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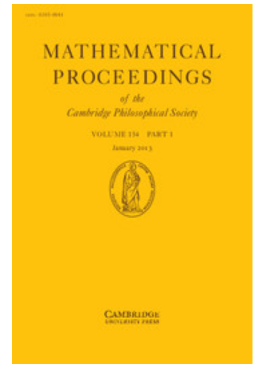


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# The Hausdorff dimension of Julia sets of entire functions III

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*Abstract*

It is known that, for a transcendental entire function $f$, the Hausdorff dimension of the Julia set of $f$ satisfies $1 \leqslant \dim J(f) \leqslant 2$. In this paper we introduce a family of transcendental entire functions $f_{p,K}$ for which the set $\{\dim J(f_{p,K}) : 0 < p, K < \infty\}$ has infemum 1 and supremum 2.

## 1. *Introduction*

Let $f: \mathbb{C} \to \mathbb{C}$ denote a transcendental entire function and $f^n, n \in \mathbb{N}$, the $n$th iterate of $f$. The set of normality, $N(f)$ is defined to be the set of points, $z \in \mathbb{C}$, such that the sequence $(f^n)_{n\in\mathbb{N}}$ forms a normal family in some neighbourhood of $z$. It is easy to see that $N(f)$ is open and has the property of complete invariance under $f$, that is $z \in N(f)$ if and only if $f(z) \in N(f)$. The complement, $J(f)$, of $N(f)$ is known as the Julia set. This set is clearly closed and completely invariant under $f$. More details of these and other basic properties of the sets $N(f)$ and $J(f)$ can be found in [**3**] and [**4**].

It was shown by Baker [**1**, corollary to theorem 1] that, if $f$ is a transcendental entire function, then $J(f)$ must contain continua and so the Hausdorff dimension of $J(f)$, $\dim J(f)$, lies in the range $1 \leqslant \dim J(f) \leqslant 2$.

If a function $f$ has the property that $J(f) = \mathbb{C}$, then clearly $\dim J(f) = 2$. An example of such a function is $f(z) = e^z$. McMullen has shown that, in fact, all functions of the form $f(z) = \lambda e^z, \lambda \in \mathbb{R}^+$, have Julia sets of dimension 2 [**6**, theorem 1·2] even though, when $0 < \lambda < 1/e$, $J(f)$ has area zero [**6**, theorem 1·3]. Similar arguments can be applied to the functions defined by

$$F_n(z) = \frac{1}{2\pi i}\int_{\gamma_n} \frac{\exp(t^n)}{t-z}\,dt - K_n,$$

where $\gamma_n$ is the boundary of

$$H_n = \{z = re^{i\theta} : |\theta| < \pi/n, \quad 0 < r < \infty\},$$

described in a clockwise direction for $z \in \mathbb{C}\backslash\bar{H}_n$ and by analytic continuation for $z \in \bar{H}_n$. Provided $K_n$ is sufficiently large, it can be shown that, for each $n \in \mathbb{N}, n \geqslant 2$, $\dim J(F_n) = 2$.

In [**7**] we gave the first examples of transcendental entire functions whose Julia sets have been proved to have dimension less than 2. More precisely, we considered the function $E$ defined by

$$E(z) = \frac{1}{2\pi i}\int_L \frac{\exp(e^t)}{t-z}\,dt,$$

where $L$ is the boundary of the region

$$G = \{z\colon \operatorname{Re}(z) > 0, \quad -\pi < \operatorname{Im}(z) < \pi\},$$

described in a clockwise direction, for $z \in \mathbb{C}\backslash \bar{G}$, and by analytic continuation for $z \in \bar{G}$. We proved the following result.

THEOREM A. *Given $\delta > 0$, there exists $K_0(\delta)$ such that, for all $K > K_0$, the function $f_K$ defined by $f_K(z) = E(z) - K$ satisfies $1 \leqslant \dim J(f_K) < 1 + \delta$.*

In [**8**] we showed further that there exists $K_0$ such that, for all $K > K_0$, $\dim J(f_K) > 1$.

These results lead us to consider the intermediate family of functions defined by

$$E_p(z) = \frac{1}{2\pi i}\int_{L_p} \frac{\exp(e^{(\log t)^{1+p}})}{t-z}\,dt,$$

where $L_p$ is the boundary of the region

$$G_p = \{z = x+iy\colon |y| \leqslant \pi x/[(1+p)(\log(x))^p], \quad x \geqslant 3\}$$

described in a clockwise direction, for $z \in \mathbb{C}\backslash \bar{G}_p$, and by analytic continuation for $z \in \bar{G}_p$. We then define $f_{p,K}(z) = E_p(z) - K$. By using a modification of McMullen's method, together with the methods used in the proof of Theorem A, we are able to prove the following.

THEOREM B. *There exists a positive real-valued increasing function $S$ defined on $(1, \infty)$ such that*

(i) *for $p \geqslant 2$, if $K > S(p)$, then* $\dim J(f_{p,K}) \leqslant 1 + 2/p$;

(ii) *for $0 < p \leqslant 1/2$, if $K > S(1/p)$, then* $\dim J(f_{p,K}) \geqslant 2 - p$.

In future work we hope to show that, for each $n \in \mathbb{N}, K > S(n), \dim J(f_{p,K})$ is a continuous function of $p$ for $p \in [1/n, n]$ and hence that, for each $d \in (1, 2]$, there exists a transcendental entire function $f$ for which $\dim J(f) = d$.

## 2. *Methods for obtaining estimates for Hausdorff dimension*

We begin this section with a formal definition for the Hausdorff dimension of a compact set $E$. If, for each $\mu > 0$, we put

$$H_\mu(E) = \lim_{\epsilon \to 0} \inf \sum_i (r_i)^\mu,$$

where the inf is taken over all possible covers of $E$ with sets of diameter $r_i < \epsilon$, then the Hausdorff dimension $d = \dim E$ of the set $E$ is defined to be the unique value satisfying

$$H_\mu(E) = \begin{cases} \infty & \text{for} \quad \mu < d \\ 0 & \text{for} \quad \mu > d. \end{cases}$$

For more details see, for example, [**5**, p. 220].

The following result is part of Frostman's Lemma and is a commonly used method for obtaining a lower bound for $\dim E$. We include the proof as it is so short.

LEMMA 2·1. *Let $E$ be a compact subset of $\mathbb{R}^n$ and $t$ be a real positive value. Suppose that there exist a measure $\mu$ supported on $E$ and constants $C, r_0 > 0$ such that every ball $B$ of radius $r < r_0$ satisfies $\mu(B) \leqslant Cr^t$. Then $\dim E \geqslant t$.*

*Proof.* Take a value $\epsilon$, $0 < \epsilon < r_0$, and a countable collection of balls $B_i$ of radius $r_i < \epsilon$ which gives a cover of $E$. Then

$$0 < M = \mu(E) \leqslant \sum_i \mu(B_i) \leqslant C \sum_i (r_i)^t$$

and so $\sum_i (r_i)^t \geqslant M/C$. Thus $H_t(E) \geqslant M/C > 0$ and hence $\dim E \geqslant t$.

By using Lemma 2·1, McMullen [**6**, proposition 2·2] was able to prove the following.

LEMMA 2·2. *For each $k \in \mathbb{N}$, let $\mathscr{A}_k$ be a finite collection of disjoint compact subsets of $\mathbb{R}^n$, each of which has positive $n$-dimensional measure, and define*

$$\mathfrak{A}_k = \bigcup_{A_k \in \mathscr{A}_k} A_k, \quad A = \bigcap_{k=1}^{\infty} \mathfrak{A}_k.$$

*Suppose also that, for each $A_k \in \mathscr{A}_k$, there exist $A_{k+1} \in \mathscr{A}_{k+1}$ and a unique $A_{k-1} \in \mathscr{A}_{k-1}$ such that*

$$A_{k+1} \subset A_k \subset A_{k-1}.$$

*If $\Delta_k, d_k$ are such that, for each $A_k \in \mathscr{A}_k$,*

$$\frac{\operatorname{vol}(\mathfrak{A}_{k+1} \cap A_k)}{\operatorname{vol}(A_k)} \geqslant \Delta_k$$

*and $\operatorname{diam} A_k \leqslant d_k$, $d_k \to 0$ as $k \to \infty$, then*

$$\overline{\lim_{k \to \infty}} \sum_{i=1}^{k} |\log \Delta_i| / |\log d_k| \geqslant n - \dim A.$$

We will use some of the ideas of this result in our proof of Theorem B, part (ii).

## 3. *Properties of the functions $E_p(z)$*

Consider the functions $E_{p,n}(z)$, $n \in \mathbb{N}$, $n \geqslant 3$, $0 < p < \infty$, defined by

$$E_{p,n}(z) = \frac{1}{2\pi i} \int_{L_{p,n}} \frac{\exp\left(e^{(\log t)^{1+p}}\right)}{t - z} dt,$$

where $L_{p,n}$ is the boundary of the region

$$G_{p,n} = \{z = x + iy : |y| \leqslant \pi x / [(1+p)(\log(x))^p], \quad x \geqslant n\}$$

described in a clockwise direction, for $z \in \mathbb{C} \backslash \bar{G}_{p,n}$. It follows from Cauchy's theorem that, for each $n \in \mathbb{N}$, $n \geqslant 3$, $E_{p,n+1}(z) = E_{p,n}(z)$ for $z \in \mathbb{C} \backslash \bar{G}_{p,n}$ and so the functions $E_{p,n}$, $n \geqslant 3$, give an analytic continuation of $E_{p,3}$ to a function $E_p$ defined on the whole of $\mathbb{C}$. For simplicity, we write $G_p$ for $G_{p,3}$ and $L_p$ for $L_{p,3}$.

