}{(1 + p)\log f^r(z)}\right| \prod_{r=1}^{n} 2 \cdot 9 \times 10^{152}|g(f^r(z))/f^r(z)|^p$$

$$< 1 \cdot 3 \times 10^{273}r^{p-2} \prod_{r=1}^{n} 2 \cdot 9 \times 10^{152} \frac{|f^{r+1}(z)g(f^r(z))|}{|f^r(z)|\log f^{r+1}(z)}^p \cdot (5·12)$$

As $z \in \hat{A}$, we have $|f^r(z)| \geq \hat{C}(p)$ for each $r \in \mathbb{N}$ and so it follows from Lemma 3·1 and Lemma 3·5 that, for $1 \leq r \leq n-1$,

$$|f^r(f^r(z))| > |(1 + p)\log f^r(z)| g(f^r(z)) \hat{h}(f^r(z))/f^r(z) - \hat{C}(p)$$

$$> |(1 + p)\log f^r(z)| g(f^r(z))f^{r+1}(z)/|4f^r(z)|$$

Thus, from Lemma 5·3, for $1 \leq r \leq n-1$,

$$\frac{|f^{r+1}(z)g(f^r(z))|}{|f^r(z)|\log f^{r+1}(z)}^p < |4f^r(z)/|\log f^{r+1}(z)|^p| < 1/(2 \cdot 9 \times 10^{152}).$$

It now follows from (5·12) that, for each $z \in \hat{A}, r < r_0$,

$$\mu(B(z, r)) < 1 \cdot 3 \times 10^{273}r^{p-2}$$

and hence, from Lemma 2·1, $\dim \hat{A} \geq 2 - p$.

As $\hat{A} \subset J(f)$, it follows that $\dim J(f) \geq 2 - p$. At the beginning of this section we stated that $f$ was a function of the form $f_{p,K}$, where $p \leq 1/2$ and $K \geq 2\hat{C}(p)$. If we take $S(p)$ to be the function defined just before Lemma 4·2 for $p \in [1, \infty)$ then, if $p \leq 1/2$, we have $S(1/p) > 2\hat{C}(1/p) = 2\hat{C}(p)$. Thus, with this definition of $S(p)$, for $p \leq 1/2$ and $K \geq S(1/p)$ we have $\dim J(f_{p,K}) \geq 2 - p$. This completes the proof of Theorem B.
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