A NEW CHARACTERISATION OF THE EREMENKO-LYUBICH CLASS

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Abstract. The Eremenko-Lyubich class of transcendental entire functions with a bounded set of singular values has been much studied. We give a new characterisation of this class of functions. We also give a new result regarding direct singularities which are not logarithmic.

1. Introduction

Throughout this paper we assume that f is a transcendental entire function. We say that f belongs to the Eremenko-Lyubich class, $\mathcal{B}$, if the set of finite critical and asymptotic values of f is bounded. This set coincides with the set of singular values of the inverse function f^{-1}. The class $\mathcal{B}$ was introduced to complex dynamics in [8], and has been widely studied; see, for example, the papers [16], [17], [19] and [20] on the structure of the escaping set for functions in this class; [1], [18] and [21] on the dimensions of the Julia set and the escaping set; and [5] for an example of a function in this class with a wandering domain. Papers studying the value distribution of functions in this class include [12], [13] and [15].

An important property of functions in the Eremenko-Lyubich class is that they are expanding, in the following sense. Define $D_R = \{z : |f(z)| > R\}$, for $R > 0$. If $f \in \mathcal{B}$, then it follows easily from [8, Lemma 1] that there is a constant $R_0 > 0$ such that

$$\left| z\frac{f'(z)}{f(z)} \right| \geq \frac{1}{4\pi}(\log|f(z)| - \log R_0), \quad \text{for } z \in D_{R_0}. \tag{1}$$

This property has many applications in complex dynamics and value distribution theory; for example, it was used in [8] to show that functions in the Eremenko-Lyubich class cannot have escaping Fatou components.

Define

$$\eta_f = \lim_{R\to\infty} \inf_{z\in D_R} \left| z\frac{f'(z)}{f(z)} \right|. \tag{2}$$

It follows from (1) that if f is a transcendental entire function in the Eremenko-Lyubich class, then $\eta_f = \infty$. The main result of this paper is the following, which shows that this property has a strong converse.

Theorem 1.1. *Suppose that f is a transcendental entire function. Then, either $\eta_f = \infty$ and $f \in \mathcal{B}$, or $\eta_f = 0$ and $f \notin \mathcal{B}$.*

It is clear that if f has an unbounded set of critical values, then $\eta_f = 0$. Thus the proof of Theorem 1.1 requires detailed analysis of the behaviour of functions with an unbounded set of asymptotic values. Since every asymptotic value of f gives rise to a transcendental singularity of f^{-1}, we need a number of results on singularities of the inverse function. In particular we require the following result on the density of transcendental singularities of a certain type, which may be of independent interest. Definitions of terms used in the statement of this theorem are given in Section 2.

Theorem 1.2. *Suppose that f is a transcendental entire function, with a direct non-logarithmic singularity with projection $a \in \widehat{\mathbb{C}}$. Then at least one of the following holds:*

(i) a is the limit of critical values of f;
(ii) every neighbourhood of this singularity contains a neighbourhood of another transcendental singularity of f^{-1} that is either indirect or logarithmic, and whose projection is different from a.

We observe that Theorem 1.2 is complementary to the following result of Bergweiler and Eremenko [3, Theorem 5], which has the same hypothesis although in this result the projection of the transcendental singularity must be finite.

Theorem 1.3. *Suppose that f is a transcendental entire function, with a direct non-logarithmic singularity with projection $a \in \mathbb{C}$. Then every neighbourhood of this singularity is also a neighbourhood of other direct singularities of f^{-1} with projection a.*

Taken together, these results show that if $a \in \mathbb{C}$ is the projection of a direct non-logarithmic singularity and is not the limit of critical values, then there is an infinite number of singularities both over a and over points arbitrarily close to a.

We mention two examples of transcendental entire functions with direct non-logarithmic singularities which illustrate some of the possibilities described above.

Example 1. Heins [9, p.435] gave the example $f_1(z) = e^z \sin(e^z)$, which has precisely one direct non-logarithmic singularity over ∞. Since the set of critical values of f_1 is unbounded, case (i) of Theorem 1.2 holds for this function. This example also shows that Theorem 1.3 cannot be strengthened to $a \in \widehat{\mathbb{C}}$.