LEMMA 3·1. *For each $n\in\mathbb{N}$, there exist $C(n), D(n) > 1$ such that, for each $p\in[1/n, n]$,*

(i) *if $z\in\mathbb{C}\backslash\bar{G}_p$, then $|E_p(z)| < C(n)$;*

(ii) *if $z\in\bar{G}_p$, then $|E_p(z)-\exp(e^{(\log z)^{1+p}})| < C(n)$ and hence*

$$|E'_p(z)-(1+p)(\log z)^p e^{(\log z)^{1+p}}\exp(e^{(\log z)^{1+p}})/z| < C(n);$$

(iii) *if $z = x+iy\in\bar{G}_p$ with $x < D(n)$, then $|E_p(z)| < C(n)$;*

(iv) *if $z = x+iy\in\bar{G}_p$ with $x \geqslant D(n)$, then $|(1+p)(\log z)^p e^{(\log z)^{1+p}}/z| > 8$.*

*Proof.* If $t = re^{i\theta}$, then there exists $K(n)$ such that, for $p\in[1/n, n]$, $|\theta| < \pi/2$, $r \geqslant 3$, we have

$$\exp(\log t)^{1+p} = \exp(\log r+i\theta)^{1+p} = \exp\left[(\log r)^{1+p}\left(1+\frac{i\theta}{\log r}\right)^{1+p}\right]$$
$$= \exp\left[(\log r)^{1+p}\left(1+\frac{i\theta(1+p)}{\log r}+e\right)\right]$$
$$= \exp[(\log r)^{1+p}+e_x+i(\theta(1+p)(\log r)^p+e_y)],$$

where $$|e| \leqslant \frac{K(n)\,\theta^2}{(\log r)^2} \quad \text{and hence} \quad |e_x|, \quad |e_y| \leqslant K(n)\,\theta^2(\log r)^{p-1}.$$

Given $\epsilon > 0$, it is clear that there exists $M(n,\epsilon)$ such that, for

$$t = re^{i\theta} = x+iy, \quad x \geqslant M(n,\epsilon), \quad \frac{5\pi x}{6(\log x)^p(1+p)} \leqslant y \leqslant \frac{7\pi x}{6(\log x)^p(1+p)},$$

we have

$$|\theta(1+p)(\log r)^p+e_y| \leqslant (1+\epsilon)\,|\theta|\,(1+p)(\log r)^p$$
$$\leqslant (1+\epsilon)\,|y/x|\,(1+p)(\log r)^p \leqslant (1+\epsilon)^2\, 7\pi/6,$$

and, similarly,

$$|\theta(1+p)(\log r)^p+e_y| \geqslant (1-\epsilon)^2\, 5\pi/6.$$

Thus, by taking a sufficiently small value for $\epsilon$, we find that there exists $M(n)$ such that, for

$$p\in[1/n, n], \quad t = re^{i\theta} = x+iy, \quad x \geqslant M(n), \quad \frac{5\pi x}{6(\log x)^p(1+p)} \leqslant y \leqslant \frac{7\pi x}{6(\log x)^p(1+p)},$$

we have

$$\cos[\theta(1+p)(\log r)^p+e_y] \leqslant -1/2 \quad \text{and} \quad |e_x| \leqslant (\log r)^{p-1}/2$$

so that

$$\operatorname{Re}[\exp(\log t)^{1+p}] \leqslant -\exp[(\log r)^{1+p}/2]/2 \leqslant -\exp[(\log x)^{1+p}/2]/2$$

and hence

$$|\exp[e^{(\log t)^{1+p}}]| \leqslant \exp[-(1/2)\,e^{(\log x)^{1+p}/2}]. \tag{3·1}$$

If $z\in\mathbb{C}\backslash\bar{G}_p$, then it follows from (3·1) and Cauchy's residue theorem that

$$E_p(z) = \frac{1}{2\pi i}\int_{L'_p}\frac{\exp(e^{(\log t)^{1+p}})}{t-z}\,dt,$$

where $L'_p$ is the boundary of

$$G'_p = \left\{z = x+iy\colon x \geqslant M(n), \quad |y| \leqslant \frac{5\pi x}{6(\log x)^p(1+p)}\right\}$$

described in a clockwise direction. If we choose $M(n)$ sufficiently large to ensure that $M(n) \geqslant 4$ and

$$\frac{\pi x}{6(\log x)^p (1+p)} \geqslant 2 \quad \text{for} \quad x \geqslant M(n), \quad p \in [1/n, n],$$

then it follows that

$$|E_p(z)| < \frac{1}{2\pi} \int_{L'_p} |\exp(e^{(\log t)^{1+p}})| \, |dt|.$$

On the curves described by

$$|y| = \frac{5\pi x}{6(\log x)^p (1+p)}$$

we have

$$\left|\frac{dy}{dx}\right| = \left|\frac{5\pi}{6(1+p)} [1/(\log x)^p - p/(\log x)^{1+p}]\right| < 2\pi$$

and hence $|dt| < 4\pi \, dx$. Thus, from (3·1), if $z \in \mathbb{C} \backslash \bar{G}_p$, then

$$|E_p(z)| < M(n) \exp[e^{(\log 2M(n))^{1+p}}] + 2 \int_{M(n)}^{\infty} \exp[-(1/2) \, e^{(\log x)^{1+p}/2}] \, dx. \tag{3·2}$$

If $z = x + iy \in \bar{G}_p$ and $x < M(n) + 1$, then it follows from Cauchy's residue theorem and (3·1) that

$$E_p(z) = \frac{1}{2\pi i} \int_{L''_p} \frac{\exp(e^{(\log t)^{1+p}})}{t - z} \, dt,$$

where $L''_p$ is the boundary of

$$G''_p = \left\{z = x + iy : x \geqslant M(n) + 2, \quad |y| \leqslant \frac{7\pi x}{6(\log x)^p (1+p)}\right\}$$

described in a clockwise direction and hence, by similar arguments to those used above,

$$|E_p(z)| < M(n) \exp[e^{(\log 2M(n))^{1+p}}] + 2 \int_{M(n)}^{\infty} \exp[-(1/2) \, e^{(\log x)^{1+p}/2}] \, dx \tag{3·3}$$

and hence

$$|E_p(z) - \exp[e^{(\log z)^{1+p}}]| < 2M(n) \exp[e^{(\log 2M(n))^{1+p}}] + 2 \int_{M(n)}^{\infty} \exp[-(1/2) \, e^{(\log x)^{1+p}/2}] \, dx. \tag{3·4}$$

Finally, if $z = x + iy \in \bar{G}_p$ and $x \geqslant M(n) + 1$, then it follows from Cauchy's residue theorem and (3·1) that

$$E_p(z) = \exp[e^{(\log z)^{1+p}}] + \frac{1}{2\pi i} \int_{\hat{L}_p} \frac{\exp(e^{(\log t)^{1+p}})}{t - z} \, dt,$$

where $\hat{L}_p$ is the boundary of

$$\hat{G}_p = \left\{z = x + iy : x \geqslant M(n), \quad |y| \leqslant \frac{7\pi x}{6(\log x)^p (1+p)}\right\}$$

described in a clockwise direction and so, arguing as before, we find that

$$|E_p(z)-\exp[e^{(\log z)^{1+p}}]| < M(n)\exp[e^{(\log 2M(n))^{1+p}}]+2\int_{M(n)}^{\infty}\exp[-(1/2)\,e^{(\log x)^{1+p}/2}]\,dx.$$

Combined with (3·2) and (3·4) this shows that parts (i) and (ii) of Lemma 3·1 are satisfied if we take

$$C(n) = 2M(n)\exp[e^{(\log 2M(n))^{1+p}}]+2\int_{M(n)}^{\infty}\exp[-(1/2)\,e^{(\log x)^{1+p}/2}]\,dx < \infty.$$

If we take $D(n) = M(n)+1$ then part (iii) of Lemma 3·1 follows from (3·3).

By using arguments similar to those used in proving (3·1), it can be seen that, provided $M(n)$ is chosen to be sufficiently large, part (iv) of Lemma 3·1 is also true.

We now consider the inverse function singularities of the functions $f_{p,K}$ defined by

$$f_{p,K}(z) = E_p(z)-K.$$

We put

$$S(f) = \{z\colon z \text{ is a finite singularity of } f^{-1}\},$$

$$P(f) = \{z\colon z \text{ is a finite singularity of } f^{-n} \text{ for some } n\in\mathbb{N}\} = \bigcup_{n=0}^{\infty} f^n(S(f)),$$

$$\tilde{C}(p) = \begin{cases} C([p]+1) & \text{if} \quad p \geqslant 1 \\ C([1/p]+1) & \text{if} \quad 0 < p < 1, \end{cases}$$

$$B(z,r) = \{w\colon |w-z| < r\}.$$

LEMMA 3·2. *For each* $p\in(0,\infty), K > 2\tilde{C}(p)$, *we have* $P(f_{p,K}) \subset B(-K, 2\tilde{C}(p))$.

*Proof.* The finite transcendental singularities of $(f_{p,K})^{-1}$ are the finite asymptotic values of $f_{p,K}$ on curves going to $\infty$. From Lemma 3·1, we see that these must be contained in the disc $B(-K,\tilde{C}(p))$.

The remaining singularities of $(f_{p,K})^{-1}$ are the images of points $z$ such that $(f_{p,K})'(z) = 0$. If $z\in\mathbb{C}\backslash\bar{G}_p$, then it follows from Lemma 3·1 that $f_{p,K}(z)\in B(-K,\tilde{C}(p))$. If $z\in\bar{G}_p$, then it follows from Lemma 3·1 that

$$|(1+p)(\log z)^p\, e^{(\log z)^{1+p}}\exp(e^{(\log z)^{1+p}})/z| < \tilde{C}(p)$$

and either $f_{p,K}(z)\in B(-K,\tilde{C}(p))$ or $|(1+p)(\log z)^p\, e^{(\log z)^{1+p}}/z| > 1$. It follows that either $f_{p,K}(z)\in B(-K,\tilde{C}(p))$ or $|\exp(e^{(\log z)^{1+p}})| < \tilde{C}(p)$ and hence, from Lemma 3·1, $f_{p,K}(z)\in B(-K,2\tilde{C}(p))$.

Thus $S(f_{p,K}) \subset B(-K,2\tilde{C}(p))$. As $K > 2\tilde{C}(p)$, we have $B(-K,2\tilde{C}(p)) \subset \mathbb{C}\backslash\bar{G}_p$ and so it follows from Lemma 3·1 that $f_{p,K}(B(-K,2\tilde{C}(p))) \subset B(-K,\tilde{C}(p))$. Thus, for each $n\in\mathbb{N}$, $(f_{p,K})^n(B(-K,2\tilde{C}(p))) \subset B(-K,\tilde{C}(p))$ and hence $P(f_{p,K}) \subset B(-K,2\tilde{C}(p))$.

At several points in the proof of Theorem B we use the previous result together with the following result which is known as Koebe's distortion theorem.

LEMMA 3·3 (See, for example, [**2**, p. 32]). *If $g$ is univalent in the disc $B(z,r)$ then, for* $0 < s < r$,

$$\sup_{v,w\in B(z,s)} |g'(v)/g'(w)| \leqslant L(s/r) = |(r+s)/(r-s)|^4.$$

For simplicity we denote $L(1/2)$ by $L(=81)$. We conclude this section with some results concerning the set of normality and the Julia set of $f_{p,K}$.

LEMMA 3·4. *If* $K > \tilde{C}(p)$ *then* $\mathbb{C}\backslash\bar{G}_p \subset N(f_{p,K})$.

*Proof.* It follows from Lemma 3·1 that, if $K > \tilde{C}(p)$, then

$$f_{p,K}(\mathbb{C}\backslash\bar{G}_p) \subset B(-K, \tilde{C}(p)) \subset \{z\colon \mathrm{Re}(z) < 0\} \subset \mathbb{C}\backslash\bar{G}_p$$

and hence $\mathbb{C}\backslash\bar{G}_p \subset N(f_{p,K})$.

We note that it is clear that there exists $z_0 \in B(-K, \tilde{C}(p))$ such that, for each $z$ in the component of $N(f_{p,K})$ which contains $\mathbb{C}\backslash\bar{G}_p$,

$$(f_{p,K})^n(z) \to z_0 \quad \text{as} \quad n \to \infty.$$

We put

$$E_{R,N} = \{z\colon (f_{p,K})^n(z) \in \bar{G}_{p,R} \quad \text{for} \quad 0 \leqslant n \leqslant N\},$$

$$E_R = \bigcap_{N=1}^{\infty} E_{R,N},$$

$$\tilde{D}(p) = \begin{cases} D([p]+1) & \text{if} \quad p \geqslant 1 \\ D([1/p]+1) & \text{if} \quad 0 < p < 1, \end{cases}$$

and take $D'(p)$ to be a value satisfying

$$\frac{D'(p)}{(1+p)(\log D'(p))^p} \geqslant 24L\tilde{C}(p).$$

Finally, we put

$$\hat{C}(p) = \max\{\exp[(p+1/p)\,8{\cdot}3 \times 10^{88}], \tilde{C}(p), \exp((5\times 10^5)/p)^{1/p}, \tilde{D}(p), D'(p)\}.$$

LEMMA 3·5. *If* $R \geqslant \hat{C}(p)$ *and* $K > 2\tilde{C}(p)$ *then*
(i) *for each* $z \in E_{R,N}$, $1 \leqslant n \leqslant N$, *we have* $|(f^n_{p,K})'(z)| > 3^n |f^n_{p,K}(z)|$,
(ii) $E_R \subset J(f_{p,K})$.

*Proof.* (i) If $R \geqslant \hat{C}(p)$ and $z = x+iy \in \bar{G}_{p,R}$, then

$$y \leqslant \frac{\pi x}{(1+p)(\log x)^p} \leqslant \frac{\pi x}{(1+p)(\log R)^p} \leqslant \frac{\pi x}{(1+p)\,4\pi} < x/4 < |z|/4 \qquad (3{\cdot}5)$$

and hence

$$|z| < 2x. \qquad (3{\cdot}6)$$

If, further, $K > \tilde{C}(p)$, then it follows from Lemma 3·1 that

$$\begin{aligned} |(f_{p,K})'(z)| &> |(1+p)(\log z)^p\, e^{(\log z)^{1+p}} \exp(e^{(\log z)^{1+p}})/z| - \tilde{C}(p) \\ &> 8\,|\exp(e^{(\log z)^{1+p}})| - \tilde{C}(p) \\ &\geqslant 8\,\mathrm{Re}[\exp(e^{(\log z)^{1+p}})] - \tilde{C}(p) > 8\,\mathrm{Re}[f_{p,K}(z)] - \tilde{C}(p). \end{aligned}$$

If, in addition, $f_{p,K}(z) \in G_{p,R}$, then it follows from (3·6) that

$$|(f_{p,K})'(z)| > 4|f_{p,K}(z)| - \tilde{C}(p) > 3\,|f_{p,K}(z)|.$$

Thus, for each $z \in E_{R,N}$, $1 \leqslant n \leqslant N$,

$$|(f^n_{p,K})'(z)| = \prod_{r=0}^{n-1} |(f_{p,K})'(f^r_{p,K})(z)| > 3^n \prod_{r=0}^{n-1} |(f^{r+1}_{p,K})(z)| > 3^n |f^n_{p,K}(z)|.$$

(ii) As $K > 2\tilde{C}(p)$, it follows from Lemma 3·2 that, for each $z \in E_R$, $n \in \mathbb{N}$, the branch $g_n$ of $f_{p,K}^{-n}$ that maps $f_{p,K}^n(z)$ to $z$ is univalent to $B(f_{p,K}^n(z), |f_{p,K}^n(z)|)$ and hence, from Lemma 3·3,

$$g_n(B(f_{p,K}^n(z), |f_{p,K}^n(z)|/2)) \subset B(z, 81\,|f_{p,K}^n(z)|/|2(f_{p,K}^n)'(z)|). \tag{3·7}$$

If $z \in N(f_{p,K})$ then there exists $r > 0$ such that $B(z, r) \subset N(f_{p,K})$. From (i) and (3·7) we see that, for sufficiently large values of $n$,

$$g_n(B(f_{p,K}^n(z), |f_{p,K}^n(z)|/2)) \subset B(z, r)$$

and hence

$$f_{p,K}^n(B(z,r)) \supset B(f_{p,K}^n(z), |f_{p,K}^n(z)|/2).$$

It follows from (3·5) that

$$B(f_{p,K}^n(z), |f_{p,K}^n(z)|/2) \cap \mathbb{C} \backslash \bar{G}_p \neq \varnothing$$

and so, from the comment after Lemma 3·4, if $B(z,r) \subset N(f_{p,K})$, then

$$f_{p,K}^n(z) \to z_0 \in \{z\colon \operatorname{Re}(z) < 0\} \quad \text{as} \quad n \to \infty.$$

This, however, is impossible as $z \in E_R$ and hence $f_{p,K}^n(z) \in \bar{G}_{p,R} \subset \{z\colon \operatorname{Re}(z) \geqslant 3\}$ for each $n \in \mathbb{N}$.

## 4. *Proof of Theorem B part* (i)

We take a function $f = f_{p,K}$ with $K > 2\tilde{C}(p)$, a value $x_1 \geqslant \hat{C}(p)$ and define the real values $x_n$ inductively by

$$x_{n+1} = x_n + r_n, \quad r_n = x_n/[(1+p)(\log x_n)^p].$$

We define the sets $B_n$ by

$$B_n = \{z = x + iy\colon x_n \leqslant x \leqslant x_n + R_n, |y| \leqslant R_n\},$$

where $r_n/(8L) \leqslant R_n \leqslant \pi r_n$, and define the following collection of sets inductively:

$$\mathscr{A}_1 = \{A_1\colon A_1 \text{ is a component of } f^{-1}(B_n) \text{ for some } n \in \mathbb{N}, A_1 \subset \bigcup_{n\in\mathbb{N}} B_n\},$$

$$\mathfrak{A}_k = \bigcup_{A_k \in \mathscr{A}_k} A_k,$$

and

$$\mathscr{A}_{k+1} = \{A_{k+1}\colon A_{k+1} \text{ is a component of } f^{-(k+1)}(B_n) \text{ for some } n \in \mathbb{N}, A_{k+1} \subset \mathfrak{A}_k\}.$$

LEMMA 4·1. *For each* $k \in \mathbb{N}, A_k \in \mathscr{A}_k$,

$$\operatorname{diam} A_k < 1/3^k \to 0 \quad as \quad k \to \infty.$$

*Proof*. If $A_k \in \mathscr{A}_k$, then there exists $n \in \mathbb{N}$ such that $f^k(A_k) = B_n$. As $K > 2\tilde{C}(p)$, $\mathfrak{A}_k \subset E_{x_1,k}$ and $x_1 \geqslant \hat{C}(p)$, it follows from Lemma 3·5 that

$$\operatorname{diam} A_k < \sup_{z\in A_k} \frac{\operatorname{diam} B_n}{|(f^k)'(z)|} < \sup_{z \in A_k} (1/3^k) \frac{\operatorname{diam} B_n}{|f^k(z)|} < \frac{x_n}{3^k x_n}.$$

We put

$$\hat{h}(z) = \exp(e^{(\log z)^{1+p}}),$$

$$\hat{G}_p = \{z = x+iy\colon z \in \bar{G}_p, x \leqslant \hat{C}(p)\},$$

and

$$S(p) = 2\tilde{C}(p) + \max_{z \in \hat{G}_p} |\hat{h}(z)|.$$

LEMMA 4·2. *If $K > S(p)$, $x = \hat{C}(p)$ and $R_n = \pi r_n$ then, for each $k \in \mathbb{N}$, the sets in $\mathscr{A}_k$ form a cover of $J(f_{p,K})$.*

*Proof.* If $z \in \hat{G}_p$ then, as $K > S(p)$,

$$\operatorname{Re}[f(z)] \leqslant |\hat{h}(z)| - K + \tilde{C}(p) < 0$$

and hence, from Lemma 3·4, $f(z) \in \mathbb{C} \backslash \bar{G}_p \subset N(f)$. So $J(f) \subset \bar{G}_{p,x_1}$.

If
$$z = x + iy \in \bar{G}_{p,x_1} \setminus \bigcup_{n=1}^{\infty} B_n,$$

then there exists $n \in \mathbb{N}$ such that

$$x_n \leqslant x < x_{n+1}, \quad \frac{\pi x}{(1+p)(\log x)^p} \geqslant |y| \geqslant \frac{\pi x_n}{(1+p)(\log x_n)^p} \geqslant \frac{5\pi x}{6(1+p)(\log x)^p}$$

and so it follows from (3·1) that

$$\operatorname{Re}[\exp(\log z)^{1+p}] \leqslant -\exp[(\log x)^{1+p}/2]/2 < -1/2.$$

Thus
$$\operatorname{Re}(f(z)) \leqslant |\hat{h}(z)| - K + \tilde{C}(p) < e^{-1/2} - K + \tilde{C}(p) < 0$$

and so, as previously, $z \in N(f)$.