Example 2. Herring [10] gave the example $f_2(z) = \int_0^z \exp(-e^t)\ dt$. This function has no critical points. It follows from results in [10] that f_2 has a direct non-logarithmic singularity over ∞, every neighbourhood of which contains a left half-plane. It also follows that within each set

$$A_k = \{z : \operatorname{Re}(z) > 0, |\operatorname{Im}(z) - 2k\pi| \leq \pi/2\}, \quad \text{for } k \in \mathbb{Z},$$

there is a neighbourhood of a logarithmic singularity with projection

$$\alpha_k = \alpha + 2k\pi i, \text{ where } \alpha \in \mathbb{C} \text{ is constant.}$$

Moreover, each neighbourhood of the direct non-logarithmic singularity over ∞ contains neighbourhoods of these logarithmic singularities. Hence case (ii) of Theorem 1.2 holds for f_2.

The structure of this paper is as follows. In Section 2 we give details of Iversen's classification of singularities. We then prove Theorem 1.2 in Section 3. Finally, in Section 4, we use Theorem 1.2 to prove Theorem 1.1.

2. Singularities of the inverse function

We write $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, and use $B(w, r)$ to refer to the open disc around the point $w \in \mathbb{C}$, of radius r. We also write $B(\infty, r) = \{ z : |z| > r\}$. We write $\mathbb{D}$ for the unit disc $B(0, 1)$, and $\mathbb{D}^*$ for the punctured unit disc $B(0, 1)\backslash\{0\}$. We write $\mathbb{H}$ for the left half-plane $\{z : \text{Re}(z) < 0\}$.

We recall Iversen's classification of singularities; see, for example, [2], [3], and [11]. Suppose that f is a transcendental entire function, and suppose that $a \in \widehat{\mathbb{C}}$. For each $r > 0$, we can choose a component $U(r)$ of $f^{-1}(B(a, r))$ so that $r_1 < r_2$ implies that $U(r_1) \subset U(r_2)$. Then we have two possibilities:

(a) $\cap_{r>0} U(r)$ consists of a single point w, say, or
(b) $\cap_{r>0} U(r) = \emptyset$.

In the first case, if $f'(w) = 0$, then w is a *critical point* of f, a is a *critical value* of f, and we say that the singularity is *algebraic*.

In the second case we say that the choice $r \mapsto U(r)$ defines a *transcendental singularity* of f^{-1}, and we say that a is the *projection* of the transcendental singularity or equivalently that the transcendental singularity is *over* a. Any of the sets $U(r)$ is called a *neighbourhood* of the transcendental singularity.

We say that a transcendental singularity over a point a is *direct* if there exists $r > 0$ such that $f(z) \neq a$, for $z \in U(r)$. Otherwise we call the transcendental singularity *indirect*. We call a direct transcendental singularity over a point a *logarithmic* if, for some $r > 0$, the restriction $f : U(r) \to B(a, r)\backslash\{a\}$ is a universal covering. If a transcendental singularity is direct but not logarithmic, we use the term *direct non-logarithmic*.

We call a curve $\Gamma : (0, 1) \to \mathbb{C}$ an *asymptotic curve* with *asymptotic value* a if, as $t \to 1$, we have both $\Gamma(t) \to \infty$ and $f(\Gamma(t)) \to a$. Given a transcendental singularity over a point a it is possible to construct an asymptotic curve with asymptotic value a, and *vice versa*; see [2, p.2] for details.

3. Direct non-logarithmic singularities

In this section we prove Theorem 1.2. We need the following [9, Theorem 4′].

Theorem 3.1. *Suppose that f is a transcendental entire function, $D \subset \mathbb{C}$ is a domain, and W is a component of $f^{-1}(D)$. Then either f_W, the restriction of f to W, has finite constant valence in D, or else there is at most one point of D at which the valence of f_W is finite.*

Here the *valence* of a point $a \in D$ is the number of solutions of $f(z) = a$, for $z \in W$. It follows from Theorem 3.1 that there cannot be two distinct points $a, a' \in D$ such that $f(z) \in \{a, a'\}$ has no solutions, for $z \in W$.

We also need the following result, and two straightforward corollaries of it. This seems to be well known, and follows quickly from results such as [14, Example 4.2]. See also [22, Theorem 6.1.1] for a detailed proof.