So
$$J(f) \subset \bigcup_{n=1}^{\infty} B_n$$

and hence, as $J(f)$ is forward invariant under $f$,

$$J(f) \subset \bigcup_{n=1}^{\infty} f^{-k}(B_n)$$

for each $k \in \mathbb{N}$. It follows from the arguments given above that

$$f\left(\mathbb{C} \setminus \bigcup_{n=1}^{\infty} B_n\right) \subset \{z \colon \operatorname{Re}(z) < 0\}$$

and so, if
$$f^k(z) \in \bigcup_{n=1}^{\infty} B_n,$$

we must have
$$f^{k-1}(z) \in \bigcup_{n=1}^{\infty} B_n.$$

Thus, for each $k \in \mathbb{N}$, we have

$$J(f) \subset \bigcup_{n=1}^{\infty} f^{-k}(B_n) = \mathfrak{A}_k.$$

If, for some $0 < t \leqslant 2$,

$$\sum_{\substack{A_{n+1} \cap A_n \neq \varnothing \\ A_{n+1} \in \mathscr{A}_{n+1}}} (\operatorname{diam} A_{n+1})^t \leqslant (\operatorname{diam} A_n)^t$$

for each $n\in\mathbb{N}, A_n\in\mathscr{A}_n$, then it follows from Lemma 4·1 that, for each $z\in\mathbb{C}, n\in\mathbb{N}$,

$$\sum_{\substack{A_{n+1}\cap B(z,1)\neq\varnothing\\ A_{n+1}\in\mathscr{A}_{n+1}}}(\operatorname{diam}A_{n+1})^t \leqslant \sum_{\substack{A_n\cap B(z,1+3^{-(n+1)})\neq\varnothing\\ A_n\in\mathscr{A}_n}}(\operatorname{diam}A_n)^t$$

$$\leqslant \ldots \leqslant \sum_{\substack{A_1\cap B(z,1+3^{-(n+1)}+3^{-n}+\ldots+3^{-2})\neq\varnothing\\ A_1\in\mathscr{A}_1}}(\operatorname{diam}A_1)^t < \sum_{\substack{A_1\cap B(z,2)\neq\varnothing\\ A_1\in\mathscr{A}_1}}(\operatorname{diam}A_1)^t.$$

If, further, we have $K > S(p)$ and $R_n = \pi r_n$, then it follows from Lemma 4·2 that, for each $z\in\mathbb{C}$,

$$\dim[J(f)\cap B(z,1)]\leqslant t$$

and hence $\dim J(f)\leqslant t$.

LEMMA 4·3. *If* $A_k\in\mathscr{A}_k$ *and* $f^k(A_k) = B_n$, *then*

$$\sum_{\substack{A_{k+1}\cap A_k\neq\varnothing\\ A_{k+1}\in\mathscr{A}_{k+1}}}(\operatorname{diam}A_{k+1})^t/(\operatorname{diam}A_k)^t \leqslant L^t \sum_{\substack{A_1\cap B_n\neq\varnothing\\ A_1\in\mathscr{A}_1}}(\operatorname{diam}A_1)^t/(\operatorname{diam}B_n)^t.$$

*Proof*. Clearly

$$\left.\begin{aligned}\operatorname{diam}A_{k+1}&\leqslant \operatorname{diam}f^k(A_{k+1})/[\inf_{z\in A_{k+1}}|(f^k)'(z)|],\\ \operatorname{diam}A_k&\geqslant \operatorname{diam}B_n/[\sup_{z\in A_k}|(f^k)'(z)|].\end{aligned}\right\}\qquad(4·1)$$

If $A_{k+1}\in\mathscr{A}_{k+1}$, then $f^k(A_{k+1})\in\mathscr{A}_1$ and so it follows from Lemma 4·1 that

$$f^k\Big(A_k\cup\bigcup_{\substack{A_{k+1}\cap A_k\neq\varnothing\\ A_{k+1}\in\mathscr{A}_{k+1}}}\Big)\subset B(x_n, x_n/4).$$

Thus, from Lemma 3·2 and Lemma 3·3, if $A_{k+1}\cap A_k\neq\varnothing$ and $A_{k+1}\in\mathscr{A}_{k+1}$, then

$$\sup_{v,w\in A_k\cup A_{k+1}}|(f^k)'(v)/(f^k)'(w)|\leqslant L.$$

Combined with (4·1) this gives the desired result.

Thus, if $K > S(p)$, in order to show that $\dim J(f)\leqslant t$, it is sufficient to show that, for each $n\in\mathbb{N}$,

$$\sum_{\substack{A_1\cap B_n\neq\varnothing\\ A_1\in\mathscr{A}_1}}(\operatorname{diam}A_1)^t\leqslant(\operatorname{diam}B_n)^t/L^t,\qquad(4·2)$$

when $R_n = \pi r_n$. In order to do this we need the following results. We put

$$g(z) = \exp(\log z)^{1+p}.$$

LEMMA 4·4. *If*

$$z\in\bigcup_{n=1}^{\infty}B_n\quad\textit{and}\quad r\leqslant\left|\frac{2\pi z}{(1+p)(\log z)^p}\right|\quad\textit{then, for each } w\in B(z,r),$$

(i) $1/(6·8\times10^{21})\leqslant|g(w)/g(z)|\leqslant 6·8\times10^{21}$,
(ii) $1/2\leqslant|w/z|\leqslant 2$,
(iii) $1/5\leqslant|(\log w)/(\log z)|^p\leqslant 5$,
(iv) $1/(6·8\times10^{22})\leqslant|g'(w)/g'(z)|\leqslant 6·8\times10^{22}$.

*Proof.* If $w \in B(z,r)$, where $z, r$ satisfy the above conditions, then

$$g(w) = g(z+e) = \exp[\log(z+e)]^{1+p} = \exp[\log(z)+\log(1+e/z)]^{1+p},$$

where

$$|e/z| \leqslant |r/z| \leqslant \left|\frac{2\pi}{(1+p)(\log z)^p}\right| \leqslant \frac{2\pi}{(1+p)(\log \hat{C}(p))^p} < 1/(1+p),$$

and hence

$$g(w) = \exp[\log(z)+E]^{1+p} = \exp[(\log z)^{1+p}(1+E/(\log z))^{1+p}],$$

where $|E| < |2e/z|$ and hence

$$|E/(\log z)| < \left|\frac{4\pi}{(1+p)(\log z)^{1+p}}\right| < 1/[(1+p)(\log \hat{C}(p))]$$

so that $$g(w) = \exp[(\log z)^{1+p}(1+\hat{E})],$$

where $$|\hat{E}| < |4(1+p)E/(\log z)| < 16\pi/|\log z|^{1+p}.$$

Thus $g(w) = \exp[\log z]^{1+p}\exp(\tilde{E}) = g(z)\exp(\tilde{E})$, where $|\tilde{E}| < 16\pi$ and hence

$$1/[6{\cdot}8\times 10^{21}] < \exp(\tilde{E}) < 6{\cdot}8\times 10^{21}.$$

Similar arguments to those used above show that

$$|\log z|^p/5 < |\log w|^p < 5|\log z|^p.$$

Clearly $|z|/2 < |w| < 2|z|$. As $g'(w) = (1+p)(\log w)^{1+p}g(w)/w$, part (iv) follows from parts (i), (ii) and (iii).

We put $h(z) = \exp[e^{(\log z)^{1+p}}]-K$.

LEMMA 4·5. *If*

$$z \in \bigcup_{n=1}^{\infty} B_n \quad \text{and} \quad r \leqslant \left|\frac{2\pi z}{(1+p)(\log z)^p}\right| \quad \text{then, for each } n \in \mathbb{N},$$

$$|\{w : h(w) = x_n, w \in B(z,r)\}| \leqslant 2{\cdot}8\times 10^{26}|g'(z)|\,r.$$

*Proof.* We put

$$\Gamma_m = \{t = x+iy : \log(x_1+K) \leqslant x < \infty, y = 2\pi m\}, \quad \tilde{h}(t) = e^t - K.$$

As $\tilde{h}(t) = x_n$ implies that $t \in \Gamma_m$ for some $m \in \mathbb{Z}$ and $\tilde{h}(t)$ is a strictly increasingly real-valued function of $t$ on each $\Gamma_m, m \in \mathbb{Z}$, it follows that

$$|\{w : h(w) = x_n, w \in B(z,r)\}| \leqslant |\{m \in \mathbb{Z} : g(B(z,r)) \cap \Gamma_m \neq \varnothing\}| \times \max_{t \in \mathbb{C}} |\{w : g(w) = t, w \in B(z,r)\}|. \quad (4{\cdot}3)$$

From Lemma 4·4 we know that, if $z, r$ satisfy the given conditions, then

$$g(B(z,r) \subset B(g(z), 6{\cdot}8\times 10^{22}|g'(z)|r)$$

and so

$$|\{m \in \mathbb{Z} : g(B(z,r)) \cap \Gamma_m \neq \varnothing\}| < 6{\cdot}8\times 10^{22}|g'(z)|r. \quad (4{\cdot}4)$$

We note that it follows from Lemma 4·4 that, if $w = x+iy = se^{i\theta} \in B(z,r)$, then

$$w \in B\left(x, \left|\frac{5\pi z}{(1+p)(\log z)^p}\right|\right) \subset B\left(x, \left|\frac{50\pi x}{(1+p)(\log x)^p}\right|\right).$$

So $\log w = \log(s) + i\theta = te^{i\phi}$, where

$$|\phi| < |\tan\phi| = |\theta|/(\log s) < |\tan\theta|/(\log s)$$
$$= \frac{|y|}{x(\log s)} < \left|\frac{50\pi}{(1+p)(\log x)^{1+p}}\right| < \pi/[2(1+p)], \quad (4{\cdot}5)$$

and hence $(\log w)^{1+p}$ is univalent in $B(z,r)$.

We also note that, if $w \in B(z,r)$, then it follows from (4·5) that

$$|\mathrm{Im}\,[(\log w)^{1+p}]| = |t^{1+p}\sin((1+p)\phi)| \leqslant |\log w|^{1+p}\,2(1+p)|\phi|$$
$$< |\log(x)+2|^{1+p}\,2(1+p)|\phi| = (\log x)^{1+p}\,|1+2/(\log x)|^{1+p}\,2(1+p)|\phi|$$
$$< 10(\log x)^{1+p}\,2(1+p)\left|\frac{50\pi}{(1+p)(\log x)^{1+p}}\right| < 1000\pi < 4\times 10^3$$

and so

$$\max_{t\in\mathbb{C}} |\{w : g(w) = \exp(\log w)^{1+p} = t, w \in B(z,r)\}| \leqslant 4\times 10^3.$$

Combining this with (4·3) and (4·4) gives the desired result.

The next result is the main result of this section and has two parts. The proof of Theorem B part (i) will follow easily from the first part and the second part will be of use in the proof of Theorem B part (ii).

LEMMA 4·6. *Let* $H = \{A_1 : A_1 \in \mathscr{A}_1, A_1 \cap B(z,r) \neq \varnothing\}$.

(i) *If* $z \in \mathbb{R}$, $|z| \geqslant \hat{C}(p)$, $R_n = \pi r_n$

$$r = \left|\frac{(2)^{1/2}\pi z}{(1+p)(\log z)^p}\right|,$$

$p \geqslant 2$ *and* $2 \geqslant t > 1 + 2/p$, *then*

$$\sum_{A_1\in H} (\mathrm{diam}\,A_1)^t \leqslant r/L^2 < r^t/L^t.$$

(ii) *If*

$$z, f(z) \in \bigcup_{n=1}^{\infty} B_n, \quad p \leqslant 1/2, \quad R_n = r_n/(8L), \quad x_1 \geqslant 2(\hat{C}(p)+K)$$

*and*
$$\left|\frac{f(z)}{8L(1+p)(\log f(z))^p f'(z)}\right| \leqslant r \leqslant \left|\frac{z}{8L(1+p)(\log z)^p}\right|,$$

*then*
$$\sum_{A_1\in H} (\mathrm{diam}\,A_1)^2 \leqslant 9{\cdot}6\times 10^{112} r^{2-p}/|g'(z)|^p.$$

*Proof.* As $x_1 \geqslant \hat{C}(p)$, it follows from Lemma 3·1 that, if $f(A_1) = B_n$, then

$$h(A_1) \subset B\left(x_n, \frac{4x_n}{(1+p)(\log x_n)^p}\right) = H_n. \quad (4{\cdot}6)$$

We let $A_1'$ denote the component of $h^{-1}(H_n)$ that contains $A_1$ and note that it follows from Lemma 3·1 that

$$f(A_1') \subset B\left(x_n, \frac{5x_n}{(1+p)(\log x_n)^p}\right) \subset B(x_n, x_n) \quad (4{\cdot}7)$$

and hence, from Lemma 3·2, $f$ is univalent in $A_1'$. Thus, for each $n \in \mathbb{N}$,

$$|\{A_1 : A_1 \text{ is a component of } f^{-1}(B_n), A_1 \in H\}| \leqslant |\{A_1' : A_1' \text{ is a component of } h^{-1}(H_n), A_1' \cap B(z,r) \neq \varnothing\}|. \quad (4·8)$$

Clearly the only transcendental singularity of $h^{-1}$ is at $-K$ and, as $h'(z) = 0$ implies that $z = 1$, the only critical value of $h$ is at $h(1) = e - K$. Thus, if $A_1' \cap B(z,r) \neq \varnothing$ and $f(A_1') = H_n$, the branch of $h^{-1}$ that maps $H_n$ to $A_1'$ is univalent in $B(x_n, x_n/2)$. It follows from Lemma 3·3 and Lemma 4·4 that, if $w \in B(z,r) \cap A_1'$,

$$A_1' \subset B\left(w, \frac{8Lx_n}{(1+p)(\log x_n)^p\,|h'(w)|}\right) \subset B\left(w, \left|\frac{80Lh(w)}{(1+p)(\log h(w))^p\, h'(w)}\right|\right)$$
$$\subset B(w, 5/(1+p)) \subset B\left(w, \left|\frac{z}{(1+p)(\log z)^p}\right|\right) \subset B\left(z, \left|\frac{2\pi z}{(1+p)(\log z)^p}\right|\right). \quad (4·9)$$

If $r < \operatorname{diam}(A_1')/2$, then it follows from (4·9) that

$$B(z,r) \subset B\left(w, \frac{16Lx_n}{(1+p)(\log x_n)^p\,|h'(w)|}\right).$$

As $16L^2x_n/[(1+p)(\log x_n)^p] < x_n/8$, it follows from Lemma 3·3 that

$$h(z) \in B(h(w), x_n/8) \subset B(x_n, x_n/4)$$

and hence, from Lemma 3·1, we have $f(z) \in B(x_n, x_n/2)$.

Thus, if $r < \operatorname{diam}(A_1')/[2560L^3]$, then it follows from (4·9) that

$$r < \frac{16Lx_n}{2560L^3(1+p)(\log x_n)^p\,|h'(w)|} < \left|\frac{160L^2 f(z)}{2560L^3(1+p)(\log f(z))^p\,|h'(z)|}\right|. \quad (4·10)$$

It follows from Lemma 3·1 that

$$|f'(z)| + \tilde{C}(p) > |h'(z)| > |f'(z)| - \tilde{C}(p).$$

As

$$z, f(z) \in \bigcup_{n=1}^{\infty} B_n,$$

it follows from Lemma 3·5 that

$$|f'(z)| > 3|f(z)| > 3x_1 > 3\tilde{C}(p)$$

and hence $|h'(z)| > |f'(z)|/2$.

So, if $r < \operatorname{diam}(A_1')/[2560L^3]$, it follows from (4·10) that

$$r < \left|\frac{320L^2 f(z)}{2560L^3(1+p)(\log f(z))^p\,|f'(z)|}\right| = \left|\frac{f(z)}{8L(1+p)(\log f(z))^p\,|f'(z)|}\right|.$$

This is clearly impossible and so we must have

$$\operatorname{diam} A_1' \leqslant 2560L^3 r < 1·4 \times 10^9 r. \quad (4·11)$$

It follows from (4·8), (4·9) and (4·11) that there exists

$$\hat{r} \leqslant \min\left\{\left|\frac{2\pi z}{(1+p)(\log z)^p}\right|, \quad 1·5 \times 10^9 r\right\} \quad (4·12)$$

such that, for each $n \in \mathbb{N}$,

$$\begin{aligned}&|\{A_1 : A_1 \text{ is a component of } f^{-1}(B_n), A_1 \in H\}|\\ &\quad \leqslant |\{A_1' : A_1' \text{ is a component of } h^{-1}(H_n), A_1' \subset B(z, \hat{r})\}|\\ &\quad \leqslant |w : h(w) = x_n, w \in B(z, \hat{r})\}\end{aligned}$$

and so, from Lemma 4·5 and (4·12),

$$|\{A_1 : A_1 \text{ is a component of } f^{-1}(B_n), A_1 \in H\}| \leqslant 2{\cdot}8 \times 10^{26}\, |g'(z)| \hat{r} < 4{\cdot}2 \times 10^{35} |g'(z)|\, r. \quad (4{\cdot}13)$$

If $A_1$ is a component of $f^{-1}(B_n)$ and $w \in A_1 \cap B(z, r)$, then it follows from Lemma 3·3 and Lemma 3·4 that

$$\operatorname{diam} A_1 < L \operatorname{diam}(B_n)/|f'(w)|.$$

Arguing as before, we see that $|f'(w)| > |h'(w)|/2$ and hence, as $|f(w)| \geqslant x \geqslant 2\tilde{C}(p))$, it follows from Lemma 3·1 and Lemma 4·4 that

$$\operatorname{diam} A_1 < \left|\frac{4\pi L x_n}{(1+p)(\log x_n)^p\, h'(w)}\right| < \frac{8L\pi}{(1+p)(\log x_n)^p\, |g'(w)|} < \frac{1{\cdot}4 \times 10^{26}}{(1+p)(\log x_n)^p\, |g'(z)|}. \quad (4{\cdot}14)$$

Now let $m$ denote the smallest and $M$ the largest $n \in \mathbb{N}$ for which there exists a set $A_1 \in H$ with $f(A_1) = B_n$. It follows from (4·13), (4·14) and Lemma 4·4 that

$$\begin{aligned}\sum_{A_1 \in H} (\operatorname{diam} A_1)^t &\leqslant \frac{4{\cdot}2 \times 10^{35} |g'(z)|\, r (1{\cdot}4 \times 10^{26})^2}{(1+p)^t\, |g'(z)|^t} \times \sum_{n=m}^{M} 1/(\log x_n)^{pt}\\ &< \frac{8{\cdot}3 \times 10^{87} r}{(1+p)^t\, |g'(z)|^{t-1}} \int_{x_m}^{x_{M+1}} \frac{10(1+p)}{x(\log x)^{p(t-1)}}\, dx\\ &= \frac{8{\cdot}3 \times 10^{88} r}{(1-p(t-1))\,(1+p)^{t-1}\, |g'(z)|^{t-1}} [(\log x)^{1-p(t-1)}]_{x_m}^{x_{M+1}}. \quad (4{\cdot}15)\end{aligned}$$

If we are in case (i) then, as $2 \geqslant t > 1 + 2/p$, we have $1 - p(t-1) < -1$ and so, as $\log x_m \geqslant \log \hat{C}(p) > 1$ and $z \in \mathbb{R}$, it follows from (4·15) that

$$\sum_{A_1 \in H} (\operatorname{diam} A_1)^t < 8{\cdot}3 \times 10^{88} r/|g'(z)|^{t-1} < \frac{8{\cdot}3 \times 10^{88} r z^{t-1}}{[g(z)\,(\log z)^p]^{t-1}} < 8{\cdot}3 \times 10^{88} r/(\log z)^2.$$

As $|z| \geqslant \hat{C}(p) \geqslant \exp(8{\cdot}3 \times 10^{87})$, it follows that

$$\sum_{A_1 \in H} (\operatorname{diam} A_1)^t < r/L^2 < r^t/L^t$$

as claimed.

If we are in case (ii) then, as $p \leqslant 1/2$, it follows from (4·15) that

$$\sum_{A_1 \in H} (\operatorname{diam} A_1)^2 < \frac{1{\cdot}7 \times 10^{89} r}{|g'(z)|\,(\log x_m)^p} |\log(x_{M+1}) - \log(x_m)|. \quad (4{\cdot}16)$$

As $x_m \geqslant 2(\hat{C}(p) + K)$ and $K \geqslant \hat{C}(p)$, it follows from Lemma 3·1 that, if $z_1, z_2 \in B(z, r)$ with $f(z_1) \in B_m$ and $f(z_2) \in B_M$, then

$$|\exp(e^{(\log z_1)^{1+p}})| \leqslant 2x_m, \quad |\exp(e^{(\log z_2)^{1+p}})| \geqslant x_M > x_{M+1}/2. \quad (4{\cdot}17)$$

So it follows from Lemma 4·4 that

$$|\log(x_{M+1}) - \log(x_m)| \leqslant \mathrm{Re}\,(g(z_2)) - \mathrm{Re}\,(g(z_1)) + \log 4$$
$$\leqslant \log(4) + |g(z_2) - g(z_1)| \leqslant \log(4) + 2 \times 6{\cdot}8 \times 10^{22} |g'(z)|\, r.$$

Thus, if $r\,|g'(z)| > 1$, then

$$|\log(x_{M+1}) - \log(x_m)| < 1{\cdot}4 \times 10^{23} |g'(z)|\, r$$

and so, from (4·16) and (4·17),

$$\sum_{A_1 \in H} (\mathrm{diam}\, A_1)^2 < \frac{1{\cdot}7 \times 10^{89} r}{|g'(z)|\,(\log x_m)^p} \times 1{\cdot}4 \times 10^{23} |g'(z)|\, r < 4{\cdot}8 \times 10^{112} r^2 / [\mathrm{Re}\,(g(z_1))]^p. \tag{4·18}$$