Theorem 3.2. *Suppose that $W \subset \mathbb{C}$ is a domain, and $g : W \to \mathbb{D}^*$ is an unbranched covering map. Then exactly one of the following holds:*

(i) there exists a conformal map $\phi : W \to \mathbb{H}$ such that $g = \exp \circ\, \phi$;
(ii) there exists a conformal map $\phi : W \to \mathbb{D}^*$ such that $g = (\phi)^m$, for some $m \in \mathbb{N}$.

The first corollary is similar to [22, Theorem 6.2.2], and differs from that result in that it specifies the location of the neighbourhoods of the singularities, which is necessary for the proof of Theorem 1.2. We give a proof for completeness.

Corollary 3.1. *Suppose that f is a transcendental entire function with a transcendental singularity which is not logarithmic, over a point $a \in \widehat{\mathbb{C}}$. Then at least one of the following holds:*

(i) a is the limit of critical values of f;
(ii) every neighbourhood of this singularity contains a neighbourhood of another transcendental singularity of f^{-1} whose projection is different from a.

Proof. Suppose that, contrary to the conclusion of the corollary, we can choose a sufficiently small $r > 0$ such that there are no critical points of f in

$$W = U(r)\backslash\{z : f(z) = a\},$$

and all transcendental singularities of f^{-1} with a neighbourhood contained in $U(r)$ have projection a. The restriction of f to W is, therefore, an unbranched covering of $B(a, r)\backslash\{a\}$.

Let h be a conformal map from $B(a, r)\backslash\{a\}$ to $\mathbb{D}^*$. We apply Theorem 3.2 with $g = h \circ f$. If case (i) of the theorem holds, then W is simply connected, and the singularity is logarithmic, which is a contradiction. If case (ii) of the theorem holds, then the conformal mapping ϕ has a punctured disc in the Riemann sphere as its domain, and at the puncture ϕ has a removable singularity. Hence, since $f = h^{-1} \circ (\phi)^m$, the singularity is algebraic; this is also a contradiction. □

The second corollary of Theorem 3.2 is similarly straightforward, and we omit the proof.

Corollary 3.2. *Suppose that f is a transcendental entire function with a logarithmic singularity over a point $a \in \widehat{\mathbb{C}}$. Then there exist a neighbourhood of the singularity, $W = U(r)$, and conformal maps $\phi : W \to \mathbb{H}$ and $h : B(a, r) \to \mathbb{D}^*$ such that $h \circ f = \exp \circ \phi$.*

We now prove Theorem 1.2.

Proof of Theorem 1.2. Suppose that f has a direct non-logarithmic singularity over a point $a \in \widehat{\mathbb{C}}$, and that a is not the limit of critical values of f. The existence of transcendental singularities, over points other than a, in any neighbourhood of this direct non-logarithmic singularity follows from Corollary 3.1; we need to show that in any neighbourhood of this singularity there are singularities over points other than a, which are either logarithmic or indirect.

The structure of the proof is as follows. We assume the contrary, and construct a sequence of direct non-logarithmic singularities the projections of which have a limit. We show that this limit is itself the projection of a direct non-logarithmic singularity, and use the comment after Theorem 3.1 to obtain a contradiction. Figure 1 illustrates the points and sets constructed.

Let $r_0 > 0$ be such that there are no critical values of f in $B(a, r_0)$ and also, since the transcendental singularity is assumed to be direct, such that $f(z) \neq a$, for z in the neighbourhood $U(r_0)$. Suppose also that r_0 is sufficiently small that all transcendental singularities, over points other than a and with a neighbourhood contained in $U(r_0)$, are direct non-logarithmic.