It is not difficult to deduce from the conditions on $p$, $z$ and $r$ that there exists $w \in \mathbb{R}$ with $w \geqslant \hat{C}(p)$ such that

$$z_1 \in B(z, r) \subset B\left(w, \frac{w}{3L(1+p)(\log w)^p}\right).$$

As
$$g'(w)\, L \frac{w}{3L(1+p)(\log w)^p} = g(w)/3,$$

it follows from Lemma 3·3 that $g(B(z,r)) \subset B(g(w), g(w)/3)$ and hence

$$\mathrm{Re}\,(g(z_1)) > 2g(w)/3 > |g(z)|/2.$$

So, if $r\,|g'(z)| > 1$, it follows from (4·18) that

$$\sum_{A_1 \in H} (\mathrm{diam}\, A_1)^2 < 2^p \times 4{\cdot}8 \times 10^{112} r^2 / |g(z)|^p < 9{\cdot}6 \times 10^{112} r^2 / |g(z)|^p$$
$$\leqslant 9{\cdot}6 \times 10^{112} r^{2-p} / |g(z)|^p \times \left|\frac{z}{8L(1+p)(\log z)^p}\right|^p$$
$$= 9{\cdot}6 \times 10^{112} r^{2-p} / [8L|g'(z)|]p < 9{\cdot}6 \times 10^{112} r^{2-p} / |g'(z)|^p.$$

Finally, if $r\,|g'(z)| \leqslant 1$ then, from (4·12),

$$\sum_{A_1 \in H} (\mathrm{diam}\, A_1)^2 \leqslant (1{\cdot}5 \times 10^9)^2\, r^2 < 2{\cdot}3 \times 10^{18} r^2 < 2{\cdot}3 \times 10^{18} r^{2-p} / |g'(z)|^p.$$

This completes the proof of Lemma 4·6.

We are now in a position to prove Theorem B part (i). If $R_n = \pi r_n$ then, for each $n \in \mathbb{N}$,

$$\mathrm{diam}\, B_n = (2)^{1/2}\, \pi r_n \tag{4·19}$$

and
$$B_n \subset B(x_n, (2)^{1/2} \pi r_n) = B\left(x_n, \frac{(2)^{1/2} \pi x_n}{(1+p)(\log x_n)^p}\right).$$

Thus, if $H = \{A_1 : A_1 \in \mathscr{A}_1, A_1 \cap B(x_n, (2)^{1/2}\, \pi r_n) \neq \varnothing\}$, it follows from Lemma 4·6 part (i) and (4·19) that, if $p \geqslant 2, 2 \geqslant t > 1 + 2/p$, then, for each $n \in \mathbb{N}$,

$$\sum_{A_1 \cap B_n \neq \varnothing} (\mathrm{diam}\, A_1)^t \leqslant \sum_{A_1 \in H} (\mathrm{diam}\, A_1)^t < [(2)^{1/2}\, \pi r_n]^t / L^t = (\mathrm{diam}\, B_n)^t / L^t.$$

It now follows from (4·2) that, if $p \geqslant 2, K \geqslant S(p)$, then $\dim J(f) \leqslant 1 + 2/p$.

## 5. *Proof of Theorem B part (ii)*

Take a function $f = f_{p,K}$, where $p \leqslant 1/2, K \geqslant 2\tilde{C}(p)$ and consider the sets $A_k$ defined at the beginning of section four with $x_1 \geqslant 2(K+\hat{C}(p))$ and $R_n = r_n/(8L)$. We note that, for each $k \in \mathbb{N}$, the sets in $A_k$ are disjoint and so, if $A_{k+1} \in \mathscr{A}_{k+1}$, there exists a unique $A_k \in \mathscr{A}_k$ such that $A_{k+1} \subset A_k$. We choose a set $\hat{A}_1 \in \mathscr{A}_1$ and put $\mathfrak{B}_k = \mathfrak{A}_k \cap \hat{A}_1$. If $\hat{A} = \bigcap_{k=1}^{\infty} \mathfrak{B}_k$, then $\hat{A} \subset E_{\hat{C}(p)}$ and so, from Lemma 3·5, $\hat{A} \subset J(f)$.

Following the ideas used by McMullen in his proof of Lemma 2·2, we construct a measure $\mu$ which is supported on $\hat{A}$. We first construct a sequence of measures $\mu_k$ supported on $\mathfrak{B}_k$ in the following way. If $\mu_0$ is the 2-dimensional volume measure, we have $\mu_1 = 0$ on $\mathbb{C} \backslash \hat{A}_1$ and $\mu_1 = \mu_0|_{\hat{A}_1}$ elsewhere. Then, given $\mu_k$, we define $\mu_{k+1}$ by letting $\mu_{k+1} = 0$ on $\mathfrak{B}_k \backslash \mathfrak{B}_{k+1}$ and elsewhere taking it to be the restriction of $\mu_k$ to $\mathfrak{B}_{k+1}$ scaled within each $A_k \in \mathscr{A}_k$ so that $\mu_{k+1}(A_k) = \mu_k(A_k)$. There is then a subsequence of measures $\mu_k$ which converge weakly to a measure $\mu$ supported on $\hat{A}$ (see, for example, [**9**, theorem 5·13]).

We note some properties of these measures. First, for each $k \in \mathbb{N}$ and each $A_k \in \mathscr{A}_k$, $\mu_n(A_k) = \mu_k(A_k)$ for $n \geqslant k$ and so

$$\mu(A_k) = \mu_k(A_k). \tag{5·1}$$

Secondly, if

$$\Delta(A_k) = \mathrm{vol}\,(\mathfrak{A}_{k+1} \cap A_k)/\mathrm{vol}\,(A_k), \quad \nabla(A_k) = \prod_{r=1}^{k} \Delta(A_r),$$

where
$$A_k \subset A_{k-1} \subset \ldots \subset A_1$$

then for each $r \in \mathbb{N}, A_r \in \mathscr{A}_r$ with $A_r \subset \hat{A}_1$ we have

$$\mu_{r+1}|_{(\mathfrak{A}_{r+1} \cap A_r)} = (1/\Delta(A_r))\, \mu_r|_{(\mathfrak{A}_{r+1} \cap A_r)}$$

and so, for each $k \in \mathbb{N}, A_k \in \mathscr{A}_k$ with $A_k \subset \hat{A}_1$,

$$\mu_{k+1}|_{(\mathfrak{A}_{k+1} \cap A_k)} = (1/\nabla(A_k))\, \mu_0|_{(\mathfrak{A}_{k+1} \cap A_k)}. \tag{5·2}$$

In the following results we obtain enough information on the sets $A_n$ (and hence on the measure $\mu$) to enable us to apply Lemma 2·1 to show that

$$\dim(J(f)) \geqslant \dim(\hat{A}) \geqslant 2-p.$$

We begin by getting an estimate for the quantities $\Delta(A_r)$. In order to do this, we need the following preliminary results.

LEMMA 5·1. *If $A_k \subset \hat{A}_1$ and $f^k(A_k) = B_n$, then*

$$\begin{aligned}\Delta(A_k) &= \sum_{A_{k+1} \subset A_k} (\mathrm{diam}\, A_{k+1})^2/(\mathrm{diam}\, A_k)^2 \\ &\geqslant (1/L^2) \sum_{A_1 \subset B_n} (\mathrm{diam}\, A_1)^2/(\mathrm{diam}\, B_n)^2.\end{aligned}$$

*Proof.* If $A_1 \in \mathscr{A}_1$ and $A_1 \subset B_n$, then there exists $A_{k+1} \in \mathscr{A}_{k+1}$ such that $A_{k+1} \subset A_k$ and $f^k(A_{k+1}) = A_1$. Clearly,

$$\mathrm{diam}\, A_{k+1} \geqslant \mathrm{diam}\,(A_1)/[\sup_{z \in A_{k+1}} |(f^k)'(z)|] \geqslant \mathrm{diam}\,(A_1)/[\sup_{z \in A_k} |(f^k)'(z)|],$$

and $$\operatorname{diam} A_k \leqslant \operatorname{diam}(B_n)/[\inf_{z\in A_k} |(f^k)'(z)|].$$

As $f^k(A_k) = B_n \subset B(x_n, x_n/2)$, it follows from Lemma 3·2 and Lemma 3·3 that

$$\sup_{u,v\in A_k} |(f^k)'(u)/(f^k)'(v)| \leqslant L.$$

The result now follows.

LEMMA 5·2. *For each* $n\in\mathbb{N}$,

$$\sum_{A_1\subset B_n} (\operatorname{diam} A_1)^2/(\operatorname{diam} B_n)^2 \geqslant 1/[6\cdot 3\times 10^{126}(g(x_n))^p].$$

*Proof.* We begin by noting that it follows from Lemma 3·1 that, if

$$h(z) = x'_m = x_m + R_m/2,$$

for some $m\in\mathbb{N}$, then $f(z)\in B_m$. Thus, for each point $z$ such that $h(z) = x'_m$, there exists a set $A_1(z)\in\mathscr{A}_1$ such that $z\in A_1(z)$ and $f(A_1(z)) = B_m$. We note further that $h(A_1(z))\subset B(x_m, x_m/4)$ and hence, as the singularities of $h^{-1}$ are contained in $\{z\colon \operatorname{Re}(z) < 0\}$, $h$ is univalent in $A_1(z)$. Thus, if $z_1 \neq z_2$, $A(z_1)\neq A(z_2)$.

We note from (4·14) that, if $A_1\cap B(x'_n, R_n/4)\neq\varnothing$, then

$$\operatorname{diam} A_1 < 1\cdot 4\times 10^{26}/|g'(x_n)| = 1\cdot 4\times 10^{26} r_n/g(x_n) < 1\cdot 4\times 10^{26} r_n/x_n < R_n/4$$

and hence $A_1\subset B_n$.

For each $M\in\mathbb{Z}$, we take $\Gamma_M$ to be the curve defined by

$$\Gamma_M = \{z = x+iy\colon x\geqslant \log(K+x_1), y = 2\pi M\}.$$

If $g(z)\in\Gamma_M$, for some $M\in\mathbb{Z}$, then $h(z)\in[x_1,\infty)$. Now let $\gamma_M$ be a curve which is mapped onto $\Gamma_M$ by $g$. If $\gamma_M\cap B(x'_n, R_n/8)\neq\varnothing$, then there exists a segment of $\gamma_M\cap B(x'_n, R_n/4)$ of length at least $R_n/8$ and hence, from Lemma 4·4, there exists a segment of $\Gamma_M\cap g(B(x'_n, R_n/4))$ of length at least

$$g'(x_n)R_n/(8\times 6\cdot 8\times 10^{22}) \geqslant g'(x_n)R_n/(5\cdot 5\times 10^{23}).$$

It follows that there exist $z_1, z_2\in\gamma_M\cap B(x'_n, R_n/4)$ such that

$$h(z_1) = x_{n(1)},\quad h(z_2) = x_{n(2)},\quad h(\gamma_M\cap B(x'_n, R_n/4))\supset[x_{n(1)}, x_{n(2)}]$$

and $$\operatorname{Re} g(z_2) - \operatorname{Re} g(z_1) \geqslant g'(x_n)R_n/(5\cdot 5\times 10^{23}) - 4. \tag{5·3}$$

As $B_m\subset B(x_m, x_m/4)$, for each $m\in\mathbb{N}$, it follows from Lemma 3·2 and Lemma 3·3 that, if $f(A_1) = B_m$, $z\in A_1$ and $h(z) = x'_m$, then

$$\operatorname{diam} A_1 \geqslant \operatorname{diam}(B_m)/(L|f'(z)|) = x_m/[4(2)^{1/2}(1+p)(\log x_m)^p L^2|f'(z)|].$$

As $x_m > 2(K+\tilde{C}(p))$, it follows from Lemma 3·1 and Lemma 4·4 that

$$\begin{aligned}\operatorname{diam} A_1 &> x_m/[16(1+p)(\log x_m)^p L^2|h'(z)|]\\ &= x_m/[16(1+p)(\log x_m)^p L^2|g'(z)(h(z)+K)|]\\ &> 1/[32\times 6\cdot 8\times 10^{22}(1+p)(\log x_m)^p L^2 g'(x_n)]\\ &> 1/[1\cdot 5\times 10^{28}(1+p)(\log x_m)^p g'(x_n)].\end{aligned}$$

Combined with earlier observations, this shows that, if $\gamma_M \cap B(x'_n, R_n/8) \neq \varnothing$, then

$$\sum_{\substack{A_1 \subset B_n \\ A_1 \cap \gamma_m \neq \varnothing}} (\operatorname{diam} A_1)^2 \geqslant 1/[1{\cdot}5 \times 10^{28}(1+p)\, g'(x_n)]^2 \sum_{x_{n(1)}}^{x_{n(2)}} 1/[\log x_m]^{2p}$$

$$\geqslant 1/[1{\cdot}5 \times 10^{28}(1+p)\, g'(x_n)]^2 \int_{x_{n(1)}}^{x_{n(2)}} (1+p)/[x \log x)^p]\, dx$$

$$> 1/[2{\cdot}3 \times 10^{56}(1-p^2)\,(g'(x_n))^2]\,[(\log x)^{1-p}]_{x_{n(1)}}^{x_{n(2)}}$$

$$> [\log(x_{n(2)}) - \log(x_{n(1)})]/[2{\cdot}3 \times 10^{56}(g'(x_n))^2 (\log x_{n(2)})^p].$$

As $\exp(g(z_1)) = x_{n(1)} + K$ and $\exp(g(z_2)) = x_{n(2)} + K < 2x_{n(2)}$, it follows from Lemma 4·4 that

$$\operatorname{Re}(g(z_2)) - 2 \leqslant \log x_{n(2)} \leqslant \operatorname{Re}(g(z_2)) \leqslant |g(z_2)| \leqslant 6{\cdot}8 \times 10^{21} |g(x_n)|$$

and

$$\log(x_{n(1)}) \leqslant \operatorname{Re}(g(z_1)).$$

Thus, if $\gamma_M \cap B(x'_n, R_n/8) \neq \varnothing$, it follows from (5·3) that

$$\sum_{\substack{A_1 \subset B_n \\ A_1 \cap \gamma_M \neq \varnothing}} (\operatorname{diam} A_1)^2 > [\operatorname{Re}(g(z_2) - g(z_1)) - 2]/[2{\cdot}3 \times 10^{56}(g'(x_n))^2\, 6{\cdot}8 \times 10^{21}(g(x_n))^p]$$

$$> [g'(x_n)\, R_n/(5.5 \times 10^{23}) - 4]/[1{\cdot}6 \times 10^{78}(g'(x_n))^2 (g(x_n))^p].$$

As

$$g'(x_n)\, R_n/(5{\cdot}5 \times 10^{23}) = g(x_n)/(8L \times 5{\cdot}5 \times 10^{23}) > x_n/(8L \times 5{\cdot}5 \times 10^{23}) > 8,$$

it follows that

$$\sum_{\substack{A_1 \subset B_n \\ A_1 \cap \gamma_M \neq \varnothing}} (\operatorname{diam} A_1)^2 > R_n/[2 \times 5{\cdot}5 \times 10^{23} \times 1{\cdot}6 \times 10^{78} g'(x_n)\,(g(x_n))^p]$$

$$> R_n/[1{\cdot}8 \times 10^{102} g'(x_n)\,(g(x_n))^p]. \quad (5{\cdot}4)$$

Clearly $g(x'_n) \in \Gamma_0$. As we know from Lemma 4·4 that

$$g(B(x'_n, R_n/8)) \supset B(g(x'_n), g'(x_n)\, R_n/(5{\cdot}5 \times 10^{23}),$$

it follows that there are at least $g'(x_n)\, R_n/(\pi \times 5{\cdot}5 \times 10^{23})$ values of $M \in \mathbb{Z}$ for which there exists a curve $\gamma_M$ satisfying $\gamma_M \cap B(x'_n, R_n/8) \neq \varnothing$. If a set $A_1 \in \mathscr{A}_1$ meets two curves $\gamma_{M(1)}, \gamma_{M(2)}$, where $M(1) \neq M(2)$, then there exist $w \in A_1, M \in \mathbb{N}$ such that $\operatorname{Im}(g(w)) = (2M+1)\pi$ and hence $\operatorname{Re}(h(w)) < 0$. This, however, is impossible and so it follows from (5·4) that

$$\sum_{A_1 \subset B_n} (\operatorname{diam} A_1)^2 \geqslant (R_n)^2/[1{\cdot}8 \times 10^{102}\pi \times 5{\cdot}5 \times 10^{23}(g(x_n))^p]$$

$$> (\operatorname{diam} B_n)^2/[6{\cdot}3 \times 10^{126}(g(x_n))^p].$$

It follows from Lemma 5·1 and Lemma 5·2 that, if $f^k(A_k) = B_n$, then

$$\Delta(A_k) \geqslant 1/[4{\cdot}2 \times 10^{130}(g(x_n))^p]. \quad (5{\cdot}5)$$

Before we can complete the proof of Theorem B part (ii), we need a few more preliminary results. Recall that $\hat{h}(z) = h(z) + K$.

LEMMA 5·3. *If* $z, f(z) \in \bigcup_{n=1}^{\infty} B_n$, *then*

(i) $|\hat{h}(z)| > |f(z)| > |\hat{h}(z)|/2$,

(ii) $|\log f(z)|^p > 4(2{\cdot}9 \times 10^{152})^{1/p}\,|z|$.

*Proof.* (i) It is not difficult to see that, if $z, f(z) \in \bigcup_{n=1}^{\infty} B_n$, then $|\hat{h}(z)| > |f(z)|$. As $|f(z)| \geqslant x_1 > 2(K+\tilde{C}(p))$, it follows from Lemma 3·1 that

$$|f(z)| > |\hat{h}(z)| - K - \tilde{C}(p) > |\hat{h}(z)|/2.$$

(ii) It follows from part (i) that

$$\log|f(z)| > \log|\hat{h}(z)/2| > [\log|\hat{h}(z)|]/2 = [\mathrm{Re}\,(g(z))]/2.$$

As $z \in B_n \subset B(x_n, r_n/(4L))$, for some $n \in \mathbb{N}$, and $g'(x_n)\,r_n L/(4L) = g(x_n)/4$, it follows from Lemma 3·3 that $g(z) \in B(g(x_n), g(x_n)/4)$ and hence, as $x_n \geqslant \hat{C}(p)$,

$$\begin{aligned} |\log(f(z))|^p &> [\mathrm{Re}\,(g(z))/2]^p > (g(x_n)/4)^p > g(x_n)^p/4 \\ &= \exp[p(\log x_n)^{1+p}]/4 = (x_n)^{p(\log x_n)^p}/4 > (x_n)^2/4 > x_n|z|/8 \\ &> \exp[8{\cdot}3 \times 10^{87}/p]\,|z|/8 > 4(2{\cdot}9 \times 10^{152})^{1/p}\,|z|. \end{aligned}$$

LEMMA 5·4. *If* $z \in B_n$ *and* $r \leqslant |z/[4L(1+p)(\log z)^p]|$, *then* $B(z,r) \cap B_m = \varnothing$, *for each* $m \in \mathbb{N}, m \neq n$.

*Proof.* If $z \in B_n$, then $x_n \leqslant |z| < 2x_n$ and hence, from Lemma 4·4.

$$\begin{aligned} r \leqslant |z|/[4L(1+p)(\log|z|)^p] &< 2x_n/[4L(1+p)(\log x_n)^p] \\ &= r_n/(2L) < 10x_{n-1}/[2L(1+p)(\log x_{n-1})^p] < r_{n-1}/2. \end{aligned}$$

Thus
$$\mathrm{Re}\,(z) < x_n + r_n/(8L) + r_n/(2L) < x_n + r_n = x_{n+1}$$
and
$$\mathrm{Re}\,(z) > x_n - r_{n-1}/2 = x_{n-1} + r_{n-1}/2$$
and hence $z \notin \bigcup_{m \neq n} B_m$.

By using Lemmas 5·4 and 4·6, we are able to prove the following.

LEMMA 5·5. *There exists* $r_0 > 0$ *such that, for each* $z \in \hat{A}$, $r < r_0$, *there exists* $n \in \mathbb{N}$ *and a set* $A_n \in \mathscr{A}_n$ *such that*

(i) *if* $A_{n+1} \in H = \{A_{n+1} \in \mathscr{A}_{n+1} : A_{n+1} \cap B(z,r) \neq \varnothing\}$, *then* $A_{n+1} \subset A_n$,

(ii) $\sum_{A_{n+1} \in H} (\mathrm{diam}\, A_{n+1})^2 \leqslant 4{\cdot}2 \times 10^{120} r^{2-p}/|(f^n)'(z)\, g'(f^n(z))|^p$.