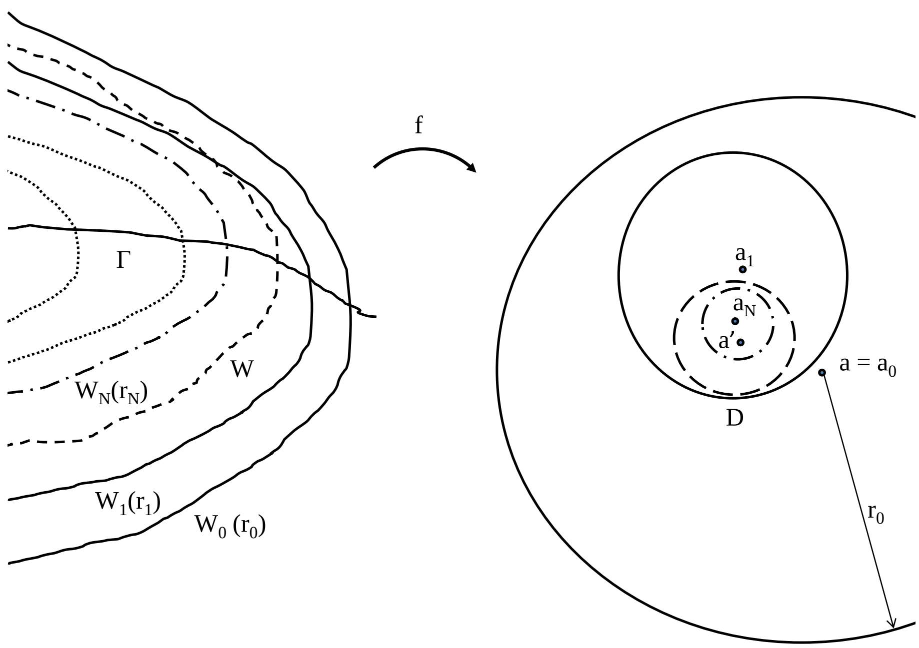

FIGURE 1. The construction used in the proof of Theorem 1.2

Let (R_n) be any increasing sequence of positive real numbers such that $R_n \to \infty$ as $n \to \infty$. We construct a sequence of neighbourhoods of direct non-logarithmic singularities $W_n(r_n)$, with projection a_n say, such that, for $n \geq 0$,

- $W_{n+1}(r_{n+1}) \subset W_n(r_n) \subset U(r_0)$;
- $\overline{W_{n+1}(r_{n+1})} \cap B(0, R_{n+1}) = \emptyset$;
- $\overline{B(a_{n+1}, r_{n+1})} \subset B(a_n, r_n)$;
- $a_{n+1} \neq a$;
- the equation $f(z) = a_n$ has no solutions for $z \in W_n(r_n)$;
- $r_n \to 0$ as $n \to \infty$.

We set $W_0(r_0) = U(r_0)$, and then construct this sequence inductively. By assumption, $W_n(r_n)$ is a neighbourhood of a direct non-logarithmic singularity, so we can use Corollary 3.1 to choose a transcendental singularity with projection a_{n+1} say and with neighbourhoods $W_{n+1}(r)$, $r > 0$, such that $W_{n+1}(r'_{n+1}) \subset W_n(r_n)$, for some $r'_{n+1} > 0$, and such that $0 < |a_{n+1} - a_n| < r_n/2$.

Next, we choose $r''_{n+1} > 0$ such that $W_{n+1}(r''_{n+1}) \cap B(0, R_{n+1}) = \emptyset$. By assumption $W_{n+1}(r'_{n+1})$ is a neighbourhood of a direct singularity with projection a_{n+1}. Hence, there exists r_{n+1} with

$$0 < r_{n+1} < \min\{r'_{n+1}, r''_{n+1}, |a_{n+1} - a_n|/4\}$$

such that $f(z) = a_{n+1}$ has no solutions for $z \in W_{n+1}(r_{n+1})$. Finally, both $\overline{B(a_{n+1}, r_{n+1})} \subset B(a_n, r_n)$ and $a_{n+1} \neq a$, by the choice of a_{n+1} and r_{n+1}. This completes the construction.

Let $a' = \lim_{n\to\infty} a_n$, which exists by our choice of r_n. Note that, by construction, $a' \neq a$. Let Γ be a curve produced inductively by joining a point in $W_n(r_n)$ to a point in $W_{n+1}(r_{n+1})$ using a curve lying in $W_n(r_n)$. By construction, Γ is an asymptotic curve with asymptotic value a'. Hence a' is the projection of a transcendental singularity of f^{-1} which, by assumption, is direct non-logarithmic.

Choose $r > 0$ sufficiently small that $D = B(a', r) \subset B(a, r_0)$, and such that $f(z) = a'$ has no solutions in W, where W is the component of $f^{-1}(D)$ which has unbounded intersection with Γ.