*Proof.* Take $z \in \hat{A}$, $r < \sup_{z \in \hat{A}} |f(z)/[8L^2(1+p)(\log f(z))^p f'(z)]|$ and the smallest value of $n \in \mathbb{N} \cup \{0\}$ for which

$$|(f^{n+1})'(z)|\, r \geqslant |f^{n+1}(z)/[8L^2(1+p)(\log f^{n+1}(z))^p]|. \tag{5·6}$$

The existence of such an $n$ follows from Lemma 3·5 and, due to the choice of $r$, we must have $n \geqslant 1$.

As $z \in \hat{A}$, we know that $f^n(z) \in \bigcup_{m=1}^{\infty} B_m$ and so, from Lemma 3·2, the branch of $f^{-n}$ that maps $f^n(z)$ to $z$ is univalent in $B(f^n(z), |f^n(z)|/2)$. As $|(f^n)'(z)|\, rL < |f^n(z)|/8$, it follows from Lemma 3·3 that

$$f^n(B(z,r)) \subset B(f^n(z), |(f^n)'(z)|\, rL) \subset B(f^n(z), |f^n(z)|/8). \tag{5.7}$$

It follows from Lemma 4·1 that

$$\sup_{A_1 \in \mathscr{A}_1} \mathrm{diam}\, A_1 \leqslant 1/3$$

and hence, if $A_1 \in \mathscr{A}_1$ and $A_1 \cap B(f^n(z), |(f^n)'(z)|\, rL) \neq \varnothing$, it follows from (5·6) that

$$\begin{aligned} A_1 &\subset B(f^n(z), |(f^n)'(z)|\, rL + 1/3) \\ &\subset B(f^n(z), |f^n(z)/[4L(1+p)(\log f^n(z))^p]|) \subset B(f^n(z), |f^n(z)|/4). \end{aligned} \tag{5·8}$$

Clearly $f^n(z) \in B_m$, for some $m \in \mathbb{N}$. It follows from (5·8) and Lemma 5·4 that, if $A_1 \cap B(f^n(z), |(f^n)'(z)|\, rL) \neq \varnothing$, then $A_1 \subset B_m$.

If $A_{n+1} \in H$, then $f^n(A_{n+1}) \in \mathscr{A}_1$ and it follows from (5·7) that,

$$f^n(A_{n+1}) \cap B(f^n(z), |(f^n)'(z)|\, rL) \neq \varnothing.$$

Thus, if $f_n$ is the branch of $f^{-n}$ that maps $f^n(z)$ to $z$, then $A_{n+1} \subset f_n(B_m) \in \mathscr{A}_n$. This shows that part (i) is true and it now follows from Lemma 3·2 and Lemma 3·3 that

$$\sum_{A_{n+1} \in H} (\operatorname{diam} A_{n+1})^2 \leqslant (L^2/|(f^n)'(z)|^2) \sum_{\substack{A_1 \in \mathscr{A}_1 \\ A_1 \cap B(f^n(z), |(f^n)'(z)|\, rL) \neq \varnothing}} (\operatorname{diam} A_1)^2.$$

It follows from (5·6) that

$$|(f^n)'(z)|\, rL < |f^n(z)/[8L(1+p)(\log f^n(z))^p]| \tag{5·9}$$

and that

$$\begin{aligned} |(f^n)'(z)|\, rL &= |(f^{n+1})'(z)\, rL/f'(f^n(z))| \\ &\geqslant |(f^{n+1})(z)/[8L(1+p)(\log f^{n+1}(z))^p f'(f^n(z))]|. \end{aligned} \tag{5·10}$$

As $z \in \hat{A}$, we know that $f^k(z) \in \hat{A} \subset \bigcup_{m=1}^{\infty} B_m$ for each $k \in \mathbb{N}$ and so it follows from (5·9) and (5·10) that $B(f^n(z), |(f^n)'(z)|\, rL)$ satisfies the conditions of Lemma 4·6 part (ii). Thus

$$\begin{aligned} \sum_{A_{n+1} \in H} (\operatorname{diam} A_{n+1})^2 &\leqslant (L^2/|(f^n)'(z)|^2)\, 9{\cdot}6 \times 10^{112} (|(f^n)'(z)|\, rL)^{2-p}/|g'(f^n(z)|^p \\ &< 4{\cdot}2 \times 10^{120} r^{2-p}/|(f^n)'(z)\, g'(f^n(z))|^p. \end{aligned}$$

We are now in a position to complete the proof of Theorem B part (ii).

LEMMA 5·6. *There exists* $r_0 > 0$ *such that, for each* $z \in \hat{A}, r < r_0$,

$$\mu(B(z,r)) < 1{\cdot}9 \times 10^{269} r^{2-p}$$

*and hence, from Lemma* 2·1, $\dim \hat{A} \geqslant 2 - p$.

*Proof.* We take a value of $r_0$ which satisfies the conditions of Lemma 5·5 and then take $z \in \hat{A}, r < r_0$ and a value $n \in \mathbb{N}$ which satisfies the conclusions of Lemma 5·5. As $\mu$ is supported on $\hat{A} \subset \mathfrak{A}_{n+1}$, it follows from (5·1) that

$$\mu(B(z,r)) \leqslant \sum_{A_{n+1} \in H} \mu(A_{n+1}) = \sum_{A_{n+1} \in H} \mu_{n+1}(A_{n+1}).$$

We know from Lemma 5·5 that there exists a unique set $A_n \in \mathscr{A}_n$ such that, if $A_{n+1} \in H$, then $A_{n+1} \subset A_n$. If $A_n \cap \hat{A}_1 = \varnothing$, then $\mu(A_n) = 0$ and hence $\mu(B(z,r)) = 0$. If $A_n \subset \hat{A}_1$, then it follows from (5·2) that

$$\mu(B(z,r)) \leqslant \sum_{A_{n+1} \in H} \mu_{n+1}(A_{n+1}) = [1/\nabla(A_n)] \sum_{A_{n+1} \in H} (\operatorname{diam} A_{n+1})^2. \tag{5.11}$$

Now
$$\nabla(A_n) = \prod_{r=1}^{n} \Delta(A_r) = \prod_{r=1}^{n} \operatorname{vol}(\mathfrak{A}_{r+1} \cap A_r)/\operatorname{vol}(A_r),$$

where $A_n \subset A_{n-1} \subset \ldots \subset A_2 \subset \hat{A}_1$. We denote $f^r(A_r)$, $1 \leqslant r \leqslant n$, by $B_{m(r)}$. It follows from (5·5) that

$$\Delta(A_r) \geqslant 1/[4{\cdot}2 \times 10^{130}(g(x_{m(r)}))^p].$$

As $z \in \hat{A}$, we must have $z \in A_n \subset \ldots \subset A_r \subset \ldots \subset \hat{A}_1 \subset B_{m(0)}$ and so it follows from Lemma 4·4 that, for $0 \leqslant r \leqslant n$,

$$|g(f^r(z))| \geqslant g(x_{m(r)})/(6{\cdot}8 \times 10^{21})$$

and hence $$\Delta(A_r) \geqslant 1/[2{\cdot}9 \times 10^{152}|g(f^r(z)|^p].$$

Thus $$\nabla(A_n) \geqslant \prod_{r=1}^{n} 1/[2{\cdot}9 \times 10^{152}|g(f^r(z)|^p],$$

and so, from (5·11), Lemma 5·5 and Lemma 3·5,

$$\begin{aligned}\mu(B(z,r)) &\leqslant \frac{4{\cdot}2 \times 10^{120} r^{2-p}}{|(f^n)'(z)\, g'(f^n(z))|^p} \prod_{r=1}^{n} [2{\cdot}9 \times 10^{152}|g(f^r(z)|^p] \\ &< 1{\cdot}3 \times 10^{273} r^{2-p} \frac{|g(f^n(z)|^p}{|f'(z)\, g'(f^n(z))|\,|^p} \prod_{r=1}^{n-1} 2{\cdot}9 \times 10^{152}|g(f^r(z))/f'(f^r(z))|^p \\ &< 1{\cdot}3 \times 10^{273} r^{2-p} \frac{|f^n(z)|^p}{|(1+p)(\log f^n(z))^p|^p} \prod_{r=1}^{n-1} 2{\cdot}9 \times 10^{152}|g(f^r(z))/f'(f^r(z))|^p \\ &< 1{\cdot}3 \times 10^{273} r^{2-p} \prod_{r=1}^{n-1} 2{\cdot}9 \times 10^{152} \left|\frac{f^{r+1}(z)\, g(f^r(z))}{f'(f^r(z))\,(\log f^{r+1}(z))^p}\right|^p. \end{aligned} \tag{5·12}$$

As $z \in \hat{A}$, we have $|f^r(z)| \geqslant \hat{C}(p)$ for each $r \in \mathbb{N}$ and so it follows from Lemma 3·1 and Lemma 3·5 that, for $1 \leqslant r \leqslant n-1$,

$$\begin{aligned}|f'(f^r(z))| &> |(1+p)(\log f^r(z))^p\, g(f^r(z))\, \hat{h}(f^r(z))/f^r(z)| - \tilde{C}(p) \\ &> |(1+p)(\log f^r(z))^p\, g(f^r(z)) f^{r+1}(z)/[4f^r(z)]| \\ &> |g(f^r(z)) f^{r+1}(z)/[4f^r(z)]|.\end{aligned}$$

Thus, from Lemma 5·3, for $1 \leqslant r \leqslant n-1$,

$$\left|\frac{f^{r+1}(z)\, g(f^r(z))}{f'(f^r(z))\,(\log f^{r+1}(z))^p}\right|^p < |4f^r(z)/[\log f^{r+1}(z)]^p|\, p < 1/(2{\cdot}9 \times 10^{152}).$$

It now follows from (5·12) that, for each $z \in \hat{A}$, $r < r_0$,

$$\mu(B(z,r)) < 1{\cdot}3 \times 10^{273} r^{2-p}$$

and hence, from Lemma 2·1, $\dim \hat{A} \geqslant 2-p$.

As $\hat{A} \subset J(f)$, it follows that $\dim J(f) \geqslant 2-p$. At the beginning of this section we stated that $f$ was a function of the form $f_{p,K}$, where $p \leqslant 1/2$ and $K \geqslant 2\tilde{C}(p)$. If we take $S(p)$ to be the function defined just before Lemma 4·2 for $p \in [1,\infty)$ then, if $p \leqslant 1/2$, we have $S(1/p) > 2\tilde{C}(1/p) = 2\tilde{C}(p)$. Thus, with this definition of $S(p)$, for $p \leqslant 1/2$ and $K \geqslant S(1/p)$ we have $\dim J(f_{p,K}) \geqslant 2-p$. This completes the proof of Theorem B.

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