Now, by construction, $a' \in B(a_n, r_n)$, for $n \in \mathbb{N}$, and so there is an $N > 0$ such that $a' \in B(a_N, r_N) \subset D$ and also $a' \neq a_N$. Note that $W_N(r_N) \subset W$, since the intersection of Γ and $W_N(r_N)$ is unbounded. Then a' and a_N are two distinct points in $B(a_N, r_N)$ such that $f(z) \in \{a', a_N\}$ has no solutions in $W_N(r_N)$, which is contrary to Theorem 3.1. □

4. The Eremenko-Lyubich Class

In this section we prove Theorem 1.1. We need the following, [7, Theorem I.2.2].

Theorem 4.1. *Suppose that $W \subset \widehat{\mathbb{C}}$ is simply connected and ∂W has more than one point. Suppose that ψ maps W conformally to $\mathbb{D}$. Let Γ be a Jordan arc in W with endpoint $z_0 \in \partial W$. Then the curve $\psi(\Gamma)$ terminates in a point $s_0 \in \partial\mathbb{D}$, and $\psi^{-1}(s) \to z_0$ as $s \to s_0$ inside any Stolz angle at s_0.*

Here a *Stolz angle* at $s_0 \in \partial\mathbb{D}$ is a set of the form;

$$\{s \in \mathbb{D} : |\arg(1 - \overline{s_0}s)| < \alpha, |s - s_0| < d\}, \quad \text{for } 0 < \alpha < \pi/2,\ d < 2\cos\alpha.$$

We also need the following result, which is a version of [2, Theorem 1] that includes some assertions that appear only in the proof of that result; see also, [22, Theorem 6.2.3].

Theorem 4.2. *Suppose that f is a transcendental entire function with an indirect singularity with projection $a \in \widehat{\mathbb{C}}$. Suppose that a is not the limit of critical values of f. Then there exists a sequence of asymptotic values (a_n), which converge to a, a sequence of disjoint unbounded simply connected domains (U_n) such that $D_n = f(U_n)$ is a disc with $a_n \in \partial D_n$, and a sequence of asymptotic curves (Γ_n) such that $\Gamma_n \subset U_n$, $f(\Gamma_n)$ is a radius of D_n ending at a_n, and f is univalent in U_n.*

Finally, we need the following lemma.

Lemma 4.1. *Let f be a transcendental entire function. Suppose that for every $R > 0$ there exist $r > 0$, $a_0 \in \mathbb{C}$ with $|a_0| > R$, an asymptotic curve Γ' with asymptotic value a_0, W a simply connected neighbourhood of Γ', and an analytic map ϕ, univalent on W, such that $\phi(\Gamma')$ is an interval $(-\infty, x_0)$, and*

$$f(z) = re^{\phi(z)} + a_0, \quad \text{for } z \in W. \tag{3}$$

Then $\eta_f = 0$.

Proof. Suppose that $\eta_f \neq 0$. Then there exist $\epsilon, R > 0$ such that

$$\left| z\frac{f'(z)}{f(z)} \right| > \epsilon, \quad \text{for } |f(z)| > R. \tag{4}$$

Choose a_0 such that $|a_0| > 2R$, let $h = \phi^{-1}$ and put $t = \phi(z)$. Then, as $z \to \infty$ along Γ', by (3) and (4),

$$\epsilon < \left| z \frac{\phi'(z) r e^{\phi(z)}}{r e^{\phi(z)} + a_0} \right| = \left| h(t) \frac{\phi'(h(t)) r e^t}{r e^t + a_0} \right| = \left| \frac{h(t) r e^t}{h'(t)(r e^t + a_0)} \right| \sim \left| \frac{h(t) r e^t}{h'(t) a_0} \right|. \tag{5}$$

Hence, for sufficiently large negative values of t,

$$\left| \frac{h'(t)}{h(t)} \right| < \frac{2 r e^t}{\epsilon |a_0|}. \tag{6}$$

Without loss of generality we can assume that (6) applies for all $t \in (-\infty, x_0)$. Since W is simply connected, and since we can assume that $0 \notin W$, we can define a branch of the logarithm, L, in W. Then, by (6),

$$\left| \frac{d}{dt} L(h(t)) \right| < \frac{2 r e^t}{\epsilon |a_0|}. \tag{7}$$

We set $\zeta = L(h(t))$ and integrate (7), to obtain

$$\frac{2r}{\epsilon |a_0|} \int_{-\infty}^{x_0} e^t \, dt > \int_{-\infty}^{x_0} \left| \frac{d}{dt} L(h(t)) \right| \, dt > \left| \int_{-\infty}^{x_0} \frac{d}{dt} L(h(t)) \, dt \right| = \left| \int_{L(\Gamma')} d\zeta \right|. \tag{8}$$

Now, $L(\Gamma')$ is an unbounded curve, and so the right-hand side of (8) is unbounded. However, the left-hand side of (8) is bounded. This contradiction completes the proof. □

We now prove Theorem 1.1.

Proof of Theorem 1.1. As mentioned in the introduction, it is clear that if $f \in \mathcal{B}$ then $\eta_f = \infty$. Suppose, then, that $\eta_f \neq 0$. It is immediate from (2) that the set of critical values of f is bounded. To complete the proof, we show that f cannot have an unbounded set of finite asymptotic values, and so $f \in \mathcal{B}$, and hence $\eta_f = \infty$. To achieve this we show first that f cannot have an unbounded set of projections of logarithmic singularities. We then show that f cannot have an unbounded set of projections of indirect singularities. Finally, we show that f cannot have an unbounded set of projections of direct non-logarithmic singularities.

Our first claim then is that f cannot have an unbounded set of projections of logarithmic singularities. Suppose that, for every $R > 0$, f has a logarithmic singularity with projection $a_0 \in \mathbb{C}$, such that $|a_0| > R$. Noting that a_0 is finite, we apply Corollary 3.2 to obtain a simply connected neighbourhood, $W = U(r)$, of the singularity, and a conformal map $\phi : W \to \mathbb{H}$ such that (3) holds for some $r > 0$. Let Γ be an asymptotic curve in W associated with the logarithmic singularity.

Put $t = \phi(z)$ and let

$$s = \sigma(t) = \frac{1-t}{1+t}.$$

Then $\psi = \sigma \circ \phi$ is a conformal mapping of W to $\mathbb{D}$. Moreover $\psi(\Gamma)$ is a curve in $\mathbb{D}$ tending to -1.

We now construct another curve to ∞ in W which satisfies the hypotheses of Lemma 4.1. Let $\Gamma' = \phi^{-1}((-\infty, 0))$. Then $\gamma = \psi(\Gamma')$ is a curve in $\mathbb{D}$ tending to -1 within a Stolz angle. By Theorem 4.1, $\psi^{-1}(s) \to \infty$ as $s \to -1$ along γ, and so Γ' is an asymptotic curve. Moreover, $re^t + a_0 \to a_0$ as $t \to -\infty$ along $\phi(\Gamma')$, and so Γ' has asymptotic value a_0. A contradiction follow by Lemma 4.1, since we have assumed that $|a_0| > R$. This establishes our initial claim.

We next show that f cannot have an unbounded set of projections of indirect singularities. Suppose that, for every $R > 0$, f has an indirect singularity with projection $a \in \mathbb{C}$, such that $|a| > 2R$. By Theorem 4.2, f has an asymptotic value a_0, with $|a_0| > R$, an asymptotic curve Γ' associated with a_0, and an unbounded simply connected domain W containing Γ' such that f is univalent in W. Moreover, $f(W)$ is a disc, D, with $a_0 \in \partial D$, and $f(\Gamma')$ is a radius in D ending at a_0.

Without loss of generality, by composing with a rotation if necessary, assume that the centre of D is at $a_0 + e^{x_0}$, for some $x_0 \in \mathbb{R}$. Define a branch of the logarithm, L_1, such that $\psi(w) = L_1(w - a_0)$ is a univalent map on D. Let ϕ be the univalent map $\phi = \psi \circ f$. Note that $\phi(\Gamma') = (-\infty, x_0)$, and (3) holds with $r = 1$. A contradiction follows by Lemma 4.1, since we have assumed that $|a_0| > R$. This establishes our second claim.

Finally we show that the projections of direct non-logarithmic singularities are bounded. This follows immediately from the fact that the projections of other types of transcendental singularities are bounded and from Theorem 1.2. This completes the proof. □

Remark: It seems possible to generalise the result of Theorem 1.1 to transcendental meromorphic functions with direct tracts (see, for example, [4] for more background on this concept). We have not done this here, for reasons of simplicity. However, the proof seems to work similarly, although a number of results used in this paper need to be generalised. In addition, we need to replace Theorem 3.1 with [6, Corollary 1], and [8, Lemma 1] with [4, Lemma 6.3].

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